

IB Optimisation

Martin von Hodenberg (mjv43@cam.ac.uk)

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Optimisation concerns problems of maximising or minimising certain functions. This extends to several different types of functions and is applicable in many areas of mathematics, particularly computational mathematics.

This article constitutes my notes for the ‘IB Optimisation’ course, held in Easter 2021 at Cambridge. This course was lectured by Prof. Varun Jog.

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§0 Introduction

The course is roughly divided into four parts:

1. Convex optimisation (Lectures 1-3)
2. Lagrange method (Lectures 4-5)
3. Linear programming (Lectures 6-8)
4. Applications of linear programming (Lectures 9-12)

Definition 0.1 (Optimisation problem)

The basic structure of an optimisation problem is as follows:

minimise $f(x)$ for $x \in X$, where in this course we take $X \subset \mathbb{R}^n$. The problem can have 'constraints' $h(x) = b$, where $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We will look at minimising functions WLOG.

1. f is called the objective function.
2. The components of x are called decision variables.
3. $h(x) = b$ is called a functional constraint.
4. $x \in X$ is called a regional constraint.
5. $\{x : x \in X\}$ is called the feasible set $X(b)$.
6. Problem is feasible if $X(b) \neq \emptyset$, and bounded if the minimum on $X(b)$ is bounded.
7. A point $x^* \in X(b)$ is optimal if it minimises f over $X(b)$. The value $f(x^*)$ is called the optimal set.

§1 Convex optimisation

§1.1 Convexity

Definition 1.1

A set $S \in \mathbb{R}^n$ is **convex** if for all $x, y \in S$, and all $\lambda \in [0, 1]$, $(1 - \lambda)x + \lambda y \in S$. (The line segment joining x and y lies in S .)

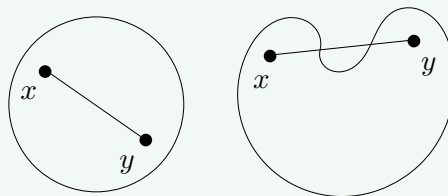


Figure 1: Two subsets of \mathbb{R}^2 : the left one is convex but the right one is not.

Definition 1.2

A function $f : S \rightarrow \mathbb{R}$ is **convex** if S is convex, and for all $x, y \in S$, and $\lambda \in [0, 1]$, we have

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

Intuitively, all the tangent lines to the function are below the function. We define f to be concave if $-f$ is convex.

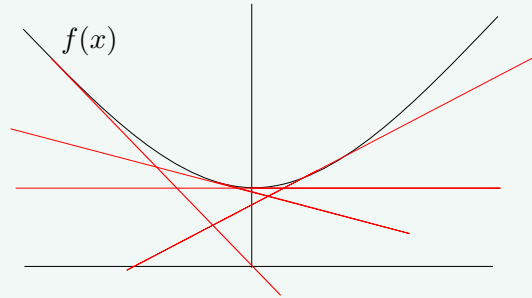


Figure 2: A convex function in \mathbb{R}^2 .

Remark. Linear functions are always both convex and concave, since we get equality in the above equation.

§1.2 Unconstrained optimisation

We want to find $\min f(x)$ where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function. This is a particularly simple case of optimisation, since convex functions have the important property that *local information extends globally*. We will now look at a couple of ways to prove convexity, called the first- and second-order conditions.

§1.2.1 First order conditions for convexity

Intuitively, for a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, we know that all the tangent lines to the function are below the function (see Figure 2). We can express this as

$$f(y) \geq f(x) + (y - x)f'(x).$$

We can generalise this to higher dimensions:

Theorem 1.3 (First-order conditions)

A differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex iff for $x, y \in \mathbb{R}^n$, we have

$$f(y) \geq f(x) + \nabla f(x)^T(y - x).$$

Remark. If $\nabla f(x) = 0$, then $f(y) \geq f(x) \implies x$ minimises f .

Proof. **Convexity of $f \implies$ F-O conditions hold:**

Let $n = 1$. For $x, y \in \mathbb{R}^n$, $t \in [0, 1]$ we have

$$f((1 - t)x + ty) = f(x + t(y - x)) \leq (1 - t)f(x) + tf(y).$$

Therefore

$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t}.$$

Taking $t \rightarrow 0$, we conclude

$$f(y) \geq f(x) + f'(x)(y - x).$$

For the general case, define a function $g(t) = f((1 - t)x + ty)$, i.e $g : [0, 1] \rightarrow \mathbb{R}$. Since f is convex, g is also convex. We can calculate

$$g'(t) = \nabla f((1 - t)x + ty)^T(y - x).$$

Since g is convex, by the above argument for $n = 1$, we have $g(1) \geq g(0) + g'(0)(1 - 0)$.

$$\implies f(y) \geq f(x) + \nabla f(x)^T(y - x).$$

F-O conditions hold \implies convexity:

Define $x_t = (1 - t)x + ty$. We have the two equations

$$f(x) \geq f(x_t) + \nabla f(x_t)^T(x - x_t) \tag{1}$$

$$f(y) \geq f(x_t) + \nabla f(x_t)^T(y - x_t) \tag{2}$$

Now we add $(1 - t) \times$ equation (1) to $t \times$ equation 2. Noting that $x - x_t = tx - ty$ and $y - x_t = (1 - t)y - (1 - t)x$, we get

$$(1 - t)f(x) + tf(y) \geq f(x_t) + \nabla f(x_t)^T(0).$$

This concludes the proof. □

§1.2.2 Second-order conditions for convexity

Next we look at second-order conditions. In one dimension, we would expect that if f' is increasing, i.e $f''(x) \geq 0$ would give us convexity. We generalise this to n dimensions using the Hessian matrix, which we saw in IA Differential Equations. It has entries $H_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

Definition 1.4 (Semidefinite matrix)

A matrix A is called **positive semidefinite** if for all $x \in \mathbb{R}^n$ we have $x^T A x \geq 0$. Equivalently, all eigenvalues of A are non-negative.

Remark. The Taylor expansion in n dimensions is

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x) + \dots$$

Theorem 1.5 (Second-order condition)

A twice differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\nabla^2 f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Proof. Using the Taylor expansion of f we have

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x).$$

where $z = (1 - \lambda)x + \lambda y$ for some $\lambda \in [0, 1]$. (IA Analysis).

We have $f(y) \geq f(x) + \nabla f(x)^T(y - x)$. Then by the first-order condition theorem, f is convex. \square

§1.3 The gradient descent algorithm

Recall we have an unconstrained optimisation problem:

$$\text{minimise } f(x), f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is a convex function.}$$

In order to solve this, we could apply a 'greedy method' where we pick an initial point x_0 and iteratively pick points x_1, x_2, \dots near this point so that the gradient of f at x_n is decreasing. In which direction should we decrease x_n ? Note that by Taylor's expansion, with small $\epsilon > 0$,

$$f(x - \epsilon \nabla f(x)) \approx f(x) - \epsilon \nabla f(x)^T \nabla f(x) = f(x) - \epsilon \|\nabla f(x)\|^2 \leq f(x).$$

We call the direction $-\nabla f(x)$ the descending direction. (Note we could also choose any other direction v so that $f(x)^T v < 0$.) We now formalise this process in an algorithm.

Definition 1.6 (Gradient descent method)

Define an algorithm:

1. Start at some point x_0 . Set $t = 0$.
2. Repeat the following:
 - Find a descending direction v_t (e.g. $-\nabla f(x)$)
 - Choose a step size η_t (can depend on t)
 - Update $x_{t+1} = x_t + \eta_t v_t$
3. Stop when some criterion is met (e.g. $\nabla f(x_t) = 0$, t is large enough)

For the scope of this course, we need to make assumptions about our convex function.

§1.3.1 Smoothness assumption

Definition 1.7 (Smoothness assumption)

We say that a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is β -smooth or β -Lipschitz if

$$\|\nabla f(x) - \nabla f(y)\| \leq \beta \|x - y\|.$$

Those who have taken IB Analysis and Topology will be familiar with this concept.

Remark. If f is twice-differentiable, then β -smoothness implies that $\nabla^2 f(x) \leq \beta I$. Equivalently, all eigenvalues of $\nabla^2 f(x)$ are $\leq \beta$. β -smoothness can be summed up by the fact that *the linear approximation is close to f in a small neighbourhood around x .*

Proposition 1.8

If f is β -smooth and convex, then

$$f(x) + \nabla f(x)^T(y - x) \leq f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2}\|x - y\|^2.$$

Proof. The left-hand inequality follows by convexity.

For the right hand inequality, by Taylor's theorem,

$$\begin{aligned} f(y) &= \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x) \\ &\leq f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T(\beta I)(y - x) \\ &= f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2}\|x - y\|^2 \end{aligned}$$

□

Corollary 1.9

At any point x ,

$$f(x - \frac{1}{\beta} \nabla f(x)) \leq f(x) - \frac{1}{2\beta} \|\nabla f(x)\|^2.$$

Proof. Let's look at

$$f(x) + \nabla f(x)^T(y - x) + \frac{\beta}{2}\|x - y\|^2,$$

and try to minimise it over y for a fixed x .

$$\nabla_y (f(x)^T(y - x) + \frac{\beta}{2}\|x - y\|^2) = \nabla f(x) - \beta(x - y) = 0.$$

$$\implies \nabla \frac{\nabla f(x)}{\beta} = x - y \implies y = x - \frac{1}{\beta} \nabla f(x).$$

Plug this value of y into the above claim:

$$\begin{aligned} f(x - \frac{1}{\beta} \nabla f(x)) &\leq f(x) + \nabla f(x)^T(\frac{-1}{\beta} \nabla f(x)) + \frac{\beta}{2} \|\frac{1}{\beta} \nabla f(x)\|^2 \\ &= f(x) - \frac{1}{\beta} \|\nabla f(x)\|^2 + \frac{1}{2\beta} \|\nabla f(x)\|^2 \\ &= f(x) - \frac{1}{2\beta} \|\nabla f(x)\|^2 \end{aligned}$$

□

Corollary 1.10 (Improved first-order condition)

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{1}{2\beta} \|\nabla f(x) - \nabla f(y)\|^2.$$

Proof. For any z , we have

$$f(x) + \nabla f(x)^T(z - x) \leq f(z) \leq f(y) + \nabla f(y)^T(z - y) + \frac{\beta}{2}\|z - y\|^2.$$

This implies

$$f(x) - f(y) \leq \nabla f(x)^T(x - z) + \nabla f(y)^T(z - y) + \frac{\beta}{2}\|z - y\|^2.$$

To minimise the RHS, set $\nabla_z = 0$. We get $-\nabla f(x) + \nabla f(y) + \beta(z - y) = 0$, which implies

$$z = \frac{f(x) - f(y)}{\beta} + y.$$

Subbing into the RHS, we get

$$f(x) - f(y) \leq \nabla f(x)^T(x - y) - \frac{1}{2\beta}\|\nabla f(x) - \nabla f(y)\|^2.$$

□

§1.3.2 Strong convexity assumption

We now introduce a new type of convexity we assume. In essence, it tells us that *if the gradient is small, we are close to the optimum*.

Definition 1.11 (Strong convexity assumption)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is α -strongly convex if

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2}\|x - y\|^2.$$

If f is twice differentiable, then

$$\nabla^2 f(x) \geq \alpha I \quad \forall x.$$

Proposition 1.12

Let f be α -strongly convex. Let p^* be the optimal cost. Then for any x we have

$$p^* \geq f(x) - \frac{1}{2\alpha}\|\nabla f(x)\|^2.$$

Remark. If $\|\nabla f(x)\| \leq \sqrt{2\alpha\epsilon}$ then $p^* \geq f(x) - \epsilon$, i.e

$$p^* \leq f(x) \leq p^* + \epsilon.$$

Proof. The α -strong convexity assumption gives us

$$\min_y f(y) \geq \min_y f(x) + \nabla f(x)^T(y - x) + \frac{\alpha}{2}\|x - y\|^2.$$

Setting ∇_y of the RHS=0, we get $y = x - \frac{\nabla f(x)}{\alpha}$.

Plugging this value of y in the RHS, we get

$$f(x) + \nabla f(x)^T \left(\frac{-\nabla f(x)}{\alpha} \right) + \frac{\alpha}{2} \left\| \frac{\nabla f(x)}{\alpha} \right\|^2 = f(x) - \frac{1}{2\alpha} \|\nabla f(x)\|^2.$$

□

We might also want to ask: how close is the true optimiser x^* to our current point x ? It turns out α -strong convexity gives us an answer to this too.

Proposition 1.13

Let $x \in S$, and x^* be the true optimiser. Then

$$\|x - x^*\| \leq \frac{2}{\alpha} \|\nabla f(x)\|.$$

Proof.

$$\begin{aligned} f(x^*) &\geq f(x) + \nabla f(x)(x^* - x) + \frac{\alpha}{2} \|x - x^*\|^2 \\ &\geq f(x) - \|\nabla f(x)\| \|x^* - x\| + \frac{\alpha}{2} \|x - x^*\|^2 \text{ by Cauchy-Schwartz} \end{aligned}$$

We already know that $f(x^*) \leq f(x)$. Therefore

$$0 \geq f(x^*) - f(x) \geq -\|\nabla f(x)\| \|x^* - x\| + \frac{\alpha}{2} \|x - x^*\|^2.$$

So $\|\nabla f(x)\| \|x - x^*\| \leq \frac{\alpha}{2} \|\nabla f(x)\|^2$, and thus

$$\|x - x^*\| \leq \frac{2}{\alpha} \|\nabla f(x)\|.$$

□

§1.3.3 Efficiency of gradient descent

Theorem 1.14

Gradient descent with step size $\frac{1}{\beta}$ satisfies:

$$f(x_t) - f(x^*) \leq \left(1 - \frac{\alpha}{\beta}\right)^t (f(x_0) - f(x^*)) \leq e^{\frac{-\alpha t}{\beta}} (f(x_0) - f(x^*)) \leq e^{\frac{-\alpha t}{\beta}} \frac{\beta}{2} \|x^* - x_0\|^2.$$

Note this is very efficient; the error decreases exponentially!

Proof.

$$\begin{aligned}
 f(x_{t+1}) - f(x^*) &\leq f(x_t) - f(x^*) - \frac{1}{2\beta} \|\nabla f(x_t)\|^2 \\
 &\leq f(x_t) - f(x^*) - \frac{\alpha}{\beta} (f(x_t) - f(x^*)) \text{ by strong convexity} \\
 &\leq (1 - \frac{\alpha}{\beta}) (f(x_t) - f(x^*))
 \end{aligned}$$

Using induction, conclude that

$$f(x_t) - f(x^*) \leq (1 - \frac{\alpha}{\beta})^t (f(x_0) - f(x^*)).$$

Because f is β -smooth, we have

$$f(x_0) \leq f(x^*) + \underbrace{\nabla f(x)^t (x_0 - x^*)}_{=0} + \frac{\beta}{2} \|x_0 - x^*\|^2.$$

□

Remark. With this algorithm, the number of steps needed to ensure error $\leq \epsilon$ is

$$\frac{\beta}{\alpha} \log\left(\frac{f(x_0) - f(x^*)}{\epsilon}\right).$$

The $\log(\frac{1}{\epsilon})$ dependence is called 'linear convergence'.

§1.4 Alternatives to gradient descent

§1.4.1 Newton's method

When we use gradient descent, the factor $(1 - \frac{\alpha}{\beta})$ controls the speed of convergence. We call $\frac{\beta}{\alpha}$ the **condition number** of f (note we always have $\beta > \alpha$). If the condition number is large, then convergence will be slow and we may want to consider other algorithms. Let's look at an example:

Example 1.15

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{1}{2}(x^2 + 100y^2).$$

Now we have

$$\nabla^2 f(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}.$$

Note that $I \leq \nabla^2 f(x, y) \leq 100I$. Our condition number is 100, which is very large! The contours of f are very stretched ellipses in \mathbb{R}^2 . If we use gradient descent on a point, it will comparatively overshoot a very large amount in the y direction. Therefore convergence will be slow.

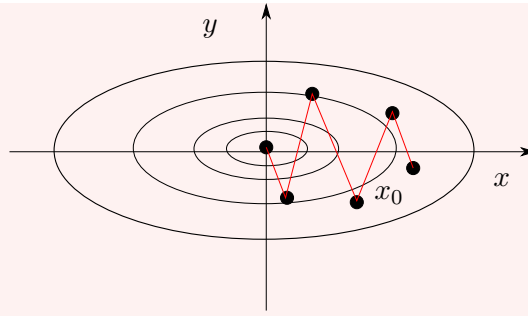


Figure 3: The path oscillates as it overshoots each time in the y direction.

We need another method in these cases. This is given by Newton's method:

Definition 1.16 (Newton's method)

Here we use a modified version of gradient descent, with

$$x_{t+1} = x_t - (\nabla^2 f(x))^{-1} \nabla f(x_t).$$

The derivation of this comes from the second order Taylor approximation of f at x_t :

$$f(x) \approx f(x_t)^T(x - x_t) + \frac{1}{2}(x - x_t)^T \nabla^2 f(x_t)(x - x_t).$$

If we minimise the RHS wrt. x , then we get $x = x_t - (\nabla^2 f(x))^{-1} \nabla f(x)$. This method of minimising the second-order approximation is shown in the figure below:

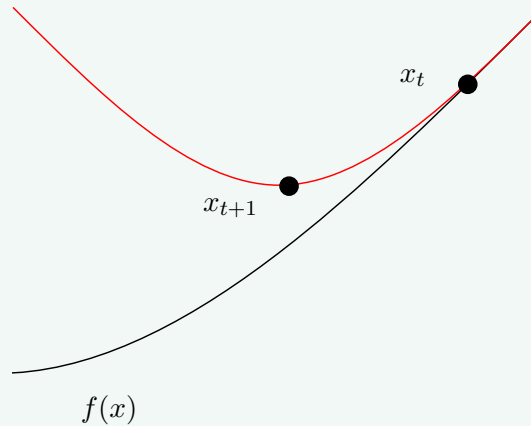


Figure 4: The second-order approximation, shown in red, is repeatedly minimised as we increment t .

Remark. Intuitively, Newton's method is excellent for functions with accurate second-order approximations.

§1.4.2 Convergence of Newton's method

We will not do the analysis in this course, but Newton's method satisfies

$$\|x_{t+1} - x^*\| \leq C\|x_t - x^*\|^2, \text{ as long as } x_t \text{ is close enough to } x^*.$$

In one dimension, Newton's method is a root-finding algorithm:

Take $f : \mathbb{R} \rightarrow \mathbb{R}$, and let $f' = g$. Write

$$x_{t+1} = x_t - \frac{f'(x_t)}{f''(x_t)} = x_t - \frac{g(x_t)}{g'(x_t)}.$$

If we set $f'(x_{t+1}) = g(x_{t+1}) = 0$, i.e looking for a stationary point of f (which is the same as a root of g if g is convex) then we can write $g(x_{t+1}) \approx g(x_t) + g'(x_t)(x_{t+1} - x_t) = 0$, and this gives the previous equation as expected. The root-finding algorithm is illustrated below:

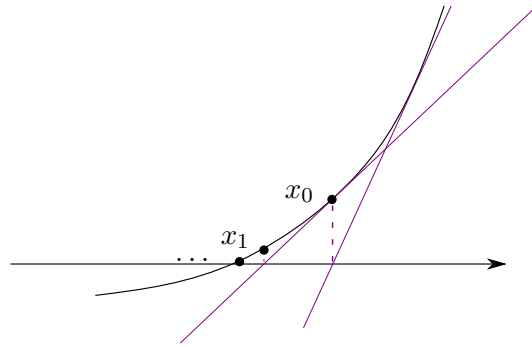


Figure 5: Newton's method applied in one dimension as a root-finding algorithm. Note the speed of convergence displayed.

§1.4.3 Barrier methods

What if the function we had to minimise involved constraints? We can write the optimisation problem as

$$\text{minimise } f(x) \text{ subject to } f_i(x) \leq 0 \quad \forall 1 \leq i \leq n.$$

The way we approach this is to transform the constrained problem into an unconstrained one. The way we do this is by defining some new functions for the new optimisation problem:

$$\text{minimise } f(x) + \sum_{i=1}^n \phi(f_i(x)), \quad \phi = \begin{cases} \infty & f_i(x) > 0 \text{ for any } 1 \leq i \leq m \\ 0 & f_i(x) \leq 0 \text{ for all } 1 \leq i \leq m \end{cases}$$

This may still be difficult to deal with, so we can define a **logarithmic barrier function**:

$$\text{minimise } tf(x) - \sum \log(-f_i(x)).$$

Here we are taking $\phi(x) = -\log(-x)$, so we are using an approximation: however we can make this negligible in practice by choosing suitable t .

Definition 1.17 (Barrier method)

We define an algorithm:

1. Find a point x in the feasible set. Set $t > 0$.
2. Repeat:
 - Solve $\min tf(x) + \sum_{i=1}^n \phi(f_i(x))$ with x as the starting point, use Newton's method.

- Set $x = x^*(t)$.
- Stop if t is large enough
- Increase $t = \alpha t$ for $\alpha > 1$.

§2 The Lagrange multiplier method

Suppose we want to minimise $f(x)$ for $x \in X$ subject to constraints $h(x) = b$ where $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

The **Lagrangian** associated with this problem is

$$L(x, \lambda) = f(x) - \lambda^T(h(x) - b).$$

Note that $\lambda \in \mathbb{R}^m$. We want to minimise $L(x, \lambda)$ over $x \in X$ for some value of λ .

§2.1 Lagrange sufficiency

Theorem 2.1 (Lagrange sufficiency theorem)

Suppose we can find a λ^* such that:

1. $\min_{x \in X} L(x, \lambda^*) = L(x^*, \lambda^*)$ for some $x^* \in X$.
2. $x^* \in X(b) = \{x : x \in X, h(x) = b\}$, the feasible set for our constrained problem.

Then x^* is optimal for 1, i.e. $\min_{x \in X(b)} f(x) = f(x^*)$.

Proof. Condition 2 says that $f(x^*) \geq \min_{x \in X(b)} f(x)$ because x^* is feasible. We have

$$\begin{aligned} \min_{x \in X(b)} f(x) &= \min_{x \in X(b)} f(x) - (\lambda^*)^T(h(x) - b) \\ &\geq \min_{x \in X} f(x) - (\lambda^*)^T(h(x) - b) \\ &= L(x^*, \lambda^*) \\ &= f(x^*) - (\lambda^*)^T(h(x^*) - b) \\ &= f(x^*). \end{aligned}$$

□

Let's look at an example.

Example 2.2

Minimise $-x_1 - x_2 + x_3$ subject to $x_1^2 + x_2^2 = 4, x_1 + x_2 + x_3 = 1$. Here we have

$$h(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1 + x_2 + x_3 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Our Lagrangian is

$$L(x, \lambda) = (-x_1 - x_2 + x_3) - \lambda_1(x_1^2 + x_2^2 - 4) - \lambda_2(x_1 + x_2 + x_3 - 1).$$

We can rewrite this as

$$L(x, \lambda) = (-(1 + \lambda_2)x_1 - \lambda_1 x_1^2) = (-(1 + \lambda_2)x_2 - \lambda_1 x_2^2) + (1 - \lambda_2)x_3 + 4\lambda_1 + \lambda_2.$$

Now fix λ , and try to find $\min_{x \in \mathbb{R}^n} L(x, \lambda)$. We'll only be interested in (λ_1, λ_2) so that the minimum is finite.

- If $\lambda_1 > 0$, the infimum is $-\infty$.
- If $\lambda_2 \neq 1$, the infimum is $-\infty$.

Let's look at $\lambda_1 \leq 0, \lambda_2 = 1$.

$$\frac{d}{dx_1} (-(1 + \lambda_2)x_1 - \lambda_1 x_1^2) = -(1 + \lambda_2) - 2\lambda_1 x_1 = 0 \implies x_1 = \frac{-1}{\lambda_1}.$$

Similarly, differentiating wrt. x_2 gives $x_2 = \frac{-1}{\lambda_1}$. Can we pick λ_1 so that (x_1, x_2) satisfy the constraints? We need $x_1^2 + x_2^2 = 4, x_1 + x_2 + x_3 = 1$. Therefore $x_1^2 = x_2^2 = 2 \implies x_1 = x_2 = \sqrt{2}$. Since $\lambda_1 \leq 0$, our optimiser is

$$x = (\sqrt{2}, \sqrt{2}, 1 - 2\sqrt{2}).$$

Let's formalize the above method for solving

$$\min_{x \in X} f(x), h(x) \leq b.$$

1. Add a slack variable s to transform the problem into

$$\min_{x \in X} f(x), h(x) + s = b, s \geq 0.$$

2. Set $L(x, \lambda, s) = f(x) - \lambda^T(h(x) + s - b)$, and let $\Lambda = \{\lambda : \inf_{x \in X, s \geq 0} L(x, \lambda, s) \geq -\infty\}$.
3. For each $\lambda \in \Lambda$, find $x^*(\lambda), s^*(\lambda)$ such that

$$\min_{x \in X, s \geq 0} L(x, \lambda, s) = L(x^*(\lambda), \lambda, s^*(\lambda)).$$

4. Find a $\lambda^* \in \Lambda$ such that $(x^*(\lambda^*), s^*(\lambda^*))$ is feasible; i.e. $h(x^*(\lambda^*)) = b$ and $s^*(\lambda^*) \geq 0$.

§2.2 Complementary slackness

Let's suppose we want to minimise $f(x) - \lambda^T(h(x) + s - b)$ subject to $x \in X, s \geq 0$. Suppose for $\lambda \in \Lambda$, we solve to arrive at $x^*(\lambda), s^*(\lambda)$. Note we must have all the components $\lambda_i \leq 0$ in order to keep $\lambda \in \Lambda$. We have

$$-\lambda^T s = -\sum \lambda_i s_i = \sum_{i=1}^m (\lambda_i) s_i.$$

$\min -\lambda^T s = 0$, which means $\lambda_i s_i = 0 \forall 1 \leq i \leq m$. Consider our inequalities $h_i(x) + s_i = b_i$. If any of these inequalities are not equalities, then for that i we must have $\lambda_i = 0$. This is called **complementary slackness**.

Example 2.3

Problem:

$$\min x_1 - 3x_2 \text{ subject to } x_1^2 + x_2^2 \leq 4, x_1 + x_2 \leq 2.$$

Let's apply our algorithm.

1. Add slack variables: our problem is now

$$\min x_1 - 3x_2 \text{ subject to } x_1^2 + x_2^2 + s_1 = 4, x_1 + x_2 + s_2 = 2, s_1, s_2 \geq 0.$$

2. Work out

$$L(x, \lambda, s) = ((1 - \lambda_2)x_1 - \lambda_1 x_1^2) + ((-3 - \lambda_2)x_2 - \lambda_1 x_2^2) - \lambda_1 s_1 - \lambda_2 s_2 + (4\lambda_1 + 2\lambda_2).$$

3. We must have $\lambda_1, \lambda_2 \leq 0$ by complementary slackness. At the optimum, $\lambda_1 s_1 = \lambda_2 s_2 = 0$. By differentiating L , we get the system of equations

$$\begin{aligned} (1 - \lambda_2) - 2\lambda_1 x_1 &= 0. \\ -3 - \lambda_2 - 2\lambda_1 x_2 &= 0. \end{aligned}$$

These show we can't have $\lambda_1 = 0$ and thus $\lambda_1 < 0$ and $s_1 = 0$. There are two more cases:

- If $\lambda_2 < 0$, then $s_2 = 0$. We have the two equations above, and then by complementary slackness we also know that

$$x_1^2 + x_2^2 = 4, x_1 + x_2 = 2.$$

Solving these gives $(x_1, x_2) = (0, 2)$ or $(2, 0)$. But with both of these solutions the first two equations give one of the $\lambda_i > 0$. So we don't have any solutions in this case.

- The last case is $\lambda_1 < 0, \lambda_2 = 0, s_1 = 0$. Here we get the two equations above and $x_1^2 + x_2^2 = 4$. Here we get

$$x_1 = \frac{1}{2\lambda_1}, x_2 = \frac{-3}{2\lambda_1}.$$

Therefore $\lambda_1^2 = \frac{5}{8}$ and thus

$$(x_1, x_2) = \left(-\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}\right).$$

So we are done by Lagrange sufficiency.

§2.3 Strong duality and shadow prices

For some optimisation problems that we solve using the Lagrange method, for *every* λ we can find a feasible x^* such that

$$\min_{x \in X} L(x, \lambda) = L(x^*, \lambda).$$

This property of an optimisation problem is called **strong duality**. Before we do this, however, let's start with a weaker theorem.

Theorem 2.4 (Weak duality)

For a Lagrange problem define

$$g(\lambda) = \inf_{x \in X} L(x, \lambda).$$

If $x \in X(b)$, and $\lambda \in \Lambda$, then $f(x) \geq g(\lambda)$. In particular,

$$\inf_{x \in X(b)} f(x) \geq \sup_{\lambda \in \Lambda} g(\lambda).$$

Proof. For all $\lambda \in \Lambda$,

$$\begin{aligned} \inf_{x \in X(b)} f(x) &= \inf_{x \in X(b)} (f(x) - \lambda^T(h(x) - b)) \\ &\geq \inf_{x \in X} (f(x) - \lambda^T(h(x) - b)) \\ &= \inf_{x \in X} L(x, \lambda) \\ &= g(\lambda). \end{aligned}$$

□

The problem of maximising $g(\lambda)$ subject to $\lambda \in \Lambda$ is called the **dual problem**, and the problem of minimising $f(\lambda)$ subject to $x \in X(b)$ is called the **primal problem**.

The **duality gap** is

$$\inf_{x \in X(b)} f(x) - \sup_{\lambda \in \Lambda} g(\lambda).$$

If the gap is 0, we say that **strong duality** holds.

Definition 2.5 (Hyperplane)

A function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to have a supporting hyperplane at b if there exists a $\lambda \in \mathbb{R}^m$ such that for all $c \in \mathbb{R}^m$, the following holds:

$$\phi(c) \geq \phi(b) + \lambda^T(c - b).$$

This is easy to visualise in one dimension: There exists a tangent to the curve ϕ at $(b, \phi(b))$ such that the tangent is below the curve.

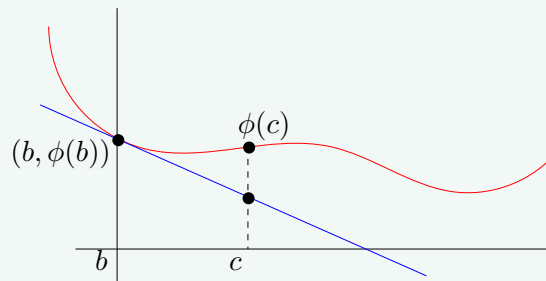


Figure 6: The hyperplane in one dimension is a tangent below the curve.

Definition 2.6 (Value function)

Define the **value function** $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ associated with the primal problem as follows:

$$\phi(c) = \inf_{x \in X(c)} f(x).$$

Theorem 2.7 (Strong duality)

Strong duality holds if and only if the value function ϕ has a supporting hyperplane at b .

Proof. We prove both directions. **Supporting hyperplane \implies strong duality:** There exists λ such that $\phi(c) \geq \phi(b) + \lambda^T(c - b)$. Then we have

$$\begin{aligned} g(\lambda) &= \inf_{x \in X} (f(x) - \lambda^T(h(x) - b)) \\ &= \inf_c \inf_{x \in X} (f(x) - \lambda^T(h(x) - c) - \lambda^T(c - b)) \\ &= \inf_c (\phi(c) - \lambda^T(c - b)) \\ &\geq \phi(b). \end{aligned}$$

Weak duality gives $g(\lambda) \leq \phi(b)$, and so $g(\lambda) = \phi(b)$; i.e., strong duality holds.

Strong duality holds \implies supporting hyperplane exists at b : There exists λ such that $g(\lambda) = \phi(b)$.

$$\begin{aligned} \phi(b) &= g(\lambda) = \inf_{x \in X} (f(x) - \lambda^T(h(x) - b)) \\ &= \inf_{x \in X} (f(x) - \lambda^T(h(x) - c) - \lambda^T(c - b)) \\ &\leq \phi(c) - \lambda^T(c - b) \text{ by weak duality} \end{aligned}$$

This means that λ gives the supporting hyperplane at b . □

§2.3.1 When does $\phi(b)$ have a supporting hyperplane?**Theorem 2.8** (Supporting hyperplanes and convex functions)

A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if every point $b \in \mathbb{R}^n$ has a supporting hyperplane.

Proof. We state this theorem without proof. □

Theorem 2.9 (Convexity conditions)

Consider the problem of minimising $f(x)$ with $x \in X, h(x) \leq b$. Then ϕ is convex if:

1. X is convex
2. f is convex
3. h is convex

Proof. This is left to be proved on Example Sheet 1. □

§2.3.2 Economics interpretations

Let's say we have a factory owner. He makes n types of products using m types of raw materials (which have a finite supply). Let's say that with $x = (x_1, \dots, x_m)$ raw materials he makes a profit of $f(x)$.

Further, let $h_j(x)$ is the raw material consumed of type j . Our problem is to maximise $f(x)$ such that $h_i(x) \leq b_i$ for $1 \leq i \leq m$, and $x \geq 0$.

If $(\epsilon_1, \dots, \epsilon_m)$ amount of raw material is offered to the factory owner, how much is it worth? For small enough ϵ , we have

$$\phi(b + \epsilon) - \phi(b) \approx \sum_{j=1}^m \frac{\partial \phi(b)}{\partial b_j} \epsilon_j.$$

The vector of prices $(\frac{\partial \phi(b)}{\partial b_1}, \dots, \frac{\partial \phi(b)}{\partial b_m}) = \nabla \phi(b)$. These are called "shadow prices".

Theorem 2.10

If ϕ is differentiable at b and has a separating hyperplane given by λ , then $\lambda = \nabla \phi(b)$.

Proof. Let $a = (a_1, \dots, a_m)$ be an arbitrary vector. Pick $\delta > 0$.

$$\begin{aligned} \frac{\phi(b + \delta a) - \phi(b)}{\delta} &\geq \lambda^T a \\ \implies \nabla \phi(b)^T a &\geq \lambda^T a \text{ for every vector } a. \\ \implies \nabla \phi(b) &= \lambda. \end{aligned}$$

□

The Lagrange multiplier at b is the vector of partial derivatives, i.e the vector of shadow prices. The intuitive interpretation of 'not wanting to pay any more' when you have enough of a certain raw material (that shadow price being 0) is an interpretation of complementary slackness.

Now let's look at the dual problem. If the raw material seller charges a price λ , and the finished product is consumed when he sells it, then

$$\lambda^T (h(x) - b) - f(x)$$

is the Lagrangian he is trying to minimise. But the factory producer is producing $x^*(\lambda)$ to maximise

$$f(x) - \lambda^T (h(x) - b).$$

Let's call $g(\lambda) = \lambda^T (h(x^*(\lambda)) - b) - f(x^*(\lambda))$. The seller is trying to maximise g . This provides a good example of the dual problem.