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MAE557: Simulation and Modeling of Fluid Flows
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Mini-Project 1 Report

Disclaimer: I collaborated with Philip Satterthwaite and Michael Schroeder. I received assistance from Trevor Fush and Israel Bonilla. I used Google AI Studio (Gemini 2.5 Pro) for debugging.

1. Problem 1: Linear Advection-Diffusion Equation

The one-dimensional linear advection-diffusion equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

The solution exists on the domain $x \in [0, 2\pi)$ with periodic boundary conditions. The domain is discretized using N points, such that $x_j = j\Delta x$ for $j = 0, 1, \ldots, N-1$, with a uniform grid spacing of $\Delta x = 2\pi/N$. The initial condition is given as $u = \sin(x)\exp[-(x-\pi)^2]$.

a. Two Proposed Numerical Schemes. Two distinct numerical schemes are proposed to solve the one-dimensional linear advection-diffusion equation. The schemes are chosen to be one explicit and one implicit, utilizing different spatial discretizations for the advection term. For both schemes, a second-order centered difference is used for the diffusive (second derivative) term. A centered scheme is chosen for the diffusion term because it is second-order accurate and correctly reflects the physically isotropic nature of diffusion, whereas a one-sided scheme would introduce artificial directionality. This choice is motivated by the need to avoid numerical instabilities. Non-centered schemes for second-order derivatives can introduce imaginary components into the eigenvalues of the discretization matrix, leading to non-physical, spurious oscillations in the solution.

1) **First-Order Upwind (Advection) + Explicit Forward Euler (Time).** This explicit scheme assumes the wave speed $c > 0$, so the upwind direction is in the negative x-direction.

- **Temporal Discretization (Forward Euler):**

$$\frac{\partial u}{\partial t} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

- **Advection Discretization (First-Order Upwind):**

$$c\frac{\partial u}{\partial x} \approx c\frac{u_j^n - u_{j-1}^n}{\Delta x}$$

- **Diffusion Discretization (Second-Order Centered):**

$$\nu\frac{\partial^2 u}{\partial x^2} \approx \nu\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

Combining these approximations yields the fully discretized equation:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{\Delta x} = \nu\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

The equation is rearranged to solve for the solution at the next timestep, u_j^{n+1}:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + \frac{\nu\Delta t}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

Let the Courant number be $\mathrm{CFL} = c\Delta t/\Delta x$ and the diffusion number (or Fourier number) be $\mathrm{DN} = \nu\Delta t/(\Delta x)^2$.

$$u_j^{n+1} = \mathrm{DN}\, u_{j+1}^n + (1 - \mathrm{CFL} - 2\mathrm{DN})u_j^n + (\mathrm{CFL} + \mathrm{DN})u_{j-1}^n$$

The semi-discrete matrix form is $d\mathbf{u}/dt = \mathbb{A}\mathbf{u}$. The matrix $\mathbb{A}$ for this scheme is a circulant matrix where the non-zero entries on the row corresponding to grid point j are:

- $\mathbb{A}_{j,j-1} = \frac{c}{\Delta x} + \frac{\nu}{(\Delta x)^2}$
- $\mathbb{A}_{j,j} = -\frac{c}{\Delta x} - \frac{2\nu}{(\Delta x)^2}$
- $\mathbb{A}_{j,j+1} = \frac{\nu}{(\Delta x)^2}$

The indices are taken modulo N to enforce the periodic boundary conditions. In terms of the cell Péclet number $\mathrm{Pe} = c\,\Delta x/\nu$,

$$\begin{aligned}
\frac{\partial u_j}{\partial t} &= -c\frac{u_j - u_{j-1}}{\Delta x} + \nu\frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} \\
\frac{\partial \mathbf{u}}{\partial t} &= \left(-\frac{c}{\Delta x}\mathbb{T}(0,1,-1) + \frac{\nu}{(\Delta x)^2}\mathbb{T}(1,-2,1)\right)\mathbf{u} \\
\frac{\partial \mathbf{u}}{\partial t} &= \frac{\nu}{(\Delta x)^2}\mathbb{T}\left(1,-2-\mathrm{Pe},1+\mathrm{Pe}\right)\mathbf{u}
\end{aligned}$$

and $\mathbb{T}$ is a tridiagonal matrix. Given the periodic boundary conditions, $\mathbb{T}$ is modified to be a circulant matrix with non-zero elements (ones) in the upper-right and lower-left corners to enforce periodicity.

2) **Second-Order Centered (Advection) + Implicit Backward Euler (Time)** This scheme is implicit in nature, as the spatial derivatives are evaluated at the future time level, $n+1$.

- **Temporal Discretization (Backward Euler):**
$$\frac{\partial u}{\partial t} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$
- **Advection Discretization (Second-Order Centered):**
$$c\frac{\partial u}{\partial x} \approx c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$
- **Diffusion Discretization (Second-Order Centered):**
$$\nu\frac{\partial^2 u}{\partial x^2} \approx \nu\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}$$

The combination of these terms results in the following discrete equation:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = \nu\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}$$

Rearranging the equation to group terms at time level $n+1$ on the left-hand side yields a linear system for the unknown solution vector $\mathbf{u}^{n+1}$:

$$-\left(\frac{c\Delta t}{2\Delta x} + \frac{\nu\Delta t}{(\Delta x)^2}\right)u_{j-1}^{n+1} + \left(1 + \frac{2\nu\Delta t}{(\Delta x)^2}\right)u_j^{n+1} + \left(\frac{c\Delta t}{2\Delta x} - \frac{\nu\Delta t}{(\Delta x)^2}\right)u_{j+1}^{n+1} = u_j^n$$

where again substitutions for the Courant number $\mathrm{CFL} = c\Delta t/\Delta x$ and the diffusion number (or Fourier number) $\mathrm{DN} = \nu\Delta t/(\Delta x)^2$ may be made. The semi-discrete form is $d\mathbf{u}/dt = \mathbb{B}\mathbf{u}$, which after temporal discretization leads to the linear system $(I - \Delta t\mathbb{B})\mathbf{u}^{n+1} = \mathbf{u}^n$. The matrix $\mathbb{B}$ has non-zero entries:

- $\mathbb{B}_{j,j-1} = \frac{c}{2\Delta x} + \frac{\nu}{(\Delta x)^2}$
- $\mathbb{B}_{j,j} = -\frac{2\nu}{(\Delta x)^2}$
- $\mathbb{B}_{j,j+1} = -\frac{c}{2\Delta x} + \frac{\nu}{(\Delta x)^2}$

In terms of the cell Péclet number $\mathrm{Pe} = c\,\Delta x/\nu$ and the tridiagonal matrix $\mathbb{T}$,

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\nu}{(\Delta x)^2}\mathbb{T}\left(1 - \frac{\mathrm{Pe}}{2}, -2, 1 + \frac{\mathrm{Pe}}{2}\right)\mathbf{u}$$

b. Theoretical Stability Limits. A von Neumann stability analysis is employed to determine the theoretical stability limits. For this analysis, the trial solution $u_j^n = G^n \exp(ikj\Delta x)$ is substituted into the discretized equations, where k is the wavenumber and G is the amplification factor. The scheme is stable if the magnitude of the amplification factor $|G|$ is less than or equal to one for all wavenumbers.

Upwind Explicit. Substitution of the trial solution into the discrete equation for yields an expression for the amplification factor G_{UE}:

$$G_{\text{UE}} = 1 - \text{CFL}(1 - e^{-i\theta}) + \text{DN}(e^{i\theta} - 2 + e^{-i\theta})$$

where $\theta = k\Delta x$, CFL is the Courant number, and DN is the diffusion number (Fourier number). Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin\theta$, this becomes:

$$G_{\text{UE}} = [1 - \text{CFL}(1 - \cos\theta) - 2\text{DN}(1 - \cos\theta)] - i[\text{CFL}\sin\theta]$$

For stability, $|G_{\text{UE}}|^2 \leq 1$. This condition is met if $\text{CFL} + 2\text{DN} \leq 1$. In terms of the physical parameters, the stability condition is:

$$\frac{c\Delta t}{\Delta x} + \frac{2\nu\Delta t}{(\Delta x)^2} \leq 1 \quad \rightarrow \quad \Delta t \leq \frac{(\Delta x)^2}{c\Delta x + 2\nu}$$

In terms of the Péclet number, this can be written as:

$$1 = [1 - \text{Pe}\cdot\text{DN} + \text{Pe}\cdot\text{DN}\cos(k\Delta x) - 2\text{DN}(1 - \cos(k\Delta x))]^2 + (\text{Pe}\cdot\text{DN}\sin(k\Delta x))^2$$

- For $(c, \nu) = (1.0, 0.01)$: The condition is $\Delta t \leq (\Delta x)^2/(\Delta x + 0.02)$. For small Δx, the diffusion term dominates the constraint.
- For $(c, \nu) = (1.0, 1.0)$: The condition is $\Delta t \leq (\Delta x)^2/(\Delta x + 2)$. The stability is heavily restricted by the diffusion term, requiring Δt to be proportional to $(\Delta x)^2$.

Centered Implicit. Substitution of the trial solution gives the amplification factor G_{CI} for the implicit scheme:

$$G_{\text{CI}}\left[1 + \frac{c\Delta t}{2\Delta x}(e^{i\theta} - e^{-i\theta}) - \frac{\nu\Delta t}{(\Delta x)^2}(e^{i\theta} - 2 + e^{-i\theta})\right] = 1$$

$$G_{\text{CI}} = \frac{1}{[1 + 2\text{DN}(1 - \cos\theta)] + i[\text{CFL}\sin\theta]}$$

The magnitude squared of the amplification factor is:

$$|G_{\text{CI}}|^2 = \frac{1}{[1 + 2\text{DN}(1 - \cos\theta)]^2 + [\text{CFL}\sin\theta]^2}$$

In terms of the Péclet number, this can be written as:

$$1 = (1 - \text{Pe}\cdot\text{DN}\sin(k\Delta x))^2 - [2\text{DN}\,(1 - \cos(k\Delta x))]^2$$

Since the diffusion number $\text{DN} \geq 0$ and the term $(1 - \cos\theta) \geq 0$, the first term in the denominator is always greater than or equal to one. The second term is always non-negative. Therefore, the denominator is always greater than or equal to one. This leads to the conclusion that $|G_{\text{CI}}|^2 \leq 1$ for all values of $\Delta t, \Delta x, c,$ and ν. The scheme is therefore **unconditionally stable** for both sets of parameters.

i. For $(c, \nu) = (1.0, 0.01)$, the flow is convection-dominated. The stability analysis, depicted in Figure 1 (left), shows that with a low diffusion coefficient, the stable grid spacing is very low as a function of the Péclet number and indicates that the stability of the explicit scheme is highly restrictive. A stable solution requires a very fine grid spacing (small Δx), which in turn, due to the Courant-Friedrichs-Lewy (CFL) and diffusion number constraints, necessitates a very small timestep (Δt) to prevent numerical divergence. With a small spatial resolution, smaller time steps are also needed to remain stable.

ii. For $(c, \nu) = (1.0, 1.0)$, diffusive and convective effects are of comparable magnitude. The stability region, shown in Figure 1 (right), is considerably larger. This allows for the use of a coarser spatial grid and consequently larger time steps while maintaining numerical stability, as the strong diffusion has a stabilizing effect on the numerical scheme. This means that the solution will also remain stable for larger time steps as a result of the larger amount of diffusion.

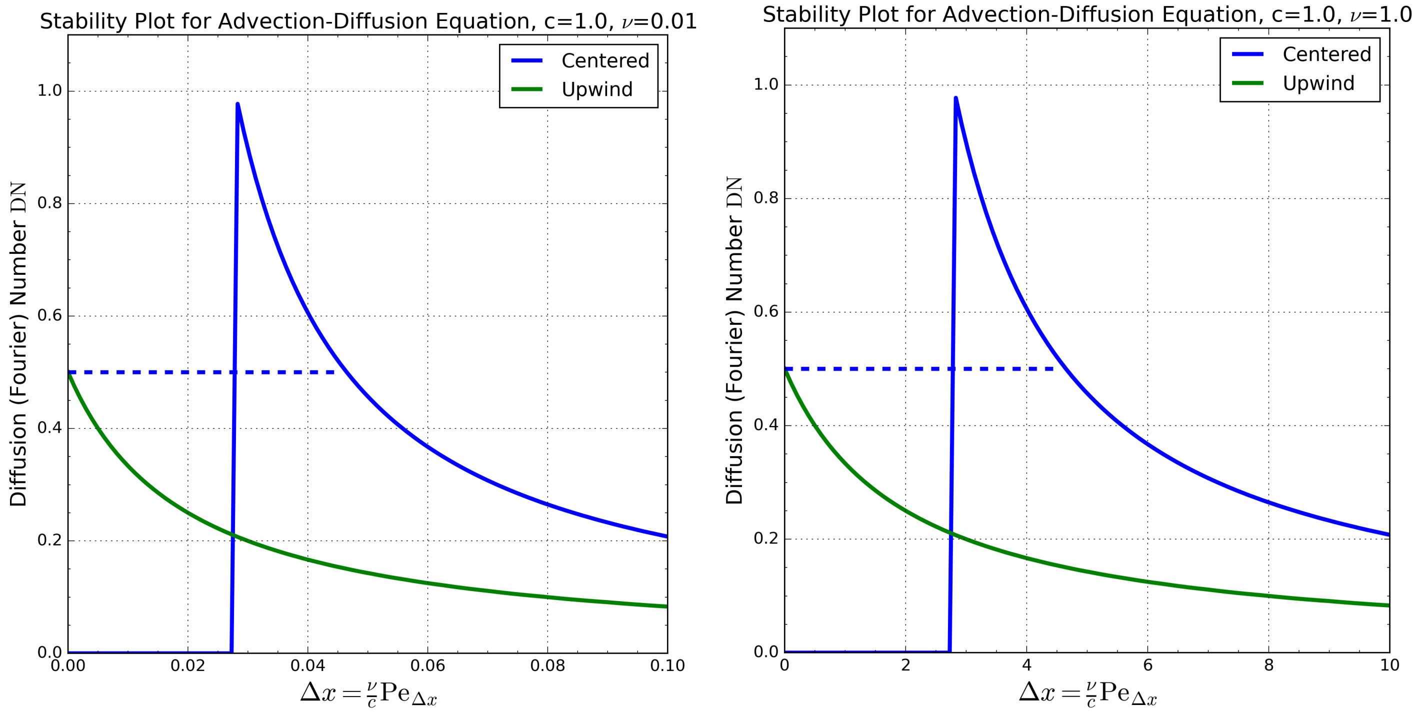

FIGURE 1. Stability plot for centered and upwind schemes for $(c, \nu) = (1.0, 0.01)$ and $(c, \nu) = (1.0, 1.0)$. Note the difference in scale of the x-axis.

c. Conservation Properties (for $\nu = 0$). To analyze the conservation properties of the advection schemes, the diffusion term is neglected ($\nu = 0$), as physical diffusion is inherently a non-conservative process. The analysis focuses on the pure linear advection equation:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

Primary conservation requires conservation of the integral of the solution, $\int u\,dx$, which corresponds to the discrete sum $\sum_j u_j$. Thus, $\sum \partial u_j/\partial t$ must be zero (or only dependent on boundary terms). Secondary conservation requires conservation of the integral of the solution squared, $\int u^2\,dx$, which corresponds to the discrete sum $\sum_j u_j^2$. Thus, $\sum u_j\,\partial u_j/\partial t$ must be zero.

For the upwind scheme, considering a few discretized terms:

$$\begin{aligned}
\frac{\partial u_{j-1}}{\partial t} &= -c\frac{u_{j-1} - u_{j-2}}{\Delta x} \\
\frac{\partial u_j}{\partial t} &= -c\frac{u_j - u_{j-1}}{\Delta x} \\
\frac{\partial u_{j+1}}{\partial t} &= -c\frac{u_{j+1} - u_j}{\Delta x} \\
\rightarrow \sum_{j=0}^{n} \frac{\partial u_j}{\partial t} &= -c\sum_{j=0}^{n} \frac{u_j - u_{j-1}}{\Delta x} = \text{B.T.}
\end{aligned}$$

The terms in the middle of the domain will cancel out (such as the first u_{j-1} term in the $\partial u_j/\partial t$ equation and the second term in the $\partial u_{j-1}/\partial t$ equation that are equal and opposite), leaving only boundary terms on the edges of the simulation domain. For periodic boundary conditions, this is a telescoping sum that evaluates to zero. Hence, the upwind scheme **satisfies primary conservation**. However, for secondary conservation with the upwind scheme:

$$
\begin{aligned}
u_{j-1}\frac{\partial u_{j-1}}{\partial t} &= -cu_{j-1}\frac{u_{j-1}-u_{j-2}}{\Delta x} \\
u_j\frac{\partial u_j}{\partial t} &= -cu_j\frac{u_j-u_{j-1}}{\Delta x} \\
u_{j+1}\frac{\partial u_{j+1}}{\partial t} &= -cu_{j+1}\frac{u_{j+1}-u_j}{\Delta x} \\
\rightarrow \sum_{j=0}^{n} u_j\frac{\partial u_j}{\partial t} &= -c\sum_{j=0}^{n} u_j\frac{u_j-u_{j-1}}{\Delta x} \neq \text{B.T.}
\end{aligned}
$$

The summation does not result in a telescoping series and is not guaranteed to be zero. The scheme possesses inherent numerical dissipation, causing the total "energy" $\sum u_j^2$ to decay. Therefore, it **does not satisfy secondary conservation**.

For the centered scheme:

$$
\begin{aligned}
\frac{\partial u_{j-1}}{\partial t} &= -c\frac{u_j-u_{j-2}}{2\Delta x} \\
\frac{\partial u_j}{\partial t} &= -c\frac{u_{j+1}-u_{j-1}}{2\Delta x} \\
\frac{\partial u_{j+1}}{\partial t} &= -c\frac{u_{j+2}-u_j}{2\Delta x} \\
\rightarrow \sum_{j=0}^{n} \frac{\partial u_j}{\partial t} &= -c\sum_{j=0}^{n} \frac{u_{j+1}-u_{j-1}}{2\Delta x} = \text{B.T.}
\end{aligned}
$$

This sum is also a telescoping sum that is zero for periodic boundaries. Thus, the centered scheme **satisfies primary conservation**. And analyzing secondary conservation:

$$
\begin{aligned}
u_{j-1}\frac{\partial u_{j-1}}{\partial t} &= -cu_{j-1}\frac{u_j-u_{j-2}}{2\Delta x} \\
u_j\frac{\partial u_j}{\partial t} &= -cu_j\frac{u_{j+1}-u_{j-1}}{2\Delta x} \\
u_{j+1}\frac{\partial u_{j+1}}{\partial t} &= -cu_{j+1}\frac{u_{j+2}-u_j}{2\Delta x} \\
\rightarrow \sum_{j=0}^{n} u_j\frac{\partial u_j}{\partial t} &= -c\sum_{j=0}^{n} u_j\frac{u_{j+1}-u_{j-1}}{2\Delta x} = \text{B.T.}
\end{aligned}
$$

Both sums telescope due to the symmetry of the stencil. This sum can be shown to be zero for periodic boundary conditions. The spatial operator matrix for this scheme is skew-symmetric, which is the condition for secondary conservation. Thus, the scheme **satisfies secondary conservation**.

i. Approaches that do not satisfy conservation. Schemes lacking primary conservation fail to conserve the total quantity $\sum u_j$ over time, leading to numerical dissipation where the solution amplitude decays artificially because the flux between cells is not being conserved at each timestep. Schemes lacking secondary conservation fail to conserve energy $(\sum u_j^2)$ within the simulation volume which is nonphysical, and can manifest as incorrect wave speeds or amplitudes. The upwind scheme does not satisfy secondary conservation, which indicates the presence of numerical dissipation. Over long integration times, this dissipation will cause the solution amplitude to decay and will smear sharp gradients, leading to a loss of accuracy. This degradation can be diminished by using a higher-order accurate upwind scheme, which would reduce the magnitude of the dissipative error, or by refining the grid spacing Δx (or by using a method that does possess the desired conservation).

ii. Approaches that do satisfy conservation. Schemes that satisfy conservation properties preserve the integrated quantities more accurately over long integration times. However, these schemes are often more susceptible to dispersion errors, where different wave components travel at incorrect speeds, leading to oscillations. The centered difference scheme satisfies both primary and secondary conservation, meaning the discrete mass and energy of the system are preserved over time. The trade-off is that this scheme is purely dispersive and lacks any numerical dissipation. For solutions containing sharp gradients, this can lead to the formation of spurious, non-physical oscillations that can contaminate the solution. The long-time degradation manifests as a corruption by high-frequency numerical noise. The long-term degradation due to these errors can be diminished by adding a small amount of artificial diffusion to the scheme, applying a numerical filter to remove the high-frequency oscillations, refining the mesh, or employing higher-order spatial schemes (or compact schemes) and temporal discretization schemes.

iii. Properties for Secondary Conservation. A scheme satisfies secondary conservation when the scheme is symmetrical about the point the derivative is being calculated (the spatial discretization stencil). The symmetry allows inner spatial derivative terms to cancel which conserves the information being passed between spatial locations. For a semi-discretized equation of the form $\frac{d\mathbf{u}}{dt} = \mathbb{A}\mathbf{u}$, secondary conservation is achieved if the discretization matrix $\mathbb{A}$ is skew-symmetric ($\mathbb{A} = -\mathbb{A}^T$). This property ensures that the energy, defined by the L_2-norm of the solution, is conserved (the rate of change of the squared norm of the solution is zero). A temporal scheme that preserves this property, such as the Crank-Nicolson method, is also required for the fully discrete scheme to be conservative.

2. Problem 2: Burgers' Equation

Consideration is now given to the one-dimensional Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

a. Effect of the Nonlinear Term. The nonlinear advection term $u\,\partial u/\partial x$ introduces significant challenges.

1) **Stability:** For explicit schemes, the stability now depends on the solution itself, as the advection speed is no longer constant. The stability constraint becomes dependent on the maximum velocity, $|u|_{\max}$, which can change in time.
2) **Convergence:** For implicit schemes, the discretization results in a system of nonlinear algebraic equations at each timestep. Solving this system requires an iterative method, such as a Newton-Raphson iteration, which increases computational cost and may fail to converge if the timestep is too large.
3) **Physics:** The introduction of the nonlinear advection term fundamentally alters the equation's behavior, leading to the formation of shock waves from smooth initial conditions. The term causes wave steepening, where parts of the wave with higher values of u travel faster. This can lead to the formation of very sharp gradients or shocks, which are difficult to resolve numerically.

With the addition of a nonlinear term, implicit time integration schemes are anticipated to retain their advantage of larger stability regions. For spatial discretization, the upwind scheme, due to its inherent numerical dissipation, is expected to be more robust in capturing the steep gradients associated with shocks, suppressing the spurious oscillations that typically plague non-dissipative schemes like the centered difference method. Implicit schemes should converge with larger time steps than the explicit method; the upwind scheme should be more stable near the shock that will occur, however converge slower due to its lower order compared to the centered method.

b. Implementation and Convergence Verification. Numerical experiments confirm the theoretical predictions regarding stability. The spatial accuracy was assessed by running the simulation on a series of progressively refined grids while keeping the timestep sufficiently small to ensure that spatial errors dominated. Figure 2 shows a log-log plot L_2 norm of error versus grid spacing, Δx. The norm was calculated using a reference value $u^\star$ from the highest resolution case. A similar analysis was performed to assess temporal convergence. The grid was held fixed at a high resolution, and the timestep Δt was systematically reduced. Figure 3 shows a log-log plot of the error versus the timestep. The explicit (Forward Euler) scheme demonstrated extreme sensitivity to the timestep size, requiring very small Δt to avoid divergence. In contrast, the implicit (Backward Euler) scheme remained stable for significantly larger time steps. The convergence analysis for the implicit and explicit methods, shown in Figure 3, indicates a global order of accuracy of approximately one in time. This is consistent with the first-order accuracy of the Forward and Backward Euler schemes. The convergence of the spatial discretization schemes is shown in Figure 2. For the upwind case, the scheme convergences with approximately order one, as expected. The centered scheme demonstrates spatial order of accuracy of approximately two, as expected, thus verifying the correct implementation of the central difference scheme.

For the nonlinear problem, the stability and convergence characteristics will differ from the linear theoretical predictions.

- **Upwind Explicit:** The advection term can be discretized based on the sign of the local velocity u_j^n. The stability will be governed by a Courant number based on the maximum velocity in the domain at each timestep.
- **Centered Implicit:** The nonlinear system of equations arising at each timestep must be solved iteratively. While the underlying linear stability analysis suggests unconditional stability, the nonlinear solver will likely impose a practical limit on the timestep size to ensure convergence.

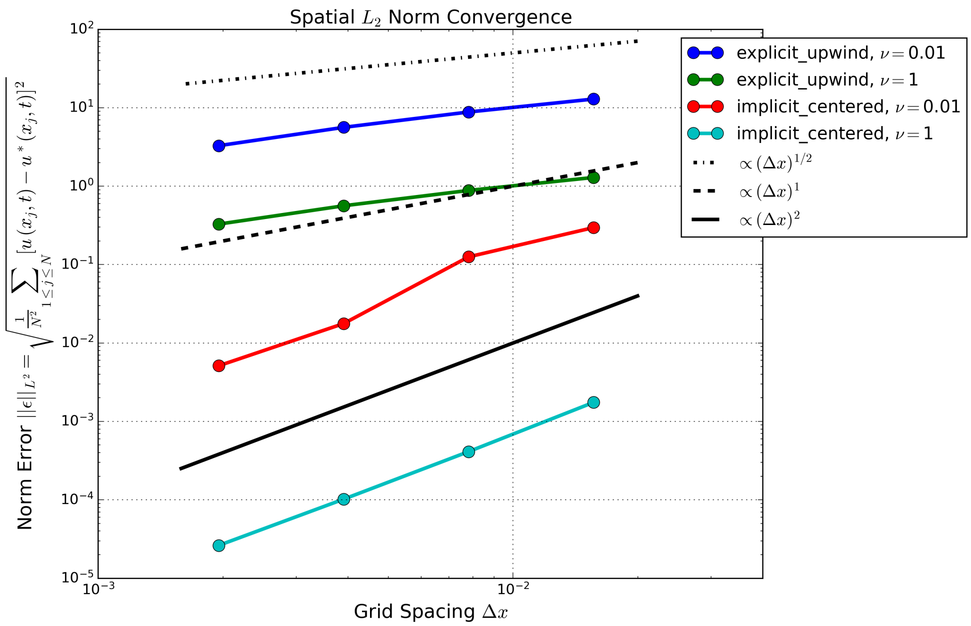

FIGURE 2. L_2 norm of simulation error as a function of grid spacing.

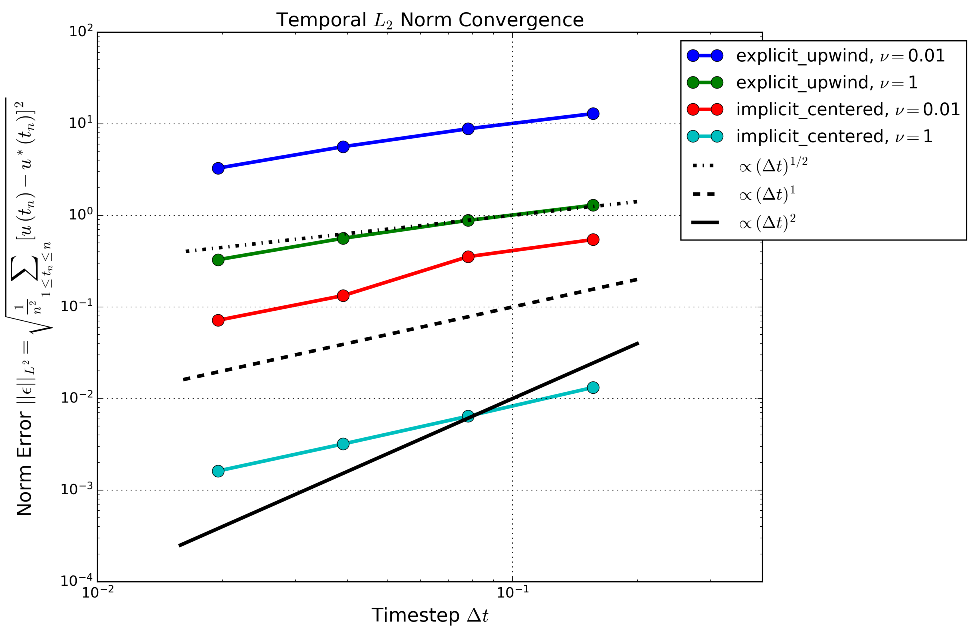

FIGURE 3. L_2 norm of simulation error as a function of timestep.

c. Conservation Properties in the Nonlinear Problem (for $\nu = 0$). An analysis of the semi-discrete form of Burgers' equation reveals a change in the conservation properties. For the upwind discretization of $u\,\partial u/\partial x$, for primary conservation:

$$\frac{\partial u_{j-1}}{\partial t} = -u_{j-1}\frac{u_{j-1} - u_{j-2}}{\Delta x}$$
$$\frac{\partial u_j}{\partial t} = -u_j\frac{u_j - u_{j-1}}{\Delta x}$$
$$\frac{\partial u_{j+1}}{\partial t} = -u_{j+1}\frac{u_{j+1} - u_j}{\Delta x}$$
$$\rightarrow \sum_{j=0}^{n}\frac{\partial u_j}{\partial t} = -\sum_{j=0}^{n} u_j\frac{u_j - u_{j-1}}{\Delta x} \neq \text{B.T.}$$

The summation no longer telescopes, indicating a loss of primary conservation. Consequently, it also lacks secondary conservation.

$$\begin{aligned}
u_{j-1}\frac{\partial u_{j-1}}{\partial t} &= -u_{j-1}^2\frac{u_{j-1}-u_{j-2}}{\Delta x}\\
u_j\frac{\partial u_j}{\partial t} &= -u_j^2\frac{u_j-u_{j-1}}{\Delta x}\\
u_{j+1}\frac{\partial u_{j+1}}{\partial t} &= -u_{j+1}^2\frac{u_{j+1}-u_j}{\Delta x}\\
\rightarrow \sum_{j=0}^{n} u_j\frac{\partial u_j}{\partial t} &\neq \text{B.T.}
\end{aligned}$$

For primary conservation of the centered scheme:

$$\begin{aligned}
\frac{\partial u_{j-1}}{\partial t} &= -u_{j-1}\frac{u_j-u_{j-2}}{2\Delta x}\\
\frac{\partial u_j}{\partial t} &= -u_j\frac{u_{j+1}-u_{j-1}}{2\Delta x}\\
\frac{\partial u_{j+1}}{\partial t} &= -u_{j+1}\frac{u_{j+2}-u_j}{2\Delta x}\\
\rightarrow \sum_{j=0}^{n}\frac{\partial u_j}{\partial t} &= -\sum_{j=0}^{n} u_j\frac{u_{j+1}-u_{j-1}}{2\Delta x} = \text{B.T.}
\end{aligned}$$

The centered scheme retains primary conservation. However, for secondary conservation:

$$\begin{aligned}
u_{j-1}\frac{\partial u_{j-1}}{\partial t} &= -u_{j-1}^2\frac{u_j-u_{j-2}}{2\Delta x}\\
u_j\frac{\partial u_j}{\partial t} &= -u_j^2\frac{u_{j+1}-u_{j-1}}{2\Delta x}\\
u_{j+1}\frac{\partial u_{j+1}}{\partial t} &= -u_{j+1}^2\frac{u_{j+2}-u_j}{2\Delta x}\\
\rightarrow \sum_{j=0}^{n} u_j\frac{\partial u_j}{\partial t} &= -\sum_{j=0}^{n} u_j^2\frac{u_{j+1}-u_{j-1}}{2\Delta x} \neq \text{B.T.}
\end{aligned}$$

The centered scheme loses its secondary conservation property for the nonlinear equation. This is consistent with the numerical simulations where $\nu \rightarrow 0$, as the upwind continues to dissipate because of the lack of primary conservation, however the solution with the centered scheme is conserved for much longer times.

i. Primary and Secondary Conservation. The conservation properties are different from the linear case. Neither of the previously defined discretizations, $u_j(u_j - u_{j-1})/\Delta x$ (upwind) nor $u_j(u_{j+1} - u_{j-1})/(2\Delta x)$ (centered), are in a form that guarantees secondary conservation for the nonlinear equation. Both methods lose an order of conservation from the translation to a nonlinear problem: the upwind scheme is no longer conservative, while the centered scheme retains only primary conservation. Summing these expressions over the domain does not result in a telescoping sum that cancels to zero. This represents a significant departure from the linear problem. Secondary conservation (energy) is not a property of the continuous inviscid Burgers' equation in the presence of shocks and is not expected to be satisfied by the numerical schemes.

ii. Ensuring Conservation. To ensure primary conservation, the conservative form of the equation must be discretized. The inviscid Burgers' equation can be written in a conservation law form,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

The spatial flux term $\frac{1}{2}u^2$ term must be discretized as a flux difference, $\frac{du_j}{dt} = -\frac{1}{\Delta x}(F_{j+1/2} - F_{j-1/2})$, where $F_{j+1/2}$ is a numerical flux function that approximates the flux at the cell interface (a centered difference of the flux that ensures that $\sum u_j$ is conserved). By writing the scheme in this form, the sum over the domain will always telescope to zero for periodic boundaries, thus guaranteeing primary conservation. Achieving secondary conservation requires a specific skew-symmetric formulation of the convective term.

d. Long-time Behavior for $(c, \nu) = (1.0, 0.01)$**.** In this regime, the strong nonlinear advection will cause the initial wave to steepen, forming a sharp gradient, or a viscous shock, whose thickness is scaled by ν. The numerical challenge is to resolve this feature accurately in Burgers' equation.

i. Discretization and Spurious Oscillations. Symmetric, non-dissipative spatial discretizations, such as the **centered difference scheme**, are known to produce spurious oscillations (Gibbs phenomenon) near sharp gradients (this method may attempt to get information ahead of shock for some grid points). These oscillations arise from the inability of a finite grid to resolve the high-frequency content of the shock. These oscillations are unphysical and can destroy the quality of the solution. It should be noted that spurious oscillations were never observed in any implementation, however extremely small time steps were taken in the cases shown.

ii. Eliminating Spurious Oscillations. Schemes with inherent numerical dissipation can damp these high-frequency oscillations, providing more stable and monotonic (non-oscillatory) solutions across shocks (only attempting to gather information from upwind instead of potentially jumping across a discontinuity). The **first-order upwind scheme** is highly dissipative and will prevent oscillations, but it will also excessively smear the shock profile. More advanced methods, such as **TVD (Total Variation Diminishing)** or **WENO (Weighted Essentially Non-Oscillatory)** schemes, are specifically designed to provide high-order accuracy in smooth regions while adaptively adding dissipation near shocks to prevent oscillations.

iii. Sacrificed Numerical Properties. When a scheme such as first-order upwind is used to eliminate spurious oscillations, the numerical property that is sacrificed is accuracy. The scheme is formally first-order, and its inherent numerical dissipation can smear sharp features, effectively widening the shock profile. This can be detrimental in simulations where physical dissipation is low or zero.

e. Explicit vs. Implicit Schemes: Advantages and Disadvantages. This project highlights the fundamental trade-offs between explicit and implicit time integration schemes.

- **Explicit Schemes:** The primary advantage is simplicity of implementation and low computational cost per timestep because no system of equations is solved. However, stability is conditional and often restricted by a severe timestep limitation (like the CFL condition), which can be extremely small for fine grids or problems with high diffusion, making long-time simulations inefficient.
- **Implicit Schemes:** The main advantage is superior stability properties (unconditionally stable for this problem), which allows for much larger time steps than explicit methods. This makes them highly efficient for stiff problems, where stability, not accuracy, limits the timestep. The principal disadvantage is the higher computational cost per timestep, as it requires the solution of a large, coupled system of algebraic equations (which is nonlinear for Burgers' equation) at each timestep. This increases the computational cost per step and introduces the complexity of an iterative nonlinear solver. The implementation is more complex. When it is implemented in an inefficient manner it can be significantly more computationally intensive.

The selection between explicit and implicit methods constitutes a trade-off between implementation complexity and per-step computational cost versus stability and the ability to use larger time steps.