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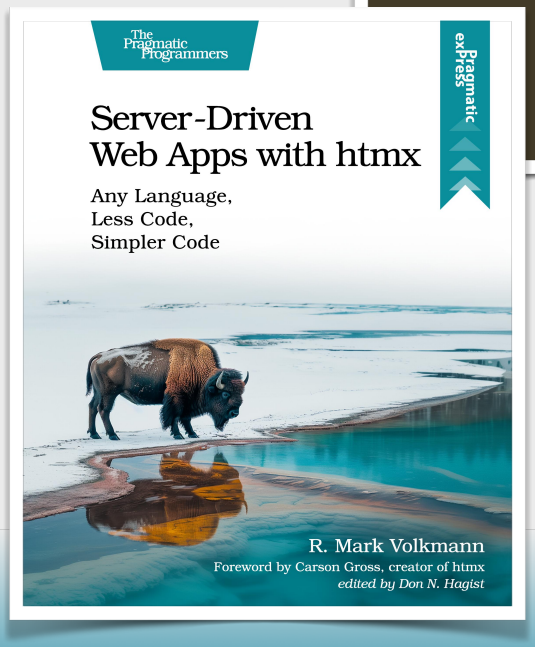
Slides at **https://github.com/mvolkmann/talks/**

About Me

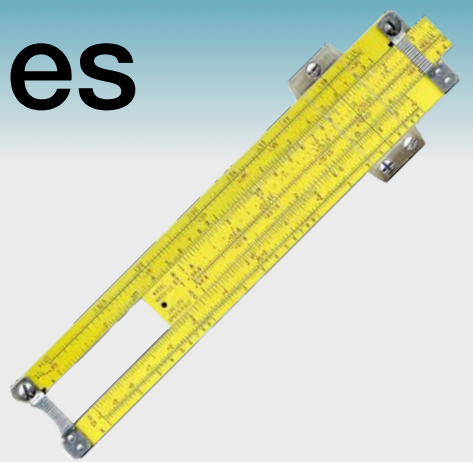
- Partner and Distinguished Software Engineer at Object Computing, Inc. in St. Louis, Missouri USA
- 44 years of professional software development experience
- Writer and speaker
- Blog at https://mvolkmann.github.io/blog/
- Author of Manning book “Svelte ... in Action”
- Author of Pragmatic Bookshelf book “Server-Driven Web Apps with htmx”



λ-calculus Overview

- Describes concepts that are fundamental to **functional programming**
  - **first-class functions** take other functions as arguments and can return functions
  - **currying** implements a function with multiple parameters as a sequence of functions that each have a single parameter
- **Purpose**
  - study how functions can interact with each other, not to calculate results in a useful or efficient way
- **Turing complete**
  - capable of performing any calculation or solving any computational problem, given enough time and memory

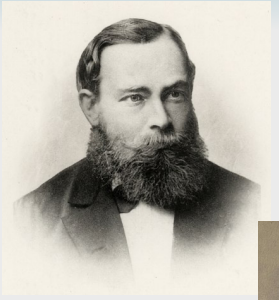
Compared to Slide Rules



- Calculators are an excellent replacement for slide rules
- It's not necessary to understand how to use a slide rule in order to use a calculator
- But it's fascinating to learn how slide rules work (logarithms)

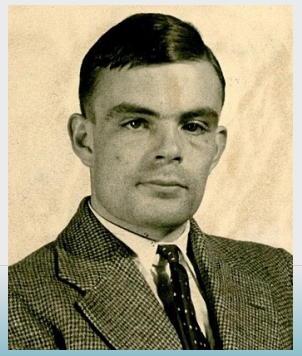
- Likewise, it's not necessary to understand λ-calculus in order to be productive in modern programming languages
- But it's fascinating to learn how much can be accomplished within the constraints of λ-calculus

History



- **Gottlob Frege** (1848-1925)
  - studied use of functions in logic in 1893

- **Moses Schönfinkel** (1888-1942)
  - studied how combinators can be applied to formal logic in the 1920s
  - “Combinator" has two meanings, both of which describe a kind of function.
    The first describes functions that have no free variables.
    It combines only its arguments to produce a result.
    The second describes functions that take other functions
    and combine them to create a new function.



- **Alonzo Church** (1903-1985)
  - invented Lambda Calculus in the 1930s
  - was PhD advisor of Alan Turing (1912-1954)

Concepts

- **Variable**
  - placeholder for a term represented by a single-letter name
  - two kinds of variables, bound and free ... discussed later
- **Lambda Abstraction**
  - defines an anonymous function that has exactly one parameter

λ-calculus	JavaScript
λ<parameter>.<body>	<parameter> => <body>

λx.a b c x is evaluated as **λx.(((a b) c) x)**

expressions like **λx.λy.λz.a b c** are sometimes written in the shorthand form **λxyz.a b c**

- **Application**
  - calls a function (or sequence of them) with arguments

λ-calculus	JavaScript
(λ<parameter>.<body>) <arguments>	(<parameter> => <body>)(<arguments>)
(λxyz.<body>) a b c	(x => y => z => <body>)(a)(b)(c)

<arguments> is a whitespace-separated list of expressions

λ-calculus Does Not Define

- Syntax for values such as Booleans, numbers, and strings
- Operators on those types
- Any built-in functions

- However, alternatives to those can be defined using only concepts on previous slide, which is the amazing thing about λ-calculus

Bound vs. Free Variables

- Bound variables
  - bound by a specific abstraction (function)
  - appear as function parameters and represent an input value
- Free variables
  - appear in function definitions and are not parameters
  - can represent any value from the “environment”

pretending to have a + function for now; will define later

λ-calculus	JavaScript	Bound Variables	Free Variables
`λx.(+ x 1)`	`x => x + 1`	`x`	none
`λx.(+ y 1)`	`x => y + 1`	none	`y`
`λx.x λx.(+ x 1)`	`x => x((x => x + 1))`	rename 2nd `x` as shown below	
`λx.x λy.(+ y 1)`	`x => x((y => y + 1))`	`x` and `y`	none

Evaluation Rules

- λ-calculus defines four evaluation rules
  - α**-conversion** (alpha)
  - **β-reduction** (beta)
  - **δ-rule** (delta)
  - **η-conversion** (eta)

α-conversion

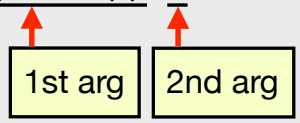
- Changes names of bound variables, resulting in equivalent functions
- Examples
  - function **`λx.x`** is equivalent to **`λy.y`**
  - function **`λfx.f (+ x 1)`** is equivalent to **`λgy.g (+ y 1)`**

δ-rule (delta)

- Evaluates functions that are assumed to be built-in
- Example
  - `(+ 1 2)` can be evaluated to `3`

β-reduction

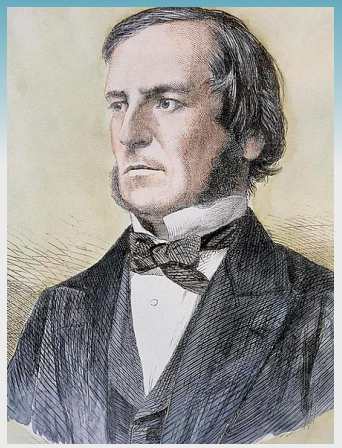
- Applies arguments to a function
- Result is determined by substituting argument values for all occurrences of function parameter in body
- Examples
  - `(λx.(+ x 3)) 2` evaluates to `(+ 2 3)` which evaluates to `5`
  - `(λfx.f (+ x 1))` takes two arguments, a function and a number
    - apply two arguments with `(λfx.f (+ x 1))` `(λx.(* x 2))` `3`
    - apply β-reduction to obtain `(λx.(* x 2)) (+ 3 1)`
    - apply δ-rule to second term to get `(λx.(* x 2)) 4`
    - apply β-reduction again to obtain `(* 4 2)`
    - apply δ-rule to obtain `8`



η-conversion

- Replaces function that has an explicit parameter with one that takes an implicit parameter, creating a **point-free** version of the function
- Example
  - `λx.(+ x 1)` is equivalent to `(+ x)` because + is a a function that takes two arguments, but only one is supplied

Boolean Values



George Boole
(1815-1864)

- **True** - function that takes two arguments and always returns the first

λ-calculus	JavaScript
λt. λf. t	t => f => t

parameter names are arbitrary

- **False** - function that takes two arguments and always returns the second

λ-calculus	JavaScript
λt. λf. f	t => f => f

```
// Adding underscores to avoid
// conflicting with JavaScript keywords.
const true_ = x => y => x;
const false_ = x => y => y;
```

Not

- Function to return “not” of a Boolean value, where **b** is either the true or false function

λ-calculus	JavaScript
λb.b false true	b => b(false_)(true_)

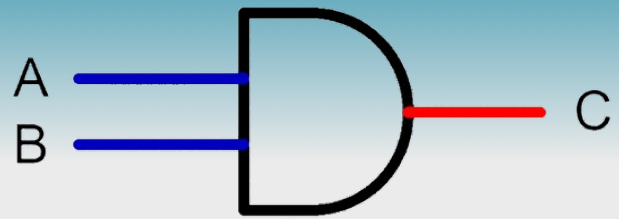
```
const not = b => b(false_)(true_);
```

- Examples
  - **(λb.b false true) (λt. λf. t) (λt. λf. t) false true false**
  - **(λb.b false true) (λt. λf. f) (λt. λf. f) false true true**

function that represents true

function that represents false

And



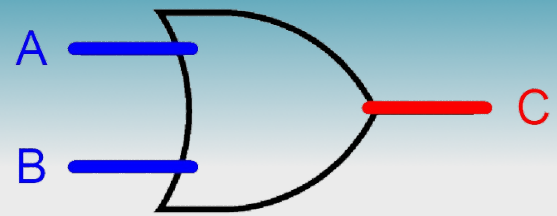
- Function to return “and” of two Boolean values, where **x** and **y** the true or false function

λ-calculus	JavaScript
λx. λy. x y false	x => y => x(y)(false_)

```
const and = x => y => x(y)(false_);
```

- If first argument is false, that is result
- Otherwise second argument is result

Or



- Function to return “or” of two Boolean values, where **x** and **y** the true or false function

λ-calculus	JavaScript
λx. λy. x true y	x => y => x(true_)(y)

```
const or = x => y => x(true_)(y);
```

- If first argument is true, that is result
- Otherwise second argument is result

Church Numerals

Alonzo Church
(1903-1985)

- Represent natural numbers by functions that take another function and a value
- The passed function is called some number of times

Number	λ term	JavaScript
0	λfx.x	f => x => x
1	λfx.f x	f => x => f(x)
2	λfx.f (f x)	f => x => f(f(x))
3	λfx.f (f (f x))	f => x => f(f(f(x)))

same as function for false

```
const zero  = f => x => x;
const one   = f => x => f(x);
const two   = f => x => f(f(x));
const three = f => x => f(f(f(x)));
```

- Also used to repeat an operation **n** times

Successor

- Function to return next number after a given number
- Applies function passed one more time

λ-calculus	JavaScript
λn (λf. λx. f (n f x))	n => f => x => f(n(f)(x))

- Example
  - `succ(two)` is `three`
  - from `f(f(x))` we get `f(f(f(x)))`
- O(n)

```
const succ = n => f => x => f(n(f)(x));
```

n times

one more time

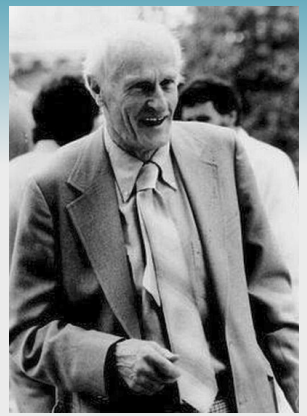
Peano numbers are numbers that include zero and the results of repeatedly applying the successor function.

Giuseppe Peano (1858-1932)

Predecessor ...

- Function to return previous number before a given number
- Alonzo Church couldn't find a solution, but his student Stephen Kleene did
- Recipe
  - start with pair `(0, 0)`
  - to get predecessor of **`n`**,
    create a new pair from previous pair **`n`** times
    - 1st number in new pair is 2nd number in previous pair
    - 2nd number in new pair is successor of its 1st number
  - take first number of final pair

while in a dentist chair waiting to have wisdom teeth removed



Stephen Kleene (1909-1994)

Predecessor of 5?

```
initial: (0, 0)
    1st: (0, 1)
    2nd: (1, 2)
    3rd: (2, 3)
    4th: (3, 4)
    5th: (4, 5)
```

works because of delay in incrementing the first number

... Predecessor ...

- Requires 4 helper functions
  - represent a pair (**`pair`**)

λ-calculus	JavaScript
λx.λy.λf.f x y	x => y => f => f(x)(y)

  - get 1st element of pair (**`fst`**)

λ-calculus	JavaScript
λp.p TRUE	p => p(true_)

  - get 2nd element of pair (**`snd`**)

λ-calculus	JavaScript
λp.p FALSE	p => p(false_)

  - create new pair from existing (**`phi`**)

λ-calculus	JavaScript
λp.pair (snd p) (succ (snd p))	p => pair(snd(p))(succ(snd(p)))

```
const pair = x => y => f => f(x)(y);
const fst = p => p(true_);
const snd = p => p(false_);
const phi = p => pair(snd(p))(succ(snd(p)));
```

`true_` becomes value of **`f`** in **`pair`** function

... Predecessor

- Putting it all together

λ-calculus	JavaScript
λn.fst (n phi (pair zero zero))	n => fst(n(phi)(pair(zero)(zero)))

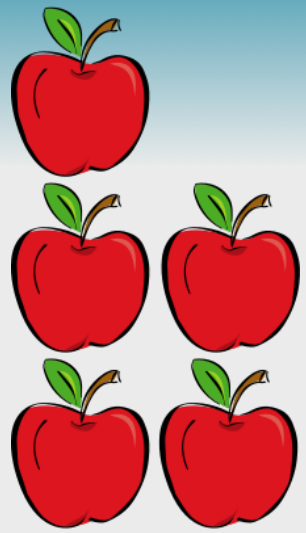
```
const pred = n => fst(n(phi)(pair(zero)(zero)));
```

applies **phi** function **n** times to the pair **(0, 0)**

- O(n)
- Another way to write this that is harder to follow, but doesn't need helper functions

```
const pred = n => f => x => n(g => h => h(g(f)))(u => x)(u => u);
```

Addition



- Can be seen as iterated successors
- To find **`m + n`**, start with **`n`** and call **`succ`** on it **`m`** times

λ-calculus	JavaScript
λmn. (m succ) n	m => n => m(succ)(n)

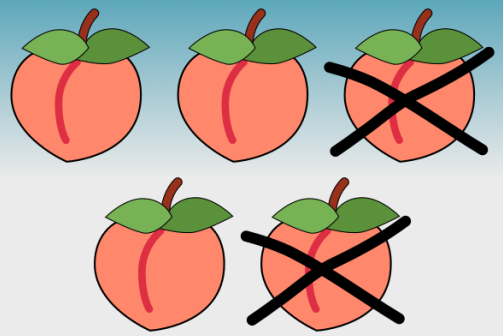
- O(n)

```
const add = m => n => m(succ)(n);
```

3 + 2 ?

```
add(three)(two)
three(succ)(two)
succ(succ(succ(two)))
succ(succ(three))
succ(four)
five
```

Subtraction



- Can be seen as iterated predecessors
- To find **`m - n`**, start with **`m`** and call **`pred`** on it **`n`** times

λ-calculus	JavaScript
λmn. (n pred) m	`m => n => n(pred)(m)`

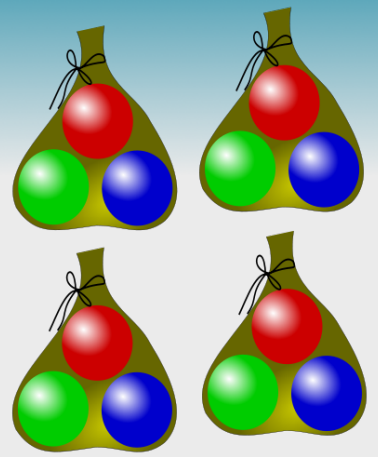
- O(n^2)
  - because **`pred`** is O(n)

```
const sub = m => n => n(pred)(m);
```

5 - 2 ?

```
sub(five)(two)
two(pred)(five)
pred(pred(five))
pred(four)
three
```

Multiplication



- Can be seen as iterated addition
- To find **`m * n`**, start with **`zero`** and call **`add(n)`** to it **`m`** times

λ-calculus	JavaScript
λmn. m (add n) 0	m => n => m(add(n))(zero)

- O(n^2)

```
const mul = m => n => m(add(n))(zero);
```

```
4 * 3 ?

mul(four)(three)
four(add(three))(zero)
(add(three)(add(three)(add(three)(add(three)))))(zero)
(add(three)(add(three)(add(three))))(three)
(add(three)(add(three)))(six)
(add(three))(nine)
twelve
```

Exponentiation

- Can be seen as iterated multiplication
- To find **`m ^ n`**, start with **`one`** and call **`mul(m)`** on it **`n`** times

λ-calculus	JavaScript
λmn. n (mul m) 1	m => n => n(mul(m))(one)

```
const exp = m => n => n(mul(m))(one);
```

2^3 ?

```
three(mul(two))(one)
(mul(two)(mul(two)(mul(two))))(one)
(mul(two)(mul(two)))(two)
(mul(two))(four)
eight
```

- O(n^3)
- Alternative definition

```
const exp = m => n => n(m);
```

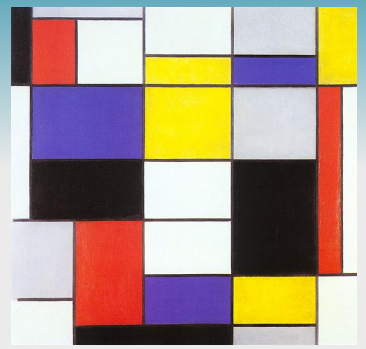
passes a number function
to another number function

2^3 ?

applies the function for two
a total of three times

```
1st: f(f(x))
- replace each f with two calls to f
2nd: f(f(f(f(x))))
- replace each f with two calls to f
3rd: f(f(f(f(f(f(f(f(x)))))))
- result is function for 8
```

Function Composition



- Function that combines other functions
- Example
  - composing a function that adds 2 with a function that multiplies by 3

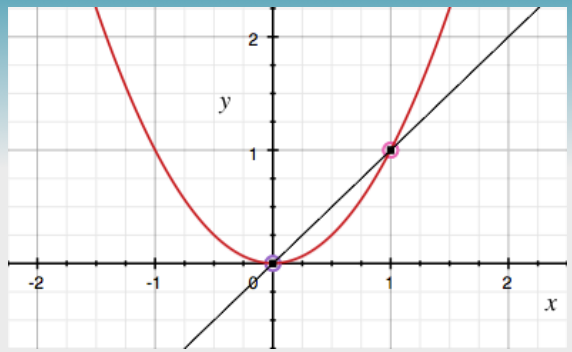
λ-calculus	JavaScript
λfgx.f (g x)	f => g => x => f(g(x))

```
const compose = f => g => x => f(g(x));
```

- When functions representing two numbers are composed, the result is their product (multiplication)
  - so this function can be used in place of the one we saw for multiplication

When **f** and **g** are number functions, this returns a function that calls a given function **g** times and repeats this **f** times resulting in **f * g** calls which is the number function for their product.

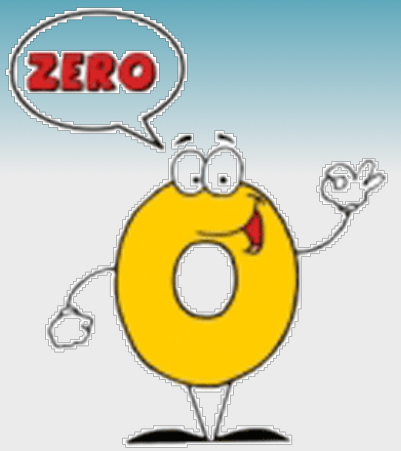
Fixed Points



x = f(x) = y

- Some functions have a fixed point
  - value that can be passed to it that results in same value
  - ex. `sqrt(1) = 1` and `cos(0.739085) ≅ 0.739085`
- λ-calculus function `λx.x x` has a fixed point of `λx.x x`
  - `(λx.x x)(λx.x x)` - left term is a function and right is argument
  - substituting argument for all `x` in function body gives same thing, so evaluating this never ends
  - called “Omega Combinator” (Ω)

Is Zero



- Function to determine if a given number is zero
- Recall that **`zero`** function is same as **`false`** function, which always returns its second argument

λ-calculus	JavaScript
λn.n (λx.FALSE) TRUE	n => n(x => false_)(true_)

```
const iszero = n => n(x => false_)(true_);
```

If **`n`** is zero, this will return the 2nd argument which is **`true_`**. Otherwise it will call 1st argument function **`n`** times and every call will return **`false_`**.

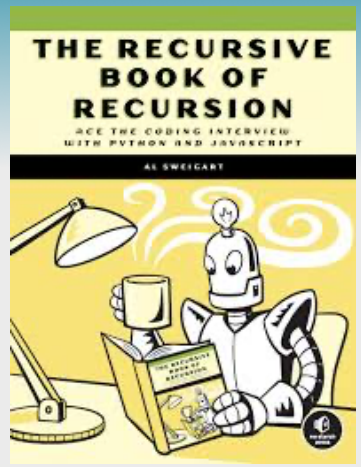
- O(n)
- We need this to implement recursion on next slide

Recursion

- λ-calculus functions do not have names, so they can't call themselves recursively
- Enter **Y Combinator**!
  - function that adds a function parameter to Omega combinator in order to call a provided function repeatedly
  - discovered by Haskell Curry

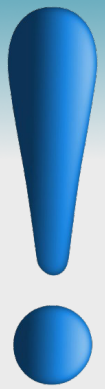
```
λf.(λx.x x) (λx.f (x x))
```

```
const Y = f => (x => x(x))(x => f(y => x(x)(y)));
```



Haskell Curry (1900-1982)

Factorial



- Can use Y Combinator to implement a factorial function

```
const facgen = f => n => iszero(n)(() => one)(() => mul(n)(f(pred(n))))();

const factorialY = Y(facgen);
```

tests `n` and returns either the 1st or 2nd function

1st function

2nd function

calls selected function

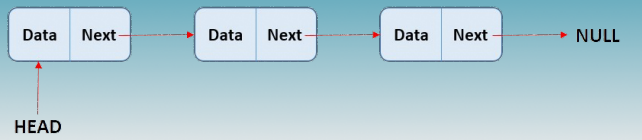
3! ?

```
mul(three)(f(pred(three))
mul(three)(mul(two)(f(pred(two))))
mul(three)(mul(two)(f(one)))
mul(three)(mul(two)(mul(one)(f(pred(one)))))
mul(three)(mul(two)(mul(one)(f(zero))))
mul(three)(mul(two)(mul(one)(() => one))))
mul(three)(mul(two)(one))
mul(three)(two)
six
```

Technically this uses **Z Combinator** rather than **Y Combinator** so 1st and 2nd functions are called **lazily** rather than eagerly.

Haskell is an example of a programming language that automatically performs lazy evaluation and can use Y Combinator.

Linked Lists



- Can simulate linked lists with "cons cells" (short for "construct")
  - each cell holds a pair of values
  - **`cons`** function returns a cons cell
  - **`car`** function takes a cons cell and returns first element
  - **`cdr`** function takes a cons cell and returns last element
  - **`nil`** function is used to mark end of a linked list

```
const cons = a => b => f => f(a)(b);
const car = p => p(true_);
const cdr = p => p(false_);
const nil = f => x => null;
```

When `car` or `cdr` are called, `true_` or `false_` becomes the value of `f` here

- Examples

code from unit tests

```
const pair = cons(one)(two);
expect(car(pair)).toBe(one);
expect(cdr(pair)).toBe(two);

const list = cons(one)(cons(two)(cons(three)(nil)));
expect(car(list)).toBe(one);
expect(car(cdr(list))).toBe(two);
expect(car(cdr(cdr(list)))).toBe(three);
expect(cdr(cdr(cdr(list)))).toBe(nil);
```

Testing

- It's useful to convert λ-calculus representations to actual boolean values and numbers
- Can be accomplished with following JavaScript functions where
  - `b` is a λ-calculus function that represents true or false
  - `n` is a λ-calculus function that represents a natural number

```
const jsbool = b => b(true)(false);
const jsnum = n => n(x => x + 1)(0);
```

Resources

- **Lambda calculus page on Wikipedia**
  - https://en.wikipedia.org/wiki/Lambda_calculus
- **Learn X in Y minutes Where X=Lambda Calculus**
  - https://learnxinyminutes.com/docs/lambda-calculus/
- **Intro to hacking with the lambda calculus - blog post by L Rudolf L**
  - https://www.lesswrong.com/posts/D4PYwNtYNwsgoixGa/intro-to-hacking-with-the-lambda-calculus
- **Fundamentals of Lambda Calculus & Functional Programming in JavaScript - YouTube talk by Gabriel Lebec**
  - https://www.youtube.com/watch?v=3VQ382QG-y4

Wrap Up

- You don't need to know this
- But it's super cool to see how much can be represented and implemented using only functions!
- Use this knowledge to impress your programmer friends
- See my runnable code at https://github.com/mvolkmann/lambda-calculus/
  - in file **`lambda-calculus.test.ts`**
  - install Bun and enter **`bun test`**