Equational theories of semigroups with involution

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Abstract

We employ the techniques developed in an earlier paper to show that involutory semigroups arising in various contexts do not have a finite basis for their identities. Among these are partition semigroups endowed with their natural inverse involution, including the full partition semigroup $\mathfrak{C}_n$ for $n \geq 2$, the Brauer semigroup $\mathfrak{B}_n$ for $n \geq 4$ and the annular semigroup $\mathfrak{A}_n$ for $n \geq 4$, n even or a prime power. Also, all of these semigroups, as well as the Jones semigroup $\mathfrak{J}_n$ for $n \geq 4$, turn out to be inherently nonfinitely based when equipped with another involution, the 'skew' one. Finally, we show that similar techniques apply to the finite basis problem for existence varieties of locally inverse semigroups.

Keywords: (non)finitely based algebraic structure, involutory semigroup, partition semigroup, existence variety
2010 MSC: 20M07, 03C05

Introduction

One of the most fundamental and widely studied questions of general (universal) algebra is whether the equational theory $\mathrm{Eq}\,\mathcal{A}$ of an algebraic structure $\mathcal{A}$ is finitely axiomatizable. Let Σ be a set of identities holding in $\mathcal{A}$ such that every identity from $\mathrm{Eq}\,\mathcal{A}$ is a consequence of Σ; such Σ is called an *(equational) basis* of $\mathcal{A}$. So, the question just formulated (usually referred to as the *finite basis problem*) asks if there is a finite basis for the identities of $\mathcal{A}$. If this is indeed the case, then $\mathcal{A}$ is said to be *finitely based*, while otherwise it is *nonfinitely based*. Being very natural by itself, the finite basis problem has also revealed a number of interesting and unexpected relations to many issues of theoretical and practical importance ranging from feasible algorithms for membership in certain classes of formal languages (see [1]) to classical number-theoretic conjectures such as the Twin Prime, Goldbach, existence of odd perfect numbers and the infinitude of even perfect numbers (see [36] where it is shown that each of these conjectures is equivalent to the finite axiomatizability of the equational theory of a particular groupoid).

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Perhaps the most influential question which motivated the development of the area was formulated by Alfred Tarski, who asked if there is an algorithm to determine whether a finite algebra in a finite signature is finitely based. The *Tarski problem*—as it became known later—was first reduced to the case of groupoids (algebras with a single binary operation) by McKenzie in [28] and later solved in the negative [29]. This negative solution makes the finite basis problem for various types of algebras particularly interesting and adds to its significance. Some classes of algebras have the property that all of its finite members are finitely based: for example, such are groups [34], associative rings [18, 22], lattices [27] and commutative semigroups [35]. On the other hand, there are important classes of algebraic structures —such as semigroups—in which the instance of the Tarski problem is yet unsolved: it is not known if the set of isomorphism types of finite semigroups with a finite basis is recursive. For an overview of the rather diversified landscape of the finite basis problem for finite semigroups (as of the year 2000) we direct the reader's attention to the survey article [39] by the third author.

This paper builds upon a previous publication [7] of the authors, where 'unary versions' of two classical approaches to the finite basis problem for semigroups were developed, namely, the critical semigroup method and the method of inherently nonfinitely based semigroups (both presented in the survey [39]). The main field of application in [7] was concerned with matrix semigroups over finite fields endowed with matrix transposition as unary operation. The authors were able to show that no such involutory semigroup (aside from the obvious trivial cases) does have a finite equational basis and, moreover, they completely classified the cases when these are inherently nonfinitely based. In this paper we exhibit further applications of the methods from [7]. These methods will be briefly reviewed in the next section. Our main theme of application, presented in Section 2, involve partition semigroups endowed with their natural inverse involution as a fundamental operation. Considered are the Brauer semigroup [8], the full partition semigroup [24] and the annular semigroup [15]. In addition, these semigroups as well as the Jones semigroup [38] are also studied when equipped with another—'skew'—involution. All of these semigroups originally arose in representation theory and gained much attention recently among semigroup theorists. Section 3 contains some other applications, including joins of involutory semigroup varieties. Finally, in Section 4, we demonstrate how the approach of critical semigroups applies to so-called existence varieties of locally inverse semigroups.

1. Preliminaries

1.1. Identity bases

Throughout the paper we assume the reader's familiarity with the most basic concepts and results of the theory of varieties such as the HSP-theorem, see, e.g., [9, Chapter II]. As far as semigroup theory is concerned, we adopt the standard terminology and notation from [10].

By an *involutory semigroup* we mean an algebraic structure $\mathcal{S} = \langle S, \cdot, ^* \rangle$ of type $(2, 1)$ such that the binary operation $\cdot$ is associative, while the unary operation * satisfies the identities

$$(xy)^* = y^* x^*, \qquad (x^*)^* = x\,.$$

In other words, the unary operation $x \mapsto x^*$ is an involutory anti-automorphism of the semigroup $\langle S, \cdot \rangle$. If, in addition, the identity $x = xx^*x$ holds, $\mathcal{S}$ is said to be a *regular $*$-semigroup*. Each group, subject to its inverse operation $x \mapsto x^{-1}$, is an involutory semigroup, even a regular $*$-semigroup; throughout the paper, any group is considered as a unary semigroup with respect to this inverse unary operation.

In order to conveniently formalize the notions related to identities of involutory semigroups, we employ the *free involutory semigroup* $\mathcal{FI}(X)$ on a given alphabet X. It can be constructed as follows. Let $\overline{X} = \{x^* \mid x \in X\}$ be a disjoint copy of X and define $(x^*)^* = x$ for all $x^* \in \overline{X}$. Then $\mathcal{FI}(X)$ is the free semigroup $(X \cup \overline{X})^+$ endowed with an involution * defined by

$$(x_1 \cdots x_m)^* = x_m^* \cdots x_1^*$$

for all $x_1, \ldots, x_m \in X \cup \overline{X}$. Elements of $\mathcal{FI}(X)$ are referred to as *involutory words* over X. By an *involutory semigroup identity over* X we mean a formal expression $u = v$ where $u, v \in \mathcal{FI}(X)$. An involutory semigroup $\mathcal{S} = \langle S, \cdot, ^* \rangle$ *satisfies* the identity $u = v$ if the equality $\varphi(u) = \varphi(v)$ holds in $\mathcal{S}$ under all possible homomorphisms $\varphi : \mathcal{FI}(X) \to \mathcal{S}$. Given $\mathcal{S}$, we let $\mathrm{Eq}\,\mathcal{S}$ be the set of all involutory semigroup identities it satisfies. For any collection Σ of involutory semigroup identities, we say that an identity $u = v$ *follows* from Σ or that Σ *implies* $u = v$ if every involutory semigroup satisfying all identities of Σ satisfies the identity $u = v$ as well.

With such a notion of the consequence relation between identities, the definitions of a finitely based and a nonfinitely based involutory semigroup $\mathcal{S}$ apply (as sketched in the introduction), depending on whether there is a finite set $\Sigma \subseteq \mathrm{Eq}\,\mathcal{S}$ such that all identities in $\mathrm{Eq}\,\mathcal{S}$ follow from Σ. Analogous expressions can be introduced for *varieties* of involutory semigroups. The class of all involutory semigroups satisfying all identities from a given set Σ of involutory semigroup identities is called the *variety defined by* Σ. A variety $\mathbf{V}$ is *finitely based* if it can be defined by a finite set of identities, otherwise it is *nonfinitely based*. Given an involutory semigroup $\mathcal{S}$, the variety defined by $\mathrm{Eq}\,\mathcal{S}$ is the *variety generated by* $\mathcal{S}$; we denote this variety by $\mathsf{var}\,\mathcal{S}$. From the HSP-theorem it follows that every member of $\mathsf{var}\,\mathcal{S}$ is a homomorphic image of an involutory subsemigroup of a direct product of several copies of $\mathcal{S}$. Observe also that an involutory semigroup and the variety it generates are simultaneously finitely or nonfinitely based.

1.1.1. Critical semigroups

To formulate the first of the two tools from [7] that will be utilized here, we first need the ‘involutory version’ of the well-known Rees matrix construction (see [10, Section 3.1] for a description of the construction in the plain semigroup case). Let $\mathcal{G} = \langle G, \cdot, ^{-1} \rangle$ be a group, 0 a symbol beyond G, and I a non-empty set. We formally set $0^{-1} = 0$. Given an $I \times I$-matrix $P = (p_{ij})$ over $G \cup \{0\}$ such that $p_{ij} = p_{ji}^{-1}$ for all $i, j \in I$, we define a multiplication $\cdot$ and an involution * on the set $(I \times G \times I) \cup \{0\}$ by the following rules:

$$\begin{gathered} a \cdot 0 = 0 \cdot a = 0 \ \text{ for all } a \in (I \times G \times I) \cup \{0\}; \\ (i, g, j) \cdot (k, h, \ell) = \begin{cases} (i, g p_{jk} h, \ell) & \text{if } p_{jk} \neq 0, \\ 0 & \text{if } p_{jk} = 0; \end{cases} \\ (i, g, j)^* = (j, g^{-1}, i), \ 0^* = 0. \end{gathered}$$

It can be easily checked that $\langle (I \times G \times I) \cup \{0\}, \cdot, ^* \rangle$ becomes an involutory semigroup; it will be a regular $*$-semigroup precisely when $p_{ii} = e$ (the identity element of the group $\mathcal{G}$) for all $i \in I$. We denote this unary semigroup by $\mathcal{M}^0(I, \mathcal{G}, I; P)$ and call it the *unary Rees matrix semigroup over* $\mathcal{G}$ *with the sandwich matrix* P. If the involved group $\mathcal{G}$ happens to be the trivial group $\mathcal{E} = \{e\}$, then we usually shall ignore the group entry and represent the non-zero elements of such a Rees matrix semigroup by the pairs (i, j) with $i, j \in I$.

One particular 10-element unary Rees matrix semigroup plays a key role here. It is defined over the trivial group $\mathcal{E} = \{e\}$ with the sandwich matrix

$$\begin{pmatrix} e & e & e \\ e & e & 0 \\ e & 0 & e \end{pmatrix}.$$

We denote this involutory semigroup by $\mathcal{K}_3$. Thus, subject to the convention mentioned above, $\mathcal{K}_3$ consists of the nine pairs (i, j), $i, j \in \{1, 2, 3\}$, and the element 0, and the operations restricted to its non-zero elements can be described as follows:

$$(i, j) \cdot (k, \ell) = \begin{cases} (i, \ell) & \text{if } (j, k) \neq (2, 3), (3, 2), \\ 0 & \text{otherwise;} \end{cases} \tag{1.1}$$
$$(i, j)^* = (j, i).$$

For any involutory semigroup $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ we denote by $\mathrm{H}(\mathcal{S})$ the involutory subsemigroup of $\mathcal{S}$ which is generated by all elements of the form xx^*, where $x \in S$. We call $\mathrm{H}(\mathcal{S})$ the *Hermitian subsemigroup* of $\mathcal{S}$. For any variety $\mathbf{V}$ of involutory semigroups, let $\mathrm{H}(\mathbf{V})$ be the subvariety of $\mathbf{V}$ generated by all Hermitian subsemigroups of members of $\mathbf{V}$. As is easy to verify (see [7, Lemma 2.1]), for every involutiory semigroup $\mathcal{S}$ we have $\mathrm{H}(\mathsf{var}\,\mathcal{S}) = \mathsf{var}\,\mathrm{H}(\mathcal{S})$.

The next result is Theorem 2.2 in [7], specialized to the case of involutory semigroup varieties (whereas the original theorem holds for more general varieties of *unary* semigroups).

Theorem 1.1. *Let* $\mathbf{V}$ *be any involutory semigroup variety such that* $\mathcal{K}_3 \in \mathbf{V}$. *If* $\mathbf{V}$ *contains a group which is not in* $\mathrm{H}(\mathbf{V})$*, then* $\mathbf{V}$ *has no finite basis of identities.*

1.1.2. Inherently nonfinitely based structures

Our second tool from [7] used in this paper involves a sufficient condition on a finite involutory semigroup to be *inherently nonfinitely based* [30]. Namely, a variety $\mathbf{V}$ is said to be *locally finite* if every finitely generated member of $\mathbf{V}$ is finite. A finite involutory semigroup is called inherently nonfinitely based (INFB) if it is not contained in any finitely based locally finite variety. Since the variety generated by a finite involutory semigroup is locally finite (this is an easy consequence of the HSP-theorem, see [9, Theorem 10.16]), the property of being inherently nonfinitely based implies the property of being nonfinitely based; in fact, the former property is much stronger.

A particularly important example of an inherently nonfinitely based involutory semigroup and, in fact our main tool in this context, is the *twisted Brandt monoid* $\mathcal{TB}_2^1 = \langle B_2^1, \cdot, {}^\sigma \rangle$, where B_2^1 is the set of the following six 2×2-matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.2}$$

the binary operation $\cdot$ is the usual matrix multiplication and the unary operation ${}^\sigma$ fixes the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and swaps each of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the other one. The following result can be found as Corollary 2.7 in [7].

Theorem 1.2. *The twisted Brandt monoid $\mathcal{TB}_2^1$ is inherently nonfinitely based.*

1.2. Partition semigroups

For each positive integer n we are going to define:

- the *partition semigroup* $\mathfrak{C}_n$,
- the *Brauer semigroup* $\mathfrak{B}_n$,
- the *partial* Brauer semigroup $P\mathfrak{B}_n$,
- the *annular semigroup* $\mathfrak{A}_n$,
- the *partial* annular semigroup $P\mathfrak{A}_n$,
- the *Jones semigroup* $\mathfrak{J}_n$,
- the *partial* Jones semigroup $P\mathfrak{J}_n$.

The semigroups $\mathfrak{C}_n$, $\mathfrak{B}_n$, $\mathfrak{A}_n$ and $\mathfrak{J}_n$ arise as vector space bases of certain associative algebras which are relevant in representation theory [8, 41, 15, 11]. The semigroup structure and related questions for $\mathfrak{C}_n$, $P\mathfrak{B}_n$ and $\mathfrak{B}_n$ have been studied recently by Mazorchuk et al., see, for example, [25, 26, 19, 20, 23, 21].

We start with the definition of $\mathfrak{C}_n$, as given in [40]. For each positive integer n let

$$[n] = \{1, \ldots, n\}, \quad [n]' = \{1', \ldots, n'\}, \quad [n]'' = \{1'', \ldots, n''\}$$

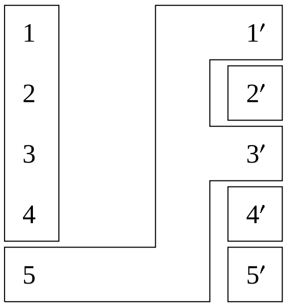
be three pairwise disjoint copies of the set of the first n positive integers and put

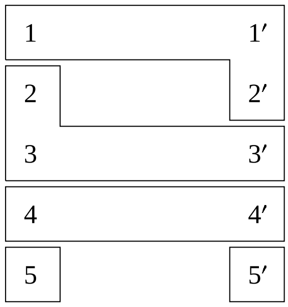
$$\widetilde{[n]} = [n] \cup [n]'.$$

The base set of the partition semigroup $\mathfrak{C}_n$ is the set of all partitions of the set $\widetilde{[n]}$; throughout, we consider a partition of a set and the corresponding equivalence relation on that set as two different views of the same thing and without further mention we freely switch between these views, whenever it seems to be convenient. For $\xi, \eta \in \mathfrak{C}_n$, the product $\xi\eta$ is defined (and computed) in four steps:

1. Consider the $'$-analogue of η: that is, define η' on $[n]' \cup [n]''$ by
$$x' \ \eta' \ y' :\Leftrightarrow x \ \eta \ y \text{ for all } x, y \in \widetilde{[n]}.$$
2. Let $\langle \xi, \eta \rangle$ be the equivalence relation on $\widetilde{[n]} \cup [n]''$ generated by $\xi \cup \eta'$, that is, set $\langle \xi, \eta \rangle := (\xi \cup \eta')^t$ where ^{t} denotes the transitive closure.
3. Forget all elements having a single prime $'$: that is, set
$$\langle \xi, \eta \rangle^\circ := \langle \xi, \eta \rangle |_{[n] \cup [n]''}.$$
4. Replace double primes with single primes to obtain the product $\xi\eta$: that is, set
$$x \ \xi\eta \ y :\Leftrightarrow f(x) \ \langle \xi, \eta \rangle^\circ \ f(y) \text{ for all } x, y \in \widetilde{[n]}$$
where $f : \widetilde{[n]} \to [n] \cup [n]''$ is the bijection
$$x \mapsto x, x' \mapsto x'' \text{ for all } x \in [n].$$

For example, let $n = 5$ and

$\xi =$ 

,
$\eta =$ 

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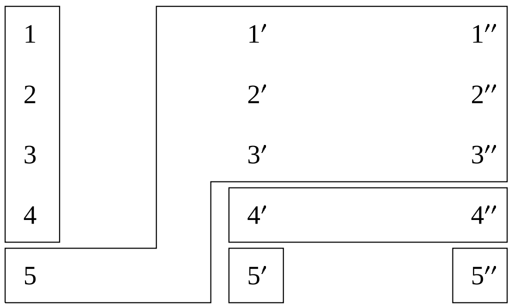
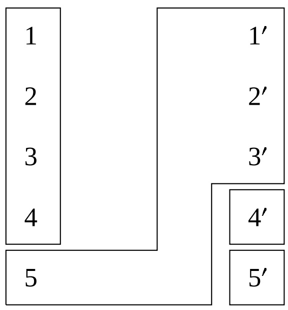
Then

$\langle \xi, \eta \rangle =$ 

and

$\xi\eta =$ 

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This multiplication is associative making $\mathfrak{C}_n$ a semigroup with identity 1 where

$$1 = \{\{k, k'\} \mid k \in [n]\}.$$

The group of units of $\mathfrak{C}_n$ is the symmetric group $\mathfrak{S}_n$ (acting on $[n]$ on the right) with canonical embedding $\mathfrak{S}_n \hookrightarrow \mathfrak{C}_n$ given by

$$\sigma \mapsto \{\{k, (k\sigma)'\} \mid k \in [n]\} \text{ for all } \sigma \in \mathfrak{S}_n.$$

More generally, the semigroup of all (total) transformations of $[n]$ acting on the right is also naturally embedded in $\mathfrak{C}_n$ by

$$\phi \mapsto \{\{k'\} \cup k\phi^{-1} \mid k \in [n]\}. \tag{1.3}$$

If k is not in the image of ϕ then $\{k'\}$ forms by definition a singleton class. The equivalence classes of some $\xi \in \mathfrak{C}_n$ are usually referred to as *blocks*; the *rank* $\operatorname{rk}\xi$ is the number of blocks of ξ whose intersection with $[n]$ as well as with $[n]'$ is not empty—this coincides with the usual notion of rank of a mapping on $[n]$ in case ξ is in the image of the embedding (1.3). It is known that the rank characterizes the $\mathscr{D}$-relation in $\mathfrak{C}_n$ [25, 20]: for any $\xi, \eta \in \mathfrak{C}_n$, one has $\xi \mathrel{\mathscr{D}} \eta$ if and only if $\operatorname{rk}\xi = \operatorname{rk}\eta$.

The semigroup $\mathfrak{C}_n$ admits a natural involution making it a regular $*$-semigroup: consider first the permutation * on $\widetilde{[n]}$ that swaps primed with unprimed elements, that is, set

$$k^* = k',\ (k')^* = k \text{ for all } k \in [n].$$

Then define, for $\xi \in \mathfrak{C}_n$,

$$x\,\xi^*\,y :\Leftrightarrow x^*\,\xi\,y^* \text{ for all } x, y \in \widetilde{[n]}.$$

That is, ξ^* is obtained from ξ by interchanging in ξ the primed with the unprimed elements. It is easy to see that

$$\xi^{**} = \xi,\ (\xi\eta)^* = \eta^*\xi^* \text{ and } \xi\xi^*\xi = \xi \text{ for all } \xi, \eta \in \mathfrak{C}_n. \tag{1.4}$$

The elements of the form $\xi\xi^*$ are called *projections*. They are idempotents (as one readily sees from the last equality in (1.4)) and have the following transparent structure. If k is the rank of $\xi\xi^*$ (equal to the rank of ξ), then there is some $t \in \{0, 1, \ldots, n-k\}$ and a partition of $[n]$ into $k+t$ blocks:

$$[n] = A_1 \cup \cdots \cup A_k \cup B_1 \cup \cdots \cup B_t$$

such that

$$\xi\xi^* = \{A_1 \cup A_1', \ldots, A_k \cup A_k', B_1, B_1', \ldots, B_t, B_t'\}.$$

Fig. 1 shows a typical projection in $\mathfrak{C}_8$; here $k = 3$, $t = 2$ and $A_1 = \{3, 4\}$, $A_2 = \{5, 8\}$, $A_3 = \{7\}$ while $B_1 = \{1, 2\}$, $B_2 = \{6\}$.

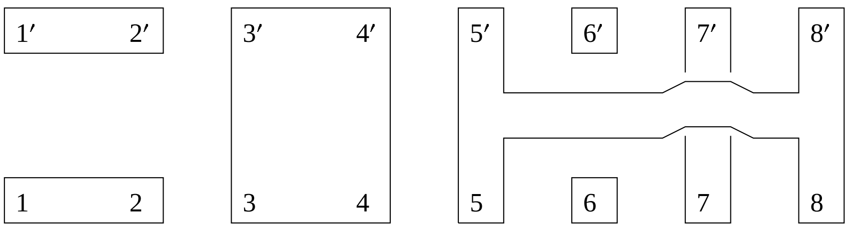

Figure 1: A rank 3 projection in $\mathfrak{C}_8$

From this it follows easily that the maximal subgroup of $\mathfrak{C}_n$ with identity $\xi\xi^*$ is isomorphic to the symmetric group $\mathfrak{S}_k$ where the isomorphism between $\mathfrak{S}_k$ and the group $\mathscr{H}$-class of $\xi\xi^*$ is given by

$$\sigma \mapsto \{A_1 \cup A_{1\sigma}', \ldots, A_k \cup A_{k\sigma}', B_1, B_1', \ldots, B_t, B_t'\},\ \sigma \in \mathfrak{S}_k. \tag{1.5}$$

We note that in the group $\mathscr{H}$-class of any projection, the involution * coincides with the inverse operation in that group. Since $\xi \mathrel{\mathscr{R}} \xi\xi^*$, *each* maximal subgroup of $\mathfrak{C}_n$ is isomorphic to the symmetric group $\mathfrak{S}_k$ for some $k \leq n$.

Apart from the involution *, $\mathfrak{C}_n$ admits another—'skew'—involution which is defined as follows: let α be the permutation of $[n]$ that reverses the order, considered as element of the group of units of $\mathfrak{C}_n$. More precisely, let α be the partition

$$\alpha = \{\{1, n'\}, \{2, (n-1)'\}, \dots, \{n, 1'\}\}$$

and define the unary operation $^\rho : \mathfrak{C}_n \to \mathfrak{C}_n$ by

$$\xi^\rho := \alpha\xi^*\alpha.$$

Since $\alpha^* = \alpha$ and $\alpha^2 = 1$ we get that $^\rho$ is indeed an involution on $\mathfrak{C}_n$. The intuitive meaning of $^\rho$ is that its application rotates the picture of a partition (as in the previous examples) by the angle of π while the application of * is the reflection along the axis between $[n]$ and $[n]'$. The involution $^\rho$ has not yet been studied in the literature but has turned out to be of significant use in the first author's recent paper [5].

The Brauer semigroup and the partial Brauer semigroup can be conveniently defined as subsemigroups of $\mathfrak{C}_n$: namely, $\mathfrak{B}_n$ [respectively $P\mathfrak{B}_n$] consists of all elements of $\mathfrak{C}_n$ all of whose blocks have size 2 [at most 2]. Both semigroups are closed under both involutions * and $^\rho$. In both types of semigroups, the group $\mathscr{H}$-class of a projection $\xi\xi^*$ of rank k is isomorphic (as a regular $*$-semigroup) with the symmetric group $\mathfrak{S}_k$. For completeness we note that $\mathfrak{B}_1$ is the trivial semigroup and $P\mathfrak{B}_1 \cong \mathfrak{C}_1$ is isomorphic to the 2-element semilattice semigroup $\{0, 1\}$ (endowed with trivial involution).

Next we define the annular semigroup $\mathfrak{A}_n$ [15]. It will be realized as a certain subsemigroup of the Brauer semigroup. For this purpose it is convenient to first represent the elements of $\mathfrak{B}_n$ as *annular diagrams*. Consider an annulus A in the complex plane, say $A = \{z \mid 1 < |z| < 2\}$ and identify the elements of $\widetilde{[n]}$ with certain points of the boundary of A via

$$k \mapsto 2e^{\frac{2\pi i(k-1)}{n}} \quad \text{and} \quad k' \mapsto e^{\frac{2\pi i(k-1)}{n}} \quad \text{for all} \quad k \in [n].$$

For $\xi \in \mathfrak{B}_n$ take a copy of A and link any $x, y \in \widetilde{[n]}$ with $\{x, y\} \in \xi$ by a path (called *string*) running entirely in A (except for its endpoints). For example, the element $\xi \in \mathfrak{B}_4$ given by

$$\xi = \{\{1, 1'\}, \{2, 4\}, \{3, 2'\}, \{3', 4'\}\}$$

is then represented by the annular diagram in Fig. 2. Paths representing blocks of the form $\{x, y'\}$ [$\{x, y\}$ and $\{x', y'\}$, respectively] for some $x, y \in [n]$ are called *through strings* [*outer* and *inner strings*, respectively]. The *annular semigroup* $\mathfrak{A}_n$ by definition consists of all elements of $\mathfrak{B}_n$ that have a representation as an annular diagram any two of whose strings have empty intersection—this will be referred to as the *annular condition* later in the text. One can compose annular diagrams in an obvious way, modelling the multiplication in $\mathfrak{B}_n$—from this it follows that $\mathfrak{A}_n$ is closed under the multiplication of $\mathfrak{B}_n$. Clearly, $\mathfrak{A}_n$ is closed under both involutions * and $^\rho$, as well.

Analogously to the partial Brauer semigroup $P\mathfrak{B}_n$, one can also define the *partial* annular semigroup $P\mathfrak{A}_n$ by considering all elements of $P\mathfrak{B}_n$ which admit a representation by an annular diagram in which any two distinct strings have empty intersection. Again each $P\mathfrak{A}_n$ is closed under both involutions * and $^\rho$.

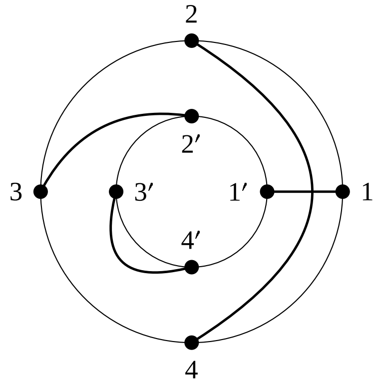

Figure 2: Annular diagram representation of a member of $\mathfrak{B}_4$

We note that the rank characterizes the $\mathscr{D}$-relation in $\mathfrak{A}_n$, as well. Let $\xi \in \mathfrak{B}_n$ be of rank t with through strings

$$\{k_1, l'_1\}, \ldots, \{k_t, l'_t\}, \text{ for some } k_i, l_i \in [n].$$

Then $\{k_1, \ldots, k_t\}$ respectively $\{l'_1, \ldots, l'_t\}$ is the *domain* $\operatorname{dom}\xi$ respectively *range* $\operatorname{ran}\xi$ of ξ. For any projection ε we obviously have $\operatorname{ran}\varepsilon = (\operatorname{dom}\varepsilon)'$.

In order to show that the rank function characterizes the $\mathscr{D}$-relation it is sufficient to show that any two projections ε, η of the same rank t are $\mathscr{D}$-related. Let ε and η be arbitrary projections of rank t with $a_1 < a_2 < \cdots < a_t$ the domain of ε and $b'_1 < b'_2 < \cdots < b'_t$ the range of η; define α to be the element having the same outer strings as ε, the same inner strings as η and the through strings

$$\{a_1, b'_1\}, \ldots, \{a_t, b'_t\}.$$

Then $\alpha \in \mathfrak{A}_n$, $\varepsilon = \alpha\alpha^*$ and $\eta = \alpha^*\alpha$.

Finally, we shall give the definition of the *Jones semigroup* $\mathfrak{J}_n$[1] [38, 21] and the *partial Jones semigroup* $P\mathfrak{J}_n$. For this purpose, consider the rectangle $R = [0,1] \times [1,n]$ in the Euclidean plane, identify the elements $i \in [n]$ with the points $(0,i)$, the elements $i' \in [n]'$ with the points $(1,i)$ on the boundary of R. Analogously to annular diagrams, the members of $\mathfrak{B}_n$ also admit representations by *rectangular diagrams* which are defined in an obvious way—if $\{x,y\}$ is a block of $\xi \in \mathfrak{B}_n$ then the corresponding points on the boundary of R are connected by a *string* running entirely in the interior of R. The set $\mathfrak{J}_n$ then consists by definition of all members of $\mathfrak{B}_n$ that admit a representation as a rectangular diagram any two of whose strings have empty intersection. Similarly, $P\mathfrak{J}_n$ may be defined to consist of those members of $P\mathfrak{B}_n$ that have a representation as a rectangular diagram any two of whose strings have empty intersection. Both sets $\mathfrak{J}_n$ and $P\mathfrak{J}_n$ are closed under multiplication and both involutions * and $^\rho$. It is well known that $\mathfrak{J}_n$ is aperiodic (that is, all subgroups are trivial) [21] and easy to see that the same is true for $P\mathfrak{J}_n$.

[1] This is also called *Temperley-Lieb semigroup*; following [21], we use the term *Jones semigroup*.

For each n we have the following inclusions among the various involutory semigroups:

$$\begin{array}{ccccccc} P\mathfrak{J}_n & \longrightarrow & P\mathfrak{A}_n & \longrightarrow & P\mathfrak{B}_n & \longrightarrow & \mathfrak{C}_n \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathfrak{J}_n & \longrightarrow & \mathfrak{A}_n & \longrightarrow & \mathfrak{B}_n & & \end{array}$$

Here an arrow denotes an injective mapping respecting multiplication and both involutions, and the diagram can be assumed to commute. Moreover, for each n and $\mathfrak{K} \in \{\mathfrak{J}, P\mathfrak{J}, \mathfrak{A}, P\mathfrak{A}, \mathfrak{B}, P\mathfrak{B}, \mathfrak{C}\}$ we also have the inclusion $\mathfrak{K}_n \leq \mathfrak{K}_{n+2}$ as involutory semigroup with respect to * as well as $^\rho$. With respect to * we have even $\mathfrak{K}_n \leq \mathfrak{K}_{n+1}$ for all n and all $\mathfrak{K}$ except $\mathfrak{K} = \mathfrak{A}$.

2. The finite basis problem for partition semigroups with involution

2.1. The involution *

In this subsection we assume that each partition semigroup considered is endowed with the involution *. In particular, for $\mathfrak{K} \in \{\mathfrak{J}, P\mathfrak{J}, \mathfrak{A}, P\mathfrak{A}, \mathfrak{B}, P\mathfrak{B}, \mathfrak{C}\}$, we let $\mathsf{var}\,\mathfrak{K}_n$ denote the variety of involution semigroups generated by $\langle \mathfrak{K}_n, \cdot, ^* \rangle$. It follows from Proposition 2.9 in [7] that a regular $*$-semigroup is never inherently nonfinitely based. Hence, for involution semigroups of this type the tool presented in subsection 1.1.2 cannot be applied at all. So we are left with tool presented in subsection 1.1.1 and we are going to examine the cases which this tool can be applied to. We need to check two things:

(i) Does there exist a (non trivial) group in $\mathsf{var}\,\mathfrak{K}_n \setminus \mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$?
(ii) Is $\mathcal{K}_3$ contained in $\mathsf{var}\,\mathfrak{K}_n$?

We immediately see that for $\mathfrak{K} \in \{\mathfrak{J}, P\mathfrak{J}\}$ condition (i) fails because $\mathsf{var}\,P\mathfrak{J}_n$ and thus also $\mathsf{var}\,\mathfrak{J}_n$ contains only trivial groups. Altogether we see that none of our tools can be applied to these cases. Direct inspection shows that $\langle P\mathfrak{J}_1, \cdot, ^* \rangle$ and $\langle \mathfrak{J}_n, \cdot, ^* \rangle$ are finitely based for $n \leq 3$. So we are left with the following open problem.

Problem 2.1. 1. Is the regular *-semigroup $P\mathfrak{J}_n$ nonfinitely based for each $n \geq 2$?
2. Is the regular *-semigroup $\mathfrak{J}_n$ nonfinitely based for each $n \geq 4$?

2.1.1. Groups in $\mathsf{var}\,\mathfrak{K}_n \setminus \mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$

We are going to check condition (i) mentioned above and shall distinguish between the cases $\mathfrak{K} \in \{\mathfrak{B}, P\mathfrak{B}, \mathfrak{C}\}$ on the one hand and $\mathfrak{K} \in \{\mathfrak{A}, P\mathfrak{A}\}$ on the other. The first case turns out to be easy. Let $\mathfrak{K}_n$ be any of $\mathfrak{C}_n$, $P\mathfrak{B}_n$ or $\mathfrak{B}_n$. Each projection different from the identity of $\mathfrak{K}_n$ has rank less than n, whence the semigroup $\mathrm{H}(\mathfrak{K}_n)$ contains, apart from the identity element, only elements of rank strictly less than n. This implies the following result.

Proposition 2.1. *For each $n \geq 2$ and each $\mathfrak{K}_n \in \{\mathfrak{C}_n, P\mathfrak{B}_n, \mathfrak{B}_n\}$ there exists a group in $\mathsf{var}\,\mathfrak{K}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$.*

Proof. The group in question is the symmetric group $\mathfrak{S}_n$ that is the group of units in $\mathfrak{K}_n$ and thus belongs to $\mathsf{var}\,\mathfrak{K}_n$. By the argument of Kim and Roush [17], each group in the variety $\mathsf{var}\,\mathrm{H}(\mathfrak{K}_n)$ belongs to the group variety generated by the subgroups of the semigroup $\mathrm{H}(\mathfrak{K}_n)$. As observed above, each subgroup of $\mathrm{H}(\mathfrak{K}_n)$ embeds into the symmetric group $\mathfrak{S}_{n-1}$ whence it remains to check that $\mathfrak{S}_n$ does not belong to the group variety generated by $\mathfrak{S}_{n-1}$. This follows from [33, Theorem 51.2] because the group $\mathfrak{S}_n$ always has a chief factor of order larger than the maximum order of chief factors in $\mathfrak{S}_{n-1}$. □

In contrast, the cases $\mathfrak{K} = \mathfrak{A}$ and $\mathfrak{K} = P\mathfrak{A}$ turn out to be much more complicated and require quite a bit of effort and space. We start with $\mathfrak{K} = \mathfrak{A}$. It was observed by Jones [15] that the maximal subgroup of $\mathfrak{A}_n$ whose identity is a projection ε of rank t is a cyclic group of order t. Indeed, suppose that $k_1 < k_2 < \cdots < k_t$ are the elements of $\operatorname{dom}\varepsilon$ and let $\xi \in \mathfrak{A}_n$ be $\mathscr{H}$-related with ε. In ξ there exists a unique through string of the form $\{k_1, k'_\ell\}$. From the annular condition it follows that the remaining $t-1$ through strings of ξ are precisely

$$\{k_2, k'_{\ell+1}\}, \ldots, \{k_{t-\ell+1}, k'_t\}, \{k_{t-\ell+2}, k'_1\} \ldots, \{k_t, k'_{\ell-1}\}$$

while the inner and outer strings of ξ are those of ε. Taking into account (1.5), we observe that $\xi = \tau^{\ell-1}$, where τ consists of the through strings

$$\{k_1, k'_2\}, \{k_2, k'_3\}, \ldots, \{k_t, k'_1\}$$

together with the inner and outer strings of ε. Obviously, $\tau^t = \varepsilon$ and the order of τ is t.

In the following we shall obtain some facts about $\mathrm{H}(\mathfrak{A}_n)$ in case n is even. Let $\alpha \in \mathfrak{A}_n$ be an element of rank r and let $a_1 < a_2 < \cdots < a_r$ and $b'_1 < b'_2 < \cdots < b'_r$ be the elements of $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. Then the numbers a_i are alternately even and odd, and likewise are the numbers b_i. This is because the nodes between a_i and a_{i+1} as well as between b'_i and b'_{i+1} are entirely involved in outer strings respectively inner strings. A through string $\{i, j'\}$ of α is *even* if $i - j$ is even, and otherwise it is *odd*. Suppose that $\{a_1, b'_{s+1}\}$ is a through string of α. By the annular condition, the other through strings of α are exactly the strings $\{a_i, b'_{s+i}\}$ (where the sum $s + i$ has to be taken $\bmod\ r$). It follows that either all through strings of α are even or all are odd. Let the element α be *even* if all of its through strings are even (or equivalently, if α has no odd through string)—note that the even members of $\mathfrak{A}_n$ coincide with the *oriented diagrams* in [15]. All diagrams of rank 0 are even, by definition. Let $\alpha, \beta \in \mathfrak{A}_n$ and suppose that $\mathbf{s} = \underset{k}{\bullet}\!\!-\!\!\!-\!\!\underset{l'}{\bullet}$ is a through string in $\alpha\beta$. By definition of the product in $\mathfrak{A}_n$ there exist a unique number $s \geq 1$ and pairwise distinct $u_1, v_1, u_2, \ldots, v_{s-1}, u_s \in [n]$ such that $\mathbf{s}$ is obtained as the concatenation of the strings

$$\underset{k}{\bullet}\!\!-\!\!\!-\!\!\underset{u'_1}{\bullet}\;\underset{u_1}{\bullet}\!\!-\!\!\!-\!\!\underset{v_1}{\bullet}\;\underset{v'_1}{\bullet}\!\!-\!\!\!-\!\!\underset{u'_2}{\bullet}\cdots\underset{u_{s-1}}{\bullet}\!\!-\!\!\!-\!\!\underset{v_{s-1}}{\bullet}\;\underset{v'_{s-1}}{\bullet}\!\!-\!\!\!-\!\!\underset{u'_s}{\bullet}\;\underset{u_s}{\bullet}\!\!-\!\!\!-\!\!\underset{l'}{\bullet}$$

where $\mathbf{u} := \underset{k}{\bullet}\!\!-\!\!\!-\!\!\underset{u'_1}{\bullet}$ is a through string of α, $\mathbf{v} := \underset{u_s}{\bullet}\!\!-\!\!\!-\!\!\underset{l'}{\bullet}$ is a through string of β, all $\underset{u_i}{\bullet}\!\!-\!\!\!-\!\!\underset{v_i}{\bullet}$ are outer strings of β and all $\underset{v'_i}{\bullet}\!\!-\!\!\!-\!\!\underset{u'_{i+1}}{\bullet}$ are inner strings of α. It follows that $u_i \not\equiv v_i \not\equiv u_{i+1} \bmod 2$ and therefore $u_i \equiv u_{i+1} \bmod 2$ for all i whence $u_1 \equiv u_s \bmod 2$. Consequently, $\mathbf{s}$ is even if and only of $\mathbf{u}$ and $\mathbf{v}$ are both even or both odd while $\mathbf{s}$ is odd if and only if exactly one of $\mathbf{u}$ and $\mathbf{v}$ is even. In particular, the set $\mathfrak{E}\mathfrak{A}_n$ of all even members of $\mathfrak{A}_n$ forms a submonoid of $\mathfrak{A}_n$. Since each projection is even, $\mathfrak{E}\mathfrak{A}_n$ contains the Hermitian subsemigroup $\mathrm{H}(\mathfrak{A}_n)$.

Next we are going to determine the maximal subgroups of $\mathfrak{E}\mathfrak{A}_n$. For this purpose we first note that the rank characterizes Green's $\mathscr{D}$-relation in $\mathfrak{E}\mathfrak{A}_n$. The argument is a refinement of the one given above for $\mathfrak{A}_n$: let ε and η be arbitrary projections of rank t with $a_1 < a_2 < \cdots < a_t$ the domain of ε and $b'_1 < b'_2 < \cdots < b'_t$ the range of η; define γ to be the element having the same outer strings as ε, the same inner strings as η and the through strings $\{a_1, b'_1\}, \ldots, \{a_t, b'_t\}$ in case $a_1 \equiv b_1 \bmod 2$ while in case $a_1 \not\equiv b_1 \bmod 2$ the through strings of γ can be chosen to be $\{a_1, b'_2\}, \{a_2, b'_3\} \ldots, \{a_t, b'_1\}$. Then $\gamma \in \mathfrak{E}\mathfrak{A}_n$, $\varepsilon = \gamma\gamma^*$ and $\eta = \gamma^*\gamma$. In order to determine the maximal subgroups of $\mathfrak{E}\mathfrak{A}_n$ it therefore suffices to determine, for each even $r \leq n$, **some** maximal subgroup of the $\mathscr{D}$-class of all rank r elements of $\mathfrak{E}\mathfrak{A}_n$.

Lemma 2.2. *For each even $r \leq n$, the maximal subgroups in the $\mathscr{D}$-class of all rank r elements of $\mathfrak{E}\mathfrak{A}_n$ are cyclic of order $\frac{r}{2}$.*

Proof. Consider the rank-r-projection ε_r depicted in Fig. 3. The element ζ_r depicted in Fig. 4

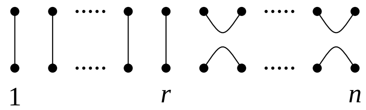

Figure 3: The rank r-projection ε_r

is a generator of the group $\mathscr{H}$-class of ε_r in $\mathfrak{A}_n$. But ζ_r is odd and therefore is definitely **not**

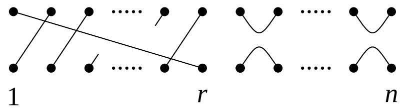

Figure 4: The element ζ_r

contained in $\mathfrak{E}\mathfrak{A}_n$. On the other hand, ζ_r^2 is even and hence belongs to $\mathfrak{E}\mathfrak{A}_n$. It is easy to see that ζ_r^2 generates a group of order $\frac{r}{2}$. $\square$

Since the group of units of $\mathrm{H}(\mathfrak{A}_n)$ is trivial we note that the subgroups of $\mathrm{H}(\mathfrak{A}_n)$ are cyclic of order at most $\frac{n}{2} - 1$. Altogether we are able to detect a group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$:

Corollary 2.3. *For each even number n there exists a group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$.*

Proof. We have seen that all cyclic groups in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ have even order less than $\frac{n}{2}$. On the other hand, each cyclic group of even order up to n belongs to $\mathsf{var}\,\mathfrak{A}_n$. In order to find a (cyclic) group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ it suffices to find an even number $k \leq n$ that does not divide the least common multiple of all even numbers less than $\frac{n}{2}$. For such k we may take the largest power of 2 which is less than or equal to n. $\square$

Since, for any n, all maximal subgroups of $\mathrm{H}(\mathfrak{A}_n)$ have order at most $n - 2$, by the same reasoning as in Corollary 2.3, the next result is immediate.

Corollary 2.4. *Let n be a prime power; then the cyclic group of order n belongs to $\mathsf{var}\,\mathfrak{A}_n$ but not to $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$.*

Finally, we show that the cases n even or a prime power are the only ones for which there is a group in $\mathsf{var}\,\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$. Therefore, our methods are applicable precisely in these cases.

Proposition 2.5. *If n is odd and not a prime power then the groups in $\mathsf{var}\,\mathfrak{A}_n$ and $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ are the same.*

Proof. Every element of $\mathfrak{A}_n$ has odd rank. Let $t < n$ be odd. We define projections $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{t+1}$ as follows. Each ε_i, $i = 0, 1, \ldots, t+1$, has the outer strings $\{t+3, t+4\}, \ldots, \{n-1, n\}$ and the corresponding inner strings; besides those, the projection ε_0 has the outer string $\{1, t+2\}$ and

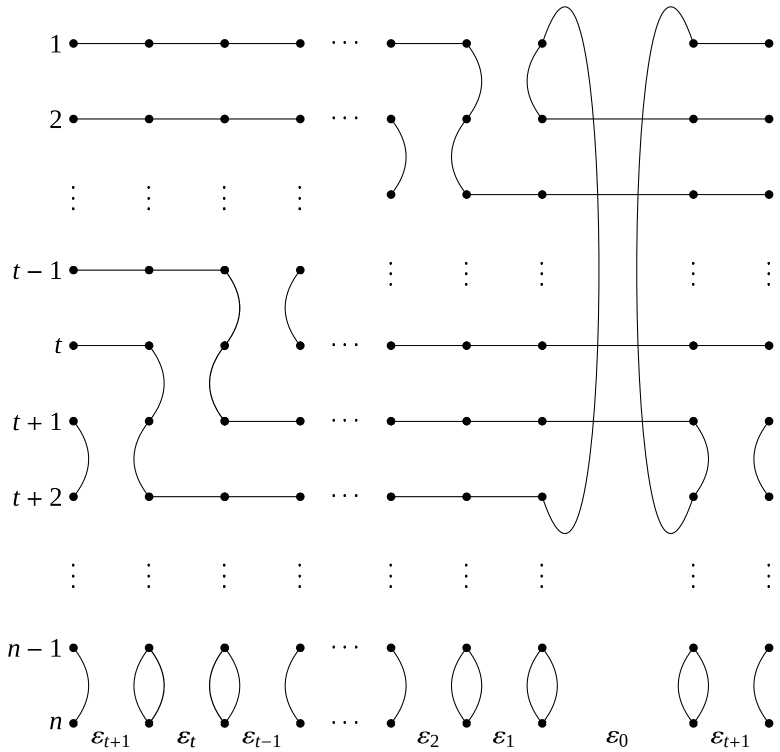

Figure 5: A cycle of order t as a product of projections in $\mathfrak{A}_n$

the inner string $\{1', (t+2)'\}$ and each of the projections ε_i, $i = 1, \dots, t+1$, has the outer string $\{i, i+1\}$ and the outer string $\{i', (i+1)'\}$. Finally, all remaining elements of $\widetilde{[n]}$ are involved in the through strings $\{k, k'\}$.

One then readily verifies (see Fig. 5) that $\alpha = \varepsilon_{t+1}\varepsilon_t \cdots \varepsilon_0\varepsilon_{t+1}$ has the same inner and outer strings as ε_{t+1} hence $\alpha \mathrel{\mathscr{H}} \varepsilon_{t+1}$. Moreover, the through strings of α are

$$\{1, 3'\}, \{2, 4'\}, \dots, \{t-1, 1'\}, \{t, 2'\}.$$

Via (1.5), α realizes the cyclic permutation $k \mapsto k+2 \pmod t$ which has order t since t is odd. Thus, for each odd $t < n$, the semigroup $\mathrm{H}(\mathfrak{A}_n)$ contains a cyclic group of order t as an involutory subsemigroup. The maximal subgroups of $\mathfrak{A}_n$ are precisely the cyclic groups of odd order at most n. So, we have already shown that $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ contains each maximal subgroup of $\mathfrak{A}_n$, with the possible exception of the group of units of $\mathfrak{A}_n$ which is cyclic of order n. Since n is not a prime power, $n = k\ell$ for some co-prime numbers k, ℓ. As already pointed out, the cyclic groups C_k and C_ℓ of orders k and ℓ, respectively, belong to $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ whence so does the cyclic group of order n which is isomorphic to $C_k \times C_\ell$. □

Corollary 2.6. *There exists a group in* $\mathsf{var}\,\mathfrak{A}_n$ *that is not in* $\mathsf{var}\,\mathrm{H}(\mathfrak{A}_n)$ *if and only if n is even or a prime power.*

We can also characterize the even numbers n for which there exists a group in $\mathsf{var}\,\mathfrak{C}\mathfrak{A}_n$ which is not in $\mathsf{var}\,\mathrm{H}(\mathfrak{C}\mathfrak{A}_n)$. Indeed, it was shown in [6] that for even n—recall that $\mathfrak{C}\mathfrak{A}_n$ is only defined for even n—, the semigroup $\mathrm{H}(\mathfrak{A}_n) = \mathrm{H}(\mathfrak{E}\mathfrak{A}_n)$ consists of the identity element 1 together with all elements of $\mathfrak{E}\mathfrak{A}_n$ whose rank is strictly smaller than n. It follows that the maximal subgroups of $\mathfrak{E}\mathfrak{A}_n$ are the cyclic groups of orders $\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1$ while the maximal subgroups of $\mathrm{H}(\mathfrak{E}\mathfrak{A}_n)$ are the cyclic groups of orders $\frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1$. This shows the next result.

Corollary 2.7. *Let n be even; then there exists a group in* $\mathsf{var}\,\mathfrak{E}\mathfrak{A}_n$ *which is not in* $\mathsf{var}\,\mathrm{H}(\mathfrak{E}\mathfrak{A}_n)$ *if and only if $\frac{n}{2}$ is a prime power.*

In contrast to the ordinary annular case, it is no longer true that there is a group in $\mathsf{var}\,P\mathfrak{A}_n$ that is not in $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$ for each even n. The fact that some elements of $\widetilde{[n]}$ need not be involved in any string gives the projections more freedom to gain cyclic permutations in $\mathrm{H}(P\mathfrak{A}_n)$, as the following result demonstrates.

Proposition 2.8. *For each $n \geq 5$ and $t \leq n-3$ there exists a cyclic subgroup of order t in* $\mathrm{H}(P\mathfrak{A}_n)$.

Proof. For odd t this follows immediately from Proposition 2.5. For even t it can be shown that the element α consisting of the through strings

$$\{1, 2'\}, \{2, 3'\}, \ldots, \{t-1, t'\}, \{t, 1'\}$$

along with the outer string $\{t+2, t+3\}$ and the inner string $\{(t+2)', (t+3)'\}$, and else having no other strings can be written as a product of $\frac{5t}{2} + 4$ projections, see Fig. 6. Clearly, α realizes a cyclic permutation of order t. ☐

From this we obtain:

Corollary 2.9. *If $n \notin \{p^k, p^k + 1, 2^k + 2\}$ for each prime p and each $k \geq 1$, then* $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$ *and* $\mathsf{var}\,P\mathfrak{A}_n$ *contain the same groups.*

Proof. We may assume that $n \geq 15$. As already mentioned, the variety of all groups in $\mathsf{var}\,P\mathfrak{A}_n$ is generated by all cyclic groups of orders at most n. By Proposition 2.8, all cyclic groups of orders at most $n-3$ belong to $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$. Since n is not a prime power it can be factored as $n = k\ell$ with k, ℓ co-prime and $k, \ell \leq n-3$. Since the cyclic groups of order k and ℓ belong to $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$, so does the cyclic group of order n. The same reasoning applies to the cyclic group of order $n-1$. Consider finally the case of $n-2$. By assumption, $n-2$ either is an odd prime power or has at least two distinct prime factors. In the former case the claim follows from the proof of Proposition 2.5 and in the latter case the argument is the same as for n and $n-1$. ☐

On the other hand, the converse of Corollary 2.9 also holds.

Proposition 2.10. *If $n \in \{p^k, p^k + 1, 2^k + 2\}$ for some prime p and some positive integer k, then there exists a group in* $\mathsf{var}\,P\mathfrak{A}_n$ *which is not in* $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$.

Proof. The case $n = p^k$ is obvious. Since the product of any two *distinct* projections of rank $n-1$ has rank less than $n-1$, the group $\mathscr{H}$-class in $\mathrm{H}(P\mathfrak{A}_n)$ of any projection of rank $n-1$ is trivial, implying the claim for the case $n = p^k + 1$.

Finally, in case $n = 2^k + 2$ we show that the cyclic group of order $2^k = n-2$ is not in $\mathsf{var}\,\mathrm{H}(P\mathfrak{A}_n)$. Let ε be a projection of rank $n-2$ containing the outer string $\{i, i+1\}$ and the inner string $\{i', (i+1)'\}$ and let ε° be the projection obtained from ε by removing these two strings. If η is a projection of rank $n-1$ such that $\eta\varepsilon$ has rank $n-2$, then $\eta\varepsilon = \varepsilon^\circ\varepsilon$ (and likewise $\varepsilon\eta = \varepsilon\varepsilon^\circ$). Hence, if α is a product of projections and of rank $n-2$ then we may assume that all these

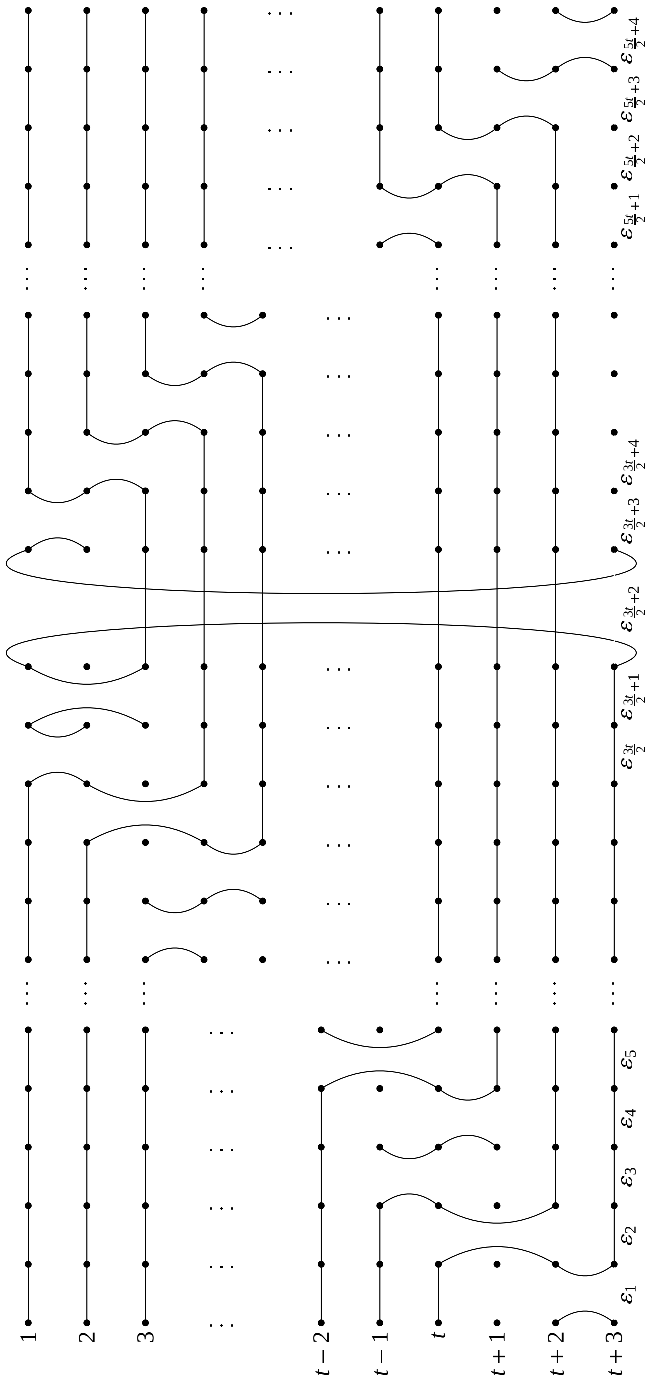

Figure 6: A cycle of order t as a product of projections in $P\mathfrak{A}_n$

projections have rank $n-2$. Moreover, any product of two distinct projections of rank $n-2$ that have only through strings has rank less than $n-2$. Finally, let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be projections of rank $n-2$ such that ε_1 and ε_3 have outer and inner strings but ε_2 does not. If $\varepsilon_1\varepsilon_2\varepsilon_3$ has rank $n-2$ then $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_1\varepsilon_3 = \varepsilon_1$.

Let α be of rank $n-2$ and assume that it is a product of projections: $\alpha = \varepsilon_0\varepsilon_1\cdots\varepsilon_r$. The observations in the preceding paragraph imply that in addition we may assume that ε_i has rank $n-2$ for each $i = 0, \ldots, r$ and that $\varepsilon_1, \ldots, \varepsilon_{r-1}$ have inner and outer strings, that is, $\varepsilon_1, \ldots, \varepsilon_{r-1}$ belong to $\mathfrak{A}_n$. Assume further that α is contained in a subgroup of $\mathrm{H}(P\mathfrak{A}_n)$. We intend to prove that the order of α is at most $\frac{n-2}{2} = 2^{k-1}$. For this purpose we may assume that α is $\mathscr{H}$-related to a projection ε. This implies immediately that $\varepsilon_0 = \varepsilon = \varepsilon_r$.

Now consider two cases: (i) ε has an inner and an outer string, that is, ε belongs to $\mathfrak{A}_n$, and (ii) ε has only through strings, that is, ε does not belong to $\mathfrak{A}_n$. In the first case, α belongs to $\mathrm{H}(\mathfrak{A}_n)$ and so the order of α is at most $\frac{n-2}{2}$ by Lemma 2.2.

In the second case, we get $\varepsilon_1 = \varepsilon_{r-1}$ and $\varepsilon = \varepsilon_1^\circ$ since $\varepsilon\varepsilon_1$ as well as $\varepsilon_{r-1}\varepsilon$ have rank $n-2$. From this it follows that the set $\{\varepsilon, \varepsilon_1, \varepsilon\varepsilon_1, \varepsilon_1\varepsilon\}$ forms a 2×2-rectangular band under multiplication. In particular, ε and ε_1 are $\mathscr{D}$-related in $\mathrm{H}(P\mathfrak{A}_n)$. Green's Lemma implies that the order of α is the same as the order of $\varepsilon_1\alpha\varepsilon_1 = \varepsilon_1\cdots\varepsilon_{r-1}$. The latter element belongs to $\mathrm{H}(\mathfrak{A}_n)$, so its order is at most $\frac{n-2}{2}$, again by Lemma 2.2. Since no group element of rank less than $n-2$ can have order $n-2$ we actually have shown that $\mathrm{H}(P\mathfrak{A}_n)$ does not contain a group element of order $n-2 = 2^k$.

Altogether, the cyclic group of order 2^k belongs to $\mathsf{var}\, P\mathfrak{A}_n$ but not to $\mathsf{var}\, \mathrm{H}(P\mathfrak{A}_n)$, just as required. $\square$

Having thus completely examined when condition (i) mentioned in the introduction to Section 2.1 is fulfilled we turn to condition (ii).

2.1.2. Membership of $\mathcal{K}_3$

In order to complete the results which make an application of Theorem 1.1 possible, we need to check membership of $\mathcal{K}_3$.

Proposition 2.11. *The regular *-semigroup $\mathcal{K}_3$ is contained in*

1. $\mathsf{var}\,\mathfrak{C}_n$ *for each* $n \geq 2$,
2. $\mathsf{var}\, P\mathfrak{A}_n \subseteq \mathsf{var}\, P\mathfrak{B}_n$ *for each* $n \geq 3$,
3. $\mathsf{var}\,\mathfrak{A}_n \subseteq \mathsf{var}\,\mathfrak{B}_n$ *for each* $n \geq 4$ *and* $\mathsf{var}\,\mathfrak{C}\mathfrak{A}_n$ *for each even* $n \geq 4$.

Proof. In the first case, consider the regular *-subsemigroup $\mathcal{U}_1$ of $\mathfrak{C}_2$ generated by the projections of rank 1—these are

$$\{\{1, 1', 2, 2'\}\}, \{\{1, 1'\}, \{2\}, \{2'\}\}, \{\{1\}, \{1'\}, \{2, 2'\}\}.$$

It is easy to calculate that $\mathcal{U}_1$ contains 13 partitions: 9 of rank 1 and 4 of rank 0. The $\mathscr{D}$-class of $\mathcal{U}_1$ consisting of the partitions of rank 1 is shown in Fig. 7 where the idempotents are marked with $\star$.

Now it is clear that if one factors $\mathcal{U}_1$ by the ideal of all elements of rank 0, then the resulting regular *-semigroup is isomorphic to $\mathcal{K}_3$. Thus, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{C}_2$, and hence, to the variety $\mathsf{var}\,\mathfrak{C}_n$ for each $n \geq 2$.

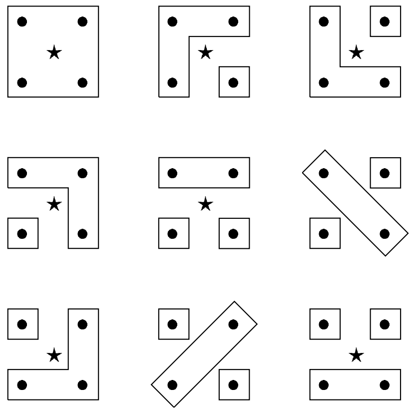

Figure 7: The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_1$ of $\mathfrak{C}_2$

For the second case consider the involutory subsemigroup $\mathcal{U}_2$ of $P\mathfrak{A}_3$ generated by the projections

$$\{\{1, 1'\}, \{2, 3\}, \{2', 3'\}\},$$
$$\{\{1, 2\}, \{1', 2'\}, \{3, 3'\}\},$$
$$\{\{1, 1'\}, \{2\}, \{2'\}, \{3\}, \{3'\}\}.$$

Again it is easy to calculate that $\mathcal{U}_2$ contains 9 partitions of rank 1 and 9 partitions of rank 0. The partitions of rank 1 form a regular $\mathscr{D}$-class depicted in Fig. 8. As above, the idempotents are marked with $\star$.

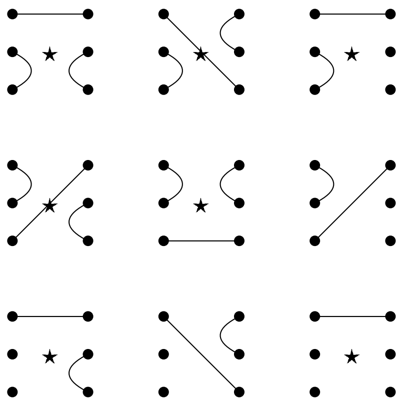

Figure 8: The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_2$ of $P\mathfrak{A}_3$

Again, it follows that the quotient of $\mathcal{U}_2$ by the ideal of all elements of rank 0 is isomorphic to $\mathcal{K}_3$. We see that $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\, P\mathfrak{A}_3$, and hence, to the varieties $\mathsf{var}\, P\mathfrak{A}_n$ and $\mathsf{var}\, P\mathfrak{B}_n$ for each $n \geq 3$.

Finally, for the third case consider the involutory subsemigroup $\mathcal{U}_3$ of $\mathfrak{C}\mathfrak{A}_4$ generated by the projections

$$\{\{1, 1'\}, \{2, 3\}, \{2', 3'\}, \{4, 4'\}\},$$
$$\{\{1, 1'\}, \{2, 2'\}, \{3, 4\}, \{3', 4'\}\},$$
$$\{\{1, 2\}, \{1', 2'\}, \{3, 3'\}, \{4, 4'\}\}.$$

It can be easily shown that $\mathcal{U}_3$ contains 13 partitions: 9 of rank 2 and 4 of rank 0. (Actually, one can observe that $\mathcal{U}_3$ is isomorphic to $\mathcal{U}_1$ where the isomorphism is induced by the following mapping of the base sets: $1, 2 \mapsto 1$; $3, 4 \mapsto 2$; $1', 2' \mapsto 1'$ and $3', 4' \mapsto 2'$.) Fig. 9 presents the top $\mathscr{D}$-class of $\mathcal{U}_3$ consisting of the partitions of rank 2.

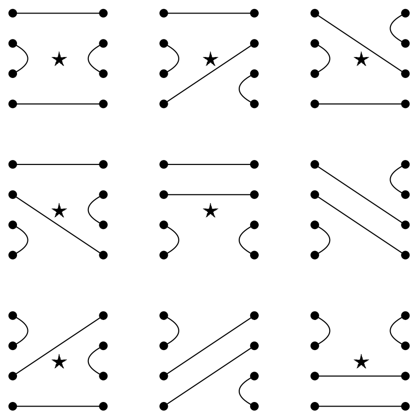

Figure 9: The upper $\mathscr{D}$-class of the subsemigroup $\mathcal{U}_3$ of $\mathfrak{A}_4$

Thus, factoring $\mathcal{U}_3$ by the ideal of all elements of rank 0, one gets a regular *-semigroup isomorphic to $\mathcal{K}_3$. Therefore, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{E}\mathfrak{A}_4$, and hence, to $\mathsf{var}\,\mathfrak{E}\mathfrak{A}_n$ and therefore also to $\mathsf{var}\,\mathfrak{A}_n$ for each even $n \geq 4$ (recall that there is an embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+2}$ of regular $*$-semigroups which restricts to an embedding $\mathfrak{E}\mathfrak{A}_n \hookrightarrow \mathfrak{E}\mathfrak{A}_{n+2}$) and to $\mathsf{var}\,\mathfrak{B}_n$ for each $n \geq 4$.

It remains to verify that $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathfrak{A}_5$ (as then it also belongs to all the varieties $\mathsf{var}\,\mathfrak{A}_n$ with odd $n \geq 5$). Here an obvious modification of the above construction works, namely, we add to each of the 13 partitions forming $\mathcal{U}_3$ the new through string $\{5, 5'\}$. It is easy to see that the resulting 13 partitions lie in $\mathfrak{A}_5$ and form an involutory subsemigroup isomorphic to $\mathcal{U}_3$. □

We can summarize the results obtained so far in this section as follows.

Theorem 2.12. *The following regular $*$-semigroups are not finitely based:*

1. $\mathfrak{C}_n$ *for* $n \geq 2$,
2. $P\mathfrak{B}_n$ *for* $n \geq 3$,
3. $\mathfrak{B}_n$ *for* $n \geq 4$,
4. $\mathfrak{A}_n$ *for* $n \geq 4$, *n even or a prime power,*
5. $\mathfrak{E}\mathfrak{A}_n$ *for n even and* $\frac{n}{2}$ *a prime power,*
6. $P\mathfrak{A}_n$ *for* $n \geq 3$, *n of the form* $2^k + 2$, p^k *or* $p^k + 1$ *for a prime p and* $k \geq 1$.

Proof. From Proposition 2.1, Corollaries 2.3 and 2.4, and Propositions 2.10 and 2.11, it follows that Theorem 1.1 applies in each case. □

Given this result, the question arises what happens in the cases not covered by Theorem 2.12. First of all, we may formulate

Problem 2.2.
1. Is the regular *-semigroup $\mathfrak{A}_n$ nonfinitely based, where n is odd, not a prime power?
2. Is the regular $*$-semigroup $\mathfrak{E}\mathfrak{A}_{2n}$ nonfinitely based for n not a prime power?
3. Is the regular *-semigroup $P\mathfrak{A}_n$ nonfinitely based for $n \notin \{2^k + 2, p^k, p^k + 1\}$ (p prime, $k \geq 1$)?

Remaining are now only some cases for small n. In case $n = 1$ we have: $\mathfrak{B}_1 \cong \mathfrak{A}_1$ is the trivial semigroup which is of course finitely based and $P\mathfrak{B}_1 \cong P\mathfrak{A}_1$ is the two element semilattice (with trivial involution) which is also finitely based. In case $n = 2$, $\mathfrak{B}_2 \cong \mathfrak{A}_2$ is a Clifford semigroup (a cyclic group of order 2 with zero adjoined) which is finitely based, $\mathfrak{E}\mathfrak{A}_2$ is again a two element semilattice (with trivial involution), and $P\mathfrak{B}_2 \cong P\mathfrak{A}_2$ which turns out to be an ideal extension of a 2×2 rectangular band (with involution) by the symmetric inverse semigroup of rank 2—we do not know if this is finitely based. Finally, in case $n = 3$ we observe that $\mathfrak{B}_3$ is an ideal extension of a 3×3 rectangular band (with involution) by the symmetric group $\mathfrak{S}_3$ and $\mathfrak{A}_3$ is an ideal extension of a 3×3 rectangular band (with involution) by the cyclic group of order 3. Kudryavtseva (unpublished) has verified that $\mathfrak{A}_3$ is finitely based while the case of $\mathfrak{B}_3$ remains unsettled so far. So we may formulate

Problem 2.3. Are the regular $*$-semigroups $P\mathfrak{B}_2 \cong P\mathfrak{A}_2$ and $\mathfrak{B}_3$ finitely based?

2.2. The 'skew' involution $^\rho$

In the case of the 'skew' involution $^\rho$ we are in the lucky position to be able to apply the tool presented in subsection 1.1.2 (except for a few cases with small n).

Theorem 2.13.
1. *For each $n \geq 2$ the involutory semigroups $P\mathfrak{J}_n, P\mathfrak{A}_n, P\mathfrak{B}_n, \mathfrak{C}_n$ are inherently nonfinitely based (with respect to $^\rho$).*
2. *For each $n \geq 4$ the involutory semigroups $\mathfrak{J}_n, \mathfrak{A}_n, \mathfrak{B}_n$ are inherently nonfinitely based and so is $\mathfrak{E}\mathfrak{A}_n$ for each even $n \geq 4$ (with respect to $^\rho$).*

Proof. (1) Since $P\mathfrak{J}_n \leq P\mathfrak{A}_n \leq P\mathfrak{B}_n \leq \mathfrak{C}_n$ and $P\mathfrak{J}_n \leq P\mathfrak{J}_{n+2}$ for all n, Theorem 1.2 implies that it is sufficient to verify that $P\mathfrak{J}_2$ as well as $P\mathfrak{J}_3$ contains an involutory subsemigroup isomorphic with the twisted Brandt monoid. For the case $n = 2$ consider the elements of $P\mathfrak{J}_2$ depicted in Fig. 10.

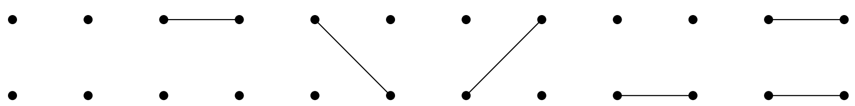

Figure 10: The twisted Brandt monoid in $P\mathfrak{J}_2$

This set forms an involutory subsemigroup of $P\mathfrak{J}_2$ with respect to $^\rho$. The mapping that sends each of these elements in the given order to the matrices given in (1.2) (in the order given there) turns out to be an isomorphism of involutory semigroups. For the case $n = 3$ the same can be done with the members of $P\mathfrak{J}_3$ depicted in Fig. 11.

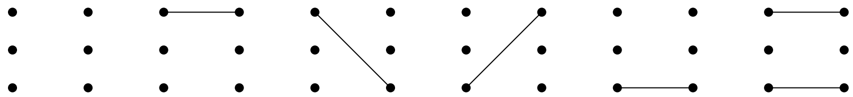

Figure 11: The twisted Brandt monoid in $P\mathfrak{J}_3$

(2) Since $\mathfrak{J}_n \leq \mathfrak{A}_n \leq \mathfrak{B}_n$, $\mathfrak{J}_n \leq \mathfrak{J}_{n+2}$ for all n and $\mathfrak{J}_n \leq \mathfrak{E}\mathfrak{A}_n$ for all even n, by Theorem 1.2 it suffices to show that $\mathfrak{J}_4$ as well as $\mathfrak{J}_5$ contains an involutory subsemigroup or divisor isomorphic with the twisted Brandt monoid; here "divisor" means a homomorphic image of an involutory subsemigroup. In the case $n = 4$, the same observation as in (1) applies to the elements of $\mathfrak{J}_4$ depicted in Fig. 12.

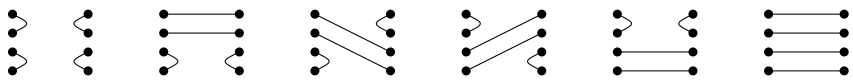

Figure 12: The twisted Brandt monoid in $\mathfrak{J}_4$

In case $n = 5$ consider the involutory subsemigroup of $\mathfrak{J}_5$ generated by the elements of $\mathfrak{J}_5$ depicted in Fig. 13, add the identity element of $\mathfrak{J}_5$ and factor by the ideal consisting of all

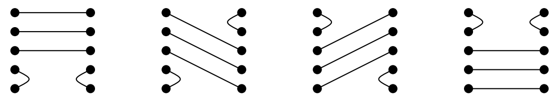

Figure 13: The twisted Brandt monoid in $\mathsf{var}\,\mathfrak{J}_5$

elements of rank one. The result is again an involutory semigroup isomorphic with $\mathcal{TB}_2^1$. $\square$

For the remaining cases of small n we note that $P\mathfrak{J}_1 \cong P\mathfrak{A}_1 \cong P\mathfrak{B}_1 \cong \mathfrak{C}_1 \cong \mathfrak{J}_2 \cong \mathfrak{E}\mathfrak{A}_2$ is a two element semilattice with trivial involution which is finitely based. Moreover, $\mathfrak{J}_1 \cong \mathfrak{A}_1 \cong \mathfrak{B}_1$ is the trivial involutory semigroup; $\mathfrak{A}_2 \cong \mathfrak{B}_2$ is a cyclic group of order 2 with an extra zero element adjoined with trivial involution; $\mathfrak{J}_3$ is a 2×2 rectangular band with an extra identity adjoined, the involution satisfying the identity $x = xx^{\rho}x$—all of these are known to be finitely based. So we are left with the following two involutory semigroups:

Problem 2.4. Are the involutory semigroups $\mathfrak{A}_3$ and $\mathfrak{B}_3$ (with respect to $^{\rho}$) finitely based?

3. Further applications

3.1. Unary Rees matrix semigroups

We had used in [7] unary Rees matrix semigroups as a tool in the proof of Theorem 1.1; in turn, here we shall show that this theorem allows one to solve the finite basis problem for a large family of unary Rees matrix semigroups.

An $I \times I$-matrix $P = (p_{ij})$ over $\mathcal{G} \cup \{0\}$, where $\mathcal{G}$ is a group, is called *block-diagonalizable* if there exists a partition π of the set I such that $p_{ij} \neq 0$ if and only if $i \mathrel{\pi} j$. If one defines a graph $\Gamma(P)$ on the set I in which two distinct vertices i and j are adjacent if and only if $p_{ij} \neq 0$, then it is clear that block-diagonalizable matrices correspond to graphs whose connected components are cliques (i.e. complete graphs). We say that P is $*$-*regular* if for all $i, j \in I$, one has $p_{ji} = p_{ij}^{-1}$

whenever $p_{ij} \in \mathcal{G}$ and $p_{ii} = e$, where e is the identity element of $\mathcal{G}$. (Recall that this property ensures that the unary Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is a regular $*$-semigroup.)

Theorem 3.1. *Let P be an $I \times I$-matrix over $\mathcal{G} \cup \{0\}$, where $\mathcal{G}$ is a group. Suppose that P is $*$-regular and not block-diagonalizable. If $\mathcal{G}$ does not belong to the group variety $\mathsf{var}\,\mathcal{H}$, where $\mathcal{H}$ is the subgroup generated by the non-zero entries of P, then the involutory Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is not finitely based.*

Proof. Let $P = (p_{ij})$. Since P is not block-diagonalizable, there is a connected component C in $\Gamma(P)$ which is not a clique. Let Q be a maximal clique in C. As C is connected, there exist $i_0 \in Q$ and $j_0 \in C \setminus Q$ such that i_0 and j_0 are adjacent. At the same time, there should be a vertex $k_0 \in Q$ such that j_0 and k_0 are not adjacent—otherwise $Q \cup \{j_0\}$ would make a larger clique in C. Thus, the submatrix P_0 of P corresponding to the set $I_0 = \{i_0, j_0, k_0\}$ is of the form

$$P_0 = \begin{pmatrix} e & g & h \\ g^{-1} & e & 0 \\ h^{-1} & 0 & e \end{pmatrix}$$

where $g = p_{i_0 j_0}, h = p_{i_0 k_0}$ belong to $\mathcal{G}$. The unary Rees matrix semigroup $\mathcal{M}^0(I_0, \mathcal{G}, I_0; P_0)$ is then a unary subsemigroup in $\mathcal{M}^0(I, \mathcal{G}, I; P)$, and the obvious homomorphism $\mathcal{G} \to \mathcal{E}$, where $\mathcal{E} = \{e\}$ is the trivial group, extends to a unary semigroup homomorphism from $\mathcal{M}^0(I_0, \mathcal{G}, I_0; P_0)$ onto $\mathcal{K}_3$. Thus, $\mathcal{K}_3$ belongs to the variety $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{G}, I; P)$.

The Hermitian subsemigroup $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ of $\mathcal{M}^0(I, \mathcal{G}, I; P)$ is generated by the elements (i, e, i) where i runs over I. This implies that the group coordinates of triples (i, g, j) in $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ belong to the subgroup $\mathcal{H}$ generated by the non-zero entries of P. Hence $\mathrm{H}(\mathcal{M}^0(I, \mathcal{G}, I; P))$ is a unary subsemigroup of the unary Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{H}, I; P)$. It is not hard to see that each group in the variety $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{H}, I; P)$ belongs to the group variety $\mathsf{var}\,\mathcal{H}$. Since $\mathcal{G}$ does not belong to $\mathsf{var}\,\mathcal{H}$ but obviously belongs to $\mathsf{var}\,\mathcal{M}^0(I, \mathcal{G}, I; P)$, we are in a position to apply Theorem 1.1. □

A comprehensive treatment of the finite basis problem for unary Rees matrix semigroups forms the subject of a paper by Jackson and the third author [14].

3.2. Varietal joins

Recall that the *join* $\mathbf{V} \vee \mathbf{W}$ of two varieties $\mathbf{V}$ and $\mathbf{W}$ is the least variety containing both $\mathbf{V}$ and $\mathbf{W}$. We show how Theorem 1.1 can be used to produce interesting examples of non-finitely based joins of varieties of involutory semigroups.

Denote by $\mathbf{CSR}^*$ the variety generated by the regular *-semigroup $\mathcal{K}_3$ (that is, the variety of all *combinatorial strict regular $*$-semigroups*, see [2]) and let $\mathbf{I}$ be the variety of all inverse semigroups.

Theorem 3.2. *Let $\mathbf{K}$ and $\mathbf{A}$ be varieties of involutory semigroups such that:*

1. $\mathbf{K}$ *contains* $\mathbf{CSR}^*$*,*
2. $\mathbf{A}$ *consists of inverse semigroups and contains a group not contained in* $\mathrm{H}(\mathbf{K})$.

Then no variety in the interval $[\mathbf{CSR}^* \vee \mathbf{A}, \mathbf{K} \vee \mathbf{I}]$ *is finitely based.*

Proof. Let $\mathcal{S} \in \mathbf{K} \vee \mathbf{I}$; then there exist $\mathcal{K} \in \mathbf{K}$, $\mathcal{I} \in \mathbf{I}$ such that $\mathcal{S}$ *divides* (that is, $\mathcal{S}$ is a homomorphic image of a substructure of) $\mathcal{K} \times \mathcal{I}$ whence $\mathrm{H}(\mathcal{S})$ divides $\mathrm{H}(\mathcal{K}) \times \mathrm{H}(\mathcal{I})$. Observe that $\mathrm{H}(\mathcal{I})$ is a semilattice (with trivial involution). Further, since $\mathrm{H}(\mathcal{K}_3) = \mathcal{K}_3$ and $\mathbf{CSR}^* \subseteq \mathbf{K}$ we have $\mathbf{CSR}^* = \mathrm{H}(\mathbf{CSR}^*) \subseteq \mathrm{H}(\mathbf{K})$ so that $\mathrm{H}(\mathbf{K})$ contains all semilattices with trivial involution since $\mathbf{CSR}^*$ does so. Altogether, we have $\mathrm{H}(\mathcal{S}) \in \mathrm{H}(\mathbf{K})$, that is $\mathrm{H}(\mathbf{K} \vee \mathbf{I}) = \mathrm{H}(\mathbf{K})$, and thus, for any variety $\mathbf{V}$ in the interval $[\mathbf{CSR}^* \vee \mathbf{A}, \mathbf{K} \vee \mathbf{I}]$, we have $\mathrm{H}(\mathbf{V}) \subseteq \mathrm{H}(\mathbf{K})$. By assumption (2), there exists a group in $\mathbf{A} \subseteq \mathbf{V}$ that is not in $\mathrm{H}(\mathbf{K}) \supseteq \mathrm{H}(\mathbf{V})$. Thus, Theorem 1.1 applies to the variety $\mathbf{V}$. $\square$

The conditions of Theorem 3.2 are obviously fulfilled if $\mathbf{CSR}^* \subseteq \mathbf{K}$ and $\mathbf{A}$ contains a group that is not in $\mathbf{K}$; so, for example $\mathbf{K} = [x^m = x^{m+n}]$ for fixed $n \geq 1$ and $m \geq 2$ and $\mathbf{A} = \mathbf{G}$ (the variety of all groups) meet the requirements. We mention that Theorem 3.2 holds more generally for varieties of *unary* semigroups—it is not really required that the unary operation in question be an involution.

Recall that a variety $\mathbf{V}$ of algebraic structures is a *Cross variety* if

1) $\mathbf{V}$ is generated by a finite structure,
2) $\mathbf{V}$ contains only finitely many subvarieties,
3) $\mathbf{V}$ is finitely based.

For an interesting treatment of Cross varieties of plain semigroups consult Sapir [37]. The variety $\mathbf{CSR}^*$ is a Cross variety, see [2, Theorems 5.1 and 5.2, Corollary 5.4]. Now let $\mathbf{A}_p$ denote the variety of all abelian groups of exponent p (p is a prime number); clearly, $\mathbf{A}_p$ is a a Cross variety. By [2, Corollaries 5.4 and 6.5], the join $\mathbf{CSR}^* \vee \mathbf{A}_p$ contains only fourteen subvarieties; however, by the above remark, the join is not finitely based and therefore is not a Cross variety. We thus have a simple example of two Cross varieties whose join is not a Cross variety. A plain semigroup example of this kind found in [37, Corollary 2.1] is much more involved (with 39 subvarieties).

4. Existence varieties of locally inverse semigroups

In this section we give an application to existence varieties of the method of proof of Theorem 1.1. Recall that an *existence variety* (shortly *e-variety*) of regular semigroups is a class of regular semigroups closed under taking direct products, *regular* subsemigroups and homomorphic images. This section assumes the reader's acquaintance with some basics of the theory of regular semigroups.

While research into the structure of regular semigroups was particularly active in the 1970s and early 1980s, a universal algebra approach for regular semigroups has been introduced at the end of the 1980s by Kaďourek and Szendrei [16] for orthodox semigroups, and, independently, by Hall [12, 13] for regular semigroups in general. We shall recall the basic definitions and results necessary to understand the following treatment. For further information consult the papers [16, 12, 13, 42, 3, 4].

A regular semigroup $\mathcal{S} = \langle S, \cdot \rangle$ is *locally inverse* if for each idempotent e of $\mathcal{S}$, the *local submonoid* eSe is an inverse semigroup. The class $L\mathbf{I}$ of all (regular) locally inverse semigroups is a typical example of an existence variety. Observe that Rees matrix semigroups over groups are locally inverse (moreover, in such a semigroup each local submonoid is a group with 0 adjoined or the trivial group).

It is known [31, Theorem 7.6] that a regular semigroup $\mathcal{S}$ is locally inverse if and only if for any two $x, y \in \mathcal{S}$ the set $xV(yx)y$ is a singleton (as usual, $V(z)$ denotes the set of all inverses of the element z). This gives rise to the *sandwich operation* $\wedge$ that can be defined on any locally inverse semigroup by setting $x \wedge y$ to be the unique element of $xV(yx)y$, so that in this context, locally inverse semigroups are treated as algebras of type $(2, 2)$.

As explained in [3, 4], the adequate concept of equational theory for e-varieties of locally inverse semigroups is based on the signature $\{\cdot, \wedge\}$ and is with respect to a doubled alphabet $X \cup X'$. Here X is, as usual, a countably infinite set of variables and $X' = \{x' \mid x \in X\}$ is a disjoint copy of X; the elements of X' are devoted to represent inverses of the elements which are represented by the elements of X. The terms are over this extended alphabet and are in the signature $\{\cdot, \wedge\}$ where $\cdot$ stands for the associative operation of multiplication and $\wedge$ for the sandwich operation. Given a term $w(x_1, \ldots, x_n, x_1', \ldots, x_n')$ of this kind, a value of that term in the locally inverse semigroup $\mathcal{S}$ is obtained by substituting the variables x_i, x_i' by elements s_i, s_i' in $\mathcal{S}$ in such a way that each s_i' is an inverse of s_i. (In this evaluation it definitely may happen that distinct variables x, y, say, will be substituted with the same element s, while the formal inverses x' and y', respectively, are substituted with distinct inverses $s^\sharp$ and $s^\flat$, say, of s.) Given this notion of evaluation of terms in a locally inverse semigroup, it is clear what it means that a locally inverse semigroup $\mathcal{S}$ *satisfies* a *bi-identity* $u = v$ of terms of that kind. The following Birkhoff type theorem then holds [3].

Theorem 4.1. *A class* $\mathbf{V}$ *of locally inverse semigroups is an e-variety if and only if it is definable by bi-identities, that is,* $\mathbf{V}$ *consists of all locally inverse semigroups that satisfy a certain set of bi-identities.*

Now call a set B of bi-identities a *basis* of $\mathbf{V}$ if a locally inverse semigroup $\mathcal{S}$ is a member of $\mathbf{V}$ if and only if $\mathcal{S}$ satisfies all bi-identities of B. This semantic notion of basis is equivalent to a syntactic one: B is a basis of $\mathbf{V}$ if and only if B axiomatizes the *bi-equational theory* which is the set of all bi-identities over a (fixed) countable infinite set X of variables satisfied by all members of $\mathbf{V}$. The latter means that each bi-identity satisfied by all members of $\mathbf{V}$ can be derived, using natural deduction rules, from the bi-identities of $B \cup B(L\mathbf{I})$ where $B(L\mathbf{I})$ is a basis for the bi-equational theory of the class of all locally inverse semigroups. A set consisting of four independent bi-identities which may serve as $B(L\mathbf{I})$ has been found in [4]. For more analogues between the theory of e-varieties of regular semigroups and varieties of universal algebras see [3, 4, 16, 42].

The objective of this section is to obtain an analogue of Theorem 1.1 giving a sufficient condition for an e-variety $\mathbf{V}$ of locally inverse semigroups to have no finite basis of bi-identities.[2] Let $\mathcal{S}$ be a locally inverse semigroup and $a_1, \ldots, a_k \in \mathcal{S}$. For each i take an element $a_i' \in V(a_i)$. Then the closure of the set $\{a_1, \ldots, a_k, a_1', \ldots, a_k'\}$ under multiplication and sandwich operation is a locally inverse subsemigroup of $\mathcal{S}$, and is the least locally inverse subsemigroup of $\mathcal{S}$ containing the set $\{a_1, \ldots, a_k, a_1', \ldots, a_k'\}$ (by [42]). We call such a subsemigroup a *k-generated* locally inverse subsemigroup of $\mathcal{S}$. Define the *content* $c(t)$ of a term t inductively by $c(x) = c(x') = \{x\}$ and $c(uv) = c(u \wedge v) = c(u) \cup c(v)$. In order to prove that an e-variety $\mathbf{V}$ has no finite basis of bi-identities it is sufficient to prove for each natural number k the existence of a locally inverse semigroup $\mathcal{T}_k$ such that $\mathcal{T}_k \notin \mathbf{V}$ but $\mathcal{T}_k$ satisfies each bi-identity $u = v$ that holds in $\mathbf{V}$ and for which $|c(u) \cup c(v)| \leq k$. The latter is equivalent to the property that each k-generated locally inverse subsemigroup (as defined above) is contained in $\mathbf{V}$.

[2]The reader may note that by an argument similar to that in the proof of Proposition 2.9 in [7] it can be shown that there do not exist inherently nonfinitely based e-varieties of locally inverse semigroups.

For each e-variety $\mathbf{V}$ denote by $\mathrm{C}(\mathbf{V})$ the sub-e-variety of $\mathbf{V}$ generated by all idempotent generated members of $\mathbf{V}$. We also need the 5-element Rees matrix semigroup $\mathcal{A}_2$ with the sandwich matrix

$$\begin{pmatrix} 0 & e \\ e & e \end{pmatrix}. \tag{4.1}$$

We are ready to formulate an e-variety analogue of Theorem 1.1.

Theorem 4.2. *Let* $\mathbf{V}$ *be a locally inverse e-variety containing the semigroup* $\mathcal{A}_2$*. If* $\mathbf{V}$ *contains a group which is not in* $\mathrm{C}(\mathbf{V})$ *then* $\mathbf{V}$ *has no finite basis for its bi-identities.*

Proof. This can be proved in a manner similar to the proof of Theorem 2.2 in [7]. As mentioned above, we have to prove, for each k, the existence of a locally inverse semigroup $\mathcal{T}_k$ such that $\mathcal{T}_k \notin \mathbf{V}$ but each k-generated locally inverse subsemigroup of $\mathcal{T}_k$ is contained in $\mathbf{V}$.

Let $\mathcal{G}$ be a group in $\mathbf{V}$ that is not contained in $\mathrm{C}(\mathbf{V})$. Since there must be a bi-identity which holds in $\mathrm{C}(\mathbf{V})$ but fails in $\mathcal{G}$, we may assume that $\mathcal{G}$ is generated by finitely many elements, say $g_1, \dots, g_m$, and for convenience we may assume that this set of generators is closed under taking inverse elements and so generates $\mathcal{G}$ as a semigroup. Next, let $\mathcal{T}_k$ be the Rees matrix semigroup in the proof of [7, Theorem 2.2], but with $n = 2k + 1$ being replaced with $n = 4k + 1$. In that proof it has been shown that $(1, g_j, mn) \in \langle E(\mathcal{T}_k)\rangle$ for $j = 1, \dots, m$ (here $\langle E(\mathcal{T}_k)\rangle$ denotes the idempotent generated subsemigroup of $\mathcal{T}_k$). It follows that $\{1\} \times \mathcal{G} \times \{mn\} \subseteq \langle E(\mathcal{T}_k)\rangle$ whence $\langle E(\mathcal{T}_k)\rangle$ contains a subgroup isomorphic to $\mathcal{G}$. Consequently, $\langle E(\mathcal{T}_k)\rangle \notin \mathrm{C}(\mathbf{V})$ which implies that $\mathcal{T}_k \notin \mathbf{V}$.

Finally, consider any k-generated locally inverse subsemigroup of $\mathcal{T}_k$, that is, choose elements $a_1, \dots, a_k, a'_1, \dots, a'_k \in \mathcal{T}_k$ such that $a'_i \in V(a_i)$ for each i. Let $\mathcal{T}$ be the locally inverse subsemigroup generated by $\{a_1, \dots, a_k, a'_1, \dots, a'_k\}$ (that is, the closure of that set under multiplication and sandwich operation). It is clear that at most $4k$ indices of $\{1, \dots, nm\}$ can occur in the triple representation of the elements a_i and a'_i. Therefore, analogously to the unary case proved in [7, Theorem 2.2], there exist numbers $\lambda_1, \dots, \lambda_m$ such that

$$1 \leq \lambda_1 \leq n < \lambda_2 \leq 2n < \dots (m-1)n < \lambda_m \leq mn$$

and $\mathcal{T}$ is contained in the semigroup $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$. Again as in Section 1, we can show that $\mathcal{T}_k(\lambda_1, \dots, \lambda_m)$ is isomorphic to a homomorphic image of the direct product $\mathcal{G} \times \mathcal{U}_k$ of the group $\mathcal{G}$ and a completely 0-simple semigroup $\mathcal{U}_k$ with trivial subgroups. Now $\mathcal{G} \in \mathbf{V}$ by our assumption and $\mathcal{U}_k \in \mathbf{V}$ by a result of Hall [13] because $\mathbf{V}$ contains $\mathcal{A}_2$. This completes the proof. $\square$

Theorem 4.2 can be in particular applied to certain joins of e-varieties. Here is an example. Denote by **CSR** the e-variety generated by the semigroup $\mathcal{A}_2$ and by **GI** the e-variety of all orthodox locally inverse semigroups (these semigroups are often called *generalized inverse*). The proof of the next corollary is analogous to that of Theorem 3.2 and is left to the reader.

Corollary 4.3. *Let* $\mathbf{K}$ *and* $\mathbf{A}$ *be locally inverse e-varieties such that:*

(1) $\mathbf{K}$ *contains* **CSR**,
(2) $\mathbf{A}$ *consists of orthodox semigroups and contains a group not contained in* $\mathrm{C}(\mathbf{K})$.

Then no e-variety in the interval $[\mathbf{CSR} \vee \mathbf{A}, \mathbf{K} \vee \mathbf{GI}]$ *is finitely based.*

Acknowledgements. The second author was supported by Grant No. 174019 of the Ministry of Education and Science of the Republic of Serbia, and by a grant (Contract 114–451–2002/2011) of the Secretariat of Science and Technological Development of the Autonomous Province of Vojvodina. The third author acknowledges support from the Russian Foundation for Basic Research, grant 10-01-00524.

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