



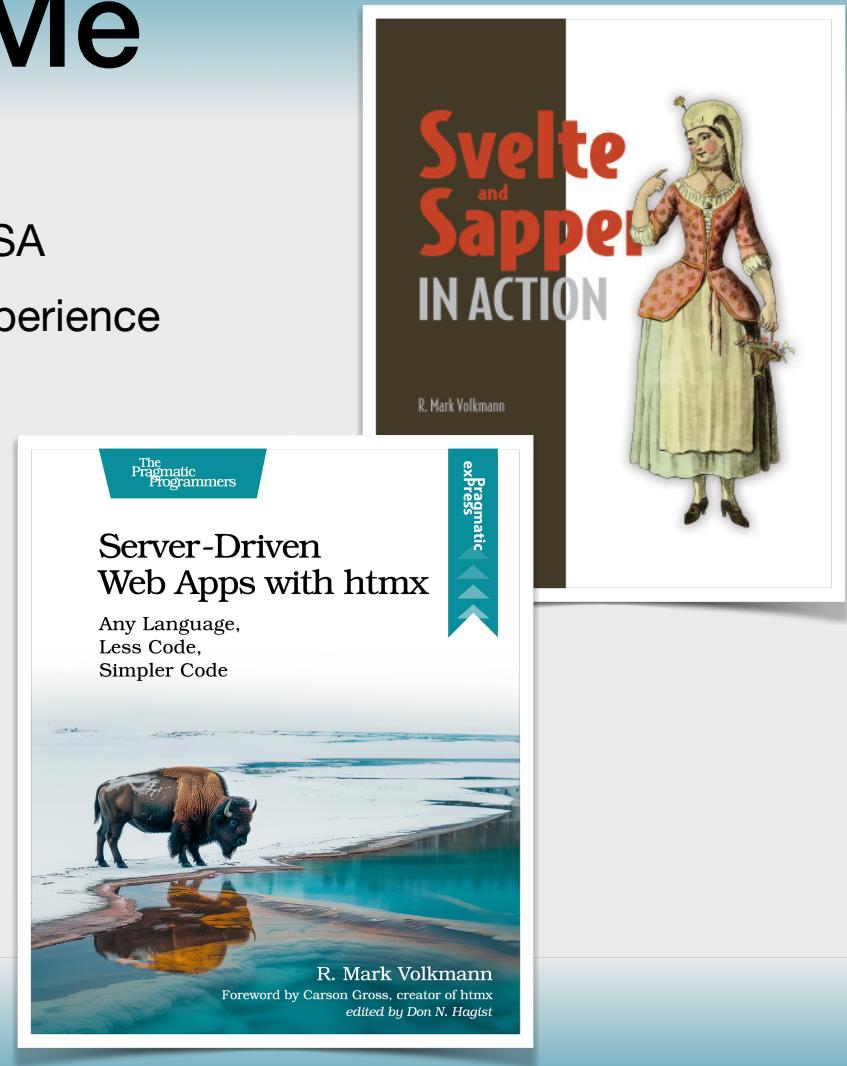
R. Mark Volkmann
 <https://mvolkmann.github.io>
 r.mark.volkmann@gmail.com



Slides at <https://github.com/mvolkmann/talks/>

About Me

- Partner and Distinguished Software Engineer at Object Computing, Inc. in St. Louis, Missouri USA
- 45 years of professional software development experience
- Writer and speaker
- Blog at <https://mvolkmann.github.io/blog/>
- Author of Manning book “Svelte ... in Action”
- Author of Pragmatic Bookshelf book “Server-Driven Web Apps with htmx”

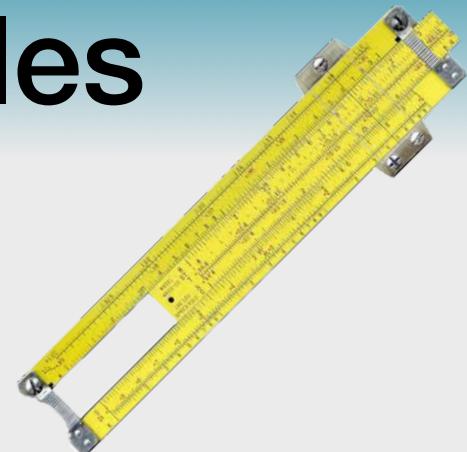


λ -calculus Overview



- Describes concepts that are fundamental to **functional programming**
 - **first-class functions** take other functions as arguments and can return functions
 - **currying** implements a function with multiple parameters as a sequence of functions that each have a single parameter
- **Purpose**
 - study how functions can interact with each other, not to calculate results in a useful or efficient way
- **Turing complete**
 - capable of performing any calculation or solving any computational problem, given enough time and memory

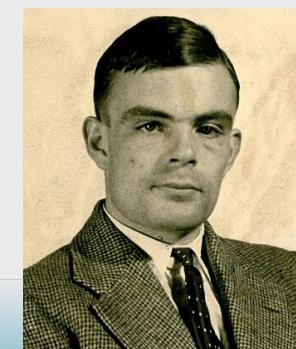
Compared to Slide Rules



- Calculators are an excellent replacement for slide rules
 - It's not necessary to understand how to use a slide rule in order to use a calculator
 - But it's fascinating to learn how slide rules work (logarithms)
-
- Likewise, it's not necessary to understand λ -calculus in order to be productive in modern programming languages
 - But it's fascinating to learn how much can be accomplished within the constraints of λ -calculus

History

- **Gottlob Frege** (1848-1925)
 - studied use of functions in logic in 1893
- **Moses Schönfinkel** (1888-1942)
 - studied how combinators can be applied to formal logic in 1920s
 - “Combinator” has two meanings, both of which describe a kind of function.
The first describes functions that have no free variables.
It combines only its arguments to produce a result.
The second describes functions that take other functions
and combine them to create a new function.
- **Alonzo Church** (1903-1985)
 - invented Lambda Calculus in the 1930s
before computers existed!
 - was PhD advisor of Alan Turing (1912-1954)



Concepts



- **Variable**
 - placeholder for a term, represented by a single-letter name
 - two kinds of variables, bound and free ... discussed later
- **Lambda Abstraction**
 - defines an anonymous function that has exactly one parameter

λ -calculus	JavaScript
$\lambda<\text{parameter}>.<\text{body}>$	$<\text{parameter}> => <\text{body}>$

$\lambda x. a \ b \ c \ x$ is evaluated as $\lambda x. (((a \ b) \ c) \ x)$

expressions like $\lambda x. \lambda y. \lambda z. a \ b \ c$ are sometimes written in the shorthand form $\lambda xyz. a \ b \ c$

- **Application**
 - calls a function (or sequence of them) with arguments

λ -calculus	JavaScript
$(\lambda<\text{parameter}>.<\text{body}>) \ <\text{arguments}>$	$(<\text{parameter}> => <\text{body}>)(<\text{arguments}>)$
$(\lambda xyz.<\text{body}>) \ a \ b \ c$	$(x => y => z => <\text{body}>)(a)(b)(c)$

$<\text{arguments}>$ is a whitespace-separated list of expressions

λ -calculus Does Not Define

- Does not define
 - syntax for values such as Booleans, numbers, and strings
 - operators on those types
 - any built-in functions
- However, alternatives to those can be defined using only concepts on previous slide, which is the amazing thing about λ -calculus!
- All you have are functions
 - they take a single argument that is a function
 - they return a single function



Bound vs. Free Variables



- Bound variables
 - bound by a function
 - appear as parameters and represent an input value
- Free variables
 - appear in function bodies and are not parameters
 - can represent any value from the “environment” (scope)

pretending to have a
+ function for now;
will define later

function bodies are
underlined in blue

λ-calculus	JavaScript	Bound Variables	Free Variables
$\lambda x. (+ x 1)$	$x \Rightarrow x + 1$	x	none
$\lambda x. (+ y 1)$	$x \Rightarrow y + 1$	none	y
$\lambda x. x \ \underline{\lambda x. (+ x 1)}$	$x \Rightarrow x((x \Rightarrow x + 1))$	rename 2nd x as shown below	
$\lambda x. x \ \underline{\lambda y. (+ y 1)}$	$x \Rightarrow x((y \Rightarrow y + 1))$	x and y	none

Evaluation Rules

- λ -calculus defines four evaluation rules
 - **α -conversion** (alpha)
 - **β -reduction** (beta)
 - **δ -rule** (delta)
 - **η -conversion** (eta)



α -conversion

α

- Changes the name of a bound variable, resulting in an equivalent function
- Examples
 - function $\lambda x . x$ is equivalent to $\lambda y . y$
 - function $\lambda f x . f (+ x 1)$ is equivalent to $\lambda g y . g (+ y 1)$

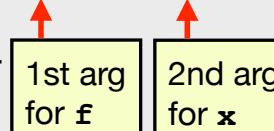
δ -rule (delta)

δ

- Evaluates functions that are assumed to be built-in
- Example
 - `(+ 1 2)` can be evaluated to 3

β -reduction

β

- Applies arguments to a function
- Result is determined by substituting argument value for all occurrences of a function parameter in body
- Examples
 - $(\lambda x. (+ x 3)) 2$ evaluates to $(+ 2 3)$ which evaluates to 5
 - $(\lambda f x. f (+ x 1))$ evaluates two arguments, a function and a number
 - apply both arguments with $(\lambda f x. f \underline{(+ x 1)}) \underline{(\lambda x. (* x 2))} \underline{3}$
 - apply β -reduction to obtain $(\lambda x. (* x 2)) \underline{(+ 3 1)}$ 
 - apply δ -rule to second term to get $(\lambda x. (* x 2)) 4$
 - apply β -reduction again to obtain $(* 4 2)$
 - apply δ -rule to obtain 8

η -conversion

n

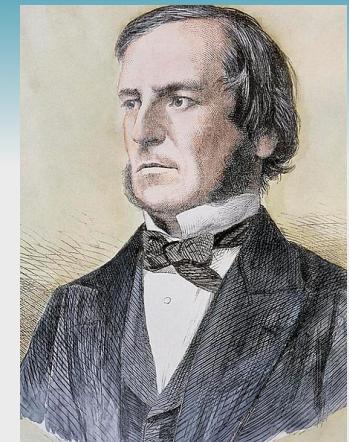
- Replaces function that has an explicit parameter with one that takes an implicit parameter, creating a **point-free** version of the function
 - a way of defining a function without explicitly mentioning arguments
- Example
 - $\lambda x. (+ 1 x)$ is equivalent to $(+ 1)$ because + is a function that takes two arguments, but only one is supplied

Boolean Values

- **True** - function that takes two arguments and always returns the first

λ-calculus	JavaScript
$\lambda t. \lambda f. t$	$t \Rightarrow f \Rightarrow t$

parameter names
are arbitrary



George Boole
(1815-1864)

- **False** - function that takes two arguments and always returns the second

λ-calculus	JavaScript
$\lambda t. \lambda f. f$	$t \Rightarrow f \Rightarrow f$

```
// Adding underscores to avoid
// conflicting with JavaScript keywords.
const true_ = t => f => t;
const false_ = t => f => f;
```

Not



- Function to return “not” of a Boolean value, where **b** is either the true or false function

λ -calculus	JavaScript
$\lambda b.b \text{ false true}$	<code>b => b(false_)(true_)</code>

```
const not = b => b(false_)(true_);
```

- Examples

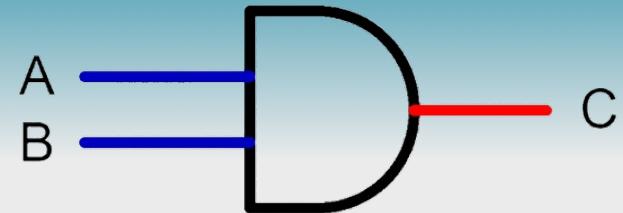
- $(\lambda b.b \text{ false true}) (\lambda t. \lambda f. t)$
 $(\lambda t. \lambda f. t) \underline{\text{false}} \text{ true}$
 false

function that
represents true

- $(\lambda b.b \text{ false true}) (\lambda t. \lambda f. f)$
 $(\lambda t. \lambda f. f) \underline{\text{false}} \text{ true}$
 true

function that
represents false

And



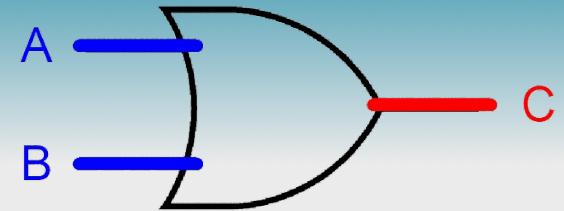
- Function to return “and” of two Boolean values, where **x** and **y** the true or false function

λ -calculus	JavaScript
$\lambda x. \lambda y. x \ y \ \text{false}$	<code>x => y => x(y)(false_)</code>

```
const and = x => y => x(y)(false_);
```

- If first argument is false, that is result
 - returns second value
- Otherwise second argument is result
 - returns first value

Or



- Function to return “or” of two Boolean values, where **x** and **y** the true or false function

λ -calculus	JavaScript
$\lambda x. \lambda y. x \text{ true } y$	$x \Rightarrow y \Rightarrow x(\text{true_})(y)$

```
const or = x => y => x(true_)(y);
```

- If first argument is true, that is result
 - returns first value
- Otherwise second argument is result
 - returns second value

Church Numerals



- Represent whole numbers non-negative integers by functions that take another function and a value
- The passed function is called some number of times

Number	λ term	JavaScript
0	$\lambda f x. x$	$f \Rightarrow x \Rightarrow x$ ←
1	$\lambda f x. f \ x$	$f \Rightarrow x \Rightarrow f(x)$
2	$\lambda f x. f \ (f \ x)$	$f \Rightarrow x \Rightarrow f(f(x))$
3	$\lambda f x. f \ (f \ (f \ x))$	$f \Rightarrow x \Rightarrow f(f(f(x)))$

same as function for false

```
const zero = f => x => x;
const one = f => x => f(x);
const two = f => x => f(f(x));
const three = f => x => f(f(f(x)));
```

Alonzo Church
(1903-1985)

- Also used to repeat an operation n times

Successor

- Function to return next number after a given number
- Applies function passed one more time

λ -calculus	JavaScript
$\lambda n \ (\lambda f. \ \lambda x. \ f \ (n \ f \ x))$	$n \Rightarrow f \Rightarrow x \Rightarrow f(n(f)(x))$

```
const succ = n => f => x => f(n(f)(x));
```

Diagram illustrating the application of the successor function:

- A yellow box labeled "n times" has a red arrow pointing down to the argument `n` in the code.
- A yellow box labeled "one more time" has a red arrow pointing up to the argument `x` in the code.

- Example
 - `succ(two)` is `three`
 - from `f(f(x))` we get `f(f(f(x)))`
- $O(n)$

Peano numbers are numbers that include zero and the results of repeatedly applying the successor function.

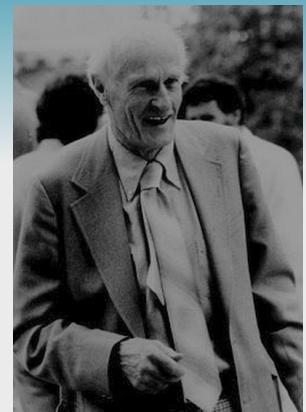


Giuseppe Peano
(1858-1932)

Predecessor ...

- Function to return previous number before a given number
- Alonzo Church couldn't find a solution,
but his student Stephen Kleene did
- Recipe
 - start with pair $(0, 0)$
 - to get predecessor of n ,
create a new pair from previous pair n times
 - 1st number in new pair is 2nd number in previous pair
 - 2nd number in new pair is successor of its 1st number
 - take first number of final pair

while in a dentist chair
waiting to have
wisdom teeth removed



Stephen Kleene
(1909-1994)

Predecessor of 5?

initial: $(0, 0)$
1st: $(0, 1)$
2nd: $(1, 2)$
3rd: $(2, 3)$
4th: $(3, 4)$
5th: $(4, 5)$

works because
of delay in
incrementing
the first number

... Predecessor ...



- Requires 4 helper functions
 - represent a pair (**pair**)

λ -calculus	JavaScript
$\lambda x. \lambda y. \lambda f. f \ x \ y$	$x \Rightarrow y \Rightarrow f \Rightarrow f(x)(y)$

- get 1st element of pair (**fst**)

λ -calculus	JavaScript
$\lambda p. p \text{ TRUE}$	$p \Rightarrow p(\text{true_})$

- get 2nd element of pair (**snd**)

λ -calculus	JavaScript
$\lambda p. p \text{ FALSE}$	$p \Rightarrow p(\text{false_})$

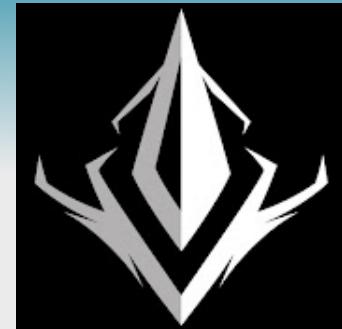
- create new pair from existing (**phi**)

λ -calculus	JavaScript
$\lambda p. \text{pair} (\text{snd } p) (\text{succ} (\text{snd } p))$	$p \Rightarrow \text{pair}(\text{snd}(p))(\text{succ}(\text{snd}(p)))$

```
const pair = x => y => f => f(x)(y);
const fst = p => p(true_);
const snd = p => p(false_);
const phi = p => pair(snd(p))(succ(snd(p)));
```

true_ becomes value
of f in pair function

... Predecessor



- Putting it all together

λ -calculus	JavaScript
$\lambda n.fst (n\ \phi\ (pair\ zero\ zero))$	$n \Rightarrow fst(n(phi)(pair(zero)(zero)))$

```
const pred = n => fst(n(phi)(pair(zero)(zero)));
```

applies **phi** function **n** times
to the pair (0 , 0)

- $O(n)$
- Another way to write this that is harder to follow,
but doesn't need helper functions

```
const pred = n => f => x => n(g => h => h(g(f)))(u => x)(u => u);
```

Addition

- Can be seen as iterated successors
- To find $m + n$, start with n and call **succ** on it m times

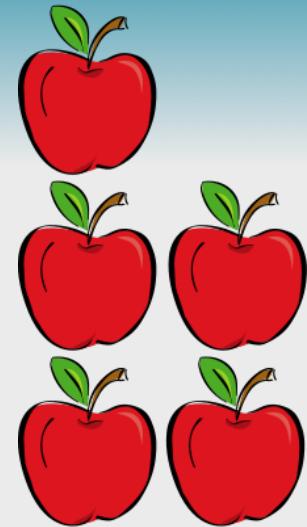
λ -calculus	JavaScript
$\lambda mn. (m \text{ succ})\ n$	$m \Rightarrow n \Rightarrow m(\text{succ})(n)$

```
const add = m => n => m(succ)(n);
```

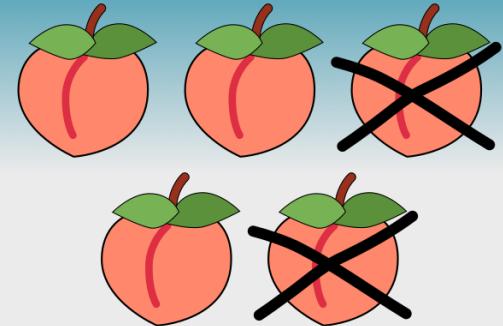
- $O(n)$

3 + 2 ?

```
add(three)(two)
three(succ)(two)
succ(succ(succ(two)))
succ(succ(three))
succ(four)
five
```



Subtraction



- Can be seen as iterated predecessors
- To find $m - n$, start with m and call **pred** on it n times

λ -calculus	JavaScript
$\lambda mn. (n \text{ pred}) m$	$m \Rightarrow n \Rightarrow n(\text{pred})(m)$

```
const sub = m => n => n(pred)(m);
```

- $O(n^2)$
 - because **pred** is $O(n)$ and is called n times

```
5 - 2 ?  
  
sub(five)(two)  
two(pred)(five)  
pred(pred(five))  
pred(four)  
three
```

Multiplication

- Can be seen as iterated addition
- To find $m * n$, start with **zero** and call **add(n)** on it m times

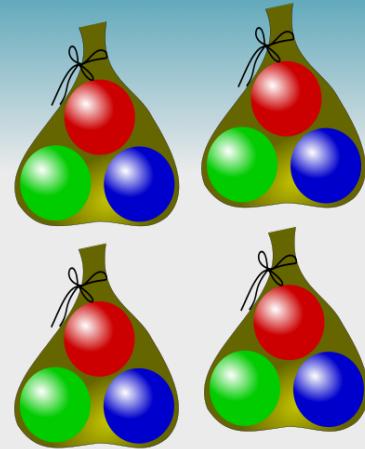
λ -calculus	JavaScript
$\lambda mn. m (add\ n)\ 0$	$m \Rightarrow n \Rightarrow m(\text{add}(n))(\text{zero})$

```
const mul = m => n => m(add(n))(zero);
```

- $O(n^2)$

$4 * 3 ?$

```
mul(four)(three)
four(add(three))(zero)
(add(three)(add(three)(add(three)(add(three)))))(zero)
(add(three)(add(three)(add(three))))(three)
(add(three)(add(three)))(six)
(add(three))(nine)
twelve
```



Exponentiation

5^3
base exponent

- Can be seen as iterated multiplication
- To find m^n , start with **one** and call **mul (m)** on it **n** times

λ -calculus	JavaScript
$\lambda mn. n (\text{mul } m) \ 1$	$m \Rightarrow n \Rightarrow n(\text{mul}(m))(\text{one})$

- $O(n^3)$
- Alternative definition

```
const exp = m => n => n(mul(m))(one);
```

$2^3 ?$

```
three(mul(two))(one)
(mul(two)(mul(two)(mul(two))))(one)
(mul(two)(mul(two)))(two)
(mul(two))(four)
eight
```

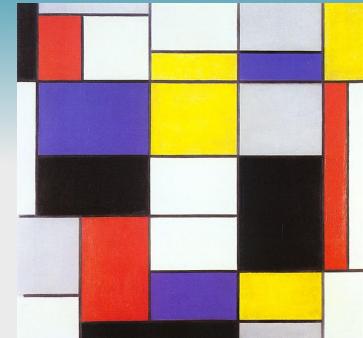
$2^3 ?$

applies the function for two
a total of three times

1st: $f(f(x))$
 - replace each f with two calls to f
 2nd: $f(f(f(f(x))))$
 - replace each f with two calls to f
 3rd: $f(f(f(f(f(f(f(f(x))))))))$
 - result is function for eight

passes a number function
to another number function

Function Composition



- Function that combines other functions
- Example
 - composing a function that adds 2 with a function that multiplies by 3

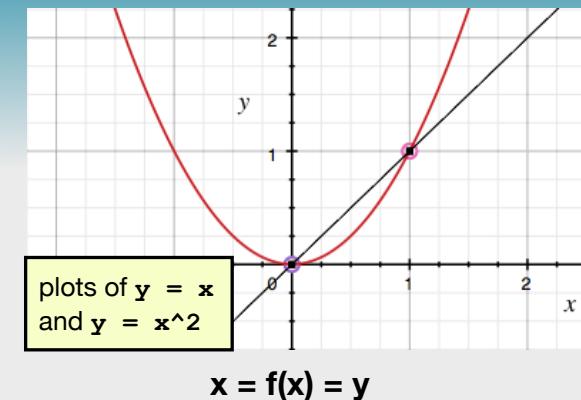
λ-calculus	JavaScript
$\lambda f g x. f (g x)$	<code>f => g => x => f(g(x))</code>

```
const compose = f => g => x => f(g(x));
```

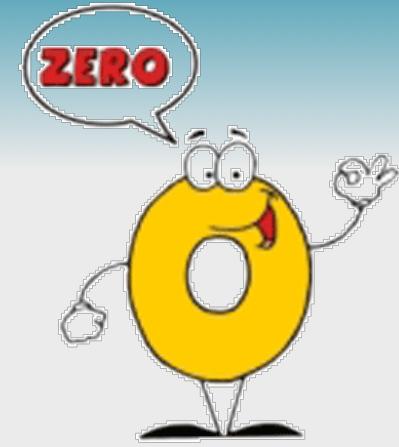
- When functions representing two numbers are composed, the result is their product (multiplication)
 - so this function can be used in place of the one we saw for multiplication

Fixed Points

- Some functions have a fixed point
 - value that can be passed to it that results in same value
 - ex. `sqrt(1) = 1` and `cos(0.739085) ≈ 0.739085`
- λ -calculus function $\lambda x.x \ x$ has a fixed point of $\lambda x.x \ x$
 - $(\lambda x.x \ x)(\lambda x.x \ x)$ - left term is a function and right is argument
 - substituting argument for all x in function body gives same thing, so evaluating this never ends
 - called “Omega Combinator” (Ω)
- We need this to implement recursion



Is Zero



- Function to determine if a given number is zero
- Recall that **zero** function is same as **false** function, which always returns its second argument

λ -calculus	JavaScript
$\lambda n. n (\lambda x. \text{FALSE}) \text{ TRUE}$	<code>n => n(x => false_)(true_)</code>

```
const iszero = n => n(x => false_)(true_);
```

If **n** is zero, this will return the 2nd argument which is **true_**. Otherwise it will call 1st argument function **n** times and every call will return **false_**.

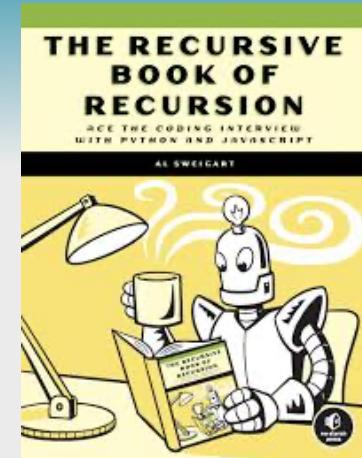
- $O(n)$
- We need this to implement recursion

Recursion

- λ-calculus functions do not have names, so they can't call themselves recursively
- Enter **Y Combinator!**
 - function that adds a function parameter to **Omega combinator** in order to call a provided function repeatedly
 - discovered by Haskell Curry

```
λf. (λx.x x) (λx.f (x x))
```

```
const Y = f => (x => x(x)) (x => f(y => x(x)(y))) ;
```



Haskell Curry
(1900-1982)

Factorial

- Can use Y Combinator to implement a factorial function

```
const facgen = f => n => iszero(n) () => one() => mul(n)(f(pred(n)))();  
const factorialY = Y(facgen);
```

defined on
previous slide

tests n and returns
either the 1st
or 2nd function

1st
function

2nd
function

calls
selected
function

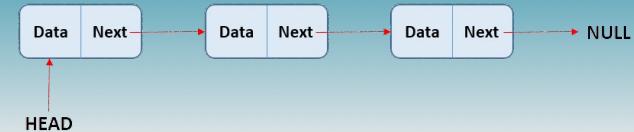
3! ?

```
mul(three)(f(pred(three))  
mul(three)(mul(two)(f(pred(two))))  
mul(three)(mul(two)(mul(one)(f(pred(one))))  
mul(three)(mul(two)(mul(one)(f(zero))))  
mul(three)(mul(two)(mul(one)(one)))  
mul(three)(mul(two)(one))  
mul(three)(two)  
six
```

Technically this uses **Z Combinator**
rather than **Y Combinator**
so 1st and 2nd functions are
called **lazily** rather than eagerly.

Haskell is an example of a programming language
that automatically performs lazy evaluation
and can use Y Combinator.

Linked Lists



- Can simulate linked lists with “cons cells” short for “construct”
 - each cell holds a pair of values
 - **cons** function returns a cons cell
 - **car** function takes a cons cell and returns first element
 - **cdr** function takes a cons cell and returns last element
 - **nil** function is used to mark end of a linked list
- Examples

```
const pair = cons(one)(two);  
expect(car(pair)).toBe(one);  
expect(cdr(pair)).toBe(two);
```

code from unit tests

```
const list = cons(one)(cons(two)(cons(three)(nil)));  
expect(car(list)).toBe(one);  
expect(car(cdr(list))).toBe(two);  
expect(car(cdr(cdr(list)))).toBe(three);  
expect(cdr(cdr(cdr(list)))).toBe(nil);
```

```
const cons = a => b => f => f(a)(b);  
const car = p => p(true_);  
const cdr = p => p(false_);  
const nil = f => x => null;
```

When **car** or **cdr** are called,
true_ or **false_** becomes the
value of **f** here

Testing



- It's useful to convert λ -calculus representations to actual boolean and number values
- Can be accomplished with following JavaScript functions where
 - **b** is a λ -calculus function that represents true or false
 - **n** is a λ -calculus function that represents a whole number

```
const jsbool = b => b(true)(false);
const jsnum = n => n(x => x + 1)(0);
```

Wrap Up

- You don't need to know this
- But it's super cool to see how much can be represented and implemented using only functions!
- Use this knowledge to impress your programmer friends
- See my **Runnable code** at
<https://github.com/mvolkmann/lambda-calculus/>
 - in file `lambda-calculus.test.ts`
 - install Bun and enter `bun test`
 - **amazingly fast!** only 95ms to run all the tests

See Smalltalk version at
<https://github.com/mvolkmann/Cuis-Smalltalk-LambdaCalculus>



Resources

- **Lambda calculus page on Wikipedia**
 - https://en.wikipedia.org/wiki/Lambda_calculus
- **Learn X in Y minutes Where X=Lambda Calculus**
 - <https://learnxinyminutes.com/docs/lambdacalculus/>
- **Intro to hacking with the lambda calculus - blog post by L Rudolf L**
 - <https://www.lesswrong.com/posts/D4PYwNtYNwsogoixGa/intro-to-hacking-with-the-lambda-calculus>
- **Fundamentals of Lambda Calculus & Functional Programming in JavaScript - YouTube talk by Gabriel Lebec**
 - <https://www.youtube.com/watch?v=3VQ382QG-y4>

