Slowly synchronizing automata and digraphs*

D. S. Ananichev, V. V. Gusev, and M. V. Volkov

Department of Mathematics and Mechanics,
Ural State University, 620083 Ekaterinburg, RUSSIA
Dmitry.Ananichev@usu.ru, vl.gusev@gmail.com, Mikhail.Volkov@usu.ru

Abstract. We present several infinite series of synchronizing automata for which the minimum length of reset words is close to the square of the number of states. These automata are closely related to primitive digraphs with large exponent.

1 Background and overview

A *complete deterministic finite automaton* (DFA) is a triple $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, where Q and Σ are finite sets called the *state set* and the *input alphabet* respectively, and $\delta : Q \times \Sigma \to Q$ is a totally defined function called the *transition function*. Let Σ^* stand for the collection of all finite words over the alphabet Σ, including the empty word. The function δ extends to a function $Q \times \Sigma^* \to Q$ (still denoted by δ) in the following natural way: for every $q \in Q$ and $w \in \Sigma^*$, we set $\delta(q, w) = q$ if w is empty and $\delta(q, w) = \delta(\delta(q, v), a)$ if $w = va$ for some $v \in \Sigma^*$ and $a \in \Sigma$. Thus, via δ, every word $w \in \Sigma^*$ acts on the set Q.

A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called *synchronizing* if the action of some word $w \in \Sigma^*$ resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter at which state in Q it is applied: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$. Any such word w is said to be a *reset word* for the DFA. The minimum length of reset words for $\mathscr{A}$ is called the *reset length* of $\mathscr{A}$.

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applications (coding theory, robotics, testing of reactive systems) and also reveal interesting connections with symbolic dynamics and other parts of mathematics. For a brief introduction to the theory of synchronizing automata we refer the reader to the recent surveys [15, 22]. Here we focus on the so-called Černý conjecture that constitutes a major open problem in this area.

In 1964 Černý [5] constructed for each $n > 1$ a synchronizing automaton $\mathscr{C}_n$ with n states whose reset length is $(n - 1)^2$. Soon after that he

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conjectured that these automata represent the worst possible case, that is, every synchronizing automaton with n states can be reset by a word of length $(n-1)^2$. This simply looking conjecture resists researchers' efforts for more than 40 years. Even though the conjecture has been confirmed for various restricted classes of synchronizing automata (cf., e.g., [9, 6, 11, 19, 20, 2, 23]), no upper bound of magnitude $O(n^2)$ for the reset length of n-state synchronizing automata is known in general. The best upper bound achieved so far is $\frac{n^3-n}{6}$, see [13].

One of the difficulties that one encounters when approaching the Černý conjecture is that there are only very few *extreme* automata, that is, n-state synchronizing automata with reset length $(n-1)^2$. In fact, the Černý series $\mathscr{C}_n$ is the only known infinite series of extreme automata. Besides that, only a few isolated examples of such automata have been found, see [22] for a complete list. Moreover, even *slowly* synchronizing automata, that is, automata with reset length close to the Černý bound are very rare. This empirical observation is supported also by probabilistic arguments. For instance, the probability that a composition of $2n$ random self-maps of a set of size n is a constant map tends to 1 as n goes to infinity [10]. In terms of automata, this result means that the reset length of a random automaton with n states and at least $2n$ input letters does not exceed $2n$. For further results of the same flavor see [16]. Thus, there is no hope to find new examples of slowly synchronizing automata by a lucky chance or via a random sampling experiment.

We therefore have designed and performed a set of exhaustive search experiments. Our experiments are briefly described in Section 5 while the main body of the paper is devoted to a theoretical analysis of their outcome. We concentrate on two principal issues. In Section 3 we discuss a similarity between the distribution of reset lengths of synchronizing automata and the distribution of exponents of primitive digraphs. Section 4 presents a few series of slowly synchronizing automata. Most of these series have been expanded from new examples discovered in the course of our experiments. In our opinion, the proof technique is also of interest; in fact, we provide a transparent and uniform approach to all sufficiently large slowly synchronizing automata with 2 input letters, both new and already known ones.

2 Preliminaries

We start with recalling two elementary and well-known number-theoretic results.

Lemma 1 ([14, Theorem 1.0.1]). *If $k_1, \ldots, k_m$ are positive integers whose greatest common divisor is equal to 1, then there exists an integer N such that every integer larger than N is expressible as a non-negative integer combination of $k_1, \ldots, k_m$.*

The question of how the least N with the property stated in Lemma 1 depends on the integers $k_1, \ldots, k_m$ is known as the *diophantine Frobenius problem* and in general is highly non-trivial, see [14]. There is, however, a simple special case which we will need in Section 4.

Lemma 2 ([14, Theorem 2.1.1]). *If k_1, k_2 are relatively prime positive integers, then $k_1k_2 - k_1 - k_2$ is the largest integer that is not expressible as a non-negative integer combination of k_1 and k_2.*

A *directed graph* (digraph) is a pair $D = \langle V, E\rangle$ where V is a finite set and $E \subseteq V \times V$. We refer to elements of V and E as *vertices* and *edges*. Observe that our definition allows loops but excludes multiple edges. If $v, v' \in V$ and $e = (v, v') \in E$, the edge e is said to be *outgoing* for v. We assume the reader's acquaintance with basic notions of the theory of directed graphs such as (directed) path, cycle, isomorphism etc.

Given a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$, its *underlying digraph* $D(\mathscr{A})$ has Q as the vertex set and $(q, q') \in Q \times Q$ is an edge of $D(\mathscr{A})$ if and only if $q' = \delta(q, a)$ for some $a \in \Sigma$. It is easy to see that a digraph D is isomorphic to the underlying digraph of some DFA if and only if each vertex of D has at least one outgoing edge. In the sequel, we always consider only digraphs satisfying this property. Every DFA $\mathscr{A}$ such that $D \cong D(\mathscr{A})$ is called a *coloring* of D. Thus, every coloring of D is defined by assigning non-empty sets of labels (colors) from some alphabet Σ to edges of D such that the label sets assigned to the outgoing edges of each vertex form a partition of Σ. Fig. 1 shows a digraph and two of its colorings by $\Sigma = \{a, b\}$.

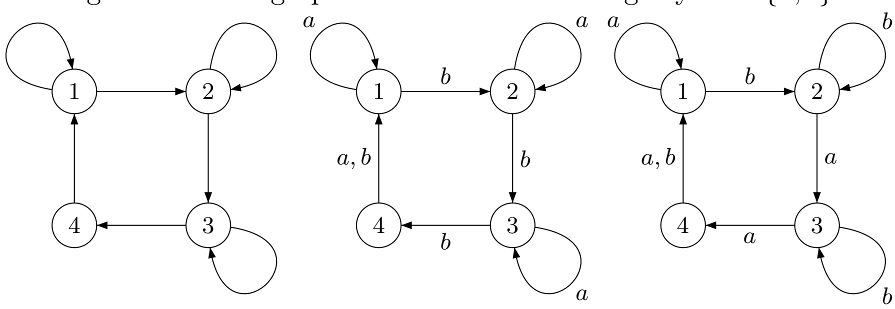

Fig. 1. A digraph and two of its colorings

The *matrix* of a digraph $D = \langle V, E\rangle$ is just the incidence matrix of the edge relation, that is, a $V \times V$-matrix whose entry in the row v

and the column v' is 1 if $(v, v') \in E$ and 0 otherwise. For instance, the matrix of the digraph in Fig. 1 (with respect to the chosen numbering of its vertices) is $\begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 1&0&0&0 \end{pmatrix}$. Conversely, given an $n \times n$-matrix $P = (p_{ij})$ with non-negative real entries, we assign to it a digraph $D(P)$ on the set $\{1, 2, \dots, n\}$ as follows: (i, j) is an edge of $D(P)$ if and only if $p_{ij} > 0$. This "two-way" correspondence allows us to formulate in terms of digraphs several important for the sequel notions and results which originated in the classical Perron–Frobenius theory of non-negative matrices.

Recall that a digraph $D = \langle V, E\rangle$ is said to be *strongly connected* if for every pair $(v, v') \in V \times V$, there exists a path from v to v'. By the t^{th} *power* of D we mean the digraph D^t with the same vertex set V, such that $(v, v') \in V \times V$ is an edge of D^t if and only if there is a path in D from v to v' of length precisely t. If M is the matrix of D, then the digraph D^t can be equivalently defined as $D(M^t)$, where M^t is the usual t^{th} power of M.

A strongly connected digraph D is called *primitive* if the greatest common divisor of the lengths of all cycles in D is equal to 1. (In the literature such graphs are sometimes called *aperiodic*.) Lemma 1 readily implies that if D is a primitive digraph, then in some power D^t of D every pair of vertices constitutes an edge, i.e., D^t is a complete digraph with loops. (This is equivalent to saying that every entry of the matrix M^t, where M is the matrix of D, is positive.) The least t with this property is called the *exponent* of the digraph D and is denoted by $\gamma(D)$. We need some results on exponents of digraphs summarized as follows.

Theorem 1. (a) (Wielandt's theorem, see [24, 7], [8, Theorem 1]) *If a primitive graph D has n vertices, then $\gamma(D) \le (n-1)^2 + 1$.*

(b) [8, Theorem 6 and Corollary 4] *Up to isomorphism, there is exactly one primitive digraph D on $n > 2$ vertices with $\gamma(D) = (n-1)^2 + 1$, and exactly one with $\gamma(D) = (n-1)^2$. The matrices of the digraphs are*

$$\begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots\dots \\ 0\,0\,0\dots0\,1 \\ 1\,1\,0\dots0\,0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots\dots \\ 1\,0\,0\dots0\,1 \\ 1\,1\,0\dots0\,0 \end{pmatrix} \quad \textit{respectively.} \tag{1}$$

(c) [8, Theorem 7] *If $n > 4$ is even, then there is no primitive digraph D on n vertices such that $n^2 - 4n + 6 < \gamma(D) < (n-1)^2$, and, up to isomorphism, there are either* 3 *or* 4 *primitive digraphs D on n vertices with $\gamma(D) = n^2 - 4n + 6$, according as n is or is not a multiple of* 3.

(d) [8, Theorem 8] *If $n > 3$ is odd, then there is no primitive digraph D on n vertices such that $n^2 - 3n + 4 < \gamma(D) < (n-1)^2$, and, up to isomorphism, there is exactly one primitive digraph D on n vertices with $\gamma(D) = n^2 - 3n + 4$, exactly one with $\gamma(D) = n^2 - 3n + 3$, and exactly two with $\gamma(D) = n^2 - 3n + 2$. The matrices of these digraphs are:*

$$\begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots \\ 0\,0\,0\dots1\,0 \\ 0\,0\,0\dots0\,1 \\ 1\,0\,1\dots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots \\ 0\,0\,0\dots1\,0 \\ 0\,1\,0\dots0\,1 \\ 1\,0\,1\dots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots \\ 1\,0\,0\dots1\,0 \\ 0\,1\,0\dots0\,1 \\ 1\,0\,1\dots0\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\dots0\,0 \\ 0\,0\,1\dots0\,0 \\ \dots\dots\dots \\ 1\,0\,0\dots1\,0 \\ 0\,0\,0\dots0\,1 \\ 1\,0\,1\dots0\,0 \end{pmatrix}. \qquad (2)$$

(e) [8, Theorem 8] *If $n > 3$ is odd, then there is no primitive digraph D on n vertices such that $n^2 - 4n + 6 < \gamma(D) < n^2 - 3n + 2$, and, up to isomorphism, there are either 3 or 4 primitive digraphs D on n vertices with $\gamma(D) = n^2 - 4n + 6$, according as n is or is not a multiple of 3.*

3 Exponents of digraphs vs lengths of reset words

As mentioned in Section 1, this paper has grown from certain observations made when we analyzed experimental results. One such observation has been a similarity between the "upper parts" of two sequences: the sequence of possible reset lengths of 2-letter synchronizing automata with n states and the sequence of possible exponents of primitive digraphs with n vertices. As it is clear from Theorem 1, the upper part of the latter sequence has certain gaps whose sizes and positions depend on the parity of n; our experiments have revealed a similar pattern of gaps in the upper part of the former sequence. Table 1 illustrates this observation for $n = 9$.

Table 1. Exponents of primitive digraphs with 9 vertices vs lengths of shortest reset words for 2-letter synchronizing automata with 9 states

N	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51
Number of non-isomorphic primitive digraphs with exponent N	1	1	0	0	0	0	0	1	1	2	0	0	0	0	4
Number of non-isomorphic 2-letter synchronizing automata with reset length N	0	1	0	0	0	0	0	1	2	3	0	0	0	4	4

The data in the second row of Table 1 are calculated from Theorem 1, while the data in the third row come from our experiments.

Concerning gaps in the upper part of the sequence of possible reset lengths of 2-letter synchronizing automata with a given number of states, we notice that the first gap was registered in earlier experiments. (Namely, according to [17, 18], for $n = 7, 8, 9, 10$ there exists no 2-letter synchronizing automata with n states with reset lengths between $n^2 - 2n$ and $n^2 - 3n + 5$.) However, to the best of our knowledge, the second gap as seen in Table 1 has not been reported in the literature up to now.

We strongly believe that the observed similarity is more than a coincidence. Clearly, there are deep connections between primitive digraphs and synchronizing automata. Indeed, it is well known (see [1]) that if the underlying digraph of a synchronizing automaton is strongly connected that the digraph must be primitive; on the other hand, as follows from Trahtman's proof [21] of the so-called Road Coloring conjecture by Adler, Goodwyn, and Weiss [1], every primitive digraph admits a synchronizing coloring. This, however, does not suffice to explain similarities such as in Table 1 because many of slowly synchronizing automata "responsible" for non-zero entries in the third row cannot be obtained as colorings of primitive digraphs with large exponents corresponding to non-zero entries in the second row. In the next section we demonstrate some new connections between primitive digraphs with large exponents and slowly synchronizing automata with two input letters. In this way, we derive all known series of such automata and construct many new ones.

4 Some series of slowly synchronizing automata

Due to space limitations, we present here only a part of our results on slowly synchronizing automata. Namely, we restrict ourselves to series derived from three of the primitive digraphs whose matrices are listed in Theorem 1. These series, in particular, ensure that the "island" of reset lengths between $n^2 - 3n + 2$ and $n^2 - 3n + 4$ exists for each n.

We start with the digraph W_n corresponding to the first matrix in (1). The digraph (more precisely, its matrix) first appeared in Wielandt's seminal paper [24]. It has n vertices $1, 2, \dots, n$, say, and the following $n + 1$ edges: $(i, i + 1)$ for $i = 1, \dots, n - 1$, $(n, 1)$, and $(n, 2)$.

It is easy to see that, up to isomorphism and renaming of letters, there exists a unique coloring of the digraph W_n by two letters. Let $\mathscr{W}_n$ denote this coloring. Fig. 2 shows the digraph W_n and the DFA $\mathscr{W}_n$.

Theorem 2. *The automaton $\mathscr{W}_n$ is synchronizing and its reset length is $n^2 - 3n + 3$.*

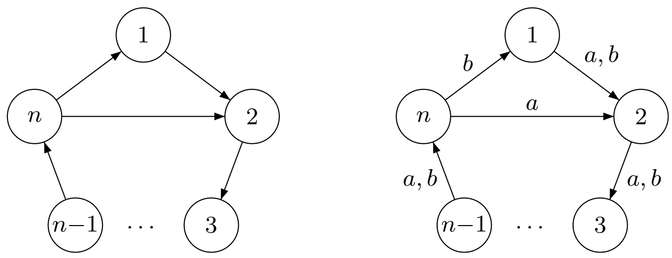

Fig. 2. The digraph W_n and its unique coloring $\mathscr{W}_n$

Proof. It is routine to verify that the word $(ab^{n-2})^{n-2}a$, whose length is $(n-1)(n-2)+1 = n^2-3n+3$, is a reset word for $\mathscr{W}_n$.

Now let w be a reset word for $\mathscr{W}_n$ and assume that the length of w (denoted $|w|$) is minimal. Let $j \in Q = \{1, 2, \dots, n\}$ be the state to which the action of w brings $\mathscr{W}_n$. Then from every state in Q there is a path to j labelled w. It is clear that for each $j \neq 2$ all paths ending at j share the last edge. Therefore, if $j \neq 2$, removing the last letter from the word w produces a word that still would be a reset word for $\mathscr{W}_n$. We conclude that $j = 2$ because $|w|$ is minimal.

If $u \in \{a, b\}^*$, the word uw also is a reset word and it also brings the automaton to the state 2. Hence, for every $\ell \geq |w|$, there is a path of length ℓ in W_n from any given vertex i to 2. In particular, setting $i = 2$, we conclude that for every $\ell \geq |w|$ there is a cycle of length ℓ in W_n. The digraph W_n has only two simple cycles: one of length n and one of length $n-1$. Each cycle of W_n must consist of these two cycles traversed several times whence each number $\ell \geq |w|$ must be expressible as a non-negative integer combination of n and $n-1$. Here we invoke Lemma 2 which implies that $|w| > n(n-1) - n - (n-1) = n^2 - 3n + 1$. Suppose that $|w| = n^2 - 3n + 2$. Then there should be a path of this length from the vertex 1 to the vertex 2. The only outgoing edge of 1 is $(1, 2)$, and thus, in the path it must be followed by a cycle of length $n^2 - 3n + 1$. No cycle of such length may exist by Lemma 2. Hence $|w| \geq n^2 - 3n + 3$.

The series $\mathscr{W}_n$ was discovered by the first author in 2008 (unpublished). His rather involved proof of Theorem 2 used a technique developed in [4].

As mentioned in Section 3, Trahtman's recent result [21] implies that every primitive digraph admits a synchronizing coloring. This gives rise to the following natural question: given a primitive digraph on n vertices, what is the minimum length of reset words for its synchronizing colorings? Observe that in general underlying digraphs of slowly synchronizing automata may admit colorings with rather short reset words. Fig. 1 illustrates this phenomenon: the first coloring of the 4-vertex digraph in Fig. 1 is the Černý automaton $\mathscr{C}_4$ with shortest reset word of length 9 while the second coloring can be reset of the word a^3 of length 3. Wielandt's digraphs W_n, however, can be colored in an essentially unique way, whence Theorem 2 gives the lower bound $n^2 - 3n + 3$ for the value in question. We strongly believe that this lower bound is in fact tight, in other words, we suggest a conjecture that is in a sense parallel to the Černý one.

Conjecture 1. Every primitive digraph on n vertices admits a synchronizing coloring that can be reset by a word of length $n^2 - 3n + 3$.

We observe that while there is a clear analogy between Conjecture 1 and the Černý conjecture, the validity of none of them immediately implies the validity of the other.

Now we discuss a less straightforward way to get a slowly synchronizing series from Wielandt's digraphs W_n. Namely, we aim to show that the Černý automata $\mathscr{C}_n$ are closely related to these digraphs. First, recall the definition of $\mathscr{C}_n$. We may assume that the state set of $\mathscr{C}_n$ is $Q = \{1, 2, \dots, n\}$ and the letters a and b act on Q as follows:

$$\delta(i, a) = \begin{cases} i & \text{if } i < n, \\ 1 & \text{if } i = n; \end{cases} \quad \delta(i, b) = \begin{cases} i+1 & \text{if } i < n, \\ 1 & \text{if } i = n. \end{cases}$$

The automaton $\mathscr{C}_n$ is shown in Fig. 3 on the left.

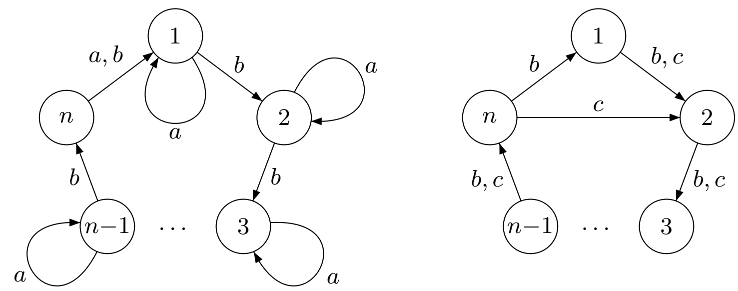

Fig. 3. The automaton $\mathscr{C}_n$ and the automaton induced by the actions of b and $c = ab$

Now we present a new simple proof for the following classic result.

Theorem 3 ([5, Lemma 1]). *The automaton $\mathscr{C}_n$ is synchronizing and its reset length is $(n-1)^2$.*

Proof. It is easy to see that the word $(ab^{n-1})^{n-2}a$ of length $n(n-2)+1 = (n-1)^2$ is a reset word for $\mathscr{C}_n$.

Now let w be a reset word of minimum length for $\mathscr{C}_n$. Since the letter b acts on Q as a cyclic permutation, the word w cannot end with b. (Otherwise removing the last letter gives a shorter reset word.) Thus, we can write w as $w = w'a$ for some $w' \in \{a,b\}^*$ such that the image of Q under the action of w' is precisely the set $\{1,n\}$.

Since the letter a fixes each state in its image $\{1,2,\dots,n-1\}$, every occurrence of a in w except the last one is followed by an occurrence of b. (Otherwise a^2 occurs in w as a factor and reducing this factor to just a results in a shorter reset word.) Therefore, if we let $c = ab$, then the word w' can be rewritten into a word v over the alphabet $\{b,c\}$. The actions of b and c induce a new automaton on the state set Q; this induced automaton (shown in Fig. 3 on the right) is obviously isomorphic to the automaton $\mathscr{W}_n$. Since w' and v act on Q in the same way, the word vc is a reset word for the induced automaton. By Theorem 2 the length of vc (as a word over $\{b,c\}$) is at least n^2-3n+3. Since the action of b on any set S of states cannot change the cardinality of S and the action of c can decrease the cardinality by 1 at most, the word vc must contain at least $n-1$ occurrences of c. Hence the length of v over $\{b,c\}$ is at least n^2-3n+2 and v contain at least $n-2$ occurrences of c. Since each occurrence of c in v corresponds to an occurrence of the factor ab in w', we conclude that the length of w' over $\{a,b\}$ is at least $n^2-3n+2+n-2 = n^2-2n$. Thus, $|w| = |w'a| \geq n^2-2n+1 = (n-1)^2$.

We have found two more series of slowly synchronizing automata related to Wielandt's digraphs W_n: a series with reset length n^2-3n+2 and another one with reset length n^2-4n+6. These two series will be presented in an extended version of the paper.

Now we discuss a few series related to the digraph D_n defined by the second matrix in (1). The digraph is obtained from W_n by adding the edge $(n-1,1)$. Fig. 4 shows the digraph D_n and its colorings $\mathscr{D}'_n$ and $\mathscr{D}''_n$.

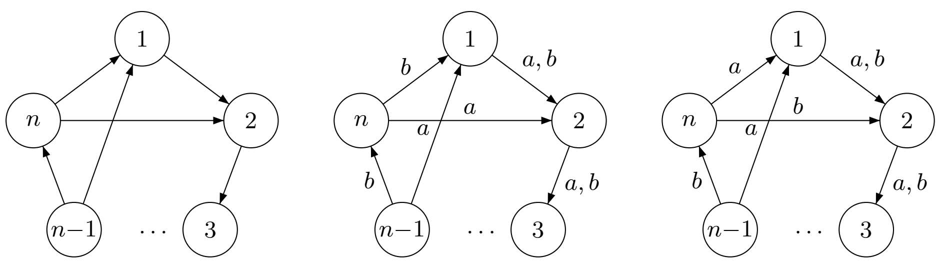

Fig. 4. The digraph D_n and its colorings $\mathscr{D}'_n$ and $\mathscr{D}''_n$

Theorem 4. *The automata $\mathscr{D}'_n$ and $\mathscr{D}''_n$ are synchronizing with reset lengths $n^2 - 3n + 4$ and $n^2 - 3n + 2$ respectively.*

The proof of Theorem 4 is similar to that of Theorem 2. It will be presented in an extended version of this paper. The series $\mathscr{D}'_n$ is of interest because for $n > 6$ it yields the maximum known value of reset length beyond the Černý series $\mathscr{C}_n$ and also the maximum known value of reset length for synchronizing automata without loops. The series $\mathscr{D}''_n$ also enjoys an extremal property: it provides the maximum known value of reset length for synchronizing automata in which no letter acts as a permutation.

One more series of slowly synchronizing automata related to the digraphs D_n has reset length $n^2 - 4n + 6$. It will be presented in an extended version of this paper.

Except the Černý series $\mathscr{C}_n$, the only infinite series of 2-letter slowly synchronizing automata published so far was the series $\mathscr{B}_n$ ($n > 3$ is odd) constructed in [4]. The automaton $\mathscr{B}_n$ has $Q = \{1, 2, \dots, n\}$ as its state set, and its input letters a and b act on Q as follows:

$$\delta(i,a) = \begin{cases} i & \text{if } i < n-1, \\ 1 & \text{if } i = n-1, \\ 2 & \text{if } i = n; \end{cases} \qquad \delta(i,b) = \begin{cases} i+1 & \text{if } i < n, \\ 1 & \text{if } i = n. \end{cases}$$

The automaton $\mathscr{B}_n$ is shown in Fig. 5 on the left.

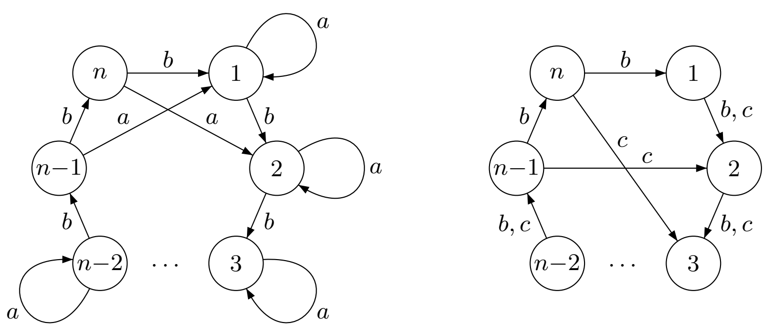

Fig. 5. The automaton $\mathscr{B}_n$ and the automaton induced by the actions of b and $c = ab$

Theorem 5 ([4, Theorem 1.1]). *If $n > 3$ is odd, then the automaton $\mathscr{B}_n$ is synchronizing and its reset length is $n^2 - 3n + 2$.*

The proof of Theorem 5 in [4] is quite involved. Now we can easily prove this result using an argument similar to that in our proof of Theorem 3. The key observation is that the automaton induced by the actions of b and $c = ab$ on the set Q as shown in Fig. 5 on the right is nothing but a coloring of one of the digraphs with exponent $n^2 - 3n + 2$, namely, of the digraph defined by the second matrix in (2). The details of the proof will appear in an extended version of this paper.

5 Experiments

Here we briefly describe the settings of our experiments. Recall that a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is said to be *initially-connected* if there exists a state $q_0 \in Q$ from which every state $q \in Q$ is reachable, that is, $q = \delta(q_0, w)$ for some $w \in \Sigma^*$. In general a synchronizing automaton need not be initially-connected. However, it is well known that when studying the Černý conjecture, we may restrict ourselves to DFA whose underlying digraphs are strongly connected because the validity of the conjecture can be easily reduced to this case (see [12] for example). Clearly, DFA with strongly connected underlying digraphs are initially-connected.

We used a convenient string representation of initially-connected DFA (ICDFA) developed in [3] to generate all such DFA with up to 9 states and 2 input letters. Each ICDFA was tested for synchronizability and then for each synchronizing automaton its reset length was calculated. For these tasks, we implemented standard algorithms (see [15, 22]) in C.

The main difficulty that had to be overcome is that the number of ICDFA dramatically grows with the number of states. (For 9 states, there are about 700 billions ICDFA with 2 input letters.) The problem, however, can be efficiently parallelized. For this, a dedicated processor was programmed to generate ICDFA in portions (slices in terminology of [3]) that were fed to other processors for synchronization tests etc. The management program was written in C with MPI. Calculations organized this way took less than a day of running a small size computer grid based on a number of AMD Opteron 2.6 GHz processors.

All slowly synchronizing automata that we found were double-checked by running on them the package TESTAS developed by Trahtman [17].

Our experiments have also produced some interesting statistical results that will be discussed elsewhere.

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