IB Complex Analysis

Martin von Hodenberg (mjv43@cam.ac.uk)

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Contents

§1 Basic notions

Recall definitions in $\mathbb{C}$ from IA courses. Note that $d(z,w) = |z-w|$ defines a metric on $\mathbb{C}$ (the standard metric). For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : \ |z-a| < r\}$ for the open ball with centre a and radius r.

> **Definition** (Open subset of $\mathbb{C}$)
> A subset $U \subset \mathbb{C}$ is open wrt. the standard metric if for all $a \in U$, there exists an r such that $D(a,r) \in U$.

Remark. This is equivalent to being open wrt. the Euclidean metric on $\mathbb{R}^2$.

This course is about complex valued functions of a single variable, i.e functions

$$f : A \to \mathbb{C}, \quad \text{where } A \subset \mathbb{C}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, we can write $f = u + iv$ for real functions u, v and thus define $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$. Almost exclusively we'll focus on differentiable functions f. But first let's recall continuity.

> **Definition** (Continuous function on $\mathbb{C}$)
> The function f (as above) is continuous at a point $w \in A$ if
>
> $$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall z \in A, \quad |z-w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Remark. This is equivalent to saying that $\lim_{z \to w} f(z) = f(w)$.

§1.1 Complex differentiation

Let $f : U \to \mathbb{C}$, where U is open.

> **Definition** (Differentiability)
> f is differentiable at $w \in U$ if the limit
>
> $$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}.$$
>
> exists at a complex number.

> **Definition** (Holomorphic function)
> f is **holomorphic**[1] at $w \in U$ if there is $\varepsilon > 0$ such that $D(w, \varepsilon) \subset U$ and f is differentiable at every point in $D(w,\varepsilon)$.
>
> Equivalently, f is holomorphic in U if f is holomorphic at every point in U, or equivalently, f is differentiable at every point in U.

[1] Sometimes we use "analytic" to mean holomorphic.

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (can easily be checked).

Proposition 1.1.1

The chain rule for composite functions also holds: if $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ with $f(U) \subset V$, and $h = g \circ f : U \to \mathbb{C}$. If f is differentiable at $w \in U$ and g is differentiable at $f(w)$, then h is differentiable at w with

$$h'(w) = (g \circ f)'(w) = f'(w)(g' \circ f)(w).$$

Proof. Omitted; analagous to the proof for the real case. □

We might ask ourselves a question:

Write $f(z) = u(x, y)+iv(x, y)$, $z = x+iy$. Is differentiability of f at a point $w = c+id \in U$ is the same as differentiability of u and v at (c, d)?

Recall from IB Analysis & Topology that $u : U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a "good linear approximation of u at (c, d)". We can show that if u is differentiable at c, d then L (the derivative of u at (c, d)) is uniquely defined, and we write $L = Du(c, d)$; moreover, L is given by the partial derivatives of u, i.e

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d)\right) x + \left(\frac{\partial u}{\partial y}(c, d)\right) y.$$

The answer to the above question is **no** (otherwise complex analysis would be useless!). Now we want to characterise differentiability of f in terms of u and v.

Theorem 1.1.2 (Cauchy-Riemann equations)

This theorem states that $f = u + iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if

u, v are differentiable at $(c, d) \in U$ **and** u, v satisfy the Cauchy-Riemann equations at (c, d):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d).$$

There are three other such expressions following from the Cauchy-Riemann equations.

Proof. f is differentiable at w with derivative $f'(w) = p + iq$:

$$\iff \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = p + iq$$

$$\iff \lim_{z \to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0$$

Writing $f = u + iv$ and separating real and imaginary parts, the above holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - p(x-c) + q(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$
$$\text{and} \quad \lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - q(x-c) + p(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

This is precisely the statement that u is differentiable at (c,d) with $Du(c,d)(x,y) = px - qy$, and v is differentiable at (c,d) with $Dv(c,d)(x,y) = qx + py$.

So $\iff$ u, v are differentiable at (c,d) and $u_x(c,d) = p = v_y(c,d)$, $u_y(c,d) = -q = -v_x(c,d)$, i.e the Cauchy-Riemann equations hold at (c,d).

We also get from the above that if f is differentiable at w, then $f'(w) = p + iq = u_x(c,d) + iv_x(c,d)$. □

Remark. u, v satisfying the Cauchy-Riemann equations at a point does **not** guarantee differentiability of f on its own. We also can proceed in a more simple way if we simply want to show the reverse implication, by writing

$$f'(w) = \lim_{z\to w} \frac{f(z) - f(w)}{z - w} = \lim_{h\to 0} \frac{f(w+h) - f(w)}{h},$$

and then choosing $h = t \in \mathbb{R}$ and $h = it$ (since we can choose any direction we want for h) in order to get that u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations.

Example (Differentiability (?) of conjugation map)

Let $f(z) = \overline{z} = x - iy$. For this, $u = x, v = -1$, so $u_x = 1, v_y = -1$ and so the C-R equations are not satisfied and f is not differentiable at any point.

Corollary 1.1.3

Let $f = u + iv : U \to \mathbb{C}$. If u, v have continuous partial derivatives at $(c,d) \in U$ and satisfy the C-R equations there, then f is differentiable at $w = c + id$.

In particular, if u, v are $\mathbb{C}^1$ functions on U (i.e have continuous partial derivatives in U) satisfying the C-R equations in U, then f is holomorphic in U.

Proof. Continuity of partial derivatives of u implies that u is differentiable, and similarly for v (IB Analysis & Topology). So the corollary follows from the C-R theorem. □

Complex differentiability is much more restrictive than real differentiability of real and imaginary parts (because of the additional requirement that the Cauchy-Riemann equations must hold). This leads to surprising theorems compared to the real case, for example

(a) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant! (Liouville's theorem). This is false for real functions; for example $\sin(x) : \mathbb{R} \to \mathbb{R}$.

(b) If $f : U \to \mathbb{C}$ is holomorphic, then f is infinitely differentiable on U.

We will prove these later on. Note that (b) implies that partial derivatives of u, v of all orders exists. So we can differentiate the Cauchy-Riemann equations to get:

$$(u_x)_x = (v_y)_x \implies u_{xx} = v_{yx} \text{ and}$$
$$(u_y)_y = (-v_x)_y \implies u_{yy} = -v_{xy}.$$

This gives $\nabla^2 u = u_{xx} + u_{yy} = 0$ in U. Similarly $\nabla^2 v = 0$ in U.

This means that real and imaginary parts of a holomorphic function are harmonic. This gives a deep connection between harmonic functions and complex analysis; some theorems can be viewed as giving results about harmonic functions.

Now we need some definitions before the next corollary.

Definition

1. A curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a closed interval. We say γ is a C^1 curve if γ' exists and is continuous on $[a, b]$.
2. An open set $U \subset \mathbb{C}$ is path-connected if for any two points $z, w \in U$, there is a curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.
3. A domain is a non-empty, open, path-connected subset of $\mathbb{C}$.

Corollary 1.1.4

If $U \in C$ is a domain and $f : U \to \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant.

Proof. Write $f = u + iv$. By the C-R equations, $f' = 0 \implies Du = Dv = 0$ in U. Since U is a domain, this means (IA Analysis and Topology) that u and v are constant, i.e f is constant. □

Now we want to look at some examples of holomorphic functions other than polynomials on $\mathbb{C}$ and rational functions on their domains. We now look at power series, which will give us a wealth of examples.

§1.2 Power series

Recall the next theorem from IA Analysis:

Definition (Radius of convergence)

If $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers, then there is a unique number $R \in [0, \infty]$ such that the power series

$$\sum_{n=0}^{\infty} c_n (z - a)^n \qquad z, a \in \mathbb{C}.$$

converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly wrt z on the compact disk $D_r = \{z \in \mathbb{C} : |z - a| < R\}$.

We call R the radius of convergence of the power series. Note that when $z = R$, we cannot say anything in general about convergence.

Theorem 1.2.1

Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R > 0$. Fix $a \in \mathbb{C}$, and define $f : D(a,R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then

1. f is holomorphic on $D(a,R)$.
2. The derived series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has radius of convergence R, and

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1} \quad \forall z \in D(a,R).$$

3. f has derivatives of all orders on $D(a,R)$, and $c_n = \frac{f^{(n)}(a)}{n!}$.
4. If f vanishes on $D(a,\varepsilon)$ for some $\varepsilon > 0$, then $f \equiv 0$ on $D(a,R)$.

Proof. Parts (i) and (ii): By considering $g(z) = f(z+a)$, we assume WLOG that $a = 0$. So $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in D(0,R)$.

The derived series $\sum_{n=1}^{\infty} nc_n z^{n-1}$ will have some radius of convergence $R_1 \in [0,\infty]$. Now let $z \in D(0,R)$ be arbitrary. Choose ρ such that $|z| < \rho < R$. Then since $n|\frac{z}{\rho}|^{n-1} \to 0$ as $n \to \infty$, we have

$$n|c_n||z|^{n-1} = n|c_n||\frac{z}{\rho}|^{n-1}\rho^{n-1} \leq |c_n|p^{n-1}.$$

for sufficiently large n. Since $\sum |c_n|\rho^n$ converges, it follows that $\sum_{n=1}^{\infty} nc_n z^{n-1}$ converges. Thus $D(0,R) \subset D(0,R_1)$, i.e $R_1 \geq R$. Since

$$|c_n||z|^n \leq n|c_n||z|^n = |z|\left(|c_n||z|^{n-1}\right),$$

if $\sum n|c_n||z|^{n-1}$ converges then so does $\sum |c_n||z|^n$, so $R_1 \leq R$. So $R_1 = R$.

To prove that f is differentiable with $f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$, fix $z \in D(0,R)$. The key idea is that this is equivalent to continuity at z of the function

$$g : D(0,R) \to \mathbb{C}, \quad g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ \sum_{n=1}^{\infty} nc_n z^{n-1} & w = z. \end{cases}$$

By subbing in f we can write $g(w) = \sum_{n=1}^{\infty} h_n(w)$ where

$$h_n(w) = \begin{cases} \frac{c_n(w^n - z^n)}{w-z} & w \neq z \\ nc_n z^{n-1} & w = z. \end{cases}$$

Now h_n is continuous on $D(0,R)$ (since $w \to w^n$ is differentiable with derivative nw^{n-1}). Using $\frac{w^n-z^n}{w-z} = \sum_{j=0}^{n-1} z^j w^{n-1-j}$, we get that for any r with $|z| < r < R$ and any $w \in D(0,r)$, $|h_n(w)| \leq n|c_n|r^{n-1} \equiv M_n$. Since $\sum M_n < \infty$, by the Weierstrass M-test that $\sum h_n$ converges uniformly on $D(0,r)$. But a uniform limit of continuous functions is continuous, so $g = \sum h_n$ is continuous in $D(0,r)$ and in particular at z.

Part (iii): Repeatedly apply (ii). The formula $c_n = \frac{f^{(n)(a)}}{n!}$ follows by differentiating the series n times and setting $z = a$.

Part (iv): If $f = 0$ in $D(a, \varepsilon)$, then $f^{(n)}(a) = 0$ for all n, so $c_n = 0$ for all n and hence $f = 0$ in $D(a, R)$. □

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, we say f is entire.

Proposition 1.2.2

The complex exponential function is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(i) e^z is entire, with $(e^z)' = e^z$.

(ii) $e^z \neq 0$ and $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

(iii) $e^{x+iy} = e^x(\cos(y) + isin(y))$ for $x, y \in \mathbb{R}$.

(iv) $e^z = 1$ iff $z = 2n\pi i$ for some $n \in \mathbb{Z}$.

(v) Let $z \in \mathbb{C}$. There exists $w \in \mathbb{C}$ such that $e^w = z$ iff $z \neq 0$.

Proof. Part (i): The r.o.c of the series is ∞. To see ($e^z = z'$), differentiate the series term by term using the previous theorem.

Part (ii): Fix any $w \in \mathbb{C}$ and set $F(z) = e^{z+w}e^{-z}$. Then $F'(z) = -e^{z+w}e^{-z} + e^{z+w}e^{-z} = 0$, so $F(z)$ is a constant. Thus $F(z) = F(0) = e^w$ for all $z \in \mathbb{C}$. Thus

$$e^{z+w}e^{-z} = e^w \quad \forall z, w \in \mathbb{C}.$$

Taking $w = 0$, $e^z e^{-z} = 1$. So $e^z \neq 0$. Multiplying by e^z, we get $e^{z+w} = e^z e^w$.

Part (iii): $e^{x+iy} = e^x e^{iy}$ by (ii). Now use the definition of e^{iy}, and the series for $\sin(y), \cos(y)$ for $y \in \mathbb{R}$.

Part (iv) and (v): Follow from (iii). (Exercise) □

Definition (Logarithm)

Given $z \in \mathbb{C}$, we say a complex $w \in \mathbb{C}$ is a **logarithm** of z if $e^w = z$.

By Proposition 1.2.2(v), z has a logarithm iff $z \neq 0$. By (ii) and (iv), if $z \neq 0$ then z has infinitely many logarithms, with any two differing from each other by $2n\pi i$ for some integer n.

If w is a logarithm of z, then $e^{\mathrm{Re}(w)} = |z|$, so $\mathrm{Re}(w) = \log |z|$ (the real logarithm of the positive number $|z|$); in particular, this is well-defined.

Definition (Branch of a logarithm)

Let $U \subset \mathbb{C} \setminus \{0\}$ be open. Then a branch of logarithm on U is a continuous function $\lambda : U \to \mathbb{C}$ such that $e^{\lambda(z)} = z$ for each $z \in U$.

Proposition 1.2.3

If λ is a branch of log on U then λ is automatically holomorphic in U, with $\lambda'(z) = \frac{1}{z}$.

Proof. If $w \in U$ then

$$\begin{aligned}
\lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{z - w} &= \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z)} - e^{\lambda(w)}}{\lambda(z) - \lambda(w)}\right)} \\
&= \frac{1}{e^{\lambda(w)}} \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z) - \lambda(w)} - 1}{\lambda(z) - \lambda(w)}\right)} \\
&= \frac{1}{e^{\lambda(w)}} \lim_{h \to 0} \frac{1}{\left(\frac{e^h - 1}{h}\right)} \qquad \text{since } \lambda \text{ is continuous} \\
&= \frac{1}{e^{\lambda(w)}} = \frac{1}{w}.
\end{aligned}$$

$\square$

Definition (Principal branch of logarithm)

The principal branch of logarithm is the function

$$\mathrm{Log} : U_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : \ x \leq 0\} \to \mathbb{C}.$$

defined by

$$\mathrm{Log}(z) = \log|z| + i \arg(z).$$

where $\arg(z)$ is the unique argument of $z \in U_1$ in $(-\pi, \pi)$.

Remark. Log is a branch of logarithm in U_1 : to check continuity of Log, note that $z \mapsto \log|z|$ is continuous on $\mathbb{C}\setminus\{0\}$ (by continuity of $z \mapsto |z|$ and continuity of $r \mapsto \log r$ for $r > 0$); also, $z \mapsto \arg(z)$ is continuous, since $\theta \mapsto e^{i\theta}$ is a homeomorphism $(-\pi, \pi) \to \mathbb{S}^1 \setminus \{-1\}$ (as can be checked directly, where $\mathbb{S}^1 = \{z : |z| = 1\}\big)$, and $z \mapsto \frac{z}{|z|}$ is continuous on $\mathbb{C}\setminus\{0\}$. So $z \mapsto \log(z)$ is continuous on U_1. We also have

$$e^{\mathrm{Log}(z)} = e^{\ln|z| + i \arg(z)} = e^{\ln|z|} \cdot e^{i \ \arg\ (z)} = |z|(\cos\arg(z) + i \sin\arg(z)) = z.$$

So Log is a branch of logarithm in U_1.

Remark. Log does not have a continuous extension to $\mathbb{C}\setminus\{0\}$ since $\arg(z) \to \pi$ as $z \to -1$ with $\mathrm{Im}(z) > 0$, and $\arg(z) \to -\pi$ as $z \to -1$ with $\mathrm{Im}(z) < 0$.

Proposition 1.2.4 (i) Log is holomorphic on U_1 with $\mathrm{Log}'(z) = \frac{1}{z}$.

(ii) For $|z| < 1$, we have

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof. (i) is the remark above.

(ii) To see this, note that the R.O.C of the series is 1, and $|z| < 1 \implies 1+z \in U_1$, so both sides are defined on $|z| < 1$.

Let $F(z) = \mathrm{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{n}$ for $|z| < 1$. Then

$$F'(z) = \frac{1}{1+z} - \sum_{n=1}^{\infty}(-z)^{n-1} = 0.$$

So $F(z) = \text{ constant } = F(0) = 0$.

□

Using exp and Log we can define further useful functions.

1. For any $\alpha \in \mathbb{C}$, define
$$z^\alpha = e^{\alpha \,\mathrm{Log}(z)}, \quad z \in U_1.$$
This is the principal branch of z^α. It's holomorphic on U_1 with derivative $\alpha z^{\alpha-1}$.

2. We can define the familiar functions
   - $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
   - $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$
   - $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$
   - $\sinh(z) = \frac{e^{z}-e^{-z}}{2}$

   These are all entire since exp is entire, with derivatives given by the familiar expressions from real variables.

§1.3 Conformality

Let $f : U \to \mathbb{C}$ be holomorphic ($U \subset C$ is open). Let $w \in U$ and suppose that $f'(w) \neq 0$. Take two C^1 curves $\gamma_1, \gamma_2 : [-1,1] \to U$ such that $\gamma_1(0) = \gamma_2(0) = w$ and with nonzero derivative. Then $f \circ \gamma_i$ are C^1 curves passing through $f(w)$. Moreover, $(f \circ \gamma_i)'(0) = f'(w)\gamma_i'(0) \neq 0$. Thus

$$\frac{(f\circ\gamma_1)'(0)}{(f\circ\gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}.$$

Hence

$$\arg(f\circ\gamma_1)'(0) - \arg(f\circ\gamma_2)'(0) = \arg\gamma_1'(0) - \arg\gamma_2'(0).$$

This means that the angle that the curves γ_1, γ_2 make at w is the same as the angle their images make at $f(w)$. We say f is 'angle-preserving at w', whenever $f'(w) \neq 0$.

Remark. If f is a C^1 map on U, the converse of this also holds. See Example Sheet 1.

Definition (Conformal map)

A holomorphic function $f : U \to \mathbb{C}$ on an open set U is said to be **conformal** at a point $w \in U$ if $f'(w) \neq 0$.

Definition (Conformal equivalence)

Let $U, \widetilde{U}$ be domains in $\mathbb{C}$. A map $f : U \to \widetilde{U}$ is said to be a conformal equivalence between U and $\widetilde{U}$ if f is a bijective holomorphic map with $f'(z) \neq 0$ for every $z \in U$.

Remark.

- If f is holomorphic and injective, then $f'(z) \neq 0$ for each z. We will prove this later. So in the above definition $f'(z) \neq 0$ is redundant.
- It is automatic that the inverse $f^{-1} : U \to \widetilde{U}$ is holomorphic. This can be proved using the holomorphic inverse function theorem, which you will prove on Example Sheet 1.

Example

Let's look at some examples of conformal equivalence.

1. Mobius maps are defined for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (see IA Groups):

$$f(z) = \frac{az+b}{cz+d}.$$

Mobius maps sometimes serve as explicit conformal equivalences between sub-domains of $\mathbb{C}$. For example, let $\mathbb{H}$ be the open upper half plane. Then

$$\begin{aligned} z \in \mathbb{H} &\iff |z-i| < |z+i| \\ &\iff \left|\frac{z-i}{z+i}\right| < 1. \end{aligned}$$

Thus $g(z) = \frac{z-i}{z+i}$ maps $\mathbb{H}$ onto $D(0,1)$, so g is a conformal equivalence.

2. Consider $f : z \to z^n$ where $n \in \mathbb{N}$ where $f : \left\{z \in C \setminus \{0\} : 0 < \arg(z) < \frac{\pi}{n}\right\} \to \mathbb{H}$. This is a conformal equivalence with inverse $f(z) = z^{1/n}$ (the principal branch).

3. We have that

$$\exp : \{z \in \mathbb{C} : \ -\pi < \mathrm{Im}(z) < \pi\} \to \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}.$$

Aside:

Theorem 1.3.1 (Riemann Mapping Theorem)

Any simply connected domain $U \subset \mathbb{C}$ with $U \neq \mathbb{C}$ is conformally equivalent to $D(0,1)$.

Proof. This is beyond the scope of the course.[2] □

[2] See Rudin's *Real and Complex Analysis*.

§2 Complex Integration: Part I

§2.1 Definitions and basic properties

We aim to extend Riemann integration to complex functions $f : U \to \mathbb{C}$ along curves in U. First we take a look at complex functions of a real variable.

Definition

If $f : [a, b] \subset \mathbb{R} \to C$ is a complex function and if f is Riemann integrable, define

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \mathrm{Re}\, f(t)\mathrm{d}t + \int_a^b \mathrm{Im}\, f(t)\mathrm{d}t.$$

In particular,

$$\int_a^b if(t)\mathrm{d}t = i\int_a^b f(t)\mathrm{d}t.$$

We can then directly calculate for any $w \in \mathbb{C}$ that

$$\int_a^b wf(t)\mathrm{d}t = w\int_a^b f(t)\mathrm{d}t.$$

Proposition 2.1.1

If $f : [a, b] \to \mathbb{C}$ is continuous, then

$$|\int_a^b f(t)\mathrm{d}t| \leq \int_a^b |f(t)|\mathrm{d}t \leq (b-a) \sup_{t\in[a,b]} |f(t)|,$$

with equality iff f is constant.

Proof. If $\int_a^b f(t)\mathrm{d}t$ then we are done. Else write $\int_a^b f(t)\mathrm{d}t = re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and let $M = \sup_{t\in[a,b]} |f(t)|$. Then

$$\begin{aligned}|\int_a^b f(t)\mathrm{d}t| = r = e^{-i\theta}\int_a^b f(t)\mathrm{d}t &= \int_a^b e^{-i\theta}f(t)\mathrm{d}t \\ &= \int_a^b \mathrm{Re}(e^{-i\theta})f(t)\mathrm{d}t + \int_a^b \mathrm{Im}(e^{-i\theta})f(t)\mathrm{d}t.\end{aligned}$$

Since the LHS is real,

$$|\int_a^b f(t)\mathrm{d}t| = \int_a^b \mathrm{Re}\, f(t)\mathrm{d}t \leq \int_a^b |e^{-i\theta}f(t)|\mathrm{d}t = \int_a^b |f(t)|\mathrm{d}t \leq (b-a)M.$$

Equality holds iff $|f(t)| = M$ and $\mathrm{Re}(e^{-i\theta}f(t)) = M$ for all $t \in [a, b]$, i.e iff $|f(t)| = M$ and $\arg f(t) = \theta$ for all t; iff f constant. □

Definition (Integral along a curve)

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to U$ be a C^1 curve.

Then the **integral of f along γ** is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

Proposition 2.1.2 (Basic properties of the integral)

If we have the integral of f along γ, we have the following properties:

1. Invariance under reparametrisation: Let $\varphi : [a_1, b_1] \to [a, b]$ be C^1 and injective with $\varphi(a_1) = a, \varphi(b_1) = b$. Let $\delta = \gamma \circ \varphi : [a_1, b_1] \to U$. Then we have

$$\int_\delta f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z.$$

2. Linearity:

$$\int_\gamma c_1 f_1(z) + c_2 f_2(z)\mathrm{d}z = c_1 \int_\gamma f_1(z)\mathrm{d}z + c_2 \int_\gamma f_2(z)\mathrm{d}z.$$

3. Additivity: If γ is our C^1 curve and $a < c < b$, then

$$\int_\gamma f(z)\mathrm{d}z = \int_{\gamma|_{[a,c]}} f(z)\mathrm{d}z + \int_{\gamma|_{[c,b]}} f(z)\mathrm{d}z.$$

4. Inverse path: Define the inverse path $(-\gamma) : [-b, -a] \to U$ by $(-\gamma)(t) = \gamma(-t)$ for $-b \le t \le -a$. Then

$$\int_{(-\gamma)} f(z)dz = -\int_\gamma f(z)\mathrm{d}z$$

Proof. For 1, we have

$$\begin{aligned}\int_\delta f(z)\mathrm{d}z &= \int_{a_1}^{b_1} f(v \circ \varphi(t))\gamma'(\varphi(t))\varphi'(t)\mathrm{d}t \\ &= \int_a^b f(\gamma(s))\gamma'(s)\mathrm{d}s = \int_\gamma f(z)\mathrm{d}z \quad \text{by change of vars. } s = \varphi(t).\end{aligned}$$

2,3, and 4 are all easy to check from the definition. □

Definition (Length of a curve)

Definition: Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve. The length of γ is defined by

$$\text{length}(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Definition

A **piecewise C^1 curve** is a continuous map $\gamma : [a, b] \to \mathbb{C}$ such that there exists a finite subdivision

$$a = a_0 < a_1 < \ldots < a_{n-1} < a_n = b$$

with the property that $\gamma_j = \gamma|_{[a_{j-1},a_j]} : [a_{j-1}, a_j] \to \mathbb{C}$ is C^1 for all $1 \le j \le n$. We define

$$\int_\gamma f(z)\ \mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\ \mathrm{d}z.$$

and

$$\mathrm{length}(\gamma) = \sum_{j=1}^{n} \mathrm{length}(\gamma_j) = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)|\ \mathrm{d}t.$$

Remark. From now on, by a 'curve' we shall mean a piecewise C^1 curve.

Definition

If $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are curves with $\gamma_1(b) = \gamma_2(c)$, we define the sum of of γ_1 and γ_2 to be the curve

$$(\gamma_1 + \gamma_2) : [a, b + d - c] \to \mathbb{C},$$

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t - b + c) & b \le t \le b + d - c \end{cases}$$

Proposition 2.1.3

For any continuous function $f : U \to \mathbb{C}$ and any curve $\gamma : [a, b] \to \mathbb{C}$, we have that

$$\left| \int_\gamma f(z) dz \right| \le \ \mathrm{length}\ (\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| = \sup_{t\in[a,b]} |f(\gamma(t))|$.

Proof. If γ is C^1, then

$$\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \le \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \le \sup_{t\in[a,b]} |f(\gamma(t))|\, \mathrm{length}(\gamma).$$

If γ is piecewise C^1 then the result follows from the definition $\int_\gamma f(z)dz = \sum_{j=1}^n \int_{\gamma_j} f(z)dz$ where γ_j is C^1, and the triangle inequality. $\square$

§2.2 Fundamental theorem of calculus

We can now look at the complex version of the FTC.

Theorem 2.2.1 (Fundamental Theorem of Calculus)

Suppose that $f : U \to \mathbb{C}$ is continuous, $U \subset \mathbb{C}$ open. If there is a function $F : U \to \mathbb{C}$

such that $F'(z) = f(z)$ for all $z \in U$, then for any curve $\gamma : [a, b] \to U$,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

If additionally γ is a closed curve, i.e. $\gamma(b) = \gamma(a)$, then $\int_\gamma f(z)dz = 0$.

Proof. This follows immediately:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a)).$$

□

Remark. Such F as in Theorem 2.3 is called an **anti-derivative** of f.

We shall see later (by infinite differentiability of holomorphic functions) that if $F'(z) = f(z)$, then f is automatically continuous.

Example

Let $\int_\gamma z^n \, \mathrm{d}z$ for $n \in \mathbb{Z}$, where $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = Re^{2\pi it}$ for some $R > 0$. (The image of γ is the circle of radius R centred at 0).

For $n \neq 1$, $\frac{z^{n+1}}{n+1}$ is an antiderivative of z^n in $\mathbb{C} \backslash \{0\}$, so by the FTC, $\int_\gamma z^n \, \mathrm{d}z = 0$ since γ is a closed curve.

For $n = -1$, use the definition of the integral:

$$\int_\gamma \frac{1}{z} \, \mathrm{d}z = \int_0^1 \frac{\gamma'(t)}{\gamma(t)} \, \mathrm{d}t = \int_0^1 \frac{2\pi i Re^{2\pi it}}{Re^{2\pi it}} \, \mathrm{d}t = 2\pi i.$$

Since $\int_\gamma \frac{1}{z} dz \neq 0$, we can conclude that for any $R > 0$, $\frac{1}{z}$ has no anti-derivative in any open set containing the circle $\{|z| = R\}$.

In particular, since for any branch $\lambda(z)$ of logarithm the derivative $\lambda'(z) = \frac{1}{z}$, there is no branch of logarithm on $\mathbb{C}^\star = \mathbb{C} \backslash \{0\}$.

Theorem 2.2.2 (Converse to FTC)

Let $U \subset \mathbb{C}$ be a domain. If $f : U \to \mathbb{C}$ is continuous and if $\int_\gamma f(z) \, \mathrm{d}z = 0$ for every closed curve γ in U, then f has an antiderivtive, i.e there is a holomorphic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for each $z \in U$.

Proof. Fix $a_0 \in U$. For $w \in U$, define

$$F(w) \int_{\gamma_w} f(z) \, \mathrm{d}z.$$

where $\gamma_w : [0, 1] \to \mathbb{C}$ is a curve with $\gamma_w(0) = a_0$, $\gamma_w(1) = w$ (exists since U is path-connected). The definition of F is independent of the chocie of γ_w by our hypothesis that $\int_\gamma f(z) \, \mathrm{d}z = 0$ for every closed curve γ in U. So $F : U \to \mathbb{C}$ is well-defined.

Fix $w \in U$. Since U is open, there is $r > 0$ such that $D(w, r) \subset U$. For $h \in \mathbb{C}$ with $0 < |h| < r$, let δ_h be the straight line path $t \mapsto w + th$ for $t \in [0, 1]$. Let

$$\gamma = \gamma_w + \delta_h + (-\gamma_{w+h}).$$

γ is closed so $\int_\gamma f \, \mathrm{d}z = 0$. Thus

$$\int_{\gamma_{w+h}} f(z) \, \mathrm{d}z = \int_{\gamma_w} f(z) \, \mathrm{d}z + int_{\delta_h} f(z) \, \mathrm{d}z.$$

In terms of F, this says that

$$F(w+h) = F(w) + int_{\delta_h} f(z) \, \mathrm{d}z = F(w) + hf(w) + int_{\delta_h} f(z) - f(w) \, \mathrm{d}z.$$

So

$$\begin{aligned} \left| \frac{F(w+h) - F(w)}{h} - f(w) \right| &= \frac{1}{|h|} |int_{\delta_h} f(z) - f(w) \, \mathrm{d}z| \\ &\le \frac{1}{|h|} \operatorname{length}(\delta_h) \sup_z |f(z) - f(w)| \\ &= \sup_z |f(z) - f(w)| \\ &\to 0 \text{ as } h \to 0 \qquad \text{since } f \text{ is continuous.} \end{aligned}$$

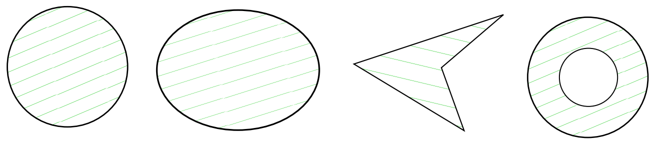
Therefore F is differentiable at w with $F'(w) = f(w)$. $\square$

The FTC is true for any domain, but next we restrict ourselves to a smaller subset of domains and prove some results for domains of certain shapes.

Definition (Star-shaped domain)

A domain U is **star-shaped** if there is $a_0 \in U$ such that got each $w \in U$, the straight-line segment $[a_0, w] \subset U$.

Remark. U is a disk $\implies$ U is convex $\implies$ U is star-shaped $\implies$ U is path-connected. None of the reverse implications hold.



Disk $\implies$ convex $\implies$ star-shaped $\implies$ path-connected

Definition (Triangle)

A **traingle** in $\mathbb{C}$ is the convex hull of three points in $\mathbb{C}$. The three points are the vertices of the triangle.

Notation. For a triangle T, we write $\int_{\partial T} f(z) \, \mathrm{d}z$ to denote the integral of f along the piecewise affine closed curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_1, \gamma_2, \gamma_3$ parameterise the three straight lines that are the sides of T with the interior of T to the left of them.

Corollary 2.2.3 (of Theorem 2.2.2)

If U is star-shaped, $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\ \mathrm{d}z = 0$ for any triangle $T \subset U$, then f has an antiderivative in U.

Proof. Suppose U is star-shaped with respect to a point $a_0 \in U$ and let $w \in U$ be some point. Let γ_w be the function parameterising $[a_0, w]$ and let $F(w) = \int_{\gamma_w} f(z)\ \mathrm{d}z$. With h, δ_h and γ as in the proof of Theorem 2.2.2, we then have that $\int_\gamma f(z)dz = \pm \int_{\partial T} f(z)dz$ for a triangle $T \subset U$ (with the - sign if T lies to the right of the directed boundary segments). Since $\int_{\partial T} f(z)dz = 0$ by hypothesis, we have that $\int_\gamma f(z)dz = 0$. We can now proceed exactly as in the proof of Theorem 2.2.2. □

We will soon see that the validity of $\int_\gamma f(z)\ \mathrm{d}z = 0$ for any holomorphic f on U and any curve γ in U has important consequences.

One of our goals in the course will be to characterise these domains U. We are going to build towards Cauchy's theorem, which states that this is true for simply connected U.

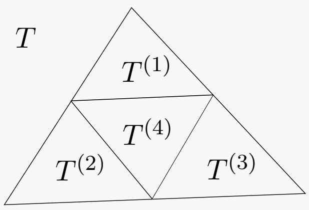
The next object of discussion is an elegant proof of Cauchy's theorem restricted to the special case of triangles in $\mathbb{C}$.

Theorem 2.2.4 (Cauchy's theorem for triangles)

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then $\int_{\partial T} f(z)\ \mathrm{d}z = 0$ for any triangle $T \subset U$.

Proof. Let $\eta(T) = \int_{\partial T} f(z)\ \mathrm{d}z$.

First key idea: subdivide T into four smaller triangles $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$ by joining the midpoints of the sides of T:



Note that

$$\eta(T) = \int_{\partial T(1)} f(z)dz + \int_{\partial T(2)} f(z)dz + \int_{\partial T(3)} f(z)dz + \int_{\partial T^{(4)}} f(z)dz.$$

So by the triangle inequality,

$$\left| \int_{\partial T^{(j)}} f(z)dz \right| \geq \frac{|\eta(T)|}{4}$$

for some $j \in \{1, 2, 3, 4\}$. Let $T_1 = T^{(j)}$ for this j, and write $T_0 = T$. So $|\eta(T_1)| \geq \frac{1}{4} |\eta(T_0)|$. Also, length $(\partial T_1) = \frac{1}{2}$ length (∂T_0). Now repeat the process: subdivide T_1 and choose a new triangle $T_2 \subset T_1$ exactly the same way. Doing this indefinitely generates a sequence of triangles $T_0 \supset T_1 \supset T_2 \supset \dots$ satisfying, for $n = 1, 2, 3, \dots$,

$$|\eta(T_n)| \geq \frac{1}{4}|\eta(T_{n-1})| \text{ and } \operatorname{length}(\partial T_n) = \frac{1}{2}\operatorname{length}(\partial T_{n-1}).$$

Iterating, we get

$$|\eta(T_n)| =\geq \frac{1}{4^n}|\eta(T_0)| \text{ and } \operatorname{length}(\partial T_n) = \frac{1}{2^n}\operatorname{length}(\partial T_0).$$

Since T_n are non-empty, nested closed subsets with $\operatorname{diam}(T_n) \to 0$ and $\mathbb{C}$ is a complete metric space, we have that $\bigcup_{n=1}^{\infty} T_n = \{z_0\}$ for some $z_0 \in \mathbb{C}$ (Analysis & Topology sheet 3).

Now let $\varepsilon > 0$. Since f is differentiable at z_0, there is $\delta > 0$ such that for all $z \in U$,

$$|z - z_0| < \delta \implies |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon|z - z_0|.$$

Second key idea: observe that for any n,

$$\int_{\partial T_n} f(z)dz = \int_{\partial T_n} \left(f(z) - f\left(z_0\right) - f'\left(z_0\right)\left(z - z_0\right)\right) dz$$

(since $\int_{\partial T_n} 1dz = \int_{\partial T_n} zdz = 0$, by the FTC).

So choosing n with $T_n \subset D\left(z_0, \delta\right)$ (which is possible since $z_0 \in T_n$ for all n and $\operatorname{diam}\left(T_n\right) \to 0$),

$$\begin{aligned}
\frac{1}{4^n}\left|\eta\left(T_0\right)\right| \leq \left|\eta\left(T_n\right)\right| &= \left|\int_{\partial T_n} f(z)dz\right| \\
&= \left|\int_{\partial T_n} \left(f(z) - f\left(z_0\right) - f'\left(z_0\right)\left(z - z_0\right)\right) dz\right| \\
&\leq \left(\sup_{z \in \partial T_n} \left|f(z) - f\left(z_0\right) - f'\left(z_0\right)\left(z - z_0\right)\right|\right) \operatorname{length}\left(\partial T_n\right) \\
&=\leq \epsilon \left(\sup_{z \in \partial T_n} \left|z - z_0\right|\right) \operatorname{length}\left(\partial T_n\right) \leq \epsilon\left(\operatorname{length}\left(\partial T_n\right)\right)^2 \\
&= \frac{\epsilon}{4^n}\left(\operatorname{length}\left(\partial T_0\right)\right)^2.
\end{aligned}$$

Cancel $\frac{1}{4^n}$ on both sides and let $\epsilon \to 0$. This gives $\eta\left(T_0\right) = 0$. □

For later applications, we want to generalise this theorem to continuous functions which are assumed holomorphic *except at a finite number of points.*

Theorem 2.2.5

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $S \subset U$ be a finite set and suppose that f is holomorphic on $U \backslash S$. Then

$$\int_{\partial T} f(z) \, \mathrm{d}z = 0 \quad \text{for every triangle } T \subset U.$$

Proof. Subdivide T into a total of $N = 4^n$ smaller triangles by the iterative procedure before, where at each step we join up the midpoints of the sides of the triangles at the previous step. This time we keep all the smaller triangles; call them $T_1, T_2, \ldots, T_N$. (Note the notational difference from before.) Then since the integrals along the sides of the smaller triangles that are interior to T cancel, we get that

$$\int_{\partial T} f(z)\ \mathrm{d}z = \sum_{j=1}^{N} \int_{\partial T_j} f(z)\ \mathrm{d}z.$$

Note by the previous theorem that $\int_{\partial T_j} f(z)\ \mathrm{d}z = 0$ unless $T_j \cap S \neq \emptyset$. So letting $I = \{j : T_j \cup S = \emptyset\}$, we have that $\int_{\partial T} f(z)\ \mathrm{d}z = \sum_{j\in I} \int_{\partial T_j} f(z)\ \mathrm{d}z$. Since any point can be in at most 6 smaller triangles and $\text{length}(\partial T_j) = \frac{1}{2^n}\text{length}(\partial T)$, we get that

$$\left|\int_{\partial T} f(z)\ \mathrm{d}z\right| \leq 6\,|S| \left(\sup_{z\in T} |f(z)|\right) \frac{\text{length}(\partial T)}{2^n}.$$

Then let $n \to \infty$ and we are done. □

Corollary 2.2.6

(Convex Cauchy) Let $U \subset C$ be convex, or more generally, a star domain. Let $f : U \to \mathbb{C}$ be continuous and holomorphic in $U\backslash S$ where S is a finite set. Then $\int_\gamma f(z)\ \mathrm{d}z = 0$ for any closed curve γ in U.

Proof. By Theorem 2.2.4, $\int_{\partial T} f(z)\ \mathrm{d}z = 0$ for any triangle $T \subset U$. Since U is a star domain and f is continuous, this means that by Corollary 2.2.3 that f has an antiderivative in U. The result now follows from the FTC. □

Now we are ready to draw a series of very nice corollaries of "convex Cauchy". The main corollary is a representation formula known as the *Cauchy integral formula*, from which the other results will follow.

§2.3 Cauchy integral formula

Notation. For a disk $D(a, \rho)$ we will write $\int_{\partial D(a,\rho)} f(z)\ \mathrm{d}z$ to mean $\int_\gamma f(z)\ \mathrm{d}z$ where γ is the curve $\gamma(t) = a + \rho e^{2\pi i t}$ (which parameterises the boundary of the disk with positive orientation, i.e so that the disk lies to the left of the directed boundary circle).

Theorem 2.3.1 (Cauchy integral formula for a disk)

Let $D = D(a, r)$ and let $f : D \to \mathbb{C}$ be holomorphic. Then for any ρ with $0 < \rho < r$ and any $w \in D(a, \rho)$ wehave that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z - w}\ \mathrm{d}z.$$

In particular (taking $w = a$),

$$f(a) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z - a}\ \mathrm{d}z = \int_0^1 f(a + \rho e^{2\pi i t})\ \mathrm{d}t.$$

This is called the *mean value property* for holomorphic functions.

For the proof we will need the following fact:

Lemma 2.3.2

If $\gamma : [a,b] \to \mathbb{C}$ is a curve and (f_n) is a sequence of continuous complex functions on image (γ) converging uniformly to a function f on image (γ), then $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$. This is true because

$$\begin{aligned}\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &= \left|\int_\gamma (f_n(z) - f(z))\, dz\right| \\ &\le \sup_{z\in \text{image}(\gamma)} |f_n(z) - f(z)| \,\text{length}(\gamma)\end{aligned}$$

Now let's prove the Cauchy integral formula.

Proof. Fix $w \in D(a,\rho)$ and define $h : D \to C$ by

$$h(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Then h is continuous on D and holomorphic in $D\backslash\{w\}$, so by "convex Cauchy",

$$\int_{\partial D(a,\rho)} h(z)\ \mathrm{d}z = 0.$$

Substituting for h, we get

$$f(w)\int_{\partial D(a,\rho)} \frac{1}{z-w}\ \mathrm{d}z = \int_{\partial D(a,\rho)} \frac{f(z)}{z-w}\ \mathrm{d}z.$$

Now we just have to show that $\int_{\partial D(a,\rho)} \frac{1}{z-w}dz = 2\pi i$. To do this, note that $\frac{1}{z-w} = \frac{1}{z-a+a-w} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = \sum_{j=0}^{\infty} \frac{(w-a)^j}{(z-a)^{j+1}}$, where the convergence is uniform for $z \in \partial D(a,\rho)$ by the Weierstrass M-test (since $\left|\left(\frac{w-a}{z-a}\right)^j\right| = \left(\frac{|w-a|}{\rho}\right)^j \equiv M_j$, and $\sum M_j < \infty$.) Therefore, by the above fact, we can interchange summation and integration to get,

$$\int_{\partial D(a,\rho)} \frac{dz}{z-w} = \sum_{j=0}^{\infty}(w-a)^j \int_{\partial D(a,\rho)} \frac{1}{(z-a)^{j+1}}dz$$

Now for $j \ge 1$, the function $\frac{1}{(z-a)^{j+1}}$ has an anti-derivative $\left(= -\frac{1}{j(z-a)^j}\right)$ in a neighbourhood of $\partial D(a,\rho)$, so by FTC, all integrals on the right for $j \ge 1$ are zero. For $j = 0$, by direct computation $\int_{\partial D(a,\rho)} \frac{1}{z-a}dz = 2\pi i$. So $\int_{\partial D(a,\rho)} \frac{1}{z-w}dz = 2\pi i$, which concludes the proof. □

This finally gives us the tools to prove two very deep results: Liouville's theorem and the Fundamental Theorem of Algebra.

Theorem 2.3.3 (Liouville's theorem)

If $f : \mathbb{C} \to \mathbb{C}$ is entire (holomorphic on $\mathbb{C}$) and bounded, then f is constant. More generally, if f is entire with sub-linear growth (i.e there are constants $K > 0$ and $\alpha < 1$ such that $|f(z)| \leq K(1 + |z|^\alpha)$) for all $z \in \mathbb{C}$, then f is constant.

Proof. For any given $w \in \mathbb{C}$ and any $\rho > |w|$, we have by CIF that $f(w) = \frac{1}{2\pi i}\int_{\partial D(0,\rho)} \frac{f(z)}{z-w}dz$ and $f(0) = \frac{1}{2\pi i}\int_{\partial D(0,\rho)} \left(\frac{f(z)}{z}dz\right.$. Thus

$$\begin{aligned}|f(w) - f(0)| &= \frac{1}{2\pi}\left|\int_{\partial D(0,\rho)} \frac{wf(z)}{z(z-w)}dz\right| \\ &\leq \frac{|w|}{2\pi}\sup_{z\in\partial D(0,\rho)} \frac{|f(z)|}{|z|||z|-|w||}\operatorname{length}(\partial D(0,\rho)) \\ &\leq \frac{|w|K(1+\rho^\alpha)}{2\pi\rho(\rho-|w|)}2\pi\rho = \frac{|w|K(1+\rho^\alpha)}{\rho-|w|}\end{aligned}$$

Let $\rho \to \infty$ in this, keeping w fixed to conclude that $f(w) = f(0)$. So we are done. □

Theorem 2.3.4 (Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has a complex root.

Proof. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a complex polynomial of degree $n \geq 1$. Then $a_n \neq 0$, and for $z \neq 0$ we can write $p(z) = z^n\left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right)$. So by the triangle inequality, $|p(z)| \geq \left(|z|^n\left(|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n}\right)\right)$. This implies that we can find $R > 0$ such that $|p(z)| \geq 1$ for $|z| > R'$ (in fact $|p(z)| \to \infty$ as $|z| \to \infty$). Now if $p(z) \neq 0$ for all z, then $g(z) = \frac{1}{p(z)}$ is entire. By the above, $|g(z)| \leq 1$ for $|z| > R$. By continuity of g, we also have that $|g(z)|$ bounded from above on the compact set $\{|z| \leq R\}$. Thus g is a bounded entire function, so by Liouville's theorem g is constant. Since p is non-constant, this is impossible. So p must have a zero. □

Theorem 2.3.5 (Local maximum modulus principle)