Basics of backstepping

Inverted pendulum:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{J}\left(mgl \sin x_1 + u\right), \quad \underset{\text{torque}}{u - M} \ [Nm],$$

Introduce electric motor dynamics:

$$\dot{M} = \frac{1}{\tau}\left(-M + M^*\right)$$

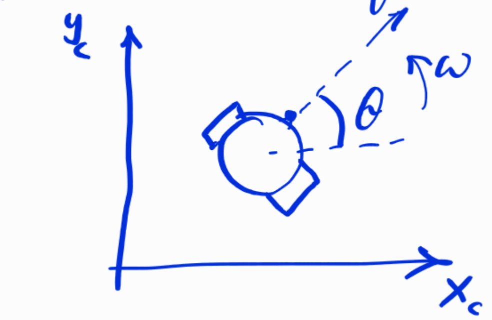
motor time const, e.g., 50 - 300 ms

„New" $u := M^*$

$x_3 := M$

$$\Rightarrow \quad \dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{J}\left(mgl \sin x_1 + x_3\right)$$
$$\dot{x}_3 = \frac{1}{\tau}\left(u - x_3\right)$$

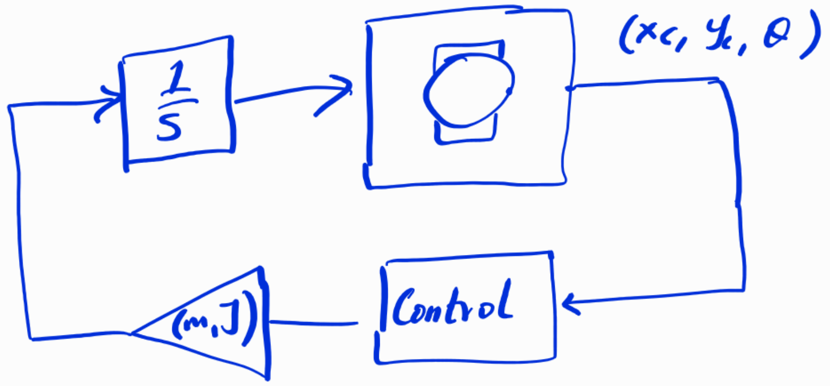
Another example: differential-drive robot



$$u = \begin{pmatrix} v \\ \omega \end{pmatrix}, \quad x = \begin{pmatrix} x_c \\ y_c \\ \theta \end{pmatrix}$$

$$\dot{x}_c = v \cos\theta$$
$$\dot{y}_c = v \sin\theta$$
$$\dot{\theta} = \omega$$

$$\dot{v} = \frac{1}{m} F$$
$$\dot{\omega} = \frac{1}{J} M$$



What should we do if we know how to control the „base" system?

We had a pair L, ρ s.t. L is an LF for the closed loop under ρ.

$$L_c := L + \frac{1}{2} \| v - \rho(x) \|^2$$

General case:

$$\begin{aligned} \dot{x} &= f(x) + g(x) v \\ \dot{v} &= u \end{aligned} \Big\} \text{ control-affine}$$

$$\dot{L}_c = \dot{L} + (v - \rho(x))^T \left(u - \frac{d}{dt}\rho(x)\right)$$

$$\boxed{\text{Lie derivative: } \quad \mathcal{L}_z F = \langle \nabla F, z \rangle}$$

$$\Rightarrow \dot{L} = \mathcal{L}_f L + \mathcal{L}_g L v + (v - \rho(x))^T (u - \nabla\rho \cdot \dot{x})$$

Remark: $\mathcal{L}_f L + \mathcal{L}_g L \rho(x) < 0$

Recall the base case:

$$\dot{x} = f(x) + g(x)v$$

$$L$$

$$\dot{L} = \mathcal{L}_f L + \mathcal{L}_g L \underbrace{v}_{\rho(x)} < 0$$

Back to our business:

$$\dot{L} = \mathcal{L}_f L + \mathcal{L}_g L v + (v - \rho(x))^T (u - \nabla\rho \cdot \dot{x})$$

$$= \mathcal{L}_f L + \mathcal{L}_g L v + (v - \rho(x))^T (u - \cancel{\mathcal{L}_f \rho} - \mathcal{L}_g \rho v)$$

First suggestion for the policy is:

$$u := \underbrace{-K(v - \rho(x))}_{①} + \underbrace{\mathcal{L}_f \rho}_{② \text{ cancels}} + \underbrace{\mathcal{L}_g \rho v}_{③ \text{ cancels}}$$

$$-K\|v - \rho(x)\|^2$$

yields decay term

$$\Rightarrow \boxed{\dot{L}_c = \mathcal{L}_f L + \mathcal{L}_g L v - K\|v - \rho(x)\|^2}$$

Can we do better?

Alternative assignment:

$$u := -K(v - \rho(x)) + L_{f+gv}\rho + \eta \quad ?$$

$$\dot{L}_c = L_f L + L_g L v - K\|v - \rho(x)\|^2 + (v - \rho(x))^T \eta$$

Suggestion: $\eta := -L_g L$

$$\Rightarrow \dot{L}_c = L_f L + L_g L \rho - K\|v - \rho(x)\|^2$$

$$< 0$$

Simplified approach:

$$u := -K(v - \rho(x))$$

$$\Rightarrow \dot{L}_c = -K\|v - \rho(x)\|^2 + \dots$$