Q1) BAYESIAN INFERENCE **(Total 3 points)**

Let i.i.d. samples $X_1, X_2, \dots, X_n \sim$Bernoulli(p). The p.d.f. of the Beta(α, β) distribution (with parameters α and β) is $f(x) = C \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}$, for $x \in [0,1]$, where C is a constant.

(a) Show that Beta(1, 1) is the Uniform(0, 1) distribution. Do not ignore C. (1 point)

(b) If the prior for p is Beta(1, 1), show that the posterior distribution looks like a Beta. Find its parameters. Feel free to ignore any constants. Clearly show all steps. (2 points)

(a) pdf of Beta (1, 1) : $f(x) = C \cdot x^0 \cdot (1-x)^0$

$= C$

$\because \int_0^1 f(x)\,dx = 1$, we have $\int_0^1 C \cdot dx = 1$

$\Rightarrow C \cdot 1 = 1$

$\Rightarrow \underline{C = 1}$ (-1/2)

$\therefore$ Beta(1, 1) has pdf $\underline{f(x) = 1}$ in $[0, 1]$

same as Unif (0,1) $\because \frac{1}{1-0} = \underline{1}$

(b) posterior, $f(p \mid \vec{x}) \propto L(p) \cdot \underline{f(p)}$

prior : $f(p) = 1$
$\because$ Unif(0,1)

$= L(p)$

$= \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$

$= p^{\Sigma x_i} (1-p)^{\underline{n} - \Sigma x_i}$

$\therefore$ posterior for $f(p)$ is Beta$(\Sigma x_i \underline{+1}, n - \Sigma x_i \underline{+1})$

(-1/2)

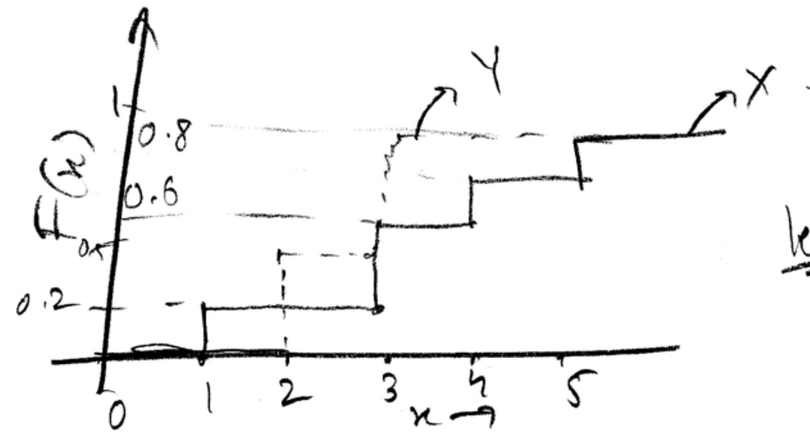
Q2) K-S TEST **(Total 4 points)**

Let X = {3, 1, 3, 5, 4} and Y = {3, 2}.

(a) Clearly draw the eCDF for X and Y (in the same figure). (1 point)

(b) Use K-S test to check whether X and Y are from the same distribution or not. Reject if the max difference statistic > 0.35. Show all relevant values as a table, as in class, including difference to the left and right of each point. (3 points)

(a)



lenient

(b)

x	$F_X^-(x)$	$F_X^+(x)$	$F_Y^-(x)$	$F_Y^+(x)$	$\lvert F_X^- - F_Y^- \rvert$	$\lvert F_X^+ - F_Y^+ \rvert$
2	0.2	0.2	0	1/2	0.2	0.3
3	0.2	0.6	1/2	1	0.3	0.4 > 0.35

0.4 > 0.35, so Reject.

-1/2

-1/2

Q3) MODIFIED WALD'S TEST **(Total 5 points)**

Consider a variant of Wald's Test where we reject null if $W > z_{\alpha/2}$ (instead of $|W| > z_{\alpha/2}$). Suppose the null hypothesis is H_0: $\theta = \theta_0$, but the true value of θ is $\theta_* \neq \theta_0$ (meaning null is not true). Under this variant of Wald's test, derive the probability of rejecting the null for an Asymptotically Normal estimator $\hat{\theta}$ with estimated standard error $\widehat{se}$. Also draw a figure to indicate the area under the curve that represents this probability. Show all steps clearly.

Pr(reject Null | Null is not true).

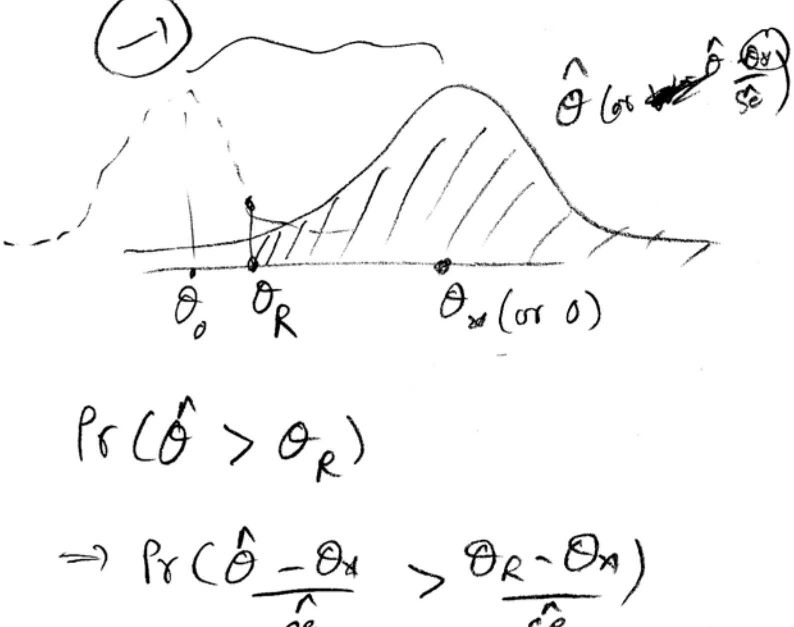
We reject Null if $W > Z_{\alpha/2}$

$$\frac{\hat{\theta} - \theta_0}{\hat{se}} > Z_{\alpha/2}$$

$$\Rightarrow \hat{\theta} > \underbrace{\theta_0 + Z_{\alpha/2} \cdot \hat{se}}_{\theta_R}$$

-1

θ_* not θ_0



$\hat{\theta}$ is Asym Nor.

$\Rightarrow \hat{\theta} \approx N(\theta_*, \hat{se})$ -1/2

$$\Rightarrow \frac{\hat{\theta} - \theta_*}{\hat{se}} \approx N(0,1)$$

-1/2

$$Pr(\hat{\theta} > \theta_R)$$

$$\Rightarrow Pr\left(\frac{\hat{\theta} - \theta_*}{\hat{se}} > \frac{\theta_R - \theta_*}{\hat{se}}\right)$$

$$\Rightarrow 1 - Pr\Big(\underbrace{\frac{\hat{\theta} - \theta_*}{\hat{se}}}_{\sim N(0,1)} \leq \frac{\theta_R - \theta_*}{\hat{se}}\Big)$$

$$\Rightarrow 1 - \Phi\left(\frac{\theta_R - \theta_*}{\hat{se}}\right) = 1 - \Phi\left(\frac{\theta_0 - \theta_*}{\hat{se}} + Z_{\alpha/2}\right).$$

Q4) METHOD OF MOMENTS ESTIMATOR (MME) WITH DATA SAMPLES (Total 6 points)

Let $X = \begin{cases} 1 & with\ prob\ \theta \\ 3 & otherwise \end{cases}$, where θ is unknown. Let D = {1, 3, 1} be drawn i.i.d. from X. For all parts below, your final answer should be numeric (fractions or square roots are fine).

(a) Derive $\hat{\theta}_{MME}$ using D as the sample data. Clearly show all your steps. (2 points)

(b) Derive $\widehat{se}(\hat{\theta}_{MME})$. Specifically, first derive $se(\hat{\theta}_{MME})$ in terms of θ, and then estimate $\widehat{se}(\hat{\theta}_{MME})$, as in class. Show all your steps clearly. (3 points)

(c) Derive a (1-α) confidence interval for $\hat{\theta}_{MME}$. Explain your steps. (1 point)

(a) $K=1$

$$\hat{\alpha}_1 = \sum_{i=1}^{\hat{}} X_i \hat{p}(X_i) = \frac{1}{n}\sum X_i \quad \Big\}\ -1/2$$

$$\alpha_1(\theta) = E[X(\theta)] = 1\cdot\theta + 3\cdot(1-\theta) = \underline{3-2\theta} \quad \Big\}\ -1/2$$

$$\text{MME} \Rightarrow \alpha_1(\hat{\theta}) = \hat{\alpha}_1 \Rightarrow 3-2\hat{\theta} = \frac{1}{n}\sum X_i$$

-1

$$\Rightarrow \hat{\theta} = \underline{\frac{(3-\frac{1}{n}\sum X_i)}{2}} = \frac{3-(\frac{1}{3}\cdot(1+3+1))}{2} = \frac{3-\frac{5}{3}}{2}$$

$$= \underline{\underline{2/3}}$$

(b) $$se(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{Var\left(\frac{3-\frac{1}{n}\sum x_i}{2}\right)}$$

$$= \sqrt{Var\left(\frac{1/n\sum x_i}{2}\right)} \overset{iid}{=} \sqrt{\frac{1}{4}\cdot\frac{1}{n^2}\cdot n\cdot Var(X_i)} = \frac{1}{2}\sqrt{\frac{Var(X_i)}{n}}$$

-1

Now, $E[X^2] = 1\cdot\theta + 9\cdot(1-\theta) = 9-8\theta$

$$Var(X) = (9-8\theta) - (3-2\theta)^2 = (9-8\theta)-(9-12\theta+4\theta^2)$$
$$= 4\theta(1-\theta) \quad (-1)$$

$$\therefore \underline{se(\hat{\theta})} = \frac{1}{2}\sqrt{\frac{4(\theta)(1-\theta)}{3}} = \sqrt{\frac{\theta(1-\theta)}{3}}$$

$-1/2$

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{3}} = \sqrt{\frac{2/3\cdot 1/3}{3}} = \sqrt{\frac{2}{27}} \quad \Big\}\ -1/2$$

-1/2

(C) $\because$ MME is Asym Normal, we have

$(1-\alpha)$ CI as $\hat{\theta} \pm z_{\alpha/2}\, \hat{se}$

$$= \frac{2}{3} \pm z_{\alpha/2} \sqrt{\frac{2}{27}}$$

Q5) MLE++ **(Total 7 points)**

Let i.i.d. samples $X_1, X_2, \dots, X_n \sim$Binomial(m, p), with n ≠ m.

(a) Derive $\hat{p}_{MLE}$, the MLE of p, assuming that m is a constant. (3 points)

(b) Derive E[$\hat{p}_{MLE}$] and show that $\hat{p}_{MLE}$ is unbiased. (1 point)

(c) Derive $se(\hat{p}_{MLE})$ in terms of p, m, and n. (1 point)

(d) Using only (b) and (c) above, show that $\hat{p}_{MLE}$ is consistent. (1 point)

(e) Find a consistent estimator for $se(\hat{p}_{MLE})$. Why is your estimator consistent? (1 point)

(a) likelihood, $L(p) = \prod_{i=1}^{n} \underbrace{{}^{m}C_{X_i}}_{\text{independent of } p} \cdot p^{X_i} (1-p)^{m-X_i}$

log likelihood $\ell(p) = \sum_{i=1}^{n} \{ \log({}^{m}C_{X_i}) + X_i \log p$

$+ (m-X_i)\log(1-p) \}$

$$\frac{d\ell(p)}{dp} = 0 = \sum_{i=1}^{n} \left\{ 0 + \frac{X_i}{p} - \frac{(m-X_i)}{1-p} \right\}$$

$$= \frac{\sum X_i}{p} - \frac{m \cdot n - \sum X_i}{1-p} = 0$$

$$\Rightarrow \sum X_i - \cancel{p \sum X_i} = m \cdot n \cdot p - \cancel{p \sum X_i}$$

$$\Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} X_i}{m \cdot n}$$

(b) $E[\hat{p}] = \dfrac{\sum_{i=1}^{n} E[X_i]}{mn} = \dfrac{E[X_i]}{m} = \dfrac{\cancel{m} \cdot p}{\cancel{m}} = p$

$\therefore$ bias $= E[\hat{p}] - p = p - p = \underline{\underline{0}}$

(c) $se(\hat{p}) = \sqrt{Var(\hat{p})} = \sqrt{Var\left(\frac{\sum X_i}{mn}\right)} = \sqrt{\frac{\cancel{n} \cdot Var(X_i)}{m^2 n^{\cancel{2}}}}$

$$= \sqrt{\frac{\cancel{m} \cdot p(1-p)}{m^{\cancel{2}} n}} = \sqrt{\frac{p(1-p)}{mn}}$$

(d) Unbiased & $se(\hat{p}) \to 0$ as $n \to \infty$

$\therefore$ consistent

(e) $\hat{se}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{mn}}$

$\cancel{= \sum X_i (\Sigma}$

$$= \sqrt{\frac{\frac{\sum X_i}{mn}\left(1 - \frac{\sum X_i}{mn}\right)}{mn}}$$

consistent $\because$ $\hat{p}$ is MLE, so by equi-variance, $\hat{se}(\hat{p}_{MLE})$ is also an MLE, and thus consistent

$-1/2$