Slowly synchronizing automata and digraphs*

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Abstract. We present several infinite series of synchronizing automata for which the minimum length of reset words is close to the square of the number of states. These automata are closely related to primitive matrices with large exponent.

1 Background and overview

A *complete deterministic finite automaton* (DFA) is a triple $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, where Q and Σ are finite sets called the *state set* and the *input alphabet* respectively, and $\delta : Q \times \Sigma \to Q$ is a totally defined function called the *transition function*. Let Σ^* stand for the collection of all finite words over the alphabet Σ, including the empty word. The function δ extends to a function $Q \times \Sigma^* \to Q$ (still denoted by δ) in the following natural way: for every $q \in Q$ and $w \in \Sigma^*$, we set

$$\delta(q, w) = \begin{cases} q & \text{if } w \text{ is empty,} \\ \delta(\delta(q, v), a) & \text{if } w = va \text{ for some } v \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

Thus, via δ, every word $w \in \Sigma^*$ acts on the set Q.

A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called *synchronizing* if the action of some word $w \in \Sigma^*$ resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter at which state in Q it is applied: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$. Any such word w is said to be a *reset word* for the DFA.

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applications (coding theory, robotics, testing of reactive systems) and also reveal interesting connections with symbolic dynamics and other parts of mathematics. For a brief introduction to the theory of synchronizing automata we refer the reader to the recent surveys [14, 20]. Here we focus on the so-called Černý conjecture that constitutes a major open problem in this area.

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In 1964 Černý [4] constructed for each $n > 1$ a synchronizing automaton $\mathscr{C}_n$ with n states whose shortest reset word has length $(n-1)^2$. Soon after that he conjectured that these automata represent the worst possible case, that is, every synchronizing automaton with n states can be reset by a word of length $(n-1)^2$. This simply looking conjecture resists researchers' efforts for more than 40 years. Even though the conjecture has been confirmed for various restricted classes of synchronizing automata (cf., e.g., $[8, 5, 10, 18, 2, 21]$), no upper bound of magnitude $O(n^2)$ for the minimum length of reset words for n-state synchronizing automata is known in general—the best upper bound achieved so far is $\frac{n^3-n}{6}$, see [12].

One of the difficulties that one encounters when approaching the Černý conjecture is that there are only very few examples of *extreme* synchronizing automata, that is, n-state synchronizing automata whose shortest reset words have length $(n-1)^2$. In fact, the Černý series $\mathscr{C}_n$, $n = 2, 3, \dots$, is the only known infinite series of extreme synchronizing automata. Besides that, only a few isolated examples of such automata have been found, see [20] for a complete list. Moreover, even *slowly* synchronizing automata, that is, synchronizing automata whose shortest reset words have length close to the Černý bound are very rare. This empirical observation is supported also by probabilistic arguments. For instance, Higgins [9] has shown that the probability that a composition of $2n$ random self-maps of a set of size n is a constant map tends to 1 as n goes to infinity. In terms of automata, Higgins's result means that a random automaton with n states and at least $2n$ input letters has a reset word of length $2n$. For further results of the same flavor see [15]. Thus, there is no hope to find new examples of slowly synchronizing automata by a lucky chance or via a random sampling experiment.

We therefore have designed and performed a set of exhaustive search experiments. Our experiments are briefly described in Section 5 while the main body of the paper is devoted to a theoretical analysis of their outcome. We concentrate on two principal issues. In Section 3 we discuss a similarity between the distribution of lengths of shortest reset words for synchronizing automata and the distribution of exponents of primitive digraphs. Section 4 collects several new series of slowly synchronizing automata. The "initial" examples in the series were found in the course of the experiments; then each example was expanded to a series of automata that have been proved to be slowly synchronizing. In our opinion, the proof technique is also of interest; in fact, we provide a transparent and uniform approach to all sufficiently large slowly synchronizing automata with 2 input letters, both new and already known ones.

2 Preliminaries on digraphs and their exponents

A *directed graph* (digraph) is a pair $D = \langle V, E\rangle$ where V is a finite set and $E \subseteq V \times V$. We refer to elements of V and E as *vertices* and *edges.* Observe that our definition allows loops but excludes multiple edges. If $v, v' \in V$ and $e = (v, v') \in E$, the edge e is said to be *outgoing* for v. We assume the reader's acquaintance with basic notions of the theory of directed graphs such as (directed) path, cycle, isomorphism etc.

Given a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$, its *underlying digraph* $D(\mathscr{A})$ has Q as the vertex set and $(q, q') \in Q\times Q$ is an edge of $D(\mathscr{A})$ if and only if $q' = \delta(q, a)$ for some $a \in \Sigma$. It is easy to see that a digraph D is isomorphic to the underlying digraph of some DFA if and only if each vertex of D has at least one outgoing edge. In the sequel, we always consider only digraphs satisfying this property. Every DFA $\mathscr{A}$ such that $D \cong D(\mathscr{A})$ is called a *coloring* of D. Thus, every coloring of D is defined by assigning non-empty sets of labels (colors) from some alphabet Σ to edges of D such that the label sets assigned to the outgoing edges of each vertex form a partition of Σ. Fig. 1 shows a digraph and two of its colorings by $\Sigma = \{a, b\}$.

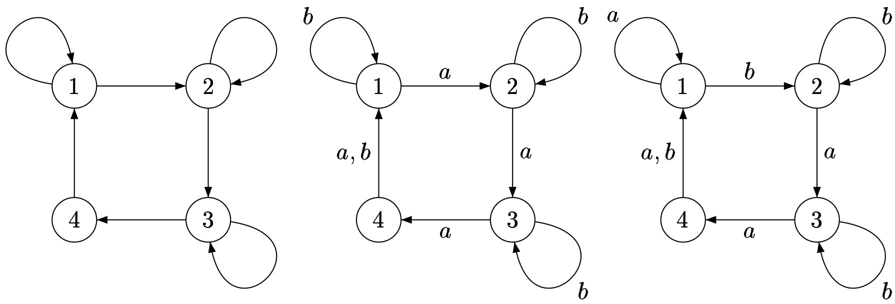

Fig. 1. A digraph and two of its colorings

The *matrix* of a digraph $D = \langle V, E\rangle$ is just the incidence matrix of the edge relation, that is, a $V \times V$-matrix whose entry in the row v and the column v' is 1 if $(v, v') \in E$ and 0 otherwise. For instance, the matrix of the digraph in Fig. 1 (with respect to the chosen numbering of its vertices) is $\begin{pmatrix} 1&1&0&0\\ 0&1&1&0\\ 0&0&1&1\\ 1&0&0&0 \end{pmatrix}$. Conversely, given an $n \times n$-matrix $P = (p_{ij})$ with non-negative real entries, we assign to it a digraph $D(P)$ on the set $\{1, 2, \ldots, n\}$ as follows: (i, j) is an edge of $D(P)$ if and only if $p_{ij} > 0$. This "two-way" correspondence allows us to formulate in terms of digraphs several important for the sequel notions and results which originated in the classical Perron–Frobenius theory of non-negative matrices.

Recall that a digraph $D = \langle V, E\rangle$ is said to be *strongly connected* if for every pair $(v, v') \in V \times V$, there exists a path from v to v'. When studying the Černý conjecture, we may restrict ourselves to automata whose underlying digraphs are strongly connected because the validity of the conjecture can be easily reduced to this case (see [11] for example). By the t^{th} *power* of D we mean the digraph D^t with the same vertex set V, such that $(v, v') \in V \times V$ is an edge of D^t if and only if there is a path in D from v to v' of length precisely t. If M is the matrix of D, then the digraph D^t can be equivalently defined as $D(M^t)$, where M^t is the usual t^{th} power of M.

A strongly connected digraph D is called *primitive* if the greatest common divisor of the lengths of all cycles in D is equal to 1. (In the literature such graphs are sometimes called *aperiodic.*) The following elementary result is well known:

Lemma 1 ([13, Theorem 1.0.1]). *If $k_1, \dots, k_m$ are positive integers whose greatest common divisor is equal to* 1*, then there exists an integer N such that every integer larger than N is expressible as a non-negative integer combination of $k_1, \dots, k_m$.*

The question of how the least N with the property stated in Lemma 1 depends on the integers $k_1, \dots, k_m$ is known as the *diophantine Frobenius problem* and in general is highly non-trivial, see [13]. There is, however, an elementary special case which will be used in the sequel.

Lemma 2 ([13, Theorem 2.1.1]). *If k_1, k_2 are relatively prime positive integers, then $k_1k_2 - k_1 - k_2$ is the largest integer that is not expressible as a non-negative integer combination of k_1 and k_2.*

Lemma 1 readily implies that if D is a primitive digraph, then some power D^t of D has an edge between any pair of vertices, i.e., is a complete digraph with loops. (This is equivalent to saying that every entry of the matrix M^t, where M is the matrix of D, is positive.) The least t with this property is called the *exponent* of the digraph D and is denoted by $\gamma(D)$. We need some results on exponents of digraphs summarized as follows.

Theorem 1. (a) ([22, 6], [7, Theorem 1]) *If a primitive graph D has n vertices, then $\gamma(D) \le (n-1)^2 + 1$.*

(b) [7, Theorem 6 and Corollary 4] *Up to isomorphism, there is exactly one primitive graph D on $n > 2$ vertices for which $\gamma(D) = (n-1)^2 + 1$, and exactly one for which $\gamma(D) = (n-1)^2$. The matrices of these digraphs* *are*

$$\begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 0\,0\,0\ldots0\,1 \\ 1\,1\,0\ldots0\,0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 1\,0\,0\ldots0\,1 \\ 1\,1\,0\ldots0\,0 \end{pmatrix} \quad \textit{respectively.} \tag{1}$$

(c) [7, Theorem 7] *If* $n > 4$ *is even, then there is no primitive digraph* D *on* n *vertices such that*

$$n^2 - 4n + 6 < \gamma(D) < (n-1)^2, \tag{2}$$

and, up to isomorphism, there are exactly 3 *or exactly* 4 *primitive digraphs* D *on* n *vertices with* $\gamma(D) = n^2 - 4n + 6$*, according as* n *is or is not a multiple of* 3.

(d) [7, Theorem 8] *If* $n > 3$ *is odd, then there is no primitive digraph* D *on* n *vertices such that*

$$n^2 - 3n + 4 < \gamma(D) < (n-1)^2, \tag{3}$$

and, up to isomorphism, there is exactly one primitive graph D *on* n *vertices for which* $\gamma(D) = n^2 - 3n + 4$*, exactly one for which* $\gamma(D) = n^2 - 3n + 3$*, and exactly two for which* $\gamma(D) = n^2 - 3n + 2$*. The matrices of these digraphs are:*

$$\begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 0\,0\,0\ldots1\,0 \\ 0\,0\,0\ldots0\,1 \\ 1\,0\,1\ldots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 0\,0\,0\ldots1\,0 \\ 0\,1\,0\ldots0\,1 \\ 1\,0\,1\ldots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 1\,0\,0\ldots1\,0 \\ 0\,1\,0\ldots0\,1 \\ 1\,0\,1\ldots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\ldots0\,0 \\ 0\,0\,1\ldots0\,0 \\ \ldots\ldots\ldots\ldots \\ 1\,0\,0\ldots1\,0 \\ 0\,0\,0\ldots0\,1 \\ 1\,0\,1\ldots0\,0 \end{pmatrix}. \tag{4}$$

(e) [7, Theorem 8] *If* $n > 3$ *is odd, then there is no primitive digraph* D *on* n *vertices such that*

$$n^2 - 4n + 6 < \gamma(D) < n^2 - 3n + 2, \tag{5}$$

and, up to isomorphism, there are exactly 3 *or exactly* 4 *primitive digraphs* D *on* n *vertices with* $\gamma(D) = n^2 - 4n + 6$*, according as* n *is or is not a multiple of* 3.

3 Exponents of digraphs vs lengths of reset words

As mentioned in Section 1, this paper has grown from certain observations made when we analyzed experimental results. One such observation has been a similarity between the "upper parts" of two sequences: the sequence of possible lengths of shortest reset words for 2-letter synchronizing automata with n states and the sequence of possible exponents of primitive digraphs with n vertices. As it is clear from Theorem 1, the upper part of the latter sequence has certain gaps whose sizes and positions depend on the parity of n; our experiments have revealed quite a similar pattern of gaps in the upper part of the former sequence. Table 1 illustrates this observation for $n = 9$.

Table 1. Exponents of primitive digraphs with 9 vertices vs lengths of shortest reset words for 2-letter synchronizing automata with 9 states

N	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51
Number of non-isomorphic primitive digraphs with exponent N	1	1	0	0	0	0	0	1	1	2	0	0	0	0	4
Number of non-isomorphic 2-letter synchronizing automata whose shortest reset words have length N	0	1	0	0	0	0	0	1	2	3	0	0	0	4	4

The data in the second row of Table 1 are calculated from Theorem 1, while the data in the third row come from our experiments.

Concerning gaps in the upper part of the sequence of possible lengths of shortest reset words for 2-letter synchronizing automata with a given number of states, we notice that the first gap has been registered in earlier experiments in the area. (Namely, according to [16, 17], for $n = 7, 8, 9, 10$ there exists no 2-letter synchronizing automata with n states whose shortest reset words have lengths between $n^2 - 2n$ and $n^2 - 3n + 5$.) However, to the best of our knowledge, the second gap as seen in Table 1 has not been reported in the literature up to now.

We strongly believe that the observed similarity is more than a coincidence. Clearly, there are deep connections between primitive digraphs and synchronizing automata. Indeed, it is well known (see [1]) that if the underlying digraph of a synchronizing automaton is strongly connected that the digraph must be primitive; on the other hand, as follows from Trahtman's proof [19] of the so-called Road Coloring conjecture by Adler, Goodwyn, and Weiss [1], every primitive digraph admits a synchronizing coloring. This, however, does not suffice to explain similarities such as in Table 1 because many of slowly synchronizing automata "responsible" for non-zero entries in the third row cannot be obtained as colorings of primitive digraphs with large exponents corresponding to non-zero entries in the second row. In the next section we demonstrate some new connections between primitive digraphs with large exponents and slowly synchronizing automata with two input letters. In this way, we derive all known series of such automata and construct many new other ones.

4 Series of slowly synchronizing automata

Due to space limitations, we present here only a part of our results on slowly synchronizing automata. Namely, we restrict ourselves to series derived from primitive digraphs whose matrices are listed in Theorem 1. We start with the digraph W_n corresponding to the first matrix in (1). The digraph (more precisely, its matrix) first appeared in Wielandt's seminal paper [22]. It has n vertices $1, 2, \ldots, n$, say, and the following $n+1$ edges: $(i, i+1)$ for $i = 1, \ldots, n-1$, $(n, 1)$, and $(n, 2)$.

It is easy to see that, up to isomorphism and renaming of letters, there exists a unique coloring of the digraph W_n by two letters. Let $\mathscr{W}_n$ denote this coloring. Fig. 2 shows the digraph W_n and the DFA $\mathscr{W}_n$.

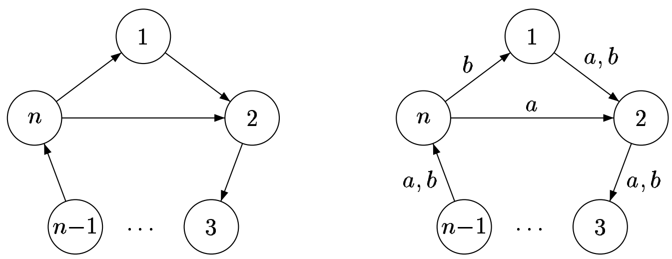

Fig. 2. The digraph W_n and its unique coloring $\mathscr{W}_n$

Theorem 2. *The automaton $\mathscr{W}_n$ is synchronizing and the minimum length of its reset words is $n^2 - 3n + 3$.*

Proof. It is routine to verify that the word $(ab^{n-2})^{n-2}a$, whose length is $(n-1)(n-2) + 1 = n^2 - 3n + 3$, is a reset word for $\mathscr{W}_n$.

Now let w be a reset word for $\mathscr{W}_n$ and assume that the length of w (denoted $|w|$) is minimal. Let $j \in Q = \{1, 2, \ldots, n\}$ be the state to which the action of w brings $\mathscr{W}_n$. Then from every state in Q there is a path to j labelled w. It is clear that for each $j \neq 2$ all paths ending at j share the last edge. Therefore, if $j \neq 2$, removing the last letter from the word w produces a word that still would be a reset word for $\mathscr{W}_n$. We conclude that $j = 2$ because $|w|$ is minimal.

If $u \in \Sigma^*$, the word uw also is a reset word and it also brings the automaton to the state 2. Hence, for every $\ell \geq |w|$, there is a path of length ℓ in W_n from any given vertex i to 2. In particular, setting $i = 2$, we conclude that for every $\ell \geq |w|$ there is a cycle of length ℓ in W_n. The digraph W_n has only two simple cycles: one of length n and one of length $n-1$. Each cycle of W_n must consist of these two cycles traversed several times whence each number $\ell \geq |w|$ must be expressible as a non-negative integer combination of n and $n-1$. Here we invoke Lemma 2 which implies that $|w| > n(n-1) - n - (n-1) = n^2 - 3n + 1$. Suppose that $|w| = n^2 - 3n + 2$. Then there should be a path of this length from the vertex 1 to the vertex 2. The only outgoing edge of 1 is $(1, 2)$, and thus, in the path it must be followed by a cycle of length $n^2 - 3n + 1$. No cycle of such length may exist by Lemma 2. Hence $|w| \geq n^2 - 3n + 3$.

Theorem 3. *For each n, there exists strongly connected synchronizing automaton with n states, such that it's shortest reset word is of length $n^2 - 2n + 1$.*

Lemma 3. *The word $(ab^{n-1})^{n-2}a$ is a reset word for the automaton $\mathscr{C}_n$.*

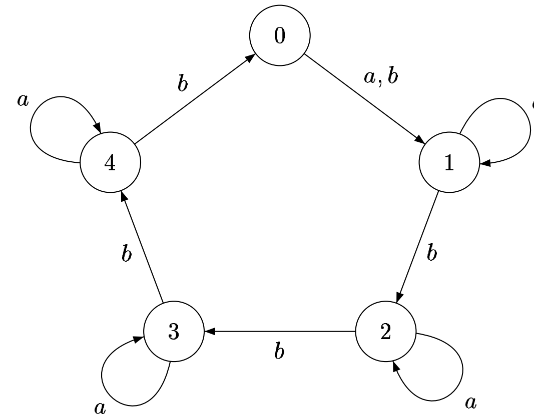

Fig. 3. The automaton $\mathscr{C}_5$

The states of the automaton $\mathscr{C}_n$ are the residues modulo n and its input letters a and b act as follows:

$$\delta(m,a) = \begin{cases} m+1 \pmod{n} & \text{for } m = 0, \\ m \pmod{n} & \text{for } m \neq 0; \end{cases} \qquad \delta(m,b) = m+1 \pmod{n}.$$

The smallest automaton in the series is shown in Fig. 3.

Proof. Let w be reset word of minimum length of $\mathscr{C}_n$. Action of b is cyclic permutation thus w ends on a. Let $w = w'a$. It is easy to see that action of a and a^2 are equal thus every occurence of a in w is followed by b except the last one. Let's introduce new letter(and transition) $d = ab$ then w' could be transformed into $v \in \{b,d\}^*$.

Lemma 4. $(ab^{n-2})^{n-2}ba$

Theorem 4. $n^2 - 3n + 4$

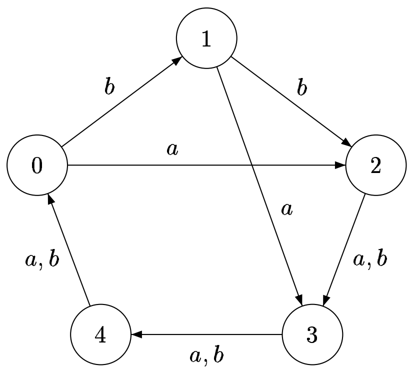

Fig. 4. The automaton $\mathscr{B}_5$

The states of the automaton $\mathscr{B}_n$ are the residues modulo n and its input letters a and b act as follows:

$$\delta(m,a) = \begin{cases} m+2 \pmod{n} & \text{for } m = 0, 1, \\ m+1 \pmod{n} & \text{for } 1 < m < n; \end{cases} \qquad \delta(m,b) = m{+}1 \pmod{n}.$$

The smallest automaton in the series is shown in Fig. 7.

Theorem 5. $n^2 - 3n + 2$

Lemma 5. $(ba^{n-1})^{n-3}ba$

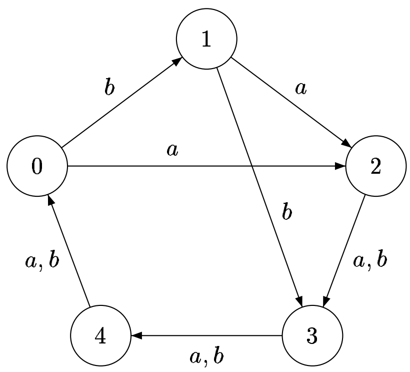

Fig. 5. The automaton $\mathscr{B}_5$

The states of the automaton $\mathscr{B}_n$ are the residues modulo n and its input letters a and b act as follows:

$$\delta(m,a) = \begin{cases} m+2 \pmod{n} & \text{for } m = 1, \\ m+1 \pmod{n} & \text{for } m \neq 0; \end{cases} \qquad \delta(m,b) = \begin{cases} m+2 \pmod{n} & \text{for } m = 1, \\ m+1 \pmod{n} & \text{for } m \neq 1; \end{cases} .$$

The smallest automaton in the series is shown in Fig. 7.

Theorem 6. $n^2 - 3n + 3$

Lemma 6. $(ab^{n-2})^{n-2}a$

The states of the automaton $\mathscr{B}_n$ are the residues modulo n and its input letters a and b act as follows:

$$\delta(m,a) = \begin{cases} m+2 \pmod{n} & \text{for } m = 0, \\ m \pmod{n} & \text{for } m \neq 0; \end{cases} \qquad \delta(m,b) = m+1 \pmod{n}.$$

The smallest automaton in the series is shown in Fig. 7.

Theorem 7. $n^2 - 3n + 2$

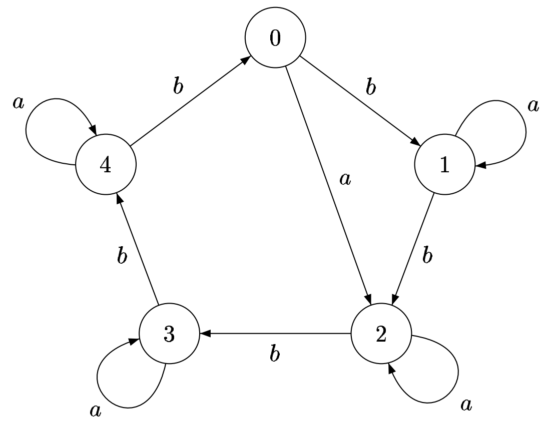

Fig. 6. The automaton $\mathscr{B}_5$

Lemma 7. $aab^{n-2})^{n-3}aa$

The states of the automaton $\mathscr{B}_n$ are the residues modulo n and its input letters a and b act as follows:

$$\delta(m,a) = \begin{cases} m+1 \pmod n & \text{for } m = 0,1, \\ m \pmod n & \text{for } 1 < m < n; \end{cases} \qquad \delta(m,b) = \begin{cases} m+2 \pmod n & \text{for } m = 0, \\ m+1 \pmod n & \text{for } m \neq 0; \end{cases}.$$

The smallest automaton in the series is shown in Fig. 7.

Theorem 8. $n^2 - 4n + 6$

Lemma 8. $b(ab^{n-2})^{n-3}ab$

Theorem 9. $n^2 - 4n + 6$

Lemma 9. $b(ab^{n-2})^{n-3}ba$

5 Experiments

During the experiments every initially-connected deterministic automata was checked on synchronizability and it's shortest reset word was calculated. This class of automata was deceided to use because it contains every strongly connected automata and has very simple and convenient way of enumeration which is presented and thoroughly described in [Almeida].

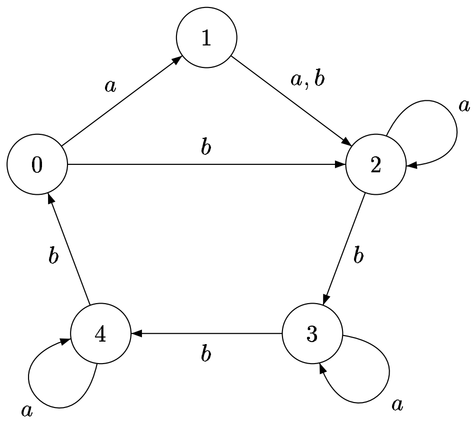

Fig. 7. The automaton $\mathscr{B}_5$

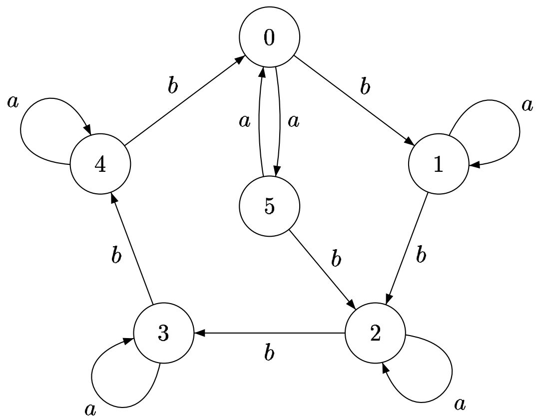

Fig. 8. The automaton $\mathscr{B}_6^{\star}$

There are about 700 billions of initially-connected deterministic automata with 9 states and it is hard task for a single computer to do. That's why this check was distributed and done within a day on 60 cores of amd opteron 2.6 Ghz processors. Program was written in C with mpi. Every obtained slowly synchronizing automaton was rechecked with package TESTAS[trahtman's site] developed by A.Trahtman. It is also worth to mention that experiments show that deterministic automata indeed tends to be synchronizible with short reset words. Almost 90% of initially-connected deterministic automata with 9 states could be synchronized with word of length not greater than 9. And as shows random sampling experiment this part becomes bigger with growth of number of states.

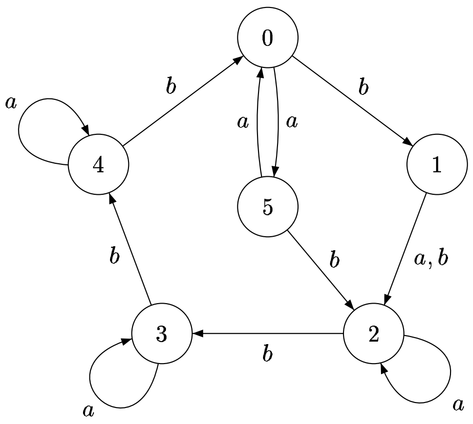

Fig. 9. The automaton $\mathscr{B}_6^\star$

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Appendix