

# IB Groups, Rings and Modules

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These are my notes for the IB course ‘Groups, Rings and Modules’, which was lectured in Lent 2022 at Cambridge by Dr R.Zhou. These notes are written in  $\text{\LaTeX}$  for revision purposes<sup>1</sup>. Any suggestions or feedback is welcome.

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<sup>1</sup>These notes are posted online on [my website](#).

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## 0 Introduction

This is a second course in abstract algebra that will extend our knowledge of groups and introduce other fundamental objects.

The course is divided into several sections:

1. Groups; this will be a continuation from IA, focusing on simple groups,  $p$ -groups, and  $p$ -subgroups. The main result in this part of the course will be the Sylow theorems.
2. Rings; these are sets where you can add, subtract and multiply (e.g  $\mathbb{Z}$  or  $\mathbb{C}[X]$ ). We will study rings of integers such as  $\mathbb{Z}[i]$ ,  $\mathbb{Z}[\sqrt{2}]$ . These also generalise to polynomial rings. We will also study fields, which are rings where you can divide (e.g  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{Z}/p\mathbb{Z}$  for  $p$  prime).
3. Modules; these are an analogue of vector spaces where the scalars belong to a ring instead of a field. We will classify modules over certain "nice" rings. This allows us to prove Jordan Normal Form, and classify finite abelian groups.

# 1 Groups

## 1.1 Recall of IA Groups

This first subsection will just recap the results seen in IA Groups; it can be skipped by anyone with a sufficient knowledge of the course.

### Definition (Group)

A group is a pair  $(G, \cdot)$  where  $G$  is a set and  $\cdot : G \times G \rightarrow G$  is a binary operation satisfying:

1.  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$  (associativity)
2.  $\exists e \in G$  such that  $e \cdot g = g \cdot e = g$  for all  $g \in G$  (identity)
3.  $\forall g \in G, \exists g^{-1} \in G$  such that  $g \cdot g^{-1} = g^{-1} \cdot g = e$  (inverses)

*Remarks.*

- In practice, one often needs to check closure in order to check that  $\cdot$  is well-defined.
- If using additive (respectively multiplicative) relations, we will often write 0 (or 1) for the identity.
- We write  $|G|$  for the number of elements in  $G$ .

### Definition (Subgroup)

A subset  $H \subseteq G$  is a subgroup (written  $H \leq G$ ) if  $H$  is closed under  $\cdot$  and  $(H, \cdot)$  is a group.

*Remark.* A non-empty subset  $H$  of  $G$  is a subgroup if  $a, b \in H \implies a \cdot b^{-1} \in H$  (see IA Groups for the proof).

### Example (Examples of groups)

Groups we have already seen include:

- Additive groups  $(\mathbb{Z}, +) \leq (\mathbb{Q}, +) \leq (\mathbb{R}, +)$ .
- Cyclic and dihedral groups  $C_n$  and  $D_{2n}$ .
- Abelian groups: those groups  $G$  such that  $a \cdot b = b \cdot a$  for all  $a, b \in G$ .
- Symmetric and alternating groups  $S_n =$  group of all permutations of  $\{1, \dots, n\}$  and  $A_n \leq S_n$ , the group of all even permutations.
- Quaternion group  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  where  $i, j, k$  are quaternions.
- General and special linear groups  $GL_n(\mathbb{R}) = n \times n$  matrices on  $\mathbb{R}$  with  $\det \neq 0$ , where the group operation is matrix multiplication. This contains the subgroup  $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$ , which is the subgroup of matrices with  $\det = 1$ .

### Definition (Direct product)

The direct product of groups  $G$  and  $H$  is the set  $G \times H$  with operation

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

**Theorem 1.1** (Lagrange's theorem)

Let  $H \leq G$ . Then the left cosets of  $H$  in  $G$  are the sets  $gH = \{gh : h \in H\}$  for  $g \in G$ . These partition  $G$ , and each has the same cardinality as  $H$ . From this we can deduce Lagrange's theorem:

If  $G$  is a finite group and  $H \leq G$ , then  $|G| = |H|[G : H]$  where  $[G : H]$  is the number of left cosets of  $H$  in  $G$  (the index of  $H$  in  $G$ ).

*Remark.* Can also carry this out with right cosets. A corollary of Lagrange's theorem is thus that the number of left cosets = number of right cosets.

**Definition** (Order of an element)

Let  $g \in G$ . If  $\exists n \geq 1$  such that  $g^n = 1$ , then the least such  $n$  is the order of  $g$  in  $G$ . If no such  $n$  exists,  $g$  has infinite order.

*Remark.* If  $g$  has order  $d$ , then

- $g^n = 1 \implies d|n$ .
- $\{1, g, \dots, g^{d-1}\} \leq G$  and so if  $G$  is finite, then  $d||G|$  (Lagrange).

**Definition** (Normal subgroup)

A subgroup  $H \leq G$  is normal if  $g^{-1}Hg = H$  for all  $g \in G$ . We write  $H \trianglelefteq G$ .

**Proposition 1.2**

If  $H \trianglelefteq G$  then the set  $G/H$  of left cosets of  $H$  in  $G$  is a group (called the quotient group) with operation

$$g_1 H \cdot g_2 H = g_1 g_2 H.$$

*Proof.* Check  $\cdot$  is well-defined:

Suppose  $g_1 H = g'_1 H$  and  $g_2 H = g'_2 H$  for some  $g_1, g'_1, g_2, g'_2 \in G$ . Then  $g'_1 = g_1 h_1$  and  $g'_2 = g_2 h_2$  for some  $h_1, h_2 \in H$ . Therefore

$$\begin{aligned} g'_1 g'_2 &= g_1 h_1 g_2 h_2 \\ &= g_1 g_2 \underbrace{(g_2^{-1} h_1 g_2)}_{\in H} \underbrace{h_2}_{\in H} \end{aligned}$$

Therefore  $g'_1 g'_2 H = g_1 g_2 H$ . Associativity is inherited from  $G$ , the identity is  $H = eH$ , and the inverse of  $gH$  is  $g^{-1}H$ .  $\square$

**Definition** (Homomorphism)

If  $G, H$  are groups, then a function  $\phi : G \rightarrow H$  is a group homomorphism if

$\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ . It has kernel

$$\ker \phi = \{g \in G : \phi(g) = e\} \leq G.$$

and image

$$\text{Im } \phi = \{\phi(g) : g \in G\} \leq H.$$

*Remark.* If  $a \in \ker \phi$  and  $g \in G$ , then

$$\begin{aligned} \phi(g^{-1}ag) &= \phi(g^{-1})\phi(a)\phi(g) \\ &= \phi(g^{-1})\phi(g) \\ &= \phi(g^{-1}g) = \phi(e) = e. \end{aligned}$$

So  $g^{-1}ag \in \ker \phi$  and hence  $\ker \phi$  is a normal subgroup of  $G$ .

### Definition (Isomorphism)

An isomorphism of groups is a group homomorphism that is also a bijection. We say  $G$  and  $H$  are isomorphic and write  $G \cong H$  if there exists an isomorphism  $\phi : G \rightarrow H$ . (Note it follows from the definition that  $\phi^{-1}$  is also a group homomorphism)

### Theorem 1.3 (First Isomorphism Theorem)

Let  $\phi : G \rightarrow H$  be a group homomorphism. Then  $\ker \phi \trianglelefteq G$  and

$$G / \ker \phi \cong \text{Im } \phi.$$

*Proof.* Let  $K = \ker \phi$ . We have already checked  $K$  is normal. Now we define  $\Phi : G/K \rightarrow \text{Im } \phi$  by

$$gK \rightarrow \phi(g).$$

To show  $\Phi$  is well defined and injective:

$$\begin{aligned} g_1K = g_2K &\iff g_2^{-1}g_1 \in K \\ &\iff \phi(g_2^{-1}g_1) = e \\ &\iff \phi(g_1) = \phi(g_2). \end{aligned}$$

To show  $\Phi$  is a group hom.:

$$\begin{aligned} \Phi(g_1K g_2K) &= \Phi(g_1g_2K) \\ &= \phi(g_1g_2) = \phi(g_1)\phi(g_2) \\ &= \Phi(g_1K)\Phi(g_2K) \end{aligned}$$

Showing  $\Phi$  is surjective:

Let  $x \in \text{Im } \phi$ , say  $x = \phi(g)$  for some  $g \in G$ . Then  $x = \phi(gK)$ . □

### Example

Let  $\phi : \mathbb{C} \rightarrow \mathbb{C}^\times = \{x \in \mathbb{C} : x \neq 0\}$  given by  $z \mapsto e^z$ .

Since  $e^{z+w} = e^z e^w$ , this is a group homomorphism from  $(\mathbb{C}, +) \rightarrow (\mathbb{C}^\times, \times)$ . We have

that

$$\ker \phi = \{z \in \mathbb{C} : e^z = 1\} = 2\pi i\mathbb{Z}$$

$$\operatorname{Im} \phi = \mathbb{C}^x \text{ by existence of } \log$$

Hence  $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}^x$ .

**Theorem 1.4 (Second Isomorphism Theorem)**

Let  $H \leq G$ , and  $K \trianglelefteq G$ . Then  $HK = \{hk : h \in H, k \in K\} \leq G$  and  $H \cap K \trianglelefteq H$ . Moreover,

$$HK/K \cong H/(H \cap K).$$

*Proof.* Let  $h_1k_1, h_2k_2 \in HK$  (so  $h_1h_2 \in H$ ,  $k_1k_2 \in K$ ). Now

$$h_1k_1(h_2k_2)^{-1} = \underbrace{h_1h_2^{-1}}_{\in H} \underbrace{(h_2k_1k_2^{-1}h_2^{-1})}_{\in K} \in HK.$$

Thus  $HK \leq G$  (by our previous remark). Let  $\phi : H \rightarrow G/K$  be given by  $h \rightarrow hK$ . This is the composite of  $H \rightarrow G$  and the quotient map  $G \rightarrow G/K$ ; hence  $\phi$  is a group homomorphism. Thus

$$\ker \phi = \{h \in H : hK = K\} = H \cap K \trianglelefteq H$$

$$\operatorname{Im} \phi = \{hK : h \in H\} = HK/K$$

Now by the First Isomorphism Theorem

$$HK/K \cong H/(H \cap K).$$

□

**Lemma 1.5**

Suppose  $K \trianglelefteq G$ . There is a bijection

$$\{\text{subgroups of } G/K\} \leftrightarrow \{\text{subgroups of } G \text{ containing } K\},$$

where  $X \mapsto \{g \in G : gK \in X\}$  and  $H/K \leftarrow H$ . This further restricts to a bijection

$$\{\text{normal subgroups of } G/K\} \leftrightarrow \{\text{normal subgroups of } G \text{ containing } K\}.$$

**Theorem 1.6 (Third Isomorphism Theorem)**

Let  $K \leq H \leq G$  be normal subgroups of  $G$ . Then

$$\frac{G/K}{H/K} \cong G/H.$$

*Proof.* Let  $\phi : G/K \rightarrow G/H$  be defined by  $gK \mapsto gH$ . If  $g_1K = g_2K$ , then  $g_2^{-1}g_1 \in K \leq H \implies g_1H = g_2H$ . Thus  $\phi$  is well-defined.

Thus  $\phi$  is a surjective homomorphism with kernel  $H/K$ . Now just apply the First Isomorphism Theorem.  $\square$

## 1.2 Simple groups

If  $K \trianglelefteq G$ , then studying the groups  $K$  and  $G/K$  gives some information about  $G$ . However, this approach is not always available. This is the case when a group is simple.

**Definition** (Simple group)

A group  $G$  is simple if  $\{e\}$  and  $G$  are its only normal subgroups.

*Remark.* It is convention to not consider the trivial group a simple group.

**Lemma 1.7**

Let  $G$  be an abelian group.  $G$  is simple iff  $G \cong C_p$  for some prime  $p$ .

*Proof.*  $\Leftarrow$  : Let  $H \leq C_p$ . Lagrange's theorem says that  $|H| \mid |C_p| = p$ . Since  $p$  is prime,  $|H| = 1$  or  $p$ . So  $H$  is the trivial group or  $C_p$ .

$\Rightarrow$  : Let  $g \in G$  where  $g \neq e$ . Consider the subgroup generated by  $g$ :

$$\langle g \rangle = \{\dots, g^{-2}, g^{-1}, e, g, g^2, \dots\}.$$

This is normal in  $G$  since  $G$  is abelian. Since  $G$  is simple,  $\langle g \rangle = G$ . If  $G$  is infinite,  $G \cong (\mathbb{Z}, +)$  and  $2\mathbb{Z} \leq \mathbb{Z}$  which gives a contradiction.

Otherwise, we now know  $G \cong C_n$  for some  $n$ . Let  $g$  be a generator. If  $m \mid n$  then  $g^{n/m}$  generates a subgroup of order  $m$  and so  $G$  simple  $\Rightarrow$  the only factors of  $n$  are 1 and  $n$ . Therefore  $n$  is prime.  $\square$

**Lemma 1.8**

If  $G$  is a finite group, then  $G$  has a composition series

$$e = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_m = G,$$

with each quotient  $G_i/G_{i-1}$  simple.

*Proof.* We induct on  $|G|$ . If  $|G| = 1$  it's obvious. If  $|G| > 1$ , let  $G_{m-1}$  be a normal subgroup of largest possible order that isn't  $G$  itself. Lemma 1.5 implies  $G/G_{m-1}$  is simple. Then apply the induction hypothesis to  $G_{m-1}$ .  $\square$

## 1.3 Group actions

**Definition** (Permutation group)

For  $X$  any set, let  $\text{Sym}(X)$  be the group of all bijections  $X \rightarrow X$  under composition. This clearly forms a group with  $e = \text{Id}_X$ .

A group  $G$  is a permutation group of degree  $n$  if  $G \leq \text{Sym}(X)$  with  $|X| = n$ .



**Example** (Examples of permutation group)

- $S_n = \text{Sym}(\{1, 2, \dots, n\})$  is a permutation group of degree  $n$ , as is  $A_n \leq S_n$ .
- $D_{2n}$  (symmetries of a regular  $n$ -gon) is a subgroup of  $\text{Sym}(\{\text{vertices of } n\text{-gon}\})$ .

**Definition** (Group action)

An action of a group  $G$  on a set  $X$  is a function  $*$  :  $G \times X \rightarrow X$  satisfying

- (i)  $e * x = x$  for all  $x \in X$
- (ii)  $(g_1 g_2) * x = g_1 * (g_2 * x)$  for all  $g_1, g_2 \in G, x \in X$ .

**Proposition 1.9**

An action of a group  $G$  on a set  $X$  is equivalent to specifying a group homomorphism  $\phi : G \rightarrow \text{Sym}(X)$ .

*Proof.* For each  $g \in G$ , let  $\phi_g : X \rightarrow X$  send  $x \mapsto g * x$ .

We have  $\phi_{g_1 g_2}(x) = (g_1 g_2) * x = g_1 * (g_2 * x) = \phi_{g_1} \circ \phi_{g_2}(x)$ . (†)

In particular,  $\phi_g \circ \phi_{g^{-1}} = \phi_{g^{-1}} \circ \phi_g = \phi_e = \text{Id}_X$ . Thus  $\phi_g \in \text{Sym}(X)$ . Then the map  $\phi : G \rightarrow \text{Sym}(X)$  given by  $g \mapsto \phi_g$  is a group homomorphism by (†).

Conversely, let  $\phi : G \rightarrow \text{Sym}(X)$  be a group homomorphism. Define  $*$  :  $G \times X \rightarrow X$  given by  $(g, x) \mapsto \phi(g)(x)$ . Then

- (i)  $e * x = \phi(e)(x) = \text{Id}_X(x) = x$ .
- (ii)  $(g_1 g_2) * x = \phi(g_1 g_2)(x) = \phi(g_1)(\phi(g_2)(x)) = g_1 * (g_2 * x)$ .

□

**Definition**

We say  $\phi : G \rightarrow \text{Sym}(X)$  is a permutation representation of  $G$ .

**Definition** (Orbit and stabiliser)

Let  $G$  act on a set  $X$ .

- (i) The orbit of  $x \in X$  is  $\text{orb}_G(x) = \{g * x : g \in G\} \subset X$
- (ii) The stabiliser of  $x \in X$  is

$$G_x = \{g \in G : g * x = x\} \leq G.$$

Recall the Orbit-Stabiliser Theorem from IA Groups: There is a bijection  $\text{orb}_G(x) \leftrightarrow$  the set of left cosets of  $G_x$  in  $G$ . In particular if  $G$  is finite, then

$$|G| = |\text{orb}_G(x)| |G_x|.$$

**Example (Example of Orbit-Stabiliser)**

Let  $G$  be the group of all symmetries of a cube, acting on the set of vertices  $X$ . We can reach any vertex from any other one, so  $|\text{orb}_G(x)| = 8$ . Some basic geometry gives  $|G_x| = 6$ . Therefore  $|G| = 48$ .

*Remark.* •  $\ker \phi = \bigcap_{x \in X} G_x$  is called the kernel of the group action.

- The orbits partition  $X$ . We say the action is transitive if there is only one orbit.
- $G_{g*x} = gG_xg^{-1}$ , so if  $x, y \in X$  belong to the same orbit, then their stabilisers are conjugate.

Later on a lot of the proofs will involve picking a nice group action. So let's look at some examples of group actions.

- (i) Let  $G$  act on itself by left multiplication, i.e.  $g * x = gx$ . The kernel of this action is

$$\{g \in G : gx = x \quad \forall x \in G\} = e.$$

Thus  $G$  is injective into  $\text{Sym}(G)$ . This proves Cayley's theorem:

**Theorem 1.10 (Cayley's theorem)**

Any finite group  $G$  is isomorphic to a subgroup of the symmetric group  $S_n$  for some  $n$ . (Take  $n = |G|$ .)

*Proof.* As above in (i). □

- (ii) Let  $H \leq G$ ; then  $G$  acts on  $G/H$  by left multiplication, i.e.  $g * xH = gxH$ . This action is transitive (since  $(x_2x_1^{-1})x_1H = x_2H$ ) with

$$\begin{aligned} G_{xH} &= \{g \in G : gxH = xH\} \\ &= \{g \in G : x^{-1}gx \in H\} \\ &= xHx^{-1} \end{aligned}$$

Thus  $\ker(\phi) = \bigcap_{x \in G} xHx^{-1}$ . This is the largest normal subgroup of  $G$  that is contained in  $H$ .

**Theorem 1.11**

Let  $G$  be a non-abelian simple group, and  $H \leq G$  a subgroup of index  $n > 1$ . Then  $n \geq 5$  and  $G$  is isomorphic to a subgroup of  $A_n$ .

*Proof.* Let  $G$  act on  $X = G/H$  by left multiplication, and let  $\phi : G \rightarrow \text{Sym}(X)$  be the associated permutation representation. As  $G$  is simple,  $\ker(\phi) = e$  or  $G$ . If  $\ker(\phi) = G$ , then  $\text{Im}(\phi) = e$ . This is a contradiction since  $G$  acts transitively on  $X$  and  $|X| > 1$ . Thus  $\ker(\phi) = e$  and  $G \cong \text{Im}(\phi) \leq S_n$ . Since  $G \leq S_n$  and  $A_n \trianglelefteq S_n$ , the second isomorphism theorem gives  $G \cap A_n \trianglelefteq G$  and  $G/(G \cap A_n) \cong GA_n/A_n \leq S_n/A_n \cong C_2$ . Since  $G$  is simple,  $G \cap A_n = e$  (this is impossible as  $G \leq C_2$  but  $G$  isn't abelian) or  $G$ . Thus  $G \leq A_n$ . Finally, if  $n \leq 4$ , then  $A_n$  has no non-abelian simple subgroups. □

- (iii) Let  $G$  act on itself by conjugation, i.e.  $g * x = gxg^{-1}$ . We define the conjugacy class of  $x \in G$  to be

$$\text{ccl}_G(x) = \text{orb}_G(x) = \{gxg^{-1} \in G : g \in G\}.$$

We also define the centraliser of  $x$  by

$$C_G(x) = G_x = \{g \in G : gx = xg\} \leq G.$$

We define the centre of  $G$  by

$$Z(G) = \text{Ker}(\phi) = \{g \in G : gx = xg \forall x \in G\}.$$

Note that the  $\phi(g) : G \rightarrow G$  given by  $h \mapsto ghg^{-1}$  satisfies

$$\phi(g)(h_1h_2) = gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = \phi(g)(h_1)\phi(g)(h_2).$$

Thus  $\phi(g)$  is a group homomorphism, and also a bijection i.e.  $\phi(g)$  is an isomorphism.

**Definition (Automorphism)**

$\text{Aut}(G) = \{\text{group isomorphisms } \zeta : G \rightarrow G\}$ . Then  $\text{Aut}(G) \leq \text{Sym}(G)$  and  $\phi : G \rightarrow \text{Sym}(G)$  has image in  $\text{Aut}(G)$ .

- (iv) Let  $X$  be the set of all subgroups of  $G$ . Then  $G$  acts on  $X$  by conjugation, i.e.  $g * H = gHg^{-1}$ . The stabiliser of  $H$  is

$$\{g \in G : gHg^{-1} = H\} = N_G(H).$$

This is also called the normaliser of  $H$  in  $G$ , and is the largest subgroup of  $G$  containing  $H$  as a normal subgroup. In particular,

$$H \trianglelefteq G \iff N_G(H) = G.$$

## 1.4 Alternating groups

From IA Groups, we know that elements in  $S_n$  are conjugate iff they have the same cycle type. For example, in  $S_5$ , we have the following:

Cycle type	Number of elements	Sign
id	1	+1
(**)	10	-1
(**)(**)	15	+1
(***)	20	+1
(**)(***)	20	-1
(****)	30	-1
(*****)	24	+1
Total:	$120=5!= S_5 $	

Let  $g \in A_n$ . Then  $C_{A_n}(g) = C_{S_n}(g) \cap A_n$ . We effectively have two cases:

- If there exists an odd permutation commuting with  $g$ , then  $|C_{A_n}(g)| = \frac{1}{2}|C_{S_n}(g)|$  and by Orbit-Stabiliser,  $|\text{ccl}_{A_n}(g)| = |\text{ccl}_{S_n}(g)|$ .
- Otherwise,  $|C_{A_n}(g)| = |C_{S_n}(g)|$  and by Orbit-Stabiliser,  $|\text{ccl}_{A_n}(g)| = \frac{1}{2}|\text{ccl}_{S_n}(g)|$ .

**Example (Conjugacy classes of  $A_5$ )**

If we take  $n = 5$ , then first consider the element  $(12)(34)$ , which commutes with  $(12)$ . Also,  $(123)$  commutes with  $(45)$ .

But if we take  $g = (12345)$ , then  $h \in C_{S_5}(g)$  means

$$\begin{aligned} (12345) &= h(12345)h^{-1} \\ &= (h(1)h(2)h(3)h(4)h(5)) \implies h \in \langle g \rangle \leq A_5. \end{aligned}$$

In this case, the conjugacy class does split.

Thus  $A_5$  has conjugacy classes of sizes 1, 15, 20, 12, 12.

**Proposition 1.12**

$A_5$  is simple.

*Proof.* If  $H \trianglelefteq A_5$ , then  $H$  is a union of conjugacy classes. Therefore

$$|H| = 1 + 15a + 20b + 12c \quad \text{for some } a, b \in \{0, 1\} \text{ and } c \in \{0, 1, 2\}.$$

Since  $H|60$ , this implies  $H = 1$  or  $60$ , i.e.  $A_5$  is simple.  $\square$

Now we move on to a more general statement about  $A_n$  being simple. Before we can do that, we will need some lemmas for the proof.

**Lemma 1.13**

$A_n$  is generated by 3-cycles.

*Proof.* Each  $\sigma \in A_n$  is a product of an even number of transpositions. Thus it suffices to write the product of any two transpositions as a product of 3-cycles. For  $a, b, c, d$  distinct, we can have

$$\begin{cases} (ab)(bc) = (abc) \\ (ab)(cd) = (acb)(acd). \end{cases}$$

$\square$

**Lemma 1.14**

If  $n \geq 5$ , then all 3-cycles in  $A_n$  are conjugate.

*Proof.* We claim that every 3-cycle is conjugate to  $(123)$ . Indeed, if  $(abc)$  is a 3-cycle, then  $(abc) = \sigma(abc)\sigma^{-1}$  for some  $\sigma \in S_n$ . If  $\sigma \notin A_n$ , then replace  $\sigma$  by  $\sigma(45)$  (using the fact that  $n \geq 5$ ).  $\square$

**Theorem 1.15**

$A_n$  is simple for all  $n \geq 5$ .

*Proof.* Let  $e \neq N \trianglelefteq A_n$ . Suffices to show that  $N$  contains a 3-cycle, since by 1.13 and 1.14 we then have  $N = A_n$ .

Take  $e \neq \sigma \in N$  and write  $\sigma$  in its disjoint cycle decomposition. Consider the cases:

1.  $\sigma$  contains a cycle of length  $r \geq 4$ . WLOG  $\sigma = (123 \dots r)\tau$ , where  $\tau$  is some product of cycles that fixes  $1, 2, \dots, r$ .

Let  $\delta = (123)$ . Then consider the element

$$\underbrace{\sigma^{-1}}_{\in N} \underbrace{\delta^{-1} \sigma \delta}_{\in N} = (r \dots 21)(132)(12 \dots r)(123) = (23r) \in N.$$

Note  $\tau$  gets cancelled as it fixes 1 to  $r$ . Therefore  $N$  contains a 3-cycle.

2.  $\sigma$  contains two 3-cycles. WLOG  $\sigma = (123)(456)\tau$ . Let  $b = (124)$ . Then

$$\sigma^{-1} \delta^{-1} \sigma \delta = (132)(465)(142)(123)(456)(124) = (12436) \in N.$$

Then we are back to Case 1, so  $N$  contains a 3-cycle..

3.  $\sigma$  contains two 2-cycles. WLOG  $\sigma = (12)(34)\tau$ . Let  $\delta = (123)$  and consider

$$\sigma^{-1} \delta^{-1} \sigma \delta = (12)(34)(132)(12)(34)(123) = (14)(23) = \pi \in N.$$

Let  $\varepsilon = (235)$ . Then

$$\pi^{-1} \varepsilon^{-1} \pi \varepsilon = (14)(23)(253)(14)(23)(235) = (253).$$

Thus  $N$  contains a 3-cycle.

To conclude the proof, we consider the remaining cases:

1. Cycle type  $(**)$   $\implies \sigma \notin A_n$ .
2. Cycle type  $(***)$   $\implies \sigma$  is a 3-cycle.
3. Cycle type  $(**)(***)$   $\implies \sigma \notin A_n$ .

This concludes the proof. □

## 1.5 $p$ -groups and $p$ -subgroups

**Definition** ( $p$ -group)

Let  $p$  be a prime. A finite group  $G$  is a  $p$ -group if  $|G| = p^n$ ,  $n \geq 1$ .

**Theorem 1.16**

If  $G$  is a  $p$ -group, then  $Z(G) \neq 1$ .

*Proof.* For  $g \in G$ , we have  $|\text{ccl}_G(g)| |C_G(g)| = |G| = p^n$ . So each conjugacy class must have size that is a power of  $p$ . Since  $G$  is a disjoint union of conjugacy classes,

$$|G| \equiv (\text{number of conjugacy classes of size 1}) \pmod{p}$$

$$\begin{aligned} &\implies 0 \equiv |Z(G)| \pmod{p} \\ &\implies Z(G) \neq 1. \end{aligned}$$

We have used the fact that the conjugacy classes of size 1 are precisely the elements of  $Z(G)$ :

$$g \in Z(G) \iff x^{-1}gx = g \ \forall x \in G \iff \text{ccl}_G(g) = \{g\}.$$

□

### Corollary 1.17

The only simple  $p$ -group is  $C_p$ .

*Proof.* Let  $G$  be a simple  $p$ -group. Since  $Z(G) \trianglelefteq G$ , we have  $Z(G) = 1$  or  $G$ . By 1.16, we must have  $Z(G) = G$ . Therefore  $G$  is abelian. Conclude by Lemma 1.7. □

### Corollary 1.18

Let  $G$  be a  $p$ -group of order  $p^n$ . Then  $G$  has a subgroup of order  $p^r$  for  $0 \leq r \leq n$ .

*Proof.* By Lemma 1.8,  $G$  has a composition series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_{m-1} \trianglelefteq G_m = G.$$

with each  $G_i/G_{i-1}$  simple. Since  $G$  is a  $p$ -group, all of the  $G_i/G_{i-1}$  must be  $p$ -groups. Therefore have by Proposition 1.17 that  $G_i/G_{i-1} \cong C_p$ .

Thus  $|G_i| = p^i$  for all  $0 \leq i \leq m$  and  $m = n$ . □

### Lemma 1.19

For  $G$  a group, if  $G/Z(G)$  is cyclic, then  $G$  is abelian (so in fact  $G/Z(G) = 1$ ).

*Proof.* Let  $gZ(G)$  be a generator for  $G/Z(G)$ . Then each coset is of the form  $g^r Z(G)$  for some  $r \in \mathbb{Z}$ . Thus

$$G = \{g^r z : r \in \mathbb{Z}, z \in Z(G)\}.$$

We now check that elements in this group always commute:

$$\begin{aligned} g^{r_1} z_1 g^{r_2} z_2 &= g^{r_1+r_2} z_1 z_2 \quad \text{since } z \in Z(G) \\ &= g^{r_1+r_2} z_2 z_1 \\ &= g^{r_2} z_2 g^{r_1} z_1. \end{aligned}$$

Therefore  $G$  is abelian. □

### Corollary 1.20

If  $|G| = p^2$  then  $G$  is abelian.

*Proof.* We know that  $|Z(G)| \in \{1, p, p^2\}$ . We can't have 1 by 1.16. If we have  $p$ , then  $|G/Z(G)| = p$  and therefore is cyclic. Now applying 1.19 we have that  $G$  is abelian. If we have  $p^2$  then  $Z(G) = G$  so  $G$  is abelian. □

## 1.6 The Sylow theorems

### Theorem 1.21 (Sylow theorems)

Let  $G$  be a finite group of order  $p^a m$  where  $p$  is a prime with  $p \nmid m$ . Then

- (i) The set  $\text{Syl}_p(G) = \{P \leq G : |P| = p^a\}$  of Sylow  $p$ -subgroups is non-empty.
- (ii) All elements of  $\text{Syl}_p(G)$  are conjugate.
- (iii) We define  $n_p = |\text{Syl}_p(G)|$ . This satisfies  $n_p \equiv 1 \pmod{p}$  and  $n_p \mid |G|$  (and so  $n_p \mid m$ ).

*Proof.* (i) Let  $\Omega$  be the set of all **subsets** of  $G$  of order  $p^a$ . We know that

$$|\Omega| = \binom{p^a m}{p^a} = \binom{p^a m}{p^a} \binom{p^a m - 1}{p^a - 1} \cdots \binom{p^a m - p^a + 1}{1}.$$

For  $0 \leq k \leq p^a$ , the number  $p^a m - k$  and  $p^a - k$  are divisible by the same power of  $p$ .

Therefore  $|\Omega|$  is coprime to  $p$ . (†)

Let  $G$  act on  $\Omega$  by left-multiplication, i.e for  $g \in G$  and  $x \in \Omega$  we have

$$g * X = \{gx : x \in X\} \in \Omega.$$

For any  $X \in \Omega$  we have

$$|G_x| \mid |\text{orb}_G(X)| = |G| = p^a m.$$

By (†), there exists  $X$  such that  $|\text{orb}_G(X)|$  is coprime to  $p$ . This is because the orbits give a partition of  $\Omega$ , and so they can't all divide  $p$ . Thus

$$p^a \mid |G_x| \tag{1}$$

On the other hand, if  $g \in G$  and  $x \in X$ , then  $g \in (gx^{-1}) * X$  and hence

$$\begin{aligned} G &= \bigcup_{g \in G} g * X = \bigcup_{y \in \text{orb}_G(X)} Y \\ \implies |G| &\leq |\text{orb}_G(X)| \cdot |X| \quad \text{since } |Y| = |X| \\ \implies |G_x| &= \frac{|G|}{|\text{orb}_G(X)| = |X| = p^a}. \end{aligned} \tag{2}$$

By 1 and 2,  $|G_x| = p^a$ , i.e  $G_x \in \text{Syl}_p(G)$ .

- (ii) We prove a stronger result. We claim that if  $P \in \text{Syl}_p(G)$  and  $Q \leq G$  is a  $p$ -subgroup, then  $Q \leq gP^{-1}g^{-1}$  for some  $g \in G$ .

The proof is as follows: let  $Q$  act on the set of left cosets (not a group!)  $G/P$  by left multiplication, i.e  $q * gP = qgP$ . By Orbit-Stabiliser, each orbit has size  $|Q|$ , so either 1 or a multiple of  $p$ . Since  $|G/P| = m$  by definition, and  $m$  is coprime to  $p$ , there must exist some orbit of size 1, i.e

$$\exists g \in G : \quad qgP = gP \quad \forall q \in Q$$

$$\begin{aligned} &\implies g^{-1}qg \in P \quad \forall q \in Q \\ &\implies Q \leq gPg^{-1}. \end{aligned}$$

So we are done.

- (iii) Let  $G$  act on  $\text{Syl}_p(G)$  by conjugation. Sylow (ii) tells us this action is transitive. Thus orbit-stabiliser implies

$$n_p = |\text{Syl}_p(G)| \mid |G|.$$

Now to show that  $n_p \equiv 1 \pmod{p}$ , let  $P \in \text{Syl}_p(G)$  act on  $\text{Syl}_p(G)$  by conjugation. The orbits have size dividing  $|P| = p^a$ , so either 1 or a multiple of  $p$ . To show  $n_p \equiv 1 \pmod{p}$ , it suffices to show that  $\{P\}$  is the unique orbit of size 1.

If  $\{Q\}$  is another orbit of size 1, then  $P$  normalises  $Q$ , i.e.  $P \leq N_G(Q)$ . Now  $P, Q$  are both Sylow  $p$ -subgroups of  $N_G(Q)$  since  $|N_G(Q)| \leq p^a$ . Thus by (ii),  $P$  and  $Q$  are conjugate in  $N_G(Q)$  - but  $Q \trianglelefteq N_G(Q)$ , thus  $P = Q$ . This completes the proof. □

Now let's look at an application of these theorems.

### Corollary 1.22

If  $n_p = 1$ , then the unique Sylow  $p$ -subgroup is normal.

*Proof.* Let  $g \in G$  and  $P \in \text{Syl}_p(G)$ . Then  $gPg^{-1} \in \text{Syl}_p(G)$ , and so  $gPg^{-1} = P$ . Thus  $P \trianglelefteq G$ . □

This is very useful to show groups of certain orders can't be simple.

### Example

Let  $|G| = 1000 = 2^3 \cdot 5^3$ . Then  $n_5 = 1 \pmod{5}$  and  $n_5 \mid 8$  so  $n_5 = 1$ . Thus the unique Sylow 5-subgroup is normal and hence  $G$  is not simple.

### Example

Let  $|G| = 132 = 2^2 \cdot 3 \cdot 11$ . We have that  $n_{11} = 1 \pmod{11}$  and  $n_{11} \mid 12$ . So  $n_{11} = 1$  or 12. Suppose  $G$  is simple. Then  $n_{11} \neq 1$  (otherwise the Sylow 11-subgroup is normal). Hence  $n_{11} = 12$ .

Now  $n_3 \equiv 1 \pmod{3}$  and  $n_3 \mid 44$ . Thus  $n_3 \in \{1, 4, 22\}$ . But the case  $n_3 = 1$  can't occur as before. Now suppose  $n_3 = 4$ . Then letting  $G$  act on  $\text{Syl}_3(G)$  by conjugation gives a group homomorphism  $\phi : G \rightarrow S_4$ . But then  $\text{Ker } \phi \trianglelefteq G \implies \text{Ker } \phi = 1$  or  $G$ . But  $\text{Ker } \phi = 1$  would mean that  $G$  injects into  $S_4$ , which is a contradiction as  $|G| = 132 > 24 = |S_4|$ , and  $\text{Ker } \phi = G$  would be a contradiction to Sylow (ii).

Thus  $n_3 = 22$  and  $n_{11} = 12$ . Thus  $G$  has  $22 \cdot (3 - 1) = 44$  elements of order 3 and  $120 = 12 \cdot (11 - 1)$  elements of order 11. But  $44 + 120 > 132 = |G|$  which is a contradiction. Hence  $G$  is not simple.



## 1.7 Matrix groups

Matrix groups provide a wealth of examples of finite groups, and are crucial in the classification of finite simple groups. First we will recap a few groups we've seen before in IA Groups.

For a field  $F$ , let  $GL_n(F)$  be the set of  $n \times n$  invertible matrices over  $F$ .

This contains the subgroup  $SL_n = \text{Ker}(GL_n(F) \xrightarrow{\det} F^\times)$ . Here  $F^\times = F \setminus \{0\}$ .

Let  $Z \trianglelefteq GL_n(F)$  be the normal subgroup of scalar multiples of  $I$ . This is in fact the centre of  $GL_n(F)$ , but we won't prove this in the course since the proof is pretty involved.

### Definition

We define the projective general linear group by

$$PGL_n(F) = GL_n(F)/Z,$$

and the projective special linear group by

$$PGL_n(F) = \frac{SL_n(F)}{Z \cap SL_n(F)} \cong \frac{Z \cdot SL_n(F)}{Z} \leq PGL_n(F) \quad \text{by 2nd isom. theorem.}$$

### Example

Consider  $G = GL_n(\mathbb{Z}/p\mathbb{Z})$ . A list of  $n$  vectors in  $(\mathbb{Z}/p\mathbb{Z})^n$  are the columns of some  $A \in G$  iff they are linearly independent. Thus

$$\begin{aligned} |G| &= \underbrace{(p^n - 1)}_{\text{1st col.}} \underbrace{(p^n - p)}_{\text{2nd col.}} \underbrace{(p^n - p^2)}_{\text{3rd col.}} \dots \underbrace{(p^n - p^{n-1})}_{\text{last col.}} \\ &= p^{1+2+\dots+n-1} (p^{n-1} - 1)(p^{n-2} - 1) \dots (p - 1) \\ &= p^{n(n-1)/2} \prod_{i=1}^n (p^i - 1). \end{aligned}$$

So the Sylow  $p$ -subgroups have size  $p^{n(n-1)/2}$ . Let

$$U = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & M & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right\} \leq G$$

be the set of upper triangular matrices with 1's on the diagonal. Then  $U \in \text{Syl}_p(G)$ , since it has  $n(n-1)/2$  entries and each can take  $p$  values.

Just as  $PGL_2(\mathbb{C})$  acts on  $\mathbb{C} \cup \{\infty\}$  via Möbius transformations,  $PGL_2(\mathbb{Z}/p\mathbb{Z})$  acts on  $\mathbb{Z}/p\mathbb{Z} \cup \{\infty\}$  via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d}.$$

Since scalar matrices act trivially, we obtain an action of  $PGL_2(\mathbb{Z}/p\mathbb{Z})$ .

**Lemma 1.23**

The permutation representation  $PGL_2(\mathbb{Z}/p\mathbb{Z}) \rightarrow S_{p+1}$  is injective (in fact an isomorphism if  $p = 2$  or  $3$ ).

*Proof.* Suppose  $\frac{az+b}{cz+d} = z$  for all  $z \in \mathbb{Z}/p\mathbb{Z} \cup \{\infty\}$ .

- Setting  $z = 0$  gives  $b = 0$ .
- Setting  $z = \infty$  gives  $c = 0$ .
- Setting  $z = 1$  gives  $a = d$ .

So  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is a scalar matrix, hence it is trivial in  $PGL_2(\mathbb{Z}/p\mathbb{Z})$ . □

**Lemma 1.24**

If  $p$  is an odd prime, then

$$|PSL_2(\mathbb{Z}/p\mathbb{Z})| = p(p-1)(p+1)/2.$$

*Proof.* By Example 5.1,  $|GL_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2 - 1)(p - 1)$ . The homomorphism  $GL_2(\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\det} (\mathbb{Z}/p\mathbb{Z})^\times$  is surjective.

Thus  $|SL_2(\mathbb{Z}/p\mathbb{Z})| = |GL_2(\mathbb{Z}/p\mathbb{Z})|/(p-1) = p(p-1)(p+1)$ . If  $\begin{pmatrix} \lambda & 0 \\ \lambda & 0 \end{pmatrix} \in SL_2(\mathbb{Z}/p\mathbb{Z})$ , then  $\lambda^2 \equiv 1 \pmod{p} \implies \lambda \equiv \pm 1 \pmod{p}$  (since  $p$  is prime).

Thus  $Z \cap SL_2(\mathbb{Z}/p\mathbb{Z}) = \{\pm I\}$  which are distinct since  $p > 2$ . Therefore

$$|PSL_2(\mathbb{Z}/p\mathbb{Z})| = \frac{1}{2}|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p-1)(p+1)/2.$$

□

**Example**

Let  $G = PSL_2(\mathbb{Z}/5\mathbb{Z})$ . Then  $|G| = \frac{4 \cdot 5 \cdot 6}{2} = 60$ .

Let  $G$  act on  $\mathbb{Z}/5\mathbb{Z} \cup \{\infty\}$  via Mobius transformations. By Lemma 1.23, the permutation representation

$$\phi : G \rightarrow \text{Sym}(\{0, 1, 2, 3, 4\} \cup \infty) \cong S_6 \quad \text{is injective.}$$

**Claim.**  $\text{Im}(\phi) \leq A_6$ , i.e.  $\psi : G \rightarrow S_6 \xrightarrow{\text{sgn}} \{\pm 1\}$  is trivial.

**Proof.** Let  $h \in G$  have order  $2^m$ ,  $m$  odd. If  $\psi(h^m) = 1$ , then  $\psi(h)^m = 1 \implies \psi(h) = 1$ . So suffices to show  $\psi(g) = 1$  for all  $g \in G$  with order a power of 2. But we know that every such  $g$  belongs to a Sylow 2-subgroup. It then suffices to show  $\psi(H) = 1$ , for  $H$  a Sylow 2-subgroup (since  $\text{Ker } \psi$  is normal and all Sylow 2-subgroups are conjugate). Take

$$H = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} (\pm I), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\pm I) \right\rangle.$$

We compute that  $\phi \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (14)(23)$ , since it acts as  $z \mapsto -z$ . Also,  $\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (0\infty)(14)$ , since it acts as  $z \mapsto -\frac{1}{z}$ . Thus  $\psi(H) = 1$ . This proves the claim.

See Example Sheet 1 Q14 for a similar result: If  $G \leq A_6$  and  $|G| = 60$ , then  $G \cong A_5$ .

*Remarks.* (Not proved in this course)

- $PSL_n(\mathbb{Z}/p\mathbb{Z})$  is a simple group for all  $n \geq 2$  and  $p$  a prime, e.g.  $(n, p) = (2, 2), (2, 3)$  (finite groups of Lie type).
- The smallest non-abelian simple groups are  $A_5 \cong PSL_2(\mathbb{Z}/5\mathbb{Z})$  of order 60, and  $PSL_2(\mathbb{Z}/7\mathbb{Z})$  of order 168.

## 1.8 Finite abelian groups

We now investigate finite abelian groups, which we can actually characterise very effectively.

### Lemma 1.25

If  $m, n \in \mathbb{N}$  are coprime, then  $C_m \times C_n \cong C_{mn}$ .

*Proof.* Let  $g$  and  $h$  be generators of  $C_n$  and  $C_m$ . Then  $(g, h) \in C_m \times C_n$  and  $(g, h)^r = (g^r, h^r)$ . Hence

$$\begin{aligned} (g, h)^r = 1 &\iff m|r \text{ and } n|r \\ &\iff mn|r \quad \text{as } m, n \text{ coprime.} \end{aligned}$$

Thus  $(g, h)$  has order  $mn = |C_m \times C_n|$ . So

$$C_m \times C_n \cong C_{mn} \cong \langle (g, h) \rangle.$$

□

### Corollary 1.26

Let  $G$  be a finite abelian group. Then

$$G \cong C_{n_1} \times C_{n_2} \times \dots \times C_{n_k},$$

where  $n_i$  are prime powers.

*Proof.* If  $n = p_1^{a_1} \dots p_r^{a_r}$ , where  $p_1, \dots, p_r$  are distinct primes, then we just apply our previous lemma inductively to get

$$C_n \cong C_{p_1^{a_1}} \times C_{p_2^{a_2}} \times \dots \times C_{p_r^{a_r}}.$$

□

### Theorem 1.27

Every finite abelian group  $G$  is isomorphic to a product of cyclic groups.

Note such an isomorphism is not unique.

*Proof.* Immediate by applying Corollary 1.26. □

### Theorem 1.28

Let  $G$  be a finite abelian group. Then  $G \cong C_{d_1} \times C_{d_2} \times \dots \times C_{d_t}$ , for some  $d_1 | d_2 | \dots | d_t$  (they are successively divisible).

*Remark.*

This almost immediately shows that finite abelian groups are pretty easy to work with. Let's use our results to compute what the abelian groups of various orders are in an example.

**Example** (i) The abelian groups of order 8 are

$$C_8, C_2 \times C_4, C_2 \times C_2 \times C_2.$$

(ii) For the abelian groups of order 12, using 1.27 we get that they are

$$C_2 \times C_2 \times C_3, C_4 \times C_3.$$

Using 1.28 we get

$$C_2 \times C_6, C_{12}.$$

This isn't a problem as these are pairwise isomorphic.

**Definition** (Exponent of a group)

The **exponent** of a group  $G$  is the least integer  $n \geq 1$  such that  $g^n = 1$  for all  $g \in G$ , i.e the LCM of the orders of the elements of  $G$ . For example,  $A_4$  has exponent 6.

### Corollary 1.29

Every finite abelian group contains an element whose order is the exponent of the group.

*Proof.* If  $G \cong C_{d_1} \times C_{d_2} \times \dots \times C_{d_t}$  with  $d_1 | d_2 | \dots | d_t$ , then every  $g \in G$  has order dividing  $d_t$ , and  $h \in C_{d_t}$  of order  $d_t$ , then  $(1, 1, \dots, h) \in G$  has order  $d_t$ . Thus  $G$  has exponent  $d_t$ . □

## 2 Rings

### 2.1 Definitions and examples

#### Definition (Ring)

A **ring** is a triple  $(R, +, \cdot)$  consisting of a set  $R$  and two binary operations  $+: R \times R \rightarrow R, \cdot: R \times R \rightarrow R$  satisfying:

- (i) Addition:  $(R, +)$  is an abelian group, with identity element 0.
- (ii) Multiplication: the operation  $\cdot$  is associative, and has an identity 1.
- (iii) Distributivity:  $x \cdot (y + z) = x \cdot y + x \cdot z$  and  $(y + z) \cdot x = y \cdot x + z \cdot x$  for all  $x, y, z \in R$ .

We say  $R$  is a **commutative ring** if  $x \cdot y = y \cdot x$  for all  $x, y \in R$ . In this course, we will only consider commutative rings.

*Remarks.*

- (i) As in the case of groups, we should check closure!
- (ii) For any  $x \in R$ , write  $-x$  for the inverse of  $x$  under  $+$ , and abbreviate  $x + (-y)$  as  $x - y$ .
- (iii)  $0 \cdot x = (0 + 0) \cdot x = 0 = 0 \cdot x + 0 \cdot x$ . Therefore  $0 \cdot x = 0$  for all  $x \in R$ .
- (iv)  $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x$ . So  $(-1) \cdot x = -x$  for all  $x \in R$ .

#### Definition (Subring)

A subset  $S \subset R$  is a subring (written  $S \leq R$ ) if it is a ring under  $+$  and  $\cdot$ , with the same identity elements 0 and 1.

#### Example

- (i)  $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$  are all rings.
- (ii)  $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \leq \mathbb{C}$  is a ring. This is called the ring of Gaussian integers.
- (iii)  $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \leq \mathbb{R}$ .
- (iv)  $\mathbb{Z}/n\mathbb{Z} = (\text{integers modulo } n)$
- (v) If  $R, S$  are rings, we define the **product ring** to be the set  $R \times S$  via

$$\begin{aligned}(r_1, s_1) + (r_2, s_2) &= (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1) \cdot (r_2, s_2) &= (r_1 \cdot r_2, s_1 \cdot s_2) \\ 0_{R \times S} &= (0_R, 0_S) \text{ and } 1_{R \times S} = (1_R, 1_S).\end{aligned}$$

(vi) For  $R$  a ring, a **polynomial**  $f$  over  $R$  is an expression

$$f = a_0 + a_1X + a_2X^2 + \dots + a_nX^n, \quad a_i \in R.$$

" $X$ " is just a formal symbol (i.e our definition of a polynomial just means some finite sequence in  $R$ ). The degree of  $f$  is the largest  $n \in \mathbb{N}$  s.t  $a_n \neq 0$ . We write  $R[X]$  for the set of all polynomials over  $R$ .

*Remark.* We say  $f$  is a **monic** polynomial if  $a_n = 1_R$ .

If  $g = b_0 + b_1X + \dots + b_mX^m$  is another polynomial, set

$$f + g = \sum_i (a_i + b_i)X^i$$

$$f \cdot g = \sum_i \left( \sum_{j=0}^i a_j b_{i-j} \right) X^i$$

Then  $R[X]$  is a ring with identities  $0_R$  and  $1_R$ , which are constant polynomials. We identify  $R$  with the subring of constant polynomials (i.e  $a_i = 0$  for all  $i > 0$ )

#### Definition (Unit)

An element  $r \in R$  is a **unit** if it has an inverse under multiplication, i.e  $\exists s \in R : s \cdot r = 1$ .

The units in  $R$  form an abelian group  $(R^\times, \cdot)$  under multiplication. For example,  $\mathbb{Z}^\times = \{\pm 1\}$  and  $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$ .

#### Definition (Field)

A **field** is a ring with  $0 \neq 1$ , such that every non-zero element is a unit. It's "a ring where you can divide". Examples of rings are  $\mathbb{Q}$  and  $\mathbb{Z}/p\mathbb{Z}$  for  $p$  prime.

*Remark.* If  $R$  is a ring where  $0 = 1$ , then for all  $x \in R$  we have

$$x = 1 \cdot x = 0 \cdot x = 0.$$

So  $R = \{0\}$  is the trivial ring. This is why we stipulate that  $0 \neq 1$  for a ring to be a field.

#### Proposition 2.1 (Euclidean algorithm for rings)

Let  $f, g \in R[X]$ . Suppose the leading coefficient of  $g$  is a unit. Then there exist  $q, r \in R[X]$  s.t.

$$f(X) = q(X)g(X) + r(X) \quad \text{where } \deg(r) < \deg(g).$$

*Proof.* Induction on  $n = \deg(f)$ . Write

$$f(X) = a_nX^n + a_{n-1}X^{n-1} + \dots + a_0, \quad a_n \neq 0$$

$$g(X) = b_nX^n + b_{n-1}X^{n-1} + \dots + b_0, \quad b_n \neq 0$$

If  $n < m$ , then let  $q = 0$  and  $r = f$ . Done. Otherwise, we have  $n \geq m$  and we set

$$f_1(X) = f(X) - a_n b_m^{-1} g(X) X^{n-m}.$$

The coefficient of  $X^n$  is  $a_n - a_n b_m^{-1} b_m = 0$ . Thus  $\deg(f_1) < n$ .

By the inductive hypothesis,  $\exists q_1, r \in R[X]$  such that  $f_1(X) = q_1(X)g(X) + r(X)$  where  $\deg(r) < \deg(g)$ . Therefore

$$f(X) = \underbrace{q_1(X) + a_n b_m^{-1} X^{n-m}}_{q(X)} + r(X).$$

So we are done. □

### Example

Let's look at further examples of rings.

- (i) If  $R$  is a ring and  $X$  is a set, then the set of all functions  $X \rightarrow R$  is a ring under pointwise operations:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x)\end{aligned}$$

Further interesting examples appear as subrings, for example the ring of all  $C^1$  functions from  $\mathbb{R} \rightarrow \mathbb{R}$ .

- (ii) Power series ring  $R[[X]] = a_0 + a_1X + a_2X^2 + \dots$ , where the  $a_i \in R$ . We use the same operations  $+$  and  $\cdot$  as the polynomial ring. We could view this as an “infinite version” of the polynomial ring.
- (iii) Laurent polynomials:

$$R[X, X^{-1}] = \left\{ \sum_{i \in \mathbb{Z}} a_i X^i : a_i \in R, \text{ and } a_i \text{ is non-zero for finitely many } i \right\}.$$

## 2.2 Homomorphisms, ideals and quotients

We now define a homomorphism for rings, which similarly to the case of groups preserves the structure of a ring.

### Definition (Ring homomorphism)

Let  $R$  and  $S$  be rings. a function  $\phi : R \rightarrow S$  is a ring homomorphism if

- (i)  $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$  for all  $r_1, r_2 \in R$
- (ii)  $\phi(r_1 \cdot_R r_2) = \phi(r_1) \cdot_S \phi(r_2)$  for all  $r_1, r_2 \in R$
- (iii)  $\phi(1_R) = 1_S$

A ring homomorphism that is also a bijection is called an isomorphism.

The kernel of  $\phi$  is  $\text{Ker}(\phi) = \{r \in R : \phi(r) = 0\}$ .

**Lemma 2.2**

A ring homomorphism  $\phi : R \rightarrow S$  is injective iff  $\text{Ker}(\phi) = 0_R$ .

*Proof.*  $\phi : (R, +) \rightarrow (S, +)$  is a group homomorphism. □

**Definition** (Ideal)

A subset  $I \subset R$  is an **ideal**, written  $I \trianglelefteq R$  if

- (i)  $(I, +)$  is a subgroup of  $(R, +)$
- (ii) If  $r \in R$  and  $x \in I$ , then  $rx \in I$ .

We say  $I$  is proper if  $I \neq R$ .

The point of ideals is that they should arise as the kernels of ring homomorphisms; they are analogous to normal subgroups.

**Lemma 2.3**

If  $\phi : R \rightarrow S$  is a ring homomorphism, then  $\text{Ker}(\phi)$  is an ideal of  $R$ .

*Proof.*  $\phi : (R, +) \rightarrow (S, +)$  is a group homomorphism, so  $\text{Ker } \phi$  is a subgroup of  $(R, +)$ . If  $r \in R$  and  $x \in \text{Ker } \phi$ , then

$$\phi(rx) = \phi(r)\phi(x) = \phi(r)0_S = 0_S \implies rx \in \text{Ker } \phi.$$

□

*Remark.* If  $I$  contains a unit  $u$ , then  $1_R \in I$  (by multiplying by  $-u$ ). Hence  $I = R$  (multiplying any element in  $R$  by  $1_R$ ). Thus if  $I$  is a proper ideal,  $1_R \notin I$ , so  $I$  is not a subring of  $R$ .

**Lemma 2.4**

The ideals in  $\mathbb{Z}$  are  $n\mathbb{Z} = \{kn : k \in \mathbb{Z}\}$  for  $n = 0, 1, \dots$

*Proof.* Certainly  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ . Let  $I \trianglelefteq \mathbb{Z}$  be a non-zero ideal and  $n$  be the smallest positive integer in  $I$ . Then  $n\mathbb{Z} \subseteq I$ . If  $m \in I$ , then write  $m = qn + r$  with  $q, r \in \mathbb{Z}$  and  $0 \leq r < n$ . Then  $r = m - qn \in I$ . This contradicts the fact that  $n$  is the smallest positive integer in  $I$  unless  $r = 0$ . But then  $m \in n\mathbb{Z}$ , i.e.  $I = n\mathbb{Z}$ . □

**Definition** (Generated ideal)

For  $a \in R$ , write  $(a) = \{ra : r \in R\} \trianglelefteq R$ . This is the **ideal generated by  $a$** . More generally, if  $a_1, \dots, a_n \in R$ , we write

$$(a_1, \dots, a_n) = \{r_1a_1 + r_2a_2 + \dots + r_na_n : r_i \in R\} \trianglelefteq R.$$

**Definition** (Principal ideal)

Let  $I \trianglelefteq R$ . We say  $I$  is a **principal ideal** if  $I = (a)$  for some  $a \in R$ .



Now we show the converse of Lemma 2.3.

**Theorem 2.5**

If  $I \trianglelefteq R$ , then the set  $R/I$  of cosets of  $I$  in  $(R, +)$  forms a ring (called the **quotient ring**) with the operations

$$\begin{aligned}(r_1 + I) + (r_2 + I) &= r_1 + r_2 + I \\ (r_1 + I) \cdot (r_2 + I) &= r_1 \cdot r_2 + I\end{aligned}$$

and  $0_{R/I} = 0_R + I$ ,  $1_{R/I} = 1_R + I$ . Moreover, the map  $R \rightarrow \frac{R}{I}$  with  $r \mapsto r + I$  is a ring homomorphism (called the quotient map) with kernel  $I$ .

*Proof.* Already know  $(R/I, +)$  is a group by our definition of the quotient group.

If  $r_1 + I = r'_1 + I$  and  $r_2 + I = r'_2 + I$ , this means we can write  $r'_1 = r_1 + a_1$ ,  $r'_2 = r_2 + a_2$  for some  $a_1, a_2 \in I$ . Then

$$\begin{aligned}r'_1 r'_2 &= (r_1 + a_1)(r_2 + a_2) \\ &= r_1 r_2 + \underbrace{r_1 a_2 + r_2 a_1 + a_1 a_2}_{\in I}.\end{aligned}$$

Thus  $r_1 r_2 + I = r'_1 r'_2 + I$ . The remaining properties for  $R/I$  follow from those properties for  $R$ .  $\square$

**Example 2.6 (Examples of quotient rings)** 1.  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$  is an ideal, with quotient ring  $\mathbb{Z}/n\mathbb{Z}$ . The cosets are  $0 + n\mathbb{Z}, 1 + n\mathbb{Z}, \dots, (n-1) + n\mathbb{Z}$ . Addition and multiplication are carried out modulo  $n$ .

2. Consider  $(X) \trianglelefteq \mathbb{C}[X]$  be the ideal of polynomials with constant term 0.

If  $f(X) = \underbrace{a_n X^n + \dots + a_1 X}_{\in (X)} + a_0$  for  $a_i \in \mathbb{C}$ , then  $f(X) + (X) = a_0 + (X)$ .

There is a bijection  $\mathbb{C}[X]/(X) \rightarrow \mathbb{C}$  given by

$$f(X) + (X) \mapsto f(0), \quad a + (X) \mapsto a.$$

These maps are ring homomorphisms. Thus  $\mathbb{C}[X]/(X) \cong \mathbb{C}$ .

3. Consider  $(X^2 + 1) \trianglelefteq \mathbb{R}[X]$ . We want to understand

$$\mathbb{R}[X]/(X^2 + 1) = \{f(X) + (X^2 + 1) : f(X) \in \mathbb{R}[X]\}.$$

By Proposition 2.1,  $f(X) = g(X)(X^2 + 1) + r(X)$  with  $\deg r < 2$ , i.e.  $r(X) = a + bX$  for  $a, b \in \mathbb{R}$ . Thus  $\mathbb{R}[X]/(X^2 + 1) = \{a + bX + (X^2 + 1) : a, b \in \mathbb{R}\}$ . Now let's investigate if this representation is unique.

If  $a + bX + (X^2 + 1) = a' + b'X + (X^2 + 1)$ , then  $(a - a') + (b - b')X = g(X)(X^2 + 1)$  for some  $g \in \mathbb{R}[X]$ . Comparing degrees, we see  $g(X) = 0$ . Thus  $a = a'$  and  $b = b'$ .

Consider the bijection  $\mathbb{R}[X]/(X^2 + 1) \xrightarrow{\phi} \mathbb{C}$  defined by  $a + bX + (X^2 + 1) \mapsto a + bi$ . We show  $\phi$  is a ring homomorphism. It preserves addition and maps  $1 + (X^2 + 1)$  to 1.

$$\begin{aligned} & \phi((a + bX) + (X^2 + 1)) (c + dX + (X^2 + 1)) \\ &= \phi((a + bX)(c + dX)(X^2 + 1)) \\ &= \phi(ac + (ad + bc)X + bd(X^2 + 1) - bd + (X^2 + 1)) \\ &= ac - bd + (ad - bc)i \\ &= (a + bi)(c + di) \\ &= \phi((a + bX) + (X^2 + 1)) \phi(c + dX + (X^2 + 1)). \end{aligned}$$

Thus  $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$ .

We move on to the isomorphism theorems for rings; they are closely related to those for groups.

**Theorem 2.7 (First isomorphism theorem for rings)**

Let  $\phi : R \rightarrow S$  be a ring homomorphism, then  $\text{Ker } \phi \trianglelefteq R$ ,  $\text{Im } \phi \leq S$  and there exists an isomorphism  $R/\text{Ker } \phi \cong \text{Im } \phi$ .

*Proof.* We know that by Lemma 2.3  $\text{Ker } \phi \trianglelefteq R$ , and  $\text{Im } \phi$  is a subgroup of  $(S, +)$ . To check that the image is closed:

$$\phi(r_1 r_2) = \phi(r_1) \phi(r_2) \in \text{Im } \phi.$$

We also note that  $1_S = \phi(1_R)$ . Thus  $\text{Im } \phi$  is a subring of  $S$ .

Let  $K = \text{Ker } \phi$ . Define

$$\Phi : R/K \rightarrow \text{Im } \phi, \quad r + K \mapsto \phi(r).$$

By the first isomorphism theorem for groups, this is well-defined, a bijection, and a group homomorphism under  $+$ . So it remains to prove that  $\Phi$  preserves multiplication and that it maps the identity to the identity. We have  $\Phi(1_R + K) = \phi(1_R) = 1_S$ , and

$$\begin{aligned} \Phi((r_1 + K)(r_2 + K)) &= \Phi(r_1 r_2 + K) \\ &= \phi(r_1 r_2) = \phi(r_1) \phi(r_2) \\ &= \Phi(r_1 + K) \Phi(r_2 + K). \end{aligned}$$

Thus  $\Phi$  is a ring isomorphism. □

**Theorem 2.8 (Second isomorphism theorem for rings)**

Let  $R \leq S$  and  $J \trianglelefteq S$ . Then  $R \cap J \trianglelefteq R$  and  $R + J \equiv \{r + j : r \in R, j \in J\} \leq S$  and

$$R/(R \cap J) \cong (R + J)/J \leq S/J.$$

*Proof.* By the second isomorphism theorem for groups,  $R+J$  is a subgroup of  $(S, +)$  and we have  $1_S = \begin{smallmatrix} \in R+J \\ 1_S \end{smallmatrix} \in R+J$ . If  $r_1, r_2 \in R$  and  $j_1, j_2 \in J$ :

$$(r_1 + j_1)(r_2 + j_2) = \begin{smallmatrix} \in R \\ r_1 r_2 \end{smallmatrix} + \begin{smallmatrix} \in J \\ r_1 j_2 + r_2 j_1 + r_2 j_2 \end{smallmatrix} \in R + J.$$

Therefore  $R + J \leq S$ . Define

$$\phi : R \rightarrow S/J, \quad r \mapsto r + J.$$

This is the composite of inclusion  $R \subseteq S$ , and the quotient  $S \mapsto S/J$ , hence  $\phi$  is a ring homomorphism. We can calculate

$$\begin{aligned} \text{Ker } \phi &= \{r \in R : r + J = J\} = R \cap J \trianglelefteq R \\ \text{Im } \phi &= \{r + J : r \in R\} = (R + J)/J \leq S/J. \end{aligned}$$

Now apply the first isomorphism theorem for rings, which concludes the proof.  $\square$

*Remark.* Let  $I \trianglelefteq R$ . There exists a bijection

$$\{\text{Ideals in } R/I\} \leftrightarrow \{\text{Ideals of } R \text{ containing } I\},$$

given by

$$K \mapsto \{r \in R : r + I \in K\}, \quad J/I \leftarrow J.$$

### **Theorem 2.9** (Third isomorphism theorem for rings)

Let  $I \trianglelefteq R$ ,  $J \trianglelefteq R$  with  $I \subset J$ . Then  $J/I \trianglelefteq R/I$ , and

$$(R/I) / (J/I) \cong R/J.$$

*Proof.* Consider  $\phi : R/I \rightarrow R/J$  with  $r + I \mapsto r + J$ . This is well-defined surjective ring homomorphism by the third isomorphism theorem for groups (Exercise). Furthermore,

$$\text{Ker } \phi = \{r + I : r \in J\} = J/I \trianglelefteq R/I.$$

Apply the first isomorphism theorem to finish the proof.  $\square$

Now we go back to Example 2.6, and reprove the last part much faster using the isomorphism theorem.

### **Example**

There is a surjective ring homomorphism

$$\phi : \mathbb{R}[X] \rightarrow \mathbb{C}, \quad f(X) = \sum a_n X^n \mapsto f(i) = \sum a_n i^n.$$

Proposition 2.1 implies  $\text{Ker } \phi = (X^2 + 1)$ . By the first isomorphism theorem,

$$\mathbb{R}[X] / (X^2 + 1) \cong \mathbb{C}.$$

If  $R$  is a ring, there exists a unique ring homomorphism  $i : \mathbb{Z} \rightarrow R$ , given by

$$0 \mapsto 0_R$$

$$\begin{aligned} 1 &\mapsto 1_R \\ n &\mapsto \underbrace{1_R + \dots + 1_R}_{n \text{ times}}. \end{aligned}$$

Since  $\text{Ker}(i) \trianglelefteq \mathbb{Z}$ , have  $\text{Ker}(i) = n\mathbb{Z}$  for some  $n \in \{0, 1, 2, \dots\}$ . By the first isomorphism theorem,

$$\mathbb{Z}/n\mathbb{Z} \cong \text{Im}(i) \leq R.$$

This motivates our next definition:

**Definition** (Characteristic)

We call  $n$  the **characteristic** of  $R$ . For example,

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  have characteristic 0.
- $\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}[X]$  has characteristic  $p$ .

## 2.3 Integral domains, maximal ideals and prime ideals

**Definition** (Integral domain/Zero-divisor)

An **integral domain** is a ring with  $0 \neq 1$  such that for  $a, b \in R$ ,  $ab = 0 \implies a = 0$  or  $b = 0$ .

A **zero-divisor** in a ring  $R$  is a non-zero element  $a \in R$  such that  $ab = 0$  for some  $0 \neq b \in R$ . So an integral domain is a ring with no zero-divisors.

**Example** 1. All fields are integral domains (if  $ab = 0$  with  $b \neq 0$ , multiply by  $b^{-1}$  to get  $a = 0$ ).

2. Any subring of an integral domain is an integral domain, e.g.  $\mathbb{Z} \leq \mathbb{Q}, \mathbb{Z}[i] \leq \mathbb{C}$ .

3.  $\mathbb{Z} \times \mathbb{Z}$  is not an integral domain since  $(1, 0) \cdot (0, 1) = (0, 0)$ .

**Lemma 2.10**

If  $R$  is an integral domain, then  $R[X]$  is an integral domain.

*Proof.* Write

$$\begin{aligned} f(X) &= a_m X^m + \dots + a_1 X + a_0, & a_m &\neq 0 \\ g(X) &= b_n X^n + \dots + b_1 X + b_0, & b_n &\neq 0. \end{aligned}$$

$$\text{Then } f(X)g(X) = \underbrace{a_m b_n}_{\neq 0 \text{ as } R \text{ integral domain}} + \dots$$

and  $\deg(fg) = m + n = \deg f + \deg g$  and  $f \cdot g \neq 0$ . □

**Lemma 2.11**

Let  $R$  be an integral domain and  $0 \neq f \in R[X]$ . Let the set of roots of  $f$  be

$$\mathcal{R} = \{a \in R : f(a) = 0\}.$$

Then  $|R| \leq \deg(f)$ .

*Proof.* Uses Proposition 2.1; see Example Sheet 1.  $\square$

**Theorem 2.12** (Subgroups of a field are cyclic)

Let  $F$  be a field. Then any finite subgroup  $G \leq (F^\times, \cdot)$  is cyclic.

*Proof.*  $G$  is a finite abelian group. If  $G$  is not cyclic, then by Theorem , there exists  $H \leq G$  such that  $H \cong C_{d_1} \times C_{d_1}$  for some  $d_1 \geq 2$ . But then the polynomial

$$f(X) = X^{d_1} - 1 \in F[X]$$

has degree  $d_1$ , and  $\geq d_1^2$  roots which contradicts our previous lemma.  $\square$

**Example**

This theorem tells us that the field  $(\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic.

**Proposition 2.13**

Any finite integral domain is a field.

*Proof.* Let  $R$  be a finite integral domain. Let  $a \in R$ ,  $a \neq 0$ .

Consider the map  $\phi : R \rightarrow R$ :  $x \mapsto ax$ .

If  $\phi(x) = \phi(y)$  then  $a(x - y) = 0$ . Since  $R$  is an integral domain and  $a \neq 0$ , this gives  $x - y = 0$ . So  $x = y$ . Thus  $\phi$  is injective, and hence surjective since  $R$  is finite. Therefore

$$\exists b \in R : a \cdot b = 1 \implies a \text{ is a unit.}$$

Thus  $R$  is a field.  $\square$

The next theorem tells us about the converse of this statement.

**Theorem 2.14** (Field of fractions)

Let  $R$  be an integral domain. Then there exists a field  $F$  such that:

- (i)  $R \leq F$ .
- (ii) Every element of  $F$  can be written in the form  $a \cdot b^{-1}$  where  $a, b \in R$  and  $b \neq 0$ .

*Remark.* Condition (ii) guarantees our field is the “smallest” field with  $R \leq F$ .  $F$  is called the **field of fractions** of  $R$ .

*Proof.* Consider the set  $S = \{(a, b) \in R : b \neq 0\}$ , and the equivalence relation  $\sim$  on  $S$  given by

$$(a, b) \sim (c, d) \iff ad - bc = 0.$$

Clearly this is reflexive and symmetric. For transitivity, if  $(a, b) \sim (c, d) \sim (e, f)$ ,

then

$$(ad)f = (bc)f = b(cf) = b(de) \\ \implies d(af - be) = 0$$

Since  $R$  is integral and  $d \neq 0$ , this gives  $af - be = 0$ , i.e.  $(a, b) \sim (e, f)$ .

Let  $F = S/\sim$  be the set of equivalence classes and (somewhat suggestively) write  $\frac{a}{b}$  for  $[(a, b)]$ . Define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

It can easily be checked that these operations are well-defined and make  $F$  into a ring with  $0_F = \frac{0_R}{1_R}$  and  $1_F = \frac{1_R}{1_R}$ . If  $\frac{a}{b} \neq 0_F$ , then  $a \neq 0_R$ , and  $\frac{a}{b} \cdot \frac{b}{a} = \frac{1_R}{1_R} = 1_F$ . So  $F$  is a field. Now let's check the conditions we gave.

(i) Identify  $R$  with the subring  $\left\{ \frac{r}{1_R : r \in R} \right\} \leq F$ .

(ii)  $\frac{a}{b} = a \cdot b^{-1}$ .

This concludes the proof.  $\square$

*Remark.* In this proof, we have essentially just mimicked the construction of the rationals. This leads on to our first example.

**Example** 1.  $\mathbb{Z}$  is an integral domain with field of fractions  $\mathbb{Q}$ .

2.  $\mathbb{C}[X]$  has field of fractions

$$\mathbb{C}(X) \equiv \text{field of rational functions in } X.$$

Now we move on to discussing maximal and prime ideals, which we will use extensively in the next section.

**Definition** (Maximal ideal)

An ideal  $I \leq R$  is **maximal** if  $I \neq R$  and if  $I \leq J \leq R$ , then  $J = I$  or  $R$ . This is the biggest *proper* ideal in  $R$ .

**Lemma 2.15**

A (non-zero) ring  $R$  is a field iff its only ideals are  $\{0\}$  and  $R$ .

*Proof.* If  $R$  is a field and  $I \leq R$  is nontrivial, then  $I$  contains a unit and hence  $I = R$ .

Conversely, suppose the only ideals in  $R$  are  $\{0\}$  and  $R$ . Take  $x \in R$ ,  $x \neq 0$ , then the ideal  $(x)$  is non-zero, hence  $(x) = R$ . So  $\exists y \in R : x \cdot y = 1$ . So  $x$  is a unit.  $\square$

**Proposition 2.16**

Let  $I \leq R$  be an ideal.  $I$  is maximal iff  $R/I$  is a field.

*Proof.*  $R/I$  is a field  $\iff I/I$  and  $R/I$  are the only ideals in  $R/I$

$\iff I$  and  $R$  are the only ideals in  $R$  which contain  $I$

$\iff I \trianglelefteq R$  is maximal. □

**Definition** (Prime ideal)

An ideal  $I \trianglelefteq R$  is prime if  $I \neq R$  and whenever  $a, b \in R$  with  $ab \in I$ , we have  $a \in I$  or  $b \in I$ .

**Example**

The ideal  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$  is a prime ideal iff  $n = 0$  or  $n = p$  is prime.

Checking: If  $ab \in p\mathbb{Z}$ , then  $p|ab$ , so  $p|a$  or  $p|b$ , i.e.  $a \in p\mathbb{Z}$  or  $b \in p\mathbb{Z}$ .

Conversely, if  $n = uv$  with  $u, v > 1$ , then  $uv \in n\mathbb{Z}$ , but  $u \notin n\mathbb{Z}$ , but  $u \notin n\mathbb{Z}$ ,  $v \notin n\mathbb{Z}$ .

*Remark.* The motivation of a prime ideal is to generalise the notion of prime numbers to arbitrary rings.

**Proposition 2.17**

Let  $I \trianglelefteq R$  be an ideal. Then

$$I \text{ is prime} \iff R/I \text{ is an integral domain.}$$

*Proof.*

$I$  is prime

$\iff$  Whenever  $a, b \in R$  with  $ab \in I$ , we have  $a \in I$  or  $b \in I$ .

$\iff$  Whenever  $a + I, b + I \in R/I$  with  $(a + I)(b + I) = 0 + I$ , we have  $a + I = 0 + I$  or  $b + I = 0 + I$ .

$\iff R/I$  is an integral domain. □

*Remark.* Proposition 2.16 and Proposition 2.17 show that if  $I$  is a maximal ideal, it is also a prime ideal (since all fields are integral domains).

*Remark.* If  $\text{char}(R) = n$ , then  $\mathbb{Z}/n\mathbb{Z} \leq R$ . So if  $R$  is an integral domain, then  $\mathbb{Z}/n\mathbb{Z}$  is an integral domain. This implies that  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$  is a prime ideal, so  $n = 0$  or  $n = p$  prime.

In particular, a field has characteristic 0 (and so contains  $\mathbb{Q}$ ) or has characteristic  $p$ , in which case it contains  $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ .

## 2.4 Factorisation in integral domains

In this section we investigate the following property: we know that in  $\mathbb{N}$  we can factorise any number as a product of prime factors, and we want to know to which extent this is true in integral domains.

*Remark.* In this section we will always take  $R$  to be an integral domain.

**Definition**

We will quickly establish some key definitions that will help us work with the objects of this chapter.

- (i) Recall that  $a \in R$  is a unit if there exists  $b \in R$  with  $ab = 1$  (equivalently  $(a) = R$ ).
- (ii)  $a \in R$  **divides**  $b \in R$  (written  $a|b$ ) if there exists  $c \in R$  such that  $b = ac$  (equivalently  $(b) = (a)$ ).
- (iii)  $a, b \in R$  are **associates** if  $a = bc$  for some unit  $c \in R$ .
- (iv)  $r \in R$  is **irreducible** if  $r \neq 0$ ,  $r$  is not a unit, and  $r = ab \implies a$  or  $b$  is a unit.
- (v)  $r \in R$  is **prime** if  $r \neq 0$ ,  $r$  is not a unit and  $r|ab \implies r|a$  or  $r|b$ .

*Remark.* These properties of an element depend on the ambient ring  $R$ : for example 2 is prime and irreducible in  $\mathbb{Z}$ , but not in  $\mathbb{Q}$ .

Another example is that  $2X$  is irreducible in  $\mathbb{Q}[X]$ , but not in  $\mathbb{Z}[X]$  (since then we can factor it as  $2 \cdot X$ , and 2 is a unit in  $\mathbb{Z}$ ).

Now we want to chase some definitions to make sure our definitions of prime element and prime ideal agree.

**Lemma 2.18**

Consider  $(r) \trianglelefteq R$ . This is a prime ideal iff  $r = 0$  or  $r$  is a prime element.

*Proof.* Suppose  $(r)$  is prime and  $r \neq 0$ . Since prime ideals are proper,  $(r) \neq R$ . Therefore  $r$  is not a unit. If  $r|ab$ , then  $ab \in (r)$ , so  $a \in (r)$  or  $b \in (r)$ . Thus  $r|a$  or  $r|b$ . So  $r$  is prime.

Conversely,  $\{0\} \trianglelefteq R$  is a prime ideal since  $R$  is an integral domain. Let  $r \in R$  be a prime.  $(r) \neq R$  since  $r \in R^\times$ . If  $ab \in (r)$  then  $r|ab$ .

$$\begin{aligned} &\implies r|a \text{ or } r|b \\ &\implies r \in (a) \text{ or } r \in (b), \text{ i.e } r \text{ is a prime ideal.} \end{aligned}$$

□

**Lemma 2.19**

If  $r \in R$  is prime, then it is irreducible.

*Proof.* Since  $r$  is a prime,  $r \neq 0$  and  $r \notin R^\times$ . Suppose  $r = ab$ . Then  $r|ab$  so  $r|a$  or  $r|b$ . WLOG assume  $r|a$ , so  $a = rc$  for some element  $c \in R$ .

Then  $r = ab = rbc \implies r(1 - bc) = 0 \implies bc = 1$ . This uses the fact that  $R$  is an integral domain and  $r \neq 0$ . So  $b$  is a unit. □

The converse of this lemma does not hold in general.



**Example**

Let

$$R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \leq C, \quad R \cong \mathbb{Z}[X]/(X^2 + 5).$$

$R$  is a subring of a field, so it's an integral domain. Define a function (called the **norm**)

$$N : R \rightarrow \mathbb{Z}_{\geq 0}, \quad a + b\sqrt{-5} \mapsto a^2 + 5b^2.$$

Note that  $N(z_1 z_2) = N(z_1)N(z_2)$ .

*Claim.*  $R^\times = \{\pm 1\}$ .

*Proof.* If  $r \in R^\times$ , i.e.  $rs = 1$  for some  $s \in R$ , then

$$N(r)N(s) = N(1) = 1 \implies N(r) = 1 \text{ WLOG.}$$

But the only integer solutions to  $a^2 + 5b^2 = 1$  are  $(a, b) = (1, 0)$  or  $(-1, 0)$ . □

*Claim.*  $2 \in R$  is irreducible.

*Proof.* Suppose  $2 = rs$  where  $r, s \in R$ . We want to show one of these elements is a unit. Then

$$4 = N(2) = N(r)N(s).$$

Since  $a^2 + 5b^2 = 2$  has no integer solutions,  $R$  has no elements of norm 2. Thus  $N(r) = 1$  and  $N(s) = 4$  WLOG. But if  $N(r) = 1$ , this implies that  $r$  is a unit. □

Similarly  $3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$  are all irreducible (as there are no elements of norm 3). Now  $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3$ . Thus  $2 \mid (1 + \sqrt{-5})(1 - \sqrt{-5})$ , but  $2 \nmid 1 + \sqrt{-5}$  or  $1 - \sqrt{-5}$  (check by taking norms).

So 2 is irreducible but not prime in  $R$ .

*Remarks.* We now know that being irreducible does not imply being prime. Importantly, in the above example  $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$  gives two distinct factorisations into irreducibles. Since  $R^\times = \{\pm 1\}$ , these irreducibles are not associates.

This example tells us if we want to have a good definition of factorisation in a ring, we should impose some extra properties.

**Definition** (Principal ideal domain)

An integral domain  $R$  is a **principal ideal domain** (PID) if any ideal  $I \trianglelefteq R$  is principal. In other words, any ideal  $I \trianglelefteq R$  is generated by some element  $r \in R$ .

For example,  $\mathbb{Z}$  is a PID by Lemma 2.4.

**Proposition 2.20**

Let  $R$  be a PID. Then every irreducible element of  $R$  is prime.

*Proof.* Let  $r \in R$  be irreducible. We already know that  $r \neq 0$  and  $r$  is not a unit. Assume  $r \mid ab$  and  $r \nmid a$ .

Since  $R$  is a PID,  $(a, r) = (d)$  for some  $d \in R$ . In particular,  $r = cd$  for some  $c \in R$ . Since  $r$  is irreducible, either  $c$  or  $d$  is a unit. If  $c$  is a unit, then  $(a, r) = (r)$ . So  $r \mid a$ .

Contradiction. If  $d$  is a unit, then  $(a, r) = R$ .

So there exists  $s, t \in R$  such that  $sa + tr = 1$ . Then  $b = sab + trb$ . So  $r|b$  since  $r|ab$ . Thus  $r$  is prime.  $\square$

### Lemma 2.21

Let  $R$  be a PID, and  $0 \neq r \in R$ . Then  $r$  is irreducible iff  $(r)$  is a maximal ideal.

*Proof.* If  $r$  is irreducible,  $r \notin R^\times$  so  $(r) \neq R$ . Suppose  $(r) \subset J \subset R$  where  $J \trianglelefteq R$ . Since  $R$  is a PID,  $J = (a)$  for some  $a \in R$ . Therefore  $r = ab$  for some  $b \in R$ . Since  $r$  is irreducible, either  $a \in R^\times \implies J = R$  or  $b \in R^\times \implies J = (r)$ . Thus  $(r)$  is maximal.

Conversely, suppose  $(r)$  is maximal. So  $r \notin R^\times$ . Suppose  $r = ab$ . Then  $(r) \subset (a) \subset R$ . Since  $(r)$  is maximal, either  $(a) = (r)$  or  $R$ . If  $(a) = (r)$ , then  $a$  and  $r$  are associates and so  $b$  is a unit. If  $(a) = R$  then  $a$  is a unit. Thus  $r$  is irreducible.

*Remark.* This proof of the converse implication actually holds for a general integral domain  $R$ ; it doesn't assume  $R$  is a PID.  $\square$

*Remark.* Let  $R$  be a PID, and take a nonzero  $r \in R$ . Then  $(r)$  is maximal iff  $r$  is irreducible, which is equivalent to  $r$  being prime. This is again equivalent to  $(r)$  being prime. We can use this to deduce that there exists a bijection

$$\{ \text{non-zero prime ideals} \} \leftrightarrow \{ \text{non-zero maximal ideals} \}.$$

This is a nice property of PIDs. We can cook up some examples of PIDs using the following definition.

### Definition (Euclidean domain)

An integral domain is a **Euclidean domain** (ED) if there is a function  $\phi : R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$  (a Euclidean function) such that

- (i) If  $a|b$ , then  $\phi(a) \leq \phi(b)$ .
- (ii) If  $a, b \in R$  with  $b \neq 0$ , then there exist  $q, r \in R$  with  $a = bq + r$  and either  $r = 0$ , or  $\phi(r) < \phi(b)$ .

### Example

$\mathbb{Z}$  is an ED with Euclidean function  $\phi(n) = |n|$ .

### Proposition 2.22

If  $R$  is a Euclidean domain, then it is a principal ideal domain.

*Proof.* Let  $R$  have Euclidean function  $\phi$ . Let  $I \trianglelefteq R$  be non-zero. Choose  $b \in I \setminus \{0\}$  with  $\phi(b)$  minimal. Then  $(b) \in I$ .

For  $a \in I$ , write  $a = bq + r$  with  $q, r \in R$  and either  $r = 0$  or  $\phi(r) < \phi(b)$ . Since

$r = bq - a \in I$ , cannot have  $\phi(r) < \phi(b)$  as we chose  $b$  to be minimal. So  $r = 0$ , and  $a = bq$ . Hence  $I = (b)$ .  $\square$

*Remark.* We only used property (ii) of Euclidean domains in this proof. We include property (i) in the definition as it allows us to describe the units in  $R$  as

$$R^\times = \{u \in R \setminus \{0\} : \phi(u) = \phi(1)\}.$$

**Example** 1. Let  $F$  be a field.  $F[X]$  is an ED with Euclidean function  $\phi(f) = \deg f$  for  $f \in F[X]$ .  $\phi$  is Euclidean by Proposition 2.1.

2.  $R = \mathbb{Z}[i]$  is an ED with Euclidean function

$$\phi(a + ib) = N(a + ib) = |a + ib|^2 = a^2 + b^2.$$

Since  $N$  is multiplicative, this tells us that property (i) is satisfied.

For property (ii), let  $z_1, z_2 \in \mathbb{Z}[i]$  with  $z_2 \neq 0$ . Consider  $\frac{z_1}{z_2} \in \mathbb{C}$ . This has distance less than 1 from the nearest element of  $\mathbb{Z}[i]$ , i.e there exists  $q \in \mathbb{Z}[i]$  such that  $\left| \frac{z_1}{z_2} - q \right| < 1$ . [todo- lattice picture] Set  $r = z_1 - z_2 q \in \mathbb{Z}[i]$ . Then  $z_1 = qz_2 + r$  and

$$\phi(r) = |r|^2 = |z_1 - z_2 q|^2 < \underbrace{|z_2|^2}_{\text{by (†)}} = \phi(z_2).$$

Thus Proposition 2.22 implies that both  $\mathbb{Z}[i]$  and  $F[X]$  for  $F$  a field are PIDs.

The first application we will see involves the construction of the minimal polynomial of a matrix.

**Example**

Let  $A$  be an  $n \times n$  matrix over a field  $F$ . Let  $I = \{f \in F[X] : f(A) = 0\}$ .

*Claim.*  $I$  is an ideal in  $F$ .

*Proof.* If  $f, g \in I$ , then  $(f - g)(A) = f(A) - g(A) = 0 \implies f - g \in I$ . So  $(I, +) \leq (F, +)$ .

If  $f \in F[X]$  and  $g \in I$ , then  $(f \cdot g)(A) = f(A) \cdot g(A) = 0$ . So  $f \cdot g \in I$ .  $\square$

So  $I = (f)$  for some  $f \in F[X]$  since  $F[X]$  is a PID. Of course, this  $f$  isn't unique. We may assume  $f$  is monic by multiplying by a unit in  $F$ . Then for  $g \in F[X]$ ,

$$g(A) = 0 \iff g \in I \iff g \in (f) \iff f|g.$$

So  $f$  is the minimal polynomial of  $A$ ; we've derived it using rings (see IB Linear Algebra)

The next example will be of interest to those who want to study Part II Galois Theory.

**Example (Field of order 8)**

Let  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ . Now let  $f(X) = X^3 + X + 1 \in \mathbb{F}_2[X]$ . If  $f(X) = g(X)h(X)$  with

$g, h \in \mathbb{F}_2[X]$  and  $\deg g, \deg h > 0$ , then  $\deg g + \deg h = 3$ . Thus either  $\deg g = 1$  or  $\deg h = 1$ , and so  $f$  has a root.

But  $f(0) \neq 0$  and  $f(1) \neq 0$  in  $\mathbb{F}_2$ . Thus  $f$  is irreducible. Since  $\mathbb{F}_2$  is a PID, Lemma 2.21 implies  $(f) \trianglelefteq \mathbb{F}_2[X]$  is a maximal ideal. Hence

$$\mathbb{F}_2[X]/(f) = \{aX^2 + bX + c : a, b, c \in \mathbb{F}_2\}$$

is a field of order 8.