The module perspective on representation theory

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Modern representation theory looks at representations from a different perspective than the homomorphisms $\rho : G \to GL(V)$ we have been studying so far. Every representation turns out to be equivalent to a module over a particular algebra called the group algebra.

First let us review the structures we will be using.

Definition 1. *A* left R-module *is an abelian group M with a left R action such that for all $r, s \in R$ and $x, y \in M$*

- $r(x + y) = rx + ry$
- $(r + s)x = rx + sx$
- $(rs)x = r(sx)$
- $1x = x$

We say M is finitely generated *if there is a finite set $\{x_i\}$ in M such that every element of M is an R-linear combination of these elements.*

Many familiar structures are modules over some ring. If we take R to be a field, the above conditions are exactly the conditions of a vector space. In other words, a vector space over $\mathbb{F}$ is just an $\mathbb{F}$-module.

For any abeilan group M, M is a $\mathbb{Z}$-module via the action $nx = \underbrace{x + \cdots + x}_{n \text{ times}}$. This is why we sometimes hear abelian groups referred to as $\mathbb{Z}$-modules.

Finally, for any ring R, $R^n = R \oplus \cdots \oplus R$ is a R-module, with R acting component-wise: $rx = r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$. More generally we could consider R^n as a module over the matrix ring $M_n(R)$, where the action is matrix multiplication on the column vector in R^n. When R is a division algebra, any module over $M_n(R)$ is the direct sum of m copies of R^n, a fact we will need later.

Definition 2. *An* algebra over a field $\mathbb{F}$ *is a vector space over $\mathbb{F}$ with a bilinear product. In the case the product is associative, this algebra is also a ring.*

Every field is an algebra over itself, with the product given by normal multiplication in the ring. The complex numbers $\mathbb{C}$ are an algebra over $\mathbb{R}$, with basis $\{1, i\}$. The polynomial ring $\mathbb{F}[x]$ is an algebra over $\mathbb{F}$ with basis $\{x^0, x^1, \ldots\}$ and product given by polynomial multiplication.

Definition 3. *The* group algebra $\mathbb{F}[G]$ *is the algebra with basis $\{g_i\}$ the elements of G, and product given by extending the group operation linearly. This product is clearly associative, so $\mathbb{F}[G]$ can also be considered as a ring.*

Every element of $\mathbb{F}[G]$ is of the form $\sum_{g\in G} a_g \cdot g$ for $a_g \in \mathbb{F}$, and multiplication is given by

$$ab = (\sum_{g\in G} a_g \cdot g)(\sum_{h\in G} b_h \cdot h) = (\sum_{g\in G}\sum_{h\in G} a_g b_h \cdot gh)$$

With these definitions out of the way we can show there is a correspondence between linear representations of G and finitely generated $\mathbb{F}[G]$-modules. We will make use of the following fact.

Lemma 1. *If V is a finitely generated $\mathbb{F}[G]$-module, V can be regarded as a finite dimensional vector space over $\mathbb{F}$ via $\lambda x = (\lambda \cdot 1)x$ where 1 is the identity element of G.*

Proof. If V is generated as an $\mathbb{F}[G]$-module by $\{v_1, \ldots, v_n\}$, then V is generated as a $\mathbb{F}$-vector space by $\{gv_i \,|\, g \in G, 1 \leq i \leq t\}$. Because G is finite, we have that this is finite dimensional. $\square$

Proposition 1. *Given a field $\mathbb{F}$ and a finite group G there is a bijective correspondence between linear representations of G and $\mathbb{F}[G]$-modules.*

Proof. ($\Rightarrow$) Given a representation $\rho : G \to GL(V)$, we can give V an $\mathbb{F}[G]$-module structure through the action

$$av = \left(\sum_{g\in G} a_g g\right) v = \sum_{g\in G} a_g(\rho_g v)$$

($\Leftarrow$) Given a finitely generated $\mathbb{F}[G]$-module V, we know from the above lemma that V is finite dimensional as a $\mathbb{F}$-vector space. The action is a map $\mathbb{F}[G] \times V \to V$. This restricts to a map $G \times V \to V$ that is also linear so equivalent to a representation $\rho : G \to GL(V)$. $\square$

Considering representations as modules gives us access to some high-powered machinery from the theory of semisimple algebras. Firstly,

Theorem 1. *If $\mathbb{F}$ has characteristic 0, then $\mathbb{F}[G]$ is semisimple.*

Proof. Analogous to Maschke's Theorem. $\square$

A simple module is one with only the trivial module and itself as submodules. A simple $\mathbb{F}[G]$-module then corresponds to an irreducible representation. A semisimple module is one that decomposes as a direct sum of simple modules.

The theorem states that all $\mathbb{F}[G]$-modules decompose as direct sums of simple modules, or equivalently, all representations decompose into direct sums of irreducible representations. This is an alternative phrasing of Maschke's theorem.

$\mathbb{F}[G]$ being semisimple also gives us access to the following result.

Theorem 2 (Wedderburn's theorem)**.** *If $\mathbb{F}$ is algebraically closed, then any semisimple algebra is isomorphic to a direct sum of matrix algebras $M_i(\mathbb{F})$.*

Because of the above condition, from here on we will take our field to be $\mathbb{C}$. The theorem states that $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ as $\mathbb{C}$-algebras. The theory of semisimple algebras now gives us alternative proofs of results we have already seen.

Corollary 1. *There are exactly r isomorphism classes of simple $\mathbb{C}[G]$-modules. These correspond to the irreducible representations, and the n_i correspond to the degrees of these representations. Given representatives $S_1, \ldots, S_r$ of these isomorphism classes, any $\mathbb{C}[G]$-module can be written uniquely in the form $V \cong a_1 S_1 \oplus \cdots \oplus a_r S_r$ for some non-negative integers a_i.*

Proof. As $\mathbb{C}$ is a division algebra, $\mathbb{C}^{n_i}$ is the only simple $M_{n_i}(\mathbb{C})$-module. Furthermore, because $M_{n_i}(\mathbb{C})$ is semisimple we know that $M_{n_i}(\mathbb{C})$-modules must all be of the form $a_i \mathbb{C}^{n_i}$ for some non-negative a_i.

$\mathbb{C}[G]$ is the direct sum of such $M_{n_i}(\mathbb{C})$ so the result follows. □

Corollary 2. $\sum_i n_i^2 = |G|$

Proof.

$$|G| = \dim_{\mathbb{C}} \mathbb{C}[G] = \dim_{\mathbb{C}} \left(\bigoplus_i M_{n_i}(\mathbb{C}) \right) = \sum_i \dim_{\mathbb{C}} M_{n_i}(\mathbb{C}) = \sum_i n_i^2$$

□

The final result I will revisit is the following:

Proposition 2. *The irreducible characters of G, $\chi_i : G \to \mathbb{C}$, are linearly independent over $\mathbb{C}$.*

Proof. Let S_i be the simple $\mathbb{C}[G]$-modules and denote by e_i the identity element of $M_{n_i}(\mathbb{C})$. $\chi_i(g)$ is the trace of the linear transformation on S_i given by the action of g. We extend χ_i linearly to be a function $\mathbb{C}[G] \to \mathbb{C}$. The action of e_i on S_i is the identity, so we have that $\chi_i(e_i) = \dim_{\mathbb{C}} S_i = f_i$. For any $j \neq i$, the action of e_i on S_j is the zero map, so $\chi_j(e_i) = 0$.

Now let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ such that $\sum_j \lambda_j \chi_j = 0$. From the above we see that $0 = \sum_j \lambda_j \chi_j(e_i) = \lambda_i f_i$. Therefore, $\lambda_j = 0$ for all j, and the χ_i are all linearly independent. □