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Unary enhancements of inherently non-finitely based semigroups

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Abstract This paper is a continuation of [1], more precisely, of Subsection 2.2 of [1] dealing with inherently nonfinitely based involutory semigroups. We exhibit a simple condition under which a finite involutory semigroup whose semigroup reduct is inherently nonfinitely based is also inherently nonfinitely based as a unary semigroup. As applications, we get already known as well as new examples of inherently nonfinitely based involutory semigroups. We also show that for finite regular semigroups, our condition is not only sufficient but also necessary for the property of being inherently nonfinitely based to persist. This leads to an algorithmic description of regular inherently nonfinitely based involutory semigroups.

Keywords involutory semigroup · inherently nonfinitely based semigroup · twisted semilattice · twisted Brandt monoid · regular semigroup · matrix semigroup

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1 Background and overview

The finite basis problem, that is the problem of classifying semigroups according to the finite basability of their identities, has been intensively explored since the 1960s.

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Since the 1970s, the same problem has become investigated for semigroups endowed with an additional unary operation $x \mapsto x^*$; such structures are commonly called *unary semigroups*.

If $\mathcal{S} = \langle S, \cdot, ^* \rangle$ is a unary semigroup, then the (plain) semigroup $\langle S, \cdot \rangle$ is called the (*semigroup*) *reduct* of $\mathcal{S}$. It is quite natural to ask how the answer to the finite basis problem for a given unary semigroup relates to the finite basability of the identities of its reduct. The question turns out to be somewhat delicate. On the one hand, when we enhance the vocabulary of an equational language by adding a unary operation, the expressive power of the language increases. Hence $\mathcal{S}$ usually has more identities than $\langle S, \cdot \rangle$ so that the former may have more chance to become nonfinitely based. On the other hand, the inference power of the language increases too. Hence one can imagine the situation when some identity of $\langle S, \cdot \rangle$ does not follow from an identity system Σ as a plain identity but follows from Σ when treated as a unary identity. This indicates that $\mathcal{S}$ may be finitely based even if $\langle S, \cdot \rangle$ is not. The cumulative effect of the trade-off between increased expressivity and increased inference power is hard to predict in general, and both possible outcomes indeed occur. This means that there exist unary semigroups, even groups $\mathcal{G} = \langle G; \cdot, ^{-1} \rangle$ with inversion as the unary operation, such that $\mathcal{G}$ is finitely based [nonfinitely based] as a group while its reduct $\langle G; \cdot \rangle$ is nonfinitely based [respectively, finitely based] as a plain semigroup. See [16, Section 2] for concrete examples (known since the 1970s), references and a discussion.

Much attention has been paid to the restriction of the finite basis problem to the class of finite semigroups, in the plain as well as the unary setting, see, e.g., the survey [16]. Therefore it appears a bit surprising that the above question about the relation between the finite basability of a unary semigroup and of its reduct has not been systematically explored in the realm of finite semigroups. To the best of our knowledge, the first example of a nonfinitely based finite unary semigroup whose reduct is finitely based was constructed only in 1998, see [8]. The unary operation used in [8] was rather ad hoc, and similar examples with well behaved unary operations (including an example of a nonfinitely based finite involutory semigroup with finitely based reduct) have only recently appeared in [7]. Examples of the 'opposite' kind (of finitely based finite unary semigroups with nonfinitely based reducts) are not yet known.

For finite semigroups, the following strengthening of the property of being nonfinitely based has been successfully studied. Recall that a variety $\mathbf{V}$ of [unary] semigroups is called *locally finite* if every finitely generated [unary] semigroup in $\mathbf{V}$ is finite. A finite [unary] semigroup is said to be *inherently nonfinitely based* (INFB for short) if it is not contained in any locally finite finitely based variety of [unary] semigroups. Since the variety generated by a finite [unary] semigroup is known to be locally finite, an INFB [unary] semigroup certainly is nonfinitely based. In fact, the property of being INFB is much stronger than the property of being nonfinitely based and also behaves more regularly, see [16] for details.

Sapir [13] has given an efficient (in the algorithmic sense of the word) description of INFB semigroups. INFB unary semigroups have been investigated in [3, 1] where some sufficient and some necessary conditions for a finite involutory semigroup to be INFB have been found. Again, in this situation it is quite natural to ask what happens when one passes from a finite unary semigroup to its reduct. The aforementioned example of [8] is in fact INFB so that in general an INFB unary semigroup may have a finitely based reduct. This is however impossible for a finite involutory semigroup; indeed, it is easy to verify (see Lemma 2.1 below) that the reduct of an INFB involutory semigroup must be INFB. The converse is not true as first observed in [14], and it is this circumstance that gives rise to the specific question addressed in the present paper: when does an involution $x \mapsto x^*$ defined on an INFB semigroup $\langle S, \cdot \rangle$ preserve the property of being INFB in the sense that the resulting involutory semigroup $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ is INFB as a unary semigroup? We show (Theorem 3.1) that this is the case whenever the variety generated by $\mathcal{S}$ contains a certain 3-element involutory semigroup $\mathcal{TSL}$ (twisted semilattice). This result has several applications: first, we give new, easy and uniform proofs for some examples of INFB involutory semigroups found in [1]; second, we exhibit a further series of INFB involutory semigroups. We also show (Corollary 4.6) that if $\langle S, \cdot \rangle$ is a finite regular semigroup, then the presence of the 3-element twisted semilattice $\mathcal{TSL}$ in the variety generated by $\mathcal{S}$ is not only sufficient but also necessary for the property of being INFB to persist. Combined with Sapir's result, this leads to an efficient description of regular INFB involutory semigroups.

2 Preliminaries

We assume the reader's acquaintance with basic concepts of universal algebra such as the notion of a variety and the HSP-theorem, see, e.g., [2, Chapter II]. Section 4 also requires some knowledge of Green's relations, cf. [6, Chapter 2].

A unary semigroup $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ is called an *involutory semigroup* if it satisfies the identities

$$(xy)^* = y^* x^* \quad \text{and} \quad (x^*)^* = x, \tag{1}$$

in other words, if the unary operation $x \mapsto x^*$ is an involutory anti-automorphism of the reduct $\langle S, \cdot \rangle$.

The *free involutory semigroup* $\mathcal{FI}(X)$ on a given alphabet X can be constructed as follows. Let $\overline{X} = \{x^* \mid x \in X\}$ be a disjoint copy of X. We refer to the elements of X as *plain letters* and to the elements of $\overline{X}$ as *starred letters*. Define $(x^*)^* = x$ for all $x^* \in \overline{X}$. Then $\mathcal{FI}(X)$ is the free semigroup $(X \cup \overline{X})^+$ endowed with the involution defined by

$$(x_1 \cdots x_m)^* = x_m^* \cdots x_1^*$$

for all $x_1, \ldots, x_m \in X \cup \overline{X}$. We refer to elements of $\mathcal{FI}(X)$ as *involutory words over* X while elements of the free semigroup X^+ will be referred to as *plain words over* X.

If an involutory semigroup $\mathcal{T} = \langle T, \cdot, {}^* \rangle$ is generated by a set $Y \subseteq T$, then every element in $\mathcal{T}$ can be represented by an involutory word over Y and thus by a plain word over $Y \cup \overline{Y}$ where $\overline{Y} = \{y^* \mid y \in Y\}$. Hence the reduct $\langle T, \cdot \rangle$ is generated by the set $Y \cup \overline{Y}$; in particular, $\mathcal{T}$ is finitely generated if and only if so is $\langle T, \cdot \rangle$. This observation immediately leads to the following fact already mentioned in the introduction.

Lemma 2.1 *If an involutory semigroup $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ is inherently nonfinitely based, then so is its reduct $\langle S, \cdot \rangle$.*

Proof Arguing by contradiction, assume that $\langle S,\cdot\rangle$ is not INFB. Then $\langle S,\cdot\rangle$ belongs to a locally finite plain semigroup variety defined by a finite identity system Σ. Consider the variety $\mathbf{V}$ of involutory semigroups defined by the identities (1) and Σ. Clearly, $\mathbf{V}$ is finitely based and $\mathcal{S}\in\mathbf{V}$. If $\mathcal{T}=\langle T,\cdot,^*\rangle$ is a finitely generated involutory semigroup from $\mathbf{V}$ then the reduct $\langle T,\cdot\rangle$ is a finitely generated plain semigroup by the observation preceding the formulation of the lemma. Since the reduct satisfies the identities in Σ and Σ defines a locally finite plain semigroup variety, we conclude that the base set T is finite. Hence the variety $\mathbf{V}$ is locally finite and $\mathcal{S}$ belongs to a locally finite finitely based variety, a contradiction.

As mentioned, the converse of Lemma 2.1 is not true in general. For an example, consider the well known *Brandt monoid* $\langle B_2^1,\cdot\rangle$, where B_2^1 is the set of the following six integer 2×2-matrices:

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&1\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\1&0\end{pmatrix},\ \begin{pmatrix}0&0\\0&1\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\end{pmatrix},$$

and the binary operation $(A_1,A_2)\mapsto A_1\cdot A_2$ is the usual matrix multiplication. It is known [12, Corollary 6.1] that the Brandt monoid is INFB (this was in fact the very first example of an INFB semigroup). The Brandt monoid admits a natural involution, namely, the usual matrix transposition $A\mapsto A^T$. However, the involutory semigroup $\langle B_2^1,\cdot,^T\rangle$ is not INFB as shown in [14]. Further examples can be found in [1]: if $\mathcal{K}$ is a finite field and $\mathrm{M}_n(\mathcal{K})$ stands for the set of all $n\times n$-matrices over $\mathcal{K}$, then the semigroup $\langle\mathrm{M}_n(\mathcal{K}),\cdot\rangle$ is INFB for any $n\geq 2$ by [12, Corollary 6.2] while the involutory semigroup $\langle\mathrm{M}_2(\mathcal{K}),\cdot,^T\rangle$ is not INFB if the number of elements in $\mathcal{K}$ leaves reminder 3 when divided by 4.

Thus, not every involution defined on an INFB semigroup preserves the property of being INFB, and we are looking towards a classification of 'INFB-preserving' involutions. Our main tool is the following result from [1]. Recall the notions that appear in its formulation. Let $x_1,x_2,\dots,x_n,\dots$ be a sequence of letters. The sequence $\{Z_n\}_{n=1,2,\dots}$ of *Zimin words* is defined inductively by $Z_1=x_1$, $Z_{n+1}=Z_nx_{n+1}Z_n$. We say that an involutory word v is an *involutory isoterm for a unary semigroup* $\mathcal{S}$ if the only involutory word v' such that $\mathcal{S}$ satisfies the involutory semigroup identity $v=v'$ is the word v itself.

Theorem 2.2 ([1, Theorem 2.3]) *Let $\mathcal{S}$ be a finite involutory semigroup. If all Zimin words are involutory isoterms for $\mathcal{S}$, then $\mathcal{S}$ is inherently nonfinitely based.*

3 Main result and its applications

Recall that semigroups satisfying both $xy=yx$ and $x^2=x$ are called *semilattices*. An involutory semigroup $\mathcal{S}=\langle S,\cdot,^*\rangle$ whose reduct $\langle S,\cdot\rangle$ is a semilattice with 0 is said to be a *twisted semilattice* if 0 is the only fixed point of the involution $x\mapsto x^*$. This class of involutory semigroups was first considered in [4]. It is easy to see that the minimum non-trivial object in this class is the 3-element twisted semilattice $\mathcal{TSL}=\langle\{e,f,0\},\cdot,^*\rangle$ in which $e^2=e$, $f^2=f$ and all other products are equal to 0, while the unary operation is defined by $e^*=f$, $f^*=e$, and $0^*=0$.

If $\mathcal{S}$ is an involutory semigroup, we denote by $\mathsf{var}\,\mathcal{S}$ the variety generated by $\mathcal{S}$.

Theorem 3.1 *Let $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ be a finite involutory semigroup such that $\mathcal{TSL} \in \mathsf{var}\,\mathcal{S}$. If the reduct $\langle S, \cdot \rangle$ is inherently nonfinitely based, then so is $\mathcal{S}$.*

Proof By Theorem 2.2 we only have to show that $\mathcal{S}$ satisfies no non-trivial involutory semigroup identity of the form $Z_n = z$. If z is a plain word, we can refer to [12, Proposition 7] according to which the INFB semigroup $\langle S, \cdot \rangle$ satisfies no non-trivial plain semigroup identity of the form $Z_n = z$. Now suppose that $\mathcal{S}$ satisfies an identity $Z_n = z$ such that the involutory word z is not a plain word. This means that z contains a starred letter. Since $\mathcal{TSL} \in \mathsf{var}\,\mathcal{S}$, the identity $Z_n = z$ holds in $\mathcal{TSL}$. Substitute the element e of $\mathcal{TSL}$ for all plain letters occurring in Z_n and z. Since $e^2 = e$, the value of the word Z_n under this substitution equals e. On the other hand, since z contains a starred letter, the value of z is a product involving $e^* = f$, and every such product is equal to either f or 0. This is a contradiction.

As for applications of Theorem 3.1, we first give simplified and uniform proofs for two important results from [1]. To start with, consider the *twisted Brandt monoid* $\mathcal{TB}_2^1 = \langle B_2^1, \cdot, {}^D \rangle$ arising when one endows the Brandt monoid $\langle B_2^1, \cdot \rangle$ with the unary operation $A \mapsto A^D$ that fixes the matrices $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and swaps each of the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with the other one. We notice that this unary operation is just the reflection with respect to the secondary diagonal (from the top right to the bottom left corner). The reflection (called the *skew transposition*) makes sense for every square matrix and is in fact an involution of the semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot \rangle$; this follows from the observation that for every matrix $A \in \mathrm{M}_n(\mathcal{K})$, one has $A^D = JA^TJ$, where J is the $n \times n$-matrix with 1s on the secondary diagonal and 0s elsewhere. Moreover suppose that the field $\mathcal{K}$ is such that there exists a matrix $R \in \mathrm{M}_n(\mathcal{K})$ satisfying $R^T = R$ and $R^2 = J$ (this happens, e.g., when the characteristic of $\mathcal{K}$ is not 2 and square roots of -1 and 2 do exist in $\mathcal{K}$). Then the conjugation map $A \mapsto A\psi := R^{-1}AR$ satisfies $(A^D)\psi = (A\psi)^T$ and hence is an isomorphism between the involutory semigroups $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^D \rangle$ and $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^T \rangle$. Clearly, the set B_2^1 can be considered as a subset of $\mathrm{M}_2(\mathcal{K})$, and as such it is closed under both the usual transposition and the skew one. Therefore it appears a bit surprising that the involutory subsemigroups $\mathcal{TB}_2^1 = \langle B_2^1, \cdot, {}^D \rangle$ and $\langle B_2^1, \cdot, {}^T \rangle$ of the (isomorphic) involutory semigroups $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^D \rangle$ and respectively $\langle \mathrm{M}_2(\mathcal{K}), \cdot, {}^T \rangle$ turn out to be so much different. Indeed, $\langle B_2^1, \cdot, {}^T \rangle$ is not INFB (see Section 2) while $\mathcal{TB}_2^1$ is, as the following corollary reveals.

Corollary 3.2 ([1, Corollary 2.7]) *The twisted Brandt monoid $\mathcal{TB}_2^1$ is inherently nonfinitely based.*

Proof As already mentioned, the reduct $\langle B_2^1, \cdot \rangle$ of $\mathcal{TB}_2^1$ is INFB by [12, Corollary 6.1]. The matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ form an involutory subsemigroup in $\mathcal{TB}_2^1$ and, obviously, this subsemigroup is isomorphic to the 3-element twisted semilattice $\mathcal{TSL}$. Thus, Theorem 3.1 applies.

Now consider the matrix involutory semigroups $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^T \rangle$ where $\mathcal{K}$ is a finite field.

Corollary 3.3 ([1, Theorems 3.9 and 3.10]) *The involutory semigroup $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^T \rangle$, where $\mathcal{K}$ is a finite field, is inherently nonfinitely based if $n \geq 3$ or if $n = 2$ and the number of elements in $\mathcal{K}$ is not of the form $4k + 3$.*

Proof The reduct $\langle \mathrm{M}_n(\mathcal{K}), \cdot \rangle$ is INFB for each $n \geq 2$ and each finite field $\mathcal{K}$ by [12, Corollary 6.2]. To invoke Theorem 3.1, it only remains to show that, under the condition of the corollary, the 3-element twisted semilattice $\mathcal{TSL}$ belongs to the variety $\mathsf{var}\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^T \rangle$.

First let $n \geq 3$. By the Chevalley-Warning theorem [15, Corollary 2 in §1.2], the field $\mathcal{K}$ contains some elements x, y satisfying $1 + x^2 + y^2 = 0$. Then the $n \times n$-matrices

$$e = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ x & 0 & 0 & \cdots & 0 \\ y & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & x & y & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \text{and} \quad g = \begin{pmatrix} 1 & x & y & \cdots & 0 \\ x & x^2 & xy & \cdots & 0 \\ y & xy & y^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

satisfy

$$e^2 = e, \ f^2 = f, \ ef = g, \ fe = 0, \ e^T = f, \ f^T = e, \ \text{and} \ g^T = g.$$

Therefore the set $\{e, f, g, 0\}$ forms an involutory subsemigroup in $\langle \mathrm{M}_n(\mathcal{K}), \cdot, {}^T \rangle$, the set $\{g, 0\}$ is an ideal of this subsemigroup and is closed under transposition. It remains to observe that the Rees quotient of the involutory semigroup $\langle \{e, f, g, 0\}, \cdot, {}^T \rangle$ over the ideal $\{g, 0\}$ is isomorphic to the 3-element twisted semilattice $\mathcal{TSL}$.

Now let $n = 2$ and let the number of elements in $\mathcal{K}$ be not of the form $4k + 3$. Then the field $\mathcal{K}$ contains a square root of -1, see, e.g., [9, Theorem 3.75]. Now the argument of the previous paragraph applies, with the 2×2-matrices

$$e' = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}, \quad f' = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad g' = \begin{pmatrix} 1 & x \\ x & -1 \end{pmatrix}$$

in the roles of e, f, and g, respectively, where x denotes some fixed square root of -1.

We could have continued along the same lines to show that in fact all examples of INFB involutory semigroups found in [3, 1] can be similarly deduced from Theorem 3.1. However we think that Corollaries 3.2 and 3.3 are representative enough. Now we present a new application.

Let $\mathcal{K}$ be a finite field and let $\mathrm{T}_n(\mathcal{K})$ stand for the set of all upper-triangular $n \times n$-matrices over $\mathcal{K}$. The set $\mathrm{T}_n(\mathcal{K})$ forms an involutory semigroup under the usual matrix multiplication and the skew transposition. The following result classifies all INFB involutory semigroups of the form $\langle \mathrm{T}_n(\mathcal{K}), \cdot, {}^D \rangle$.

Theorem 3.4 *The involutory semigroup $\langle \mathrm{T}_n(\mathcal{K}), \cdot, {}^D \rangle$, where $\mathcal{K}$ is a finite field, is inherently nonfinitely based if and only if $n \geq 4$ and $\mathcal{K}$ contains at least 3 elements.*

Proof In [5] it is shown that the reduct $\langle \mathrm{T}_n(\mathcal{K}), \cdot \rangle$ is INFB if and only if $n \geq 4$ and $\mathcal{K}$ contains at least 3 elements. Therefore, the 'only if' part of our theorem follows from Lemma 2.1 and the 'if' part will follow from Theorem 3.1 as soon as we shall verify that $\mathcal{TSL} \in \mathsf{var}\langle \mathrm{T}_n(\mathcal{K}), \cdot, {}^D \rangle$. Indeed, for every $n \geq 2$ the matrix units e_{11} and e_{nn} belong to $\mathrm{T}_n(\mathcal{K})$ and satisfy

$$e_{11}^2 = e_{11}, \ e_{nn}^2 = e_{nn}, \ e_{11}e_{nn} = e_{nn}e_{11} = 0, \ e_{11}^D = e_{nn}, \ \text{and} \ e_{nn}^D = e_{11}.$$

Hence the set $\{e_{11}, e_{nn}, 0\}$ forms an involutory subsemigroup in $\langle \mathrm{T}_n(\mathcal{K}), \cdot, {}^D \rangle$ and this involutory subsemigroup is isomorphic to the 3-element twisted semilattice $\mathcal{TSL}$.

Observe that in [5] it is shown that for any n and $\mathcal{K}$, the Brandt monoid $\langle B_2^1, \cdot\rangle$ does not belong to the semigroup variety generated by the semigroup $\langle \mathrm{T}_n(\mathcal{K}), \cdot\rangle$. Hence the twisted Brandt monoid $\mathcal{TB}_2^1$ does not belong to the involutory semigroup variety $\mathsf{var}\langle \mathrm{T}_n(\mathcal{K}), \cdot, {}^D\rangle$. Thus, Theorem 3.4 provides a series of examples of INFB involutory semigroups whose varieties do not contain $\mathcal{TB}_2^1$. Such examples have not been known before.

4 Regular semigroups

In this section we show that the presence of the 3-element twisted semilattice $\mathcal{TSL}$ in the variety generated by a finite involutory semigroup $\mathcal{S}$ is (not only sufficient but also) necessary for $\mathcal{S}$ in order to inherit the property of being INFB from its semigroup reduct, provided that $\mathcal{S}$ is regular. As a preliminary result we present a criterion whether or not $\mathcal{TSL}$ belongs to $\mathsf{var}\,\mathcal{S}$ (Proposition 4.2).

We shall use two classical results concerning Green's relations, the first of which is often referred to as the Lemma of Miller and Clifford (see [6, Proposition 2.3.7]), while the property formulated in the second one is usually called the *stability* of Green's relations (see [11, Proposition 3.1.4 (2)]).

Lemma 4.1 *1. Let a, b be elements of a $\mathscr{D}$-class of an arbitrary semigroup. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*
2. Let S be a finite semigroup and $a, b \in S$; then $a \mathrel{\mathscr{J}} ab$ implies $a \mathrel{\mathscr{R}} ab$ and $b \mathrel{\mathscr{J}} ab$ implies $b \mathrel{\mathscr{L}} ab$.

The above mentioned criterion for membership of $\mathcal{TSL}$ in $\mathsf{var}\,\mathcal{S}$ is clarified by the following key result.

Proposition 4.2 *For a finite involutory semigroup $\mathcal{S}$ exactly one of the two following assertions is true.*

(A) There exists an idempotent e of $\mathcal{S}$ satisfying $e >_{\mathscr{J}} e^*e$.
(B) There exists a positive integer N such that $\mathcal{S}$ satisfies the identity

$$x^N = (x^N(x^N)^*)^N x^N. \tag{2}$$

Proof It is clear that the conditions (A) and (B) exclude each other. Let us assume that the assertion (A) does not hold for $\mathcal{S} = \langle S, \cdot, {}^*\rangle$. We have to prove that $\mathcal{S}$ satisfies (B). For each idempotent e of $\mathcal{S}$ we have $e \mathrel{\mathscr{J}} e^*e$ and therefore $e \mathrel{\mathscr{L}} e^*e$ by Lemma 4.1 (2). Since the involution * is an anti-isomorphism, we also have $e^* \mathrel{\mathscr{R}} e^*e$ for each idempotent e. Swapping the roles of e and e^* we also get $e \mathrel{\mathscr{R}} ee^* \mathrel{\mathscr{L}} e^*$. In other words,

$$e \mathrel{\mathscr{R}} ee^* \mathrel{\mathscr{L}} e^* \mathrel{\mathscr{R}} e^*e \mathrel{\mathscr{L}} e$$

holds for each idempotent e of $\mathcal{S}$.

By the 'only if' part of Lemma 4.1 (1), the fact that the product e^*e belongs to $L_e \cap R_{e^*}$ implies that the $\mathscr{H}$-class $H_{ee^*} = R_e \cap L_{e^*}$ contains an idempotent g and hence, by Green's theorem [6, Theorem 2.2.5], this class is a subgroup of $\langle S, \cdot\rangle$ having g as its identity element. Since $g \mathrel{\mathscr{R}} e$ we have that $ge = e$. Now take any common multiple

n of the exponents of all subgroups of $\mathcal{S}$; then $(ee^*)^n = g$ and hence $(ee^*)^n e = e$. Finally, choose a positive integer N for which $\mathcal{S}$ satisfies the identity $x^N = x^{2N}$. Then N is a common multiple of the exponents of all subgroups of $\mathcal{S}$ and each element of the form x^N is idempotent. Consequently $\mathcal{S}$ satisfies the identity (2).

It is now easy to see that $\mathcal{TSL} \in \mathsf{var}\,\mathcal{S}$ if and only if $\mathcal{S}$ is of type (A). Indeed suppose that $\mathcal{S}$ has an idempotent e satisfying $e >_{\mathscr{J}} e^*e$ and let $\mathcal{T} = \langle T, \cdot, ^* \rangle$ be the involutory subsemigroup of $\mathcal{S}$ generated by e; then $e \neq e^*$ and neither of the idempotents e and e^* is contained in the ideal $I := Tee^*T \cup Te^*eT$. It follows that $\mathcal{TSL}$ is isomorphic to the Rees quotient $\mathcal{T}/I$. In other words, $\mathcal{TSL}$ is a homomorphic image of an involutory subsemigroup of $\mathcal{S}$, that is, $\mathcal{TSL}$ *divides* $\mathcal{S}$ and in particular $\mathcal{TSL} \in \mathsf{var}\,\mathcal{S}$.

Conversely, if $\mathcal{S}$ is of type (B) then $\mathcal{S}$ satisfies the identity (2) for some positive integer N. Obviously, $\mathcal{TSL}$ does not satisfy this identity and, hence, it does not belong to $\mathsf{var}\,\mathcal{S}$.

Altogether we have proved:

Corollary 4.3 *For a finite involutory semigroup $\mathcal{S}$, the following are equivalent:*

1. *$\mathcal{S}$ is of type (A).*
2. *$\mathcal{TSL}$ divides $\mathcal{S}$.*
3. *$\mathcal{TSL} \in \mathsf{var}\,\mathcal{S}$.*

A finite involutory semigroup $\mathcal{S}$ of type (A) with INFB semigroup reduct $\langle S, \cdot \rangle$ is INFB as an involutory semigroup (Theorem 3.1). At the time of writing, the authors were not (yet) aware of an example of an INFB involutory semigroup of type (B).

We note that finite involutory semigroups of type (B) can be characterized in various ways; for example, as those in which each regular element admits a Moore–Penrose inverse, and likewise, as those in which each regular $\mathscr{L}$-class (and/or each regular $\mathscr{R}$-class) contains a *projection* (that is, an idempotent fixed under the involution) [10]. Another equivalent condition is that each involutory subsemigroup $\langle g \rangle$ generated by a single idempotent g is completely simple. Moreover, the class of all finite involutory semigroups of type (B) forms a pseudovariety of involutory semigroups, namely the one defined by the pseudoidentity

$$x^{\omega} = (x^{\omega}(x^{\omega})^*)^{\omega} x^{\omega}.$$

As usual, $s \mapsto s^{\omega}$ denotes the unary operation that assigns to each element s of a finite semigroup the unique idempotent in the cyclic subsemigroup generated by s.

Recall that an element x of a semigroup $\langle S, \cdot \rangle$ is said to be *regular* if $x = xyx$ for some $y \in S$. A [unary] semigroup is *regular* if all of its elements are regular. We shall refine the proof of Proposition 4.2 and show that a **regular** involutory semigroup $\mathcal{S}$ of type (B) satisfies an identity that guarantees that $\mathcal{S}$ is **not** INFB, thanks to the following result from [1].

Proposition 4.4 ([1, Proposition 2.9]) *Let $\mathcal{S} = \langle S, \cdot, ^* \rangle$ be a finite involutory semigroup. If there exists an involutory word $\iota(x)$ in one variable x such that $\mathcal{S}$ satisfies the identity $x = x\iota(x)x$, then $\mathcal{S}$ is not inherently nonfinitely based.*

We get the following consequence:

Corollary 4.5 *A finite regular involutory semigroup $\mathcal{S}$ of type (B) is not inherently nonfinitely based.*

Proof We are going to sharpen the proof of Propositon 4.2. Let $\mathcal{S}$ be an involutory semigroup of type (B) (not necessarily regular at this point). Fix an arbitrary regular element $x \in S$ and take an element $y \in S$ such that $x = xyx$. Then $e = xy$ and $f = yx$ are idempotents and $e \mathscr{R} x \mathscr{L} f$. Since the involution is an anti-isomorphism of $\langle S, \cdot \rangle$, we also have $e^* \mathscr{L} x^* \mathscr{R} f^*$. We have already seen in the proof of Proposition 4.2 that $ee^* \mathscr{R} e \mathscr{L} e^*e \mathscr{R} e^* \mathscr{L} ee^*$ and $ff^* \mathscr{R} f \mathscr{L} f^*f \mathscr{R} f^* \mathscr{L} ff^*$. All listed relations are graphically represented in Fig. 1 that shows an appropriate fragment of the eggbox picture for the $\mathscr{D}$-class of x and x^*.

f^*f		f^*	x^*
	e^*e		e^*
f		ff^*	
x	e		ee^*

Fig. 1 A fragment of the eggbox picture for the $\mathscr{D}$-class of the elements x and x^*

As in the proof of Propostion 4.2, the $\mathscr{H}$-class $H_{ee^*} = R_e \cap L_{e^*}$ contains an idempotent g and again this class is a subgroup of $\langle S, \cdot \rangle$ having g as its identity element. Observe that $R_e = R_x$ and $L_{e^*} = L_{x^*}$ whence $H_{ee^*} = R_x \cap L_{x^*}$. Also observe that $gx = x$ since g is an idempotent and $g \mathscr{R} x$.

Similarly, $ff^* \in R_f \cap L_{f^*}$ implies that $H_{f^*f} = R_{f^*} \cap L_f$ contains an idempotent. However, $R_{f^*} = R_{x^*}$ and $L_f = L_x$. Now, by the 'if' part of Lemma 4.1 (1), the fact that $L_x \cap R_{x^*} = H_{f^*f}$ contains an idempotent implies that the product xx^* lies in $R_x \cap L_{x^*} = H_{ee^*}$. Let n be the least common multiple of the exponents of the subgroups of $\langle S, \cdot \rangle$. We then have $(xx^*)^n = g$ whence $x = gx = (xx^*)^n x$.

Consequently, if $\mathcal{S}$ is regular then $\mathcal{S}$ satisfies the identity $x = (xx^*)^n x$ and hence $x = x\imath(x)x$ for $\imath(x) = x^*(xx^*)^{n-1}$, which by Proposition 4.4 implies that $\mathcal{S}$ is not INFB.

We can now easily deduce various characterizations of regular INFB involutory semigroups.

Corollary 4.6 *Let $\mathcal{S} = \langle S, \cdot, {}^* \rangle$ be a finite regular involutory semigroup. Then the following are equivalent:*

(i) *$\mathcal{S}$ is inherently nonfinitely based;*
(ii) *the reduct $\langle S, \cdot \rangle$ is inherently nonfinitely based and the* 3*-element twisted semilattice $\mathcal{TSL}$ belongs to* $\mathsf{var}\,\mathcal{S}$*;*
(iii) *the reduct $\langle S, \cdot \rangle$ is inherently nonfinitely based and there exists an idempotent e satisfying $e >_{\mathscr{J}} e^*e$;*
(iv) *all Zimin words are involutory isoterms for $\mathcal{S}$.*

Proof (i) $\to$ (ii) follows from Lemma 2.1, Proposition 4.2 and Corollary 4.5.

(ii) → (iv) has been verified in the course of the proof of Theorem 3.1.
(iv) → (i) is Theorem 2.2.
(ii) ↔ (iii) follows from Proposition 4.2 and Corollary 4.3.

We observe that the condition (iii) in Corollary 4.6 is algorithmically verifiable. Indeed, given a finite regular involutory semigroup $\mathcal{S}$, we can check whether or not its reduct is INFB by using Sapir's algorithm from [13], and the condition on the idempotents is obviously decidable.

Corollary 4.7 *There exists an algorithm which decides, when given a finite regular involutory semigroup $\mathcal{S}$, whether or not $\mathcal{S}$ is inherently nonfinitely based.*

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