Persistence Modules and Stability

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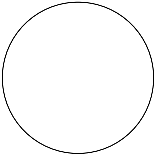
Singular Homology

Choose a dimension n, then

$$H_n(-) : \mathbf{Top} \to \mathbf{Ab}$$

describes the n-dimensional 'holes'.



$$H_0(\mathbb{S}^1) = \mathbb{Z}$$
$$H_1(\mathbb{S}^1) = \mathbb{Z}$$
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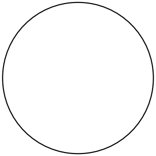

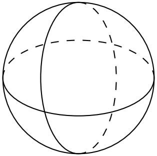

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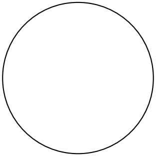

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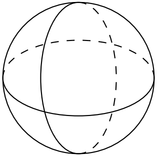
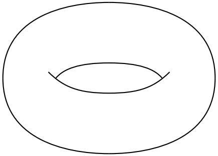


$H_0(\mathbb{S}^2) = \mathbb{Z}$
$H_1(\mathbb{S}^2) = \{1\}$
$H_2(\mathbb{S}^2) = \mathbb{Z}$



$H_0(\mathbb{T}^2) = \mathbb{Z}$
$H_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$
$H_2(\mathbb{T}^2) = \mathbb{Z}$

Singular Homology

Theorem

A continuous map $f : X \to Y$ induces a homomorphism $f_* : H_k(X) \to H_k(Y)$ for all k.

Singular Homology

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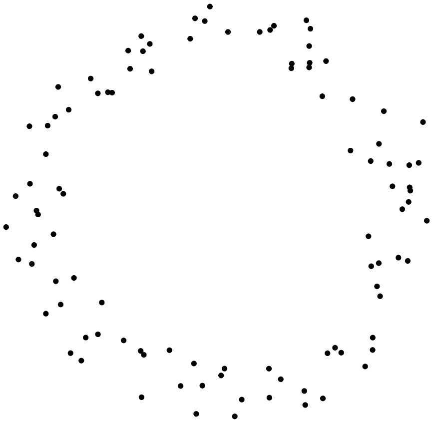
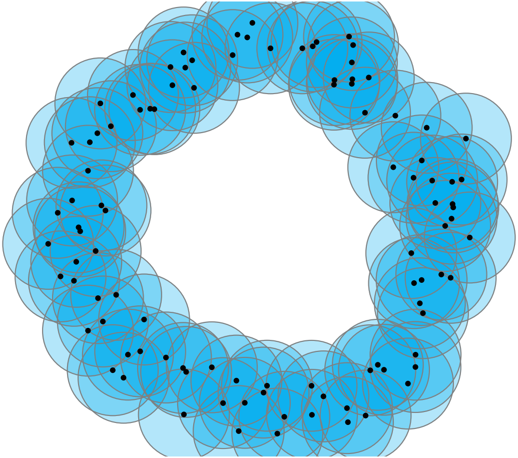
$$H_n(-; \mathbf{k}) : \mathbf{Top} \to \mathbf{Vect}_{\mathbf{k}}$$

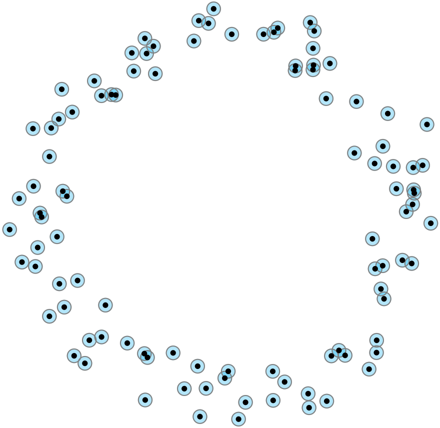


$r = 2$



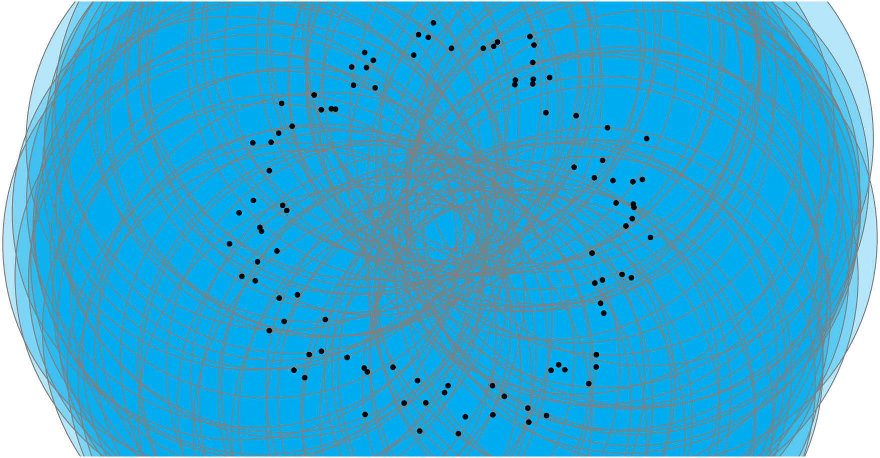
$$H_0(X) = \mathbb{Z}$$
$$H_1(X) = \mathbb{Z}$$

$r = 0.3$



$$H_0(X) = \mathbb{Z}^{73}$$
$$H_1(X) = \{1\}$$

$r = 10$



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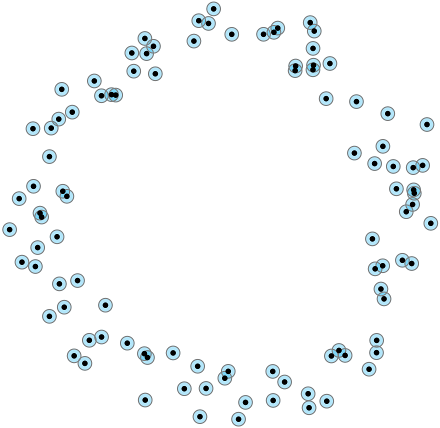
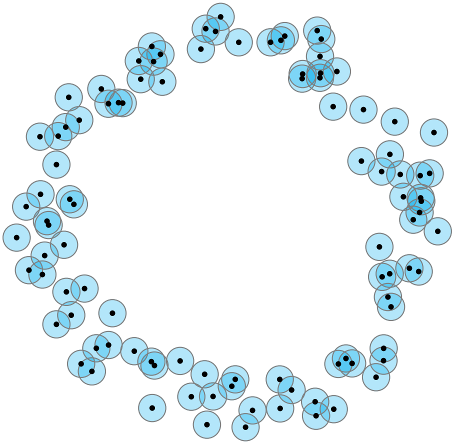
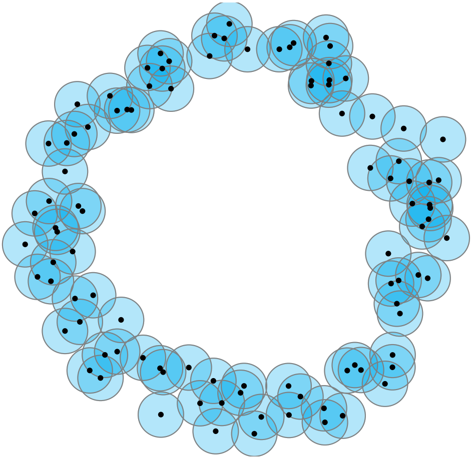


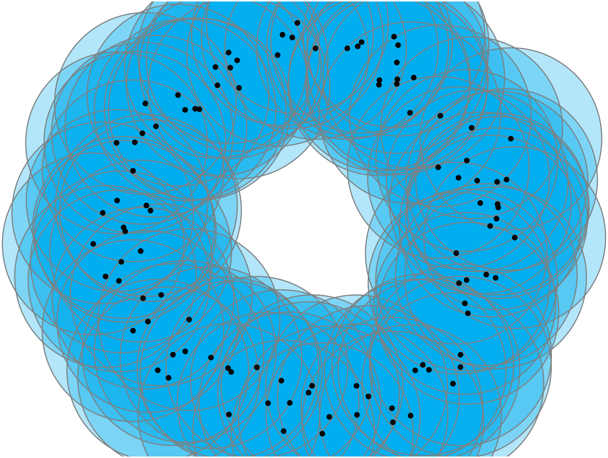
X_1



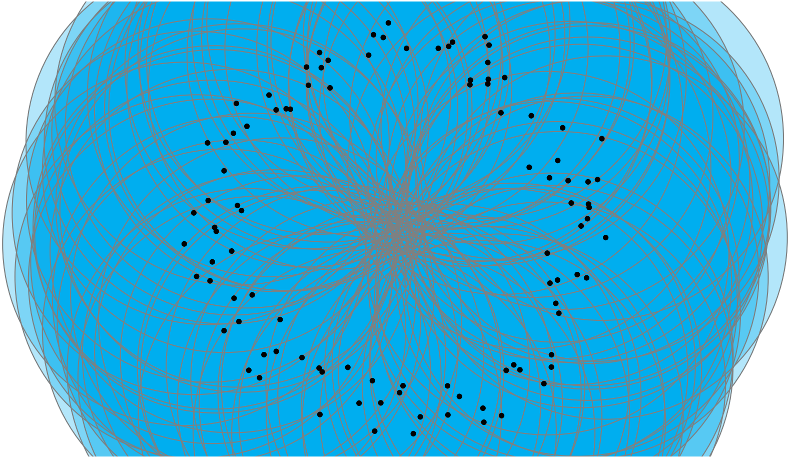
$X_1 \subset X_2$



$$X_1 \subset X_2 \subset X_3$$



$$X_1 \subset X_2 \subset X_3 \subset X_4$$



$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5$$

We have a filtration:

$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5$$

This gives us a diagram of spaces with inclusion maps:

$$X_1 \to X_2 \to X_3 \to X_4 \to X_5$$

Now find the homology of each space, giving a diagram of groups and homomorphisms:

$$H(X_1) \to H(X_2) \to H(X_3) \to H(X_4) \to H(X_5)$$

Zomorodian and Carlsson (2005)

If the ground ring is a field **k**, we have a diagram of **k**-vector spaces.

Theorem

Any finite diagram of finite dimensional vector spaces decomposes into a sum of intervals.

Zomorodian and Carlsson (2005)

$$H_0(X_1) \to H_0(X_2) \to H_0(X_3) \to H_0(X_4) \to H_0(X_5)$$

$$\|$$

$$\mathbf{k} \to \mathbf{k} \to \mathbf{k} \to \mathbf{k} \to \mathbf{k}$$

$$\bigoplus$$

$$\mathbf{k} \to \mathbf{k} \to 0 \to 0 \to 0$$

$$\bigoplus$$

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$$\bigoplus$$

$$\mathbf{k} \to 0 \to 0 \to 0 \to 0$$

$$\vdots$$

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$$\oplus$$

$$\mathbf{k} \to \mathbf{k} \to 0 \to 0 \to 0$$

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$$\mathbf{k} \to 0 \to 0 \to 0 \to 0$$

$$\oplus$$

$$\mathbf{k} \to 0 \to 0 \to 0 \to 0$$

$$\vdots$$

$$\mathcal{B}_0 = \{[1,5], [1,3), [1,2), [1,2), \dots\}$$

Zomorodian and Carlsson (2005)

$$H_1(X_1) \to H_1(X_2) \to H_1(X_3) \to H_1(X_4) \to H_1(X_5)$$

$$\|$$

$$0 \to \mathbf{k} \to \mathbf{k} \to \mathbf{k} \to 0$$

$$\bigoplus$$

$$0 \to \mathbf{k} \to 0 \to 0 \to 0$$

$$\bigoplus$$

$$0 \to 0 \to \mathbf{k} \to 0 \to 0$$

$$\vdots$$

$$\mathcal{B}_1 = \{[2,5), [2,3), [2,3), \dots\}$$

Zomorodian and Carlsson (2005)

Definition

A *barcode* (or *persistence diagram*) is a multiset of points in the half plane

$$\mathcal{H} = \{(p, q) \in \mathbb{R}^2 : p < q\}$$

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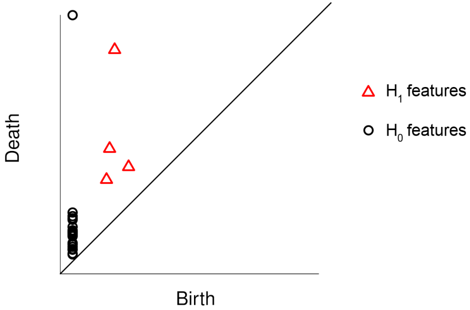

The Story So Far

- Begin with a data set $K \subset \mathbb{R}^n$

$$K = \{(0.125, 0.72), (0.627, 0.92), \ldots\}$$

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- Find features with long intervals

Applications

- Analysis of treatment response in breast cancer patients (DeWoskin et al, 2010)
- Natural language processing (Zhu, 2013)
- Computer vision (Lamar-León et al, 2012)

Persistence modules

Definition

A *persistence module* $\mathbb{V}$ over a poset P is a collection of vector spaces $\{V_i\}_{i\in P}$ and linear maps $\{v_s^t : V_s \to V_t\}_{s\leq t}$ such that

$$v_r^t = v_s^t \circ v_r^s \text{ for all } r \leq s \leq t$$

Here we are interested in persistence modules over the real numbers $\mathbb{R}$.

Sublevel persistence modules

Let $f : X \to \mathbb{R}$ be a function and define $X_a = f^{-1}((-\infty, a])$.

This gives us a filtration $\{X_a\}_{a\in\mathbb{R}}$, and therefore a persistence module $\mathbb{V}$ with

$$V_t = H(X_t)$$
$$v_s^t = \eta_s^t$$

The spaces we created above can be defined this way. Let $X = \mathbb{R}^n$, and $f(x) =$ distance from x to closest point in K.

Persistence modules can be wild!

Example (Crawley-Boevey, 2012)

Consider the persistence module:

$$\mathbb{V} = \prod_{n=1}^{\infty} \mathbb{I}[0, \tfrac{1}{n}]$$

$\mathbb{V}$ does not admit an interval decomposition.

We can still define a barcode when the module is ‘q-tame’, this is true in most settings.

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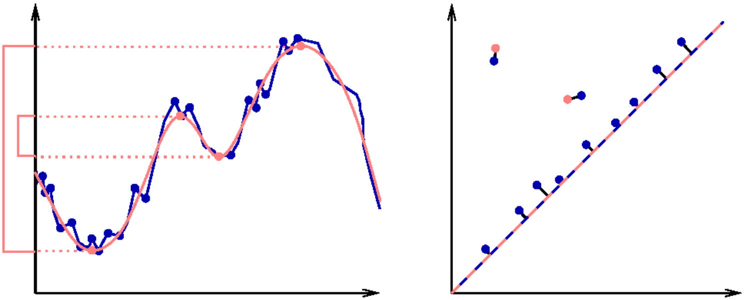
LHS: distance between two functions $f, g : X \to \mathbb{R}$

RHS: distance between two barcodes $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}$



Bottleneck distance

Definition

The bottleneck distance $d_b(\mathcal{A}, \mathcal{B})$ is the smallest $\delta \in \mathbb{R}$ such that there exists a partial matching $\mathcal{A} \longleftrightarrow \mathcal{B}$ where

- if $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are matched, then $d^\infty(a, b) \leq \delta$;
- if $a \in \mathcal{A}$ is unmatched, then $d^\infty(a, \Delta) \leq \delta$; and,
- if $b \in \mathcal{B}$ is unmatched, then $d^\infty(b, \Delta) \leq \delta$; and,

The stability theorem – Chazal et al. (2009)

Theorem

Let $f, g : X \to \mathbb{R}$ be functions on a topological space X and let $\mathbb{U} = H(X^f), \mathbb{V} = H(X^g)$ be the sublevel persistence modules. If $\mathbb{U}$ and $\mathbb{V}$ are q-tame, then:

$$d_b(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V})) \leq \|f - g\|_\infty$$

Frédéric Chazal et al. "The structure and stability of persistence modules". In: *arXiv preprint arXiv:1207.3674* (2012).

David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. "Stability of persistence diagrams". In: *Discrete & Computational Geometry* 37.1 (2007), pp. 103–120.

Mikael Vejdemo-Johansson. "Sketches of a platypus: persistent homology and its algebraic foundations". In: *arXiv preprint arXiv:1212.5398* (2012).

Afra Zomorodian and Gunnar Carlsson. "Computing persistent homology". In: *Discrete & Computational Geometry* 33.2 (2005), pp. 249–274.