IB Linear Algebra (from lecture 18)

Martin von Hodenberg (mjv43@cam.ac.uk)

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Linear algebra description etc

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Contents

§1 Bilinear forms

Lemma 1.1

We have a bilinear form $\phi : V \times V \to \mathbb{F}$, where V is a finite vector space and B, B' are bases of V. Let

$$\phi = [\mathrm{Id}]_{B,B'}.$$

Then

$$[\phi]_{B'} = P^T[\phi]_B P.$$

Proof. This is just a special case of the general change of basis formula; see that proof. □

Definition 1.2 (Congruent matrices)

Two square matrices A, B are said to be **congruent** if there exists an invertible square matrix P such that

$$A = P^T BP.$$

Remark. This defines an equivalence relation.

Definition 1.3 (Symmetric bilinear form)

A bilinear form on V is said to be **symmetric** if

$$\phi(u, v) = \phi(v, u)\ \forall u, v \in V.$$

Remark. We have encountered this definition before in IA Vectors and Matrices.

1. If A is a square matrix, we say that A is symmetric if $A^T = A$. Equivalently, $A_{ij} = A_{ji}$.
2. ϕ is symmetric iff $[\phi]_B$ is symmetric in *any* basis B.
3. To be able to represent ϕ by a diagonal matrix in some basis B, it is necesssary that ϕ is symmetric:
$$P^T A P = D = D^T = P A^T P^T \implies A = A^T \implies \phi \text{ is symmetric.}$$

Definition 1.4 (Quadratic form)

A map $Q : V \to F$ is said to be a **quadratic form** if there exists a bilinear form $\phi : V \times V \to F$ such that
$$\forall u \in V,\ Q(u) = \phi(u, u).$$

Remark (Computation in a basis)**.** Let $B = (e_i)_{1 \leq i \leq n}$ be a basis of V, and let $A = [\phi]_B$. Let $u = \sum_{i=1}^n u_i e_i$, then
$$Q(u) = \phi(u, u) = \phi\left(\sum_{i=1}^n u_i e_i, \sum_{j=1}^n u_j e_j\right) = \sum_{i,j=1}^n u_i u_j \phi(e_i, e_j) = \sum_{i,j=1}^n a_{ij} u_i u_j.$$
(by bilinearity of ϕ) Therefore we essentially have
$$Q(u) = U^T A U, \text{ where } U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Remark. We can note that
$$Q(u) = U^T A U = \sum_{i,j=1}^n a_{ij} u_i u_j = \sum_{i,j=1}^n \left(\frac{a_{ij} + a_{ji}}{2}\right) u_i u_j = U^T \left(\frac{A + A^T}{2}\right) U.$$
So this representation of A is not necessarily unique.

Proposition 1.5

If $Q : V \to F$ is a quadratic form, then there exists a unique symmetric bilinear form $\phi : V \times V \to F$ such that
$$Q(u) = \phi(u, u)\ \forall u \in V.$$

Proof. (Polarisation identity)
Proof of existence
Let ψ be a bilinear form on V such that
$$\forall u \in V,\ Q(u) = \psi(u, u).$$

Let $\phi(u, v) = \frac{1}{2}(\psi(u, v) + \psi(v, u))$. Thus we have that:

- ϕ is a bilinear form

- ϕ is symmetric
- $\phi(u,u) = \psi(u,u) = Q(u)$.

This concludes the proof of existence.
Proof of uniqueness
Let ϕ be a symmetric bilinear form such that

$$\forall u \in V,\ \phi(u,u) = Q(u).$$

Then

$$\begin{aligned} Q(u+v) &= \phi(u+v, u+v) \\ &= \phi(u,u) + \phi(u,v) + \phi(v,u) + \phi(v,v) \text{ by bilinearity} \\ &= Q(u) + 2\phi(u,v) + Q(v) \text{ by symmetry} \end{aligned}$$

From this we get that

$$\phi(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v)).$$

□

Theorem 1.6 (Diagonalisation of symmetric bilinear forms)
Let $\phi : V \times V \to F$ be a symmetric bilinear form. (dim V=n). Then there exists a basis B of V such that $[\phi]_B$ is diagonal.

Proof. We proceed by induction on the dimension of V. For $n = 1$ it is trivially true. Suppose the theorem holds for all dimensions $< n$: then

- If $\phi(u,u) = 0 \forall u \in V$, then $\phi = 0$ by the polarisation identity (ϕ is symmetric).
- If $\phi \neq 0$, then there exists a $u \in V\backslash\{0\}$ such that $\phi(u,u) \neq 0$. Let us call $u = e_1$.
- Let U be the 'orthogonal' of e_1:

$$\begin{aligned} U &= \{v \in V :\ \phi(e_1, v) = 0\} \\ &= \ker\theta \text{ where } \theta : V \to F \text{ is given by } \theta(v) = \phi(e_1, v). \end{aligned}$$

Since it is a kernel of a linear map $V \to F$, U is a vector subspace of V. By the Rank-Nullity theorem, we have

$$\dim V = n = r(\theta) + \operatorname{null}\theta = \dim U + 1.$$

We now claim that $U+ < e_1 >= U\oplus < e_1 >$. Indeed,

$$v =< e_1 > \cap U \implies v = \lambda e_1 \text{ and } \phi(e_1, v) = 0.$$

$$\implies 0 = \phi(e_1, v) = \phi(e_1, \lambda e_1) = \lambda\phi(e_1, e_1) \implies \lambda = 0 \implies v = 0.$$

$$\implies U+ < e_1 >= U\oplus < e_1 > .$$

Therefore $V = U \oplus < e_1 >$, and pick a basis $B' = (e_2, \dots, e_n)$ such that $(e_1, e_2, \dots, e_n)$ is a basis of V (since the sum is direct). So

$$[\phi]_B = (\phi(e_i, e_j))_{1 \le i,j \le n} = \left(\begin{array}{c|c} \phi(e_1, e_1) & 0 \\ \hline 0 & A' \end{array}\right).$$

Therefore $(A')^T = A'$, and $A' = [\phi|_U]_{B'}$ where $\phi|_U$ is the restriction of ϕ onto U. Now we apply the induction hypothesis to find a basis $(e_1', \dots, e_n;)$ of V such that $[\phi|U]_B$ is diagonal. So

$$\hat{B} = (e_1, e_2', \dots, e_n')$$

is a basis of V, and finally we have that $[\phi]_{\hat{B}}$ is diagonal.

□

Example 1.7

Let $V = \mathbb{R}^3$, and

$$\begin{aligned} Q(x_1, x_2, x_3) &= x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3 \\ &= x^T A x \text{ where } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \end{aligned}$$

There are two ways to diagonalise our quadratic form:

1. Diagonalise using the algorithm we developed in the previous proof.
2. Complete the square.

Let's complete the square:

$$\begin{aligned} Q(x_1, x_2, x_3) &= (x_1 + x_2 + x_3)^2 + x_3^2 - 4x_2x_3 \\ &= (x_1 + x_2 + x_3)^2 + (x_3 - 2x_1)^2 - (2x_2)^2 \\ &= (x_1')^2 + (x_2')^2 + (x_3')^2. \end{aligned}$$

Therefore

$$P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To find P, note that

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

§1.1 Sylvester's law and sesquilinear forms

Recall our theorem that a symmetric bilinear form has a diagonal basis.

Corollary 1.8

Let $F = \mathbb{C}$, and ϕ be a symmetric bilinear form of rank r on $V \times V$ where $\dim V = n$. Then there exists a basis B of V such that the matrix

$$[\phi]_B = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Proof. Pick $E = (e_1, \dots, e_n)$ such that

$$[\phi]_E = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

Order the a_i such that

$$\begin{cases} a_i \neq 0 & 1 \leq i \leq r \\ a_i = 0 & i > r \end{cases}$$

For $i \leq r$, let $\sqrt{a_i}$ be a choice of complex root for a_i. Let

$$\begin{cases} v_i = \frac{e_i}{\sqrt{a_i}} & 1 \leq i \leq r \\ v_i = e_i & i > r \end{cases}$$

Then $B = (v_1, \dots, v_r, v_{r+1}, \dots, v_n)$ is a basis of V and we can check that we have diagonalised and normalised ϕ:

$$[\phi]_B = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

□

Corollary 1.9

Every symmetric matrix of $M_n(\mathbb{C})$ is congruent to a *unique* matrix of the form

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Proof. Immediate. □

Corollary 1.10

Let $F = \mathbb{R}$. Let ϕ be a symmetric bilinear form on $V \times V$ where $\dim V = n$. Then there exists a basis $B = (v_1, \dots, v_n)$ basis of V such that for some $p - q \geq 0$ with $p + q = r(\phi)$,

$$[\phi]_B = \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & -I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Proof. Pick $E = (e_1, \dots, e_n)$ such that

$$[\phi]_E = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

Order the a_i such that

$$\begin{cases} a_i > 0 & 1 \le i \le p \\ a_i < 0 & p+1 \le i \le p+q \\ a_i = 0 & i > p+q \end{cases}$$

Similarly to before we define

$$\begin{cases} v_i = \frac{e_i}{\sqrt{a_i}} & 1 \le i \le p \\ v_i = \frac{e_i}{\sqrt{-a_i}} & p+1 \le i \le p+q \\ v_i = e_i & i > p+q \end{cases}$$

Then this basis will do the job. □

Definition 1.11 (Signature of a quadratic form)

For $F = \mathbb{R}$, we define the **signature** of ϕ

$$s(\phi) = p - q.$$

This is also the signature of the associated quadratic form Q. We will soon see that this is well-defined.

Definition 1.12 (Positive definite quadratic/bilinear form)

Let ϕ be a symmetric bilinear form on a real vector space V. We say that

(i) ϕ is positive definite if

$$\phi(u,u) > 0 \; \forall u \in V \backslash \{0\}.$$

(ii) ϕ is positive semidefinite if

$$\phi(u,u) \ge 0 \; \forall u \in V \backslash \{0\}.$$

(iii) ϕ is negative definite if

$$\phi(u,u) < 0 \; \forall u \in V \backslash \{0\}.$$

(iv) ϕ is negative semidefinite if

$$\phi(u,u) \le 0 \; \forall u \in V \backslash \{0\}.$$

Example 1.13

The matrix

$$\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right)$$

is

- positive definite for $p = n$
- positive semidefinite for $1 \leq p \leq n$.

Theorem 1.14 (Sylvester's law of inertia)

The signature of a quadratic form is well defined.
Formally, if a real symmetric bilinear form is represented in two different ways by two identity blocks of size p, q and p', q' then we have

$$p = p', q = q'.$$

Proof. In order to prove uniqueness of p, it is enough to show that p is the largest dimension of a subspace of V on which ϕ is positive definite. Say $B = (v_1, \dots, v_n)$ and

$$[\phi]_B = I_p - I_q.$$

Let $X =< v_1, \dots, v_p$. Then ϕ is positive definite on X. If we take a $u \in X$ with $u = \sum_{i=1}^p \lambda_i v_i$, then

$$Q(u) = \phi(u,u) = \phi(\sum_{i=1}^p \lambda_i v_i, \sum_{j=1}^p \lambda_j v_j) = \sum_{i,j=1}^p \lambda_i \lambda_j \phi(v_i, v_j) = \sum_{i=1}^n \lambda_i^2.$$

This is ¿0 as long as $u \neq 0$. Suppose that ϕ is positive definite on another subspace X. Let

$$X =< v_1, \dots, v_p >, Y =< v_{p+1}, \dots, v_n > .$$

Then arguing as above, looking at $[\phi]_B$ we know that ϕ is negative semidefinite on Y. This implies that

$$Y \cap X' = \{0\}.$$

Indeed, if $y \in Y \cap X$, then $Q(y) \leq 0$ since $y \in Y$, but this implies that $y = 0$ since $y \in X$. Therefore

$$Y + X = Y \oplus X \implies n = \dim V \geq \dim(Y+X) = \dim Y + \dim X \implies n \geq n - p + \dim X \implies \text{dimX} \leq p.$$

Similarly, we show that q is the largest dimension of a subspace on which ϕ is negative definite. So we have a *geometric characterisation* of p and q, which concludes the proof. □

Definition 1.15 (Kernel of a bilinear form)

Define the kernel of the bilinear form

$$K = \{v \in V : \ \forall u \in V, \phi(u,v) = 0\}.$$

Remark. $\dim K + r(\phi) = n$.

One can show using the above notation that there is a subspace T of dimension $m-(p+q)+\min p, q$ such that $\phi_T = 0$. The subspace we want to take consists of all the basis vectors that are killed by the lower corner of $[\phi]_B$, but it also includes the 'cancellations' between I_p and $-I_q$.
Moreover, one can show that the dimension of T is the largest possible dimension of such a subspace.

§1.1.1 Sesquilinear forms

Recall that the standard inner product on $\mathbb{C}^n$ is

$$< x, y >=< \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i \bar{y_i}.$$

However note that this map is **not** a bilinear form, since the conjugation of the second coordinate prevents this. However, it is still bilinear 'in a way'. We now explore this further.

Definition 1.16 (Sesquilinear form)
Let V, W be vector spaces over $\mathbb{C}$. A sesquilinear form on $V \times W$ is a function

$$\phi : V \times W \to \mathbb{C}.$$

such that

1. $\phi(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \phi(v_1, w) + \lambda_2 \phi(v_2, w)$
2. $\phi(\lambda_1 v_1 + \lambda_2 v_2, w) = \bar{\lambda_1} \phi(v_1, w) + \bar{\lambda_2} \phi(v_2, w)$

For B a basis of V, C a basis of W, we define the matrix

$$[\phi]_{B,C} = (\phi(v_i, w_j))_{1 \le i \le m, 1 \le j \le n}.$$

Lemma 1.17
If B basis for V, C a basis for W,

$$\phi(v, w) = [v]_B^T [\phi]_{B,C} [\bar{w}]_C.$$

Further, if B', C' are other bases for V, W respectively, and $P = [Id]_{B',B}, Q = [Id]_{C',C}$ then

$$[\phi]_{B',C'} = P^T [\phi]_{B,C} \bar{Q}.$$

Proof. Same proofs as for bilinear forms, just slightly modified. □

§1.2 Hermitian forms and skew symmetric forms

We will now generalise symmetric bilinear forms to sesquilinear forms.

Definition 1.18 (Hermitian form)

Let V be a finite dimensional vector space and $\phi : V \times V \to C$ be a sesquilinear form. We call ϕ **hermitian** if for all $(u, v) \in V \times V$ we have

$$\phi(u, v) = \bar{\phi(v, u)}.$$

Remark. If ϕ is Hermitian, then $\phi(u, u) = \bar{\phi(u, u)} \implies \phi(u, u) \in \mathbb{R}$. Moreover,

$$\phi(\lambda u, \lambda u) = |\lambda|^2 \phi(u, u).$$

This allows us to talk about negative/ positive definite Hermitian form.

Lemma 1.19

A sesquilinear form $\phi : V \times V \to \mathbb{C}$ is hermitian if and only if for every basis B of V,

$$[\phi]_B = [\bar{\phi}]_B^T.$$

Proof. Let $A = [\phi]_B$ have entries $a_{ij} = \phi(e_i, e_j)$ □