APPLICATION OF DERIVATIVE

TANGENT & NORMAL : (Define) ; $\tan\phi = \left.\frac{dy}{dx}\right|_P$

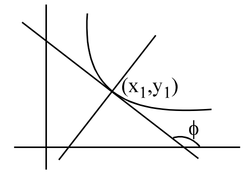
(1) Equation of a tangent at P (x_1, y_1)

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x_1,y_1}(x - x_1)$$

(2) Equation of normal at (x_1, y_1)

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x_1,y_1}}(x - x_1)$$

NOTE : *



* If $\left.\frac{dy}{dx}\right]_{x_1,y_1}$ exists. However in some cases $\frac{dy}{dx}$ fails to exist but still a tangent can be drawn ***e.g.*** case of vertical tangent. Also (x_1, y_1) must le on the tangent, normal line as well as on the curve.

Immediate examples

(1) A line is drawn touching the curve $y - \frac{2}{3-x} = 0$. Find the line if its slope/gradient is 2.
[Ans. $y - 2x + 2 = 0$; $y - 2x + 10 = 0$]

(2) Find the tangent and normal for $x^{2/3} + y^{2/3} = 2$ at (1, 1).
[Ans. T : $x + y = 2$ and N : $x - y = 0$]

(3) If the equation of the curve is represented parametrically

i.e. $x = f(t)$ and $y = g(t)$ where $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$, $\{f'(t) \neq 0\}$ then

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$ and the equation of the tangent is $y - g(t) = \frac{dy/dt}{dx/dt}\{(x - f(t)\}$.

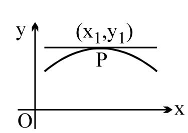
e.g. Find tangent to $x = a\sin^3 t$ and $y = a\cos^3 t$ at $t = \pi/2$. [Ans. y=0 or $x = at^2$ & $y = 2at$]

Note that the point (x_1, y_1) must lie on the equation of the curve the tangent and normal.

IMPORTANT NOTES TO REMEMBER:

(a) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is parallel to x-axis and converse.

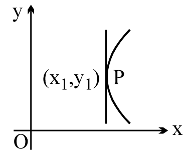
If tangent is parallel to $ax + by + c = 0 \Rightarrow \frac{dy}{dx} = -\frac{a}{b}$



(b) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} \to \infty$ or $\left.\frac{dx}{dy}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is perpendicular to x-axis.

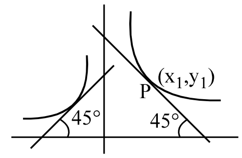
If tangent with a finite slope is perpendicular to $ax + by + c = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} \cdot \left(-\frac{a}{b}\right) = -1.$$

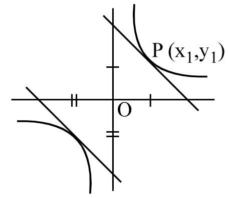


(3) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} = \pm 1.$$



(4) If the tangent makes equal non zero intercept on the coordinate axes then $\left.\frac{dy}{dx}\right|_{x_1,y_1} = -1$



(5) If tangent cuts off from the coordinate axes equal distance from the origin $\Rightarrow \frac{dy}{dx} = \pm 1$.

(6) Which of the following cases the function $f(x)$ has a vertical tangent at $x = 0$.

(i) $f(x) = x^{1/3}$ (ii) $f(x) = \text{sgn } x$ (iii) $f(x) = x^{2/3}$

(iv) $f(x) = \sqrt{|x|}$ (v) $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

[Sol. **Vertical Tangent:**

Concept: $y = f(x)$ has a vertical tangent at the point $x = x_0$ if

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \infty \text{ or } -\infty \text{ but not both}$$

for example the functions

$f(x) = x^{1/3}$ and $f(x) = \text{sgn } x$ both have a vertical tangent at $x = 0$

but $f(x) = x^{2/3}$; $f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$ have no vertical tangent

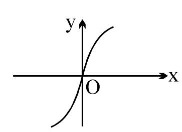
Explanation:

(i) $f(x) = x^{1/3}$

$$f'(0^+) = \lim_{h\to 0} \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} \to \infty$$

$$f'(0^-) = \lim_{h\to 0} \frac{(-h)^{1/3}}{-h} = \frac{1}{h^{2/3}} \to \infty$$

$\Rightarrow x = 0$ is a vertical tangent



(ii) $f(x) = \text{sgn } x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$f'(0^+) = \lim_{h\to 0} \frac{1-0}{h} \to \infty$$

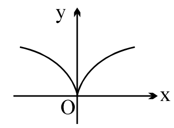
$$f'(0^-) = \lim_{h\to 0} \frac{-1}{-h} \to \infty$$

$\Rightarrow x = 0$ is a vertical tangent

(iii) $f(x) = x^{2/3}$

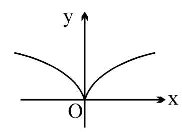
$$f'(0^+) = \lim_{h\to 0} \frac{h^{2/3}}{h} \to \infty$$

$$f'(0^-) = \lim_{h \to 0} \frac{(-h)^{2/3}}{-h} = -\frac{1}{h^{2/3}} \to -\infty \quad \Bigg] \text{ no vertical tangent at } x = 0$$



(iv) $f(x) = \sqrt{|x|} = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{\sqrt{h} - 0}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{\sqrt{h}}{-h} \to -\infty \end{aligned}\right] \text{ no vertical tangent at } x = 0$$

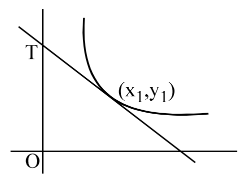


(v) $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{1-1}{h} = 0 \\ f'(0^-) &= \lim_{h \to 0} \frac{0-1}{h} = -\infty \end{aligned}\right] \text{ no vertical tangent at } x = 0$$

]

(7) OT is called the initial ordinate of the tangent



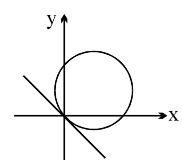
$$Y - y = \frac{dy}{dx}(X - x)$$

put $X = 0$ to get

$\therefore \quad Y = OT = y - x\dfrac{dy}{dx}$ (It is the y intercept of a tangent at P)

(8) If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating to zero the lowest degree terms appearing in the equation of the curve.

e.g. in



(i) $x^2 + y^2 + 2gx + 2fy = 0$

equation of tangent is $gx + fy = 0$

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

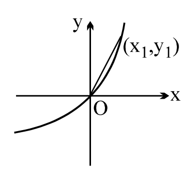
equation of tangents at origin are $x^2 - y^2 = 0$

(iii) Equation of tangents to $x^3 + y^2 - 3xy = 0$ are $xy = 0$

☞ **Note:** If the curve is $x^4 + y^4 = x^2 + y^2$, then the equation of the tangent would be $x^2 + y^2 = 0$ which would indicate that the origin is an isolated point on the graph.

Proof : Let the equation of the curve be

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 = 0 \qquad \text{....(1)}$$



$$\text{Tangent} : y - 0 = \lim_{\substack{x_1 \to 0 \\ y_1 \to 0}} \frac{y_1}{x_1}(x - 0)$$

Now equation (1) becomes

$$a_1 + b_1\frac{y_1}{x_1} + a_2x_1 + b_2\frac{y_1}{x_1} \cdot x_1 + c_2\frac{y_1}{x_1} \cdot y_1 = 0 \qquad \text{....(2)}$$

at $x_1 \to 0$ & $y_1 \to 0$, $\dfrac{y_1}{x_1} \to m$

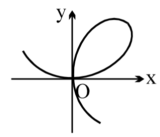
from equation (2)

$$a_1 + b_1m = 0 \qquad \left[\ \therefore\ m = -\frac{a_1}{b_1}\ \right]$$

Hence tangent is

$$y = -\frac{a_1}{b_1}x \quad \Rightarrow \quad a_1x + b_1y = 0$$

(9) Same line could be the tangent as well as normal to a given curve at a given point.
e.g. in , $x^3 + y^3 - 3xy = 0$ (Folium of descartes)
the line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.



(10) SOME COMMON PARAMETRIC COORDINATES ON A CURVE:

(a) for $x^{2/3} + y^{2/3} = a^{2/3}$ take parametric coordinate $x = a\cos^3\theta$ & $y = a\sin^3\theta$.

(b) for $\sqrt{x} + \sqrt{y} = \sqrt{a}$ take $x = a\cos^4\theta$ & $y = a\sin^4\theta$.

(c) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$ taken $x = a(\sin\theta)^{2/n}$ & $y = b(\sin\theta)^{2/n}$.

(d) for $c^2(x^2 + y^2) = x^2y^2$ take $x = c\sec\theta$ and $y = c\,\mathrm{cosec}\theta$.

(e) for $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

☞ **Note:** The tangent at P meeting the curve again at Q.

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1}$$

Consider the examples $y^2 = x^3$ find $\frac{m_{OP}}{m_{OQ}}$.

Take P (t^2, t^3)

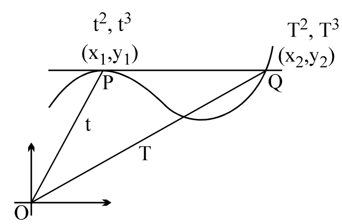


ILLUSTRATION ON TANGENT AND NORMAL

1. Find the equation of a tangent and normal at $x = 0$ if they exist on the curve $y = x^{1/3}(1 - \cos x)$

[Hint : $f'(0) = \lim\limits_{h\to 0}\frac{h^{1/3}(1-\cos h)}{h} = \lim\limits_{h\to 0} h^{4/3}\frac{(1-\cos h)}{h^2} = 0$]

Concept: $F(x)=f(x)\cdot g(x)$ are such that $f(x)$ is continuous at $x = a$ and $g(x)$ is differentiable at $x = a$ with $g(a)=0$ then the product function $f(x)\cdot g(x)$ is diffenentiable at $x = a$]

2.(a) Equation of the normal to the curve $x^2 = 4y$ which passes through $(1, 2)$.
[Ans. $x + y = 3$] [$2x + y = 4$ is using]

(b) normals to the curve $x^2 = 4y$ which passes through $(4, -1)$.

(c) Find the equation of tangent and normal to the curve $f(x) = \left[\begin{array}{ll} x-2 & \text{if } x<1 \\ x^2-x-1 & \text{if } x\ge 1\end{array}\right.$ at $x = 1$ if it exists. [Ans. T : $x - y - 2 = 0$; $x + y = 0$]
(f is continuous and differentiable at $x = 1$ and $f'(1) = 1$)

3. A curve in the plane is defined by the parametric eqautions $x = e^{2t} + 2e^{-t}$ and $y = e^{2t} + e^t$. An equation for the line tangent to the curve at the point $t = \ln 2$ is
(A) $5x - 6y = 7$ (B) $5x - 3y = 7$ (C*) $10x - 7y = 8$ (D) $3x - 2y = 3$

[Sol. when $t = \ln 2$ $x = e^{\ln 4} + 2e^{-\ln 2} = 4 + 1 = 5$
and $y = e^{\ln 4} + e^{\ln 2} = 6$
$\therefore$ point $(5, 6)$

now $y = e^{2t} + e^t \Rightarrow \frac{dy}{dt} = 2e^{2t} + e^t$

$x = e^{2t} + 2e^{-t} \Rightarrow \frac{dx}{dt} = 2(e^{2t} - e^{-t})$

$$\frac{dy}{dx} = \frac{2e^{2t}+e^{t}}{2(e^{2t}-e^{-t})} = \frac{2(4)+2}{2(4-(1/2))} = \frac{10}{7}$$

$\therefore$ Tangent $\quad y - 6 = \frac{10}{7}(x - 5);\ 7y - 42 = 10x - 50;\ 10x - 7y = 8$ **Ans.**]

4. Tangent to the curve $y = \sin^{-1}\frac{2x}{1+x^2}$ at $x = \sqrt{3}$

$$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{if } -1 < x < 1 \\ -\dfrac{2}{1+x^2} & \text{if } x > 1 \text{ or } x < -1 \end{cases}$$

Asking: Tangent to the curve $y = \sin^{-1}\frac{2x}{1+x^2}$ at $x = \sqrt{3}$ is inclined at $\cot^{-1}(k)$ where $k =$ ___

[Ans. 2]

5. Find the equation of the tangent to the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis. [Ans. $\frac{x}{a}+\frac{y}{b}=1$]

6. Prove that all points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ at which the tangent is parallel to the x-axis lie on a parabola.

7. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their point of contacts lie on the curve $x^2y^2 = x^2 - y^2$.

8. If $y = 1+\frac{x^2}{a^3}$ and $y = 4\sqrt{x}$ have only one point in common. Find a. [Ans.$(-\infty, 0) \cup \left\{\frac{1}{3}\right\}$]

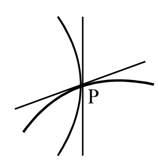
9. Show that the portion of the tangent to the curve $x = a\cos^3\theta$ and $y = a\sin^3\theta$ intercepted between the coordinate axes is constant. [Ans. $l = a$]

10. If $y = e^x$ and $y = kx^2$ touches each other, find k. [Ans. $k = \frac{e^2}{4}$]

Home work after 1st lecture: Exercise-6.3 (NCERT, Part-I)

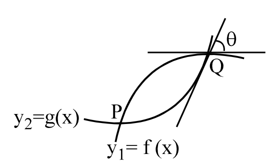
(11) ANGLE OF INTERSECTION OF TWO CURVES :

Definition **:** The angle of intersection of two curves at a point P is defined as the angle between the two tangents to the curve at their point of intersection.



If the curves are orthogonal then

$$\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right) = -1 \text{ everywhere whereever they intersect.}$$



If $\left(\frac{dy_1}{dx}\right)_P\left(\frac{dy_2}{dx}\right)_P = -1$ but $\left(\frac{dy_1}{dx}\right)_Q\left(\frac{dy_2}{dx}\right)_Q \neq -1$

then the two curves are orthogonal at P but not at Q hence they are not orthogonal.

e.g. $y^2 = 4ax$ & $y = e^{-x/2a}$; $xy = a^2$ & $x^2 - y^2 = b^2$ and $y = ax$ & $x^2 + y^2 = c^2$ are orthogonal but $y^2 = 4ax$ and $x^2 = 4by$ are not orthogonal.

Note: If the curves touch at P then $\theta = 0$ hence $f'(x_1) = g'(x_1)$.

ILLUSTRATION ON ANGLE OF INTERSECTION OF TWO CURVES :

1. Find the acute angle between the curves

(i) $y = \sin x$ & $y = \cos x$. [Ans. $\tan^{-1}(2\sqrt{2})$ or $\sec^{-1}(3)$]

(ii) If θ is the angle between $y = x^2$ and $6y = 7 - x^3$ at (a, a). Find θ. [Ans. $\theta = \frac{\pi}{2}$]

[Hint: solve $x^2 = 7 - x^3$ to get the point as (1, 1)]

2. Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$

[Sol. Solving $2y^2 = x^3$ and $y^2 = 32x$

we get (0, 0) ; (8, 16) and (8, – 16)

for $\sqrt{2}\,y = x\sqrt{x}$ or $\sqrt{2}\,y = -x\sqrt{x}$

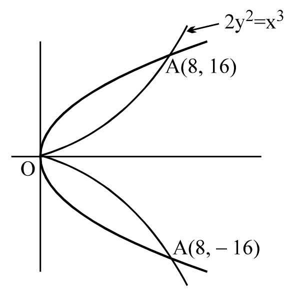
at (0, 0) $\left.\frac{dy}{dx}\right|_{(0,0)} = 0$

for I

at (0, 0) $\left.\frac{dy}{dx}\right|_{(0,0)} = \infty$

for II

hence angle = 90°



now $\left.\frac{dy}{dx}\right|_{I} = \frac{3x^2}{4y} = \frac{3}{4}\cdot\frac{64}{16} = 3$

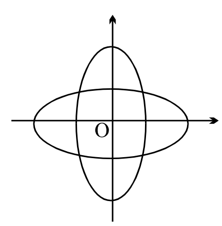
$\left.\frac{dy}{dx}\right|_{II} = \frac{32}{2y} = \frac{16}{16} = 1$ $\quad\therefore\quad \tan\theta = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$

Note: Also see that tangent at the origin on $2y^2 = x^3$ is $y^2 = 0$ i.e. $y = 0$ i.e. x-axis and tangent to $y^2 = 32x$ at the origin is $x = 0$ $\Rightarrow$ angle between the two curves at the origin is 90°]

3. Find the condition for the two concentric ellipses $a_1x^2 + b_1y^2 = 1$ and $a_2x^2 + b_2y^2 = 1$ to intersect orthogonally. [Ans. $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{b_2} - \frac{1}{b_1}$]

[Sol. $a_1x^2 + b_1y^2 = 1$ & $a_2x^2 + b_2y^2 = 1$

differentiating, $2a_1x + 2b_1y\cdot\frac{dy}{dx} = 0$



$\frac{dy}{dx} = -\frac{a_1x}{b_1y}$

$\left.\frac{dy}{dx}\right|_{x_1,y_1}^{I} = -\frac{a_1x_1}{b_1y_1}$ ....(1)

|||ly $\left.\frac{dy}{dx}\right|_{x_1,y_1}^{II} = -\frac{a_2x_1}{b_2y_1}$ ....(2)

If the two curves are orthogonal

$$\frac{a_1x_1}{b_1y_1}\cdot\frac{a_2x_1}{b_2y_1} = -1$$

$$\frac{a_1a_2}{b_1b_2}\frac{x_1^2}{y_1^2} = -1 \qquad ....(3)$$

now $a_1x_1^2 + b_1y_1^2 = 1$ & $a_2x_1^2 + b_2y_1^2 = 1$ (as x_1y_1 satisfy the given equations)

subtracting $(a_1 - a_2)\,x_1^2 + (b_1 - b_2)\,y_1^2 = 0$

$$\frac{x_1^2}{y_1^2} = -\frac{b_1 - b_2}{a_1 - a_2}$$

putting in (3) $\dfrac{a_1a_2}{b_1b_2}\,\dfrac{b_1 - b_2}{a_1 - a_2} = 1$

$$\frac{a_1 - a_2}{a_1a_2} = \frac{b_1 - b_2}{b_1b_2} \qquad \Rightarrow \qquad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{b_2} - \frac{1}{b_1} \qquad]$$

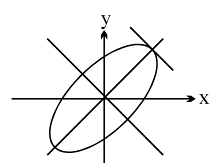
4. Find the condition for the line $y = mx$ to cut at right angles the conic $ax^2 + 2hxy + by^2 = 1$. Hence find the direction of the axes of the conic.
[Ans. $m^2h + m(a - b) - h = 0$. The roots of the quadratic equation are the slope of the axes]

[Sol. $ax^2 + 2hxy + by^2 = 1 \qquad ....(1)$

$$\cancel{2}ax + \cancel{2}h\left[x\frac{dy}{dx} + y\right] + \cancel{2}by\frac{dy}{dx} = 0$$

$$(xh + by)\frac{dy}{dx} = -(ax + hy)$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$



$$\therefore \qquad +m\left(\frac{ax + hy}{hx + by}\right) = +1$$

$$m\left[\frac{ax + h\cdot mx}{hx + b\cdot mx}\right] = 1$$

$$m\,(a + hm) = h + bm$$

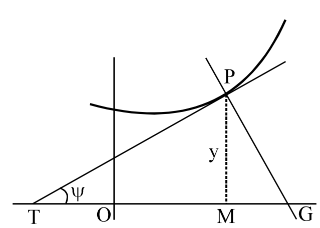
$$m^2h + (a - b)m - h = 0 \quad]$$

(12) CARTESIAN TANGENT & NORMALS ON STANDARD CURVES: (Not in CBSE)

LENGTH OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL:

(i) **Tangent :**

$$PT = MP\ \text{cosec}\ \psi = y\sqrt{1 + \cot^2\psi} = \left|\frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$



(ii) **Subtangent :** TM = MP cot $\psi = \left|\frac{y}{(dy/dx)}\right|$

(iii) **Normal :** GP = MP sec $\psi = y\sqrt{1+\tan^2\psi} = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$

(iv) **Subnormal :** MG = MP tan y $= \left|y\left(\frac{dy}{dx}\right)\right|$

ILLUSTRATION

1. Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal.

[Hint: TPT $\left(\frac{y}{(dy/dx)}\right)^2 = k \cdot y\left(\frac{dy}{dx}\right)$ or $y = k\left(\frac{dy}{dx}\right)^3$

now $2by\frac{dy}{dx} = 3(x+a)^2 \Rightarrow \frac{dy}{dx} = \frac{3(x+a)^2}{2by}$

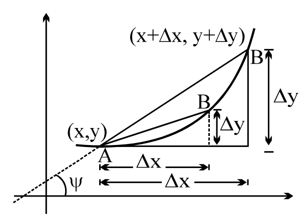
hence $\frac{y}{(dy/dx)^3} = \frac{y\,8b^3y^3}{2y(x+a)^6} = \frac{8b^3}{2y}\left(\frac{y^2}{(x+a)^3}\right)^2 = \frac{8b^3}{27}\cdot\frac{1}{b^2} = \frac{8b}{27}$ Ans.]

2. Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

APPROXIMATION AND DIFFERENTIALS :

For the figure it is clear that if Δx and Δy are sufficiently small quantities then

$$\frac{\Delta y}{\Delta x} = \tan\psi \cong \frac{dy}{dx} = f'(x)$$



Hence approximate change in the value of y, called its differential is given by

$$\Delta y = f'(x) \cdot \Delta x \quad ....(1)$$

ILLUSTRATION (ONLY CBSE)

1. Use differential to approximate (i) $\sqrt{101}$; (ii) $(25)^{1/3}$

$f(x) = y = \sqrt{x}$; $y + \Delta y = \sqrt{x+\Delta x}$

$\therefore$ $\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = f'(x) \cdot \Delta x$ {using (1) }

put $x = 100$; $\Delta x = 1$

$\sqrt{101} = 10 + \frac{1}{20} \times 1 = 10 + 0.05 = 10.05$ Ans.

similiarly for $(25)^{1/3}$ consider $y = x^{1/3}$.

2.$_{99/02}$ In an acute triangle ABC if sides a, b be constants and the base angles A and B vary,

show that $\frac{dA}{\sqrt{a^2 - b^2\sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2\sin^2 B}}$

[Sol. $\frac{a}{\sin A} = \frac{b}{\sin B}$

or b Sin A = a sin B

b cos A dA = a cos B dB

$$\frac{dA}{a\cos B} = \frac{dB}{b\cos A} \quad \Rightarrow \quad \frac{dA}{a\sqrt{1-\sin^2 B}} = \frac{dB}{b\sqrt{1-\sin^2 A}}$$

$$\frac{dA}{a\sqrt{1-\frac{b^2\sin^2 A}{a^2}}} = \frac{dB}{b\sqrt{1-\frac{a^2\sin^2 B}{b^2}}}; \quad \frac{dA}{\sqrt{a^2-b^2\sin^2 A}} = \frac{dB}{\sqrt{b^2-a^2\sin^2 B}} \;]$$

Home work: Refer Exercise-6.4 NCERT

RATE MEASURE:

1. If the area of circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely at its radius.

[Sol. $A = \pi r^2 \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = c$

$P = 2\pi r \quad \Rightarrow \quad \frac{dP}{dt} = 2\pi\frac{dr}{dt}$

$\therefore \quad \frac{dA}{dt} = r \cdot \frac{dP}{dt} = c; \quad \frac{dP}{dt} = \frac{c}{r}$]

2. If the side of an equilateral triangle increases at the rate of $\sqrt{3}$ cm/sec and area increase at the rate 12 cm^2/sec then the side of the equilateral triangle is ______. [Ans. 8]

[Sol. $\frac{dA}{dt} = 12; \; \frac{dx}{dt} = \sqrt{3}$ cm/sec; x = ?

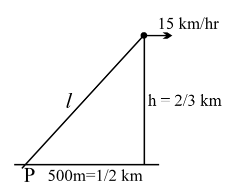
now $A = \frac{\sqrt{3}}{4}x^2; \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4}\cdot 2x \cdot \frac{dx}{dt}; \qquad 12 = \frac{\sqrt{3}}{2}\cdot\sqrt{3}\cdot x = \frac{3x}{2} \Rightarrow x = 8$ Ans.]

3. The total revenue in rupees required from the sales of *x* units of a product is given by
$R(x) = 3x^2 + 36x + 5.$
Find the marginal revenue when x = 5. [Ans. 66]

Note: Marginal revenue means, instantaneous rate of change of the total revenue w.r.t the number of items sold at any output.
|||ly marginal cost is defined.

4. An aeroplane is flying horizontally at a height of $\frac{2}{3}$ km with a velocity of 15 km/hr. Find the rate at which it is receding from a fixed point on the ground which it passed over 2 minutes ago. [Ans. 9 km/hr]

[Sol. $l^2 = h^2 + x^2$



$2l\frac{dl}{dt} = 0 + 2x\frac{dx}{dt} \quad \Rightarrow \quad \frac{dl}{dt} = \frac{x}{l}\cdot\frac{dx}{dt}$

where $x = \frac{1}{2}$km, $h = \frac{2}{3}$ km then $l = \frac{1}{4} + \frac{4}{9} = \frac{5}{6}$ km

$\therefore \quad \frac{dl}{dt} = \frac{1}{2}\cdot\frac{6}{5}\cdot 15 = 9$ km/ hr]

5. The height *h* of a right circular cone is 20 cm and is decreasing at the rate of 4 cm/sec. At the same time, the radius r is 10 cm and is increasing at the rate of 2 cm/sec. Find the rate of change of the volume in cm^3/sec. [Ans. $\frac{400\pi}{3}$]

[Hint: $V = \frac{1}{3}\pi r^2 h$.   differentiating w.r.t. time

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right);$$ when $h = 20$, $\frac{dh}{dt} = -4$; $r = 10$ and $\frac{dr}{dt} = 2$; to find $\frac{dV}{dt}$

$$= \frac{\pi}{3}[100(-4) + 2(10)(20)2] = \frac{\pi}{3}[400]$$

MONOTONOCITY

(Significance of the sign of the first order derivative)

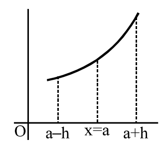
1. GENERAL INTRODUCTION :

The most useful element taken into consideration amongst the total post mortuam activities of functions, is their monotonic behaviour.

Functions are said to be monotonic if they are either increasing or decreasing in their entire domain ***e.g.*** $f(x) = e^x$; $f(x) = \ln x$ & $f(x) = 2x + 3$ are some of the examples of functions which are increasing whereas $f(x) = -x^3$; $f(x) = e^{-x}$ and $f(x) = \cot^{-1}(x)$ are some of the examples of the functions which are decreasing.

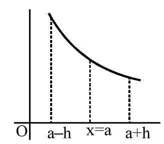
Functions which are increasing as well as decreasing in their domain are said to be non monotonic ***e.g.*** $f(x) = \sin x$; $f(x) = ax^2 + bx + c$ and $f(x) = |x|$, however in the interval $\left[0, \frac{\pi}{2}\right]$, $f(x) = \sin x$ will be said to be increasing.

2. MONOTONOCITY OF A FUNCTION AT A POINT :



A function is said to be monotonic increasing at x = a if $f(x)$ satisfies

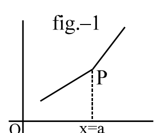
$$\left.\begin{array}{l} f(a+h) > f(a) \\ \text{and} \quad f(a-h) < f(a) \end{array}\right] \textit{for a small positive h.} \quad ....(1)$$



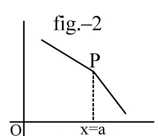
and monotonic decreasing at x = a if

$$\left.\begin{array}{l} f(a+h) < f(a) \\ \text{and} \quad f(a-h) > f(a) \end{array}\right] \quad ....(2)$$

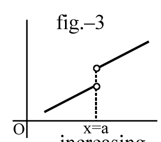
<u>It should be noted that we can talk of monotonocity of $f(x)$ at x = a only if x = a lies in the domain of f, without any consideration of continuity or differentiability of $f(x)$ at x = a.</u>



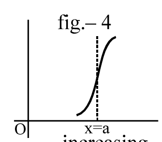
increasing at x = a



decreasing at x = a



increasing at x = a



increasing at x = a

3. MONOTONICITY IN AN INTERVAL :

For an increasing function in some interval,

if $\Delta x > 0 \iff \Delta y > 0$ or $\Delta x < 0 \iff \Delta y < 0$

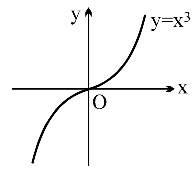
then f is said to be monotonic (strictly) increasing in that interval. In other words if Δy and Δx have the same sign *i.e.* $\frac{dy}{dx} > 0$, for increasing function. Hence if $\frac{dy}{dx} > 0$ in some J (interval) then y is said to be increasing function in that J and conversely if $f(x)$ is increasing in some J then $\frac{dy}{dx} > 0$ in that J.

Similarly if $\frac{dy}{dx} < 0$ in some J then y is decreasing in that J and conversely.

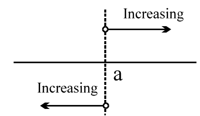
Hence to find the intervals of monotonocity for a function $y = f(x)$ one has to find $\frac{dy}{dx}$ and solve the inequality, $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$. The solution of this inequality gives the interval of monotonocity.

It should however be noted that

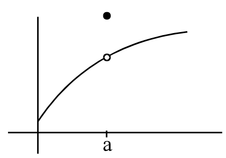
(a) $\frac{dy}{dx}$ at some point may be equal to zero but $f(x)$ may still be increasing at x = a. Consider $f(x) = x^3$ which is increasing at x = 0 although $f'(x)=0$. This is because $f(0 + h) > f(0)$ and $f(0 - h) < f(0)$. At all such points where $\frac{dy}{dx} = 0$ but y is still increasing or decreasing are known as **point of inflection**, which indicate the change of concavity of the curve.



(b) If f is increasing for x > a and f is also increasing for x < a then f is also increasing as x = a provided $f(x)$ is continuous at x = a.

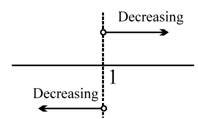


If f (x) is discontinuous at x = a then one can draw the graph as shown



x = a is the point of maxima

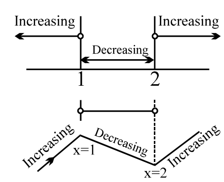
Similarly if f is decreasing for x > a and f is also decreasing for x < a then f is also decreasing for x = a provided $f(x)$ is continuous at x = a.



However if $f(1)$ is not defined then monotonocity will not be indicated at x = 1

e.g. $f(x) = \frac{1}{x-1}$ is decreasing for $x \in (-\infty, 1) \cup (1, \infty)$.

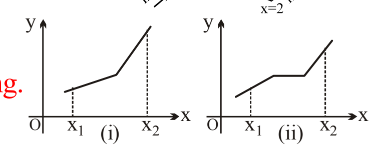
However if f is increasing and decreasing as shown then at x=1 and x=2, f will have extremum values, being maximum at x = 1 and minimum at x =2.



(c) If $x_1, x_2 \in$ domain of f and if

(i) $x_1 > x_2 \Leftrightarrow f(x_1) > f(x_2)$, f is strictly increasing.

(ii) $x_1 > x_2 \Leftrightarrow f(x_1) \geq f(x_2)$, f is non decreasing.



Increasing   non decreasing

Note:

(1) if a function is monotonic at x = a it can not have extremum point at x = a and vice versa *i.e.* A point on the curve can not simultaneously be an extremum as well as monotonic point.

(2) If f is increasing then nothing definite can be said about the function $f'(x)$ w.r.t. its increasing or decreasing behaviour.

ILLUSTRATIONS (Refer Exercise-6.2 Q. 5, 6, 7, 8, 9, 12, 14, 15, 18,19 NCERT (Part-I)

1. Compute the intervals of monotonicity for the following :

(a) $f(x) = x^2 \cdot e^{-x}$ [Ans: ↑ in (0, 2) and ↓ in $(-\infty, 0) \cup (2, \infty)$]

(b) $f(x) = x + ln(1 - 4x)$ [Ans: ↑ in $(-\infty, -3/4)$ and ↓ in $(-3/4, 1/4)$, (**Asking**)]

(c) $f(x) = ax - \sin x$ [Ans: ↑ for $a \geq 1$ & ↓ for $a \leq -1$ (**Asking**)]

(d) $f(x) = \frac{x}{ln\, x}$ [Ans: ↑ in (e, ∞) and ↓ in $(0, 1) \cup (1, e)$]

2. $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$ in $[0, \pi]$

[Hint: $f'(x) = -\cos x \cdot \sin x(2\cos x + 1)(\cos x + 2)$
increasing in $(\pi/2, 2\pi/3)$ and decreasing in $[0, \pi/2) \cup (2\pi/3, \pi]$]

3.(a) If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is always decreasing $\forall\ x \in R$, find 'a'.
[Ans : $(-\infty, -3]$]

3.(b) Find the range of values of 'a' for which the function
$f(x) = x^3 + (2a + 3)x^2 + 3(2a + 1)x + 5$ is monotonic in R. Hence find the set of values of 'a' for which f (x) in invertible. [**Ans.** $0 \le a \le 3/2$]

4. Prove that $f(x) = \frac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is always increasing.

5. Prove that $f(x) = \int_{x^2}^{x^3} \frac{dt}{\ln t}$, $(x > 0)$ is always an increasing function of x.

(Note that $f(0)$ and $f(1)$ not defined.) [**Ans.** $\uparrow$ in $(0, 1) \cup (1, \infty)$]

[Hint: $f'(x)\ \frac{3x^2}{\ln x^3} - \frac{1\cdot 2x}{\ln x^2} = \frac{3x^2}{3\ln x} - \frac{2x}{2\ln x} = \frac{x^2}{\ln x} - \frac{x}{\ln x} = \frac{x^2 - x}{\ln x} = \frac{x(x-1)}{\ln x}$

for $(0, 1) \cup (1, \infty)$ $f'(x) > 0 \rightarrow$ f is increasing]

6.218/01 The set of integral value(s) of 'b' for which $f(x) = \sin 2x - 8(b + 2)\cos x - (4b^2 + 16b + 6)x$ is monotonic decreasing for $\forall\ x \in R$ and has no critical point, is

(A*) $\{-10, -9, 2, 3\}$ (B) $\{-7, -8, -1, 0\}$
(C) $\{-8, 1, 5, 6\}$ (D) $\{-100, -200, 100, 200\}$

[Sol.: $f'(x) = 2\cos 2x + 8(b + 2)\sin x - (4b^2 + 16b + 6)$
$= 2(1 - 2\sin^2 x) + 8(b + 2)\sin x - (4b^2 + 16b + 6)$
$= -4[\sin^2 x - 2(b + 2)\sin x + (b^2 + 4b + 1)]$

for monotonic decreasing and no critical points $f'(x) < 0 \quad \forall \quad x \in R$

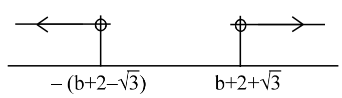
hence $\sin^2 x - 2(b + 2)\sin x + b^2 + 4b + 1 > 0$

$[\sin x - (b + 2)]^2 - (b + 2)^2 + 4b + b^2 + 1 > 0$

$[\sin x - (b + 2)]^2 - \left(\sqrt{3}\right)^2 > 0$

$[\sin x - b - 2 - \sqrt{3}]\ [\sin x - b - 2 + \sqrt{3}] > 0$

$[\sin x - (b + 2 + \sqrt{3})]\ [\sin x - (b + 2 - \sqrt{3})] > 0$



Hence $\sin x > b + (2 + \sqrt{3})$

or $b + (2 + \sqrt{3}) < \sin x \quad \forall\ x \in R$

$\therefore \quad b + (2 + \sqrt{3}) < -1$

$b < -(3 + \sqrt{3}) \Rightarrow$ **b** $\in \left(-\infty, -\left(3 + \sqrt{3}\right)\right)$

|| ly $\sin x < (b + 2 - \sqrt{3})$

or $(b + 2 - \sqrt{3}) > \sin x \quad \forall\ x \in R.$

$b + 2 - \sqrt{3} > 1$

$b > -1 + \sqrt{3}$

$b > \sqrt{3} - 1 \Rightarrow$ **b** $\in \left((\sqrt{3} - 1), \infty\right)$. Now finally $\left(-\infty, -(3+\sqrt{3})\right) \cup \left(\sqrt{3}-1, \infty\right)$]

7. Consider the function,

$f(x) = x^3 - 9x^2 + 15x + 6$ for $1 \le x \le 6$ and $g(x) = \begin{cases} \min . f(t) & \text{for } 1 \le t \le x, 1 \le x \le 6 \\ x - 18 & \text{for } x > 6 \end{cases}$

then which of the following hold(s) good?

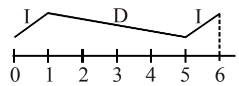
(A*) g(x) is differentiable at x = 1 (B*) g(x) is discontinuous at x = 6
(C*) g(x) is continuous and derivable at x = 5 (D*) g(x) is monotonic in (1, 5)

[Sol. $f(x) = x^3 - 9x^2 + 15x + 6$ **[13th, 28-12-2008,P-2]**

$f(t) = t^3 - 9t^2 + 15t + 6$

$f'(t) = 3t^2 - 18t + 15 = 3[t^2 - 6t + 5] = 3(t - 5)(t - 1)$

hence f is increasing in (5, 6) and f is decreasing in (1, 5)



now $g(x) = \begin{cases} f(x) = x^3 - 9x^2 + 15x + 6 & 1 \le x < 5 \\ f(5) = -19 & 5 \le x \le 6 \\ x - 18 \quad \text{if} & x > 6 \end{cases}$

$\therefore \quad g(x) = \begin{cases} x^3 - 9x^2 + 15x + 6 & \text{if} \quad 1 \le x < 5 \\ -19 & \text{if} \quad 5 \le x \le 6 \\ x - 18 & \text{if} \quad x > 6 \end{cases}$

Hence g is continuous and differentiable at x = 1
g is continuous and differentiable at x = 5
g is neither continuous nor derivable at x = 6]

8. Given $f : [0, \infty) \to R$ be a strictly increasing function such that the functions $g(x) = f(x) - 3x$ and $h(x) = f(x) - x^3$ are both strictly increasing function. Then the function $F(x) = f(x) - x^2 - x$ is
(A) increasing in (0, 1) and decreasing in $(1, \infty)$
(B) decreasing in (0, 1) and increasing in $(1, \infty)$
(C*) increasing throughout $(0, \infty)$ (D) decreasing throughout $(0, \infty)$

[Sol. $3F(x) = 3[f(x) - x^2 - x] = \underbrace{2[f(x) - 3x]}_{g(x)} + \underbrace{\left(f(x) - x^3\right)}_{h(x)} + x^3 - 3x^2 + 3x - 1 + 1$

$3F(x) = 2g(x) + h(x) + (x - 1)^3 + 1 \quad \Rightarrow \quad F(x)$ is increasing $\forall\ x \in [0, \infty)$

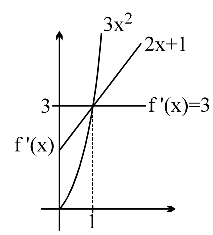
Alternatively: Given f (x) is increasing $\Rightarrow \quad f'(x) > 0$

$g(x) = f(x) - 3x$

$g'(x) = f'(x) - 3 > 0 \quad \Rightarrow \quad f'(x) > 3$

also $h(x) = f(x) - x^3$

$h'(x) = f'(x) - 3x^2 > 0 \quad \Rightarrow \quad f'(x) > 3x^2$



to prove that $F(x) = f(x) - x^2 - x$ is increasing

i.e $F'(x) = f'(x) - 2x - 1 > 0$

$F'(x) > 0$

$f'(x) > 2x + 1$ **[13th, 23-11-2008, P-1]**

for in [0, 1), obviously

$f'(x) > 3 > 2x + 1$ and in (1, 2), $3x^2 > 2x + 1$ hence proved.]

9.$_{98/4}$ Let $f : (0, \infty) \to (0, \infty)$ be a derivable function and F (x) is the premitve of f (x) such that $2(F(x) - f(x)) = f^2(x)$ for any real positive x.

(A*) f is strictly increasing (B*) $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 1$
(C) f is strictly decreasing (D) f is non monotonic

[Sol. (a) Given $2(F(x) - f(x)) = f^2(x)$ and $\frac{dF}{dx} = f(x)$

$\therefore \quad F(x) = \frac{f^2(x)}{2} + f(x)\,;\, F'(x) = f(x) \cdot f'(x) + f'(x)$

$\therefore \quad f(x) = f'(x)\,(1 + f(x))$

$\therefore \quad f'(x) = \frac{f(x)}{1 + f(x)} = 1 - \frac{1}{1 + f(x)} > 0 \qquad$ (as $f(x) > 0$)

Hence f is strictly increasing.

(b) $\lim\limits_{x\to\infty} \frac{f(x)}{x}$

or $\quad \lim\limits_{x\to\infty} f'(x)$ using L'Hospital's Rule

$$= \lim_{x\to\infty}\left(1-\frac{1}{1+f(x)}\right) \text{ as } x\to\infty;\ f(x)\to\infty \ ;\ \therefore\ \lim_{x\to\infty}\frac{f(x)}{x} = 1 \qquad]$$

4. GREATEST AND LEAST VALUE OF A FUNCTION :

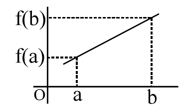


figure -1

If a continuous function $y = f(x)$ is strictly increasing in the closed interval [a, b] then

$f(a)$ is the least value. (figure - 1)

|||ly $\quad f(b)$ is the greatest value of $f(x)$ in [a, b]

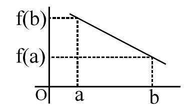


figure -2

If $f(x)$ is decreasing in [a, b] then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b].(figure - 2)

However if $f(x)$ is non monotonic in [a, b] and is continuous then the greatest and least value of $f(x)$ in [a, b] are those where $f'(x) = 0$ **or** $f'(x)$ does not exist or at the extreme values. (figure - 3)

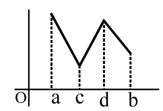


figure -3

ILLUSTRATIONS :

Find the greatest and the least values of the continuous function (1 to 4) given below in the indicated intervals.

1. $f(x) = e^{x^2-4x+3}$ in $[-5, 5]$ [Ans. M at x = – 5 & f (–5) = e^{48}; m at x = 2 & f (2) = $\frac{1}{e}$]

2.(a) $y = \int_{\frac{5\pi}{4}}^{x}(3\sin t + 4\cos t)dt$ in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ [Hint : note that y is a ↓ function in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$]

[Ans. $\frac{3}{2}+\frac{1}{\sqrt{2}}-2\sqrt{3}$ at $x = \frac{4\pi}{3}$ which is the least value ; greatest value = zero]

(b) $y = \int_{\frac{5\pi}{3}}^{x}(6\cos u - 2\sin u)du$ in $\left[\frac{5\pi}{3}, \frac{7\pi}{4}\right]$

[Hint: $\frac{dy}{dx} > 0 \Rightarrow$ y is ↑ hence max. occurs at $\frac{7\pi}{4} = 3\sqrt{3} - 2\sqrt{2} - 1$. Min. Value = 0]

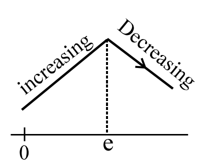
3. f (x) = cos 3x – 15 cos x + 8 in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

[Hint : $\frac{dy}{dx} = 0$ when x = π. Max. at x = π = 22 & Min. at x = $\frac{\pi}{3} = -\frac{1}{2}$]

4. **(a)** Use the function $f(x) = x^{\frac{1}{x}}$ (x > 0) to ascertain whether π^e or e^π is greater.

(b) Find min. x^x (x > 0) [Ans. $(1/e)^{1/e}$]

[Sol.**(a)** $f(x) = x^{\frac{1}{x}}$; $\quad f(x) = e^{\frac{\ln x}{x}} \Rightarrow f'(x) = x^{\frac{1}{x}}\left[\frac{1-\ln x}{x^2}\right]$



for $0 < x < e$, $f'(x) > 0$ and $x > e$, $f'(x) < 0$

$f(x)$ attains the maximum at x = e and max. value $e^{1/e}$

$\therefore \quad f(e) > f(x)$ for all $x > 0 \Rightarrow f(e) = f(\pi)$; $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$; $\therefore \quad e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$; $e^\pi > \pi^e$]

General Note:

f (x) = sin(cos x) in (0, π/2) is decreasing and g (x) = cos(cos x) in (0, π/2) is increasing.

h (x) = f [g(x)]

h ' (x) = f ' [g (x)] · g ' (x)

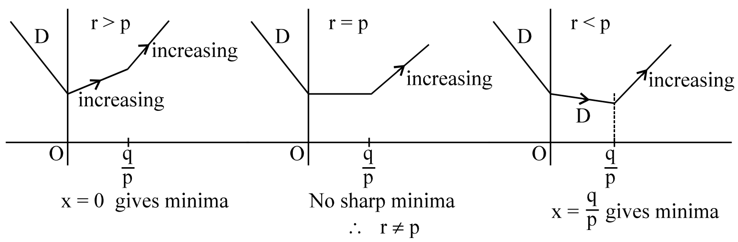
if f is increasing and g is increasing
or f is decreasing and g is decreasing
$\Rightarrow$ h is increasing
if f is increasing and g is decreasing then h is decreasing.

5. If $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$, p, q, r > 0, assumes its minimum value only at one point then find the relation between p, q, r.

[Sol. Let $x \ge \frac{q}{p}$; $f(x) = px - q + rx$; $f'(x) = p + r > 0$ $\Rightarrow$ f(x) increasing for $x \ge \frac{q}{p}$

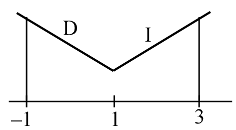
$$0 \le x < \frac{q}{p} \quad f(x) = q - px + rx \quad ; \quad f'(x) = r - p \quad \Rightarrow \begin{cases} f(x) \text{ is } \uparrow \text{ if } r > p \\ \text{constant} \quad r = p \\ f(x) \text{ is } \downarrow \text{ if } r < p \end{cases}$$

$x < 0$, $f(x) = q - px - rx$; $f'(x) = -(p + r)$ $\Rightarrow$ $f(x)$ is decreasing



6. Find the image of interval [–1, 3] under the mapping specified by the function
$f(x) = 4x^3 - 12x$ Ans. [–8, 72]

[Hint: $f'(x) = 0 \Rightarrow x = \pm 1$
Find $f(-1)$, $f(1)$ and $f(3)$ and interpret.



7.$_{94/4}$ Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & , 0 \le x < 1 \\ 2x - 3 & , 1 \le x \le 3 \end{cases}$

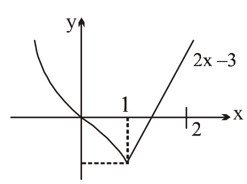
Find all possible real values of b such that f(x) has the smallest value at x = 1.
[JEE '93, 5] [Ans: $b \in (-2, -1) \cup [1, \infty)$]

[Sol. Obviously f is decreasing in $0 \le x < 1$ and increasing in [1, 3]. Also f (1) = – 1
Note that $f(1) \le f(1 - h)$ in order that f (x) has the smallest value at x = 1

$$\lim_{h \to 0} f(1-h) \ge f(1)$$

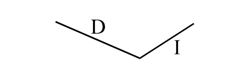
$$\lim_{h \to 0} -(1-h)^3 + \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge -1$$

or $\dfrac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0 \Rightarrow b \in (-2, -1) \cup [1, \infty)$]



8. Find the minimum value of the function $f(x) = 8^x + 8^{-x} - 4(4^x + 4^{-x})$, $\forall x \in R$.
[Ans. – 10]

[Sol. $f(x) = (2^x + 2^{-x})^3 - 3(2^x + 2^{-x}) - 4[(2^x + 2^{-x})^2 - 2]$ **[13th quiz]**
$2^x + 2^{-x} = t \ge 2$, $x \in R$
$g(t) = t^3 - 3t - 4t^2 + 8 = t^3 - 4t^2 - 3t + 8$
$g'(t) = 3t^2 - 8t - 3$
$= (3t + 1)(t - 3)$
Hence $f_{min} = g(3) = -10$]



Note: – 6 is wrong answer

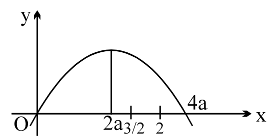
9. Find all possible values of 'a' for which $f(x) = \log_a(4ax - x^2)$ is monotonically increasing for every $x \in \left[\frac{3}{2}, 2\right]$. [Ans. $\left(\frac{1}{2}, \frac{3}{4}\right] \cup (1, \infty)$]

[Sol. Case-I:

If $0 < a < 1$ (obviously 'a' can not be < 0) **[Quiz-61, 12th 2008]**

then for $f(x)$ to be increasing

$4ax - x^2$ should be decreasing in $\left(\frac{3}{2}, 2\right)$



$\Rightarrow \quad \frac{3}{2} \ge 2a \quad$ and $\quad 2 < 4a$

$\therefore \quad a \le \frac{3}{4}$ and $a > \frac{1}{2} \Rightarrow \quad a \in \left(\frac{1}{2}, \frac{3}{4}\right]$

Caser-II: If $a > 1$ then for $f(x)$ to be increasing

$4ax - x^2$ increasing in $\left(\frac{3}{2}, 2\right)$

$\therefore \quad 2a \ge 2 \Rightarrow \quad a \ge 1$ but $a \ne 1$; $\quad \therefore \quad a > 1$

$\therefore \quad$ final answer is $\left(\frac{1}{2}, \frac{3}{4}\right] \cup (1, \infty)$ Ans.]

10. In which of the following ordered pair(s) of (a, b) the inequality a > b hold(s) good?

(A) (π^e, e^π) (B*) $\left((2008)^{2009}, (2009)^{2008}\right)$

(C*) $\left(e^{5/2}, \left(\frac{5}{2}\right)^e\right)$ (D*) $\left(\left(\frac{7}{4}\right)^{5/3}, \left(\frac{5}{3}\right)^{7/4}\right)$ **[Quiz-61, 12th 2008]**

[Hint: Consider $f(x) = x^{1/x}$ which is maximum at x = e, intercept the result]

5. ESTABLISHING INEQUALITIES :

Notion of monotonocity helps is establishing variety of inequalities involving algebraic and transcidental function with much greater ease.

ILLUSTRATIONS

1.$_{106/03}$ $2\sin x + \tan x \ge 3x \quad (0 \le x < \frac{\pi}{2})$

[Sol. Consider , $f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x} = \frac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x}$$

Hence $f'(0) > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right) \quad \Rightarrow \quad f(x)$ is $\uparrow$ in $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x) > f(0) \ \forall x$ in $\left(0, \frac{\pi}{2}\right)$; but $f(0) = 0$

$\therefore \quad f(x) \ge 0$; as equaly holds

Hence $f(x) > 0$ in $\left[0, \frac{\pi}{2}\right) \Rightarrow$ the result]

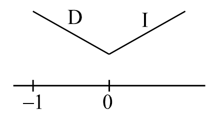
2. Find the set of values of x for which $ln(1 + x) > \frac{x}{1+x}$ [Ans. $(-1, 0) \cup (0, \infty)$]

[Hint: $f(x) = ln(1+x) > \dfrac{x}{1+x}$

$f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2} \quad \Rightarrow \quad \uparrow$ for $x > 0$ and $\downarrow$ $-1 < x < 0$

$\Rightarrow$ f (0) is the minimum value of f (x) for all $x > -1$]

3. $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \dfrac{\pi}{2}$. **[T/S]**



[Sol. We have to show that

$$\frac{\tan x \sin x - x^2}{x \sin x} > 0 \text{ for } 0 < x < \pi/2$$

Since $x \sin x > 0$ for $0 < x < \pi/2$, it is enough to show that

$\tan x \cdot \sin x - x^2 > 0, \ 0 < x < \pi/2$

Let $f(x) = \tan x \sin x - x^2$ for $0 < x < \pi/2$

$f'(x) = \sin x \sec^2 x + \tan x \cos x - 2x = \sin x \sec^2 x + \sin x - 2x$

$f''(x) = \cos x \sec^2 x + \sin x \,.\, 2\sec^2 x \tan x + \cos x - 2$

$= \sec x + \cos x - 2 + 2 \sin x \ \tan x \sec^2 x$

$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2 \sin x \tan x \sec^2 x > 0$ for $0 < x < \pi/2$

$\therefore$ f ' is strictly increasing in $[0, \pi/2)$. Also $f'(0) = 0$

$\Rightarrow$ $f'(x) > 0$ for $0 < x < \pi/2$

$\Rightarrow$ f is strictly increasing in $[0, \pi/2)$. Also $f(0) = 0$

$\Rightarrow$ $f(x) > 0$ for $0 < x < \pi/2$.

$\Rightarrow$ $\tan x \sin x - x^2 > 0$ for $0 < x < \pi/2$.

$\therefore$ $\dfrac{\tan x \sin x - x^2}{x \sin x} > 0, \quad 0 < x < \pi/2$.

$\Rightarrow$ $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \dfrac{\pi}{2}$. Henced proved.]

4.${}_{97/02}$ Establish the inequality given below by examining the sign of the derivative of an appropriate function: $\dfrac{2}{2x+1} < \ell n\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$ for $x > 0$

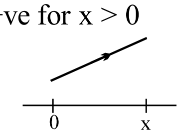
[Sol. consider $f(x) = \dfrac{2}{2x+1} - \ell n\left(1+\dfrac{1}{x}\right)$; $f'(x) = \dfrac{-4}{(2x+1)^2} - \dfrac{1}{1+(1/x)} \cdot \left(-\dfrac{1}{x^2}\right)$

$$= \frac{1}{x(x+1)} - \frac{4}{(2x+1)^2} = \frac{1}{x(x+1)(2x+1)^2}$$

which is always +ve for $x > 0$

hence $f(x)$ is $\uparrow$ for $x > 0$

i.e. $f(x) < \lim\limits_{x \to \infty} f(x)$ **(note carefully)**



but $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \left\{\dfrac{2}{2x+1} - ln\left(1+\dfrac{1}{x}\right)\right\} = 0$

so $f(x) < 0$ i.e.

$\dfrac{2}{2x+1} < \ell n\left(1+\dfrac{1}{x}\right)$ Proved

Similarly consider $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x}$; $g'(x) = \frac{1}{x^2} - \frac{1}{x(x+1)} = \frac{1}{x^2(x+1)}$

$g'(x)$ is $\uparrow$ i.e.

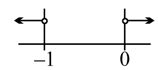
so $g(x) < \lim\limits_{x\to\infty} g(x)$ but $\lim\limits_{x\to\infty} f(x) = 0$

so $g(x) < 0$ hence proved]

5. Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is always an increasing function for all x in its domain. State its range and also plot the graph of the function.

[Sol. $f(x) = y = \left(1+\frac{1}{x}\right)^x = e^{x\, ln\left(1+\frac{1}{x}\right)}$

Hence for domain $\frac{x+1}{x} > 0$, $\Rightarrow$ $x \in (-\infty, -1) \cup (0, \infty)$



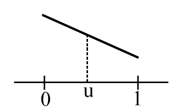
now $\lim\limits_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$; Also $ln\, y = x\, ln\left(1+\frac{1}{x}\right)$

$$\therefore \quad \lim_{x\to 0^+} \frac{ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \left(\frac{\infty}{\infty}\right) = \frac{x}{x+1}\left(\frac{-\frac{1}{x^2}}{-\frac{1}{x^2}}\right) = 0 \Rightarrow y = e^0 = 1 \; ; \; \therefore \lim_{x\to 0^+} f(x) = 1$$

$\lim\limits_{x\to -1^-} f(x) = \infty$ (obviously)

$$\text{Now} \quad f'(x) = \left(1+\frac{1}{x}\right)^x \left[x\left(\frac{1}{1+x} - \frac{1}{x}\right) + ln\frac{1+x}{x}\right] = \left(1+\frac{1}{x}\right)^x \left[\frac{x}{1+x} - 1 + ln\frac{1+x}{x}\right]$$

Let $\frac{x}{1+x} = u$ [for $x > 0$, $u \in (0, 1)$]



consider a function

$g(u) = u - 1 - ln\, u$

$g'(u) = 1 - \frac{1}{u} < 0$ which is negative for $u \in (0, 1)$

Hence g (u) is decreasing in (0, 1]

$\therefore$ $g(u) > g(1)$ $\forall$ $u \in (0, 1]$

But $g(1) = 0$

$\therefore$ $g(u) > 0$ $\forall$ $u \in (0, 1)$

$\therefore$ $\frac{x}{1+x} - 1 + ln\frac{1+x}{x} > 0$ $\forall$ $x > 0$

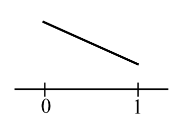
$\therefore$ $f'(x) > 0$

$\therefore$ f (x) is increasing for x > 0 **....(1)**

again for $x < -1$

$f'(x) = \left(1+\frac{1}{x}\right)^x \left[ln\frac{1+x}{x} + \frac{x}{1+x} - 1\right]$ put $\frac{1+x}{x} = u$

consider $g(u) = ln\, u + \frac{1}{u} - 1$ as $x \in (-\infty, -1)$, $u \in (0, 1)$



$g'(u) = \frac{1}{u} - \frac{1}{u^2} = \frac{u-1}{u^2} < 0$ for $u \in (0, 1)$

g (u) is decreasing in (0, 1)

$\therefore\ g(u) > g(1)$

$g(u) > 0$

$$\ln\frac{1+x}{x} + \frac{x}{1+x} - 1 > 0$$

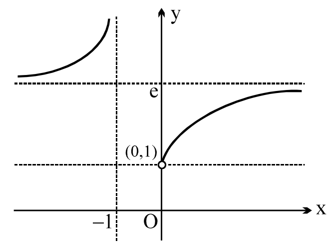
$\therefore\ f'(x) > 0$ for $x \in (-\infty, -1)$

$\therefore\ f(x)$ is increasing in $(-\infty, -1)$ **....(2)**

from (1) and (2)

f (x) is increasing for all x in the domain of f (x).

range is $(1, e) \cup (e, \infty)$



Graph of f (x)

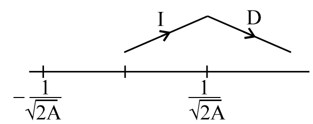
6.$_{108/4}$ Find the smallest positive constant A such that $\ln x \le Ax^2$ for all $x > 0$. [Ans. $\frac{1}{2e}$]

[Sol. Note that $A > 0$

Consider the function

$f(x) = \ln x - Ax^2$

$$f'(x) = \frac{1}{x} - 2Ax = \frac{1-2Ax^2}{x}$$



$f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2A}}$

and $f'(x) < 0$ for $x > \frac{1}{\sqrt{2A}}$; $\quad\therefore\quad \ln x < Ax^2$ for $x > 0$

this must also hold good for $x = \frac{1}{\sqrt{2A}}$

$$\therefore\quad \ln\frac{1}{\sqrt{2A}} < A\cdot\frac{1}{2A}; \qquad -\ln\sqrt{2A} < \frac{1}{2};\ \ln\sqrt{2A} > -\frac{1}{2}$$

$$\sqrt{2A} = e^{-\frac{1}{2}};\ A > \frac{1}{2e}; \qquad \therefore\quad \text{least value of } A = \frac{1}{2e} \text{ Ans.]}$$

6. ROLLE'S & MEAN VALUE THEOREM :

(a) ROLLE'S THEOREM :

Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

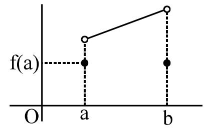
(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

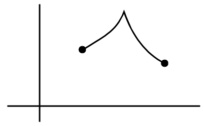
Alternative statement: Rolles theorem states that between ay two real zeroes of a differentiable real function f, lies at least one critical point of f (x).

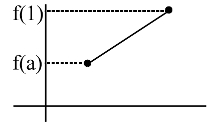
☞ **Remarks:**

1. Converse of Rolle's theorem is **Not true** i.e. $f'(x)$ may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.
2. The three conditions are sufficient but not necessary for $f'(x) = 0$ for some x in (a, b)
3. If the function $y = f(x)$ defined over [a, b] does not satisfy even one of the 3 conditions then Rolle's Theorem fails i.e. there may or may not exist point in (a, b) where $f'(x) = 0$.



f is not continuous in [a, b] and Rolles theorem failing


f is not derivable in open (a, b)



f (a) ≠ f (b)

ILLUSTRATIONS ROLLE'S THEOREM: (Refer Exercise-5.8, NCERT (Part-I)

1. Verify Rolle's Theorem for
   (a) $f(x) = x(x + 3)e^{-x/2}$ in $[-3, 0]$ [Ans. $c = -2$; $c = 3$ (rejected)]
   (b) $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$ $[c = \dfrac{\pi}{4}]$
   (c) $f(x) = x^3 - 3x^2 + 2x + 5$ in $[0, 2]$ $f(0) = 5$; $f(2) = 5$
   (d) $f(x) = 1 - x^{2/3}$ in $[-1, 1]$ ($f'(0)$ non existent)
   (e) $f(x) = |x|^3$ in $[-1, 1]$

[Hint: in (d) and (e) Rolles theorem is not applicable]

2. Let $n \in N$. If the value of c prescribed in Rolle's theorem for the function $f(x) = 2x(x - 3)^n$ on $[0, 3]$ is $3/4$ then $\boldsymbol{n}$ is equal to
   (A) 1 (B*) 3 (C) 5 (D) 7

[Sol. $f(0) = f(3)$; $f(x) = 2x(x - 3)^n$ **[13th, 09-03-2008]**
hence $f'(c) = 0$
$f'(x) = 2[(x - 3)^n + nx(x - 3)^{n-1}]$
$f'(c) = 2[(c - 3)^n + nc(c - 3)^{n-1}] = 0$
or $2(c - 3)^{n-1}[c - 3 + nc] = 0$; but $c = 3/4$
$\therefore$ $(n + 1)\dfrac{3}{4} = 3$ $\Rightarrow$ $n = 3$ **Ans.**]

3. Show that between any two roots of the equation $e^x\cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.

[Hint: Consider $f(x) = \cos x - e^{-x}$ and apply Rolle's theorem in it.]

4. Let f (x) and g (x) be differentiable functions such that $f'(x)\,g(x) \neq f(x)\,g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

[Sol. Let a, b be the solutions of $f(x)=0$
suppose g(x) is not equal to zero for any x belonging to [a, b]
now consider $h(x)= f(x)/g(x)$
since g(x) not equal to zero
h(x) is differentiable and continuous in [a, b]
$h(a) = h(b) = 0$ (as $f(a) = 0$ and $f(b) = 0$ but $g(a)$ or $g(b) \neq 0$)
applying rolles theorem for h (x) in [a, b]
$h'(c)=0$ for some c belonging to (a,b)
$f(x)g'(x)= f'(x)g(x)$
this gives the contadiction ...
hence proved]

5.$_{80/4}$ Let P (x) be a polynomial with real coefficients. Let $a, b \in R$, $a < b$, be two consecutive roots of P(x). Show that there exists 'c' such that $a \le c \le b$ and $P'(c) + 100\,P(c) = 0$.

[Sol. Consider $f(x) = e^{100x} \cdot P(x)$.
Now $f(a) = f(b) = 0$ {as $P(a) = P(b) = 0$ }
Also as P (x) is polynomial $\Rightarrow$ f (x) is continuous and differentiable in [a, b]

$\Rightarrow$ Rolle's theorem can be applied

$\Rightarrow$ $\exists$ $c \in (a, b)$ such that $f'(c) = 0$

now $f'(x) = e^{100x}(P'(x) + 100 \cdot P(x))$

$\Rightarrow$ $e^{100c}(P'(c) + 100 \cdot P(c)) = 0$, from (1)

$\Rightarrow$ $P'(c) + 100 \cdot P(c) = 0$ (as $[e^{100c} \neq 0]$) hence proved.]

6.$_{21/aod}$ Consider the function $f(x) = \begin{cases} x \sin\dfrac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ then the number of points in (0, 1) where the derivative $f'(x)$ vanishes, is

(A) 0 (B) 1 (C) 2 (D*) infinite

[Hint: f (x) vanishes at points where $\sin\dfrac{\pi}{x} = 0$ i.e. $\dfrac{\pi}{x} = k\pi$, k = 1, 2, 3, 4, ...

hence $x = \dfrac{1}{k}$. Also $f'(x) = \sin\dfrac{\pi}{x} - \dfrac{\pi}{x}\cos\dfrac{\pi}{x}$ if $x \neq 0$

Since the function has a derivative at any interior point of the interval (0, 1), also continuous in [0,1] and f (0) = f (1) hence Rolle's theorem is applicable to any one of the interval $\left[\dfrac{1}{2}, 1\right]$, $\left[\dfrac{1}{3}, \dfrac{1}{2}\right]$, ... $\left[\dfrac{1}{k+1}, \dfrac{1}{k}\right]$

hence $\exists$ some c in each of these interval where $f'(c) = 0$ $\Rightarrow$ infinite points $\Rightarrow$ (D)]

7. Let f be a continuous function on [a, b]. If $F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x - (a+b))$ then there exist some $c \in (a, b)$ such that

(A) $\int_a^c f(t)\,dt = \int_c^b f(t)\,dt$ (B*) $\int_a^c f(t)\,dt - \int_c^b f(t)\,dt = f(c)(a + b - 2c)$

(C) $\int_a^c f(t)\,dt - \int_c^b f(t)\,dt = f(c)(2c - (a+b))$ (D) $\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = f(c)(2c - (a+b))$

[Sol. Given $F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x - (a+b))$ ....(1) **[12th, 24-08-2008]**

as f is continuous hence F (x) is also continuous. Also

put $x = a$

$$F(a) = \left(-\int_a^b f(t)\,dt\right)(a-b) = (b-a)\int_a^b f(t)\,dt$$

and put $x = b$

$$F(b) = \left(\int_a^b f(t)\,dt\right)(b-a)$$

hence F (a) = F (b)

hence Rolle's Theorem is applicable to F (x)

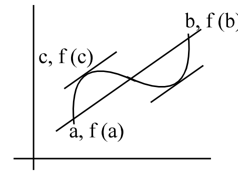
$\therefore$ $\exists$ some $c \in (a, b)$ such that $F'(c) = 0$

now $F'(x) = 2\left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right) + (2x-(a+b))[f(x)+f(x)] = 0$

$\therefore$ $F'(c) = \left(\int_a^c f(t)\,dt - \int_c^b f(t)\,dt\right) = f(c)[(a+b)-2c]$ **Ans.**]

(b) LMVT THEOREM : (LAGRANGE'S MEAN VALUE THEOREM)

Let f(x) be a function of x subject to the following conditions :



(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \neq f(b)$.

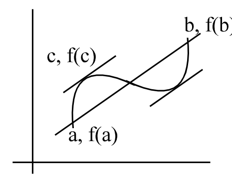
Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Geometrically, the slope of the secant line joining the curve at $x = a$ & $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$. Note the following :

☞ **Note :** Now [f (b) – f (a)] is the change in the value of function f as x changes from a to b so that [f (b) – f (a)] / (b – a) is the *average rate of change* of the function over the interval [a, b]. Also f '(c) is the instantaneous rate of change of the function at x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

☞ Rolles theorem is a special case of LMVT since



$f(a) = f(b) \Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0.$

Alternative form of LMVT :

Another form of statement of Lagrange's Mean Value Theorem. If a function f is continuous in a closed interval [a, a + h] and derivable in the open interval] a, a + h [, then there exists at least one number 'θ' ∈ (0, 1) such that

$$f(a+h) = f(a) + h f'(a+\theta h). \quad \theta \in (0, 1)$$

Proof: We write b – a = h so that *h* denotes the length of the interval [a, b] which may now be rewritten as [a, a + h]. The number, 'c' which lies between a and a + h, is greater than a and less than a + h so that we may write $c = a + \theta h$, where θ is some number between 0 and 1. Thus the equation (i) becomes

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h) \Rightarrow \quad f(a+h) = f(a) + h f'(a+\theta h)$$

Q.1 Find 'θ' in the LMVT expressed as $f(a+h) = f(a) + h f'(a+\theta h)$

(i) $f(x) = \ln x$, $a = 1$, $h = e - 1$; $\theta = \dfrac{e-2}{e-1}$

(ii) $f(x) = 2x^2 - 7x + 10$ in [2, 5]; $\theta = \dfrac{1}{2}$

(iii) $f(x) = 3x^2 - 5x + 12$ in $[0, 1]$; $\theta = \frac{2}{3}$

(iv) $f(x) = \sqrt{3}$ in $[1, 4]$; $\theta = \frac{5}{12}$

(v) $f(x) = x^2$ if $a = 1$, and $h = 1$; $\theta = \frac{1}{2}$

ILLUSTRATIONS LMVT THEOREM:

1. Find c of LMVT

(i) $f(x) = \sqrt{x-1}$ in $[1, 3]$ [Ans. $c = \frac{3}{2}$]

(ii) $f(x) = \frac{x}{x+2}$ on $[1, 4]$ [Ans. $c = 3\sqrt{2} - 2$]

(iii) If LMVT is known to be applicable for a quadratic polynomial $y = lx^2 + mx + n$ in $[a, b]$ then c of LMVT occurs at the midpoint of the interval i.e. $c = \frac{a+b}{2}$.

2.(a) Using LMVT prove that $| \cos a - \cos b | \le | a - b |$

[Hint: Consider $f(x) = \cos x$ in $[a, b]$

$$\left| \frac{\cos b - \cos a}{b - a} \right| = | -\sin c | \le 1 \quad \Rightarrow \quad | \cos b - \cos a | \le | b - a |$$

$| \cos a - \cos b | \le | a - b |$ Hence proved.]

2.(b) Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$.

[Hint: Apply the Mean Value Theorem to the function $h = f - g$]

3.(a) Find a point on the curve $f(x) = \sqrt{x-2}$ in $[2, 3]$ when the tangent is parallel to the chord joining the end points. [Ans. (9/4, 1/2)]

(b) if $a < b$, show that a real number 'c' can be found in (a, b) such that $3c^2 = a^2 + ab + b^2$

[Hint: Consider $f(x) = x^3$ and use LMVT in $[a, b]$]

(c) Use LMVT to prove that $\tan x > x$ for $x \in \left(0, \frac{\pi}{2}\right)$ **(NCERT)**

[Hint: $f(x) = \tan x$ in $[0, x]$ $\quad 0 < x < \pi/2$; Using LMVT

$$\frac{\tan x - 0}{x - 0} = \sec^2 x > 1 \quad \Rightarrow \quad \tan x > x$$]

4. If $f(x)$ is continuous on $[0, 2]$, differentiable on $(0, 2)$, $f(0) = 2$, $f(2) = 8$, and $f'(x) \le 3$ for all x in $(0, 2)$, then $f(1)$ has the value equal to

(A) 3 (B*) 5 (C) 10 (D) There is not enough information

[Sol. Applying LMVT in $[0, 1]$ for f and then in $[1, 2]$ **[12th (17-09-2006)]**

$c_1 \in (0, 1)$, $\frac{f(1) - f(0)}{1 - 0} = f'(c_1) \le 3 \quad \Rightarrow \quad f(1) \le 5$

$c_2 \in (1, 2)$, $\frac{f(2) - f(1)}{2 - 1} = f'(c_2) \le 3 \quad \Rightarrow \quad f(1) \ge 5$

Hence, $f(1) = 5$ Ans.]

5. If f (x) and g (x) are continuous on [a, b] and derivable in (a, b) then show that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} \text{ where } a < c < b$$

[Hint: Consider a function

$$F(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} \text{ which is continuous on } [a, b] \text{ and differentiable in } (a, b).]$$

6.**(a)** Prove that there exists no such function f (x) such that $f(1) = -1$; $f(4) = 7$ and $f'(x) > 3$ for all $x \in R$.

[Sol. using LMVT for f in [1, 4]
there must exist some $c \in (1, 4)$

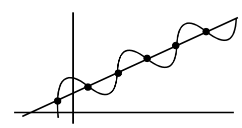
s.t. $f'(c) = \dfrac{f(4)-f(1)}{4-1} = \dfrac{7+1}{3} = \dfrac{8}{3}$

but it is given that $f'(x) > 3 \ \forall x \in R$
hence contradiction
$\therefore$ no such function exists.

Alternatively: $\int_1^4 f'(x)\,dx > \int_1^4 3\,dx = 9$; $f(4) - f(1) > 9$ as $8 > 9 \to$ nonsence $\Rightarrow$ no such f exist]

6.(b) A continuous and differentiable function y = f (x) is such that its graph cuts the line $y = px + q$ at *m* distinct points. Minimum number of points at which $f''(x) = 0$, is/are
(A) $(m-1)$ (B) $(m-2)$ (C) $(m-3)$ (D*) None

[Sol. Very obviously if line cuts the graph at *m* points then
using LMVT $\exists$ atleast $(m-1)$ points where $f'(x) = p$
(slope of line)
$\therefore$ using Rolles $\exists$ atleast $(m-2)$ points
where $f''(x) = 0$] **[TN, AOD, Rolle's] to be put**



7. If the function $f : [0, 16] \to R$ is differentiable. If $0 < \alpha < 1$ and $1 < \beta < 2$, then $\int_0^{16} f(t)\,dt$ is equal to

(A) $4\left(\alpha^3 f(\alpha^4) - \beta^3 f(\beta^4)\right)$ (B*) $4\left(\alpha^3 f(\alpha^4) + \beta^3 f(\beta^4)\right)$
(C) $4\left(\alpha^4 f(\alpha^3) + \beta^4 f(\beta^3)\right)$ (D) $4\left(\alpha^2 f(\alpha^2) + \beta^2 f(\beta^2)\right)$

[Sol. $I = \int_0^{16} f(t)\,dt$ **[12th, 07-12-2008, P-1]**

consider $g(x) = \int_0^{x^4} f(t)\,dt \Rightarrow g(0) = 0$

LMVT for g in [0, 1] gives, some $\alpha \in (0, 1)$ such that $\dfrac{g(1)-g(0)}{1-0} = g'(\alpha)$ .....(1)

|||ly LMVT in [1, 2] gives some $\beta \in (1, 2)$ such that $\dfrac{g(2)-g(1)}{2-1} = g'(\beta)$ .....(2)

(1) + (2) $g'(\alpha) + g'(\beta) = g(2) - \underbrace{g(0)}_{\text{zero}}$; but $g'(x) = f(x^4) \cdot 4x^3$

$$4\left(\alpha^3 f(\alpha^4) + \beta^3 f(\beta^4)\right) = \int_0^{16} f(t)\,dt \Rightarrow \textbf{(B)}]$$

8.$_{192/aod}$ If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0 \ \forall x \in (a, b)$, then
(A*) $f''(c) < 0$ for some $c \in (a, b)$ (B) $f''(c) > 0 \ \forall c \in (a, b)$
(C) $f(c) = 0$ for some $c \in (a, b)$ (D) none

[Hint: Applying LMVT over f (x) for $x \in \left[a, \frac{a+b}{2}\right]$ **[29-01-2006, 12&13]**

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = \frac{2}{(b-a)} \cdot f\left(\frac{a+b}{2}\right), \quad c_1 \in \left(a, \frac{a+b}{2}\right)$$

LMVT in $\left[\frac{a+b}{2}, b\right]$; |||ly $f'(c_2) = -\dfrac{2f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}}$, $c_2 \in \left(\frac{a+b}{2}, b\right)$

Applying LMVT over y = f ' (x) in $[c_1, c_2]$

$$f''(x) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{-\frac{4}{(b-a)} \cdot f\left(\frac{a+b}{2}\right)}{c_2 - c_1} < 0, \text{ where } x \in (c_1, c_2) \text{]}$$

9. Suppose f and g be two differentiable function in [a, b] where $0 < a < b$. If $g(x) \neq 0$ for all x in [a, b]. Prove that there exists 'c' such that $a < c < b$ and $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Use this result to find whether 'c' is in A.M. / G.M. / H.M. / none between 'a' and 'b' for the pair of function given below in [a, b]

(i) $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$; (ii) $f(x) = e^x$ and $g(x) = e^{-x}$; (iii) $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$

[Ans. (i) H.M.; (ii) A.M.; (iii) G.M.]

[Sol. Let A g (x) – f (x) in [a, b] **[quiz-62 12th 2008]**

where $A = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$\phi(a) = A\,g(a) - f(a)$; $\phi(b) = A\,g(b) - f(b)$

now $A\,g(a) - f(a) = A\,g(b) - f(b)$

$f(b) - f(a) = A\big(g(b) - g(a)\big)$

$\Rightarrow$ $\phi(a) = \phi(b)$

$A\,g'(c) - f'(c) = 0;$ $\dfrac{f'(c)}{g'(c)} = A = \dfrac{f(b)-f(a)}{g(b)-g(a)}$]

Generalisation of Mean Value Theorem for integrals

Statement: Let f (x) and g (x) be continuous functions and let a and b be two real numbers with $a < b$ then $\exists$ a real number $c \in (a, b)$ such that

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = f(c) \cdot (b-c) + g(c) \cdot (c-a)$$

Proof: Consider a function

$$h(x) = (x-b)(x-b)\int_a^x f(t)\,dt + (x-a)\int_x^b g(t)\,dt$$

find h ' (x). Show that h (a) = h(b) and apply Rolles Theorem for h (x).]

7. CONTRIBUTION OF MONOTONIC BEHAVIOUR IN PLOTTING THE GRAPHS OF MISCELLANEOUS FUNCTIONS :

<u>General tips for plotting the graph of a rational function</u> :

(1) Examine whether denominator has a root or not. If no, then graph is continuous and f is non monotonic. For eg. $f(x) = \dfrac{x}{x^2-5x+9}$.
If denominator has roots then f (x) is discontinuous. Such functions can be monotonic / non-monotonic.
For eg. $f(x) = \dfrac{x^2+2x-3}{x^2+2x-8} = \dfrac{(x+3)(x-1)}{(x+4)(x-2)}$

(2) If numerator and denominator has a common factor (say x = a) it would mean removable discontinuity at x = a e.g. $f(x) = \dfrac{(x-2)(x-1)}{(x+3)(x-2)}$. Such a function will always be monotonic i.e. either increasing or decreasing and removable discontinuity at x = 2 .

(3) Compute points where the curve crosses the x-axis and also where it cuts the y-axis by putting y = 0 and x = 0 respectively and mark points accordingly.

(4) Compute $\dfrac{dy}{dx}$ and find the intervals where f (x) is increasing or decreasing and also where it has horizontal tangent.

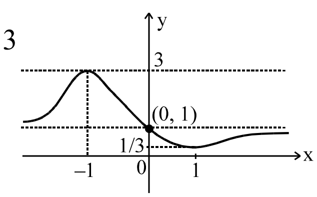
(5) In regions where curve is monotonic compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

(6) If denominator vanishes say at x = a and (x – a) is not a common factor between numerator and denominator then examine $\lim\limits_{x\to a^+}$ and $\lim\limits_{x\to a^+}$ to find whether f approaches ∞ or $-\infty$. Now plot the graphs of the following functions.

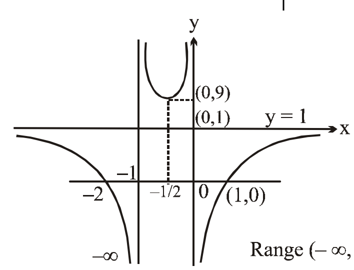
ILLUSTRATIONS GRAPHS OF MISCELLANEOUS FUNCTIONS :

1. $f(x) = \dfrac{x^2-x+1}{x^2+x+1}$; $f'(x) = \dfrac{2(x^2-1)}{(x^2+x+1)^2}$ = $f(1) = \dfrac{1}{3}$; $f(-1) = 3$

   D : $x \in R$, Range $\left[\dfrac{1}{3}, 3\right]$

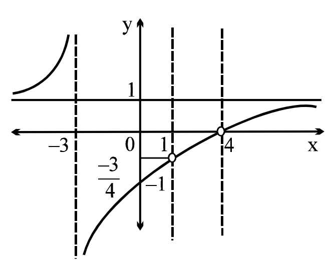


2. $f(x) = \dfrac{(x+2)(x-1)}{x(x+1)}$; $f'(x) = \dfrac{2(2x+1)}{[x(x+1)]^2}$

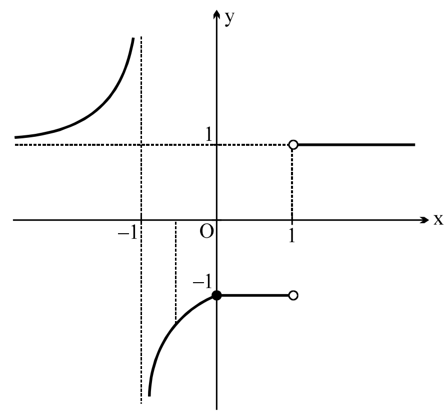


Range $(-\infty, 1) \cup (9, \infty)$

3. $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3} = \dfrac{(x-4)(x-1)}{(x+3)(x-1)}$



4. $y = \dfrac{|x-1|}{|x|-1} = \begin{cases} \dfrac{x-1}{x-1} = 1 & x > 1 \\ \text{N.D.} & \text{at } x = 1, -1 \\ \dfrac{1-x}{x-1} = -1 & 0 \le x < 1 \\ \dfrac{1-x}{-(x+1)} = \dfrac{x-1}{x+1} & x < 0, x \ne -1 \end{cases}$



Home Work :

(1) $f(x) = \dfrac{x+2}{2x^2+3x+6}$ (2) $f(x) = \dfrac{x^2+2x-11}{2(x-3)}$ (3) $f(x) = \dfrac{x^2+2x-3}{x^2+2x-8}$

(4) $f(x) = \dfrac{x^2-3x+2}{x^2+x-6}$ (5) $f(x) = \dfrac{x^2-x+1}{x^2+x+1}$ (6) $f(x) = \dfrac{x^2-1}{x^2-3x}$

ISOLATING ROOT OF AN EQUATION AND USE PROBLEM USING MONOTONOCITY:

(1) Prove that the equation $\dfrac{x^3+1}{x^2+1} = 5$ has no root in [0, 2]

[Proof: $x^3 + 1 = 5x^2 + 5 \Rightarrow x^3 - 5x - 4 = 0$

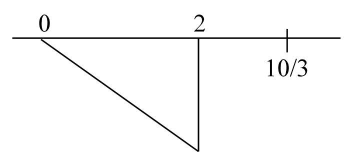
Let $f(x) = x^3 - 5x - 4$

$f'(x) = 3x^2 - 10x = x(3x - 10)$

hence f is decreasing in (0, 10/3)

$\therefore$ $f(2) < f(0) < 0$

hence no root in [0, 2] **Hence proved**]



$(2)_{304/de}$ **Statement-1:** The line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-\frac{x}{a}}$ at some point $x = x_0$.

because

Statement-2: $\dfrac{dy}{dx}$ exists at $x = x_0$.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B*) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

[Hint: Line touches the curve at (0, b) and $\left.\dfrac{dy}{dx}\right]_{x=0}$ also exists but even if $\dfrac{dy}{dx}$ fails to exist. tangents line can be drawn.] **[12th, 07-12-2008, P-2]**

$(3)_{87/aod}$ Number of roots of the function $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x$ is

(A) 0 (B) 1 (C*) 2 (D) more than 2

[Sol. $f'(x) = -\dfrac{3}{(x+1)^4} - 3 + \cos x < 0$ **[13th (24-09-2006)]**

hence $f(x)$ is always decreasing, Also as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to +\infty$

hence one positive and one negative root

Graph is as shown

MAXIMA - MINIMA

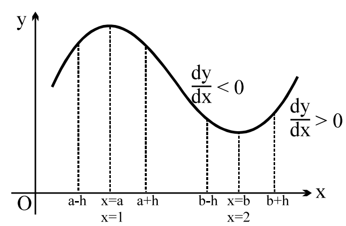
(Functions Of Single Variable)

(A) GENERAL INTRODUCTION :

The notion of optimising functions is one of the most useful application of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home. In this chapter we shall see how calculus defines the notion of maxima and minima and distinguishes it from the greatest and least value or global maxima and global minima of a function. Since most of the functions which we encounter with in practical world are differentiable hence we continue our discussion with such functions only unless otherwise stated.

(B) HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$. Symbolically



$$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a \text{ gives maxima}$$

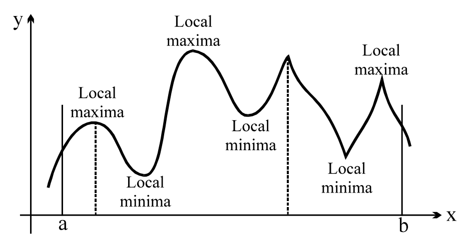
for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$. Symbolically if

$$\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive h.}$$

Note that :

(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.



(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

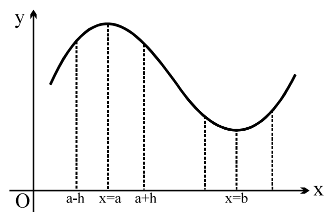
(C) NECESSARY AND SUFFICIENT CONDITIONS FOR A LOCAL MAXIMUM & MINIMUM :

<u>Fermats theorem</u>: If f has a local maxima or minima at x = 'a' and $f'(a)$ exists then $f'(a) = 0$

Consider the interval $(a-h, a)$, we find $f(x)$ is increasing $\Rightarrow \frac{dy}{dx} > 0$. Similarly for the interval $(a, a+h)$, we find $f(x)$ is decreasing $\Rightarrow \frac{dy}{dx} < 0$.

Hence at the point $x = a$ (maxima) ; $\frac{dy}{dx} = 0$

similarly $\frac{dy}{dx} = 0$ at $x = b$ which is the point of minima.



Hence $\frac{dy}{dx} = 0$ is the necessary condition for maxima or minima. These points where $\frac{dy}{dx}$ vanishes are known as stationary points as instantaneous rate of change of function momentarily ceases at this point.

However if for $x < a$, $\frac{dy}{dx} > 0$ (in the imm.nbd) and for $x > a$, $\frac{dy}{dx} < 0$ (in the imm.nbd) $\Rightarrow$ *$x = a$ is the point of local maxima $\Rightarrow$ derivative changes sign from +ve to –ve crossing over the point $x = a$ to L R $\Rightarrow$ L M*

and if for $x < b$, $\frac{dy}{dx} < 0$ (in the imm.nbd) and $x > b$, $\frac{dy}{dx} > 0$ (in the imm.nbd) $\Rightarrow$ $x = b$ is the point of local minima

Hence if

$$\left.\begin{array}{r} f'(a-h)>0 \\ \text{and } f'(a+h)<0 \end{array}\right] \Rightarrow$$ *$x = a$ is a point of local maxima, where $f'(a) = 0$ (f must be continuous at $x = a$)*

Similarly $$\left.\begin{array}{r} f'(b-h)<0 \\ f'(b+h)>0 \end{array}\right] \Rightarrow$$ *$x = a$ is a point of local minima, where $f'(b) = 0$. (f must be continuous at $x = a$)*

h is a sufficiently small positive quantity

However, if $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(c) is not an extreme value of f. ***e.g.*** $f(x) = x^3$ at $x = 0$.

(D) USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA FOR A DIFFERENTIABLE FUNCTION:

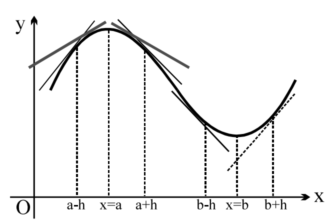
EXPLANATION :

As shown in the figure it is clear that as x increases from $a - h$ to $a + h$, the function $\frac{dy}{dx}$ continuously decreases, i.e. (+) ve for $x < a$, zero at $x = a$ and (–) ve for $x > a$.

Hence $\frac{dy}{dx}$ itself is a decreasing function. Therefore $\frac{d^2y}{dx^2} < 0$ in $(a - h, a + h)$.

Hence at local maxima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

$f'(a) = 0$ and $f''(a) < 0$



similarly at local minima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

i.e. $f'(b) = 0$ and $f''(b) > 0$

Hence if

(a) f (a) is a maximum value of the function f then f '(a) = 0 & f "(a) < 0.

(b) f (b) is a minimum value of the function f, if f '(b) = 0 & f "(b) > 0.

However, if f "(c) = 0 then the test fails. In this case f can still have a maxima or minima or point of inflection (neither maxima nor minima). In this case revert back to the first order derivative check for ascertaning the maxima or minima.

SUMMARY–WORKING RULE :

FIRST: When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. his may require some algebraic manipulations.

THIRD : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

FOURTH : Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow y$ is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow y$ is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) If $\frac{dy}{dx}$ is $\left.\begin{array}{lll} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{array}\right] \Rightarrow$ a maximum occurs at $x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum . If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called **INFLECTION POINTS**.

FIFTH : If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

SIXTH : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

ILLUSTRATIONS: (Refer Exercise-6.5, NCERT Part-I)

<u>Geometrical Problem</u>:

(Minima) :

Inventa,
Karl Fisher,
Du pont,
Barmag,
Mackenzy,
Technimont,
E.I.L.

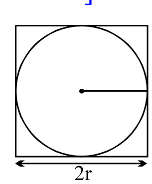
1. Of all the open right circular cylinders with a given volume, the one which is most economical i.e. requiring the least, surface area.

2. You are designing a 1728 cm^3 closed right circular cylindrical cans whose manufacture will take waste into account. There is no waste in cutting the aluminium sheet for the curved surface, but the tops and bottom of radius "*r*" will be cut from squares that measure "2r" units on a side. Find the total quantity of aluminium (in square cm) for the manufacture of a most economical can. [**Ans.** 864]

[Sol. $V = \pi r^2 h$, $V = 1728$

$S = 2\pi rh + 8r^2$

$S(r) = 8r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 8r^2 + \frac{2V}{r}$



$S'(r) = 16r - \frac{2V}{r^2} = 0$

$r^3 = \frac{2V}{16} = \frac{V}{8} = \frac{1728}{8} = 216$ **[12th, 24-08-2008]**

$S(r)|_{min} = 8 \cdot 36 + \frac{2 \cdot 1728}{6}$

$= 288 + 576 = 864$ **Ans.**]

(Maxima) : (h = 2d)

2. For a given slant height volume of conical tent is maximum if $\theta = \tan^{-1}\sqrt{2}$

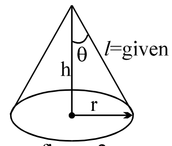


figure-2

(Minima) :

3. One corner of a rectangular sheet of paper, width 1m, is folded over so on just reach the opposite edge of the sheet. Find the minimum length of crease. [Ans. $\frac{3\sqrt{3}}{4}$]

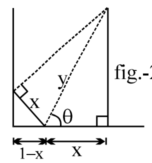


fig.-2

(Maxima) :

4. For what numbers a and b does the function $f(x) = axe^{bx^2}$ have the maximum value f (2)=1

[Sol. $f(x) = axe^{bx^2}$; $f(2) = 1$ and $f'(2) = 0$; $2ae^{4b} = 1$ ....(1)

also $f'(x) = a[xe^{bx^2} \cdot 2bx + e^{bx^2}] = axe^{bx^2}[2bx^2 + 1]$

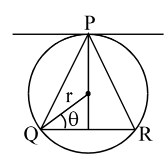
$f'(2) = ae^{4b}(8b + 1) = 0$; $a = 0$ or $b = -1/8$ but $a \neq 0$; $a = e/2$

hence $a = \frac{\sqrt{e}}{2}$ and $b = -\frac{1}{8}$]

5. A wire of length 20 cm is cut into two pieces. One piece converted into a circle and the other into a square. Where the wire is to be cut from so that the sum total of the areas of two plane figures is **(a)** minimum **(b)** maximum.

6.$_{89/02}$ A point P is given on the circumference of a circle of radius *r*. Chords QR are parallel to the tangent at P . Determine the maximum possible area of the triangle PQR if r = 2.

[Sol. Ar. of $\Delta PQR = \frac{\cancel{2}r\cos\theta(r + r\sin\theta)}{\cancel{2}}$ [Ans: $\left(\frac{3\sqrt{3}}{4}\right) \cdot r^2$] [JEE'90,4]



$A(\theta) = r^2 \cos\theta\,(1 + \sin\theta)$

$A'(\theta) = r^2 [\cos\theta \cdot \cos\theta - (1 + \sin\theta)\sin\theta]$

for maximum or minimum $A'(\theta) = 0$

$\cos^2\theta - \sin\theta\,(1 + \sin\theta) = 0 \quad = \quad 1 - \sin^2\theta - \sin\theta - \sin^2\theta = 0$

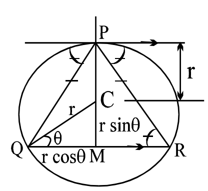
$= 2\sin^2\theta + \sin\theta - 1 = 0 \quad = \quad (\sin\theta + 1)(2\sin\theta - 1) = 0$

$\sin\theta = -1$ (not possible)

$\sin\theta = 1/2$ hence, $\theta = \pi/6$ or $5\pi/6$

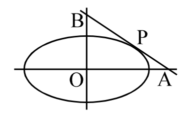
$\frac{d^2A}{d\theta^2} = -2\cos\theta\sin\theta - \cos\theta - 2\sin\theta\cos\theta$

$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\pi/6} = -\text{ve} \Rightarrow$ A is maximum



$A_{max} = 2r^2 \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)\frac{1}{2} \Rightarrow A_{max} = \frac{3\sqrt{3}}{4}r^2$]

7.(a) Find the coordinates of the point P on the curve $\frac{x^2}{8}+\frac{y^2}{18}=1$ in the 1st quadrant so that the area of the triangle formed by the tangent at P and the coordinate axes is minimum.



[Ans. (2, 3)] [Ans. $(x = 2\sqrt{2}\cos\theta,\ y = 3\sqrt{2}\sin\theta)$]

7.(b) Find the co-ordinates of all the points P on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P. [JEE '99, 10 out of 200]

[Ans. $\pm\frac{a^2}{\sqrt{a^2+b^2}}$, $\pm\frac{b^2}{\sqrt{a^2+b^2}}$]

8. For a train/steamer the cost of fuel varies as the square of its speed (in km/hr.) and the cost is Rs 24/hr when the speed is 12 km/hr. If other expenses amounts to Rs. 96 /hr, find the most economical speed and the cost of journey for 100 km.

[Ans. 24 km/hr; Rs. 800/-]

9. Find the equation of a line through (1, 8) cutting the positive semi axes at A and B if
(i) the area of ΔOAB is minimum
(ii) its intercept between the coordinate axes is minimum.
(ii) sum of its intercept on the coordinates axes is minimum.

[Ans. (i) $8x + y = 16$, (ii) $2x + y = 10$; min. intercept $5\sqrt{5}$, (iii) $2\sqrt{2}x + y = 8 + 2\sqrt{2}$]

10. Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height 'h'. [Ans. $h/3$]

11. Find the altitude of the right cone of maximum volume that can be inscribed in a sphere of radius R. [Ans. $4R/3$]

12. The corridors of width a & b meet at right angles . Show that the length of the longest pipe that can be passed round the corner horizontally is, $(a^{2/3} + b^{2/3})^{3/2}$.

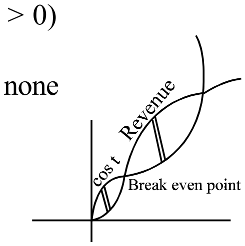
<u>Cost and Revenue function</u>:

1.198/AOD The cost function at American Gadget is

$C(x) = x^3 - 6x^2 + 15x$ (x in thousands of units and $x > 0$)

The production level at which average cost is minimum is

(A) 2 (B*) 3 (C) 5 (D) none



[Hint: $C(x)$ = cost of producing x items > 0

average cost $= \frac{C(x)}{x} = x^2 - 6x + 15 = f(x)$

$f'(x) = 2x - 3 = 0 \Rightarrow x = 3$]

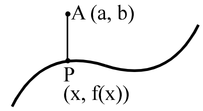
2. Manufacturer can sell x items at a price of Rs. $\left(5-\frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

[Ans. $x = 240$]

[Hint: $P = S(x) - C(x) = \left(5-\frac{x}{100}\right)x - \left(\frac{x}{5}+500\right)$; $P' = 5 - \frac{x}{50} - \frac{1}{5} \Rightarrow \frac{x}{50} = \frac{24}{5} \Rightarrow x = 240$]

GENERAL CONCEPT :

Given a fixed point A (a, b) and a moving point P (x, f (x))



on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.

Proof : $F(x) = (x-a)^2 + (f(x)-b)^2 \Rightarrow F'(x) = 2(x-a) + 2(f(x)-b)\cdot f'(x)$

$$\therefore \quad f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x)\cdot m_{AP} = -1.$$

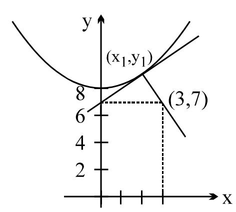
ILLUSTRATION ON GENERAL CONCEPT :

1. A straight line l passes through the points (3, 0) and (0, 4). The point A lies on the parabola $y = 2x - x^2$. Find the distance p from point A to the straight line and indicate the coordinates of the point A (x_0, y_0) on the parabola for which the distance from the parabola to the straight line is the least.

2. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B, and D are known A (–2, 3), B (–1, 1), D (2, 7). Find the coordinates of C for which the area of the quadrilateral ABCD is the greatest.
[Ans. a = b = c = 1, (1/2, 7/4)]

3. A helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier placed at (3, 7) wants to shoot down the helicopter when it is nearest to him. Find the nearest distance. [Ans. $\sqrt{5}$]

[Hint: $\left.\frac{dy}{dx}\right]_{x_1,y_1} = 2x_1$



$$\therefore \quad (2x_1)\left(\frac{7-y_1}{3-x_1}\right) = -1$$

$2x_1(7 - y_1) = x_1 - 3$

$14x_1 - 2x_1y_1 = x_1 - 3$

$13x_1 + 3 = 2x_1(x_1^2 + 7)$

$2x_1^3 + x_1 - 3 = 0 \quad \Rightarrow \quad 2x_1^3 + 3x_1 - 3 = 0$

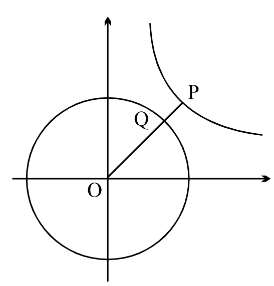
$2x_1^2(x_1 - 1) + 2x_1(x_1 - 1) + 3(x_1 - 1)$

$(x_1 - 1)(2x_1^2 + 2x_1 + 3)$

$x_1 = 1$]

4. Find the minimum distance between $x^2 + y^2 = 2$ and $xy = 9$. [Ans. $2\sqrt{2}$]

[Hint: $xy = 9 \left\langle \begin{matrix} x = 3t \\ y = 3t \end{matrix} \right.$



$$\frac{dy}{dx} = -\frac{9}{x^2}$$

slope of the normal at the point P(t)

$$\frac{x^2}{9} = t^2$$

equation of normal at P is

$$y - \frac{3}{t} = t^2(x - 3t) \quad ....(1)$$

minimum distance = PQ i.e. along common normal

$\therefore$ (1) must pass through origin

$\therefore \quad -\frac{3}{t} = -3t^2 \quad \Rightarrow \quad t^4 = 1 \Rightarrow \quad t = 1$ or -1

(t = – 1 corresponds to a point xy = 9 in the 3rd quadrant)

$\therefore \quad t = 1$

hence P is (3, 3) $\Rightarrow \quad m_{OP} = 1$

equation of OP is $y = x \quad \Rightarrow \quad Q(1, 1)$

hence $d_{min.} = \sqrt{4+4} = 2\sqrt{2}$ Ans.]

find the minimum value of $y = (x_1 - x_2)^2 + \left(\sqrt{2-x_1} - \frac{9}{x_2}\right)^2$

$x_1 \in (0, \sqrt{2})$; $x_2 \in R^+$;]

(E) USEFUL FORMULAE OF MENSURATION TO REMEMBER :

- ☞ Volume of a cuboid = $l$bh.
- ☞ Surface area of a cuboid = 2 (lb + bh + hl).
- ☞ Volume of a prism = area of the base x height.
- ☞ Lateral surface of a prism = perimeter of the base x height.
- ☞ Total surface of a prism = lateral surface + 2 area of the base
  (Note that lateral surfaces of a prism are all rectangles).
- ☞ Volume of a pyramid = $\frac{1}{3}$(area of the base) × (height).
- ☞ Curved surface of a pyramid = $\frac{1}{2}$(perimeter of the base) x slant height.
  (Note that slant surfaces of a pyramid are triangles).
- ☞ Volume of a cone = $\frac{1}{3}\pi r^2 h$.
- ☞ Curved surface of a cylinder = $2\pi rh$.
- ☞ Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
- ☞ Volume of a sphere = $\frac{4}{3}\pi r^3$.
- ☞ Surface area of a sphere = $4\pi r^2$.
- ☞ Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.

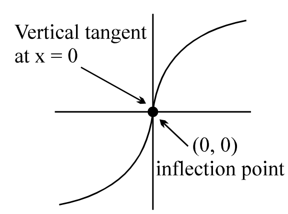
(F) SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

A point where the graph of function is continuous and has a tangent line and where the concavity changes is called point of inflection.

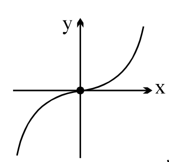
– At the point of inflection either y'' = 0 and changes sign or y '' fails to exist.
– At the point of inflection, the curve crosses its tangent at that point.
– A function can not have point of inflection and extrema at same point.

Note: If $\frac{d^2y}{dx^2} > 0$ then y is concave up and if $\frac{d^2y}{dx^2} < 0$ the y is concave down.

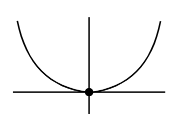
EXAMPLES:



(i) $f(x) = x^{1/5}$ at x = 0 has inflection point.
y '' D.N.E. at x = 0
Note that f (x) has a vertical tangent and the curve crosses its tangent line.

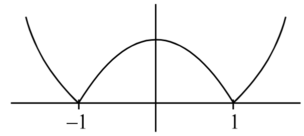


(ii) $f(x) = x^3$ at x = 0 has inflection point
y '' = 0 at x = 0 and changes sign

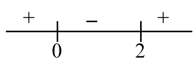


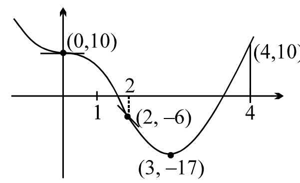
(iii) $f(x) = x^4$ at x = 0 has no inflection point though y " = 0 at x = 0 as y " does not change sign

(iv) $f(x) = |x^2 - 1|$
has no inflection point in its domain.
as no tangent can be drawn at these points.



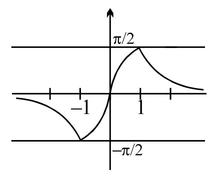
(v) $f(x) = x^4 - 4x^3 + 10$ $\quad f(1) > 0 \ \& \ f(2) < 0$
$y' = 4x^2(x - 3) \Rightarrow y' = 0$ at $x = 0 \ \& \ x = 3$
(change sign only at x = 3)
$y'' = 12x(x - 2) \Rightarrow y'' = 0$ at $x = 0 \ \& \ x = 2$
note that for $x < 0$, $\dfrac{d^2y}{dx^2} > 0$; between (0, 2), $\dfrac{d^2y}{dx^2} < 0$
or $x < 2$





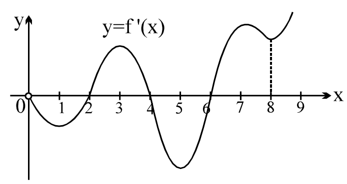
x = 0 and x = 2 are points of inflection.
x = 3 is the point of minima.

(vi) $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ has x = 0 as point inflection.
$x = \pm 1$ are not the point of inflection as no tangent line can be drawn.



EXAMPLES

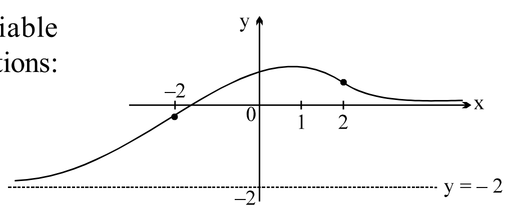
Ex.1 The graph of the first derivative f' of a function f is shown.



(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upwards or concave downwards?
(d) What are the x-coordinates of the inflection points of f?

[Ans. (a) increasing in $(2, 4) \cup (6, \infty)$ and decreasing in $(0, 2) \cup (4, 6)$
(b) x = 2 (minima), 4 (Maxima) and 6 (minima)
(c) concave up in $(1, 3) \cup (5, 7) \cup (8, \infty)$; concave down in $(0, 1) \cup (3, 5) \cup (7, 8)$
(d) x = 1, 3, 5, 7, 8]

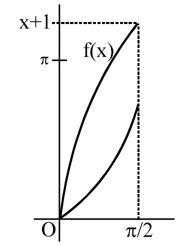
Ex.2 Sketch a possible graph of a differentiable function f that satisfies the following conditions:
(i) $f'(x) > 0$ on $(-\infty, 1)$,
$f'(x) < 0$ on $(1, \infty)$
(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$
(iii) $\lim\limits_{x \to -\infty} f(x) = -2$, $\lim\limits_{x \to \infty} f(x) = 0$



[Sol. Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.
Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$.
From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ and $y = 0$.

We first draw the horizontal asymptote y = – 2 and a dashed line. We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at x = 1 and decreasing toward the x-axis at the far right. We also make sure that the graph has inflection points when x = – 2 and 2. Notice that we made the curve bend upward for x < – 2 and x > 2, and bend downward when x is between – 2 and 2.]

Ex.3 Prove the inequality : $\sin x + 2x \ge \frac{3x(x+1)}{\pi}$ $\forall$ $x \in \left[0, \frac{\pi}{2}\right]$ [JEE 2004]



[Sol. $f(x) = \sin x + 2x$; $g(x) = \frac{3(x^2 + x)}{\pi}$

$f'(x) = \cos x + 2$ $(\uparrow)$ $\qquad g'(x) = \frac{3}{\pi}(2x+1)$

$f''(x) = -\sin x < 0$ $\qquad g''(x) = \frac{3}{\pi}(2) > 0$

$\Rightarrow$ concave down and increasing $\qquad \Rightarrow$ concave up and increasing

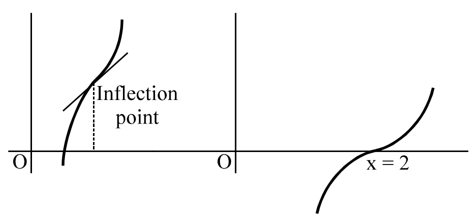
$f\left(\frac{\pi}{2}\right) = \pi + 1$ $\qquad g\left(\frac{\pi}{2}\right) = 3\frac{\pi}{2\pi}\left(\frac{\pi}{2}+1\right) = \frac{3}{2}\left(\frac{\pi}{2}+1\right) = \frac{3\pi}{4} + \frac{3}{2} = \pi + \frac{3}{4} - \frac{\pi}{4}$]

(G) DIFFERENT GRAPHS OF THE CUBIC:

$y = ax^3 + bx^2 + cx + d$

(1) One real & two imaginary roots. (always monotonic) $\forall$ $x \in R$

Condition : $f'(x) \ge 0$ or $f'(x) \le 0$ together with either $f'(x) = 0$ has no root (i.e. D < 0) or $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$.



(i) either $f'(x) = 0$ has no real root
or (ii) if $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$

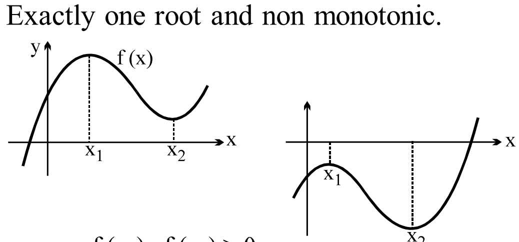
e.g. $y = x^3 - 2x^2 + 5x + 4$ $\qquad y = (x-2)^3$

$y' = 3x^2 - 4x + 5$ (D < 0) $\qquad y' = 3(x-2)^2 = 0 \Rightarrow x = 2$, also $f(2) = 0$

gives x = 2, y (2) = 0

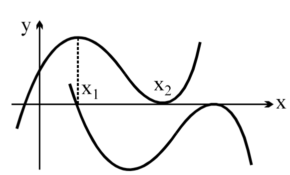
☞ **Note:** In this case if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots which is dealt separately.

(2) Exactly one root and non monotonic. **(3)** Three roots < two concident / One different



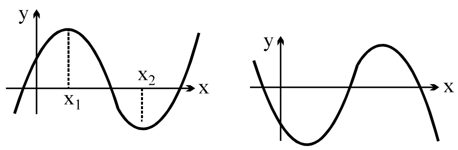
$f(x_1) \cdot f(x_2) > 0$

where x_1 & x_2 are the roots of $f'(x) = 0$

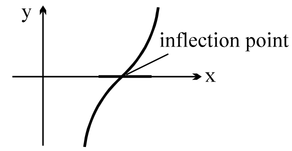


$f(x_1) \cdot f(x_2) = 0$

(4) All three distinct real roots **(5)** All three roots concident



$f(x_1) \cdot f(x_2) < 0$
where x_1 & x_2 are the roots of $f'(x) = 0$



$f'(x) \ge 0$ or $f'(x) \le 0$ & $f(\alpha) = 0$
where α is a root of $f'(x) = 0$
e.g. $y = (x - 1)^3$

☞ **Note :** (i) Graph of every cubic polynomial must have exactly one point of *inflection*.
(ii) In case (4) if $f(a)$, $f(b)$, $f(c)$ and $f(d)$ alternatively change sign.

ILLUSTRATIONS :

1. If the cubic $y = x^3 + px + q$ has 3 distinct real roots then prove that $4p^3 + 27q^2 < 0$.

[Sol. $f'(x) = 3x^2 + 0x + p$

$$x_1 + x_2 = 0 \quad \& \quad x_1 x_2 = \frac{p}{3}$$

$$(x_1^3 + px_1 + q)(x_2^3 + px_2 + q) < 0$$

$$x_1^3 \cdot x_2^3 + p x_1^3 x_2 + q x_1^3$$

$$p^2 x_1 x_2 + p x_1 x_2^3 + q x_2^3 + pqx_1 + q^2 \quad < \quad 0$$

$$+ pqx_2 \qquad]$$

2. For a cubic $f(x) = \frac{ax^3}{3} + (a + 2)x^2 + (a - 1)x + 2$. Find the value of 'a' for which it

(a) + ve point of maximum [both root of $f'(x)=0$ must be + ve $(a > 0)$]
(b) – ve point of minimum [both root of $f'(x)=0$ must be – ve $(a > 0)$]
(c) + ve point of minimum [at least one root of $f'(x)=0$ must be + ve]
(d) – ve point of maximum [at least one root of $f'(x)=0$ must be – ve]
(e) – ve point of inflection [abscissa corresponding to $\frac{d^2y}{dx^2}=0$ must be – ve]
(f) + ve point of inflection [abscissa corresponding to $\frac{d^2y}{dx^2}=0$ must be + ve]