IB Complex Analysis

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February 6, 2022

These are my notes for the IB course Complex Analysis, which was lectured in Lent 2022 at Cambridge by Prof. N.Wickramasekera. These notes are written in LaTeX for my own revision purposes. Any suggestions or feedback is welcome.

Contents

§1 Basic notions

Recall definitions in $\mathbb{C}$ from IA courses. Note that $d(z,w) = |z-w|$ defines a metric on $\mathbb{C}$ (the standard metric). For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ for the open ball with centre a and radius r.

Definition (Open subset of $\mathbb{C}$)

A subset $U \subset \mathbb{C}$ is open wrt. the standard metric if for all $a \in U$, there exists an r such that $D(a,r) \in U$.

Remark. This is equivalent to being open wrt. the Euclidean metric on $\mathbb{R}^2$.

This course is about complex valued functions of a single variable, i.e functions

$$f : A \to \mathbb{C}, \quad \text{where } A \subset \mathbb{C}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, we can write $f = u + iv$ for real functions u, v and thus define $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$. Almost exclusively we'll focus on differentiable functions f. But first let's recall continuity.

Definition (Continuous function on $\mathbb{C}$)

The function f (as above) is continuous at a point $w \in A$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall z \in A, \quad |z-w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Remark. This is equivalent to saying that $\lim_{z \to w} f(z) = f(w)$.

§1.1 Complex differentiation

Let $f : U \to \mathbb{C}$, where U is open.

Definition (Differentiability)

f is differentiable at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}.$$

exists at a complex number.

Definition (Holomorphic function)

f is **holomorphic**[1]at $w \in U$ if there is $\varepsilon > 0$ such that $D(w,\varepsilon) \subset U$ and f is differentiable at every point in $D(w,\varepsilon)$.

Equivalently, f is holomorphic in U if f is holomorphic at every point in U, or equivalently, f is differentiable at every point in U.

[1]Sometimes we use ”analytic” to mean holomorphic.

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (can easily be checked).

Proposition 1.1.1

The chain rule for composite functions also holds: if $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ with $f(U) \subset V$, and $h = g \circ f : U \to \mathbb{C}$. If f is differentiable at $w \in U$ and g is differentiable at $f(w)$, then h is differentiable at w with

$$h'(w) = (g \circ f)'(w) = f'(w)(g' \circ f)(w).$$

Proof. Omitted; analagous to the proof for the real case. □

We might ask ourselves a question:

Write $f(z) = u(x, y)+iv(x, y)$, $z = x+iy$. Is differentiability of f at a point $w = c+id \in U$ is the same as differentiability of u and v at (c, d)?

Recall from IB Analysis & Topology that $u : U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a "good linear approximation of u at (c, d)". We can show that if u is differentiable at c, d then L (the derivative of u at (c, d)) is uniquely defined, and we write $L = Du(c, d)$; moreover, L is given by the partial derivatives of u, i.e

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d)\right) x + \left(\frac{\partial u}{\partial y}(c, d)\right) y.$$

The answer to the above question is **no** (otherwise complex analysis would be useless!). Now we want to characterise differentiability of f in terms of u and v.

Theorem 1.1.2 (Cauchy-Riemann equations)

This theorem states that $f = u + iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if

u, v are differentiable at $(c, d) \in U$ **and** u, v satisfy the Cauchy-Riemann equations at (c, d):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d).$$

There are three other such expressions following from the Cauchy-Riemann equations.

Proof. f is differentiable at w with derivative $f'(w) = p + iq$:

$$\begin{aligned}
&\iff \lim_{z\to w} \frac{f(z) - f(w)}{z - w} = p + iq \\
&\iff \lim_{z\to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0
\end{aligned}$$

Writing $f = u + iv$ and separating real and imaginary parts, the above holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - p(x-c) + q(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

$$\text{and} \quad \lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - q(x-c) + p(y-d)}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

This is precisely the statement that u is differentiable at (c,d) with $Du(c,d)(x,y) = px - qy$, and v is differentiable at (c,d) with $Dv(c,d)(x,y) = qx + py$.

So $\iff$ u, v are differentiable at (c,d) and $u_x(c,d) = p = v_y(c,d)$, $u_y(c,d) = -q = -v_x(c,d)$, i.e the Cauchy-Riemann equations hold at (c,d).

We also get from the above that if f is differentiable at w, then $f'(w) = p + iq = u_x(c,d) + iv_x(c,d)$. □

Remark. u, v satisfying the Cauchy-Riemann equations at a point does **not** guarantee differentiability of f on its own. We also can proceed in a more simple way if we simply want to show the reverse implication, by writing

$$f'(w) = \lim_{z\to w} \frac{f(z) - f(w)}{z - w} = \lim_{h\to 0} \frac{f(w+h) - f(w)}{h},$$

and then choosing $h = t \in \mathbb{R}$ and $h = it$ (since we can choose any direction we want for h) in order to get that u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations.

Example (Differentiability (?) of conjugation map)

Let $f(z) = \bar{z} = x - iy$. For this, $u = x, v = -1$, so $u_x = 1, v_y = -1$ and so the C-R equations are not satisfied and f is not differentiable at any point.

Corollary 1.1.3

Let $f = u + iv : U \to \mathbb{C}$. If u, v have continuous partial derivatives at $(c,d) \in U$ and satisfy the C-R equations there, then f is differentiable at $w = c + id$.

In particular, if u, v are $\mathbb{C}^1$ functions on U (i.e have continuous partial derivatives in U) satisfying the C-R equations in U, then f is holomorphic in U.

Proof. Continuity of partial derivatives of u implies that u is differentiable, and similarly for v (IB Analysis & Topology). So the corollary follows from the C-R theorem. □

Complex differentiability is much more restrictive than real differentiability of real and imaginary parts (because of the additional requirement that the Cauchy-Riemann equations must hold). This leads to surprising theorems compared to the real case, for example

(a) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant! (Liouville's theorem). This is false for real functions; for example $\sin(x) : \mathbb{R} \to \mathbb{R}$.

(b) If $f : U \to \mathbb{C}$ is holomorphic, then f is infinitely differentiable on U.

We will prove these later on. Note that (b) implies that partial derivatives of u, v of all orders exists. So we can differentiate the Cauchy-Riemann equations to get:

$$(u_x)_x = (v_y)_x \implies u_{xx} = v_{yx} \text{ and}$$
$$(u_y)_y = (-v_x)_y \implies u_{yy} = -v_{xy}.$$

This gives $\nabla^2 u = u_{xx} + u_{yy} = 0$ in U. Similarly $\nabla^2 v = 0$ in U.

This means that real and imaginary parts of a holomorphic function are harmonic. This gives a deep connection between harmonic functions and complex analysis; some theorems can be viewed as giving results about harmonic functions.

Now we need some definitions before the next corollary.

Definition

1. A curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a closed interval. We say γ is a C^1 curve if γ' exists and is continuous on $[a, b]$.
2. An open set $U \subset \mathbb{C}$ is path-connected if for any two points $z, w \in U$, there is a curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.
3. A domain is a non-empty, open, path-connected subset of $\mathbb{C}$.

Corollary 1.1.4

If $U \in C$ is a domain and $f : U \to \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant.

Proof. Write $f = u + iv$. By the C-R equations, $f' = 0 \implies Du = Dv = 0$ in U. Since U is a domain, this means (IA Analysis and Topology) that u and v are constant, i.e f is constant. □

Now we want to look at some examples of holomorphic functions other than polynomials on $\mathbb{C}$ and rational functions on their domains. We now look at power series, which will give us a wealth of examples.

§1.2 Power series

Recall the next theorem from IA Analysis:

Definition (Radius of convergence)

If $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers, then there is a unique number $R \in [0, \infty]$ such that the power series

$$\sum_{n=0}^{\infty} c_n (z - a)^n \qquad z, a \in \mathbb{C}.$$

converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly wrt z on the compact disk $D_r = \{z \in \mathbb{C} : \ |z - a| < R\}$.

We call R the radius of convergence of the power series. Note that when $z = R$, we cannot say anything in general about convergence.

Theorem 1.2.1

Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R > 0$. Fix $a \in \mathbb{C}$, and define $f : D(a, R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then

1. f is holomorphic on $D(a, R)$.
2. The derived series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has radius of convergence R, and
$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1} \quad \forall z \in D(a, R).$$
3. f has derivatives of all orders on $D(a, R)$, and $c_n = \frac{f^{(n)}(a)}{n!}$.
4. If f vanishes on $D(a, \varepsilon)$ for some $\varepsilon > 0$, then $f \equiv 0$ on $D(a, R)$.

Proof. Parts (i) and (ii): By considering $g(z) = f(z+a)$, we assume WLOG that $a = 0$. So $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in D(0, R)$.

The derived series $\sum_{n=1}^{\infty} nc_n z^{n-1}$ will have some radius of convergence $R_1 \in [0, \infty]$. Now let $z \in D(0, R)$ be arbitrary. Choose ρ such that $|z| < \rho < R$. Then since $n|\frac{z}{\rho}|^{n-1} \to 0$ as $n \to \infty$, we have

$$n|c_n||z|^{n-1} = n|c_n||\frac{z}{\rho}|^{n-1}\rho^{n-1} \le |c_n|p^{n-1}.$$

for sufficiently large n. Since $\sum |c_n|\rho^n$ converges, it follows that $\sum_{n=1}^{\infty} nc_n z^{n-1}$ converges. Thus $D(0, R) \subset D(0, R_1)$, i.e $R_1 \ge R$. Since

$$|c_n||z|^n \le n|c_n||z|^n = |z| \left(|c_n||z|^{n-1}\right),$$

if $\sum n|c_n||z|^{n-1}$ converges then so does $\sum |c_n||z|^n$, so $R_1 \le R$. So $R_1 = R$.

To prove that f is differentiable with $f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$, fix $z \in D(0, R)$. The key idea is that this is equivalent to continuity at z of the function

$$g : D(0, R) \to \mathbb{C}, \quad g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \ne z \\ \sum_{n=1}^{\infty} nc_n z^{n-1} & w = z. \end{cases}$$

By subbing in f we can write $g(w) = \sum_{n=1}^{\infty} h_n(w)$ where

$$h_n(w) = \begin{cases} \frac{c_n(w^n - z^n)}{w-z} & w \ne z \\ nc_n z^{n-1} & w = z. \end{cases}$$

Now h_n is continuous on $D(0, R)$ (since $w \to w^n$ is differentiable with derivative nw^{n-1}). Using $\frac{w^n - z^n}{w-z} = \sum_{j=0}^{n-1} z^j w^{n-1-j}$, we get that for any r with $|z| < r < R$ and any $w \in D(0, r)$, $|h_n(w)| \le n|c_n|r^{n-1} \equiv M_n$. Since $\sum M_n < \infty$, by the Weierstrass

M-test that $\sum h_n$ converges uniformly on $D(0, r)$. But a uniform limit of continuous functions is continuous, so $g = \sum h_n$ is continuous in $D(0, r)$ and in particular at z.

Part (iii): Repeatedly apply (ii). The formula $c_n = \frac{f^{(n)}(a)}{n!}$ follows by differentiating the series n times and setting $z = a$.

Part (iv): If $f = 0$ in $D(a, \varepsilon)$, then $f^{(n)}(a) = 0$ for all n, so $c_n = 0$ for all n and hence $f = 0$ in $D(a, R)$. □

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, we say f is entire.

Proposition 1.2.2

The complex exponential function is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(i) e^z is entire, with $(e^z)' = e^z$.

(ii) $e^z \neq 0$ and $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

(iii) $e^{x+iy} = e^x(\cos(y) + isin(y))$ for $x, y \in \mathbb{R}$.

(iv) $e^z = 1$ iff $z = 2n\pi i$ for some $n \in \mathbb{Z}$.

(v) Let $z \in \mathbb{C}$. There exists $w \in \mathbb{C}$ such that $e^w = z$ iff $z \neq 0$.

Proof. Part (i): The r.o.c of the series is ∞. To see ($e^z = z'$), differentiate the series term by term using the previous theorem.

Part (ii): Fix any $w \in \mathbb{C}$ and set $F(z) = e^{z+w}e^{-z}$. Then $F'(z) = -e^{z+w}e^{-z} + e^{z+w}e^{-z} = 0$, so $F(z)$ is a constant. Thus $F(z) = F(0) = e^w$ for all $z \in \mathbb{C}$. Thus

$$e^{z+w}e^{-z} = e^w \quad \forall z, w \in \mathbb{C}.$$

Taking $w = 0$, $e^z e^{-z} = 1$. So $e^z \neq 0$. Multiplying by e^z, we get $e^{z+w} = e^z e^w$.

Part (iii): $e^{x+iy} = e^x e^{iy}$ by (ii). Now use the definition of e^{iy}, and the series for $\sin(y), \cos(y)$ for $y \in \mathbb{R}$.

Part (iv) and (v): Follow from (iii). (Exercise) □

Definition (Logarithm)

Given $z \in \mathbb{C}$, we say a complex $w \in \mathbb{C}$ is a **logarithm** of z if $e^w = z$.

By Proposition 1.2.2(v), z has a logarithm iff $z \neq 0$. By (ii) and (iv), if $z \neq 0$ then z has infinitely many logarithms, with any two differing from each other by $2n\pi i$ for some integer n.

If w is a logarithm of z, then $e^{\mathrm{Re}(w)} = |z|$, so $\mathrm{Re}(w) = \log|z|$ (the real logarithm of the positive number $|z|$); in particular, this is well-defined.

Definition (Branch of a logarithm)

Let $U \subset \mathbb{C} \setminus \{0\}$ be open. Then a branch of logarithm on U is a continuous function $\lambda : U \to \mathbb{C}$ such that $e^{\lambda(z)} = z$ for each $z \in U$.

Proposition 1.2.3

If λ is a branch of log on U then λ is automatically holomorphic in U, with $\lambda'(z) = \frac{1}{z}$.

Proof. If $w \in U$ then

$$\begin{aligned}\lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{z - w} &= \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z)} - e^{\lambda(w)}}{\lambda(z) - \lambda(w)}\right)} \\ &= \frac{1}{e^{\lambda(w)}} \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z) - \lambda(w)} - 1}{\lambda(z) - \lambda(w)}\right)} \\ &= \frac{1}{e^{\lambda(w)}} \lim_{h \to 0} \frac{1}{\left(\frac{e^h - 1}{h}\right)} \qquad \text{since } \lambda \text{ is continuous} \\ &= \frac{1}{e^{\lambda(w)}} = \frac{1}{w}.\end{aligned}$$

□

Definition (Principal branch of logarithm)

The principal branch of logarithm is the function

$$\mathrm{Log} : U_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : \ x \leq 0\} \to \mathbb{C}.$$

defined by

$$\mathrm{Log}(z) = \log|z| + i \arg(z).$$

where $\arg(z)$ is the unique argument of $z \in U_1$ in $(-\pi, \pi)$.

Remark. Log is a branch of logarithm in U_1 : to check continuity of Log, note that $z \mapsto \log|z|$ is continuous on $\mathbb{C}\setminus\{0\}$ (by continuity of $z \mapsto |z|$ and continuity of $r \mapsto \log r$ for $r > 0$); also, $z \mapsto \arg(z)$ is continuous, since $\theta \mapsto e^{i\theta}$ is a homeomorphism $(-\pi, \pi) \to \mathbb{S}^1 \setminus \{-1\}$ (as can be checked directly, where $\mathbb{S}^1 = \{z : |z| = 1\}$), and $z \mapsto \frac{z}{|z|}$ is continuous on $\mathbb{C}\setminus\{0\}$. So $z \mapsto \log(z)$ is continuous on U_1. We also have

$$e^{\mathrm{Log}(z)} = e^{\ln|z| + i\arg(z)} = e^{\ln|z|} \cdot e^{i \ \arg\ (z)} = |z|(\cos\arg(z) + i \sin\arg(z)) = z.$$

So Log is a branch of logarithm in U_1.

Remark. Log does not have a continuous extension to $\mathbb{C}\setminus\{0\}$ since $\arg(z) \to \pi$ as $z \to -1$ with $\mathrm{Im}(z) > 0$, and $\arg(z) \to -\pi$ as $z \to -1$ with $\mathrm{Im}(z) < 0$.

Proposition 1.2.4 (i) Log is holomorphic on U_1 with $\mathrm{Log}'(z) = \frac{1}{z}$.

(ii) For $|z| < 1$, we have

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof. (i) is the remark above.

(ii) To see this, note that the R.O.C of the series is 1, and $|z| < 1 \implies 1+z \in U_1$, so both sides are defined on $|z| < 1$.

Let $F(z) = \operatorname{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$ for $|z| < 1$. Then

$$F'(z) = \frac{1}{1+z} - \sum_{n=1}^{\infty} (-z)^{n-1} = 0.$$

So $F(z) = \text{ constant } = F(0) = 0$.

$\square$

Using exp and Log we can define further useful functions.

1. For any $\alpha \in \mathbb{C}$, define
$$z^\alpha = e^{\alpha \operatorname{Log}(z)}, \quad z \in U_1.$$
This is the principal branch of z^α. It's holomorphic on U_1 with derivative $\alpha z^{\alpha-1}$.

2. We can define the familiar functions
   - $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
   - $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$
   - $\cosh(z) = \frac{e^z+e^{-z}}{2}$
   - $\sinh(z) = \frac{e^z-e^{-z}}{2}$

   These are all entire since exp is entire, with derivatives given by the familiar expressions from real variables.

§1.3 Conformality

Let $f : U \to \mathbb{C}$ be holomorphic ($U \subset C$ is open). Let $w \in U$ and suppose that $f'(w) \neq 0$. Take two C^1 curves $\gamma_1, \gamma_2 : [-1,1] \to U$ such that $\gamma_1(0) = \gamma_2(0) = w$ and with nonzero derivative. Then $f \circ \gamma_i$ are C^1 curves passing through $f(w)$. Moreover, $(f \circ \gamma_i)'(0) = f'(w)\gamma_i'(0) \neq 0$. Thus

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}.$$

Hence
$$\arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0).$$

This means that the angle that the curves γ_1, γ_2 make at w is the same as the angle their images make at $f(w)$. We say f is 'angle-preserving at w', whenever $f'(w) \neq 0$.

Remark. If f is a C^1 map on U, the converse of this also holds. See Example Sheet 1.

Definition (Conformal map)

A holomorphic function $f : U \to \mathbb{C}$ on an open set U is said to be **conformal** at a point $w \in U$ if $f'(w) \neq 0$.

Definition (Conformal equivalence)

Let $U, \widetilde{U}$ be domains in $\mathbb{C}$. A map $f : U \to \widetilde{U}$ is said to be a conformal equivalence between U and $\widetilde{U}$ if f is a bijective holomorphic map with $f'(z) \neq 0$ for every $z \in U$.

Remark.

- If f is holomorphic and injective, then $f'(z) \neq 0$ for each z. We will prove this later. So in the above definition $f'(z) \neq 0$ is redundant.
- It is automatic that the inverse $f^{-1} : U \to \widetilde{U}$ is holomorphic. This can be proved using the holomorphic inverse function theorem, which you will prove on Example Sheet 1.

Example

Let's look at some examples of conformal equivalence.

1. Mobius maps are defined for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (see IA Groups):
$$f(z) = \frac{az + b}{cz + d}.$$
Mobius maps sometimes serve as explicit conformal equivalences between subdomains of $\mathbb{C}$. For example, let $\mathbb{H}$ be the open upper half plane. Then
$$\begin{aligned} z \in \mathbb{H} &\iff |z - i| < |z + i| \\ &\iff |\frac{z - i}{z + i}| < 1. \end{aligned}$$
Thus $g(z) = \frac{z-i}{z+i}$ maps $\mathbb{H}$ onto $D(0, 1)$, so g is a conformal equivalence.
2. Consider $f : z \to z^n$ where $n \in \mathbb{N}$ where $f : \left\{z \in C \setminus \{0\} : 0 < \arg(z) < \frac{\pi}{n}\right\} \to \mathbb{H}$. This is a conformal equivalence with inverse $f(z) = z^{1/n}$ (the principal branch).
3. We have that
$$\exp : \{z \in \mathbb{C} : \ -\pi < \operatorname{Im}(z) < \pi\} \to \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}.$$

Aside:

Theorem 1.3.1 (Riemann Mapping Theorem)

Any simply connected domain $U \subset \mathbb{C}$ with $U \neq \mathbb{C}$ is conformally equivalent to $D(0, 1)$.

Proof. This is beyond the scope of the course.[2] □

[2] See Rudin's *Real and Complex Analysis*.

§2 Complex Integration: Part I

We aim to extend Riemann integration to complex functions $f : U \to \mathbb{C}$ along curves in U. First we take a look at complex functions of a real variable.

Definition

If $f : [a, b] \subset \mathbb{R} \to C$ is a complex function and if f is Riemann integrable, define

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \operatorname{Re} f(t)\mathrm{d}t + \int_a^b \operatorname{Im} f(t)\mathrm{d}t.$$

In particular,

$$\int_a^b if(t)\mathrm{d}t = i\int_a^b f(t)\mathrm{d}t.$$

We can then directly calculate for any $w \in \mathbb{C}$ that

$$\int_a^b wf(t)\mathrm{d}t = w\int_a^b f(t)\mathrm{d}t.$$

Proposition 2.0.1

If $f : [a, b] \to \mathbb{C}$ is continuous, then

$$|\int_a^b f(t)\mathrm{d}t| \le \int_a^b |f(t)|\mathrm{d}t \le (b-a) \sup_{t\in[a,b]} |f(t)|,$$

with equality iff f is constant.

Proof. If $\int_a^b f(t)\mathrm{d}t$ then we are done. Else write $\int_a^b f(t)\mathrm{d}t = re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and let $M = \sup_{t\in[a,b]} |f(t)|$. Then

$$\begin{aligned} |\int_a^b f(t)\mathrm{d}t| = r = e^{-i\theta}\int_a^b f(t)\mathrm{d}t &= \int_a^b e^{-i\theta} f(t)\mathrm{d}t \\ &= \int_a^b \operatorname{Re}(e^{-i\theta})f(t)\mathrm{d}t + \int_a^b \operatorname{Im}(e^{-i\theta})f(t)\mathrm{d}t. \end{aligned}$$

Since the LHS is real,

$$|\int_a^b f(t)\mathrm{d}t| = \int_a^b \operatorname{Re} f(t)\mathrm{d}t \le \int_a^b |e^{-i\theta}f(t)|\mathrm{d}t = \int_a^b |f(t)|\mathrm{d}t \le (b-a)M.$$

Equality holds iff $|f(t)| = M$ and $\operatorname{Re}(e^{-i\theta}f(t)) = M$ for all $t \in [a, b]$, i.e iff $|f(t)| = M$ and $\arg f(t) = \theta$ for all t; iff f constant. □

Definition (Integral along a curve)

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to U$ be a C^1 curve.

Then the **integral of f along γ** is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

Proposition 2.0.2 (Basic properties of the integral)

If we have the integral of f along γ, we have the following properties:

1. Invariance under reparametrisation: Let $\varphi : [a_1, b_1] \to [a, b]$ be C^1 and injective with $\varphi(a_1) = a, \varphi(b_1) = b$. Let $\delta = \gamma \circ \varphi : [a_1, b_1] \to U$. Then we have
$$\int_\delta f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z.$$
2. Linearity:
$$\int_\gamma c_1 f_1(z) + c_2 f_2(z)\mathrm{d}z = c_1 \int_\gamma f_1(z)\mathrm{d}z + c_2 \int_\gamma f_2(z)\mathrm{d}z.$$
3. Additivity: If γ is our C^1 curve and $a < c < b$, then
$$\int_\gamma f(z)\mathrm{d}z = \int_{\gamma|_{[a,c]}} f(z)\mathrm{d}z + \int_{\gamma|_{[c,b]}} f(z)\mathrm{d}z.$$
4. Inverse path: Define the inverse path $(-\gamma) : [-b, -a] \to U$ by $(-\gamma)(t) = \gamma(-t)$ for $-b \leq t \leq -a$. Then
$$\int_{(-\gamma)} f(z)dz = -\int_\gamma f(z)\mathrm{d}z$$

Proof. For 1, we have

$$\begin{aligned}\int_\delta f(z)\mathrm{d}z &= \int_{a_1}^{b_1} f(v \circ \varphi(t))\gamma'(\varphi(t))\varphi'(t)\mathrm{d}t \\ &= \int_a^b f(\gamma(s))\gamma'(s)\mathrm{d}s = \int_\gamma f(z)\mathrm{d}z \quad \text{by change of vars. } s = \varphi(t).\end{aligned}$$

2,3, and 4 are all easy to check from the definition. □

Definition (Length of a curve)

Definition: Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve. The length of γ is defined by

$$\text{length}(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Definition

A **piecewise C^1 curve** is a continuous map $\gamma : [a,b] \to \mathbb{C}$ such that there exists a finite subdivision

$$a = a_0 < a_1 < \ldots < a_{n-1} < a_n = b$$

with the property that $\gamma_j = \gamma|_{[a_{j-1},a_j]} : \ [a_{j-1}, a_j] \to \mathbb{C}$ is C^1 for all $1 \leq j \leq n$. We define

$$\int_\gamma f(z) \ \mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z) \ \mathrm{d}z.$$

and

$$\mathrm{length}(\gamma) = \sum_{j=1}^{n} \mathrm{length}(\gamma_j) = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)| \ \mathrm{d}t.$$

Remark. From now on, by a 'curve' we shall mean a piecewise C^1 curve.

Definition

If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are curves with $\gamma_1(b) = \gamma_2(c)$, we define the sum of of γ_1 and γ_2 to be the curve

$$(\gamma_1 + \gamma_2) : [a, b+d-c] \to \mathbb{C},$$

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \leq t \leq b \\ \gamma_2(t-b+c) & b \leq t \leq b+d-c \end{cases}$$

Proposition 2.0.3

For any continuous function $f : U \to \mathbb{C}$ and any curve $\gamma : [a,b] \to \mathbb{C}$, we have that

$$\left| \int_\gamma f(z) dz \right| \leq \ \mathrm{length}\ (\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| = \sup_{t\in[a,b]} |f(\gamma(t))|$.

Proof. If γ is C^1, then

$$\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \leq \int_a^b \left| f(\gamma(t)) \| \gamma'(t) \right| dt \leq \sup_{t\in[a,b]} |f(\gamma(t))| \, \mathrm{length}(\gamma).$$

If γ is piecewise C^1 then the result follows from the definition $\int_\gamma f(z)dz = \sum_{j=1}^n \int_{\gamma_j} f(z)dz$ where γ_j is C^1, and the triangle inequality. $\square$

We can now look at the complex version of the FTC.

Theorem 2.0.4 (Fundamental Theorem of Calculus)

Suppose that $f : U \to \mathbb{C}$ is continuous, $U \subset \mathbb{C}$ open. If there is a function $F : U \to \mathbb{C}$

such that $F'(z) = f(z)$ for all $z \in U$, then for any curve $\gamma : [a, b] \to U$,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

If additionally γ is a closed curve, i.e. $\gamma(b) = \gamma(a)$, then $\int_\gamma f(z)dz = 0$.

Proof. This follows immediately:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a)).$$

$\square$

Remark. Such F as in Theorem 2.3 is called an **anti-derivative** of f.

We shall see later (by infinite differentiability of holomorphic functions) that if $F'(z) = f(z)$, then f is automatically continuous.

Example

Let $\int_\gamma z^n \mathrm{~d}z$ for $n \in \mathbb{Z}$, where $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = Re^{2\pi it}$ for some $R > 0$. (The image of γ is the circle of radius R centred at 0).

For $n \neq 1$, $\frac{z^{n+1}}{n+1}$ is an antiderivative of z^n in $\mathbb{C} \setminus \{0\}$, so by the FTC, $\int_\gamma z^n \mathrm{~d}z = 0$ since γ is a closed curve.

For $n = -1$, use the definition of the integral:

$$\int_\gamma \frac{1}{z} \mathrm{~d}z = \int_0^1 \frac{\gamma'(t)}{\gamma(t)} \mathrm{~d}t = \int_0^1 \frac{2\pi i Re^{2\pi it}}{Re^{2\pi it}} \mathrm{~d}t = 2\pi i.$$

Since $\int_\gamma \frac{1}{z}dz \neq 0$, we can conclude that for any $R > 0$, $\frac{1}{z}$ has no anti-derivative in any open set containing the circle $\{|z| = R\}$.

In particular, since for any branch $\lambda(z)$ of logarithm the derivative $\lambda'(z) = \frac{1}{z}$, there is no branch of logarithm on $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$.

Theorem 2.0.5 (Converse to FTC)