

## Cartpole system

The dynamics of the cartpole system are governed by the following set of differential equations:

$$\begin{aligned}\dot{\vartheta} &= \omega \\ \dot{x} &= v_x \\ \dot{\omega} &= g \sin \vartheta - \cos \vartheta \frac{F + m_p l \omega^2 \sin \vartheta}{\frac{4l}{3}(m_c + m_p) - l m_p \cos^2 \vartheta} \\ \dot{v}_x &= \frac{F + m_p l (\omega^2 \sin \vartheta - \dot{\omega} \cos \vartheta)}{m_c + m_p}\end{aligned}\tag{1}$$

where the variables are defined as follows:

- $\vartheta$ : pole turning angle (**state variable**) [rad]
- $x$ : x-coordinate of the cart (**state variable**) [m]
- $\omega$ : pole angular speed with respect to relative coordinate axes with cart in the origin (**state variable**) [rad/s]
- $v_x$ : absolute speed of the cart (**state variable**) [m/s]
- $F$ : pushing force (**control variable**) [N]
- $m_c$ : mass of the cart [kg]
- $m_p$ : mass of the pole [kg]
- $l$ : pole length [m]

Lagrange's equations are employed to derive the above expressions (1):

$$\bar{I}_p \dot{\omega} + \dot{v}_x m_p l \cos \vartheta - m_p g l \sin \vartheta = 0\tag{2}$$

$$(m_c + m_p) \dot{v}_x - m_p l v_x^2 \sin \vartheta + \dot{\omega} m_p l \cos \vartheta = F\tag{3}$$

with the moment of inertia  $\bar{I}_p$  given by  $\bar{I}_p = \frac{4}{3}(m_c + m_p)l^2 + m_p l^2$ .

## Exercise 1

Define the pendulum's energy as:

$$E_p = \bar{I}_p \omega^2 + m g l (\cos \vartheta - 1)$$

Consider the Lyapunov function:

$$L = \frac{1}{2}(E_p^2 + m l \lambda v_x^2)\tag{4}$$

Prove that the Lyapunov function's time derivative is:

$$\frac{dL}{dt} = -\dot{v}_x ml(E_p \omega \cos \vartheta - \lambda v_x) \quad (5)$$

**Hint.** *You will need equation (2) to derive (5).*

## Exercise 2

By substituting:

$$\dot{v}_x = k(E_p \omega \cos \theta - \lambda v_x) \quad (6)$$

into (5), we obtain:

$$\frac{dL}{dt} = -mlk(E_p \omega \cos \theta - \lambda v_x)^2 \leq 0,$$

Find such a value of control variable  $F$  that enforces the above condition (6) for  $\dot{v}_x$ . This will constitute the energy-based control law. Note that  $k \in \mathbb{R}$  is a positive constant hyperparameter.

**Hint.** *You will need equations (2) and (3) to enforce (6).*