IB Complex Methods

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January 27, 2022

These are my notes for the IB course Complex Methods, which was lectured in Lent 2022 at Cambridge by Dr U. Sperhake. These notes are written in LaTeX for my own revision purposes. Any suggestions or feedback is welcome.

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§0 Background material

§0.1 Complex numbers

Recall the definition of a complex number, its real and imaginary parts, complex conjugate, modulus, and argument. Note that $\arg z$ is only defined up to adding $2n\pi$, for $n \in \mathbb{Z}$. Recall also the definition of the principal argument ($\arg z \in [-\pi, \pi]$).

Recall the triangle inequality:

$$|z_1| + |z_2| \leq |z_1| + |z_2| \quad \forall z_1, z_2 \in \mathbb{C}.$$

By setting $z_1 = \zeta_1 + \zeta_2$ and $z_2 = -\zeta_2$ we get the reverse triangle inequality

$$||\zeta_1| - |\zeta_2|| \leq |\zeta_1 + \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{C}.$$

Recall the geometric series: for $z \in \mathbb{C}$, $z \neq 1$ and $n \in \mathbb{N}_0$: $\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$.

For $|z| < 1$, this converges for $n \to \infty$: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$

Definition 0.1 (Open set)

A set $D \subset \mathbb{C}$ is an "open set" if for all $z_0 \in D$, $\exists \varepsilon > 0$ such that the ε-sphere $|z - z_0| < \varepsilon$ lies in D. A neighbourhood of $z \in \mathbb{C}$ is an open set D that contains z.

§0.2 Trigonometric and hyperbolic functions

Recall Euler's identity, and the complex definitions of cos, sin, and their hyperbolic counterparts. Recall that $\cos(ix) = \cosh(x)$ and $\sin(ix) = i \sinh(x)$ from the definitions.

§0.3 Calculus of real functions in ≥ 1 variables

Sometimes, we regard a complex function as 2 real functions on $\mathbb{R}^2$: $f(z) = u(x, y) + iv(x, y)$. (See IB Complex Analysis notes for more on this.)

Definition 0.2

We define $C^m(\Omega)$ as the set of functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ whose partial derivatives up to order m exist and are continuous.

Remark. We need the continuity condition: consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x & y = 0 \\ y & x = 0 \\ 1 & \text{elsewhere} \end{cases}$$

Then $\frac{\partial f}{\partial x}(0, 0) = 1 = \frac{\partial f}{\partial y}(0, 0)$, but f is not even continuous at $(0, 0)$.

Definition 0.3 (Differentiable function)

$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $x \in \Omega$ if there exists a linear function $A : \mathbb{R}^n \to R$ with

$$f(x + h) - f(x) = A(x)(h) + o(||h||).$$

(See IB Analysis and Topology.) We define f to be continuously differentiable if its partial derivatives are also continuous. This generalises to vector-valued functions $f : \Omega \to \mathbb{R}^m$ by considering each component f_i separately.

Definition 0.4 (Uniform convergence)
A sequence of functions $f_k : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is uniformly convergent with limit f iff

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} : \quad \forall k \geq n, x \in \Omega : \quad |f_k(x) - f(x)| < \varepsilon.$$

See IB Analysis and Topology for more. In this course, we will use this to justify swapping limits with integrals and sums.

§1 Analytic functions

§1.1 The extended complex plane of the Riemann sphere

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ since $z \leftrightarrow (x, y)$ is bijective with $z = x + iy$.

Definition 1.1 (Extended complex domain)
We define the extended complex domain $\mathbb{C}^* = \mathbb{C} \cup \infty$. We have seen this before in IA Groups. Some things to remember:

- $z = \infty$ is a single point
- $z = -\infty$ means we approach this point along the negative real axis.
- We can visualise this as the Riemann sphere: [image] We have the projection mapping $P \leftrightarrow z$ via the line $\vec{NP}$. The south pole P is mapped to $z = 0$ and the north pole N is mapped to $z = \infty$. In practice, f has a property at $z = \infty$ if $f\left(\frac{1}{\zeta}\right)$ has this property at $\zeta = 0$.

§1.2 Complex differentiation and analytic functions

Definition 1.2 (Differentiability in $\mathbb{C}$)
$f : \mathbb{C} \to \mathbb{C}$ is "differentiable" at $z \in \mathbb{C}$ if

$$f'(z) \equiv \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists and is independent of the direction of approach.

Remark. Direction independence is a strong requirement! (See IB Complex Analysis for more). In $\mathbb{R}$, we have only 2 directions: for example, $f(x) = |x|$ is not differentiable at $x = 0$. Conversely, in $\mathbb{C}$, we have infinitely many directions.

Definition 1.3 (Analyticity)
A complex function f is analytic at $z \in \mathbb{C}$ iff there exists a neighbourhood D of z where f is differentiable.

Remark.
- In this course, we will almost exclusively consider analytic functions.
- Analyticity implies many things: an analytic function can be differentiated infinitely many times, as opposed to in $\mathbb{R}$.
- Many rules of differentiation of real functions hold for complex ones, too.

Let us consider 2 directions for $f(z) = u(x, y) + iv(x, y)$.

1. Real axis: $\delta z = \delta x$. So we have

$$\begin{aligned} f'(z) &= \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

2. Imaginary axis: $\delta z = i\delta y$. So we have

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{f(z + i\delta y) - f(z)}{i\delta y} \\ &= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

But these two expressions must agree in their real and imaginary parts.

Proposition 1.4

Any differentiable function $f(z) = u(x, y) + iv(x, y)$ must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The converse does not hold; we also need that u, v are differentiable (see IB Complex Analysis)

In practice we use

Proposition 1.5

If $f(z) = u(x, y) + iv(x, y)$ saitsfies the Cauchy-Riemann equations at $z = z_0$ and the partial derivatives are continuous in a neighbourhood of z_0, then $f(z)$ is differentiable at z_0. See IB Complex Analysis for the proof.

Proposition 1.6 (Product rule)

The product of two analytic functions f, g is analytic with

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

Proof. Let us define

$$\begin{aligned}\overline{\nu} &= \frac{f(z+h)-f(z)}{h} - f'(z)\\ \nu &= \frac{g(z+h)-g(z)}{h} - g'(z)\end{aligned}$$

so both $\overline{\nu}, \nu \to 0$ as $h \to 0$, and

$$\begin{aligned}(gf)' &= \lim_{h\to 0} \frac{g(z+h)f(z+h) - g(z)f(z)}{h}\\ &= \lim_{h\to 0} \frac{[g(z)+(g'(z)+\nu)h][f(z)+(f'(z)+\overline{\nu})h] - g(z)f(z)}{h}\\ &= \lim_{h\to 0} \frac{(g'(z)+\nu)hf(z) + ((f'(z)+\overline{\nu})hg(z)) + (g'(z)+\nu)h(f'(z)+\overline{\nu})h}{h}\\ &= g'(z)f(z) + g(z)f'(z).\end{aligned}$$

□

Proposition 1.7 (Chain rule)
The composition of two analytic functions f, g is analytic with

$$(f \circ g)'(z) = f'(g(z))g'(z).$$

Proof. Omitted. □

Example 1.8
Now let's look at some examples.

1. $f(z) = z = x + iy$ is **entire**, i.e analytic in all of $\mathbb{C}$. This is true since $\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial y}$ and they are continuous. The definition of $f'(z)$ gives us $f'(z) = 1$.
2. $e^z = e^{x+iy}$