CSE-512 HW-5

Name: Venkata Subba Narasa Bharath, Meadam

SBU ID: 112672986

Q1)

a) $y_i = x + z_i$ (x is constant)

$z_i = y_i - x$,

Given $z_i \sim N(0,1)$

$\Rightarrow$ As x is constant, $y_i - x$ will also be a $N(0,1)$.

$\therefore (y_i - x) \sim N(0,1)$

Normal distribution, $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$,

$(\mu=0, \sigma=1) \Rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

Likelihood, $L(y_i) = \prod_{i=1}^{m} e^{-\frac{(y_i-x)^2}{2}}$

Taking log on both sides, we get,

$$\log(L(y_i)) = \sum_{i=1}^{m} -\frac{(y_i-x)^2}{2}$$

For maximum, equate derivative to 0.

$$\frac{\partial}{\partial y}(\log(L(y_i))) = 0$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\sum_{i=1}^{m} -\frac{(y_i-x)^2}{2}\right) = 0$$

$$\sum_{i=1}^{m} -\frac{\cancel{2}(y_i-x)}{\cancel{2}} = 0$$

$$-\left(\sum_{i=1}^{m} y_i - mx\right) = 0$$

$$\boxed{x_{MLE} = \frac{\sum_{i=1}^{m} y_i}{m}}$$

$$\text{Bias} = E[\hat{x}] - x$$

$$E[\hat{x}] = E\left[\frac{1}{m}\sum_{i=1}^{m} y_i\right]$$

$$= \frac{1}{m} E\left[\sum_{i=1}^{m} y_i\right]$$

$$= \frac{1}{m} \times m \times E[y_i]$$

$$= E[y_i] \quad \left\{ \begin{array}{l} y_i = x + z_i \\ y_i \sim N(x, 1) \end{array} \right\} \rightarrow \text{shifted standard normal}$$

$$= x$$

$$\boxed{\text{Bias} = x - x = 0}$$

Variance, $\hat{\sigma}^2$

It can also be written as

$$\hat{\sigma}^2(\hat{\theta}) = \sigma^2\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right)$$

$$= \frac{1}{m^2}\sigma^2\left(\sum_{i=1}^{m} y_i\right)$$

$$= \frac{1}{m^{\cancel{2}}} \times \cancel{m} \times 1 = \frac{1}{m}$$

as $m \rightarrow \infty$, variance $= 0$

$$\boxed{\begin{array}{l} \underline{\text{Bias}} : 0 \\ \underline{\text{variance}} : \frac{1}{m} \\ \text{as } m \rightarrow \infty \\ \quad \text{Bias}: 0 \\ \quad \text{variance}: 0 \end{array}}$$

(b) $\underset{x}{\text{minimize}} \quad \frac{1}{m} \sum_{i=1}^{m} (y_i - x)^2 + \frac{\rho}{2} (x - \bar{x})^2$

For x_{MAP}, take derivative and equate to 0.

$$\Rightarrow \frac{1}{m} * 2 * \sum_{i=1}^{m} (y_i - x)(-1) + \frac{\rho}{2} \times 2 * (x - \bar{x})(1) = 0$$

$$\Rightarrow \frac{2}{m} \sum_{i=1}^{m} (y_i - x) = \rho (x - \bar{x})$$

$$\Rightarrow \frac{2}{m} \sum_{i=1}^{m} y_i - \frac{2}{m} \times m x = \rho x - \rho \bar{x}$$

$$\Rightarrow 2 \sum_{i=1}^{m} y_i - 2m x = \rho m x - \rho \bar{x} m$$

$$\boxed{x_{MAP} = \frac{2 \sum_{i=1}^{m} y_i + \rho m \bar{x}}{m(\rho + 2)}}$$

Bias, $\Rightarrow E[\hat{x}] - x$

$$= E\left[\frac{2 \sum_{i=1}^{m} y_i + \rho m \bar{x}}{m(\rho+2)} \right] - x$$

$$= \frac{\rho m \bar{x}}{m(\rho+2)} + \frac{2}{m(\rho+2)} E\left[\sum_{i=1}^{m} y_i \right] - x$$

$$= \frac{\rho m \bar{x}}{m(\rho+2)} + \frac{2}{\cancel{m}(\rho+2)} \times \cancel{m} \times E[y_i] - x$$

$$= \frac{\rho \cancel{m} \bar{x}}{\cancel{m}(\rho+2)} + \frac{2x}{\rho+2} - x$$

$$= \frac{\rho \bar{x} + \cancel{2x} - \rho x - \cancel{2x}}{\rho+2}$$

$$= \frac{\rho(\bar{x} - x)}{\rho+2}$$

$$\boxed{\therefore \text{Bias} = \frac{\rho(\bar{x} - x)}{\rho+2}}$$

Variance,

$$= \sigma^2 \left(\frac{2\Sigma y_i + \rho m \bar{x}}{(\rho+2)m} \right)$$

$$= \sigma^2 \left(\frac{\rho m \bar{x}}{(\rho+2)m} \right) + \sigma^2 \left(\frac{2\Sigma y_i}{(\rho+2)m} \right)$$

$$= 0 \ (\text{constant}) + \frac{4}{(\rho+2)^2 m^2} \sigma^2 (\Sigma y_i)$$

$$= \frac{4}{(\rho+2)^2 m^2} m \cdot 1 = \frac{4}{(\rho+2)^2 m}$$

$$\text{Bias} = \frac{\delta(\bar{x} - x)}{\delta + 2}$$

$$\text{Variance:-} \quad \frac{4}{(\delta+2)^2 m}$$

as $m \to \infty$

$$\text{Bias} = \frac{\delta(\bar{x} - x)}{\delta + 2}$$

$$\text{Variance:-} \quad 0$$

(c) $E[(x - \hat{x})^2] = E[(x - E(\hat{x}) + E(\hat{x}) - \hat{x})^2]$

$$= (x - E(\hat{x}))^2 - 2E[(x - E(\hat{x}))(E(\hat{x}) - \hat{x})] + E[(E(\hat{x}) - \hat{x})^2]$$

$$= (x - E(\hat{x}))^2 - 2(x - E(\hat{x}))\, E[E(\hat{x}) - \hat{x}] + E[(E(\hat{x}) - \hat{x})^2]$$

$$\boxed{E[E(\hat{x}) - \hat{x}] = 0}$$

$$= \underbrace{(x - E(\hat{x}))^2} + E[(E(\hat{x}) - \hat{x})^2]$$

$$= B^2 + V$$

Here, $B = x - E(\hat{x}) \rightarrow$ estimator bias

$V = (E(\hat{x}) - \hat{x})^2$

(d) $\text{Bias}(B) = \dfrac{\rho(x - \bar{x})}{\rho + 2}$, variance $(V) = \dfrac{4}{(\rho+2)^2 m}$

$$B^2 = \frac{\rho^2 (x - \bar{x})^2}{(\rho+2)^2}, \quad V = \frac{4}{m(\rho+2)^2}$$

$$\frac{\partial}{\partial \rho}(B^2 + V) = \frac{\partial}{\partial \rho}\left(\frac{\rho^2 (x - \bar{x})^2}{(\rho+2)^2} + \frac{4}{m(\rho+2)^2}\right)$$

Result from 1C

$$\Rightarrow \frac{(\rho+2)^2\, 2\rho(x - \bar{x})^2 - 2(\rho+2)\rho^2 (x - \bar{x})^2}{(\rho+2)^4} + \frac{-4\cancel{m}\, 2(\rho+2)}{m^{\cancel{2}} (\rho+2)^4}$$

$$\Rightarrow \frac{4\rho(x - \bar{x})^2}{(\rho+2)^3} - \frac{4}{m}\left(\frac{2}{(\rho+2)^3}\right) = 0$$

$$\Rightarrow \frac{\cancel{4}\rho\Delta^2}{\cancel{(\rho+2)^3}} = \frac{\cancel{4}}{m}\left(\frac{2}{\cancel{(\rho+2)^3}}\right)$$

$$\boxed{\rho = \frac{2}{m\Delta^2}}$$