MAE/CSE 557
M.E. Mueller
Fall 2025

MAE/CSE 557: Simulation and Modeling of Fluid Flows

Mini-Project One
Due: October 1, 2025

Directions: The compute portion of all assignments must be completed in a compiled language (C, C++, or Fortran) or another performant language (Julia). However, scripting languages may be used to manage a program as long the compute components are in a performant language, and any package including MATLAB may be used for making plots. Obviously, good programming practice (comments, etc.) are highly encouraged, and your source code must be submitted electronically with your write-up along with a readme with directions for compiling the code and running the program(s), preferably in a Linux environment (e.g., Nobel or Adroit). Your write-up should be typed and address all of the points for each problem and sub-problem, and you will need plots to support your discussion. The mini-project should be submitted by email to muellerm@princeton.edu *and* israel.bonilla@princeton.edu *by 9:34am on the due date as a single tarball or zip archive. The filename should have the format "lastname_firstname.zip" (or .tar, .tar.gz, .tgz, etc.) and should contain a report in PDF format entitled "report.pdf", the readme file entitled "readme.txt", and the source code.*

1. Consider the one-dimensional linear advection-diffusion equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

The solution exists on the domain $x \in [0,2\pi)$ with periodic boundary conditions and the initial condition $u = \sin(x)\exp[-(x-\pi)^2]$. The domain should be discretized with a uniform Δx.

a. Propose two schemes for solving the above equation. The two schemes must be distinct, utilizing different schemes for temporal discretization (one explicit and one implicit) and different schemes for spatial discretization of the first derivative operator, that is, advection (one upwinded and one centered). For both schemes, a second-order centered scheme should be used for the second derivative operator, that is, diffusion (why?). Write the resulting semi-discrete governing equation in matrix form.

b. Determine the theoretical stability limits for each of the two approaches for $(c,\nu) = (1,0.01)$ and $(c,\nu) = (1,1)$.

c. For $\nu = 0$ (why?), determine the primary and secondary conservation properties of your approaches theoretically.

i. For approaches that do not satisfy primary and/or secondary conservation, how would this affect the quality of your solution at long time? How could the long-time degradation of the solution be diminished?

ii. For approaches that do satisfy primary and/or secondary conservation, how would this affect the quality of your solution at long time (i.e., is there a trade-off)? How could the long-time degradation of the solution be diminished?

iii. What properties of spatial discretization and/or temporal scheme provide secondary conservative approaches?

2. Consider now Burgers' equation in one dimension:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Consider the same domain, boundary, and initial conditions as the linear equation considered above. The domain should be discretized with a uniform Δx.

a. Consider the same two numerical approaches you chose for the linear equation. How does the nonlinear term affect anticipated convergence and stability characteristics?

b. Implement your two numerical approaches to solve this nonlinear equation for $(c, \nu) = (1,1)$ and for $(c, \nu) = (1,0.01)$, and verify your implementation for convergence, both temporal and spatial. How do the stability characteristics differ from your theoretical predictions?

c. For $\nu = 0$, determine the primary and secondary conservation properties of your approaches theoretically and verify this numerically.

i. Are the primary and secondary conservation properties of the two approaches the same as for the linear problem? Explain the differences.

ii. Is there a different way you could solve your equations (i.e., change the equations) to ensure primary and/or secondary conservation irrespective of grid spacing and spatial discretization (assuming a conservative temporal scheme)? Note that primary and secondary conservation may require different forms of the equation.

d. For $(c, \nu) = (1,0.01)$, qualitatively discuss the nature of your solution at long time ($t > 10\pi$). What feature is causing numerical challenges?

i. What types of spatial discretization introduce spurious oscillations into your solution due to this distinct feature?

ii. What types of spatial discretization can eliminate these spurious oscillations? Note that you may need to reduce your time step in order to eliminate the oscillations.

iii. What numerical properties are sacrificed when using schemes that eliminate the spurious oscillations?

e. Discuss the relative advantages and disadvantages of explicit schemes versus implicit schemes given your experience with implementation and subsequent analysis.