

Cartpole system

The dynamics of the cartpole system are governed by the following set of differential equations:

$$\begin{aligned}
 \dot{\vartheta} &= \omega \\
 \dot{x} &= v_x \\
 \dot{\omega} &= \frac{g \sin \vartheta (m_c + m_p) - \cos \vartheta (F + m_p l \omega^2 \sin \vartheta)}{\frac{4l}{3}(m_c + m_p) - l m_p \cos^2 \vartheta} \\
 \dot{v}_x &= \frac{F + m_p l (\omega^2 \sin \vartheta - \dot{\omega} \cos \vartheta)}{m_c + m_p}
 \end{aligned} \tag{1}$$

where the variables are defined as follows:

- ϑ : pole turning angle (**state variable**) [rad]
- x : x-coordinate of the cart (**state variable**) [m]
- ω : pole angular speed with respect to relative coordinate axes with cart in the origin (**state variable**) [rad/s]
- v_x : absolute speed of the cart (**state variable**) [m/s]
- F : pushing force (**control variable**) [N]
- m_c : mass of the cart [kg]
- m_p : mass of the pole [kg]
- l : pole length [m]

Lagrange's equations are employed to derive the above expressions (1):

$$\bar{I}_p \dot{\omega} + v_x m_p l \cos \vartheta - m_p g l \sin \vartheta = 0 \tag{2}$$

$$(m_c + m_p) \dot{v}_x - m_p l \omega^2 \sin \vartheta + \dot{\omega} m_p l \cos \vartheta = F \tag{3}$$

with the moment of inertia \bar{I}_p given by $\bar{I}_p = \frac{4}{3} m_p l^2$.

Exercise 1

Define the pendulum's energy as:

$$E_p = \frac{\bar{I}_p \omega^2}{2} + mgl(\cos \vartheta - 1)$$

Consider the function:

$$L = \frac{1}{2}(E_p^2 + ml\lambda v_x^2) \quad (4)$$

Prove that the L time derivative is:

$$\frac{dL}{dt} = -\dot{v}_x ml(E_p \omega \cos \vartheta - \lambda v_x) \quad (5)$$

Hint. *You will need equation (2) to derive (5).*

Exercise 2

By substituting:

$$\dot{v}_x = k(E_p \omega \cos \theta - \lambda v_x) \quad (6)$$

into (5), we obtain:

$$\frac{dL}{dt} = -mlk(E_p \omega \cos \theta - \lambda v_x)^2 \leq 0,$$

Find such a value of control variable F that enforces the above condition (6) for \dot{v}_x . This will constitute the energy-based control law. Note that $k \in \mathbb{R}$ is a positive constant hyperparameter.

Hint. *You will need equations (2) and (3) to enforce (6).*