Financial Engineering Systems I

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Study on "A stochastic Nash equilibrium portfolio game between two DC pension funds"

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1 Introduction

Pension funds play a critical role in retirement planning by pooling contributions and investing over long horizons to deliver future benefits. This study investigates the notion of pension funds focusing particularly in the 2016 paper "A stochastic Nash equilibrium portfolio game between two DC pension funds" by Guohui Guan, Zongxia Liang. There are two main paradigms in pension design: defined-benefit (DB) plan and the defined-contribution (DC) plan. The former assumes that the contributions are adjusted to ensure a predetermined benefit at termination (time T). The latter has fixed contributions, while the eventual benefit at termination would vary with market conditions and outcome.

In his paper, *Optimal asset allocation for DC pension plans under inflation*, Han argues that "in recent years, DC plans have become popular in the pension market due to the demographic evolution and the development of the equity markets." [3]. A major challenge in DC plan management (and DB) is the multi-decade accumulation phase (20–40 years), during which inflation and interest-rate fluctuations can dramatically erode real wealth. Therefore, inflation cannot be ignored over such a long horizon. Accordingly, Han also contends that "it seems implausible to assume a constant level of interest rates and ignore the inflation in the long run" [3].

Early works such as Han and Hung [3] model inflation via a continuous-time Fisher equation and derive optimal asset allocations under stochastic interest-rate and inflation risks. These models typically rely on geometric Brownian motions and martingale or dynamic-programming methods to solve high-dimensional Hamilton–Jacobi–Bellman (HJB) equations.

Guan and Liang study the setting for two DC pension funds under inflation risk. They assume a market of cash, an inflation-indexed bond, and manager-specific stocks; model the price index via a generalized Fisher equation; and derive closed-form Nash equilibrium strategies using dynamic programming. Their numerical experiments illustrate how equilibrium allocations evolve over time and with competition weight.

Solving such stochastic games requires advanced notions from continuous-time stochastic process theory:

- **Girsanov's theorem** to change measures and incorporate market prices of risk for inflation and equity shocks;
- **Itô's formula** and multidimensional Itô–Doeblin expansions to derive SDEs for real wealth processes;
- **Hamilton–Jacobi–Bellman (HJB) equations** in high dimensions, whose first-order conditions yield feedback controls for equilibrium strategies;
- **Verification theorems** linking candidate value functions to optimality, ensuring admissibility and boundary conditions.

In this report, we focus on the simplified single-fund scenario inspired by the Guan and Liang's framework. That is, we consider a single DC manager (i.e. zero competition weight), removing the game-theoretic aspect, we reduce the problem to a classical stochastic control problem. We retain the inflation-indexed bond, cash, and a stock, model the price index as in [2], and derive the optimal CRRA allocation via HJB methods. We assume the fund manager's objective is to maximize the expected utility of the fund's terminal wealth (with a risk-averse utility function). The inflation-linked bond is used as an asset to hedge against inflation fluctuations, in addition to a traditional risk-free money market account and a stock. We will show how the manager optimally allocates wealth among these assets over time to balance growth (via stock investment) and inflation protection (via the indexed bond).

Using stochastic control methods, we derive the optimal feedback control (portfolio strategy) and the value function for the single-fund problem. In particular, we formulate the HJB partial differential equation and solve it to obtain closed-form expressions for the optimal asset allocation. The solution provides insight into how inflation risk and risk aversion affect the allocation to the inflation-indexed bond versus the stock.

The remainder of this paper is organized as follows. Section 2 introduces the financial market and model setup. Section 3 formulates the optimization problem and HJB equation. Section 4 presents analytical solutions for the optimal strategy. Section 5 reports numerical experiments, and Section 7 concludes with extensions.

2 Mathematical and Model Setup

In this study we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$. $\mathcal{F}_t$ is the information up to time t. On a fixed horizon $[0,T]$, pension managers can continuously adjust their strategies. We assume that all the processes are well defined and $\mathcal{F}_t$ adapted.

2.1 Mathematical setup

The financial market consists on three parts: (1) a risk-free asset (cash), (2) an inflation-indexed zero-coupon bond, (3) and a risky stock. We assume frictionless markets (no transaction costs or taxes), continuous market participation, and short buying is allowed. The model incorporates inflation via a price index process, and the inflation-indexed bond provides a way to hedge that risk. Below we detail each asset and the relevant stochastic processes:

For the computations moving forward, we consider one fund manager therefore the subscripts $i, (i = 1, 2)$ representing each fund manager from original paper are omitted.

Assumption 1 (**Cash**). *The risk-free asset $S_0(t)$ is the following:*

$$\frac{dS_0(T)}{S_0(t)} = r_n(t)dt, S_0(0) = S_0,$$

where $S_0 > 0$ and $r_n(t)$ denotes the nominal interest rate in the financial market.

Assumption 2 (**Characterization of Inflation Risk**). *The inflation risk is modeled by the continuous time Fisher equation:*

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_I dW_I(t),$$

where $W_I(t)$ is a standard Brownian motion under a risk-neutral measure and the risk of price index is characterized by $W_I(t)$.

Using the Girsanov theorem, the model of the stochastic price index $I(t)$ w.r.t. original measure P is obtained by the following: The market price of risk $W_I(t)$ is denoted λ_I with P-Brownian Motion: $\tilde{W_I}(t) := W_I(t) + \int_0^t \lambda_s ds$.

$$d\tilde{W}_I(t) := dW_t(t) + \lambda_t dt$$

Substitute into the SDE using P :

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_I d\tilde{W}_I(t) = (r_n(t) - r_r(t))dt + \sigma_I(\lambda_t dt + dW_t(t))$$

Hence, the stochastic price index $I(t)$ under the *original* measure P satisfies:

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_I(\lambda_t dt + dW_t(t)), I(0) = I_0$$

The inflation risk is hedged by indexed zero-bond coupon. An inflation-indexed zero-coupon bond $P(t,T)$ is a contract at time t with the final payment of real money \$1 at maturity T.

Remark 1 (**An inflation-indexed zero coupon bond**). *Unlike the general zero-coupon bond, $P(t,T)$ delivers $I(T)$ at maturity T. The price of $P(t,T)$ is:*

$$P(t,T) = \tilde{\mathbf{E}}[\exp\left(-\int_t^T r_n(s)ds\right) I(T)|\mathcal{F}_t] \Rightarrow P(t,T) = I(t)\exp\left(-\int_t^T r_n(s)ds\right)$$

The nominal interest rate in the model $r_n(t)$ is deterministic, hence the result above. P(t,T) satisfies the following stochastic differential equation:

$$\begin{cases} \dfrac{dP(t,T)}{P(t,T)} = r_n(t)\,dt + \sigma_I\left[\lambda_I\,dt + dW_I(t)\right], \\ P(T,T) = I(T) \end{cases}$$

Assumption 3 (**Stock**). *The third asset in which the fund manager can invest is the stock. The stock price $S(t)$ is as follow:*

$$\begin{cases} \dfrac{dS(t)}{S(t)} = r_n(t)dt + \sigma_{S_1}\left[\lambda_I dt + dW_I(t)\right] + \sigma_{S_2}[\lambda_S dt + dW_S(t)], \\ S(0) = S. \end{cases}$$

2.2 Model

Moving forward we fix: $r_n(t) = r, r_r(t) = \tilde{r} \quad (r, \tilde{r}) \in \mathbb{R}^2$
We assume the proportions of investment in cash, bond and stock are as follow: $u_0(t), u_p(t), u_s(t)$ and we have $1 = u_0(t) + u_p(t) + u_s(t)$. Therefore, we denote investment strategy $u(t)$ as follows:

$$u(t) = (u_p(t), u_s(t))^T$$

Assumption 4. *Under no transaction costs, no taxes and short buying allowed, the wealth of the pension manager X_t with investment behavior is as follows:*

$$\begin{cases} dX(t) = & cI(t)dt + u_0(t)X(t)\dfrac{dS_0(t)}{S_0(t)} \\ & +u_p(t)X(t)\dfrac{dP(t,T)}{P(t,T)} \\ & +u_s(t)X(t)\dfrac{dS(t)}{S(t)}, \\ X(0) = X \end{cases}$$

$X > 0$ represents the initial wealth of the pension manager.

Proposition 2.1. *Wealth of the pension manager in the more compact form is obtained by:*

$$\begin{cases} dX(t) = & cI(t)dt + rX(t)dt \\ & +u_p(t)X(t)\sigma_I[\lambda_I dt + dW_I(t)] \\ & +u_s(t)X_i(t)\sigma_{s_1}[\lambda_I dt + dW_I(t)] \\ & +u_s(t)X(t)\sigma_{s_2}[\lambda_s t + dW_s(t)], \\ X(0) = X. \end{cases}$$

Proof. Substitute $\dfrac{dS_0(t)}{S_0(t)}, \dfrac{dP(t,T)}{P(t,T)}, \dfrac{dS(t)}{S(t)}$ from Assumptions 1, 2.1 and Remark 1 in Assumption 4. □

We call $u(t)$ an *admissible strategy* if it satisfies the following conditions:

(i) $u_p(t)$ and $u_s(t)$ are progressively measurable with respect to the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.

(ii) $\mathbb{E}\left[\int_0^T \left(u_p(t)^2\sigma_i^2 \;+\; u_s(t)^2\,\sigma_{s_1}^2 \;+\; u_s(t)^2\,\sigma_{s_2}^2\right) dt\right] \;<\; +\infty.$

(iii) Proposition 2.1 admits a unique strong solution on $[0, T]$ for every initial datum $(t_0, I(0), X(0)) \in [0, T] \times (0, +\infty) \times (0, +\infty)$.

Denote by Π the set of all admissible strategies $u(t)$. Our goal is to search for the optimal investment strategy within Π.

Next, we denote the real wealth of the pension fund by: $Y(t) = \frac{X(t)}{I(t)} \Rightarrow Y(t) = X(t) \cdot I(t)^{-1}$

Proposition 2.2. *Real wealth of the pension manager Y_t with investment behavior is as follows:*

$$\begin{aligned}
dY(t) =& dt \left(c + u_p Y(t)\sigma_I \lambda_I + u_s Y(t)\sigma_{s_1}\lambda_I + u_s Y(t)\sigma_{s_2}\lambda_s + \tilde{r}Y(t) - Y(t)\sigma_I\lambda_I - \sigma_I^2 u_p Y(t) - \sigma_I\sigma_{s_1}u_s Y(t) + \sigma_I^2 Y(t)\right) \\
&+ dW_I(t)\left(u_p Y(t)\sigma_I + u_s Y(t)\sigma_{s_1} - Y(t)\sigma_I\right) \\
&+ dW_s(t) u_s Y(t)\sigma_{s_2}
\end{aligned}$$

Proof. Using the Two-Dimensional Ito-Doeblin theorem, we can derive the differential form for two Ito's processes $X(t), I(t)$ with two-dimensional Brownian motion $W_I(t), W_s(t)$ by:

$$df(t, X, I) = f_t dt + f_I dX + f_I dI + \frac{1}{2} f_{XX} dX dX + f_{XI} dX dI + \frac{1}{2} f_{II} dI dI =$$
$$= 0 + \frac{dX}{I} - \frac{X}{I^2} dI + 0 - \frac{1}{I^2} dX dI + \frac{X}{I^3} dI dI$$

We can derive the differential formulas for the processes:

$$dXdI = \sigma_I I X (u_p(t)\sigma_I + u_s(t)\sigma_{s_1})dt$$
$$dIdI = \sigma_I^2 I(t)^2 dt$$

Collecting terms and expanding $df(t, X, I)$:

$$\begin{aligned}
dY(t) =& cdt + rY(t)dt + u_p(t)Y(t)\sigma_I[\lambda_I df + dW_I(t)] + u_s(t)Y(t)\sigma_{s_1}[\lambda_I df + dW_I(t)] + u_s(t)Y(t)\sigma_{s_2}[\lambda_s dt + dW_s(t)] \\
&- Y(t)(r - \tilde{r})dt - Y(t)\sigma_I[\lambda_I dt + dW_I(t)] \\
&- \sigma_I Y(t)(u_p(t)\sigma_I + u_s(t)\sigma_{s_1})dt \\
&+ \sigma_I^2 Y(t)dt
\end{aligned}$$

Rearranging, we get to the final formula. □

3 Optimization Problem

In the previous part, we defined real wealth, which we denoted as $Y(t)$, based on our investment strategy $u(t)$. In the next part, we want to maximize the fund manager's utility from the real wealth at time T by optimizing the investment strategy.

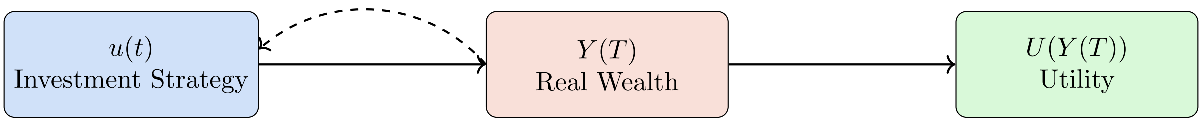

Figure 1: Closed-loop control structure: the investment strategy $u(t)$ depends on real wealth $Y(t)$

3.1 Objective Function

The objective function is as follows:

$$\max \mathbf{E}[U(Y(T))]$$
$$\text{subject to: } \begin{cases} X(t), u(t) \\ u(t) \in \Pi \\ X(T) \geq 0 \end{cases}$$

The utility function of the pension manager is defined by CRRA utility function:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1$$

This maximization problem is a classical optimization problem without competition.

4 Solutions

In the previous sections, we defined the real wealth of the fund manager and the maximization problem. In this section, we derive the Hamilton–Jacobi–Bellman equation and compute the optimal investment strategy $u(t)$.

We begin by defining a slightly unconventional notation of $dY(t)$ to help deal with the lengthy equation:

$$dY(t) = b(...u_s, u_p)dt + \rho(...u_s, u_p)dW_I(t) + \phi(...u_s, u_p)dW_s(t)$$

Each term in front of $dt, W_I(t), W_s(t)$ is shortened to accommodate our work with the Bellman Equation (HJB). Let $u_s = \alpha, u_p = \beta$. Then, the associated HJB equation for the manager is as follows:

$$\sup\left\{\tfrac{\partial V(t,y)}{\partial t} + \tfrac{\partial V(t,y)}{\partial y} \cdot b(...\alpha,\beta) + \tfrac{1}{2}\tfrac{\partial V(t,y)}{\partial y \partial y}(\rho(...\alpha,\beta)^2 + \phi(...\alpha,\beta)^2) = 0 \right. \tag{1}$$

Here

$$V(t,y) = \max_{u(t)} \mathbf{E}[U(Y(T)|Y(t) = t]$$

$V(t,y)$ represents the optimal expectation of utility given the state of financial market at time t.

Proposition 4.1. *The optimal values for $\rho(\alpha,\beta)$, $\phi(\alpha,\beta)$ are:*

$$\rho^*(\alpha,\beta) = -\frac{V_y(\lambda_I - \sigma_I)}{V_{yy}}, \quad \phi^*(\alpha,\beta) = -\frac{V_y\lambda_s}{V_{yy}}$$

Proof. Expanding the last term $\rho(...\alpha,\beta)^2 + \phi(...\alpha,\beta)^2$, we get:

$$\begin{aligned}\rho(...\alpha,\beta)^2 + \phi(...\alpha,\beta)^2 &= Y(t)^2((\beta-1)\sigma_I + \alpha\sigma_{s_1})^2 + Y(t)^2\alpha^2\sigma_{s_1}^2 = \\ &= Y(t)^2((\beta-1)^2\sigma_I + 2\alpha(\beta-1)\sigma_I\sigma_{s_1} + \alpha^2(\sigma_{s_1}^2 + \sigma_{s_2}^2))\end{aligned}$$

Now, we will look back at the 3 quantities $b(...\alpha,\beta), \rho(...\alpha,\beta), \phi(...\alpha,\beta)$ we defined earlier. We will simplify and arrange terms:

$$\begin{aligned}b(...\alpha,\beta) &= c + Y(t)\sigma_I\lambda_I\beta + Y(t)\sigma_{s_1}\lambda_i\alpha + Y(t)\sigma_{s_2}\lambda_s\alpha + \tilde{r}Y(t) - Y(t)\sigma_I\lambda_I - Y(t)\sigma_I^2\beta - Y(t)\sigma_I\sigma_{s_1}\alpha + \sigma_I^2Y(t) = \\ &= (c + \tilde{r}Y(t) - Y(t)\sigma_I\lambda_I + \sigma_I^2Y(t)) + \alpha(Y(t)\sigma_{s_1}\lambda_I + Y(t)\sigma_{s_2}\lambda_s - Y(t)\sigma_I\sigma_{s_1}) + \beta(Y(t)\sigma_I\lambda_I - Y(t)\sigma_I^2)\\ \rho(...\alpha,\beta) &= -Y(t)\sigma_I + \alpha Y(t)\sigma_{s_1} + \beta Y(t)\sigma_I\\ \phi(...\alpha,\beta) &= \alpha Y(t)\sigma_{s_2}\end{aligned}$$

Next step is to focus on the HJB equation 1 from earlier. Differentiating w.r.t. $u(t)$, we can find optimal values for α, β.

For the sake of simplicity once again, for notation, we will use $\frac{\partial V(t,y)}{\partial y} = V_y$, $\frac{\partial V(t,y)}{\partial y \partial y} = V_{yy}$.

Differentiating HJB w.r.t α:

$$V_y(Y(t)\sigma_{s_1}\lambda_I + Y(t)\sigma_{s_2}\lambda_s - Y(t)\sigma_I\sigma_{s_1}) + \frac{1}{2}V_{yy}[2\rho(\alpha,\beta)Y(t)\sigma_{s_1} + 2\phi(\alpha,\beta)Y(t)\sigma_{s_2}] = 0$$

Differentiating HJB w.r.t β:

$$V_y(Y(t)\sigma_I\lambda_I - Y(t)\sigma_I^2) + \frac{1}{2}V_{yy}[2\rho(\alpha,\beta)Y(t)\sigma_I] = 0$$

divide by $Y(t)$ and σ_I:

$$V_y(\lambda_I - \sigma_I) + V_{yy}[\rho(\alpha,\beta)] = 0$$

$$\rho(\alpha,\beta) = \frac{V_y(\lambda_I - \sigma_I)}{V_{yy}}$$

Rearranging terms and substituting $\rho(\alpha, \beta)$:

$$V_y(Y(t)\sigma_{s_1}\lambda_I + Y(t)\sigma_{s_2}\lambda_s - Y(t)\sigma_I\sigma_{s_1}) - Y(t)\sigma_{s_1}V_y(\lambda_I - \sigma_I) + V_{yy}\phi(\alpha,\beta)Y(t)\sigma_{s_2} = 0$$

Divide by $Y(t)$, rearrange and simplify terms:

$$V_y\sigma_{s_2}\lambda_s + V_{yy}\sigma_{s_2}\phi(\alpha,\beta) = 0$$

$$V_y\lambda_s + V_{yy}\phi(\alpha,\beta) = 0$$

$$\phi(\alpha,\beta) = -\frac{V_y\lambda_s}{V_{yy}}$$

□

4.1 Finding Optimal α, β

Proposition 4.2. *The optimal investment strategy* $u_0^*(t) = (1 - \alpha^* - \beta^*), u_p^* = \beta^*, u_s^* = \alpha^*$, *where:*

$$\alpha^* = -\frac{V_y}{V_{yy}} \cdot \frac{\lambda_s}{Y(t)\sigma_{s_2}}, \quad \beta^* = -\frac{V_y}{V_{yy}}\frac{1}{Y(t)\sigma_I}\left(\lambda_I - \sigma_I - \lambda_s\frac{\sigma_{s_1}}{\sigma_{s_2}}\right) + 1$$

Proof.

(i) Finding α^* with respect to the functions V_y, V_{yy}:

$$\phi(...\alpha,\beta) = \alpha Y(t)\sigma_{s_2} = -\frac{V_y\lambda_s}{V_{yy}} \Rightarrow \alpha^* = -\frac{V_y}{V_{yy}} \cdot \frac{\lambda_s}{Y(t)\sigma_{s_2}}$$

(ii) Finding β^* with respect to the functions V_y, V_{yy}:

$$\rho(\alpha,\beta) = \frac{V_y(\lambda_I - \sigma_I)}{V_{yy}} = -Y(t)\sigma_I + \alpha Y(t)\sigma_{s_1} + \beta Y(t)\sigma_I$$

$$\frac{V_y}{V_{yy}}(\lambda_I - \sigma_I) = -Y(t)\sigma_I + (-\frac{V_y}{V_{yy}} \cdot \frac{\lambda_s}{\sigma_{s_2}})\sigma_{s_1} + \beta Y(t)\sigma_I$$

$$\Leftrightarrow$$

$$(\beta - 1)Y(t)\sigma_I = -\frac{V_y}{V_{yy}}\left(\lambda_I - \sigma_I - \lambda_s\frac{\sigma_{s_1}}{\sigma_{s_2}}\right)$$

$$\Leftrightarrow$$

$$\beta^* - 1 = -\frac{V_y}{V_{yy}}\frac{1}{Y(t)\sigma_I}\left(\lambda_I - \sigma_I - \lambda_s\frac{\sigma_{s_1}}{\sigma_{s_2}}\right)$$

□

4.2 Optimal Investment Strategy

Now we're ready to define the main proposition which gives an explicit formulas for optimal utility as well as investment strategy:

Proposition 4.3. *Optimal* $V(t,y), u_0^*(t) = (1 - \alpha^* - \beta^*), u_p^* = \beta^*, u_s^* = \alpha^*$, *where:*

1. $\alpha^* = \frac{Y(t)+D(t)}{\gamma} \cdot \frac{\lambda_s}{Y(t)\sigma_{s_2}}$
2. $\beta^* = \frac{Y(t)+D(t)}{\gamma}\frac{1}{Y(t)\sigma_I}\left(\lambda_I - \sigma_I - \lambda_s\frac{\sigma_{s_1}}{\sigma_{s_2}}\right) + 1$
3. $V(t,y) = \frac{(y+D(t))^{1-\gamma}}{1-\gamma}e^{(1-\gamma)(\tilde{r}+\frac{1}{2\gamma}(\lambda_s^2+(\lambda_I-\sigma_I)^2))(T-t)}$

Proof. Optimal utility function being proposed is the following: From the boundary condition:

$$V(T,y) = \frac{y^{1-\gamma}}{1-\gamma}$$

The optimal utility function we conjecture to be:

$$V(t,y) = \frac{(y+D(t))^{1-\gamma}}{1-\gamma} f(t)$$

Where $D(t) = c\int_t^T exp[-\int_t^s \tilde{r} du]ds = c\int_t^T e^{-\tilde{r}s}e^{\tilde{r}t}ds = -\frac{c}{\tilde{r}}e^{\tilde{r}t}(e^{-\tilde{r}T} - e^{-\tilde{r}t}) = -\frac{c}{\tilde{r}}(e^{\tilde{r}(t-T)} - 1)$
Differentiating $V(t,y)$ we get:

$$V_t = \frac{(1-\gamma)(-d+\tilde{r}D(t))}{y+D(t)}V + \frac{f'(t)}{f(t)}V, \quad V_y = \frac{1-\gamma}{y+D(t)}V, \quad V_{yy} = -\frac{\gamma(1-\gamma)}{(y+D(t))^2}V$$

The optimal α^*, β^*:

$$\frac{V_y}{V_{yy}} = -\frac{y+D(t)}{\gamma}$$
$$\alpha^* = \frac{Y(t)+D(t)}{\gamma} \cdot \frac{\lambda_s}{Y(t)\sigma_{s_2}}$$
$$\beta^* - 1 = \frac{Y(t)+D(t)}{\gamma} \frac{1}{Y(t)\sigma_I} \left(\lambda_I - \sigma_I - \lambda_s \frac{\sigma_{s_1}}{\sigma_{s_2}}\right)$$

Simplifying $b(\ldots\alpha,\beta)$ in order to substitute later in HJB:

$$\begin{aligned} b(\ldots\alpha,\beta) &= (c + \tilde{r}Y(t) - Y(t)\sigma_I\lambda_I + \sigma_I^2 Y(t)) + \alpha(Y(t)\sigma_{s_1}\lambda_I + Y(t)\sigma_{s_2}\lambda_s - Y(t)\sigma_I\sigma_{s_1}) + \beta(Y(t)\sigma_I\lambda_I - Y(t)\sigma_I^2) = \\ &= c + \tilde{r}Y(t) + \frac{(Y(t)+D(t))\lambda_s}{\gamma Y(t)\sigma_{s_2}} Y(t)\sigma_{s_2}(Y(t)\sigma_{s_1}\lambda_I + Y(t)\sigma_{s_2}\lambda_s - Y(t)\sigma_I\sigma_{s_1}) \\ &+ \frac{Y(t)+D(t)}{\gamma(Y(t)\sigma_I)}\left(\lambda_I - \sigma_I - \lambda_s\frac{\sigma_{s_1}}{\sigma_{s_2}}\right)(Y(t)\sigma_I\lambda_I - Y(t)\sigma_I^2) = \\ &= c + \tilde{r}Y(t) + \frac{Y(t)+D(t)}{\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2) \end{aligned}$$

Substituting α^*, β^* and differentials of V into the HJB:

$$\begin{aligned} V_t &+ (c + \tilde{r}Y(t))\left(\frac{1-\gamma}{Y(t)+D(t)}V\right) + \frac{1-\gamma}{\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2)V - \frac{1}{2}\frac{\gamma(1-\gamma)}{(Y(t)+D(t))^2}V\frac{(Y(t)+D(t))^2}{\gamma^2}(\lambda_s^2 + (\lambda_I - \sigma_I)^2) = \\ &= \frac{(1-\gamma)(-d+\tilde{r}D(t))}{y+D(t)}V + \frac{f'(t)}{f(t)}V + \frac{1-\gamma}{Y(t)+D(t)}V(c + \tilde{r}Y(t)) + \frac{1}{2}\frac{1-\gamma}{\gamma}V(\lambda_s^2 + (\lambda_I - \sigma_I)^2) = \\ &= (1-\gamma)V\tilde{r} + \frac{f'(t)}{f(t)}V + \frac{1}{2}\frac{1-\gamma}{\gamma}V(\lambda_s^2 + (\lambda_I - \sigma_I)^2) = 0 \end{aligned}$$

Divide by $(1-\gamma)V$ to further simplify:

$$= \tilde{r} + \frac{f'(t)}{(1-\gamma)f(t)} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2) = 0$$

The explicit form of the $f(t)$ can be derived by solving the ODE above with a terminal condition $f(T) = 1$:

$$\frac{df}{f} = -(1-\gamma)(\tilde{r} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2))dt$$
$$\int_t^T \frac{df}{f} = -\int_t^T (1-\gamma)(\tilde{r} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2))ds$$
$$log(f(T)) - log(f(t)) = -(1-\gamma)(\tilde{r} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2))(T-t)$$
$$0 - log(f(t)) = -(1-\gamma)(\tilde{r} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2))(T-t)$$
$$f(t) = e^{(1-\gamma)(\tilde{r} + \frac{1}{2\gamma}(\lambda_s^2 + (\lambda_I - \sigma_I)^2))(T-t)}$$

The optimal utility function is as follow:

$$V^*(t,y) = \frac{(y+D(t))^{1-\gamma}}{1-\gamma} f(t) =$$
$$= \frac{(y+D(t))^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(\tilde{r}+\frac{1}{2\gamma}(\lambda_s^2+(\lambda_I-\sigma_I)^2))(T-t)}$$

□

5 Numerical Simulations

In this section, we simulate the evolution of both the real wealth Y(t) and the inflation index I(t) under the optimal investment strategy derived in the previous section. Our goal is to assess the practical behavior of the model over time and to validate the effectiveness of the optimal strategy in preserving and growing real wealth under inflationary risk.

5.1 Simulation Methodology

The simulation is implemented using the Euler–Maruyama discretization scheme, a widely used numerical method for solving stochastic differential equations (SDEs). We consider a pension fund manager operating over a 20-year horizon (T = 20) and a time increment Δ t = 0.01, resulting in 2000 time steps. The processes simulated include:

- I(t): the stochastic price index driven by inflation, following the generalized Fisher equation.
- Y(t): the real wealth process of the pension manager, evolving according to a controlled SDE that incorporates both inflation and stock market risk.

We use the following parameter values:

$$\begin{aligned} &r = 0.03, \quad \tilde{r} = 0.015, \quad \gamma = 2, \quad c = 1, \\ &\lambda_I = 0.2, \quad \lambda_s = 0.2, \quad \sigma_I = 0.1, \\ &\sigma_{s1} = 0.06, \quad \sigma_{s2} = 0.2 \end{aligned}$$

The Brownian motion increments dW_I and dW_s are generated as standard normal variates scaled by $\sqrt{\Delta t}$, and a fixed random seed is set to ensure reproducibility.

The simulation computes optimal strategy $\alpha(t)$ and $\beta(t)$, corresponding to the proportion of wealth allocated to the risky stock and the inflation-indexed bond, respectively. These are computed using the closed-form expressions:

$$\alpha^*(t) = \frac{Y(t)+D(t)}{\gamma Y(t)} \cdot \frac{\lambda_s}{\sigma_{s2}}, \quad \beta^*(t) = 1 + \frac{Y(t)+D(t)}{\gamma Y(t)\sigma_I}\left(\lambda_I - \sigma_I - \frac{\lambda_s \sigma_{s1}}{\sigma_{s2}}\right)$$

The wealth dynamics are then updated using these control values at each time step.

5.2 Simulation Result and Interpretation

Figure 2 shows the result of 1 simulation. From the graph we have the following observations

- The price index I(t) (orange curve) increases smoothly, due to persistent inflation modeled through the Fisher equation.
- The real wealth Y(t) (blue curve) shows greater volatility, reflecting the pension manager's exposure to market risk due to the risky stock.
- Y(t) stays above I(t) consistently, indicating that the investment strategy successfully grows real purchasing power over time.
- The wealth process experiences increasing volatility after year 10, consistent with compounding effects in both returns and inflation uncertainty.

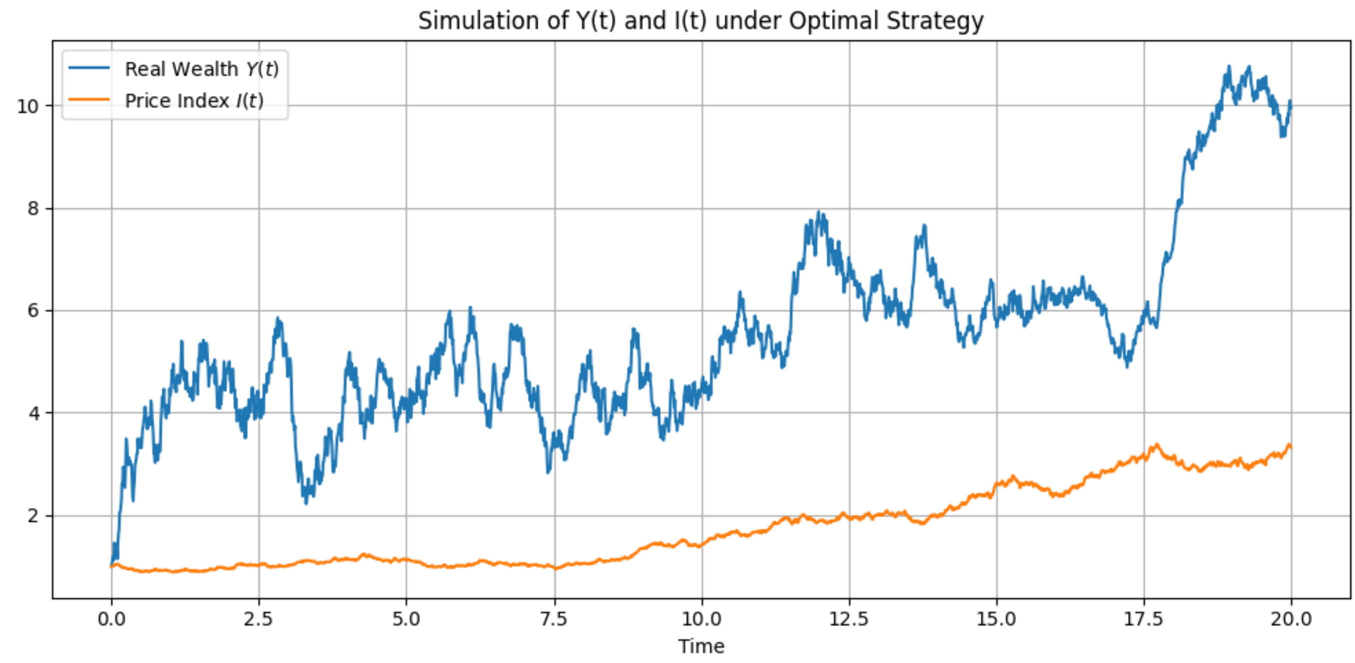

Figure 2: Simulation of Y(t) and I(t)

Then we run the simulation 1,000 times, as shown in figure 3.

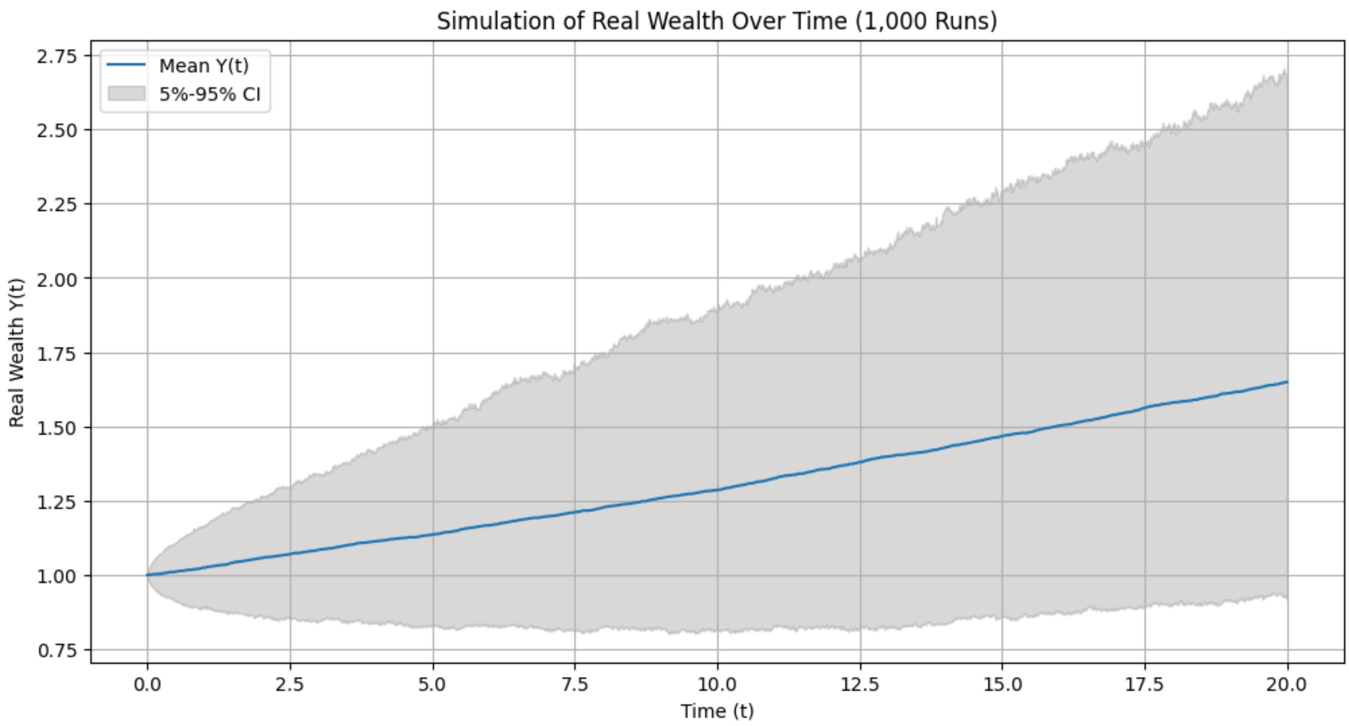

Figure 3: Evolution of Mean Wealth

The blue line represents the mean path of Y(t), which is the mean of theevolution of wealth of the fund manager, while the shaded area denotes the 90% confidence interval (from the 5th to 95th percentile) at each time point.

Several observations can be made:

- The mean path of real wealth increases over time in a convex fashion, indicating that the expected real wealth grows at an accelerating rate, consistent with the effects of compounding and dynamic risk exposure.
- The shaded region around the mean widens over time, demonstrating increasing uncertainty in future

wealth outcomes. This is expected due to the cumulative effect of stochastic elements (inflation and stock returns) over time.
- The upper tail of the wealth distribution grows more rapidly than the lower tail, suggesting that there is a small probability of significantly high real wealth — a characteristic typical of models with risky asset exposure under CRRA utility.

5.3 Observations and Discussion

The numerical results offer compelling evidence that the optimal strategy derived through stochastic control methods is effective in managing long-term inflation and investment risk in a DC pension fund.

1. Inflation Hedging via Bond Allocation

The optimal strategy effectively leverages the inflation-indexed bond to hedge against purchasing power erosion. This is evidenced by the fact that real wealth does not fall below the initial value in even the worst simulated scenarios (5th percentile), confirming that inflation risk is well-managed in the model.

2. Risk-Adjusted Growth

The growth in average real wealth suggests that the strategy enhances value over time, while the increasing spread in the confidence interval reflects the impact of controlled exposure to stock market volatility.

3. Fidelity to the Reference Model

While our model simplifies the full two-player game-theoretic setting by assuming no competition ($\theta = 0$), it retains the essential economic structure — a long-term investment plan under inflation risk, with continuous contributions and dynamic asset allocation. The simulation confirms that even under this simplification, the core insights from Guan and Liang's paper persist:

- Inflation-indexed bonds are indispensable for hedging purchasing power risk.
- Feedback strategies derived from HJB methods adapt effectively over time.
- Optimal portfolios dynamically balance risk and return while ensuring long-term real wealth growth.

This further suggests that the single-agent model is not only a useful stepping stone for understanding the full equilibrium game, but may also be more realistic in scenarios where fund managers do not face direct peer competition.

6 Extensions

While our study has focused on a simplified single-fund model with constant real and nominal rates, several natural extensions could bring the framework closer to real-world pension management:

(i) In practice, both the nominal yield curve and the real rate evolve over time. Introducing a short-rate process (e.g. Vasicek or CIR) would allow us to the term-structure dynamics of the cost of a higher-dimensional control problem.

(ii) Our current model assumes a fixed contribution rate (c). A more realistic setup would link contributions to a stochastic salary process, perhaps correlated with inflation and equity markets (as in [1]). This would introduce an additional state variable and alter the optimal hedge against both income and price level risk.

(iii) A natural and very desirable extension of our single-fund model is to incorporate competition between multiple pension managers, as in Guan and Liang (2016). In their framework, two DC pension fund managers simultaneously choose dynamic portfolio strategies to maximize a weighted utility

$$\max_{u_i(\cdot)} E\Big[\, U_i\big(Y_i(T)^{1-\theta_i} R_i(T)^{\theta_i}\big)\Big],$$

where $Y_i(T)$ is manager i's real terminal wealth, $R_i(T) = Y_i(T)/Y_j(T)$ is the relative wealth compared to the other manager, and $\theta_i \in [0, 1]$ captures how competitive (higher θ_i places more weight on outperforming the rival). By applying the dynamic programming principle to this two-player stochastic differential game, they derive Hamilton–Jacobi–Bellman equations for each fund manager and obtain closed-form expressions for the equilibrium feedback controls.

This equilibrium analysis reveals how each manager's optimal allocation to risky assets and inflation-hedging bonds depends not only on their own risk preferences but also on the competitor's strategy and the competition weights θ_1, θ_2.

7 Conclusion

This study investigated a simplified version of the stochastic control framework developed by Guan and Liang (2016), focusing on the investment strategy of a single defined-contribution (DC) pension fund manager in an inflationary environment. By removing the game-theoretic aspect (i.e., setting the competition parameter $\theta = 0$), we transformed the original two-agent differential game into a single-agent stochastic optimization problem. Despite this simplification, we kept the core features of the reference model: dynamic allocation among cash, an inflation-indexed bond, and a risky equity asset, in the presence of inflation risk.

Using Hamilton–Jacobi–Bellman (HJB) methods, we derived a closed-form solution for the optimal investment strategy under constant relative risk aversion (CRRA) utility. Our results reaffirm key insights from Guan and Liang's paper: namely, that dynamic asset allocation, incorporating inflation-hedging instruments like indexed bonds, is essential for preserving long-term real wealth.

Through numerical simulations, we demonstrated that:

- The optimal strategy consistently grows real wealth over a 20-year horizon, outperforming the inflation index in both mean and lower percentile scenarios.
- The indexed bond plays a critical role in protecting purchasing power, particularly in adverse market conditions, aligning with the original finding that inflation protection is essential for DC pension fund sustainability.
- The wealth trajectory shows increasing convexity and volatility over time, reflecting compounding effects and the balance between growth-oriented (risky) and hedge-oriented (bond) investments.

Moreover, our sensitivity analysis revealed important relationships:

- Higher discount rates encourage more aggressive early investment in risky assets.
- Larger contribution rates reduce the need for risky positions, suggesting endogenous de-risking behavior.
- As risk aversion increases, the manager shifts more capital toward the inflation-indexed bond, confirming the inverse relationship between risk tolerance and exposure to equities.

Our model captures the fundamental dynamic asset allocation behavior under inflation risk. The simplicity of the single-agent model also offers practical relevance in settings where competition is minimal or where regulatory or organizational structures isolate managers from peer benchmarks.

Finally, our work sets the stage for future extensions that reintroduce game-theoretic competition (as in Guan and Liang), stochastic interest rates, salary-linked contributions, and even mortality risk. Each of these extensions would further enhance the model's realism and bring it closer to practical pension fund management applications.

In conclusion, our findings support and complement those of Guan and Liang (2016), validating their theoretical contributions while demonstrating the effectiveness of dynamic, inflation-aware investment strategies.

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