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λ -calculus Overview



- Describes concepts that are fundamental to **functional programming**
 - **first-class functions** take other functions as arguments and can return a function
 - **currying** implements a function with multiple parameters as a sequence of functions that each have a single parameter
- **Purpose**
 - study how functions can interact with each other, not to calculate results in a useful or efficient way
- **Turing complete**
 - capable of performing any calculation or solving any computational problem, given enough time and memory

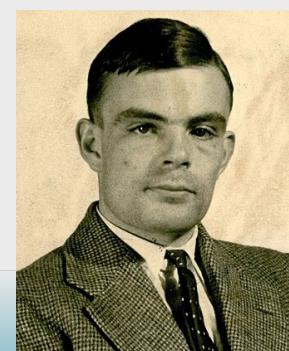
Compared to Slide Rules



- Calculators are an excellent replacement for slide rules
 - It's not necessary to understand how to use a slide rule in order to use a calculator
 - But it's fascinating to learn how slide rules work (logarithms)
-
- Likewise, it's not necessary to understand λ -calculus in order to be productive in modern programming languages
 - But it's fascinating to learn how much can be accomplished within the constraints of λ -calculus

History

- **Gottlob Frege** (1848-1925)
 - studied use of functions in logic in 1893
- **Moses Schönfinkel** (1888-1942)
 - studied how combinators can be applied to formal logic in the 1920s
 - “Combinator” has two meanings, both of which describe a kind of function.
The first describes functions that have no free variables.
It combines only its arguments to produce a result.
The second describes functions that take other functions
and combine them to create a new function.
- **Alonzo Church** (1903-1985)
 - invented Lambda Calculus in the 1930s
 - was PhD advisor of Alan Turing (1912-1954)



Concepts



- **Variable**
 - placeholder for a term represented by a single-letter name
 - two kinds of variables, bound and free ... discussed later
- **Lambda Abstraction**
 - defines anonymous functions that have exactly one parameter

λ -calculus	JavaScript
$\lambda<\text{parameter}>.<\text{body}>$	$<\text{parameter}> => <\text{body}>$

$\lambda x. a \ b \ c \ x$ is evaluated as $\lambda x. (((a \ b) \ c) \ x)$

expressions like $\lambda x. \lambda y. \lambda z. a \ b \ c$ are sometimes written in the shorthand form $\lambda xyz. a \ b \ c$

- **Application**
 - calls functions with an argument

λ -calculus	JavaScript
$(\lambda<\text{parameter}>.<\text{body}>) \ <\text{arguments}>$	$(<\text{parameter}> => <\text{body}>)(<\text{arguments}>)$
$(\lambda xyz.<\text{body}>) \ a \ b \ c$	$(x => y => z => <\text{body}>)(a)(b)(c)$

$<\text{arguments}>$ is a whitespace-separated list of expressions

λ -calculus Does Not Define

- Any built-in functions
 - Syntax for values such as Booleans, numbers, and strings
 - Operators on those types
-
- However, alternatives to those can be defined using only concepts on previous slide, which is the amazing thing about λ -calculus



Bound vs. Free Variables

- Bound variables
 - bound by a specific abstraction (function)
 - appear as function parameters and represent an input value
- Free variables
 - appear in function definitions and are not parameters
 - can represent any value from the “environment”



pretending to have a
+ function for now;
will define later

λ -calculus	JavaScript	Bound Variables	Free Variables
$\lambda x. (+ x 1)$	$x \Rightarrow x + 1$	x	none
$\lambda x. (+ y 1)$	$x \Rightarrow y + 1$	none	y
$\lambda x. x \lambda x. (+ x 1)$	$x \Rightarrow x((x \Rightarrow x + 1))$	rename 2nd x as shown below	
$\lambda x. x \lambda y. (+ y 1)$	$x \Rightarrow x((y \Rightarrow y + 1))$	x and y	none

Evaluation Rules

- λ -calculus defines four evaluation rules
 - **α -conversion** (alpha)
 - **β -reduction** (beta)
 - **δ -rule** (delta)
 - **η -conversion** (eta)



α -conversion

α

- Changes names of bound variables, resulting in equivalent functions
- Examples
 - function $\lambda x . x$ is equivalent to $\lambda y . y$
 - function $\lambda f x . f (+ x 1)$ is equivalent to $\lambda g y . g (+ y 1)$

δ -rule (delta)

δ

- Evaluates functions that are assumed to be built-in
- Example
 - `(+ 1 2)` can be evaluated to 3

β -reduction

β

- Applies arguments to a function
- Result is determined by substituting argument values for all occurrences of the function parameter
- Examples
 - $(\lambda x. (+ x 3)) 2$ evaluates to $(+ 2 3)$ which evaluates to 5
 - $(\lambda f x. f (+ x 1))$ takes two arguments, a function and a number
 - apply two arguments with $(\lambda f x. f (+ x 1)) (\underline{\lambda x. (* x 2)}) \underline{3}$
 - result is $(\lambda x. (* x 2)) (+ 3 1)$
 - apply δ rule to second term to get $(\lambda x. (* x 2)) 4$
 - apply β rule again to obtain $(* 4 2)$
 - one more use of δ -rule, gives 8

η -conversion



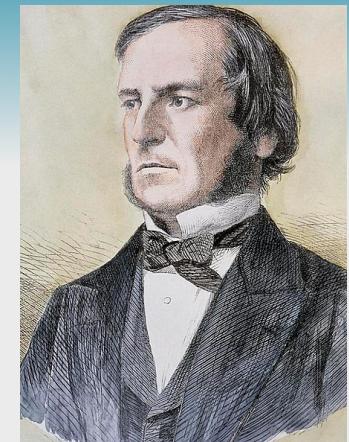
- Replaces function that has an explicit parameter with one that takes an implicit parameter, creating a **point-free** version of the function
- Example
 - $\lambda x. (+ x 1)$ is equivalent to $(+ x)$ because + is a function that takes two arguments, but only one is supplied

Boolean Values

- **True** - function that takes two arguments and always returns the first

λ-calculus	JavaScript
$\lambda t. \lambda f. t$	$t \Rightarrow f \Rightarrow t$

parameter names
are arbitrary



George Boole
(1815-1864)

- **False** - function that takes two arguments and always returns the second

λ-calculus	JavaScript
$\lambda t. \lambda f. f$	$t \Rightarrow f \Rightarrow f$

```
// Adding underscores to avoid
// conflicting with JavaScript keywords.
const true_ = x => y => x;
const false_ = x => y => y;
```

Not

- Function to return “not” of a Boolean value, where **b** is either the true or false function

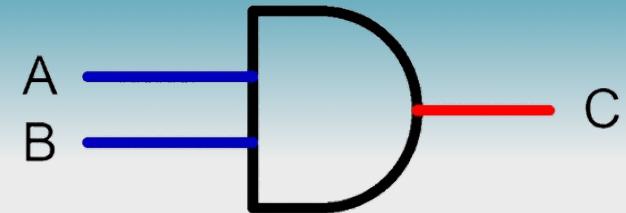
λ -calculus	JavaScript
$\lambda b.b \text{ false true}$	<code>b => b(false_)(true_)</code>

```
const not = b => b(false_)(true_);
```



- Examples
 - $(\lambda b.b \text{ false true}) (\lambda t. \lambda f. t)$ evaluates to false
 - $(\lambda b.b \text{ false true}) (\lambda t. \lambda f. f)$ evaluates to true

And



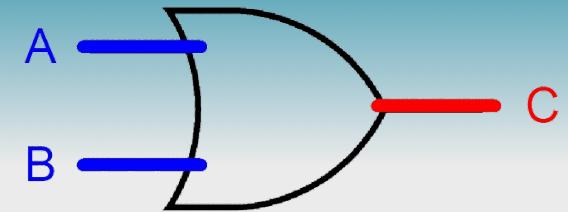
- Function to return “and” of two Boolean values, where **x** and **y** the true or false function

λ -calculus	JavaScript
$\lambda x. \lambda y. x \ y \ \text{false}$	<code>x => y => x(y)(false_)</code>

```
const and = x => y => x(y)(false_);
```

- If first argument is false, that is result
- Otherwise second argument is result

Or



- Function to return “or” of two Boolean values, where **x** and **y** the true or false function

λ -calculus	JavaScript
$\lambda x. \lambda y. x \text{ true } y$	$x \Rightarrow y \Rightarrow x(\text{true_})(y)$

```
const or = x => y => x(true_)(y);
```

- If first argument is true, that is result
- Otherwise second argument is result

Church Numerals



- Represent natural numbers by functions that take another function and a value
- The passed function is called some number of times

Number	λ term	JavaScript
0	$\lambda f x. x$	$f \Rightarrow x \Rightarrow x$ ←
1	$\lambda f x. f \ x$	$f \Rightarrow x \Rightarrow f(x)$
2	$\lambda f x. f \ (f \ x)$	$f \Rightarrow x \Rightarrow f(f(x))$
3	$\lambda f x. f \ (f \ (f \ x))$	$f \Rightarrow x \Rightarrow f(f(f(x)))$

same as function for false

```
const zero = f => x => x;
const one = f => x => f(x);
const two = f => x => f(f(x));
const three = f => x => f(f(f(x)));
const four = f => x => f(f(f(f(x))));
const five = f => x => f(f(f(f(f(x)))));
```

Alonzo Church
(1903-1985)

- Also used to repeat an operation n times

Successor

- Function to return next number after a given number
- Applies function passed to a number one more time

λ -calculus	JavaScript
$\lambda n \ (\lambda f. \ \lambda x. \ f \ (n \ f \ x))$	$n \Rightarrow f \Rightarrow x \Rightarrow f(n(f)(x))$

```
const succ = n => f => x => f(n(f)(x));
```

- Example

- **succ(two)** is **three**
- **f(f(x)) -> f(f(f(x)))**

Peano numbers are numbers that include zero and the results of repeatedly applying the successor function.

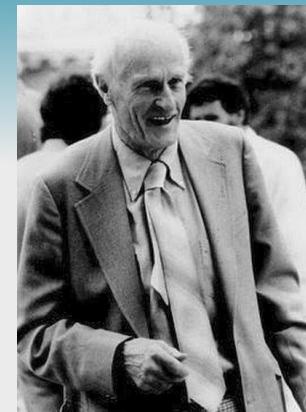


Giuseppe Peano
(1858-1932)

Predecessor ...

- Function to return previous number before a given number
- Alonzo Church couldn't find a solution,
but his student Stephen Kleene did
- Recipe
 - start with pair $(0, 0)$
 - to get predecessor of n ,
create a new pair from previous pair n times
 - 1st number in new pair is 2nd number in previous pair
 - 2nd number in new pair is successor of its 1st number
 - take first number of final pair

while in a dentist chair
waiting to have
wisdom teeth removed



Stephen Kleene
(1909-1994)

Predecessor of 5?

initial: $(0, 0)$
1st: $(0, 1)$
2nd: $(1, 2)$
3rd: $(2, 3)$
4th: $(3, 4)$
5th: $(4, 5)$

works because
of delay in
incrementing
the first number

... Predecessor ...



PREDECESSOR

- Requires 4 helper functions
 - represent a pair (**pair**)

λ -calculus	JavaScript
$\lambda x. \lambda y. \lambda f. f \ x \ y$	$x \Rightarrow y \Rightarrow f \Rightarrow f(x)(y)$

- get 1st element of pair (**fst**)

λ -calculus	JavaScript
$\lambda p. p \text{ TRUE}$	$p \Rightarrow p(\text{true}_\circ)$

- get 2nd element of pair (**snd**)

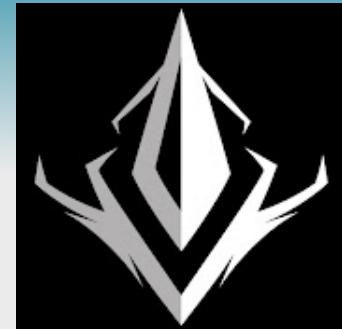
λ -calculus	JavaScript
$\lambda p. p \text{ FALSE}$	$p \Rightarrow p(\text{false}_\circ)$

- create new pair from existing (**phi**)

λ -calculus	JavaScript
$\lambda p. \text{pair } (\text{snd } p) \ (\text{succ } (\text{snd } p))$	$p \Rightarrow \text{pair}(\text{snd}(p))(\text{succ}(\text{snd}(p)))$

```
const pair = x => y => f => f(x)(y);
const fst = p => p(true_);
const snd = p => p(false_);
const phi = p => pair(snd(p))(succ(snd(p)));
```

... Predecessor



- Putting it all together

λ -calculus	JavaScript
$\lambda n.fst (n\ \phi\ (\text{pair}\ zero\ zero))$	$n \Rightarrow \text{fst}(n(\phi)(\text{pair(zero)}(\text{zero})))$

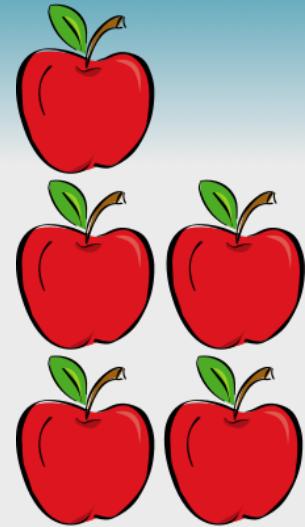
```
const pred = n => fst(n(phi)(pair(zero)(zero)));
```

applies **phi** function **n** times
to the pair (0 , 0)

- Another way to write this that is harder to follow

```
const pred = n => f => x => n(g => h => h(g(f)))(u => x)(u => u);
```

Addition



- Can be seen as iterated successors
- To find $m + n$, start with n and call **succ** on it m times

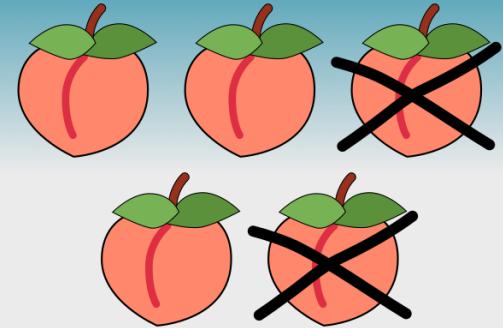
λ -calculus	JavaScript
$\lambda mn. (m \text{ succ})\ n$	$m \Rightarrow n \Rightarrow m(\text{succ})(n)$

```
const add = m => n => m(succ)(n);
```

- $O(n)$

```
3 + 2 ?  
  
add(three)(two)  
three(succ)(two)  
succ(succ(succ(two)))  
succ(succ(three))  
succ(four)  
five
```

Subtraction



- Can be seen as iterated predecessors
- To find $m - n$, start with m and call **pred** on it n times

λ -calculus	JavaScript
$\lambda mn. (n \text{ pred}) m$	$m \Rightarrow n \Rightarrow n(\text{pred})(m)$

```
const sub = m => n => n(pred)(m);
```

- $O(n)$

```
5 - 2 ?  
  
sub(five)(two)  
two(pred)(five)  
pred(pred(five))  
pred(four)  
three
```

Multiplication

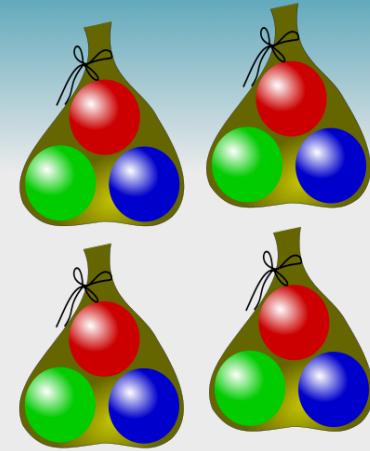
- Can be seen as iterated addition
- To find $m * n$, start with **zero** and call **add(n)** on it m times

λ -calculus	JavaScript
$\lambda mn. m (add\ n)\ 0$	$m \Rightarrow n \Rightarrow m(\text{add}(n))(\text{zero})$

```
const mul = m => n => m(add(n))(zero);
```

- $O(n^2)$

```
4 * 3 ?  
  
mul(four)(three)  
four(add(three))(zero)  
add(add(add(add(three))))(zero)  
add(add(add(three)))(three)  
add(add(three))(six)  
add(three)(nine)  
twelve
```



Exponentiation

5³
base exponent

- Can be seen as iterated multiplication
- To find m^n , start with **one** and call **mul(m)** on it **n** times

λ -calculus	JavaScript
$\lambda mn. n (\text{mul } m) \ 1$	$m \Rightarrow n \Rightarrow n(\text{mul}(m))(\text{one})$

- $O(n^3)$
- Alternative definition

```
const exp = m => n => n(m);
```

```
const exp = m => n => n(mul(m))(one);
```

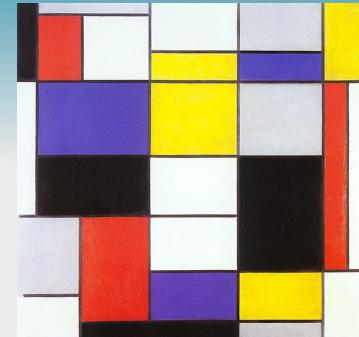
2³ ?

```
three(mul(two))(one)
(mul(two)(mul(two)(mul(two))))(one)
(mul(two)(mul(two)))(two)
(mul(two))(four)
eight
```

2³ ?

```
1st: f(f(x))
- replace each f with two calls to f
2nd: f(f(f(f(x))))
- replace each f with two calls to f
3rd: f(f(f(f(f(f(f(x)))))))
- result is function for 8
```

Function Composition



- Function that combines other functions
- Example
 - composing a function that adds 2 with a function that multiplies by 3

λ-calculus	JavaScript
$\lambda f g x. f (g x)$	<code>f => g => x => f(g(x))</code>

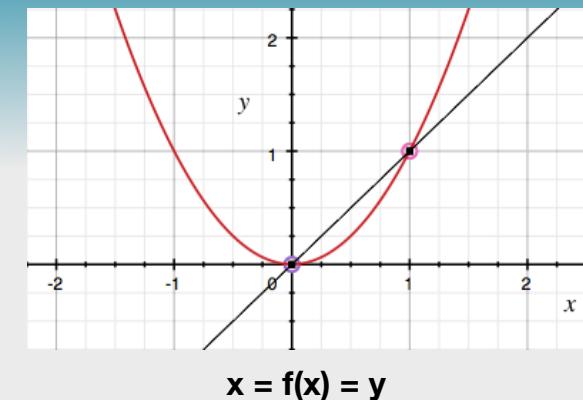
```
const compose = f => g => x => f(g(x));
```

- When functions representing two numbers are composed, the result is their product (multiplication)
 - so this function can be used in place of the one we saw for multiplication

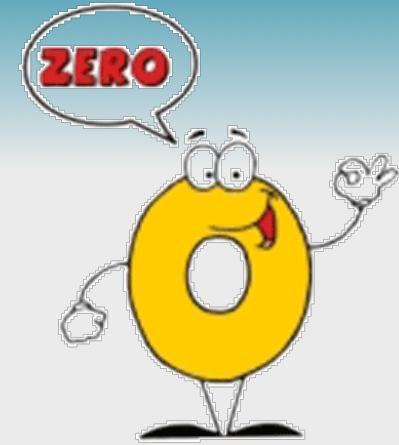
When `f` and `g` are number functions, this returns a function that calls a given function `g` times and repeats this `f` times resulting in `f * g` calls which is the number function for their product.

Fixed Points

- Some functions have a fixed point
 - value that can be passed to it that results in same value
 - ex. $\text{sqrt}(1) = 1$ and $\cos(0.739085) \approx 0.739085$
- λ -calculus function $\lambda x.x\ x$ has a fixed point of $\lambda x.x\ x$
 - $(\lambda x.x\ x)(\lambda x.x\ x)$ - left term is a function and right is argument
 - substituting argument for all x in function body gives same thing, so evaluating this never ends
 - called “Omega Combinator” (Ω)



Is Zero



- Function to determine if a given number is zero
- Recall that **zero** function is same as **false** function, which always returns its second argument

λ -calculus	JavaScript
$\lambda n. n (\lambda x. \text{FALSE}) \text{ TRUE}$	<code>n => n(x => false_)(true_)</code>

```
const iszero = n => n(x => false_)(true_);
```

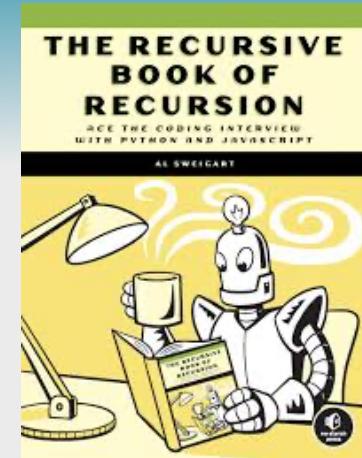
If **n** is zero, this will return the 2nd argument which is **true_**. Otherwise it will call 1st argument function **n** times and every call will return **false_**.

Recursion

- λ-calculus functions do not have names, so they can't call themselves recursively
- Enter **Y Combinator!**
 - function that adds a function parameter to Omega combinator in order to call a provided function repeatedly
 - discovered by Haskell Curry

```
λf. (λx.x x) (λx.f (x x))
```

```
const Y = f => (x => x(x)) (x => f(y => x(x)(y))) ;
```



Haskell Curry
(1900-1982)

Factorial

- Can use Y Combinator to implement a factorial function

```
const facgen = f => n => iszero(n) () => one() => mul(n)(f(pred(n))))();  
const factorialY = Y(facgen);
```

tests n and returns
either the 1st
or 2nd function

1st
function

2nd
function

calls
selected
function

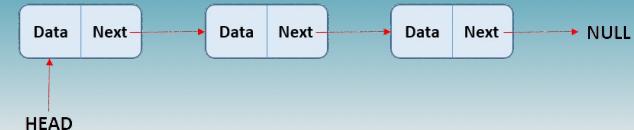
3! ?

```
mul(three)(f(pred(three))  
mul(three)(mul(two)(f(pred(two))))  
mul(three)(mul(two)(f(one)))  
mul(three)(mul(two)(mul(one)(f(pred(one))))  
mul(three)(mul(two)(mul(one)(f(zero))))  
mul(three)(mul(two)(mul(one)((() => one))))  
mul(three)(mul(two)(one))  
mul(three)(two)  
six
```

Technically this uses **Z Combinator**
rather than **Y Combinator**
so 1st and 2nd functions are
called **lazily** rather than eagerly.

Haskell is an example of a programming language
that automatically performs lazy evaluation
and can use Y Combinator.

Linked Lists



- Can simulate linked lists with “cons cells” short for “construct”
 - each cell holds a pair of values
 - **cons** function returns a cons cell
 - **car** function takes a cons cell and returns first element
 - **cdr** function takes a cons cell and returns last element
 - **nil** function is used to mark end of a linked list
- Examples

```
const cons = a => b => f => f(a)(b);  
const car = p => p(true_);  
const cdr = p => p(false_);  
const nil = f => x => null;
```

```
const pair = cons(one)(two);  
expect(car(pair)).toBe(one);  
expect(cdr(pair)).toBe(two);  
  
const list = cons(one)(cons(two)(cons(three)(nil)));  
expect(car(list)).toBe(one);  
expect(car(cdr(list))).toBe(two);  
expect(car(cdr(cdr(list)))).toBe(three);  
expect(cdr(cdr(cdr(list)))).toBe(nil);
```

code from unit tests

Testing



- It's useful to convert λ -calculus representations to actual boolean values and numbers
- Can be accomplished with following JavaScript functions where
 - **b** is a λ -calculus function that represents true or false
 - **n** is a λ -calculus function that represents a natural number

```
const jsbool = b => b(true)(false);
const jsnum = n => n(x => x + 1)(0);
```

Resources

- **Lambda calculus page on Wikipedia**
 - https://en.wikipedia.org/wiki/Lambda_calculus
- **Learn X in Y minutes Where X=Lambda Calculus**
 - <https://learnxinyminutes.com/docs/lambda-calculus/>
- **Intro to hacking with the lambda calculus - blog post by L Rudolf L**
 - <https://www.lesswrong.com/posts/D4PYwNtYNwsogoixGa/intro-to-hacking-with-the-lambda-calculus>
- **Fundamentals of Lambda Calculus & Functional Programming in JavaScript - YouTube talk by Gabriel Lebec**
 - <https://www.youtube.com/watch?v=3VQ382QG-y4>



Wrap Up

- You don't need to know this
- But it's super cool to see how much can be represented and implemented using only functions!
- Use this knowledge to impress your programmer friends
- See my runnable code at
<https://github.com/mvolkmann/lambda-calculus/>
 - in file `lambda-calculus.test.ts`
 - install Bun and enter `bun test`

