Cartpole system with friction

The dynamics of the cartpole system with friction are governed by the following set of differential equations:

$$
\begin{aligned}
\dot{\vartheta} &= \omega \\
\dot{x} &= v_x \\
\dot{\omega} &= \frac{g\sin\vartheta(m_c+m_p) - \cos\vartheta(F - bv_x + m_p l\omega^2\sin\vartheta)}{\frac{4l}{3}(m_c+m_p) - lm_p\cos^2\vartheta} \\
\dot{v}_x &= \frac{F - bv_x + m_p l\omega^2\sin\vartheta - \frac{3}{8}m_p g\sin(2\vartheta)}{m_c+m_p-\frac{3}{4}m_p\cos^2\vartheta}
\end{aligned}
\tag{1}
$$

where the variables are defined as follows:

- ϑ: pole turning angle (**state variable**) [rad]
- x: x-coordinate of the cart (**state variable**) [m]
- ω: pole angular speed with respect to relative coordinate axes with cart in the origin (**state variable**) [rad/s]
- v_x: absolute speed of the cart (**state variable**) [m/s]
- F: pushing force (**control variable**) [N]
- m_c: mass of the cart [kg]
- m_p: mass of the pole [kg]
- l: pole length [m]
- b: friction coefficient [kg/s]

Lagrange's equations are employed to derive the above expressions (6):

$$\overline{I}_p\dot{\omega} + \dot{v}_x m_p l\cos\vartheta - m_p gl\sin\vartheta = 0 \tag{2}$$

$$(m_c+m_p)\dot{v}_x - m_p l\omega^2\sin\vartheta + \dot{\omega}m_p l\cos\vartheta + bv_x = F \tag{3}$$

with the moment of inertia $\overline{I}_p$ given by $\overline{I}_p = \frac{4}{3}m_p l^2$.

Exercise 1

Theory

Define the pendulum's energy as:

$$E_p = \frac{\overline{I}_p \omega^2}{2} + m_p g l(\cos\vartheta - 1)$$

Consider the function:

$$L_1 = \frac{1}{2}(E_p^2 + m_p l \lambda v_x^2), \tag{4}$$

where $\lambda \in \mathbb{R}_{>0}$ is a positive constant hyperparameter. Prove that the L_1 time derivative is:

$$\begin{aligned}\frac{\mathrm{d}L_1}{\mathrm{d}t} &= -\dot{v}_x m_p l(E_p \omega \cos\vartheta - \lambda v_x) = \\ &= -\frac{F - b v_x + m_p l \omega^2 \sin\vartheta - \frac{3}{8} m_p g \sin(2\vartheta)}{m_c + m_p - \frac{3}{4} m_p \cos^2\vartheta} m_p l(E_p \omega \cos\vartheta - \lambda v_x)\end{aligned} \tag{5}$$

and identify such a control function $F = F_{\text{fr.comp.}}(\vartheta, \omega, v_x, b)$ that ensures

$$\frac{\mathrm{d}L_1}{\mathrm{d}t} = -m_p l k(E_p \omega \cos\vartheta - \lambda v_x)^2,$$

where $k \in \mathbb{R}_{>0}$ is a positive constant hyperparameter.

Hint. *You will need equation* (2) *to derive* (5).

Code

After determining the control function $F = F_{\text{fr.comp.}}(\vartheta, \omega, v_x, b)$, locate the following function in the code:

```
def cartpole_energy_based_control_function_friction_compensation(
    self,
    angle: float,
    angle_vel: float,
    vel: float,
    friction_coeff: float,
) -> float:
```

and implement the function body so that it computes and returns $F_{\text{fr.comp.}}(\vartheta, \omega, v_x, b)$. The variable correspondences in the code are as follows:

- `angle`$= \vartheta$
- `angle_vel`$= \omega$
- `vel`$= v_x$
- `friction_coeff`$= b$
- $k =$`self.energy_gain`
- $\lambda =$`self.velocity_gain`

Exercise 2

Theory

Determine the function $B(\vartheta, \omega, v_x)$ such that the time derivative $\frac{\mathrm{d}L_2}{\mathrm{d}t}$ of

$$L_2 = \frac{1}{2}\left(E_p^2 + m_p l \lambda v_x^2 + \frac{(\hat{b} - b)^2}{\alpha}\right), \text{ where } \hat{b} = \hat{b}(t) = \hat{b}(0) + \alpha \int_0^t B(\vartheta, \omega, v_x) d\tau$$

is equal to

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = -m_p l k (E_p \omega \cos\vartheta - \lambda v_x)^2.$$

This should be valid under the control law given by

$$F = F_{\text{fr.comp.}}(\vartheta, \omega, v_x, \hat{b})$$

In the above equations, $\hat{b}(0)$ is known and equals zero, and $\alpha \in \mathbb{R}_{>0}$ is a positive constant hyperparameter.

Hint. *Refer to the appendix for a similar deduction applied to the inverted pendulum system. The appendix should be reviewed with attention for guidance.*

Code

After deriving the function $B(\vartheta, \omega, v_x)$, locate the following function in the code:

```
def euler_update_friction_coeff_estimate(
    self,
    angle: float,
    angle_vel: float,
    vel: float,
) -> None:
```

and complete its definition to update $\hat{b}$ using the Euler method:

$$\hat{b} := \hat{b} + \alpha B(\vartheta, \omega, v_x)\Delta t$$

In the code, variable notation corresponds to:

- $\alpha =$`self.friction_coeff_est_learning_rate`
- $\hat{b} =$`self.friction_coeff_est`
- $\Delta t =$`self.sampling_time`
- $k =$`self.energy_gain`
- $\lambda =$`self.velocity_gain`

Appendix

Derivation of the Adaptive Controller for Inverted Pendulum System with Friction

Consider the inverted pendulum system with friction described by the state dynamics:

$$\begin{aligned}\dot{\vartheta} &= \omega \\ \dot{\omega} &= \frac{g}{l}\sin\vartheta + \frac{M}{ml^2} - b\omega^2 \mathrm{sgn}(\omega)\end{aligned} \tag{6}$$

where

- ϑ is the pendulum angle (**state variable**) [rad]
- ω is the pendulum angular velocity (**state variable**) [rad/s]
- M is the pendulum torque (**control variable**) [kg$\times$m^2/s^2]
- m is the pendlum mass [kg]
- l is the pendulum length [m]
- g is the gravity constant [m/s^2]
- b is the friction coefficient [m^{-2}]

We define the Lyapunov function L as:

$$L = \frac{1}{2}\left(E^2 + \frac{(\hat{b} - b)^2}{\alpha}\right)$$

where

- $E = \frac{ml^2\omega^2}{2} + mgl(\cos\vartheta - 1)$ is the energy of the system,
- $\alpha \in \mathbb{R}_{>0}$ is a positive constant hyperparameter, interpreted as a learning rate,
- $\hat{b} = \hat{b}(t)$ is an estimate of the real friction coefficient b. The function $\hat{b}$ is derived below, aiming **to find a control law and estimate for $\hat{b}$ that are independent from the actual value of b, while ensuring that $\dot{L} \leq 0$.**

The derivative of L is given by:

$$\dot{L} = E\dot{E} + \frac{\hat{b} - b}{\alpha}\dot{\hat{b}}$$

Examining $\dot{E}$ more closely:

$$\dot{E} = ml^2\omega\dot{\omega} - mgl\omega\sin\vartheta = \omega(mgl\sin\vartheta + M - bml^2\omega^2\mathrm{sgn}(\omega) - mgl\sin\vartheta) = \\ = M\omega - bml^2|\omega|^3$$

Hence,

$$\dot{L} = E(M\omega - bml^2|\omega|^3) + \frac{\hat{b} - b}{\alpha}\dot{\hat{b}} = \left\{\text{Let us add and subtract} \pm E\hat{b}ml^2|\omega|^3\right\} = \\ = E(\omega M - \hat{b}ml^2|\omega|^3) + \left(\hat{b} - b\right)\left(Eml^2|\omega|^3 + \frac{\dot{\hat{b}}}{\alpha}\right)$$

By setting:

- $\dot{\hat{b}} = -\alpha Eml^2|\omega|^3$, which implies $\hat{b}(t) = \hat{b}(0) + \alpha\int_0^t -Eml^2|\omega|^3\,\mathrm{d}\tau$
- the control law $M = -k\mathrm{sgn}(\omega E) + |\omega|\omega\hat{b}ml^2$ for some positive constant $k \in \mathbb{R}_{>0}$

we ensure that:

$$\dot{L} = -k|wE| \leq 0$$

Important Remark

It is straightforward to verify that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E^2}{2}\right) = -k|wE| \text{ for } M = -k\mathrm{sgn}(\omega E) + |\omega|\omega bml^2 \tag{7}$$

but to practically apply this control law, one must know the true value of the friction coefficient b. In contrast, we previously derived an adaptive control law of similar form,

$$M = -k\mathrm{sgn}(\omega E) + |\omega|\omega\hat{b}ml^2. \tag{8}$$

The only difference between the control laws (7) and (8) is the substitution of b with its estimate $\hat{b}(t) = \hat{b}(0) + \alpha\int_0^t -Eml^2|\omega|^3\,\mathrm{d}\tau$. **This estimate can be computed in practice using the Euler scheme, without requiring knowledge of the actual value of the friction coefficient b.**