Cartpole system

The dynamics of the cartpole system are governed by the following set of differential equations:

$$
\begin{aligned}
\dot{\vartheta} &= \omega \\
\dot{x} &= v_x \\
\dot{\omega} &= \frac{g \sin\vartheta (m_c + m_p) - \cos\vartheta (F + m_p l \omega^2 \sin\vartheta)}{\frac{4l}{3}(m_c + m_p) - l m_p \cos^2\vartheta} \\
\dot{v}_x &= \frac{F + m_p l (\omega^2 \sin\vartheta - \dot{\omega} \cos\vartheta)}{m_c + m_p}
\end{aligned}
\tag{1}
$$

where the variables are defined as follows:

- ϑ: pole turning angle (**state variable**) [rad]
- x: x-coordinate of the cart (**state variable**) [m]
- ω: pole angular speed with respect to relative coordinate axes with cart in the origin (**state variable**) [rad/s]
- v_x: absolute speed of the cart (**state variable**) [m/s]
- F: pushing force (**control variable**) [N]
- m_c: mass of the cart [kg]
- m_p: mass of the pole [kg]
- l: pole length [m]

Lagrange's equations are employed to derive the above expressions (1):

$$\overline{I}_p \dot{\omega} + \dot{v}_x m_p l \cos\vartheta - m_p g l \sin\vartheta = 0 \tag{2}$$

$$(m_c + m_p)\dot{v}_x - m_p l v_x^2 \sin\vartheta + \dot{\omega} m_p l \cos\vartheta = F \tag{3}$$

with the moment of inertia $\overline{I}_p$ given by $\overline{I}_p = \frac{4}{3} m_p l^2$.

Exercise 1

Define the pendulum's energy as:

$$E_p = \overline{I}_p\omega^2 + mgl(\cos\vartheta - 1)$$

Consider the function:

$$L = \frac{1}{2}(E_p^2 + ml\lambda v_x^2) \tag{4}$$

Prove that the L time derivative is:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\dot{v}_x ml(E_p\omega\cos\vartheta - \lambda v_x) \tag{5}$$

Hint. *You will need equation* (2) *to derive* (5)*.*

Exercise 2

By substituting:

$$\dot{v}_x = k(E_p\omega\cos\theta - \lambda v_x) \tag{6}$$

into (5), we obtain:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -mlk(E_p\omega\cos\theta - \lambda v_x)^2 \leq 0,$$

Find such a value of control variable F that enforces the above condition (6) for $\dot{v}_x$. This will constitute the energy-based control law. Note that $k \in \mathbb{R}$ is a positive constant hyperparameter.

Hint. *You will need equations* (2) *and* (3) *to enforce* (6)*.*