CMPS 102 — Quarter Spring 2017 – Homework 4

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I have read and agree to the collaboration policy.Vladoi Marian
I want to choose homework heavy option.
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Solution to Problem 2: Cycle Cover

Assume we have a directed graph G(V, E). (V= nodes, E = edges).
We transform the directed graph G(V, E) into a bipartide graph G' ($V_1 \cup V_2$, E') as following:
The nodes N of G' is made of two sets: $(V_1 == V) \cup (V_2 == V)$.
For each node $v \in V$ we create two nodes: v_1 in the set V_1 of the bipartide graph G' , and a related node $v_2 \in V_2$.
For each directed edge $e = (v_1, v_2) \in E$ of our directed graph, we create an undirected edge $e' = (v_1, v_2) \in E'$ in our bipartide graph.
Thus , each directed edge of G becomes an undirected edge in G'. We use the direction of the arc in our bipartition of vertices in G' . The set V_1 containes the tail of the edge, and the set V_2 contains the head of the edge.
I will use the algorithm bipartide matching explained in class.

The algorithm to find a cycle cover for a given graph G:
Input : Graph G (V,E)
Output : (C = cycle cover) return C or C does not exist.
1. Build the bipartide graph G'($V_1 \cup V_2$, E') from G (as I previous described)
2. Run the bipartide matching algorithm described in class.
3. If the maximum matching is perfect
4. Use the informations from G' to create a cycle cover C in G
5. Return C
6. Else if the maximum matching is not perfect
7. Return C does not exist

Claim 1. *A disjoint-cycle cover for G exist iff G' has a perfect matching.*

Proof. We have an if and only if condition to prove.
a) If G' has a perfect matching then a disjoint cycle cover exist.
Assume G' has a perfect matching. Then we know that for each node $u_1 \in V_1$ there is an edge $e' \in E'$ matching it to $v_2 \in V_2$. The edge e' corresponds to an edge $e = (u, v) \in E$. Because we have a perfect

matching , there can not be another edge that include u_1 or v_2 in our matching. Assuming this was the case, it means that we would have two edges sharing a vertex, which is impossible. Nothing in our matching can have the edge (v, u) in G. On the other hand , $v_2 \in V_2$ has a copy $v_1 \in V_1$. The node v_1 has to be connected, by an edge e', to another node in V_2.This shows an important lemma:
$Each\ vertex\ v \in V$ has indegree and outdegree 1 in the graph G' , coresponding to the perfect matching. This is true for all the vertices and it implies that: the directed subgraph in G, created by this edges, is made of directed cycles. Because of the properties of the matching, these cycles must be vertex-disjoint.
b) If we have a disjoint cycle cover , then every vertex in covered. In the same time, every vertex has indegree and outdegree 1 in the edge cover.
Assume v is such a vertex. There is an edge (u, v) $\in E$ because its indegree is one. This edge is part of the cycle cover. Also , there is an edge (v, t) $\in E$ because its outdegree is one. This edge is part of the cycle cover. Coresponding to these edges in G , we have the edges (u_1, v_2) and (v_1, t_2) in G'. Our perfect bipartide algorithm ensure that both copies of v , $(v_1,\ and\ v_2)$ are matched. Hence a cover implies a perfect matching.

The running time:
1. Building the bipartdide graph G' is done in O(m + n)
2. Computing the bipartdide matching algorithm is done in O(mn)
3. Creating the cycle cover in G is done in O(n)
The algorithm runs in O(mn) time.