Additional notes on the SVD

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1 The SVD as a "satisfaction of rules"

Definition 1. *Given $A \in \mathbb{R}^{m\times n}$ we define the "econo-mode" SVD of A as the factorizaton*

$$A = U_1 S_1 V_1^T$$

where $U_1 \in \mathbb{R}^{m\times r}$, $V_1 \in \mathbb{R}^{n\times r}$, and $S_1 \in \mathbb{R}^{r\times r}$. Furthermore,

- $U_1^T U_1 = V_1^T V_1 = I \in \mathbb{R}^{r\times r}$
- $S_1 = \mathbf{diag}(\sigma_1, ..., \sigma_r)$ *where each $\sigma_i > 0$ for $i = 1, ..., r$.*

We define the "full SVD" of A as the factorization

$$A = USV^T$$

where

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \in \mathbb{R}^{m\times m}, \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

and $UU^T = U^T U = I \in \mathbb{R}^{m\times m}$, $VV^T = V^T V = I \in \mathbb{R}^{n\times n}$. Here, U_1, S_1, and V_1 are the factors from the "econo-mode" SVD.

These rules define the SVD. When someone asks you for the SVD, they are looking for any matrices (either U_1, S_1, V_1 or U, S, V) that satisfy these conditions. MATLAB and Python by default provide the full SVD, but there are options that can turn on the econo-mode option instead.

1.1 Details

- The econo-mode SVD can also be written as

  $$A = \sum_{i=1}^{r} \sigma_i (u_i v_i^T)$$

  where

  $$U_1 = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}, \quad V_1 = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}$$

  and for any $i = 1, ..., r$, the triplet (u_i, v_i, σ_i) are *corresponding left singular vector, right singular vector, and singular values.*

- r = number of strict nonzeros in S is the rank of the matrix A.
- **Existence** The SVD always exists.
- **Uniqueness** There are several areas in which an SVD may be nonunique.

1. **Sign ambiguity.** In the economode SVD,

$$A = U_1 S_1 V_1^T = (-U_1) S_1 (-V_1^T).$$

In other words, if any (u_i, v_i, σ_i) form a singular vector/value triplet, then $(-u_i, -v_i, \sigma_i)$ also form a triplet. This is where "thinking of SVD as a set of rules" kicks in; either sign form "obeys all the rules".

2. **Repeated singular value.** Suppose that, given the singular value decomposition, there existed a repeated singular value, e.g.

$$\sigma_i = \sigma_j = \sigma, \quad i \neq j.$$

Let us rewrite the SVD such that that term is isolated, e.g.

$$A = \sum_{k=1}^{r} \sigma_k (u_k v_k^T) = \underbrace{\sigma_i u_i v_i^T + \sigma_j u_j v_j^T}_{\hat{A}} + \underbrace{\sum_{k \neq i,j} \sigma_k u_k v_k^T}_{\tilde{A}}.$$

Then

$$\hat{A} = \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} v_i^T \\ v_j^T \end{bmatrix} = \begin{bmatrix} u_i & u_j \end{bmatrix} Q^T \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} Q \begin{bmatrix} v_i^T \\ v_j^T \end{bmatrix}$$

for any $Q^T Q = I$. This is true since

$$Q^T \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} Q = \sigma Q^T I Q = \sigma I.$$

Note that this doesn't work if the two singular values were not the same.

3. **Full SVD terms not appearing in econo-mode SVD.** For the same reason as above, the singular vectors corresponding to the zero diagonal elements of S are not unique.

2 Exercises

1. Suppose $A = bc^T$ where $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. What is the SVD of A?

2. Suppose A is diagonal, e.g. $A = \mathbf{diag}(a_1, a_2, ..., a_r)$, and a_i are all positive. What is the SVD of A?

3. Suppose A is diagonal, e.g. $A = \mathbf{diag}(a_1, a_2, ..., a_r)$, but a_i can be positive, negative, or 0. What is the SVD of A?

4. Suppose

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}^T}_{\text{Given SVD}}$$

What is the SVD of

$$A' = \begin{bmatrix} 4 & 6 & 5 \\ 7 & 9 & 8 \\ 1 & 3 & 2 \end{bmatrix}$$

in terms of the factors already given?

3 Connection with range, null spaces

Definition 2. *Given $A \in \mathbb{R}^{m\times n}$, the* range *of A is given as the set*

$$\mathbf{range}(A) = \{y : Ax = y \textit{ for some } x\} \subseteq \mathbb{R}^m$$

and the null space *of A is given as the set*

$$\mathbf{null}(A) = \{x : Ax = 0\} \subseteq \mathbb{R}^n.$$

Definition 3. *The* span *of a set of vectors $\mathcal{X} = \{x_1, ..., x_s\}$ is the set of vectors "reachable" by $\mathcal{X}$, e.g.*

$$\mathbf{span}(\mathcal{X}) = \{z : z = \sum_{i=1}^{s} \alpha_i x_i\}.$$

3.1 Properties of the SVD

(You can prove these as personal exercises.)

Suppose

$$A = \underbrace{U_1 S_1 V_1}_{\text{econo-mode}} = \underbrace{\begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}}_{\text{full}}.$$

- **Subspaces**
  - The range of A is the span of the columns of U_1.
  - The nullspace of A is the span of the columns of V_2.
- **Inverses**
  - If A is square, then S is square. If $S = S_1$ (e.g. $m = n = r$), then A is invertible, and
    $$A^{-1} = US^{-1}V^T$$
    where the inverse of a diagonal matrix is just a diagonal matrix with each entry taken as a reciprocal
    $$S_{kk}^{-1} = 1/S_{kk}.$$
  - In all other cases, the matrix
    $$A^{\dagger} = U_1 S_1^T V_1^T$$
    is the Moore-Penrose Pseudoinverse of A. To prove that this is not in fact an inverse, pick any $x \in \mathbf{null}(A)$. Then
    $$A^{\dagger} A x \neq x \text{ (why?)}$$
  - **Check understanding.** What is the nullspace of A if A is invertible?
- **Projections**
  - The Euclidean projection onto the "special subspaces" can be given in terms of the SVD. Specifically,
    $$y = \mathbf{proj}_{\mathbf{range}(A)}(x) = AA^{\dagger}x = U_1 U_1^T x$$
    $$z = \mathbf{proj}_{\mathbf{null}(A)}(x) = (I - A^{\dagger}A)x = V_2 V_2^T x$$
  - **Check understanding.** From these properties, given $A = \theta^T \in \mathbb{R}^{1\times n}$, derive the solution to
    $$x = \underset{z^T\theta=0}{\operatorname{argmin}} \|x - z\|_2^2.$$
    Hint: What is the SVD of $A = \theta^T$?