Slowly synchronizing automata and digraphs*

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Extended abstract

A *complete deterministic finite automaton* (DFA) is a triple $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, where Q and Σ are finite sets called the *state set* and the *input alphabet* respectively, and $\delta : Q \times \Sigma \to Q$ is a totally defined function called the *transition function*. Let Σ^* stand for the collection of all finite words over the alphabet Σ, including the empty word. The function δ extends to a function $Q \times \Sigma^* \to Q$ (still denoted by δ) in the following natural way: for every $q \in Q$ and $w \in \Sigma^*$, we set

$$\delta(q, w) = \begin{cases} q & \text{if } w \text{ is empty,} \\ \delta(\delta(q, v), a) & \text{if } w = va \text{ for some } v \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

Thus, via δ, every word $w \in \Sigma^*$ acts on the set Q.

A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called *synchronizing* if the action of some word $w \in \Sigma^*$ resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter at which state in Q it is applied: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$. Any such word w is said to be a *reset word* for the DFA.

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applications (coding theory, robotics, testing of reactive systems) and also reveal interesting connections with symbolic dynamics and other parts of mathematics. For a brief introduction to the theory of synchronizing automata we refer the reader to the recent surveys [10, 14]. Here we focus on the so-called Černý conjecture that constitutes a major open problem in this area.

In 1964 Černý [3] constructed for each $n > 1$ a synchronizing automaton $\mathscr{C}_n$ with n states whose shortest reset word has length $(n-1)^2$. Soon after that he conjectured that these automata represent the worst possible case, that is, every synchronizing automaton with n states can be reset by a word of length $(n-1)^2$. This imply looking conjecture resists researchers' efforts for more than 40 years. Even though the conjecture has

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been confirmed for various restricted classes of synchronizing automata (cf., e.g., [5, 4, 7, 13, 1, 15]), no upper bound of magnitude $O(n^2)$ for the minimum length of reset words for n-state synchronizing automata is known in general—the best upper bound achieved so far is $\frac{n^3-n}{6}$, see [8].

One of the difficulties that one encounters when approaching the Černý conjecture is that there are only very few examples of *extreme* synchronizing automata, that is, n-state synchronizing automata whose shortest reset words have length $(n-1)^2$. In fact, the Černý series $\mathscr{C}_n$, $n = 2, 3, \dots$, is the only known infinite series of extreme synchronizing automata. Besides that, we know only a few isolated examples of such automata, see [14] for a complete list. Moreover, even *slowly* synchronizing automata, that is, synchronizing automata whose shortest reset words have length close to the Černý bound are very rare. This empirical observation is supported also by probabilistic arguments. For instance, Higgins [6] has shown that the probability that a composition of $2n$ random self-maps of a set of size n is a constant map tends to 1 as n goes to infinity. In terms of automata, Higgins's result means that a random automaton with n states and at least $2n$ input letters has a reset word of length $2n$. For further results of the same flavor see [11]. Thus, there is no hope to find new examples of slowly synchronizing automata via a random sampling experiment.

We therefore have designed and performed a set of exhaustive search experiments. Our experiments are briefly described in Section 5 while the main body of the paper is devoted to a theoretical analysis of their outcome. We concentrate on two principal issues. In Section 3 we discuss a remarkable analogy between the distribution of lengths of shortest reset words for synchronizing automata and the distribution of exponents of primitive matrices. Section 4 collects several new series of slowly synchronizing automata. The "initial" examples in the series were found in the course of the experiments; then each example was expanded to a series of automata that have been proved to be slowly synchronizing. In our opinion, the proof technique is also of interest; in fact, we provide a transparent and uniform approach to all sufficiently large slowly synchronizing automata with 2 input letters, both new and already known ones.

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