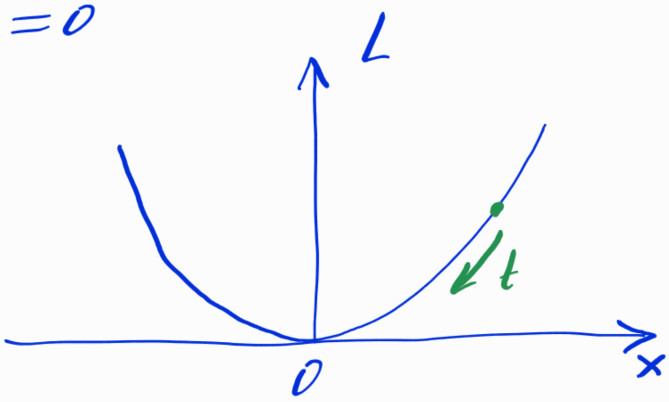
Basics of Lyapunov-based control

$\dot{x} = f(x), \; x(0) = x_0$ $\qquad x_{k+1} = f(x_k), \; k \in \mathbb{Z}_{\geq 0}$

Equilibrium state: $\quad x_e \Rightarrow f(x_e) = 0$

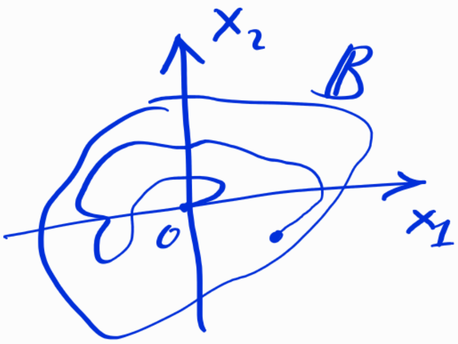
Assume $x_e = 0$



Positive-def.

Asympt. stability (local) $\quad \exists \, \mathbb{B} \ni 0$ s.t.

$x_0 \in \mathbb{B} \Rightarrow \lim_{t \to \infty} x(t) = 0$



Global: $\mathbb{B} = \mathbb{X}$

$L : \mathbb{X} \to \mathbb{R}_{\geq 0}$

(Lyapunov) decay property: $\quad \dot{L} < 0$

$L_{k+1} - L_k < 0$

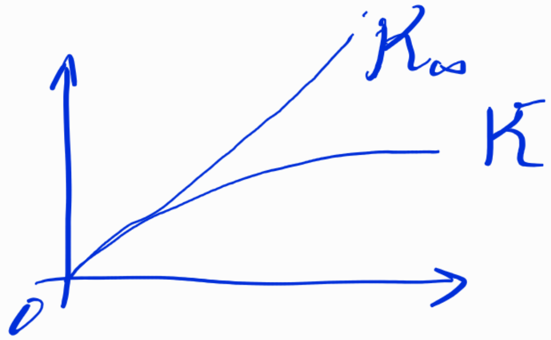
Decay rate ν : $\quad \dot{L}(x) \leq -\nu(x)$

p.-d.

\matchal K

$\mathcal{K}$, $\mathcal{K}_\infty$

↳ Functions $\mathbb{R} \to \mathbb{R}$, monoton. increasing, pos.-def., limit of the function as the arg. tends to ∞ is in turn ∞



$\exists\ K_{low}, K_{up} \in \mathcal{K}_\infty,\ \nu \in \mathcal{K}$ s.t.

$$\forall x \quad K_{low}(\|x\|) \leq L(x) \leq K_{up}(\|x\|)$$

$$\dot{L}(x) \leq -\nu(\|x\|)$$

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n$$

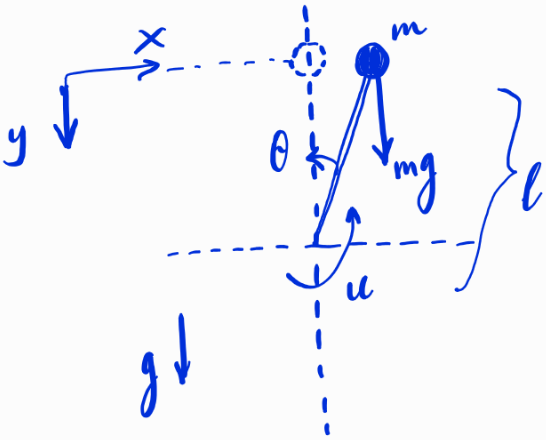
Control Lyapunov function (CLF):

$$\forall x \quad K_{low}(\|x\|) \leq L(x) \leq K_{up}(\|x\|)$$

$$\min_u \langle \nabla L, f(x,u) \rangle \leq -\nu(\|x\|)$$

Example: energy-based control of inverted pendulum

$$J = ml^2$$



$$J\ddot\theta = -mgl\sin\theta + u$$

$$\ddot\theta = -\frac{g}{l}\sin\theta + \frac{u}{ml^2}$$

$$x = \begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\dot x_1 = x_2$$
$$\dot x_2 = -\frac{g}{l}\sin x_1 + \frac{u}{ml^2}$$

Energy: $E_{tot} = E_{pot} + E_{kin}$

$$= mgl(1-\cos x_1) + \frac{m\,x_2^2 l^2}{2}$$

Want: $E_{tot} \rightarrow E^* = 0$

LF candidate: $L = \frac{1}{2}E_{tot}^2$

$$\underset{= E_{tot}\cdot\dot E_{tot}}{\dot L} = \Big(\underbrace{mgl\sin(x_1)\cdot \overset{= x_2}{\dot x_1} + ml^2 x_2 \cdot \dot x_2}_{\dot E_{tot}}\Big)\cdot E_{tot} =$$

$$= E_{tot}\big(mgl\sin(x_1)\cdot x_2 - mgl\, x_2\cdot\sin(x_1) + x_2\cdot u\big)$$

$$= E_{tot}\cdot x_2\cdot u$$

E.g. $u \leftarrow -x_2 \Rightarrow \dot{L} = -E_{tot}\, x^2$

(not a proper decay)

or $u \leftarrow -\operatorname{sign}(x_2)$