IB Linear Algebra (from lecture 18)

Martin von Hodenberg (mjv43@cam.ac.uk)

December 25, 2021

Linear algebra description etc

This article constitutes my notes for the 'IB Linear Algebra' course, held in Michaelmas 2021 at Cambridge. The course was lectured by Prof. Pierre Raphael.

Contents

§1 Bilinear forms

Lemma 1.1

We have a bilinear form $\phi : V \times V \to \mathbb{F}$, where V is a finite vector space and B, B' are bases of V. Let

$$\phi = [Id]_{B,B'}.$$

Then

$$[\phi]_{B'} = P^T [\phi]_B P.$$

Proof. This is just a special case of the general change of basis formula; see that proof. □

Definition 1.2 (Congruent matrices)

Two square matrices A, B are said to be **congruent** if there exists an invertible square matrix P such that

$$A = P^T BP.$$

Remark. This defines an equivalence relation.

Definition 1.3 (Symmetric bilinear form)

A bilinear form on V is said to be **symmetric** if

$$\phi(u, v) = \phi(v, u) \ \forall u, v \in V.$$

Remark. 1. If A is a square matrix, we say that A is symmetric if $A^T = A$. Equivalently, $A_{ij} = A_{ji}$.

2. ϕ is symmetric iff $[\phi]_B$ is symmetric in *any* basis B.
3. To be able to represent ϕ by a diagonal matrix in some basis B, it is necesssary that ϕ is symmetric:
$$P^TAP = D = D^T = PA^TP^T \implies A = A^T \implies \phi \text{is symmetric.}$$

Definition 1.4 (Quadratic form)

A map $Q : V \to F$ is said to be a **quadratic form** if there exists a bilinear form $\phi : V \times V \to F$ such that
$$\forall u \in V,\ Q(u)\phi(u,u).$$

Remark (Computation in a basis)**.** Let $B = (e_i)_{1\leq i\leq n}$ be a basis of V, and let $A = [\phi]_B$. Let $u = \sum_{i=1}^n u_ie_i$, then

$$Q(u) - \phi(u,u) = \phi(\sum_{i=1}^n u_ie_i, \sum_{j=1}^n)u_je_j = \sum_{i,j=1}^n u_iu_je_ie_j = \sum_{i,j=1}^{a_ij u_iu_j}.$$

(by bilinearity of ϕ) Therefore we essentially have

$$Q(u) = U^TAU, \text{ where } U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Remark. We can note that
$$Q(u) = U^TAU = \sum_{i,j=1}^n a_{ij}u_iu_j = \sum_{i,j=1}^n (\frac{a_{ij}+a_{ji}}{2})u_iu_j = U^T(\frac{A+A^T}{2})U.$$

So the representation of A is not necessarily unique.

Proposition 1.5

If $Q : V \to F$ is a quadratic form, then there exists a unique symmetric bilinear form $\phi : V \times V \to F$ such that
$$Q(u) = \phi(u,u)\forall u \in V.$$

Polarisation identity. Let ψ be a bilinear form on V such that
$$\forall u \in V, Q(u) = \psi(u,u).$$

Let $\phi(u,v) = \frac{1}{2}(\psi(u,v) + \psi(v,u))$. Thus we have that:

- ϕ is a bilinear form
- ϕ is symmetric
- $\phi(u,u) = \psi(u,u) = Q(u)$.

This concludes the proof of *existence.*
Proof of uniqueness

Let ϕ be a symmetric bilinear form such that

$$\forall u \in V, \phi(u,u) = Q(u).$$

Then

$$\begin{aligned} Q(u+v) &= \phi(u+v, u+v) \\ &= \phi(u,u) + \phi(u,v) + \phi(v,u) + \phi(v,v) \text{ by bilinearity} \\ &= Q(u) + 2\phi(u,v) + Q(v) \text{ by symmetry} \end{aligned}$$

From this we get that

$$\phi(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v)).$$

□

Theorem 1.6 (Diagonalisation of symmetric bilinear forms)
Let $\phi : V \times V \to F$ be a symmetric bilinear form. (dim V=n). Then there exists a basis B of V such that $[\phi]_B$ is diagonal.

Proof. We proceed by induction on the dimension of V. For $n = 1$ it is trivially true. Suppose the theorem holds for all dimensions $< n$: then

- If $\phi(u,u) = 0 \forall u \in V$, then $\phi = 0$ by the polarisation identity (ϕ is symmetric).
- If $\phi \neq 0$, then there exists a $u \in V \backslash \{0\}$ such that $\phi(u,u) \neq 0$. Let us call $u = e_1$.
- Let U be the 'orthogonal' of e_1:

  $$U = (< e_1 >) = \{v \in V : \ \phi(e_1, v) = 0\} = \ker \theta : v \to \phi(e_1, v).$$

  Since it is a kernel of a linear map $V \to F$, therefore U is a vector subspace of V. By the Rank-Nullity theorem, we have

  $$\dim V = n = R(\theta) + \operatorname{null} \theta = \dim U + 1.$$

  We now claim that $U + < e_1 > = U \oplus < e_1 >$. Indeed,

  $$v = < e_1 > \cap U \implies v = \lambda e_1, \phi(e_1, v) = 0.$$

  $$\implies 0 = \phi(e_1, v) = \phi(e_1, \lambda e_1) = \lambda \phi(e_1, e_1) \implies \lambda = 0 \implies v = 0.$$

  $$\implies U + < e_1 > = U \oplus < e_1 > .$$

  Therefore $V = U \oplus < e_1 >$, and pick a basis $B' = (e_2, \ldots, e_n)$ such that $(e_1, e_2, \ldots, e_n)$ is a basis of V (since the sum is direct). So

  $$[\phi]_B = (\phi(e_i, e_j))_{1 \leq i,j \leq n} = \begin{pmatrix} \phi(e_1, e_1) & 0 \\ 0 & A' \end{pmatrix}.$$

Therefore $(A')^T = A'$, and $A' = [\phi|_U]_{B'}$ where $\phi|_U$ is the restriction of ϕ onto U. Now we apply the induction hypothesis to find a basis $(e'_1, \ldots, e_n;)$ of V such that $[\phi|U]_B$ is diagonal. So

$$\hat{B} = (e_1, e'_2, \ldots, e'_n)$$

is a basis of V, and finally we have that $[\phi]_{\hat{B}}$ is diagonal.

□

Example 1.7

Let $V = \mathbb{R}^3$, and

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$