Cartpole system with motor

The dynamics of the cartpole system with motor are governed by the following set of differential equations:

$$\begin{aligned}
\dot{\vartheta} &= \omega \\
\dot{x} &= v_x \\
\dot{\omega} &= \frac{g \sin\vartheta (m_c + m_p) - \cos\vartheta (F + m_p l \omega^2 \sin\vartheta)}{\frac{4l}{3}(m_c + m_p) - l m_p \cos^2\vartheta} \\
\dot{v}_x &= \frac{F + m_p l \omega^2 \sin\vartheta - \frac{3}{8} m_p g \sin(2\vartheta)}{m_c + m_p - \frac{3}{4} m_p \cos^2\vartheta} \\
\dot{F} &= \frac{1}{\tau}(u - F)
\end{aligned} \tag{1}$$

where the variables are defined as follows:

- ϑ: pole turning angle (**state variable**) [rad]
- x: x-coordinate of the cart (**state variable**) [m]
- ω: pole angular speed with respect to relative coordinate axes with cart in the origin (**state variable**) [rad/s]
- v_x: absolute speed of the cart (**state variable**) [m/s]
- F: pushing force (**state variable**) [N]
- u: motor torque (**control variable**) [N]
- m_c: mass of the cart [kg]
- m_p: mass of the pole [kg]
- l: pole length [m]
- τ: motor moment [s]

Lagrange's equations are employed to derive the first 4 equations in (1):

$$\overline{I}_p \dot{\omega} + \dot{v}_x m_p l \cos\vartheta - m_p g l \sin\vartheta = 0 \tag{2}$$

$$(m_c + m_p)\dot{v}_x - m_p l \omega^2 \sin\vartheta + \dot{\omega} m_p l \cos\vartheta = F \tag{3}$$

with the moment of inertia $\overline{I}_p$ given by $\overline{I}_p = \frac{4}{3} m_p l^2$.

The variable correspondences in the code are as follows:

- $\vartheta =$`angle`
- $\omega =$`angle_vel`
- $v_x =$`vel`
- $m_c =$`mass_cart`
- $m_p =$`mass_pole`
- $l =$`length_pole`
- $g =$`grav_const`
- $\tau =$`motor_moment`
- $F =$`force`
- $u =$`control_variable`

Exercise 1

Theory

Define the pendulum's energy as:

$$E_p = \frac{\overline{I}_p \omega^2}{2} + m_p g l(\cos\vartheta - 1)$$

Consider the function:

$$L_1 = \frac{1}{2}(E_p^2 + m_p l \lambda v_x^2), \tag{4}$$

where $\lambda \in \mathbb{R}_{>0}$ is a positive constant hyperparameter. Prove that the L_1 time derivative is:

$$\begin{aligned}\frac{\mathrm{d}L_1}{\mathrm{d}t} &= -\dot{v}_x m_p l(E_p \omega \cos\vartheta - \lambda v_x) = \\ &= -\frac{F + m_p l \omega^2 \sin\vartheta - \frac{3}{8} m_p g \sin(2\vartheta)}{m_c + m_p - \frac{3}{4} m_p \cos^2\vartheta} m_p l(E_p \omega \cos\vartheta - \lambda v_x)\end{aligned} \tag{5}$$

and identify such function $F = F_{\text{en.based.}}(\vartheta, \omega, v_x)$ that ensures

$$\frac{\mathrm{d}L_1}{\mathrm{d}t} = -m_p l k (E_p \omega \cos\vartheta - \lambda v_x)^2,$$

where $k \in \mathbb{R}_{>0}$ is a positive constant hyperparameter.

Hint. *You will need equation* (2) *to derive* (5)*.*

Code

After determining the function $F = F_{\text{en.based.}}(\vartheta, \omega, v_x, b)$, locate the following function in the code:

```
def cartpole_energy_based_force_control_function(
    self,
    angle: float,
    angle_vel: float,
    vel: float,
) -> float:
```

and implement the function body so that it computes and returns $F_{\text{en.based.}}(\vartheta, \omega, v_x)$. The variable correspondences in the code are as follows:

- $\vartheta =$`angle`
- $\omega =$`angle_vel`
- $v_x =$`vel`
- $k =$`self.energy_gain`
- $\lambda =$`self.velocity_gain`
- $m_c =$`mass_cart`
- $m_p =$`mass_pole`
- $l =$`length_pole`
- $g =$`grav_const`

Exercise 2

Theory

In this homework F is not our control variable. However, we can construct a control law $u(\vartheta, \omega, v_x, F)$ based on $F_{\text{en.based}}(\vartheta, \omega, v_x)$. Using the backstepping control approach, the control law is defined as:

$$u(\vartheta, \omega, v_x, F) = F - b(F - F_{\text{en.based.}}(\vartheta, \omega, v_x)),$$

where $b \in \mathbb{R}_{>0}$ is a positive constant hyperparameter.

Code

Locate the following function definition in the code:

```
def cartpole_backstepping(
    self,
    old_energy_based_force: float,
    force: float,
) -> float:
```

Your task is to implement this function such that it calculates and returns the control law $u(\vartheta, \omega, v_x, F)$.

The variable correspondences in the code are as follows:

- $b =$`self.backstepping_gain`
- $F =$`force`
- $F_{\text{en.based.}}(\vartheta, \omega, v_x) =$`old_energy_based_force`