TEACHING NOTES

THEORY OF QUADRATIC EQUATIONS / INEQUATIONS / ALGEBRAIC EQUATIONS

Syllabus IIT JEE : Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.

1^{ST} LECTURE :

QUADRATIC POLYNOMIAL :

A polynomial of degree two in one variable of the type

$y = ax^2 + bx + c$ where $a \neq 0$, a, b, $c \in R$ is called a quadratic polynomial, where

a = leading coefficient of the trinomial

c = absolute term of the trinomial

In case $a = 0$, $y = bx + c$, is called a linear polynomial. ($b \neq 0$)

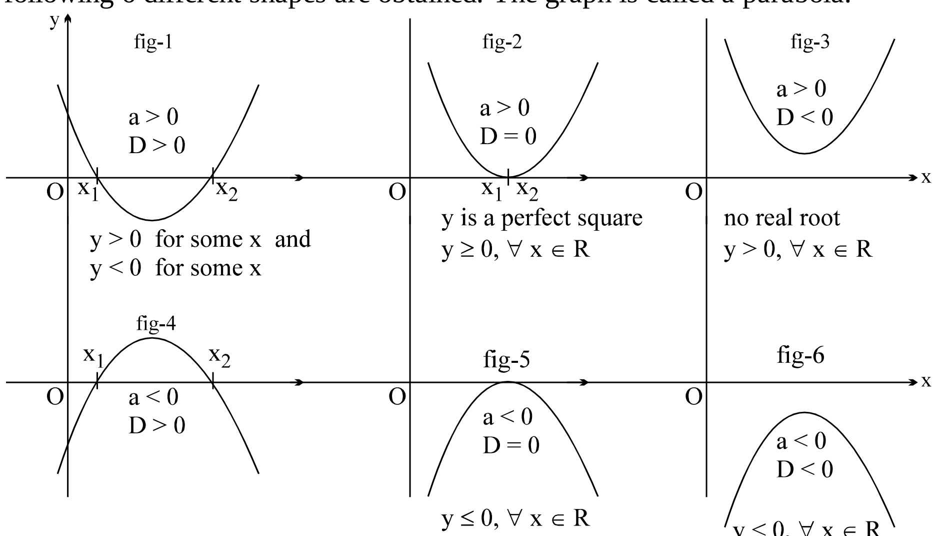
If $c = 0$ then $y = bx$ is called an odd linear polynomial

The standard appearance of a polynomial of degree n is

$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_0$ when $a_n \neq 0$ and $a_{i's} = \in R$

(**note that** a polynomial of degree 3 is called a cubic and of degree 4 is called a biquadratic polynomial)

Now for different values of a, b, c, if graph of $y = ax^2 + bx + c$ is plotted then the following 6 different shapes are obtained. The graph is called a parabola.



EXPLANATION OF ABOVE GRAPHS :

The first 3 figures are obtained when a > 0. Here the mouth of the parabola opens upwards. Shape resembles that of a cup, filling water or sign of union. Last 3 graphs are obtained when a < 0. Here the mouth of the parabola opens downwards. Shape resembles that of a cup, spilling water or sign of intersection.

Figure-1 is indicative that there are two values of x for which the value of y is zero. These values $x = x_1$ or $x = x_2$ are called the zeroes of the polynomial.

Note that for $x > x_2$ or $x < x_1$, y is positive whereas for $x_1 < x < x_2$, y is negative. In this case y can take both positive and negative values.

In figure-2 the curve touches the axis of x. Here both zeroes of the polynomial coincide.

Note that in this case the value of y is always non negative for all $x \in R$.

In figure-3 the curve completely lies above the x-axis. There is no real zero and the value of y is always greater than zero for all $x \in R$. This is an important case.

Similar explanation can be given for figure-4, 5 and 6.

Now a quadratic polynomial when equated to zero is called a quadratic equation i.e.

$ax^2 + bx + c = 0, \quad a \neq 0,\ a, b, c \in R$

Solving a quadratic equation would mean finding the value or values of x for which $ax^2 + bx + c$ vanishes and these values of x are also called the roots of the quadratic equation or zeroes of the corresponding quadratic polynomial. Two methods of solving a quadratic equation and

(1) graphical (not very useful) (2) algebraic

ALGEBRAIC METHOD :

$ax^2 + bx + c = 0 \qquad a \neq 0, \qquad a, b, c \in R$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad (a \neq 0)$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(Vietta's theorem)}$$

Hence $\alpha = \dfrac{-b + \sqrt{D}}{2a}$ and $\beta = \dfrac{-b - \sqrt{D}}{2a}$ where $D = b^2 - 4ac$

The quantity $D = b^2 - 4ac$ is called the discriminant of the quadratic equation and plays a very vital role in deciding the nature of roots of the equation without actually determining them. Now

If $D > 0$ then roots are real. (This corresponds to the figure-1 or figure-4 depending on the sign of 'a')

If $D = 0$ roots are coincident. (This corresponds to the figure-2 or figure-5)

If $D \geq 0$ roots are real

If $D < 0$ no real roots. Infact roots are complex conjugate. This corresponds to figure-3 or figure-6.

We therefore see that the condition for $y = ax^2 + bx + c$ to be + ve $\forall\ x \in R$ becomes $a > 0$ and $D < 0$. Similarly $ax^2 + bx + c < 0\ \forall\ x \in R$ if $a < 0$ and $D < 0$.

Note :

(1) In case the coefficient of quadratic equation are rational then the roots are rational if $D > 0$ and is a perfect square.

(2) Irrational roots occur in pair of conjugate surd i.e. if one root is $2 + \sqrt{3}$ the other is $2 - \sqrt{3}$.

(3) If coefficient of quadratic equation are real and one root is $\alpha + i\beta$ then other root is $\alpha - i\beta$.

RELATION BETWEEN ROOTS AND COEFFICIENT OF QUADRATIC EQUATION :

$ax^2 + bx + c = 0, \quad a \neq 0,\ a, b, c \in R$

If α, β are the roots then $\alpha + \beta = -\dfrac{b}{a}$; $\alpha\beta = \dfrac{c}{a}$

Hence we can form the quadratic equation if the sum and product of its roots are known

i.e. $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

Ex.1 Form a quadratic equation whose one root is

(a) $\cos 36°$ (b) $\tan\frac{\pi}{12}$ (c) $\tan\frac{\pi}{8}$ (d) $\cos^2\frac{\pi}{8}$

Note :

(a) If exactly one root of quadratic equation is 0, then $P = 0 \Rightarrow \frac{c}{a} = 0 \Rightarrow c = 0$ and the quadratic becomes $ax^2 + bx = 0$.

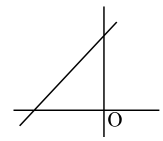
(b) If both roots of the quadratic equation are zero then $S = 0$ and $P = 0 \Rightarrow b = c = 0$ and the quadratic equation becomes $x^2 = 0$.

(c) If one root is ∞, put $x = \frac{1}{y}$ in $ax^2 + bx + c = 0$, we get

$cy^2 + by + a = 0$ must have one root zero $\Rightarrow P = 0$ i.e. $\frac{a}{c} = 0$

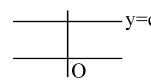
Hence , $a = 0$ and $-\frac{b}{c} \neq 0 \Rightarrow b \neq 0$.

original quadratic equation becomes $bx + c = 0$



(d) When both roots of the quadratic equation are infinity then. The quadratic equation $cy^2 + by + a = 0$ must have both roots zero.

i.e. $-\frac{b}{c} = 0$ and $\frac{a}{c} = 0 \Rightarrow b = 0 ; a = 0$ and $c \neq 0$.



In this case the equation becomes $y = c$.

e.g. if $(2p - q)x^2 + (p - 1)x + 5 = 0$ has both roots infinite. Find p and q.

[Ans. $p = 1 ; q = 2$]

(e) If $f(\alpha, \beta) = f(\beta, \alpha)$ then $f(\alpha, \beta)$ denotes symmetric functions of roots.

e.g. $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$; $f(\alpha, \beta) = \cos(\alpha - \beta)$

It is to be noted that every symmetric function in α, β can be expressed in terms of two symmetric functions $\alpha + \beta$ and $\alpha\beta$.

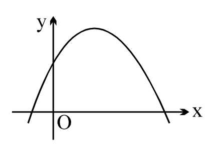
ILLUSTRATION FOR THE FIRST LECTURE :

1. Graph of $y = ax^2 + bx + c$ is as shown in the figure then

(i) $a < 0$ (ii) $D > 0$ (iii) $S > 0$ (iv) $P < 0$

(v) $-\frac{b}{a} > 0$ $(b > 0)$ (vi) $\frac{c}{a} < 0$ $(c > 0)$

(vii) b and c have the same sign and different than a.



2. The quadratic equation $ax^2 + bx + c = 0$ has no real root, then prove that

$c(a + b + c) > 0$

[Hint: both f (0) and f (1) have the same sign $\Rightarrow f(0) \cdot f(1) > 0 \Rightarrow$ result]

3. Find the set of values of 'a' for which the quadratic polynomial

(i) $(a + 4)x^2 - 2ax + 2a - 6 < 0 \ \forall \ x \in R$ [Ans. $(-\infty, -6)$]

(ii) $(a - 1)x^2 - (a + 1)x + (a + 1) > 0 \ \forall \ x \in R$ [Ans. $(5/3, \infty)$]

4. If α, β are the roots of the quadratic equation $x^2 - 2x + 5 = 0$ then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$.

[**Hint :** $\alpha^2 - 2\alpha + 5 = 0 \Rightarrow \alpha^3 + \alpha^2 - \alpha + 22 = 7$ & $\beta^2 - 2\beta + 5 = 0 \Rightarrow \beta^3 + 4\beta^2 - 7\beta + 35=5$]

5. If $f(x) = ax^2+bx+c > 0 \ \forall \ x \in R$ then prove that $g(x) = f(x) + f'(x) + f''(x) > 0$ $(\forall \ x \in R)$

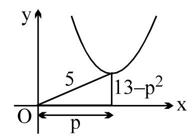
6. If $ax^2 + bx + c = 0 < \begin{matrix} x_1 \\ x_2 \end{matrix}$ then find the value of $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$.

7. If $p(q-r)x^2 + q(r-p)x + r(p-q) = 0$ has equal root prove that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$ (cyclic order) i.e. p, q, r are in H.P.

8. If a quadratic equation (in x or y) is formed from $y^2 = 4ax$ and $y = mx + c$ and has equal roots then prove that $c = a/m$.

9. If $x = 3 + \sqrt{5}$ find the value of $x^4 - 12x^3 + 44x^2 - 48x + 17$. [Ans. 1]

10. If the roots of the quadratic equation $(x-a)(x-b) - k = 0$ are c and d then prove that a and b are the roots of the quadratic equation $(x-c)(x-d) + K = 0$.



[Sol. $x^2 - (a+b)x + ab - K = 0 < \begin{matrix} c \\ d \end{matrix}$

$\therefore$ $c + d = a + b$
$cd = ab - k$ must be true

TPT $(x-c)(x-d) + K = 0 < \begin{matrix} a \\ b \end{matrix}$

$x^2 - (a+d)x + cd + K = 0$

$\therefore$ $a + b = c + d$ ....(1)
$ab = cd + K$ ....(2)

Hence Proved]

11. For what value of p the vertex of the parabola if $x^2 + 2px + 13$ lies at a distance of 5 unit from the origin.

12. Find the least value of the function $f(x) = 2bx^2 - x^4 - 3b^2$ in $[-2, 1]$ depending on the parameter b. [Ans. for $b \in (-\infty, 2]$ least value is $f(4) = 8b - 3b^2 - 16$;
for $b \in [2, \infty)$ least value is $f(0) = -3b^2$]

13. Find all numbers p for each of which the least value of the quadratic trinomial $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \le x \le 2$ is equal to 3.

[Ans. $p = 1 - \sqrt{2}$ or $5 + \sqrt{10}$]

14. If the roots of $ax^2 + 2bx + c = 0$ be possible and different, then the roots of

$$(a+c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$$

will be impossible, and vice versa.

[Hint: $4(ac - b^2)[(a-c)^2 + 4b^2]$]

Home work after 1st lecture :

Q. No. **26** to **71** Page **34** Prilepko + Exercise - 9 (a) of H & K.

2ND LECTURE

If a quadratic equation has more than 2 roots then it becomes an identity.

Proof : Let $ax^2 + bx + c = 0 < \begin{matrix} \alpha \\ \beta \\ \gamma \end{matrix}$

Hence $a\alpha^2 + b\alpha + c = 0$ ....(1)
$a\beta^2 + b\beta + c = 0$ ....(2)
$a\gamma^2 + b\gamma + c = 0$ ....(3)

From equation (1) and (2)

$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$

$\therefore$ $\alpha \ne \beta$,

hence $\quad a(\alpha + \beta) + b = 0 \quad ....(4)$

similarly $\quad a(\beta + \gamma) + b = 0 \quad ....(5)$

by subtraction, (4) and (5)

$a(\alpha - \gamma) = 0$

$\alpha \neq \gamma$, $\quad$ hence $a = 0$

if $\ a = 0 \ \Rightarrow \ b = 0 \ \Rightarrow \ c = 0$

Hence QE becomes $0x^2 + 0x + 0 = 0$ which is an identity.

EXAMPLES :

1. For what values of p, the equation

$(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$ has more than two roots.

2. $\dfrac{(x-a)(x-b)}{(c-a)(c-b)} + \dfrac{(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{(x-c)(x-a)}{(b-c)(b-a)} = 1$

3. $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2$

4. $\dfrac{(a+x)^2}{(a-b)(a-c)} + \dfrac{(b+x)^2}{(b-c)(b-a)} + \dfrac{(c+x)^2}{(c-a)(c-b)} = 1$

[Ans. $x = -a\ ; -b\ ; -c$ are the solutions]

Solving quadratic and rational inequalities. (Method of intervals)

Type-1 : Quadratic inequality involving non-repeated linear factors.

(1) $3x^2 - 7x + 6 < 0$ $\quad$ [Ans. $x \in \phi$]

(2) $(x^2 - x - 6)(x^2 + 6x) \geq 0$ (**Asking**)

(3) Solve $f'(x) \geq g'(x)$ where $f(x) = 5 - 3x + \dfrac{5}{2}x^2 - \dfrac{x^3}{3}$, $g(x) = 3x - 7$. [Ans. [2, 3]]

Type-2 : Quadratic inequality involving Repeated linear factos.

(1) $(x + 1)(x - 3)(x - 2)^2 \geq 0$. $\quad$ [Ans. $(-\infty, -1) \cup (3, \infty)$]

(2) $x(x + 6)(x + 2)^2(x - 3) > 0$ $\quad$ [Ans. $(-6, 0)(x+1)^3 - \{2\}$]

(3) $(x - 1)^2(x + 1)^3(x - 4) < 0$ $\quad$ [Ans. $(-1, 4) - \{1\}$]

(4) Number of positive integral solution of $\dfrac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \leq 0$

(A) only one $\quad$ (B) 2 $\quad$ (C*) 3 $\quad$ (D) 4

[Hint: $x \in \{1, 2, 4\}$]

Type-3 : Quadratic / algebraic inequality of the type of $\dfrac{f(x)}{g(x)}$. (Rational inequality)

(1) $\dfrac{2x-3}{3x-7}$ $\quad$ $[(-\infty, \tfrac{3}{2}) \cup (\tfrac{7}{3}, +\infty)]$ $\quad$ **(2)** $\dfrac{x^2-5x+12}{x^2-4x+5} > 3$ $\quad$ $[(\tfrac{1}{2}, 3)]$

(3) $\dfrac{x^2-5x+6}{x^2+x+1} < 0$ $\quad$ $[(2, 3)]$ $\quad$ **(4)** $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0$ $\quad$ $[(-1, 2) - \{0, 1\}]$

(5) $\dfrac{x+1}{x-1} \geq \dfrac{x+5}{x+1}$ $\quad$ $[(-\infty, -1) \cup (1, 3)]$ $\quad$ **(6)** $\dfrac{2(x-4)}{(x-1)(x-7)} \geq \dfrac{1}{x-2}$ $\quad$ $[(1, 2) \cup (7, +\infty)]$

(7) $\dfrac{x^2+6x-7}{|x+4|} < 0$ [(–7, –4) ∪ (– 4, 1)]

(8) $\dfrac{x^2+4x+4}{2x^2-x-1} > 0$ $\left[(-\infty, -2) \cup \left(-2, -\frac{1}{2}\right) \cup (-1, \infty)\right]$

Type-4 : Double inequality and biquadratic inequality.

(1) $1 < \dfrac{3x^2-7x+8}{x^2+1} \leq 2$

(2) For what value of 'x' $\sin^{-1}\left(\dfrac{3x^2-7x+8}{x^2+1}\right)$ is meaningful.

(3) $(x^2 + 3x + 1)(x^2 + 3x - 3) \geq 0$ [Ans. $(-\infty, -4] \cup [-2, -1] \cup [1, \infty)$]

(4) $(x^2 + 3x)(2x + 3) - 16\dfrac{(2x+3)}{(x^2+3x)} \geq 0.$

CONDITION OF COMMON ROOTS :

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root α.

Hence $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad \alpha = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Which is the required condition.

This is also the condition that the two quadratic functions $a_1x^2 + b_1xy + c_1y^2$ and $a_2x^2 + b_2xy + c_2y^2$ may have a common factor.

Note: If both roots of the given equations are common then $\dfrac{a_2}{a_1} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

EXAMPLES :

1. Find the value of k for which the equations $3x^2 + 4kx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ have a common root. [Ans. $k = \dfrac{7}{4}$ or $-\dfrac{11}{8}$]

[Hint: From the 2nd equation $x = \dfrac{1}{2}$ or $x = -2$. Either $x = \dfrac{1}{2}$ or $x = -2$ may be the common root]

2. If the quadratic equation $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ $(b \neq c)$ have a common root then prove that their uncommon roots are the roots of the equation $x^2 + x + bc = 0$

[Hint: $\alpha^2 + b\alpha + c = 0$

$\alpha^2 + c\alpha + b = 0$

$$\therefore \quad \frac{\alpha^2}{b^2 - c^2} = \frac{\alpha}{c - b} = \frac{1}{c - b}$$; Hence $\alpha = 1$ or $\alpha = -(b + c)$

if $\alpha = 1$ then $\alpha\beta_1 = c \Rightarrow \beta_1 = c$

and $\alpha\beta_2 = b \Rightarrow \beta_2 = b$

where β_1 and β_2 are the common root

$\therefore$ required equation $x^2 - (b + c)x + bc = 0$

but $-(b + c) = 1$

$\therefore$ $x^2 + x + bc = 0$]

Home Work after Q.No. 2.

If the equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root then show that the quadratic equation containing their other common roots is

$$a(b + c)x^2 + (b + c)x - abc = 0$$

3.(a) Find the value of p and q if the equation $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots is common. [**Hint:** $\frac{p}{3} = \frac{5}{10} = \frac{2}{q}$] [Ans. $p = \frac{3}{2}$; $q = 4$]

(b) If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$ have a common root find a and b. [**Hint:** since roots of 1st are imaginary hence both roots must be common]

4. If the equation $4x^2 \sin^2\theta - (4\sin\theta)x + 1 = 0$ and $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ have a common root and the 2nd equation has equal root find the possible values of θ in $(0, \pi)$.

[Hint: $x = 1$ is the common $\left(\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\right)$ put in equation to get $\sin\theta = \frac{1}{2}$]

5. If one root of the quadratic equation $x^2 - x + 3a = 0$ is double the root of the equation $x^2 - x + a = 0$ then find the value of 'a' $(a \neq 0)$

[Hint: $x^2 - x + 3a = 0 < \begin{matrix} 2\alpha \\ \beta \end{matrix}$; $x^2 - x + a = 0 < \begin{matrix} \alpha \\ \beta' \end{matrix}$

$\therefore$ $\alpha^2 - \alpha + a = 0$

$4\alpha^2 - 2\alpha + 3a = 0$

$$\frac{\alpha^2}{-3a + 2a} = \frac{\alpha}{4a - 3a} = \frac{1}{-2 + 4} \Rightarrow \frac{\alpha^2}{-a} = \frac{\alpha}{a} = \frac{1}{2}$$

$\therefore$ $\alpha = -1$ or $\alpha = \frac{a}{2} \Rightarrow a = -2$]

6. If each pair of the equations

$$\left.\begin{matrix} x^2 + p_1x + q_1 = 0 \\ x^2 + p_2x + q_2 = 0 \\ x^2 + p_3x + q_3 = 0 \end{matrix}\right] \text{ has exactly one root in common then show that}$$

$$(p_1 + p_2 + p_3)^2 = 4(p_1p_2 + p_2p_3 + p_3p_1 - q_1 - q_2 - q_3)$$

[Sol. Let $x^2 + p_1x + q_1 = 0 < \begin{matrix} \alpha \\ \beta \end{matrix}$

$x^2 + p_2x + q_2 = 0 < \begin{matrix} \beta \\ \gamma \end{matrix}$

$x^2 + p_3x + q_3 = 0 < \begin{matrix} \gamma \\ \alpha \end{matrix}$

Now $|(\alpha + \beta) - (\beta + \gamma)|^2 = |\alpha - \gamma|^2$

$$p_1^2 + p_2^2 - 2p_1p_2 = p_3^2 - 4q_3$$

or $\quad p_1^2 + p_2^2 - p_3^2 = 2p_1p_2 - 4q_3$

|||ly $\quad p_2^2 + p_3^2 - p_1^2 = 2p_3p_4 - 4q_1$

and $\quad p_3^2 + p_1^2 - p_2^2 = 2p_3p_1 - 4q_2$

adding $\quad p_1^2 + p_2^2 + p_3^2 = 2\sum p_1p_2 - 4\sum q_1$

$$(p_1 + p_2 + p_3)^2 = 4\left[\sum p_1p_2 - \sum q_1\right] \quad]$$

7. Find the value of 'α' for which the system of inequality $x^2 + 2x + \alpha \le 0$ and $x^2 - 4x - 6\alpha \le 0$ has a unique solution. **(Only for XIII)**

[Ans. $\alpha = 0, 1$]

8. For what value of 'a' do the curves $y = 1 + \frac{x^2}{a^3}$ and $y = 4\sqrt{x}$ possesses only one point in common. **[Only for XIII]** [Ans. $a = 1/3$ or $a < 0$]

Home Work : General inequaliteis from Prilepko.

3^RD^ LECTURE

MAXIMUM AND MINMUM VALUES OF QUADRATIC AND RATIONAL FUNCTIONS :

(1) $y = ax^2 + bx + c$ attains its maximum or minima at the point with abscissa $x = -\frac{b}{2a}$ according as $a < 0$ or $a > 0$.

(2) Maximum or minimum value can also be obtained by making a perfect square and then taking an interpretation.

Ex. $y = 2x^2 - 3x + 1$ find the minimum value.
$y = 7 + 5x - 2x^2$ find the maximum value.

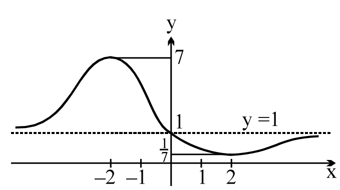
(3) For computing the maximum or minimum values of rational function consider the following examples :

EXAMPLES :

1. If x is real then $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lies from $\frac{1}{7}$ and 7.

[Sol. $\frac{dy}{dx} = \frac{6(x^2 - 4)}{(x^2 + 3x + 4)^2}$ which vanishes where $x = 2$ or -2 ;

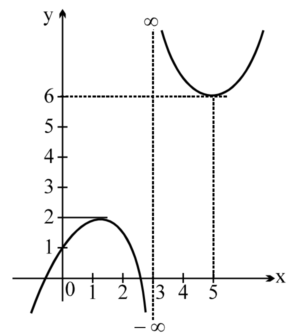
$f(2) = \frac{1}{7}$ & $f(-2) = 7$



Note that y is always > 0 as both N^r & $D^r > 0$, $\forall\ x \in R$
The graph is as follows

Note : $\left[\frac{1}{7}, 7\right]$ is also the range of the given function.]

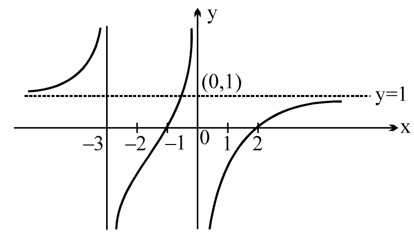
2. If x is real, prove that the expression $y = \frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except which lie between 2 and 6.



[Sol. $\frac{dy}{dx} = \frac{1}{2}\frac{x^2-6x+5}{(x-3)^2} = 0 \Rightarrow x = 1 \ \& \ x = 5$

$y|_{x=1} = 2 \ ; \ y|_{x=5} = 6$]

3. Prove that $y = \frac{(x+1)(x-2)}{x(x+3)}$ can have any value in $(-\infty, \infty)$ for $x \in R$.



[Sol. $\frac{dy}{dx} = \frac{2(2x^2+x+3)}{(x^2+3x)^2} \uparrow \ \forall \ x \in R$

$y \uparrow$ in $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$]

4. Find the maximum and minimum value of $y = \frac{x^2+14x+9}{x^2+2x+3} \ \forall \ x \in R$.

[Ans. Maximum = 4 ; minimum = – 5]

5. Find all possible values of 'a' for which the expression $\frac{ax^2-7x+5}{5x^2-7x+a}$ may be capable of all values, x being any real quantity. [Ans. $a \in (-12, 2)$]

[Hint: For common roots between N^r and D^r

$ax^2 - 7x + 5 = 0$

$5x^2 - 7x + a = 0$

$$\frac{\alpha^2}{-7a+35} = \frac{\alpha}{25-a^2} = \frac{1}{-7(a-5)}$$

$a \neq 5$

$$\frac{\alpha^2}{7} = \frac{\alpha}{a+5} = \frac{1}{7} \Rightarrow \frac{7}{a+5} = \frac{a+5}{7} \Rightarrow a = 2 \text{ or } a = -12$$

Hence for $a = 2$ or $a = -12$ we have a common factor in N^r & D^r.

for $a = 2$

$$y = \frac{2x^2-7x+5}{5x^2-7x+2} = \frac{(2x-5)(x-1)}{(x-1)(5x-2)} \qquad ....(1)$$

for $a = -12$

$$y = \frac{-12x^2-7x+5}{5x^2-7x-12} = \frac{(x+1)(5-12x)}{(x+1)(5x-12)} \qquad ....(2)$$

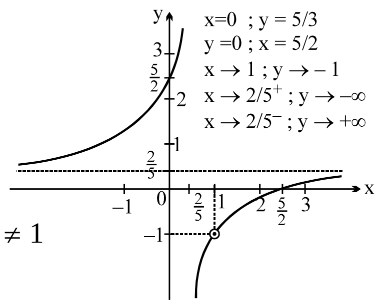
for $\quad a = 2$

in (1) when $x \to 1 \qquad y = -1$

y can take the value – 1 for any x.

in (2) when $x \to -1 \qquad y = 1;\ x \to \pm\infty,\ y \to \frac{2}{5}$

y can not take the value 1 for any x.



The graph of $y = \frac{2x-5}{5x-2}\cdot\frac{x-1}{x-1} = \frac{2x-5}{5x-2}$ $\qquad$ when $x \neq 1$

$$\frac{dy}{dx} = \frac{(5x-2)2-(2x-5)5}{(5x-2)^2} = \frac{21}{(5x-2)^2} > 0 \ \Rightarrow\ y \uparrow$$

Note that in this case y can take all values except – 1and 2/5. Similar would be the situation when a = – 12. Hence the values of a = 2 and – 12 are to be excluded.]

6. Show that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when *x* is real, if $a^2 - b^2$ and $c^2 - d^2$ have the same sign.

[Sol. TPT $(a^2 - b^2)(c^2 - d^2) > 0 \qquad$ (note that $ad \neq bc$)

$$y = \frac{adx^2-(ac+bd)x+bc}{bcx^2-(bd+ac)x+ad} \ \Rightarrow\ ybcx^2 - (bd + ac)y\,x + y\,ad = adx^2 - (ac+bd)x + bc$$

$(ybc - ad)x^2 - \left[\, y(bd + ac) - (ac + bd)\right] x + (y\,ad - bc) = 0 \qquad$ as $x \in R$

$\left[\, y(bd + ac) - (ac + bd)\right]^2 \geq 4(y\,bc - ad)(y\,ad - bc)$

$(bd + ac)^2\cdot y^2 + (ac + bd)^2 - 2(bd + ac)(ac + bd)\,y \geq 4[(abcd)y^2 - (b^2c^2 + a^2d^2)y + abcd]$

$(ac - bd)^2\cdot y^2 - 2[(ac + bd)^2 - 2(b^2c^2 + a^2d^2)]\,y + (ac - bd)^2 \geq 0$

$(ac - bd)^2\cdot y^2 - 2[(ac - bd)^2 - 2(bc - ad)^2]\,y + (ac - bd)^2 \geq 0$

$4[(ac - bd)^2 - 2(bc - ad)^2]^2 - 4[(ac - bd)^2]^2 \leq 0$

$[(ac - bd)^2 - 2(bc - ad)^2 + (ac - bd)^2]\,[(ac - bd)^2 - 2(bc - ad)^2 - (ac - bd)^2] \leq 0$

Home Work : Exercise 9 (b) of H & K.

4TH LECTURE

A. To find the condition that a quadratic function of x, y of the type
$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may be resolved into two linear factors.
The required condition is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

EXAMPLES :

1. If the expression $3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1$ can be resolved into linear factors then prove that p must be one of the roots of the equation $t^2 + 4at + 2a^2 + 6 = 0$.

[Sol. $a = 3,\ h = p,\ b = 2,\ g = a,\ f = -2,\ c = 1$

condition for this to be resolved into 2 linear factors

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$6 + (-4ap) - 12 - 2a^2 - p^2 = 0$

$-p^2 - 4ap - 2a^2 - 6 = 0$

$p^2 + 4ap + 2a^2 + 6 = 0$

$\therefore$ p is one of the roots of $t^2 + 4at + 2a^2 + 6 = 0$]

2.(a) If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x and y then prove that $1 \le x \le 2$ and $-\frac{1}{8} \le y \le \frac{1}{8}$.

[Sol. $x^2 - 3x + 16y^2 + 2 = 0$

$x \in R \quad \therefore \quad D \ge 0$

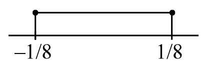
$9 - 4(16y^2 + 2) \ge 0$

$9 - 64y^2 - 8 \ge 0$

$1 - 64y^2 \ge 0$

$\therefore \quad 64y^2 - 1 \le 0$

$(8y + 1)(8y - 1) \le 0$



$\therefore \quad -\frac{1}{8} \le y < \frac{1}{8}$

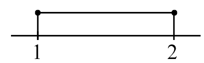
$16y^2 + x^2 - 3x + 2 = 0$

$y \in R \quad \therefore \quad D \ge 0$

$-64(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 \le 0$

$(x - 2)(x - 1) \le 0$



$\therefore \quad x \in [1, 2]$

$1 \le x \le 2$]

(b) Show that in the equation $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real value of y, and for every value of y there is a real value of x.

3. Prove that the expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorised into two linear factors. Find them.

[Sol. $a = 2, b = 1, h = 3/2, g = 3/2, f = 1, c = 1$

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$2 + \frac{9}{2} - 2 - \frac{9}{4} - \frac{9}{4}$

$\frac{9}{2} - \frac{9}{2} = 0$]

4. If $(ax^2 + bx + c)y + a'x^2 + b'x + c' = 0$ find the condition that x may be a rational function of y. [Ans. $(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$]

[Hint: Solve for x and then D must be a perfect square.]

[Sol. $(ay + a')x^2 + (by + b')x + cy + c' = 0$

$$2(ay + a')x = -(by + b') \pm \sqrt{\underbrace{(by + b')^2 - 4(ay + a')(cy + c')}_{\text{perfect sum}}}$$

$(b^2y^2 + b'^2 + 2bb'y) - 4[acy^2 + (ac' + a'c)y + a'c']$

$(b^2 - 4ac)y^2 + 2[bb' - 2(ac' + a'c)]y + (b'^2 - 4a'c') = 0$

$4[bb' - 2(ac' + a'c)]^2 = 4(b^2 - 4ac)(b'^2 - 4a'c')$

on simplifying, we get

$(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$ **Ans.**]

B. THEORY OF EQUATIONS :

$$ax^3 + bx^2 + cx + d \equiv a(x - x_1)(x - x_2)(x - x_3)$$
$$= a[x^3 - (\Sigma x_1)x^2 + (\Sigma x_1x_2)x^2 - x_1x_2x_3]$$

$$\therefore \quad x_1 + x_2 + x_3 = -\frac{b}{a}$$

$$x_1x_2 + x_2x_3 + x_3x_4 = \frac{c}{a}$$

$$\text{and} \quad x_1 x_2 x_3 = -\frac{d}{a}$$

Note: A polynomial equations of degree odd with real coefficient must have at least one real root as imaginary roots always occur in pair of conjugates.

EXAMPLES :

1. Solve the cubic $24x^3 - 14x^2 - 63x + 45 = 0$, one root being double the other.

[Sol. $24x^3 - 14x^2 - 63x + 45 = 0 \begin{cases} \alpha \\ 2\alpha \\ \beta \end{cases}$

$3\alpha + \beta = \dfrac{7}{12}$ ....(1)

$2\alpha^2 + 2\alpha\beta + \beta\alpha = -\dfrac{21}{8}$ ....(2) or $2\alpha^2 + 3\alpha\beta = -\dfrac{21}{8}$

$2\alpha^2\beta = -\dfrac{15}{8}$ ....(3) $\Rightarrow$ β must be – ve

put $\beta = \dfrac{7}{12} - 3\alpha$ in equation (2)

$$2\alpha^2 + 3\alpha\left(\frac{7}{12} - 3\alpha\right) = -\frac{21}{8}$$

$$2\alpha^2 + \frac{7}{4}\alpha - 9\alpha^2 = -\frac{21}{8}$$

$-7\alpha^2 + \dfrac{7}{4}\alpha + \dfrac{21}{8} = 0$ $\qquad$ $8\alpha^2 - 6\alpha + 4\alpha - 3$

$-\alpha^2 + \dfrac{\alpha}{4} + \dfrac{3}{8} = 0$ $\qquad$ $2\alpha(4\alpha - 3) + 4\alpha - 3$

$8\alpha^2 - 2\alpha - 3 = 0$ $\qquad$ $\alpha = \dfrac{3}{4}$ or $\alpha = -\dfrac{1}{2}$

if $\alpha = \dfrac{3}{4}$ then $\beta = \dfrac{7}{12} - \dfrac{9}{4} = \dfrac{7-27}{12} = -\dfrac{20}{12} = -\dfrac{5}{3}$

i.e. $\alpha = \dfrac{3}{4}$ or $\beta = -\dfrac{5}{3}$

If $\alpha = -\dfrac{1}{2}$ then $\beta = \dfrac{7}{12} + \dfrac{3}{2} = \dfrac{7+18}{24} = \dfrac{25}{24}$

i.e. $\alpha = \dfrac{3}{4}$ & $\beta = \dfrac{25}{24}$ (does not satisfy (3))

Hence $\alpha = \dfrac{3}{4}$ & $\beta = -\dfrac{5}{3}$ roots are $\dfrac{3}{4}, \dfrac{3}{2}, -\dfrac{5}{3}$]

Similar problem of Home Work : $24x^3 + 46x^2 + 9x - 9 = 0$

2. Find the
(i) sum of the squares and
(ii) sum of the cubes of the roots of the cubic equation $x^3 - px^2 + qx - r = 0 \begin{cases} \alpha \\ \beta \\ \gamma \end{cases}$

[Sol. (i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,\Sigma\alpha\beta$
(ii) $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$
$\therefore$ $\Sigma\,\alpha^3 = (\alpha + \beta + \gamma)[\Sigma\,\alpha^2 - \Sigma\,\alpha\beta] + 3\alpha\beta\gamma$

3. Solve the cubic $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of its two roots being equal to zero.

[Ans. $\left(\frac{3}{2}, -\frac{3}{2}, -4\right)$]

4. $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ no two of which have equal tangents, find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$. [Ans. Zero]

5. Find the cubic each of whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$. [Ans. $y^3 - 8y^2 + 19y - 1 = 0$]

[Sol. If y is one root of the required eqaution then $y = x + 1 \Rightarrow x = y - 1$.
Now put $x = y - 1$ in the given equation.]

6. Form a cubic whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

[Ans. $y^3 + 33y^2 + 12y + 8 = 0$]

[Sol. $\alpha + \beta + \gamma = -3$

$\Sigma\,\alpha\beta = 0$; $\alpha\beta\gamma = -2$

$y^3 - (\alpha^3 + \beta^3 + \gamma^3)y^2 + (\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)y + \alpha^3\beta^3\gamma^3 = 0$

$a^3 + b^3 + c^3 = (a + b + c)[a^2 + b^2 + c^2 - (ab + bc + ca)] + 3abc$

$a^3 + b^3 + c^3 = (a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)] + 3abc$

$= (-3)\,[9] + 3\,(-2)$

$= -33$

Similarly intercept $\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$]

7. Given the product p of sines of the angles of a triangle & product q of their cosines, find the cubic equation, whose coefficients are functions of p & q & whose roots are the tangents of the angles of the triangle. [REE'92, 6]

[Ans: $qx^3 - px^2 + (1 + q)\,x - p = 0$]

[Sol. Given $\sin A \sin B \sin C = p$; $\cos A \cos B \cos C = q$

Hence $\tan A \tan B \tan C = \tan A + \tan B + \tan C = p/q$

Hence equation of cubic is

$$x^3 - \frac{p}{q}x^2 + \left(\sum \tan A \tan B\right)x - \frac{p}{q} = 0 \qquad \text{...(i)}$$

now $$\sum \tan A \tan B = \frac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B}{\cos A \cos B \cos C}$$

We know that $A + B + C = \pi$

$\cos(A+B+C) = -1$

$\cos(A+B)\cos C - \sin(A+B)\sin C = -1$

$(\cos A \cos B - \sin A \sin B)\cos C - \sin C(\sin A \cos B + \cos A \sin B) = -1$

$1 + \cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$

dividing by $\cos A \cos B \cos C$

$$\frac{1+q}{q} = \sum \tan A \tan B$$

Hence (i) becomes $qx^3 - px^2 + (1+q)x - p = 0$ Ans.]

Home Work : 9 (c) of H & K.

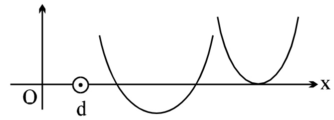
5^{TH} LECTURE

LOCATION OF ROOTS:

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :
Consider $f(x) = ax^2 + bx + c$ with $a > 0$.

Type–1 : Both roots of the quadratic equation are greater than a specified number say (d). The necessary and sufficient condition for this are :

(i) $a > 0$; (ii) $D \geq 0$; (iii) $f(d) > 0$; (iv) $-\dfrac{b}{2a} > d$



Note : If $a < 0$ then intercept accordingly.

EXAMPLES ON (TYPE–1) :

1. Find all the values of the parameter 'd' for which both roots of the equation

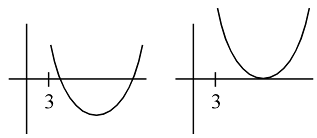
$x^2 - 6dx + (2 - 2d + 9d^2) = 0$ exceed the number 3. [Ans. $d > \dfrac{11}{9}$]

[Sol. (i) $D \geq 0$

$9t^2 - 2 + 2t - 9t^2 \geq 0$

$t - 1 \geq 0$

$t \geq 1$



(ii) $-\dfrac{b}{2a} > 3$; $3t > 3$; $t > 1$

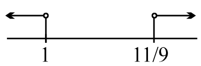
(iii) $f(3) > 0$

$9 - 18t + 2 - 2t + 9t^2 > 0$

$9t^2 - 20t + 11 > 0$

$9t^2 - 9t - 11t + 11 > 0$

$(t - 1)(9t - 11) > 0$



$\therefore$ $t \in (-\infty, 1) \cup (11/9, \infty)$

$\therefore$ Intersection of (i), (ii) and (iii) is $t > 11/9$]

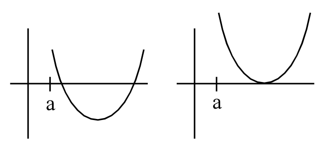
2. Find all the values of 'a' for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity 'a'. [Ans. $(-\infty, -2)$]

[Sol. (i) $D \geq 0$

$1 - 4a \geq 0$

$4a \leq 1$

$a \leq 1/4$

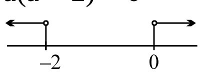


(ii) $-\dfrac{b}{2a} > a$; $-\dfrac{1}{2} > a$; $a < -\dfrac{1}{2}$

(iii) $f(a) > 0$

$a^2 + 2a > 0$

$a(a + 2) > 0$



$\therefore$ $a \in (-\infty, -2) \cup (0, \infty)$

$\therefore$ $a \in (-\infty, -2)$ Ans.]

3. Determine the values of 'a' for which both roots of the quadratic equation $(a^2 + a - 2)x^2 - (a + 5)x - 2 = 0$ exceed the number minus one.

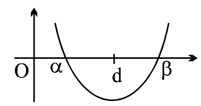
[Ans. $(-\infty, -2) \cup (-1, -1/2) \cup (1, \infty)$]

4. Find the values of $a > 0$ for which both the roots of equation $ax^2 - (a + 1)x + a - 2 = 0$ are greater than 3.

Type–2 : Both roots lie on either of a fixed number say (d). Alternatively one root is greater than 'd' and other less than 'd' or 'd' lies between the roots of the given equation.

Conditions for this

and (i) $a > 0$, (ii) $f(d) < 0$ or (i) $a < 0$, (ii) $f(d) > 0$



Note that no consideration for discriminant will be useful here.

EXAMPLES ON (TYPE–2) :

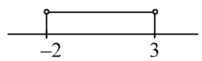
1. Find the value of k for which one root of the equation of $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2. [Ans. $k \in (-2, 3)$]

[Sol. since $a > 0$ $f(0) < 0$ $\therefore$ $f(2) < 0$

$\therefore$ $4 - 2(k + 1) + k^2 + k - 8 < 0$

$k^2 - K + 6 < 0$

$(k + 2)(k - 3) < 0$

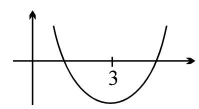


$\therefore$ $k \in (-2, 3)$]

2. Find the set of values of 'a' for which zeroes of the quadratic polynomial
$(a^2 + a + 1)x^2 + (a - 1)x + a^2$ are located on either side of 3. [Ans. ϕ]

[Sol. Leading co-efficient is always + ve

$\therefore$ $f(3) < 0$



$\therefore$ $9a^2 + 9a + 9 + 3a - 3 + a^2 < 0$

$10a^2 + 12a + 6 < 0$

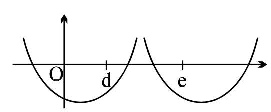
$5a^2 + 6a + 3 < 0$

This is always + ve as $5 > 0$

$\therefore$ $D < 0$

$\therefore$ $a \in \phi$ Ans.]

Type–3 : Exactly one root lies in the interval (d, e) when $d < e$.
Conditions for this are :
(i) $a \neq 0$; (ii) $f(d) \cdot f(e) < 0$



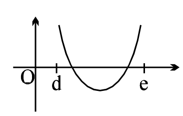
EXAMPLES ON (TYPE–3) :

1. Find the set of values of *m* for which exactly one root of the equation
$x^2 + mx + (m^2 + 6m) = 0$ lie in $(-2, 0)$ [Ans. $(-6, -2) \cup (-2, 0)$]

2. Find all possible values of 'a' for which exactly one root of the quadratic equation $x^2 - (a + 1)x + 2a = 0$ lie in the interval (0, 3). [Ans. $(-\infty, 0] \cup (6, \infty)$]
Note : In this case also check for end points. If interval is closed say [d, e] then $f(d) = 0$ or $f(e) = 0$ $\Rightarrow$ no other root should lie in (d, e)

Type–4 : When both roots are confined between the number d and e ($d < e$). Conditions for this are
(i) $a > 0$; (ii) $D \geq 0$; (iii) $f(d) > 0$; (iv) $f(e) > 0$

$$d < -\frac{b}{2a} < e$$

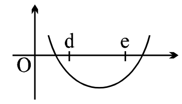


EXAMPLES ON (TYPE–4) :

1. If α, β are the roots of the quadratic equation
$x^2 + 2(k - 3)x + 9 = 0$ $(\alpha \neq \beta)$. If $\alpha, \beta \in (-6, 1)$ then find the values of k.

[Ans. $\left(6, \frac{27}{4}\right)$]

Type–5 : One root is greater than e and the other roots is less than d.
Conditions are :
(i) f (d) < 0 and f (e) < 0 if (a > 0)



EXAMPLES ON (TYPE–5) :

1. Find all the values of k for which one root of the quadratic equation $(k - 5)x^2 - 2kx + k - 4 = 0$ is smaller than 1 and the other root exceed 2. [Ans. (5, 24)]

GENERAL AND MIXED PROBLEM :

(1) For $y = f(x) = ax^2 + bx + c$
if $f(p) < 0$ and $f(q) > 0$
i.e. $f(p)\, f(q) < 0 \Rightarrow$ then the equation $ax^2 + bx + c = 0$ has one root lying between p and q.

EXAMPLES :

1. Let α be a real root of the equation $ax^2 + bx + c$ and β be a real root of the equation $-ax^2 + bx + c = 0$. Show that there exists a root γ of the equation $\frac{a}{2}x^2 + bx + c = 0$ that lie between α and β. ($\alpha, \beta \neq 0$).

[Sol. α is a root of equation $ax^2 + bx + c = 0$

$\therefore \quad a\alpha^2 + b\alpha + c = 0 \qquad ....(1)$

similarly $\quad -\alpha\beta^2 + b\beta + c = 0 \qquad ....(2)$

Let $\quad f(x) = \frac{a}{2}x^2 + bx + c \qquad ....(3)$

Now $f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c = \frac{a}{2}\alpha^2 - a\alpha^2 \quad$ {From (1)}

$= -\frac{a}{2}\alpha^2$

and $f(\beta) = \frac{a}{2}\beta^2 + b\beta + c = \frac{a}{2}\beta^2 + a\beta^2 \quad$ {From (2)}

$= \frac{3}{2}a\beta^2$

Now $f(\alpha)\, f(\beta) = -\frac{3}{4}\, a^2\alpha^2\, \beta^2 < 0 \quad [\because \ \alpha, \beta \neq 0]$

$\therefore$ f (α) and f (β) have opposite signs, therefore equation f (x) = 0 will have exactly one root between α and β if $\alpha < \beta$ or one root between β and α if $\beta < \alpha$.

2. If $a < b < c < d$, then show that the quadratic equation $(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots for all real values of λ.

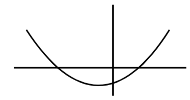
[Sol. Let $f(x) = (x - a)(x - c) + \lambda(x - b)(x - d)$
Given, $a < b < c < d$
Now, $f(b) = (b - a)(b - c) < 0 \quad [\because \ b - a > 0$ and $b - c < 0]$
and $f(d) = (d - a)(d - c) > 0$
Since $f(b)$ and $f(d)$ have opposite signs therefore, equation $f(x) = 0$ has one real root between b and d.
Since one root is real and a, b, c, d, λ are real therefore, other root will also be real.
Hence equation $f(x) = 0$ has real roots for all real values of λ.]

3.(a) Prove that for any real value of a the inequality, $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ is true for at least one negative x.

[Sol. $f(x) = (a^2 + 3)x^2 + (a + 2)x - 5$

Case-I: when $f(0) < 0$

obviously there is atleast one negative x for which $f(x) < 0$

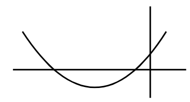


$\therefore$ $f(0) = -5$ which is always true for any $a \in R$

Case-II: If $f(0) > 0$

and $D > 0$

and $-\dfrac{b}{2a} < 0$



$\therefore$ $f(0) > 0$ is not possible. Hence f

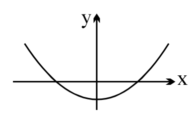
$\therefore$ $a \in R$]

3(b) If $f(x) = 4x^2 + ax + (a - 3)$ is negative for atleast one x, find all possible values of a.

[Sol. Case-I: if $f(0) < 0$

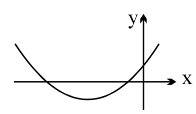
$a - 3 < 0 \Rightarrow a < 3$

$a \in (-\infty, +3)$



Case-II: if $f(0) \geq 0$ and

$D > 0$ and $-\dfrac{b}{2a} < 0$



$f(0) > 0$ gives $a - 3 \geq 0 \Rightarrow a \geq 3$ ....(1)

$D > 0$ gives $a^2 - 16(a - 3) > 0$

$a^2 - 16a - 48 > 0$

$(a - 12)(a - 4) > 0 \Rightarrow a > 12$ or $a < 4$ ....(2)

$-\dfrac{b}{2a} < 0$ gives $-\dfrac{a}{8} < 0 \Rightarrow a > 0$ ....(3)

from (1), (2) and (3)

$a \in [3, 4) \cup (12, \infty)$

finally $a \in (-\infty, 4) \cup (12, \infty)$]

4. Find the values of a for which the equation $x^2 + 2(a - 1)x + a + 5 = 0$ has at least one positive root. [Ans. $a \leq -1$]

5. Let $a, b, c \in R$ $a \neq 0$. If α and β be the roots of equation $ax^2 + bx + c = 0$, where $\alpha < -n$ and $\beta > n$, then show that $1 + \dfrac{c}{an^2} + \dfrac{1}{n}\left|\dfrac{b}{a}\right| < 0$, $n \in N$.

[Sol. $ax^2 + bx + c = 0$

$f(x) = x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

$f(x) < 0$ and $f(-x) < 0$

$\therefore$ $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} < 0$

and $x^2 - \dfrac{b}{a}x + \dfrac{c}{a} < 0$

hence $1 + \dfrac{b}{ax} + \dfrac{c}{ax^2} < 0$ ....(1)

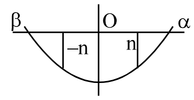
$$1 - \frac{b}{ax} + \frac{c}{ax^2} < 0$$

This two inequalities must simultaneously be true

note that the prodction of the roots is $\alpha\beta < -n^2$

$$\frac{c}{a} < -n^2 \quad \text{or} \quad \frac{c}{ax^2} + 1 < D$$

consider $E = \underbrace{1 + \frac{c}{ax^2}}_{-\text{ ve}} + \frac{1}{n}\left|\frac{b}{a}\right|$ ....(2)



if $b/a > 0$ then $E = 1 + \frac{c}{ax^2} + \frac{1}{n}\frac{b}{a} < 0$

if $b/a < 0$ then $E = 1 + \frac{c}{ax^2} - \frac{1}{n}\frac{b}{a} < 0$ from (1)

hence (2) simultaneously satisfied both (1)

$\therefore$ (1) is equivalent to

$$1 + \frac{c}{ax^2} + \left|\frac{b}{a}\right| < 0 \;]$$

6TH LECTURE

MISCELLANEOUS EQUATIONS INEQUATIONS AND LOGARITHMIC INEQUALITIES :

A. LINEAR EQUATION / INEQUATIONS INVOLVING MODULUS :

1. $|x - 3| + 2|x + 1| = 4$ Ans $\{-1\}$

2. $|x + 2| - |x - 1| < x - -\frac{3}{2}$

3. Find the least +ve integer satisfying $|x + 1| + |x - 4| > 7$.

4. Greater integer satisfying $\frac{2x+1}{3} - \frac{3x-1}{2} > 1$

B. QUADRATIC EQUATION / INEQUATION INVOLVINGMODULUS & EXPONENTIAL :

1. $|x^2 + 4x + 2| = \frac{5x+16}{3}$ Ans : $\{-2, 1\}$

2. $(|x - 1| - 3)(|x + 2| - 5) < 0$ Ans. $(-7, 2) \cup (3, 4)$

3. $|x - 5| > |x^2 - 5x + 9|$ Ans. $(1, 3)$

4. $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$

5. $\left|\frac{x^2 - 5x + 4}{x^2 - 4}\right| \le 1$ $[(0, \frac{8}{5}) \cup (\frac{5}{2}, +\infty)]$

6. $\left|\frac{x^2 - 3x - 1}{x^2 + x + 1}\right| < 3$ $[(-\infty, -2) \cup (-1, +\infty)]$

C. LOGARITHMIC EQUATION :

1. $\log_{\frac{1}{5}} \dfrac{4x+6}{x} \ge 0$ Ans. $\left[-2, \dfrac{3}{2}\right)$

2. (a) $\log_{2x+3}(x^2) < \log_{2x+3}(2x+3)$ Ans. $\left(-\dfrac{3}{2}, -1\right) \cup (-1, 3)$

(b) $\log_{x+3}(x^2 - x) < 1$ Ans. $(-3, 2) \cup (-1, 0) \cup (1, 3)$

3. $\log_7\left(\dfrac{2x-6}{2x-1}\right) > 0$ Ans. $\left(-\infty, \dfrac{1}{2}\right)$

4. $\log_3 | 3 - 4x | > 2$ Ans. $\left(-\infty, -\dfrac{3}{2}\right) \cup (3, \infty)$

5. $\log_{0.2} (x^2 - x - 2) > \log_{0.2} (-x^2 + 2x + 3)$ Ans. $\left(2, \dfrac{5}{2}\right)$

6. $(0.3)^{\log_{\frac{1}{3}} \log_2 \frac{3x+6}{x^2+2}} > 1$ Ans. (power < 0)

7. $\log_{0.5}\left(\log_6 \dfrac{x^2+x}{x+4}\right) < 0$ Ans. $(-4, -3) \cup (8, \infty)$

8. $\log_3 \dfrac{|x^2 - 4x| + 3}{x^2 + |x-5|} \ge 0$ Ans. $\left(-\infty, -\dfrac{2}{3}\right] \cup \left[\dfrac{1}{2}, 2\right]$

9. $\log_2^2\left(\dfrac{4x-3}{4-3x}\right) > -\dfrac{1}{2}$ Ans. $\left(\dfrac{3}{4}, \dfrac{4}{3}\right)$ or all domain

10. $\left(2\log_3^2 x - 3\log_3 x - 8\right)\left(2\log_3^2 x - 3\log_3 x - 6\right) \ge 3$

[Hint : Put $2\log_3^2 x - 3\log_3 x - 6 = t$]