# On the forgetting of particle filters

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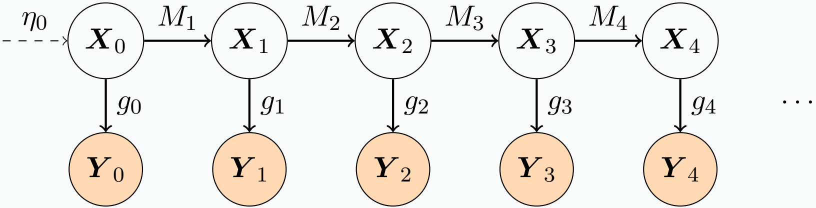
# Outline

# Introduction: Hidden Markov Model and filtering

# Hidden Markov model (a.k.a. 'state-space model')




- Hidden (unobserved) Markov state process $(\boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{X}_3, \ldots)$:
  - Initial density $\eta_0(\boldsymbol{x}_0)$
  - Transition densities $M_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$
- Observations (or measurements) $(\boldsymbol{Y}_0, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots)$:
  - Conditionally independent given $(\boldsymbol{X}_t)_{t\geq 0}$
  - Observation densities $g_t(\boldsymbol{y}_t \mid \boldsymbol{x}_t)$
  - Observed values $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots \rightsquigarrow$ potentials $G_t(\boldsymbol{x}_t) = g_k(\boldsymbol{y}_t \mid \boldsymbol{x}_t)$

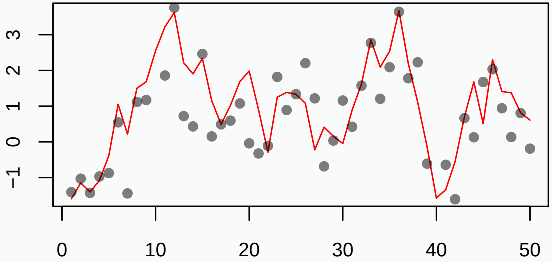
# Running example: Noisy AR(1)

$X_{0:T}$ stationary AR(1) process:

- $X_1 \sim N(0, \sigma_\eta^2/(1-\phi^2))$.
- $X_k = \phi X_{k-1} + \eta_k;\ \eta_k \sim N(0, \sigma_\eta^2)$

$Y_{0:T-1}$ noisy observations of $X_{1:T-1}$:

- $Y_k \sim N(X_k, \sigma_Y^2)$
- $G_k(x_k) = c_Y \exp\left(-\frac{1}{2\sigma_Y^2}(x_k - y_k)^2\right)$



# The filtering problem

- We are interested in:
  - predictive distributions $\eta_t = \mathrm{Law}(\boldsymbol{X}_t \mid \boldsymbol{y}_{0:t-1})$
  - filtering distributions $\pi_t = \mathrm{Law}(\boldsymbol{X}_t \mid \boldsymbol{y}_{0:t})$
- Practical example: GPS navigation
  - varying quality GPS observations $\rightsquigarrow G_t$
  - simple model of movement such as Brownian velocity $\rightsquigarrow M_t$
- Cannot determine $\eta_t$ and $\pi_t$ in a closed form

  (essentially unless $M_t$ and $G_t$ linear-Gaussian or state-space finite)
- We focus on Monte Carlo approximations, that is, sampling number of *particles* $\boldsymbol{X}_t^{1:N}$ "approximately from $\eta_t$", iteratively in $t = 0, 1, 2, \ldots$

# The ideal filter

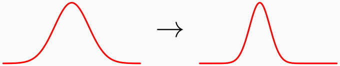
- Update $\color{red}{\eta_t} \to \color{blue}{\pi_t}$ by weighting with $G_t$:

$$\pi_t(\boldsymbol{x}_t) = \Psi_t(\eta_t)(\boldsymbol{x}_t) \qquad \text{where} \qquad \Psi_t(\mu)(\boldsymbol{x}) = \frac{\mu(\boldsymbol{x})G_t(\boldsymbol{x})}{\int G_t(\boldsymbol{z})\mu(\boldsymbol{z})\mathrm{d}\boldsymbol{z}}$$

- Mutate $\color{blue}{\pi_t} \to \color{red}{\eta_{t+1}}$ by pushing through $M_{t+1}$:

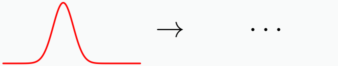
$$\eta_{t+1}(\boldsymbol{x}_{t+1}) = (\pi_t M_{t+1})(\boldsymbol{x}_{t+1}) = \int M_{t+1}(\boldsymbol{x}_{t+1} \mid \boldsymbol{x}_t)\pi_t(\boldsymbol{x}_t)\mathrm{d}\boldsymbol{x}_t$$

- We denote the composition of the above by $\Phi_t$:



$$\eta_{t+1} = \Phi_{t+1}(\eta_t) = \Psi_t(\eta_t)M_{t+1}$$

and compositions of these by $\Phi_{t,u}$:



$$\eta_u = \Phi_{t,u}(\eta_t) = \Phi_u \circ \cdots \circ \Phi_{t+1}(\eta_t)$$

# Particle filter

# Particle filter algorithm

Gordon, Salmond and Smith (*IEE Proc. F*, 1993)

PF($\eta_0, (M_t)_{t\geq 1}, (G_t)_{t\geq 0}, N$)

$X_0^i \sim \eta_0(\,\cdot\,)$    for $i \in \{1{:}N\}$

for $t = 1, 2, \ldots$ do

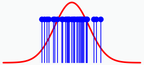
$\quad W_{t-1}^i = \dfrac{G_{t-1}(\boldsymbol{X}_{t-1}^i)}{\sum_{j=1}^N G_{t-1}(\boldsymbol{X}_{t-1}^j)}$    for $i \in \{1{:}N\}$

$\quad A_{t-1}^i \sim \mathrm{Categorical}\big(W_{t-1}^{1:N}\big)$    for $i \in \{1{:}N\}$

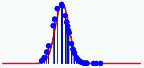
$\quad \boldsymbol{X}_t^i \sim M_t(\,\cdot \mid \boldsymbol{X}_{t-1}^{A_{t-1}^i})$    for $i \in \{1{:}N\}$

end for

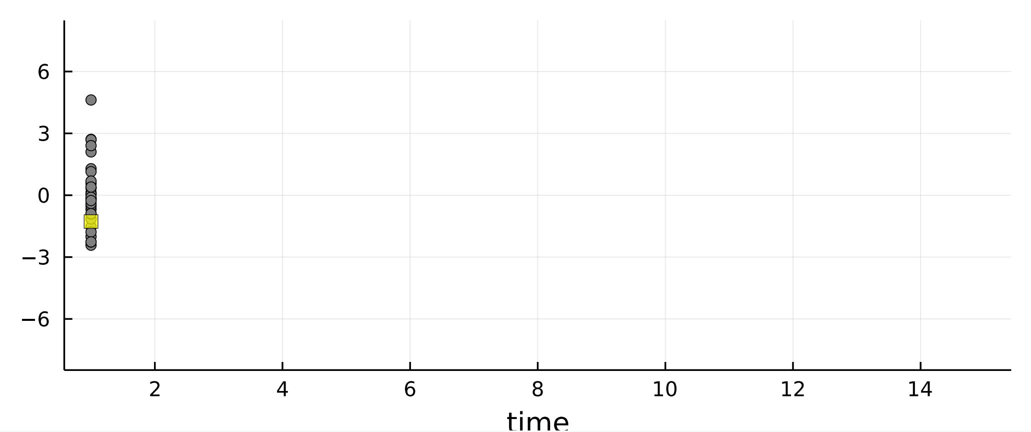
Produces empirical approximations of $\eta_t$ and $\pi_t$:



$$\eta_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\boldsymbol{X}_t^i} \qquad \text{and} \qquad \pi_t^N = \Psi_t(\eta_t^N) = \sum_{i=1}^N W_t^i \delta_{\boldsymbol{X}_t^i}$$

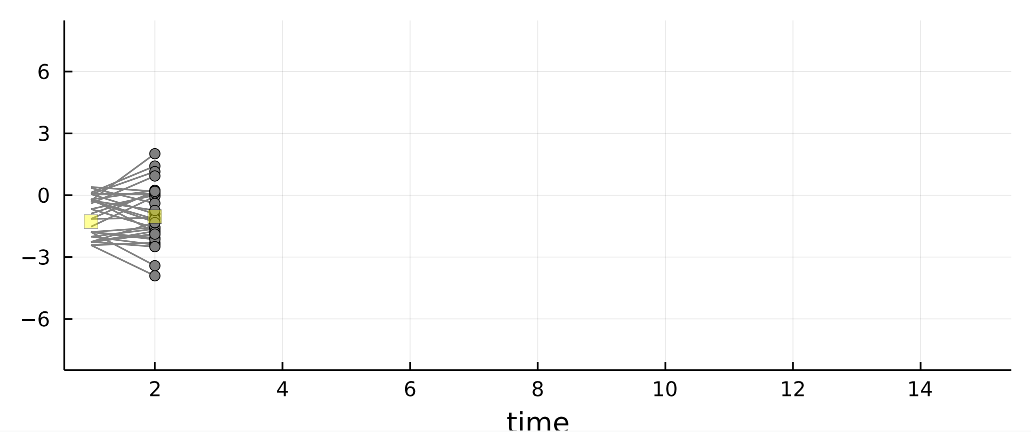


# Particle filter on noisy AR(1): Initialise



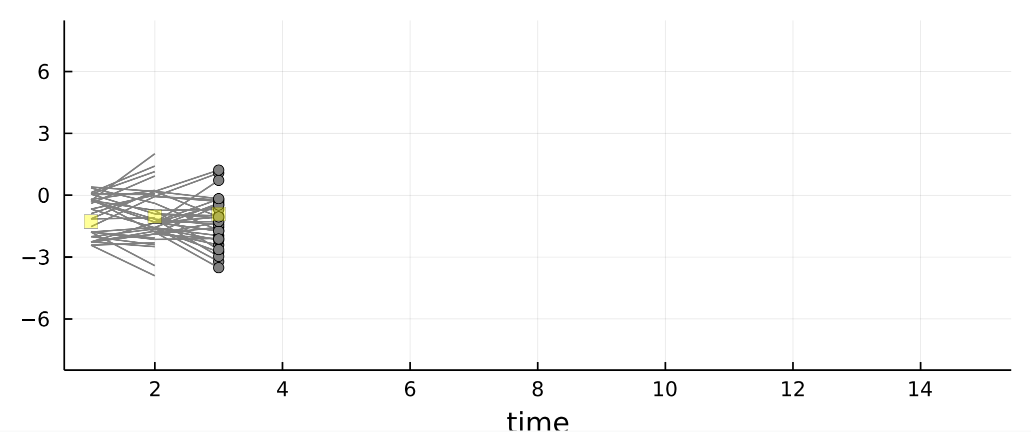

- ● Particles $X_1^{1:N} \sim M_1(\,\cdot\,)$
- ■ Observation $y_1$

# Particle filter on noisy AR(1): Resample and propagate




- ● Particles $X_t^i \sim M_t(X_{t-1}^{A_{t-1}^i}, \cdot\,)$
- ■ Observations $y_{0:t}$

# Particle filter on noisy AR(1): Resample and propagate

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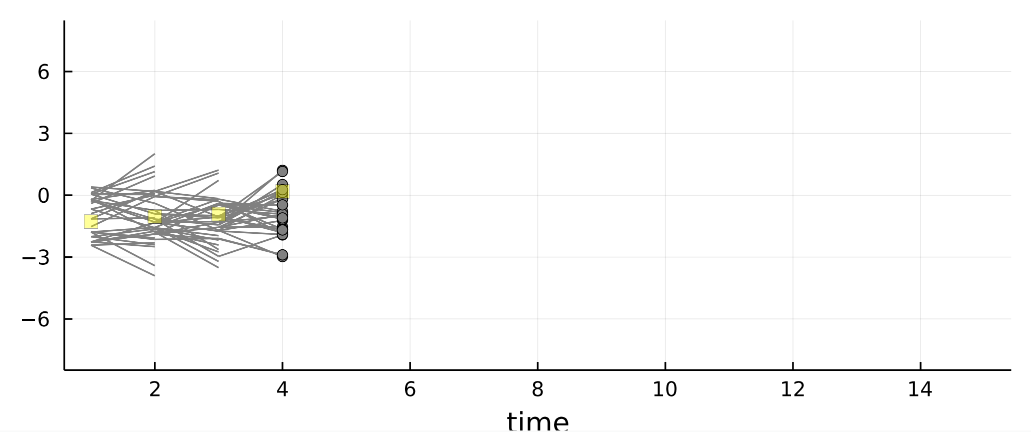
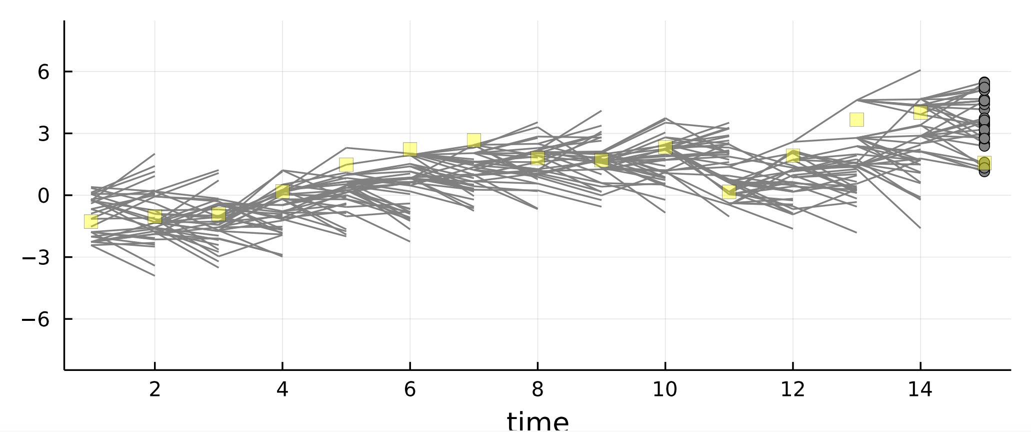



- ● Particles $X_t^i \sim M_t(X_{t-1}^{A_{t-1}^i}, \cdot\,)$
- ■ Observations $y_{0:t}$

# Particle filter on noisy AR(1): $t = 15$




- ● Particles $X_{15}^{1:N}$
- ■ Observations $y_{0:15}$

# Forgetting

# Strong mixing condition

We assume the following which is common in (quantitative) particle filter theory:

**Assumption: Strong mixing**

There exist $0 < \underline{M} \leq \overline{M} < \infty$ and $0 < \underline{G} \leq \overline{G} < \infty$ such that $\forall \boldsymbol{x}, \boldsymbol{x}', t$:

- $\underline{M} \leq M_t(\boldsymbol{x}, \boldsymbol{x}') \leq \overline{M}$
- $\underline{G} \leq G_t(\boldsymbol{x}) \leq \overline{G}$

- Typically holds only in a compact state space
- (Variants exist which e.g. extend the requirement for iterates of $M_t$)

# Ideal filter forgetting

**Theorem (Del Moral 2004, Proposition 4.3.6)**

For all probability measures $\mu$ and $\nu$, $t \geq 0$ and $k \geq 0$:

$$\sup_{\mu,\nu} \|\Phi_{t,t+k}(\mu) - \Phi_{t,t+k}(\nu)\|_{\mathrm{TV}} \leq \beta^k \qquad \text{where} \qquad \beta = 1 - \left(\frac{\underline{\underline{M}}}{\overline{\overline{M}}}\right)^2$$

- Ideal filter forgets at exponential rate
- NB: $\Phi_t$ is non-linear and generally not contracting: we might well have

$$\|\Phi_t(\mu) - \Phi_t(\nu)\|_{\mathrm{TV}} > \|\mu - \nu\|_{\mathrm{TV}}$$

# Time-uniform $L^p$ errors

**Theorem (Del Moral 2004, Theorem 7.4.4)**

For all $N \geq 2$, $n \geq 0$ and $p \geq 1$, $\operatorname{osc}(\phi) \leq 1$:

$$\|\eta_t^N(\phi) - \eta_t(\phi)\|_p \leq \frac{c}{\sqrt{N}} \qquad \text{where} \qquad c_p = 2d_p^{1/p}\left(\frac{\overline{M}}{\underline{M}}\right)^3 \frac{\overline{G}}{\underline{G}}$$

where $d_p$ has been defined in Del Moral (2004), and in particular, $d_2 = 1$.

- $\eta_t^N(\phi) = N^{-1}\sum_{i=1}^N \phi(\boldsymbol{X}_t^i) \approx \eta_t(\phi) = \int \phi(\boldsymbol{x}_t)\eta_t(\boldsymbol{x}_t)\mathrm{d}\boldsymbol{x}_t$ for large $N$ uniform in $t$ (in $L^p$ sense)

$\therefore$ Monte Carlo errors do not accumulate $\rightsquigarrow$ stability

# Forgetting of the particle filter?

In summary:

- Ideal filter is exponentially forgetting
- Particle filter is increasingly accurate approximation of the ideal filter...
- ...in a time-uniform manner

So the particle filter must also be exponentially forgetting, at least if $N$ is large enough?

# Particle filter as a Markov chain

- Unlike the ideal filter, particle filter defines a Markov chain $(\boldsymbol{X}_t^{1:N})_{t\geq 0}$
- Denote its Markov transition

$$\mathbf{M}_t(\boldsymbol{x}_{t-1}^{1:N}, \,\cdot\,) = \left( \sum_{i=1}^{N} \frac{G_{t-1}(\boldsymbol{x}_{t-1}^i)}{\sum_{j=1}^N G_{t-1}(\boldsymbol{x}_{t-1}^j)} M_t(\,\cdot\, \mid \boldsymbol{x}_{t-1}^i) \right)^{\otimes N}$$

- Is $\mathbf{M}_t$ contracting in Dobrushin sense:

$$\beta_{\mathrm{TV}}(\mathbf{M}_t) = \sup_{\boldsymbol{x}^{1:N}, \tilde{\boldsymbol{x}}^{1:N}} \|\mathbf{M}_t(\boldsymbol{x}^{1:N}, \,\cdot\,) - \mathbf{M}_t(\tilde{\boldsymbol{x}}^{1:N}, \,\cdot\,)\|_{\mathrm{TV}} < 1?$$

# An earlier forgetting result

Yes, $\mathbf{M}_t$ are contracting:

**Lemma (Tadić & Doucet, 2021)**

*For all $N \geq 1$ and $t \geq 0$:*

$$\beta_{\mathrm{TV}}(\mathbf{M}_t) \leq 1 - \epsilon^N, \qquad \textit{where} \qquad \epsilon = \left(\frac{\underline{M}}{\overline{\overline{M}}}\right)^2$$

Direct corollary of the above:

$$\beta_{\mathrm{TV}}\left(\mathbf{M}_{t,t+k}\right) \leq (1 - \epsilon^N)^k \qquad \text{where} \qquad \mathbf{M}_{t,t+k} = \mathbf{M}_{t+1}\mathbf{M}_{t+2}\cdots\mathbf{M}_{t+k}$$

$\rightsquigarrow$ forgetting in $k = O(e^N)$ time 🤔

# Forgetting result

# Particle filter forgetting

**Theorem (Karjalainen, Lee, Singh & V (2023))**

*For all $k \geq 1, t \geq 0, N \geq 2$,*

$$\beta_{\mathrm{TV}}(\mathbf{M}_{t,t+k}) \leq (1-\varepsilon)^{\lfloor k/(c \log N) \rfloor},$$

*where $\varepsilon \in (0,1)$ and $c < \infty$ only depend on the strong mixing constants.*

$\rightsquigarrow$ PF forgets in $k = O(\log N)$ time 😀

- Seems like the right order: a specific example where forgetting $\Omega(\log N)$...

# Proof sketch 1: Hellinger distance

## Definition

The squared Hellinger distance between two probability measures $P$ and $Q$ having densities $p$ and $q$ with respect to a common dominating measure $\lambda$ is

$$H^2(P,Q) = \frac{1}{2}\int \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 \lambda(\mathrm{d}x) = 1 - \int \sqrt{p(x)q(x)}\lambda(\mathrm{d}x)$$

## Lemma (Tensorisation)

$$1 - H^2(P^{\otimes N}, Q^{\otimes N}) = \left(1 - H^2(P,Q)\right)^N$$

# Proof sketch 2: Total variation & random measures

**Lemma (Le Cam's inequality)**

*For any probability measures $\mu$ and $\nu$ it holds that*

$$\|\mu - \nu\|_{\mathrm{TV}} \leq \sqrt{1 - (1 - H^2(\mu,\nu))^2}$$

Using tensorisation & Jensen's inequality:

**Corollary**

*For random product measures*

$$\|\mathbb{E}\mu^{\otimes N} - \mathbb{E}\nu^{\otimes N}\|_{\mathrm{TV}} \leq \sqrt{1 - \left(1 - \mathbb{E}H^2(\mu,\nu)\right)^{2N}}$$

# Proof sketch 3: Bounding expected Hellinger with $L^2$

**Lemma**

*Let $\mu$ and $\nu$ be two random probability measures, then*

$$\mathbb{E}H^2(\mu M, \nu M) \leq c' \sup_{\operatorname{osc}(\phi)\leq 1} \mathbb{E}(|\mu(\phi) - \nu(\phi)|^2) \qquad \textit{where} \qquad c' = \frac{1}{8}\left(\frac{\overline{M}}{\underline{M}}\right)^2$$

# Proof sketch 4: Putting things together

Let $\pi_t^N$ and $\tilde{\pi}_t^N$ be particle filters with same $M_t, G_t$ but with initial $\boldsymbol{X}_0^{1:N}$ and $\tilde{\boldsymbol{X}}_0^{1:N}$

$$\begin{aligned}\beta_{\mathrm{TV}}(\mathbf{M}_{0,k}) &= \inf_{\boldsymbol{X}_0^{1:N}, \tilde{\boldsymbol{X}}_0^{1:N}} \|\mathbb{E}(\pi_{k-1}^N M_k)^{\otimes N} - \mathbb{E}(\pi_{k-1}^N M_k)^{\otimes N}\|_{\mathrm{TV}} \\ &\leq \left(1 - \left(1 - c' \sup_{\mathrm{osc}(\phi)\leq 1} \|\pi_{k-1}^N(\phi) - \tilde{\pi}_{k-1}^N(\phi)\|_2^2\right)^{2N}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|\pi_{k-1}^N(\phi) - \tilde{\pi}_{k-1}^N(\phi)\|_2 \leq & \|\pi_{k-1}^N(\phi) - \pi_{k-1}(\phi)\|_2 && \leq cN^{-1/2} \\ & + |\pi_{k-1}(\phi) - \tilde{\pi}_{k-1}(\phi)| && \leq \beta^{k-2} \leq cN^{-1/2} \text{ if } k \geq c\log N \\ & + \|\tilde{\pi}_{t-1}(\phi) - \tilde{\pi}_{t-1}^N(\phi)\|_2 && \leq cN^{-1/2}\end{aligned}$$

in which case

$$\beta_{\mathrm{TV}}(\mathbf{M}_{0,k}) \leq \left(1 - \left(1 - \frac{c}{N}\right)^{2N}\right)^{1/2} \approx \left(1 - e^{-2c}\right)^{1/2} \leq 1 - \epsilon \qquad \square$$

# Conditional particle filter

# Conditional particle filter algorithm

Andrieu, Doucet & Holenstein (*JRSS B*, 2010)

**CPF**$(\eta_0, (M_t)_{t\geq 1}, (G_t)_{t\geq 0}, (\boldsymbol{x}_t^*)_{t\geq 0}, N)$

1: $\boldsymbol{X}_t^0 = \boldsymbol{x}_t^*$, for $t \geq 0$
2: $\boldsymbol{X}_0^i \sim \eta_0(\,\cdot\,)$ for $i \in \{1{:}N\}$
3: **for** $t = 1, 2, \ldots$ **do**
4: $\quad W_{t-1}^i = \dfrac{G_{t-1}(\boldsymbol{X}_{t-1}^i)}{\sum_{j=1}^N G_{t-1}(\boldsymbol{X}_{t-1}^j)}$ for $i \in \{0{:}N\}$
5: $\quad A_{t-1}^i \sim \mathrm{Categorical}\big(W_{t-1}^{0:N}\big)$ for $i \in \{1{:}N\}$
6: $\quad \boldsymbol{X}_t^i \sim M_t(\,\cdot \mid \boldsymbol{X}_{t-1}^{A_{t-1}^i})$ for $i \in \{1{:}N\}$
7: **end for**

- Used for smoothing, but we think of CPF as a perturbed particle filter

⇝ Empirical approximations of $\eta_t$ and $\pi_t$:

$$\hat{\eta}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\boldsymbol{X}_t^i} \qquad \text{and} \qquad \hat{\pi}_t^N = \sum_{i=1}^N W_t^i \delta_{\boldsymbol{X}_t^i}$$

# Time-uniform $L^p$ errors for CPF

**Theorem (Karjalainen, Lee, Singh & V, 2023)**

*For every $p \geq 1$, there exists a constant $c = c(p)$ depending on strong mixing constants, such that for all $\mathrm{osc}(\phi) \leq 1$, $t \geq 0$, $N \geq 1$ and $x_t^*$,*

$$\left\| \hat{\pi}_t^N(\phi) - \pi_t(\phi) \right\|_p \leq \frac{c}{\sqrt{N}}$$

- Monte Carlo errors do not accumulate
- The effect of reference $x_t^*$ remains limited, too

# Forgetting of CPF

**Theorem**

*For all* $k \geq 1, t \geq 0, N \geq 2$, *and references* $x^* = (x_t^*)_{t \geq 0}$

$$\beta_{\mathrm{TV}}(\mathbf{M}_{t,t+k}^{x^*}) \leq (1 - \varepsilon)^{\lfloor k/(c \log N) \rfloor},$$

*where* $\varepsilon$ *and* $c$ *only depend on the strong mixing constants.*

- Similar proof as for PF, using the time-uniform $L^p$ error result

# Concluding remarks

# Maximal coupling of particle filters

- Coupled particle filters (and CPFs) have been used recently for multilevel Monte Carlo (and unbiased estimation)
- Let $\mu_k = \Phi_t(\eta_k^N)$ and $\tilde{\mu}_k = \Phi_t(\tilde{\eta}_k^N)$ stand for the conditional distributions of $X_t^{1:N}$ and $\tilde{X}_t^{1:N}$, respectively.
- Jasra & Yu (2020) suggested an 'independent maximal coupling' (IMC) algorithm:

$$(X_k^i, \tilde{X}_k^i) \sim \textsc{MaxCouple}(\mu_k, \tilde{\mu}_k), \quad \text{independently for } i = 1, \ldots, N$$

- Our analysis suggests another 'joint maximal coupling' (JMC) algorithm:

$$(X_k^{1:N}, \tilde{X}_k^{1:N}) \sim \textsc{MaxCouple}(\mu_k^{\otimes N}, \tilde{\mu}_k^{\otimes N})$$

- Both are implementable, but have $O(N^2)$ complexity

# Coupling probability when $N$ increases

- Suppose $X_t^{1:N}$ and $\tilde{X}_t^{1:N}$ follow same dynamics, but have $\eta_0 \neq \tilde{\eta}_0$
- Suppose that $k > c \log N$
- Our analysis says that with JMC, for all $N \geq 1$:

$$\mathbb{P}(X_t^{1:N} \neq \tilde{X}_t^{1:N}) = \mathbb{E}\|\mu_k - \tilde{\mu}_k\|_{\mathrm{TV}} \leq \left(1 - \left(1 - \frac{c}{N}\right)^{2N}\right)^{1/2} \leq \epsilon$$

$\rightsquigarrow$ filters fully coupled after $O(\log N)$ iterations.

- In contrast, with IMC, we can only guarantee:

$$\mathbb{P}(X_t^{1:N} = \tilde{X}_t^{1:N}) = \mathbb{E}\left[(1 - \|\mu_k - \tilde{\mu}_k\|_{\mathrm{TV}})^N\right] \geq \left(1 - \frac{c}{\sqrt{N}}\right)^N,$$

which vanishes as $N \to \infty$

# Discussion

- Dobrushin forgetting is a strong form of stability
  - Complementary to $L^p$ uniform stability, which is about averages
  - Could prove useful in some PF/CPF analysis
- Forgetting entails $\log N$ ‘penalty’ term
  - Seems unavoidable
  - Drowned by $N^{-p}$ where $p > 0$
- Implications to CPF out soon...

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Thank you!