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1) (a)

fur\tail	furry	rope-like
blue	0.1	0
gray	0.1	0
brown	0	0.8

1) (b) $P(\text{blue}) = 0.1$, $P(\text{gray}) = 0.1$, $P(\text{brown}) = 0.8$

$P(\text{furry}) = 0.2$, $P(\text{rope-like}) = 0.8$

$$P(\text{blue}, \text{furry}) = \frac{1}{10}$$

$$P(\text{blue}) * P(\text{furry}) = \frac{1}{10} \times \frac{2}{10} = \frac{1}{50}$$

$$P(\text{blue}, \text{furry}) \neq P(\text{blue}) * P(\text{furry})$$

$P(\text{blue, rope line}) = 0 \neq \frac{1}{10} \times \frac{8}{10}$

$P(\text{gray, furry}) = \frac{1}{10} \neq \frac{1}{10} \times \frac{2}{10}$

$P(\text{brown, furry}) = 0 \neq \frac{1}{10} \times \frac{8}{10}$

$P(\text{brown, rope-like}) = \frac{8}{10} \neq \frac{8}{10} \times \frac{8}{10}$

∴ "Fur color" and "Tail Texture" are not independent.

(1c)

fur\tail	furry	rope-like
blue	0.5	0
gray	0.5	0
brown	0	0

1d) $P(\text{blue}) = \frac{1}{2}$

$P(\text{gray}) = \frac{1}{2}$

$P(\text{brown}) = 0$

$P(\text{rope-like}) = 0$

$P(\text{furry}) = 1$

$P(\text{blue, furry}) = \frac{1}{2} = \frac{1}{2} \times 1$

$P(\text{blue, rope line}) = 0 = \frac{1}{2} \times 0$

$P(\text{gray, furry}) = \frac{1}{2} = \frac{1}{2} \times 1$

$P(\text{gray, rope-line}) = 0 = \frac{1}{2} \times 0$

$P(\text{brown, furry}) = 0 = 0 \times 1$

$P(\text{brown, rope-like}) = 0 = 0 \times 0$

$\therefore$ "Fur color" and "Tail Texture" are independent.

Independent:-

$$P(A \cap B) = P(A) * P(B)$$

Q2)

a) <u>Given</u>:- $F(x) \rightarrow$ cdf (cummilative distribution)

Pdf, $f(x) = \frac{d}{dx}(F(x))$

$\Rightarrow f(x) = \frac{d}{dx}(1 - e^{-\lambda x}) \quad x \geq 0$

$= \frac{d}{dx}(0) \quad x \leq 0$

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}}$$

b) <u>mean</u>:-

$E(x) = \int_0^{\infty} x \, f(x) \, dx$

$= \int_0^{\infty} x \lambda e^{-\lambda x} \, dx$

Applying Integration by-Parts,

$$= \lambda x \int_0^\infty e^{-\lambda x}\, dx - \int_0^\infty \lambda e^{-\lambda x}\, dx$$

$$= \lambda x \left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^\infty - \lambda \int_0^\infty \frac{e^{-\lambda x}}{-\cancel{\lambda}}$$

$$= \int_0^\infty e^{-\lambda x}\, dx$$

$$= -\frac{1}{\lambda} \left. e^{-\lambda x} \right|_0^\infty = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}.$$

$$\boxed{\text{Mean} = \frac{1}{\lambda}}$$

$$Var(x) = E(x^2) - E(x)$$

$$E(x^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= x^2 \frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}} \Big|_0^\infty - \int 2x \cancel{\lambda} \frac{e^{-\lambda x}}{-\cancel{\lambda}} dx$$

$$= 0 + \frac{2}{\lambda} \underbrace{\int_0^\infty \lambda x e^{-\lambda x} dx}_{\text{mean}}$$

$$= 0 + \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\boxed{Var(X) = \frac{1}{\lambda^2}}$$

c) $x_1, x_2, x_3 \cdots\cdots x_m$ (delay times)

MLE of $(\hat{\lambda})$

We want to maximize the data, our goal is to choose a λ which maximizes the data.

As $x_1, x_2 \dots x_m$ are iid's.

$f(x_1, x_2, x_3 \dots\dots x_m, \lambda)$

$= f(x_1, \lambda) \cdot f(x_2, \lambda) \dots\dots f(x_m, \lambda)$

$$f(x, \lambda) = \left(\lambda e^{-\lambda x_1}\right) \cdot \left(\lambda e^{-\lambda x_2}\right) \dots\dots \left(\lambda e^{-\lambda x_m}\right)$$

$$= \lambda^m e^{-\lambda \sum_{i=1}^{m} x_i}$$

$$\log(f(x, \lambda)) = m \log \lambda - \lambda \sum_{i=1}^{m} x_i$$

To maximize, derivative should be '0'.

$$\frac{d}{d\lambda}\left(\log(f(x,\lambda))\right) = \frac{d}{d\lambda}\left(m \log \lambda - \lambda \sum_{i=1}^{m} x_i\right)$$

$$\Rightarrow 0 = \frac{m}{\lambda} - \sum_{i=1}^{m} x_i$$

$$\boxed{\hat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_i} \qquad \hat{\lambda} \to \lambda_{MLE}}$$

d) If an estimator is unbiased,

i) $E[\hat{\theta}] = \theta$

$$E\left[\frac{1}{\hat{\lambda}}\right] = E\left[\frac{\sum x_i}{m}\right]$$

$$= \frac{1}{m} E\left[\sum_{i=1}^{m} x_i\right]$$

$$= \frac{1}{\cancel{m}} \times \cancel{m} * E[x]$$ [Linearity of Expectations]

$$= \frac{1}{\lambda}$$

$$\boxed{E\left[\frac{1}{\hat{\lambda}}\right] = \frac{1}{\lambda} \Rightarrow \text{unbiased estimator}}$$

ii) $$\frac{1}{\hat{\lambda}^2} = \left(\frac{\sum x_i}{m}\right)^2$$

$$E\left[\left(\frac{1}{\hat{\lambda}}\right)^2\right] = E\left[\left(\frac{\sum x_i}{m}\right)^2\right]$$

$$= E\left[\frac{\sum x_i^2}{m^2}\right]$$

$$= \frac{1}{m^2} E\left[\sum_{i=1}^{m} x_i^2\right]$$

$$= \frac{1}{\cancel{m^2}} \times \cancel{m} \times E\left[x_i^2\right]$$ [Linearity of Expectations]

$$= \frac{1}{m} * E\left[x_i^2\right]$$

$$= \frac{1}{m} \times \frac{2}{\lambda^2}$$ (calculated in 2(b))

$$\frac{1}{\lambda^2} \neq \frac{2}{m}\left(\frac{1}{\lambda^2}\right)$$

$$\boxed{\therefore \ \frac{1}{\hat{\lambda}^2} \text{ is an UNBIASED ESTIMATE.}}$$

e) $$P_{\lambda,c}(x) = \begin{cases} 0 & \text{if } x > c \text{ or } x < 0 \\ \dfrac{\lambda e^{-\lambda x}}{1-e^{-\lambda c}} & \text{else} \end{cases}$$

Condition for Hoeffding's inequality :-

1) $x_1, x_2 \dots x_m$ are iid's

2) $0 \le x_i \le 1$

In our case, $x_1, x_2 \dots x_m$ are iid's, but $0 \le x_i \le c$

To satisfy this condition, we will introduce another random variable,

$$y = \frac{x_i}{c} \quad [0 \le y_i \le 1]$$

NORMALIZING X, SO THAT IT IS BETWEEN 0 AND 1

$$x = yc$$

Your pmf now becomes,

$$P_{\lambda,c}(y) = \begin{cases} 0 & y > 1 \text{ or } y < 0 \\ \dfrac{\lambda e^{-\lambda y c}}{1 - e^{-\lambda c}} & \text{else} \end{cases}$$

Since, x and y are both random variables, we can replace them.

$$P_{\lambda,c}(x) = \begin{cases} 0 & x > 1 \text{ or } x < 0 \\ \dfrac{\lambda e^{-\lambda x c}}{1-e^{-\lambda c}} & \text{else} \end{cases}$$

Now, we can apply Hoeffding's inequality. [$0 \leq x_i \leq 1$ and $x_i \to$ all of them are iid's]

we know,

$$m \geq \frac{\log(2/\delta)}{2e^2}$$

$$e^2 \geq \frac{\log(2/\delta)}{2m}$$

$$\boxed{e \geq \sqrt{\frac{\log(2/\delta)}{2m}}}$$

We also know,

$$\Pr\left(\frac{1}{m}\sum_{i=1}^{m} x_i - E[x] \geq t\right) \leq e^{-2mt^2}$$

(or)

$$\Pr\left(|\hat{\theta}_m - \theta^*| > \varepsilon\right) \leq 2e^{-2m\varepsilon^2}$$

Rearranging the terms we get

$$\Pr\left(-\varepsilon < E[x] - \frac{1}{m}\sum x_i < \varepsilon\right) \geq 1-\delta$$

$$\Pr\left(-\varepsilon + \frac{1}{m}\sum x_i < E[x] < \varepsilon + \frac{1}{m}\sum x_i\right) \geq 1-\delta$$

conditions :- $\varepsilon \geq \log\sqrt{\frac{2/\delta}{2m}}$

$$\lambda_{min} = -\varepsilon + \frac{1}{m}\sum x_i$$

$$\lambda_{max} = \varepsilon + \underbrace{\frac{1}{m}\sum x_i}_{\text{sample mean}}$$

Q3) (a)

N -> Normal
W -> Warped
D -> Defective
G -> Good

(i) Red Press:-

Bayes Risk =

$\Rightarrow P(D|N) * P(N) * LOSS + P(D|W) * P(W) * LOSS$

$\Rightarrow 0.9 \times 0.05 + 0.1 \times 0.03$

$\Rightarrow 0.075$

Blue Press:-

Bayes Risk $= 0.9 \times 0 + 0.1 \times 0.85$

$= 0.085$

Use Red press to minimize Bayes Risk.

ii) <u>Minimax</u> <u>Risk</u> :-

<u>Red</u> <u>Press</u> :- max loss is 1.

<u>Blue</u> <u>Press</u> :- max loss is 1.

iii) <u>Red</u> <u>Press</u> :-

Bayes Risk :- $1 \times 0.05 \times 1 = 0.05$

<u>Blue</u> <u>Press</u> :-

Bayes Risk $= 0.9 \times 0 \times 1 = 0$

iv)

<u>Minimax</u> <u>Risk</u> :-

<u>Red</u> <u>Press</u> :- max loss is 1.

<u>Blue</u> <u>Press</u> :- max loss is 0.

(b) <u>Bayes</u> <u>Reward</u>:-

Inspection cost = $100 * x$ (UPFRONT COST)

we have 3 possibilities

SOLD and Good → Revenue \$1

SOLD and Defective → $-\$1$

UNSOLD → \$0

we will consider only 2 cases:-

① Sold and good → Inspected, uninspected

② Sold and defective → Inspected, uninspected

General Formula

Reward for sold, good $= \left(P(\bar{D})\, P(N) + P(\bar{D})\, P(W) * (1-x) \right)$

Reward for Sold, bad $= \left(P(D)\,P(N) + (1-x)\,P(D)\,P(W) \right)$

Bayes Reward for red Press:-

$\Rightarrow 1\left[(0.9 \times 0.95) + (1-x) * (0.1) * 0.7\right]$

$-1\left[(0.9 \times 0.05) + (1-x) * (0.1) \times 0.3\right]$

$-100x$

$= 0.85 - 100.04x$

Blue Press:-

$\Rightarrow 1\left[(0.9 \times 1) + (1-x) * (0.1) \times 0.15\right]$

$-1\left[(0.9 \times 0) + (1-x) * (0.1)\,(0.85\right]$

$-100x$

$\Rightarrow 0.83 - 99.93x$

ii) RECOMMENDATION:-

DO NOT INSPECT ANYTHING.

I will recomend using RED PRESS over Blue Press and NO INSPECTION.

(c) Red Press:-

$$= 500\left[(0.9 \times 0.95) + (1-x) * (0.1) * 0.7\right]$$

$$- 10{,}000\left[(0.9 \times 0.05) + (1-x) * (0.1) \times 0.3\right]$$

$$- 100x$$

$$= -287.5 + 165x$$

Blue Press:-

$$\Rightarrow 500\left[(0.9 \times 1) + (1-x) * (0.1) \times 0.15\right]$$

$$- 10{,}000\left[(0.9 \times 0) + (1-x) * (0.1) (0.85)\right]$$

$$- 100x$$

$\Rightarrow -392.5 + 742.5x$

ii) RECOMMENDATION :-

As the penalty of defective and sold is very high as compared to good and sold.

I WILL INSPECT All THE GADGETS.

maximum Reward:- If I use blue risk.