CMPS 102 — Quarter Spring 2017 – Homework 4

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I have read and agree to the collaboration policy.Vladoi Marian
I want to choose homework heavy option.
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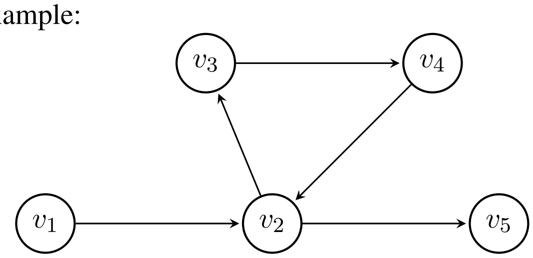
Solution to Problem 3: Flow decomposition

Assume that we have a set of paths (P) connecting source node (s) and sink node (t). Let p_i be one of the paths $\in$ P. We know that for each $p_i \in P$, $f'(p_i) \geq 0$ is a valid path solution iff for every edge $e \in E$, $\sum_{P:p_i:e\in P} f'(p_i) \leq c(e)$. The value of the flow path solution is $\sum_{p_i \in P} f'(p_i)$.
We know that we have a valid acyclic s-t flow, and f is integral.
I will give an iterative algorithm that find a valid flow paths solution .
1. We start at source node (s).
2. Use Depth-First Search from s .
3. We traverse in each iteration of the algorithm an edge $e = (u, v) \in E$ from the vertex u to v iff f(e) > 0.
4. Because the flow is acyclic we know that in a finite number of iteration we have to reach the sink node (t). At this moment we have a s-t path.
5. Let edge e' be the min $\{f(e')|e' \in p_i\}$ (this edge is the edge that is carrying the min amount of flow in path p_i).
6. We add the flow p_i to our set of path flows P, with f'(p_i) = f(e')
7. Then update f(e) = f(e) - f(e') for every edge $e \in p_i$.
8. The algorithm terminates when with a 0 flow coming out of s. (at each iteration on edge in the set P has 0 flow).

a) Suppose that some directed cycle C has positive flow on every edge. As I mentioned in my algorithm e' is min $\{f(e')|e' \in p_i\}$. Then f' is defined by : f'(e) = f(e) - e' if e $\in C$, or f'(e) = f(e) if e $\notin C$. We can prove that f' is still a flow and $|f'| = |f|$.
If f is acyclic, f(e) = 0 must hold for all $e \in E$ at termination.
Assume for contradiction that some edge e has f(e) > 0.
If we have positive flow out of s, then by flow conservation , we can traverses edges on the path and reach t. Assume that we do not have a possitive flow out of s. If e is an edge $\notin out(s)$, and has f(e) >0, we can traverse all the edges in reverse direction and create a cycle, which is a contradiction.
Since f is initialy integral, and f'(p_i) = f(e') is integral at each iteration, the updated flow f'(e) = f(e) - e' is also integral. Therefore f'$\{p_i|p_i \in P\}$ are all integral.

b)The algorithm terminates in m iterations, because at each iteration at least one edege in P is updated to have flow value = 0, and the flow value on an edge never increases.
Given an acyclic flow f , we find an s-t path $p1 \in P$ along which all flow is positive. Our algorithm decrement the flow on each edge of p_1. In the same time this operation will also decrement $|f|$. Then we will repeat the procedure for an s-t path on $p_2 \in P$. At the end , we partion f into a colection of s-t paths (P) of cardinality $|f|$.

c) So we proved that an acyclic flow f can be represented as a finite combination of path flows f'. But if the flow is not guaranteed to be acyclic, then the algorithm is not guarantee to terminate. It is not guaranteed that f(e) = 0 must hold for all $e \in E$ at termination. We might have some edge e $\notin out(s)$ that have f(e) >0. As I previous showed, this implies that our path has cycles. Therefore the flow f is a combination of path flows and cycles. It is not a finite combination of path flows, because we can heve infinite paths s - t.
Example:



Assume that v_1 is the source (s)
And v_5 is the sink (t).
Then we have an infinite number of paths from v_1 to v_5.
v_1, v_2, v_5
$v_1, v_2, v_3, v_4, v_2, v_5.$
$v_1, v_2, v_3, v_4, v_2, v_3, v_4, v_2, v_5.$
$v_1, v_2, v_3, v_4, v_2, v_3, v_4, v_2, v_3, v_4, v_2, v_5.$
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