

IB Complex Analysis

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§1 Basic notions

Recall definitions in \mathbb{C} from IA courses. Note that $d(z, w) = |z - w|$ defines a metric on \mathbb{C} (the standard metric). For $a \in \mathbb{C}$ and $r > 0$, we write $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ for the open ball with centre a and radius r .

Definition 1.1 (Open subset of \mathbb{C})

A subset $U \subset \mathbb{C}$ is open wrt. the standard metric if for all $a \in U$, there exists an r such that $D(a, r) \subset U$.

Remark. This is equivalent to being open wrt. the Euclidean metric on \mathbb{R}^2 .

This course is about complex valued functions of a single variable, i.e functions

$$f : A \rightarrow \mathbb{C}, \quad \text{where } A \subset \mathbb{C}.$$

Identifying \mathbb{C} with \mathbb{R}^2 in the usual way, we can write $f = u + iv$ for real functions u, v and thus define $u = \operatorname{Re}(f)$, $v = \operatorname{Im}(f)$. Almost exclusively we'll focus on differentiable functions f . But first let's recall continuity.

Definition 1.2 (Continuous function on \mathbb{C})

The function f (as above) is continuous at a point $w \in A$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall z \in A, \quad |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Remark. This is equivalent to saying that $\lim_{z \rightarrow w} f(z) = f(w)$.

§1.1 Complex differentiation

Let $f : U \rightarrow \mathbb{C}$, where U is open.

Definition 1.3 (Differentiability)

f is differentiable at $w \in U$ if the limit

$$f'(w) = \lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w}.$$

exists at a complex number.

Definition 1.4 (Holomorphic function)

f is holomorphic at $w \in U$ if there is $\varepsilon > 0$ such that $D(w, \varepsilon) \subset U$ and f is differentiable at every point in $D(w, \varepsilon)$.

Equivalently, f is holomorphic in U if f is holomorphic at every point in U , or equivalently, f is differentiable at every point in U .

Remark. Sometimes we use "analytic" to mean holomorphic.

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (can easily be checked).

Proposition 1.5

The chain rule for composite functions also holds: if $f : U \rightarrow \mathbb{C}$, $g : V \rightarrow \mathbb{C}$ with $f(U) \subset V$, and $h = g \circ f : U \rightarrow \mathbb{C}$. If f is differentiable at $w \in U$ and g is differentiable at $f(w)$, then h is differentiable at w with

$$h'(w) = (g \circ f)'(w) = f'(w)(g' \circ f)(w).$$

Proof. Omitted; analagous to the proof for the real case. \square

We might ask ourselves a question:

Write $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$. Is differentiability of f at a point $w = c + id \in U$ the same as differentiability of u and v at (c, d) ?

Recall from IB Analysis & Topology that $u : U \rightarrow \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a "good linear approximation of u at (c, d) ". We can show that if u is differentiable at c, d then L (the derivative of u at (c, d)) is uniquely defined, and we write $L = Du(c, d)$; moreover, L is given by the partial derivatives of u , i.e

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d) \right) x + \left(\frac{\partial u}{\partial y}(c, d) \right) y.$$

The answer to the above question is **no** (otherwise complex analysis would be useless!). Now we want to characterise differentiability of f in terms of u and v .

Theorem 1.6 (Cauchy-Riemann equations)

This theorem states that $f = u + iv : U \rightarrow \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if

u, v are differentiable at $(c, d) \in U$ **and** u, v satisfy the Cauchy-Riemann equations at (c, d) :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i \frac{\partial v}{\partial x}(c, d).$$

There are three other such expressions following from the Cauchy-Riemann equations.

Proof. f is differentiable at w with derivative $f'(w) = p + iq$:

$$\begin{aligned} &\iff \lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w} = p + iq \\ &\iff \lim_{z \rightarrow w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0 \end{aligned}$$

Writing $f = u + iv$ and separating real and imaginary parts, the above holds if and

only if

$$\lim_{(x,y) \rightarrow (c,d)} \frac{u(x,y) - u(c,d) - p(x-c) + q(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

and

$$\lim_{(x,y) \rightarrow (c,d)} \frac{v(x,y) - v(c,d) - q(x-c) + p(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

This is precisely the statement that u is differentiable at (c, d) with $Du(c, d)(x, y) = px - qy$, and v is differentiable at (c, d) with $Dv(c, d)(x, y) = qx + py$.

So $\iff u, v$ are differentiable at (c, d) and $u_x(c, d) = p = v_y(c, d)$, $u_y(c, d) = -q = -v_x(c, d)$, i.e the Cauchy-Riemann equations hold at (c, d) .

We also get from the above that if f is differentiable at w , then $f'(w) = p + iq = u_x(c, d) + iv_x(c, d)$. \square

Remark. u, v satisfying the Cauchy-Riemann equations at a point does **not** guarantee differentiability of f on its own. We also can proceed in a more simple way if we simply want to show the reverse implication, by writing

$$f'(w) = \lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w} = \lim_{h \rightarrow 0} \frac{f(w+h) - f(w)}{h},$$

and then choosing $h = t \in \mathbb{R}$ and $h = it$ (since we can choose any direction we want for h) in order to get that u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations.

Example 1.7 (Differentiability (?) of conjugation map)

Let $f(z) = \bar{z} = x - iy$. For this, $u = x, v = -y$, so $u_x = 1, v_y = -1$ and so the C-R equations are not satisfied and f is not differentiable at any point.

Corollary 1.8

Let $f = u + iv : U \rightarrow \mathbb{C}$. If u, v have continuous partial derivatives at $(c, d) \in U$ and satisfy the C-R equations there, then f is differentiable at $w = c + id$.

In particular, if u, v are \mathbb{C}^1 functions on U (i.e have continuous partial derivatives in U) satisfying the C-R equations in U , then f is holomorphic in U .

Proof. Continuity of partial derivatives of u implies that u is differentiable, and similarly for v (IB Analysis & Topology). So the corollary follows from the C-R theorem. \square

Complex differentiability is much more restrictive than real differentiability of real and imaginary parts (because of the additional requirement that the Cauchy-Riemann equations must hold). This leads to surprising theorems compared to the real case, for example

- (a) If $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and bounded, then f is constant! (Liouville's theorem). This is false for real functions; for example $\sin(x) : \mathbb{R} \rightarrow \mathbb{R}$.
- (b) If $f : U \rightarrow \mathbb{C}$ is holomorphic, then f is infinitely differentiable on U .

We will prove these later on. Note that (b) implies that partial derivatives of u, v of all orders exists. So we can differentiate the Cauchy-Riemann equations to get:

$$\begin{aligned}(u_x)_x &= (v_y)_x \implies u_{xx} = v_{yx} \text{ and} \\ (u_y)_y &= (-v_x)_y \implies u_{yy} = -v_{xy}.\end{aligned}$$

This gives $\nabla^2 u = u_{xx} + u_{yy} = 0$ in U . Similarly $\nabla^2 v = 0$ in U .

This means that real and imaginary parts of a holomorphic function are harmonic. This gives a deep connection between harmonic functions and complex analysis; some theorems can be viewed as giving results about harmonic functions.

Now we need some definitions before the next corollary.

- Definition 1.9**
1. A curve is a continuous map $\gamma : [a, b] \rightarrow \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a closed interval. We say γ is a C^1 curve if γ' exists and is continuous on $[a, b]$.
 2. An open set $U \subset \mathbb{C}$ is path-connected if for any two points $z, w \in U$, there is a curve $\gamma : [0, 1] \rightarrow U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.
 3. A domain is a non-empty, open, path-connected subset of \mathbb{C} .

Corollary 1.10

If $U \subset \mathbb{C}$ is a domain and $f : U \rightarrow \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant.

Proof. Write $f = u + iv$. By the C-R equations, $f' = 0 \implies Du = Dv = 0$ in U . Since U is a domain, this means (IA Analysis and Topology) that u and v are constant, i.e f is constant. \square

Now we want to look at some examples of holomorphic functions other than polynomials on \mathbb{C} and rational functions on their domains. We now look at power series, which will give us a wealth of examples.

§1.2 Power series

Recall the next theorem from IA Analysis:

Definition 1.11 (Radius of convergence)

If $(c_n)_{n=0}^\infty$ is a sequence of complex numbers, then there is a unique number $R \in [0, \infty]$ such that the power series

$$\sum_{n=0}^{\infty} c_n (z - a)^n \quad z, a \in \mathbb{C}.$$

converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly wrt z on the compact disk $D_r = \{z \in \mathbb{C} : |z - a| < r\}$.

We call R the radius of convergence of the power series. Note that when $|z - a| = R$, we cannot say anything in general about convergence.

Theorem 1.12

Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R > 0$. Fix $a \in \mathbb{C}$, and define $f : D(a, R) \rightarrow \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then

1. f is holomorphic on $D(a, R)$.
2. The derived series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has radius of convergence R , and

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1} \quad \forall z \in D(a, R).$$

3. f has derivatives of all orders on $D(a, R)$, and $c_n = \frac{f^{(n)}(a)}{n!}$.
4. If f vanishes on $D(a, \varepsilon)$ for some $\varepsilon > 0$, then $f \equiv 0$ on $D(a, R)$.

Proof. Parts (i) and (ii): By considering $g(z) = f(z+a)$, we assume WLOG that $a = 0$. So $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in D(0, R)$.

The derived series $\sum_{n=1}^{\infty} nc_n z^{n-1}$ will have some radius of convergence $R_1 \in [0, \infty]$. Now let $z \in D(0, R)$ be arbitrary. Choose ρ such that $|z| < \rho < R$. Then since $n|\frac{z}{\rho}|^{n-1} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$n|c_n||z|^{n-1} = n|c_n||\frac{z}{\rho}|^{n-1}\rho^{n-1} \leq |c_n|\rho^{n-1}.$$

for sufficiently large n . Since $\sum |c_n|\rho^n$ converges, it follows that $\sum_{n=1}^{\infty} nc_n z^{n-1}$ converges. Thus $D(0, R) \subset D(0, R_1)$, i.e. $R_1 \geq R$. Since

$$|c_n||z|^n \leq n|c_n||z|^{n-1} = |z|(|c_n||z|^{n-1}),$$

if $\sum n|c_n||z|^{n-1}$ converges then so does $\sum |c_n||z|^n$, so $R_1 \leq R$. So $R_1 = R$.

To prove that f is differentiable with $f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$, fix $z \in D(0, R)$. The key idea is that this is equivalent to continuity at z of the function

$$g : D(0, R) \rightarrow \mathbb{C}, \quad g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ \sum_{n=1}^{\infty} nc_n z^{n-1} & w = z. \end{cases}$$

By subbing in f we can write $g(w) = \sum_{n=1}^{\infty} h_n(w)$ where

$$h_n(w) = \begin{cases} \frac{c_n(w^n - z^n)}{w-z} & w \neq z \\ nc_n z^{n-1} & w = z. \end{cases}$$

Now h_n is continuous on $D(0, R)$ (since $w \rightarrow w^n$ is differentiable with derivative nw^{n-1}). Using $\frac{w^n - z^n}{w-z} = \sum_{j=0}^{n-1} z^j w^{n-1-j}$, we get that for any r with $|z| < r < R$ and any $w \in D(0, r)$, $|h_n(w)| \leq n|c_n|r^{n-1} \equiv M_n$. Since $\sum M_n < \infty$, by the Weierstrass M-test that $\sum h_n$ converges uniformly on $D(0, r)$. But a uniform limit of continuous functions is continuous, so $g = \sum h_n$ is continuous in $D(0, r)$ and in particular at z .

Part (iii): Repeatedly apply (ii). The formula $c_n = \frac{f^{(n)}(a)}{n!}$ follows by differentiating the series n times and setting $z = a$.

Part (iv): If $f = 0$ in $D(a, \varepsilon)$, then $f^{(n)}(a) = 0$ for all n , so $c_n = 0$ for all n and hence $f = 0$ in $D(a, R)$. \square

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on all of \mathbb{C} , we say f is entire.

Proposition 1.13

The complex exponential function is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

- (i) e^z is entire, with $(e^z)' = e^z$.
- (ii) $e^z \neq 0$ and $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.
- (iii) $e^{x+iy} = e^x(\cos(y) + i\sin(y))$ for $x, y \in \mathbb{R}$.
- (iv) $e^z = 1$ iff $z = 2n\pi i$ for some $n \in \mathbb{Z}$.
- (v) Let $z \in \mathbb{C}$. There exists $w \in \mathbb{C}$ such that $e^w = z$ iff $z \neq 0$.

Proof. **Part (i):** The r.o.c of the series is ∞ . To see $(e^z)' = e^z$, differentiate the series term by term using the previous theorem.

Part (ii): Fix any $w \in \mathbb{C}$ and set $F(z) = e^{z+w}e^{-z}$. Then $F'(z) = -e^{z+w}e^{-z} + e^{z+w}e^{-z} = 0$, so $F(z)$ is a constant. Thus $F(z) = F(0) = e^w$ for all $z \in \mathbb{C}$. Thus

$$e^{z+w}e^{-z} = e^w \quad \forall z, w \in \mathbb{C}.$$

Taking $w = 0$, $e^z e^{-z} = 1$. So $e^z \neq 0$. Multiplying by e^z , we get $e^{z+w} = e^z e^w$.

Part (iii): $e^{x+iy} = e^x e^{iy}$ by (ii). Now use the definition of e^{iy} , and the series for $\sin(y)$, $\cos(y)$ for $y \in \mathbb{R}$.

Part (iv) and (v): Follow from (iii). (Exercise) \square

Definition 1.14 (Logarithm)

Given $z \in \mathbb{C}$, we say a complex $w \in \mathbb{C}$ is a **logarithm** of z if $e^w = z$.

By Proposition 1.13(v), z has a logarithm iff $z \neq 0$. By (ii) and (iv), if $z \neq 0$ then z has infinitely many logarithms, with any two differing from each other by $2n\pi i$ for some integer n .

If w is a logarithm of z , then $e^{\operatorname{Re}(w)} = |z|$, so $\operatorname{Re}(w) = \log |z|$ (the real logarithm of the positive number $|z|$); in particular, this is well-defined.

Definition 1.15 (Branch of a logarithm)

Let $U \subset \mathbb{C} \setminus \{0\}$ be open. Then a branch of logarithm on U is a continuous function $\lambda : U \rightarrow \mathbb{C}$ such that $e^{\lambda(z)} = z$ for each $z \in U$.

Proposition 1.16

If λ is a branch of log on U then λ is automatically holomorphic in U , with $\lambda'(z) = \frac{1}{z}$.

Proof. If $w \in U$ then

$$\begin{aligned} \lim_{z \rightarrow w} \frac{\lambda(z) - \lambda(w)}{z - w} &= \lim_{z \rightarrow w} \frac{1}{\frac{e^{\lambda(z)} - e^{\lambda(w)}}{\lambda(z) - \lambda(w)}} \\ &= \frac{1}{e^{\lambda(w)}} \lim_{z \rightarrow w} \frac{1}{\left(\frac{e^{\lambda(z) - \lambda(w)} - 1}{\lambda(z) - \lambda(w)} \right)} \\ &= \frac{1}{e^{\lambda(w)}} \lim_{h \rightarrow 0} \frac{1}{\left(\frac{e^h - 1}{h} \right)} \quad \text{since } \lambda \text{ is continuous} \\ &= \frac{1}{e^{\lambda(w)}} = \frac{1}{w}. \end{aligned}$$

□

Definition 1.17 (Principal branch of logarithm)

The principal branch of logarithm is the function

$$\text{Log} : U_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\} \rightarrow \mathbb{C}.$$

defined by

$$\text{Log}(z) = \log |z| + i \arg(z).$$

where $\arg(z)$ is the unique argument of $z \in U_1$ in $(-\pi, \pi)$.

Remark. do later, why it's a branch of log

Proposition 1.18 (i) Log is holomorphic on U_1 with $\text{Log}'(z) = \frac{1}{z}$.

(ii) For $|z| < 1$, we have

$$\text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof. (i) is the remark above.

(ii) To see this, note that the R.O.C of the series is 1, and $|z| < 1 \implies 1+z \in U_1$, so both sides are defined on $|z| < 1$.

Let $F(z) = \text{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$ for $|z| < 1$. Then

$$F'(z) = \frac{1}{1+z} - \sum_{n=1}^{\infty} (-z)^{n-1} = 0.$$

So $F(z) = \text{constant} = F(0) = 0$.

□

Using exp and Log we can define further useful functions.

1. For any $\alpha \in \mathbb{C}$, define

$$z^\alpha = e^{\alpha \operatorname{Log}(z)}, \quad z \in U_1.$$

This is the principal branch of z^α . It's holomorphic on U_1 with derivative $\alpha z^{\alpha-1}$.

2. We can define the familiar functions

- $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
- $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$
- $\cosh(z) = \frac{e^z + e^{-z}}{2}$
- $\sinh(z) = \frac{e^z - e^{-z}}{2}$

These are all entire since exp is entire, with derivatives given by the familiar expressions from real variables.

§1.3 Conformality

Let $f : U \rightarrow \mathbb{C}$ be holomorphic ($U \subset \mathbb{C}$ is open). Let $w \in U$ and suppose that $f'(w) \neq 0$. Take two C^1 curves $\gamma_1, \gamma_2 : [-1, 1] \rightarrow U$ such that $\gamma_1(0) = \gamma_2(0) = w$ and with nonzero derivative. Then $f \circ \gamma_i$ are C^1 curves passing through $f(w)$. Moreover, $(f \circ \gamma_i)'(0) = f'(w)\gamma_i'(0) \neq 0$. Thus

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}.$$

Hence

$$\arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0).$$

This means that the angle that the curves γ_1, γ_2 make at w is the same as the angle their images make at $f(w)$. We say f is 'angle-preserving at w ', whenever $f'(w) \neq 0$.

Remark. If f is a C^1 map on U , the converse of this also holds. See Example Sheet 1.

Definition 1.19 (Conformal map)

A holomorphic function $f : U \rightarrow \mathbb{C}$ on an open set U is said to be **conformal** at a point $w \in U$ if $f'(w) \neq 0$.

Definition 1.20 (Conformal equivalence)

Let U, \tilde{U} be domains in \mathbb{C} . A map $f : U \rightarrow \tilde{U}$ is said to be a conformal equivalence between U and \tilde{U} if f is a bijective holomorphic map with $f'(z) \neq 0$ for every $z \in U$.

Remark. • If f is holomorphic and injective, then $f'(z) \neq 0$ for each z . We will prove this later. So in the above definition $f'(z) \neq 0$ is redundant.

- It is automatic that the inverse $f^{-1} : \tilde{U} \rightarrow U$ is holomorphic. This can be proved using the holomorphic inverse function theorem, which you will prove on Example Sheet 1.

Example 1.21

Let's look at some examples of conformal equivalence.

1. Möbius maps are defined for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (see IA Groups):

$$f(z) = \frac{az + b}{cz + d}.$$