IB Complex Analysis

Martin von Hodenberg (mjv43@cam.ac.uk)

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These are my notes for the IB course Complex Analysis, which was lectured in Lent 2022 at Cambridge by Prof. N.Wickramasekera. These notes are written in LaTeX for my own revision purposes. Any suggestions or feedback is welcome.

Contents

§1 Basic notions

Recall definitions in $\mathbb{C}$ from IA courses. Note that $d(z,w) = |z-w|$ defines a metric on $\mathbb{C}$ (the standard metric). For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : \ |z-a| < r\}$ for the open ball with centre a and radius r.

Definition 1.1 (Open subset of $\mathbb{C}$)

A subset $U \subset \mathbb{C}$ is open wrt. the standard metric if for all $a \in U$, there exists an r such that $D(a,r) \in U$.

Remark. This is equivalent to being open wrt. the Euclidean metric on $\mathbb{R}^2$.

This course is about complex valued functions of a single variable, i.e functions

$$f : A \to \mathbb{C}, \quad \text{where } A \subset \mathbb{C}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, we can write $f = u + iv$ for real functions u, v and thus define $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$. Almost exclusively we'll focus on differentiable functions f. But first let's recall continuity.

Definition 1.2 (Continuous function on $\mathbb{C}$)

The function f (as above) is continuous at a point $w \in A$ if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \forall z \in A, \quad |z-w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Remark. This is equivalent to saying that $\lim_{z \to w} f(z) = f(w)$.

§1.1 Complex differentiation

Let $f : U \to \mathbb{C}$, where U is open.

Definition 1.3 (Differentiability)

f is differentiable at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}.$$

exists at a complex number.

Definition 1.4 (Holomorphic function)

f is holomorphic at $w \in U$ if there is $\varepsilon > 0$ such that $D(w,\varepsilon) \subset U$ and f is differentiable at every point in $D(w,\varepsilon)$.

Equivalently, f is holomorphic in U if f is holomorphic at every point in U, or equivalently, f is differentiable at every point in U.

Remark. Sometimes we use "analytic" to mean holomorphic.

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (can easily be checked).

Proposition 1.5

The chain rule for composite functions also holds: if $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ with $f(U) \subset V$, and $h = g \circ f : U \to \mathbb{C}$. If f is differentiable at $w \in U$ and g is differentiable at $f(w)$, then h is differentiable at w with

$$h'(w) = (g \circ f)'(w) = f'(w)(g' \circ f)(w).$$

Proof. Omitted; analagous to the proof for the real case. □

We might ask ourselves a question:

Write $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$. Is differentiability of f at a point $w = c + id \in U$ is the same as differentiability of u and v at (c, d)?

Recall from IB Analysis & Topology that $u : U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a "good linear approximation of u at (c, d)". We can show that if u is differentiable at c, d then L (the derivative of u at (c, d)) is uniquely defined, and we write $L = Du(c, d)$; moreover, L is given by the partial derivatives of u, i.e

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d)\right) x + \left(\frac{\partial u}{\partial y}(c, d)\right) y.$$

The answer to the above question is **no** (otherwise complex analysis would be useless!). Now we want to characterise differentiability of f in terms of u and v.

Theorem 1.6 (Cauchy-Riemann equations)

This theorem states that $f = u + iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if

u, v are differentiable at $(c, d) \in U$ **and** u, v satisfy the Cauchy-Riemann equations at (c, d):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d).$$

There are three other such expressions following from the Cauchy-Riemann equations.

Proof. f is differentiable at w with derivative $f'(w) = p + iq$:

$$\begin{aligned} &\iff \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = p + iq \\ &\iff \lim_{z \to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0 \end{aligned}$$

Writing $f = u + iv$ and separating real and imaginary parts, the above holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y)-u(c,d)-p(x-c)+q(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} = 0$$
$$\text{and} \quad \lim_{(x,y)\to(c,d)} \frac{v(x,y)-v(c,d)-q(x-c)+p(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} = 0$$

This is precisely the statement that u is differentiable at (c,d) with $Du(c,d)(x,y) = px - qy$, and v is differentiable at (c,d) with $Dv(c,d)(x,y) = qx + py$.

So $\iff$ u, v are differentiable at (c,d) and $u_x(c,d) = p = v_y(c,d)$, $u_y(c,d) = -q = -v_x(c,d)$, i.e the Cauchy-Riemann equations hold at (c,d).

We also get from the above that if f is differentiable at w, then $f'(w) = p + iq = u_x(c,d) + iv_x(c,d)$. $\square$

Remark. u, v satisfying the Cauchy-Riemann equations at a point does **not** guarantee differentiability of f on its own.