A weakly informative prior for resonance frequencies

Marnix Van Soom & Bart de Boer

MaxEnt 2021

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Resonance frequencies in the natural sciences

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[Wilson+ 2014]

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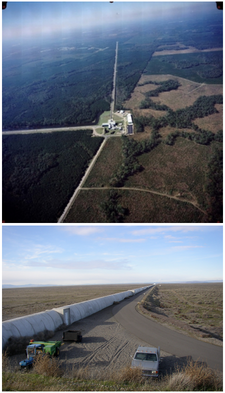

[Littenberg and Cornish 2015]

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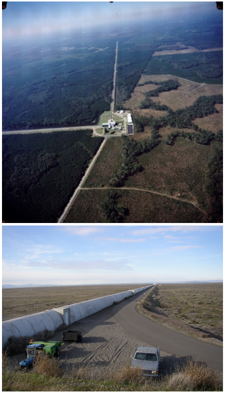

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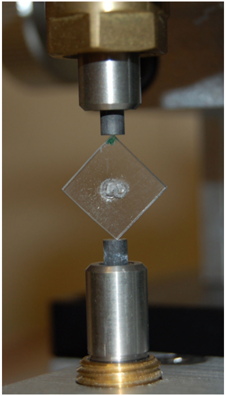

[Xu+ 2019]

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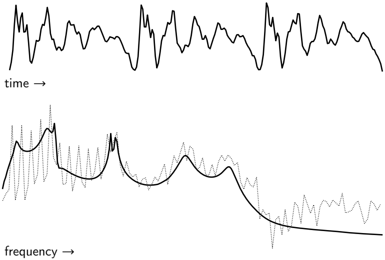
$$\boldsymbol{x} = \{x_1 \cdots x_K\}$$
$$= K \text{ resonance frequencies}$$



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$$Z(K) = \int \mathrm{d}^K\boldsymbol{x}\,\mathscr{L}(\boldsymbol{x})\pi(\boldsymbol{x})$$

First candidate: $\pi_1(\boldsymbol{x}|\cdot)$

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  - Frustrates calculating $Z(K)$ [Celeux+ 2018]

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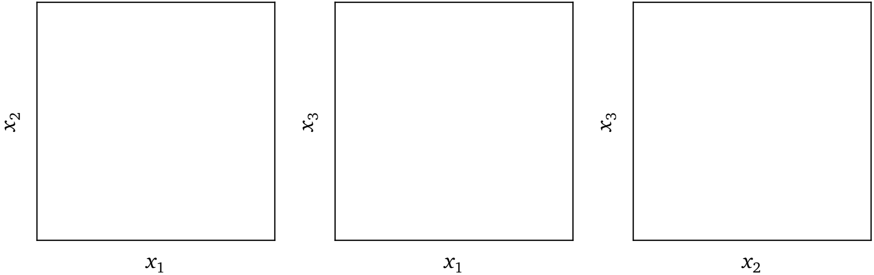

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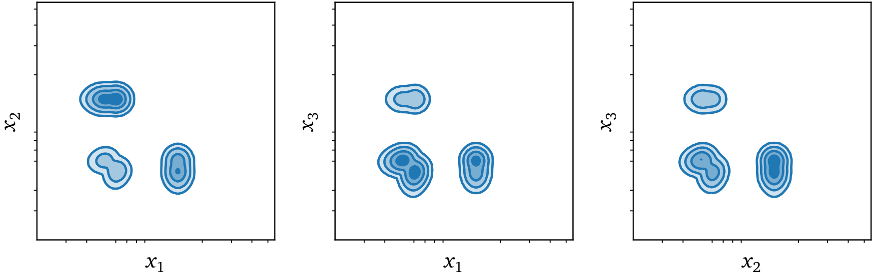

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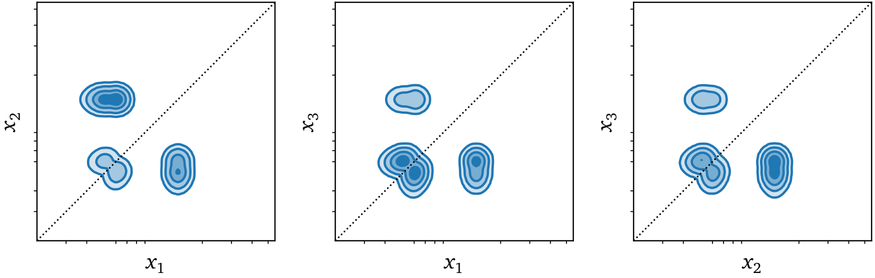

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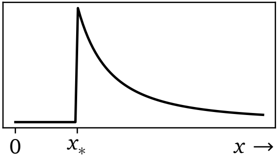
$$\mathsf{Pareto}(x|x_*, \lambda) = \frac{\lambda}{x}\left(\frac{x_*}{x}\right)^{\lambda} \qquad (x_* \leq x)$$



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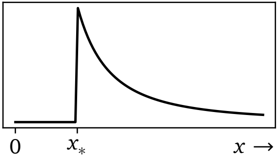
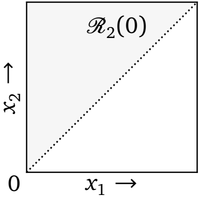


- Supported by the *ordered region*

$$\mathscr{R}_K(x_0) = \{\boldsymbol{x}|x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_K\}$$



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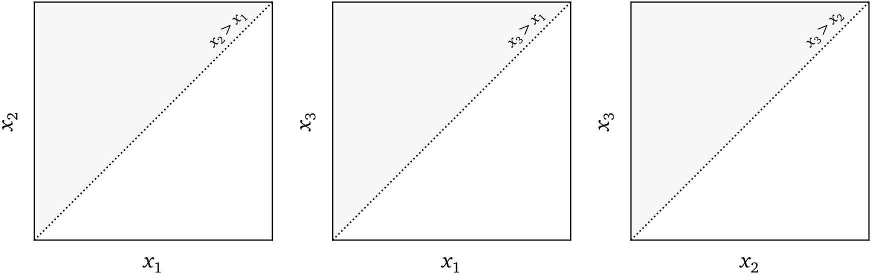

Support $\mathscr{R}_3(x_0) = \{\boldsymbol{x}|x_0 \leq x_1 \leq x_2 \leq x_3\}$

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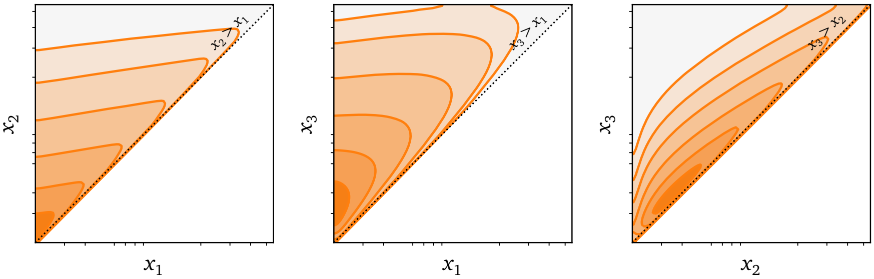

Pairwise prior distribution $\pi_3(\boldsymbol{x}|\boldsymbol{\lambda})$

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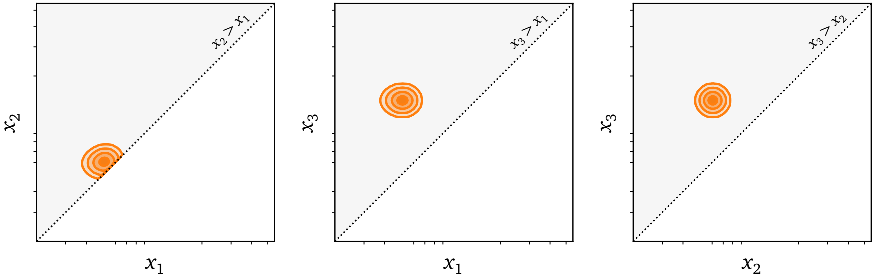

Pairwise posterior distribution $P_3(\boldsymbol{x}|\boldsymbol{\lambda}) \propto \mathscr{L}(\boldsymbol{x})\pi_3(\boldsymbol{x}|\boldsymbol{\lambda})$

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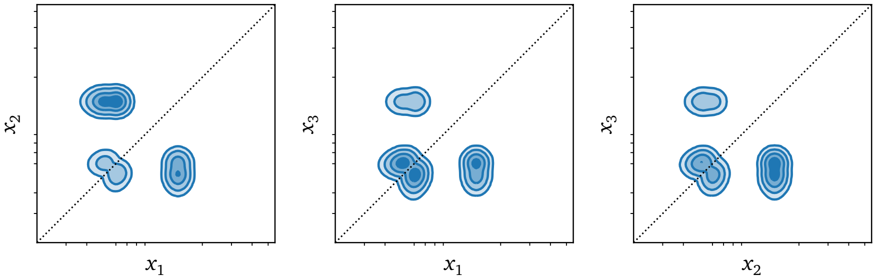

Pairwise posterior distribution $P_1(\boldsymbol{x}|a,b) \propto \mathscr{L}(\boldsymbol{x})\pi_1(\boldsymbol{x}|a,b)$

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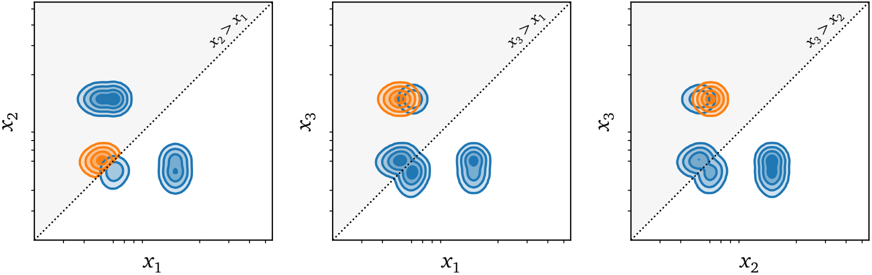

$P_1(\boldsymbol{x}|a,b)$ vs. $P_3(\boldsymbol{x}|\boldsymbol{\lambda})$

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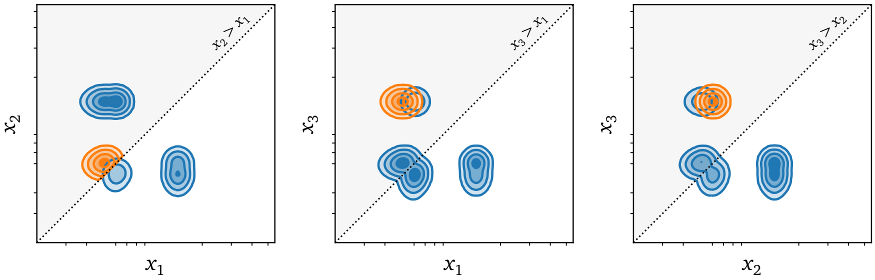

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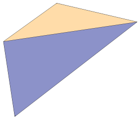
$\mathscr{R}_3(x_0)$



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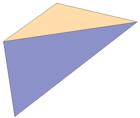

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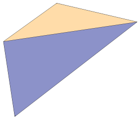

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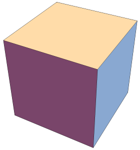
$\mathscr{R}_3(x_0)$ in $\boldsymbol{u}$ space



- $m(\boldsymbol{u}) = m(\boldsymbol{x}(\boldsymbol{u}))\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{u}}\right| = \begin{cases} 1 & \boldsymbol{u} \geq 0 \\ 0 & \text{else} \end{cases}$

Deriving $\pi_3(\boldsymbol{x}|\cdot)$

Normalize $m(\boldsymbol{u}) = \begin{cases} 1 & \boldsymbol{u} \geq 0 \\ 0 & \text{else} \end{cases}$ by constraining first moments $\langle \boldsymbol{u} \rangle := \overline{\boldsymbol{u}}$

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$$D_{\mathrm{KL}}(\pi_3|m) = \int \mathrm{d}^K\boldsymbol{u}\, \pi_3(\boldsymbol{u}) \log \frac{\pi_3(\boldsymbol{u})}{m(\boldsymbol{u})}$$

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- Solution:

$$\pi_3(\boldsymbol{u}|\boldsymbol{\lambda}) = \prod_{k=1}^{K} \mathsf{Exp}(u_k|\lambda_k) \qquad (\boldsymbol{u} \geq 0)$$

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- Equivalent to Jaynes' *principle of maximum entropy* with $m(\boldsymbol{u})$ serving as the invariant measure [Jaynes 1968]

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  - Thus:

$$\boxed{\lambda_k = \frac{\overline{x_k}}{\overline{x_k} - \overline{x_{k-1}}}}$$

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- **Convenient parametrization**: $\boxed{\pi_3(\boldsymbol{x}|\overline{\boldsymbol{x_0}}) \quad \text{where} \quad \overline{\boldsymbol{x_0}} \equiv (x_0, \overline{x_1}, \overline{x_2}, \cdots, \overline{x_K})}$

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- **Inference insensitive** to values of the hyperparameters $\overline{\boldsymbol{x_0}}$
  - Maximum entropy: weak inductive bias
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- **Scale invariant**: $\pi_3(c\boldsymbol{x}|\overline{\boldsymbol{x_0}}) = f(c)\pi_3(\boldsymbol{x}|\overline{\boldsymbol{x_0}})$ [Newman 2005]

Application

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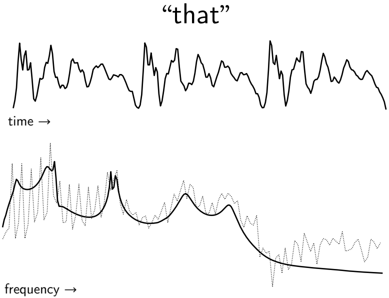
- Measure resonance frequencies of the human vocal tract
- Five representative vowel sounds taken from the CMU ARCTIC database [Kominek and Black 2004]
- $D \in \{\mathsf{shore}, \mathsf{that}, \mathsf{you}, \mathsf{little}, \mathsf{until}\}$
- Compare $Z_i(K)$ and $H_i(K)$ for each D and $i \in \{1, 2, 3\}$



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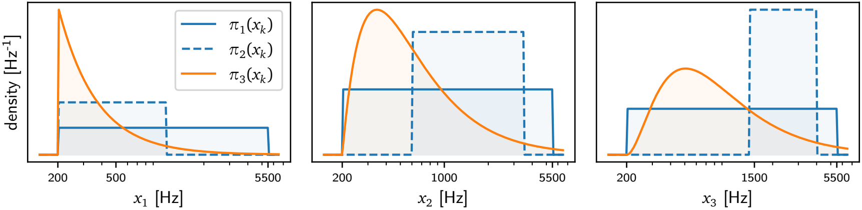

Figure: Comparison of π_1, π_2 and π_3 in terms of the marginal priors $\pi_i(x_k|\cdot)$ for the case $K := 3$. The marginal $\pi_i(x_k|\cdot)$ is obtained by integrating out the two other frequencies; for example, $\pi_i(x_1|\cdot) = \iint \mathrm{d}x_2\,\mathrm{d}x_3\,\pi_i(\boldsymbol{x}|\cdot)$. The pdfs are shown on a common log scale and are scaled by the appropriate Jacobian determinant $|\,\mathrm{d}x_k/\mathrm{d}\log x_k\,| = x_k$.

$\pi_1(\boldsymbol{x}|\cdot)$ **is excluded for** $K \geq 4$

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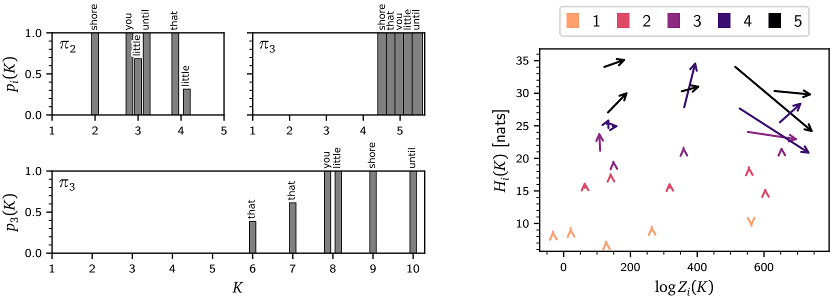

(a)

(b)

Figure: **(a)** Model selection in Experiment I (top row) and Experiment II (bottom row). **(b)** In Experiment I, π_2 and π_3 are compared in terms of evidence $[\log Z_i(K)]$ and uninformativeness $[H_i(K)]$ for each (D, K). The arrows point from π_2 to π_3 and are color-coded by the value of K. For small values of K, the arrow lengths are too small to be visible on this scale.

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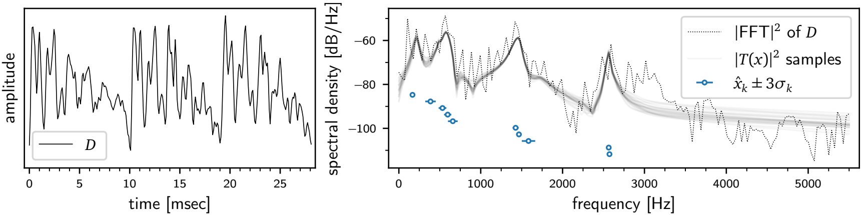

Figure: The VTR problem for the case ($D :=$ until, $K := 10$). Left panel: The data D, i.e., the quasi-periodic steady-state part consisting of 3 highly correlated pitch periods. Right panel: Inferred VTR frequency estimates $\{\hat{x}_k\}_{k=1}^{K}$ for $K := 10$ at 3 sigma. They describe the power spectral density of the vocal tract transfer function $|T(x)|^2$, represented here by 25 posterior samples and compared to the Fast Fourier Transform (FFT) of D. All $\hat{x}_k$ are well resolved and most have error bars too small to be seen on this scale.

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Conclusions:

1. π_1 can't be used for $K \geq 4$
2. π_2 dominated by π_3 in terms of **evidence** $Z_i(K)$
3. π_2 about as uninformative as π_3 in terms of the **information** $H_i(K)$
4. π_3 can push K further

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- The prior facilitates model selection problems in which the number K of resonance frequencies is unknown by enabling the use of more robust evidence-based methods, even in the presence of multiplets of arbitrary order.
    1. Solves label switching problem
    2. Solves multiplet problem

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- The prior is valid for any collection of scale variables which are intrinsically ordered.
    – Does it apply to modeling spectra directly?

References I

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