IB Complex Methods

Martin von Hodenberg (mjv43@cam.ac.uk)

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These are my notes for the IB course Complex Methods, which was lectured in Lent 2022 at Cambridge by Dr U. Sperhake. These notes are written in LaTeX for my own revision purposes. Any suggestions or feedback is welcome.

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§0 Background material

§0.1 Complex numbers

Recall the definition of a complex number, its real and imaginary parts, complex conjugate, modulus, and argument. Note that $\arg z$ is only defined up to adding $2n\pi$, for $n \in \mathbb{Z}$. Recall also the definition of the principal argument ($\arg z \in [-\pi, \pi]$).

Recall the triangle inequality:

$$|z_1| + |z_2| \leq |z_1| + |z_2| \quad \forall z_1, z_2 \in \mathbb{C}.$$

By setting $z_1 = \zeta_1 + \zeta_2$ and $z_2 = -\zeta_2$ we get the reverse triangle inequality

$$||\zeta_1| - |\zeta_2|| \leq |\zeta_1 + \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{C}.$$

Recall the geometric series: for $z \in \mathbb{C}$, $z \neq 1$ and $n \in \mathbb{N}_0$: $\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$.

For $|z| < 1$, this converges for $n \to \infty$: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$

Definition 0.1 (Open set)

A set $D \subset \mathbb{C}$ is an "open set" if for all $z_0 \in D$, $\exists \varepsilon > 0$ such that the ε-sphere $|z - z_0| < \varepsilon$ lies in D. A neighbourhood of $z \in \mathbb{C}$ is an open set D that contains z.

§0.2 Trigonometric and hyperbolic functions

Recall Euler's identity, and the complex definitions of cos, sin, and their hyperbolic counterparts. Recall that $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$ from the definitions.

§0.3 Calculus of real functions in ≥ 1 variables

Sometimes, we regard a complex function as 2 real functions on $\mathbb{R}^2$: $f(z) = u(x, y) + iv(x, y)$. (See IB Complex Analysis notes for more on this.)

Definition 0.2

We define $C^m(\Omega)$ as the set of functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ whose partial derivatives up to order m exist and are continuous.

Remark. We need the continuity condition: consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x & y = 0 \\ y & x = 0 \\ 1 & \text{elsewhere} \end{cases}$$

Then $\frac{\partial f}{\partial x}(0, 0) = 1 = \frac{\partial f}{\partial y}(0, 0)$, but f is not even continuous at $(0, 0)$.

Definition 0.3 (Differentiable function)

$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $x \in \Omega$ if there exists a linear function $A : \mathbb{R}^n \to R$ with

$$f(x + h) - f(x) = A(x)(h) + o(||h||).$$

(See IB Analysis and Topology.) We define f to be continuously differentiable if its partial derivatives are also continuous. This generalises to vector-valued functions $f : \Omega \to \mathbb{R}^m$ by considering each component f_i separately.

Definition 0.4 (Uniform convergence)
A sequence of functions $f_k : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is uniformly convergent with limit f iff

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} : \quad \forall k \geq n, x \in \Omega : \quad |f_k(x) - f(x)| < \varepsilon.$$

See IB Analysis and Topology for more. In this course, we will use this to justify swapping limits with integrals and sums.

§1 Analytic functions

§1.1 The extended complex plane of the Riemann sphere

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ since $z \leftrightarrow (x, y)$ is bijective with $z = x + iy$.