Chapter 1

Two-Dimensional Fourier Transforms in Polar Coordinates

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Contents

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1. INTRODUCTION

The Fourier transform needs no introduction, and it would be an understatement to say that it has proved invaluable in many diverse disciplines, such as engineering, mathematics, physics, and chemistry. Its applications are numerous and include a wide range of topics, such as communications, optics, astronomy, geology, image processing, signal processing, and so forth. It is known that the Fourier transform can easily be extended to n-dimensions. The strength of the Fourier transform is that it is accompanied by a toolset of operational properties that simplify the calculation of more-complicated transforms through the use of these standard rules. Specifically, the standard Fourier toolset consists of results for scaling, translation (spatial shift), multiplication, and convolution, along with the basic transforms of the Dirac delta function and complex exponential that are so essential to the derivation of the shift, multiplication, and convolution results. This basic toolset of operational rules is well known for the Fourier transform in single and multiple dimensions (Bracewell, 1999; Howell, 2000).

As is also known, the Fourier transform in two dimensions can be developed in terms of polar coordinates. (Chirikjian and Kyatkin, 2001) instead of the usual Cartesian coordinates, which is most useful when the function being transformed is naturally describable in polar coordinates. For example, this has been applied in the field of photoacoustics. (Xu *et al.*, 2002), and attempts have been made to translate ideas from the continuous domain to the discrete domain by developing numerical algorithms for such calculations (Averbuch *et al.*, 2006). However, to the best of the author's knowledge, a complete interpretation of the standard

Fourier operational toolset in terms of polar coordinates is missing from the literature. Some results are known, such as the Dirac delta function in both polar and spherical polar coordinates, but the results on shift, multiplication, and in particular, convolution, are incomplete.

This chapter thus aims to develop the Fourier operational toolset of Dirac delta, exponential, spatial shift, multiplication, and convolution for the two-dimensional (2D) Fourier transform in polar coordinates. Of particular novelty is the treatment of the shift, multiplication, and convolution theorems, which can also be adapted for the special cases of circularly symmetric functions that have no angular dependence. It is well known from the literature that 2D Fourier transforms for radially symmetric functions can be interpreted in terms of a (zeroth-order) Hankel transform. It is also known that the Hankel transforms do not have a multiplication/convolution rule, a rule that has been widely used in the Cartesian version of the transform. In this chapter, the multiplication/convolution rule is treated in detail for the curvilinear version of the transform; in particular, it is shown that the Hankel transform *does* obey a multiplication/convolution rule once the proper interpretation of convolution is applied. This chapter carefully considers the definition of convolution and derives the correct interpretation of this concept in terms of the curvilinear coordinates so that the standard multiplication/convolution rule is once again applicable.

The outline of the text is as follows. For completeness, the Hankel transform and the interpretation of the 2D Fourier transform in terms of a Hankel transform and a Fourier series are introduced in Sections 2 and 3. Sections 4 and 5 treat the special functions of the Dirac delta and complex exponential. Sections 6, 7, and 8 address the multiplication, spatial shift, and convolution operations. In particular, the nature of the spatial shift and its role in the ensuing convolution theorem are discussed. Sections 9 and 10 discuss the spatial shift and convolution operators for the special case of radially (circularly) symmetric functions. Section 11 addresses the special case of a convolution of a radially symmetric function with one that is not, since this case has many important applications. Sections 12 and 13 discuss the special cases of angular or radial convolution only—that is, not a full 2D but a special one-dimensional (1D) convolution as restricted to convolving over only one of the variables of the polar coordinates. In particular, it is shown that while the angular convolution yields a simple convolution relationship, the radial-only convolution does not. Section 14 derives the Parseval relationships. Sections 15 and 16 introduce some applications involving the Laplacian and the Helmholtz equation. Section 17 summarizes and concludes the chapter. The operational toolset as derived is summarized in a table.

2. HANKEL TRANSFORM

The nth-order Hankel transform is defined by the integral (Piessens, 2000)

$$\widehat{F}_n(\rho) = \mathbb{H}_n(f(r)) = \int_0^\infty f(r) J_n(\rho r) r \, dr, \tag{1}$$

where $J_n(z)$ is the nth-order Bessel function and the overhat indicates a Hankel transform as shown in Eq. (1). Here, n may be an arbitrary real or complex number. However, an integral transform needs to be invertible in order to be useful, which restricts the allowable values of n. If n is real and $n > -1/2$, the transform is self-reciprocating and the inversion formula is given by

$$f(r) = \int_0^\infty F_n(\rho) J_n(\rho r) \rho \, d\rho. \tag{2}$$

The inversion formula for the Hankel transform follows immediately from Hankel's repeated integral, which states that under suitable boundary conditions and subject to the condition that $\int_0^\infty f(r)\sqrt{r} dr$ is absolutely convergent, then for $n > -1/2$

$$\int_0^\infty s \, ds \int_0^\infty f(r) J_n(sr) J_n(su) \, r dr = \frac{1}{2}\left[f(u+) + f(u-)\right]. \tag{3}$$

The most important cases correspond to $n = 0$ or $n = 1$. The Hankel transform exists only if the following integral exists: $\int_0^\infty \left| r^{1/2} f(r) \right| dr$. The Hankel transform is particularly useful for problems involving cylindrical symmetry. It is useful to note that the Bessel functions satisfy an orthogonality/closure relationship given by

$$\int_0^\infty J_n(ux) J_n(vx) \, x dx = \frac{1}{u} \delta(u - v). \tag{4}$$

3. THE CONNECTION BETWEEN THE 2D FOURIER AND HANKEL TRANSFORMS

The 2D Fourier transform of a function $f(x, y)$ is defined similar to its 1D counterpart:

$$F(\vec{\omega}) = F(\omega_x, \omega_y) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) \, e^{-j(\omega_x x + \omega_y y)} dx \, dy. \tag{5}$$

The inverse Fourier transform is given by

$$f(\vec{r}) = f(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_x, \omega_y) e^{j\vec{\omega}\cdot\vec{r}} d\omega_x \, d\omega_y, \tag{6}$$

where the shorthand notation of $\vec{\omega} = (\omega_x, \omega_y)$, $\vec{r} = (x, y)$ has been used.

Polar coordinates can be introduced as $x = r\cos\theta$, $y = r\sin\theta$, and similarly in the spatial frequency domain as $\omega_x = \rho\cos\psi \quad \omega_y = \rho\sin\psi$. It then follows that the 2D Fourier transform can be written as

$$F(\rho, \psi) = \int_{0}^{\infty} \int_{-\pi}^{\pi} f(r, \theta) \, e^{-ir\rho\cos(\psi-\theta)} r dr d\theta. \tag{7}$$

Thus, in terms of polar coordinates, the Fourier transform operation transforms the spatial position radius and angle (r, θ) to the frequency radius and angle (ρ, ψ). The usual polar-coordinate relationships apply in each domain so that $r^2 = x^2 + y^2$, $\theta = \arctan\left(y/x\right)$, $\rho^2 = \omega_x^2 + \omega_y^2$, and $\psi = \arctan\left(\omega_y/\omega_x\right)$. Using $\vec{r}$ to represent (r, θ) in physical polar coordinates and $\vec{\omega}$ to denote the frequency vector (ρ, ψ) in frequency polar coordinates, the following expansions are valid (Chirikjian and Kyatkin, 2001):

$$e^{i\vec{\omega}\cdot\vec{r}} = \sum_{n=-\infty}^{\infty} i^n J_n(\rho r) \, e^{in\theta} e^{-in\psi} \tag{8}$$

$$e^{-i\vec{\omega}\cdot\vec{r}} = \sum_{n=-\infty}^{\infty} i^{-n} J_n(\rho r) \, e^{-in\theta} e^{in\psi}. \tag{9}$$

These expansions can be used to convert the 2D Fourier transform into polar coordinates.

3.1. Radially Symmetric Functions

If it is assumed that f is radially symmetric, then it can be written as a function of r only and can thus be taken out of the integration over the angular coordinate so that Eq. (7) becomes

$$F(\rho, \psi) = \int_{0}^{\infty} r f(r) dr \int_{-\pi}^{\pi} e^{-ir\rho\cos(\psi-\theta)} d\theta. \tag{10}$$

Using the integral definition of the zeroth-order Bessel function,

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\cos(\psi-\theta)} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\cos\alpha} d\alpha. \tag{11}$$

Equation (10) can then be written as

$$F(\rho) = \mathbb{F}_{2D}\{f(r)\} = 2\pi \int_0^\infty f(r) J_0(\rho r) r dr, \tag{12}$$

which can be recognized as 2π times the Hankel transform of order zero. Thus, the special case of the 2D Fourier transform of a radially symmetric function is the same as the zeroth-order Hankel transform of that function:

$$F(\rho) = \mathbb{F}_{2D}\big\{f(r)\big\} = 2\pi\, \mathbb{H}_0\left\{f(r)\right\}. \tag{13}$$

With reference to Eq. (13), $f(r)$ and $F(\rho)$ are functions with radial symmetry in a 2D regime, $\mathbb{F}_{2D}\{\cdot\}$ is an operator in a 2D regime, while $\mathbb{H}_0\{\cdot\}$ is an operator in a 1D regime.

3.2. Non–Radially Symmetric Functions

When the function $f(r,\theta)$ is not radially symmetric and is a function of both r and θ, the preceding result can be generalized. Since $f(r,\theta)$ depends on the angle θ, it can be expanded into a Fourier series

$$f(\vec{r}) = f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{jn\theta}, \tag{14}$$

where

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta)\, e^{-jn\theta} d\theta. \tag{15}$$

This transform is well suited for functions that are separable in r and θ. This case is extensively treated in the widely used text, *Introduction to Fourier Optics*, by Joseph Goodman (2004).

Similarly, the 2D Fourier transform $F(\rho,\psi)$ can also be expanded into its own Fourier series so that

$$F(\vec{\omega}) = F(\rho,\psi) = \sum_{n=-\infty}^{\infty} F_n(\rho)\, e^{jn\psi} \tag{16}$$

and

$$F_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} F(\rho,\psi) e^{-jn\psi} d\psi. \tag{17}$$

It is important to note that $F_n(\rho)$ is *not* the Fourier transform of $f_n(r)$. In fact, it is the relationship between $f_n(r)$ and $F_n(\rho)$ that we seek to define.

3.2.1. Forward Transform

Expansions (9) and (14) are substituted into the definition of the forward Fourier transform to give

$$
\begin{aligned}
F(\vec{\omega}) &= \int_{-\infty}^{\infty} f(\vec{r}) e^{-j\vec{\omega}\cdot\vec{r}} d\vec{r} \\
&= \int_0^{\infty}\int_0^{2\pi} \sum_{m=-\infty}^{\infty} f_m(r) e^{jm\theta} \sum_{n=-\infty}^{\infty} i^{-n} J_n(\rho r)\, e^{-in\theta} e^{in\psi} d\theta\, r dr \\
&= \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi} \int_0^{\infty} f_n(r) J_n(\rho r)\, r dr.
\end{aligned} \tag{18}
$$

This follows since

$$
\int_0^{2\pi} e^{in\theta} d\theta = 2\pi\, \delta_{n0} = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{19}
$$

where δ_{nm} denotes the Kronecker delta function. By comparing Eqs. (16) and (18), we obtain the expression for $F_n(\rho)$, the Fourier coefficients of the Fourier domain expansion. Using Eq. (1), this can also be interpreted in terms of a Hankel transform as follows:

$$
F_n(\rho) = 2\pi\, i^{-n} \int_0^{\infty} f_n(r) J_n(\rho r)\, r dr = 2\pi\, i^{-n} \mathbb{H}_n\{ f_n(r)\}. \tag{20}
$$

3.2.2. Inverse Transform

The corresponding 2D inverse Fourier transform is written as

$$
f(\vec{r}) = \frac{1}{(2\pi)^2} \int_0^{\infty}\int_0^{2\pi} F(\vec{\omega}) e^{j\vec{\omega}\cdot\vec{r}} d\psi\, \rho d\rho. \tag{21}
$$

Using Eq. (16) along with the expansion (8) yields

$$f(\vec{r}) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} i^n e^{in\theta} \int_0^{\infty} F_n(\rho) J_n(\rho r)\, \rho d\rho \tag{22}$$

so that

$$f_n(r) = \frac{i^n}{2\pi} \int_0^{\infty} F_n(\rho) J_n(\rho r)\, \rho d\rho = \frac{i^n}{2\pi} \mathbb{H}_n\{F_n(\rho)\}\,. \tag{23}$$

Thus, it can be observed that the nth term in the Fourier series for the original function will Hankel transform into the nth term of the Fourier series of the Fourier transform function. However, it is an nth-order Hankel transform for the nth term, so that all the terms are not equivalently transformed. Furthermore, recall that the general 2D Fourier transform for radially symmetric functions was equivalent to the zeroth-order Hankel transform. Therefore, the mapping from $f_n(r)$ to $F_n(\rho)$, which is an nth-order Hankel transform, is *not* a 2D Fourier transform.

3.2.3. Discussion

Most importantly, it can be seen that the operation of taking the 2D Fourier transform of a function is equivalent to (1) first finding its Fourier series expansion in the angular variable and (2) then finding the nth-order Hankel transform (of the spatial radial variable to the spatial frequency radial variable) of the nth coefficient in the Fourier series and appropriately scaling the result. Clearly, for functions with cylindrical-type symmetries that are naturally described by cylindrical coordinates, the operation of taking a three-dimensional (3D) Fourier transform will be equivalent to (1) a regular 1D Fourier transform in the z coordinate, then (2) a Fourier series expansion in the angular variable, and then (3) an nth-order Hankel transform (of the radial variable to the spatial radial variable) of the nth coefficient in the Fourier series. Since each of these operations involves integration over one variable only with the others considered parameters vis-à-vis the integration, the order in which these operations are performed is interchangeable.

3.3. Fourier Pairs

Thinking in terms of operators and paired functions is fairly common with Fourier transforms but less common with Fourier *series*. Fourier *series* are typically introduced as an equality $f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{jn\theta}$ instead of as an operator. We thus propose here that a Fourier series should be considered a transform/operator in the same way that we consider a Fourier transform and use the symbol $\mathbb{F}_S\left\{f(\cdot)\right\}$ to represent this transform. That is, we reinterpret Eqs. (14) and (15) as forward and inverse Fourier series transforms so that we define the forward Fourier series transform as the operation of finding the Fourier series coefficients:

$$f_n = \mathbb{F}_S\left\{f(\theta)\right\} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-jn\theta} d\theta, \tag{24}$$

and the inverse transform is the one where the original function is returned from the coefficients by constructing the Fourier series itself:

$$f(\theta) = \mathbb{F}_S^{-1}\left\{f_n\right\} = \sum_{n=-\infty}^{\infty} f_n\, e^{jn\theta}. \tag{25}$$

Hence f_n and $f(\theta)$ are a Fourier (series) pair $f(\theta) \Leftrightarrow f_n$.

In working with Fourier transforms, we often refer to *Fourier pairs*, meaning a pair of functions where one is the Fourier transform of the other. For full 2D Fourier transforms, we denote a Fourier pair as $f(x,y) \Leftrightarrow F(\omega_x, \omega_y)$, which means that $F(\omega_x, \omega_y) = \mathbb{F}_{2D}(f(x,y))$. Similarly, the notation $f(r,\theta) \Leftrightarrow F(\rho,\psi)$ implies that $F(\rho,\psi) = \mathbb{F}_{2D}(f(r,\theta))$. These functions are "paired" via a Fourier transform, and this notation is captured with the $\Leftrightarrow$ arrow. In terms of the Fourier series expansion of the function and its transform, Eq. $F(\rho,\psi) = \mathbb{F}_{2D}(f(r,\theta))$ becomes

$$\sum_{n=-\infty}^{\infty} f_n(r)\, e^{jn\theta} \quad \Leftrightarrow \quad \sum_{n=-\infty}^{\infty} F_n(\rho)\, e^{jn\psi}. \tag{26}$$

Since both sides of Eq. (26) contain an infinite series, where essentially θ is replaced with ψ and vice versa, it is actually far more convenient to denote the Fourier pair of expression (26) as

$$f_n(r) \quad \Leftrightarrow \quad F_n(\rho), \tag{27}$$

where (1) the series is implied but dropped for brevity and, more importantly, (2) the reader must recall that the relationship between the Fourier pair of expression (27) is not that of a Fourier transform, as a naive interpretation of (27) would imply, but rather is given by Eqs. (20) and (23).

4. THE DIRAC DELTA FUNCTION AND ITS TRANSFORM

The unit-mass Dirac delta function in 2D polar coordinates is defined as

$$f(\vec{r}) = \delta(\vec{r} - \vec{r}_0) = \frac{1}{r}\delta(r - r_0)\,\delta(\theta - \theta_0). \tag{28}$$

To find the Fourier transform, the Fourier series expansion is required, followed by a Hankel transform, as previously discussed. Thus for the Dirac delta function, the Fourier series expansions terms are

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r}\delta(r-r_0)\,\delta(\theta-\theta_0)\,e^{-in\theta}\,d\theta = \frac{1}{2\pi r}\delta(r-r_0)e^{-in\theta_0}. \tag{29}$$

so that Eq. (28) can be written as

$$f(\vec{r}) = \delta(\vec{r}-\vec{r}_0) = \frac{1}{r}\delta(r-r_0)\,\delta(\theta-\theta_0) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi r}\delta(r-r_0)\,e^{-in\theta_0}e^{in\theta}. \tag{30}$$

Then the full transform is given by

$$\begin{aligned} F(\vec{\omega}) &= \sum_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi} = \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi}\int_0^{\infty} f_n(r)J_n(\rho r)\, rdr \\ &= \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi}\int_0^{\infty}\frac{\delta(r-r_0)}{2\pi r}e^{-in\theta_0}J_n(\rho r)\, rdr \\ &= \sum_{n=-\infty}^{\infty} i^{-n}J_n(\rho r_0)e^{-in\theta_0}\, e^{in\psi} = e^{-i\vec{\omega}\cdot\vec{r}_0}, \end{aligned} \tag{31}$$

where Eq. (31) is the 2D linear exponential function. The Fourier transform of the Dirac delta function is the exponential function, as would be expected from the results in Cartesian coordinates. However, the more important result that we seek is that the *coefficients* of the Fourier transform are

$$F_n(\rho) = i^{-n}J_n(\rho r_0)e^{-in\theta_0}. \tag{32}$$

Hence the coefficients of the Fourier pair for the Dirac delta function are given by

$$\begin{aligned} f_n(r) = [\delta(\vec{r}-\vec{r}_0)]_n &= \frac{1}{2\pi r}\delta(r-r_0)\,e^{-in\theta_0} \\ \Leftrightarrow\ F_n(\rho) &= i^{-n}J_n(\rho r_0)e^{-in\theta_0} = \left[e^{-i\vec{\omega}\cdot\vec{r}_0}\right]_n. \end{aligned} \tag{33}$$

This Fourier pair is included in Table 1.

TABLE 1 Summary of Fourier Transform Relationships in Polar Coordinates

$f(\vec{r})$	$f_n(r)$	$F_n(\rho)$	$F(\vec{\omega})$
$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}$	$\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-jn\theta}d\theta$	$2\pi\, i^{-n}\int_0^{\infty} f_n(r)J_n(\rho r)\, rdr$	$F(\rho,\psi) = \sum_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi}$
$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}$	$\frac{i^n}{2\pi}\int_0^{\infty} F_n(\rho)J_n(\rho r)\rho d\rho$	$\frac{1}{2\pi}\int_0^{2\pi} F(\rho,\psi)e^{-jn\psi}d\psi$	$F(\rho,\psi) = \sum_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi}$
$\delta(\vec{r}-\vec{r}_0) = \frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0)$	$\frac{\delta(r-r_0)}{2\pi r}e^{-in\theta_0}$	$i^{-n}J_n(\rho r_0)e^{-in\theta_0}$	$e^{-i\vec{\omega}\cdot\vec{r}_0}$
$e^{i\vec{\omega}_0\cdot\vec{r}}$	$i^n J_n(\rho_0 r)\, e^{-in\psi_0}$	$2\pi\, e^{-in\psi_0}\frac{1}{\rho}\delta(\rho-\rho_0)$	$(2\pi)^2\delta(\vec{\omega}-\vec{\omega}_0)$
$f(\vec{r}-\vec{r}_0)$	$\frac{i^n}{2\pi}\int_0^{\infty} F_n(\rho)J_n(\rho r)\rho d\rho = \sum_{m=-\infty}^{\infty} e^{-im\theta_0}\int_0^{\infty} f_{n-m}(u)S^n_{n-m}(u,r,r_0)\ udu$	$\left[i^{-n}J_n(\rho r_0)e^{-in\theta_0}\right] * F_n(\rho) = \sum_{m=-\infty}^{\infty} i^{-m}J_m(\rho r_0)\, e^{-im\theta_0}F_{n-m}(\rho)$	$e^{-i\vec{\omega}\cdot\vec{r}_0}F(\vec{\omega})$
$h(\vec{r})g(\vec{r})$	$f_n(r) = h_n(r) * g_n(r) = \sum_{m=-\infty}^{\infty} h_{n-m}(r)g_m(r)$	$2\pi\, i^{-n}\int_0^{\infty} f_n(r)J_n(\rho r)\, rdr$	$\frac{G(\vec{\omega}) ** H(\vec{\omega})}{(2\pi)^2}$
$h(\vec{r}) ** g(\vec{r})$	$\frac{i^n}{2\pi}\int_0^{\infty} F_n(\rho)J_n(\rho r)\rho d\rho$	$F_n(\rho) = H_n(\rho) * G_n(\rho) = \sum_{m=-\infty}^{\infty} G_{n-m}(\rho)H_m(\rho)$	$G(\vec{\omega})H(\vec{\omega})$

(Continued)

TABLE 1 *(Continued)*

$f\,(\vec{r})$	$f_n(r)$	$F_n(\rho)$	$F(\vec{\omega})$
$h(\vec{r}) *_\theta g(\vec{r})$	$f_n(r) = h_n(r)g_n(r)$	$2\pi\, i^{-n} \int\limits_0^\infty f_n(r)J_n(\rho r)\, rdr$	$\sum\limits_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi}$
$f\left(r - \vec{r}_0\right)$ $\left(f \text{ symmetric}\right)$	$\frac{e^{-in\theta_0}}{2\pi} \int\limits_0^\infty F(\rho)J_n(\rho r_0)J_n(\rho r)\rho d\rho$	$i^{-n}e^{-in\theta_0}F(\rho)J_n(\rho r_0)$	$e^{-i\vec{\omega}\cdot\vec{r}_0}F(\rho)$
$g(\vec{r}) ** h(r)$ $(h \text{ symmetric})$	$f_n(r) = \int\limits_0^\infty g_n(r_0)\varphi_n\,(r - r_0)\; r_0 dr_0$ $\varphi_n\,(r - r_0)$ $= \int\limits_0^\infty H(\rho)J_n(\rho r_0)J_n(\rho r)\rho d\rho$	$2\pi\, i^{-n} \int\limits_0^\infty f_n(r)J_n(\rho r)\, rdr$	$F(\rho, \psi)$ $= \sum\limits_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi}$
$g(\vec{r}) ** h(r)$ $(h \text{ symmetric})$	$\frac{i^n}{2\pi} \int\limits_0^\infty F_n(\rho)J_n(\rho r)\rho d\rho$	$F_n(\rho) = G_n(\rho)H(\rho)$	$\sum\limits_{n=-\infty}^{\infty} F_n(\rho)e^{jn\psi}$
$\nabla^2 f(r, \theta)$ $f = \sum\limits_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}$	$\nabla_n^2 f_n = \frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} - \frac{n^2}{r^2}f_n$	$-\rho^2 F_n(\rho)$	$-\rho^2 F(\rho, \psi)$

4.1. Dirac Delta Function at the Origin

If $\vec{r}_0$ is at the origin, then it is multiply covered by the angular variable and the Dirac delta function in two dimensions is given by

$$\delta(\vec{r}) = \frac{1}{2\pi r}\delta(r), \tag{34}$$

where the difference in notation between $\delta(\vec{r})$ and $\delta(r)$ is emphasized. The notation $\delta(\vec{r})$ is used to represent the Dirac delta function in the appropriate multidimensional coordinate system, whose actual form may vary. The notation $\delta(r)$ denotes the standard 1D scalar version of the Dirac delta function that is most familiar. Either Eq. (12) or Eq. (32) can be used to calculate the Fourier transform since they both yield the correct transform for the Dirac delta function at the origin, which is 1. Because the function is radially symmetric, the series consists of only the zeroth-order terms—that is, $f_n(r) = f(r)\delta_{n0}$ and $F_n(\rho) = F(\rho)\delta_{n0}$. The Fourier pair is thus given by

$$f_n(r) = \frac{1}{2\pi r}\delta(r)\delta_{n0} \quad \Leftrightarrow \quad F_n(\rho) = i^{-n}J_n(0) = \delta_{n0}, \tag{35}$$

or more compactly as $\frac{1}{2\pi r}\delta(r) \Leftrightarrow 1$.

4.2. Ring Delta Function

An often-used function is the *ring* delta function given by

$$f(r) = \frac{1}{2\pi r}\delta(r - r_0). \tag{36}$$

The function given in Eq. (36) is nonzero only on the ring of radius r_0. The 2D Fourier transform of the ring delta is most easily found from Eq. (12) and is given by

$$F(\rho) = 2\pi \int_0^\infty \left\{ \frac{1}{2\pi r}\delta(r - r_0) \right\} J_0(\rho r) r dr = J_0(\rho r_0). \tag{37}$$

Because the function is radially symmetric, the full 2D transform is simply a Hankel transform of order zero and the Fourier transform is also only radially symmetric. The series consists of only the zeroth-order terms—that is, $f_n(r) = f(r)\delta_{n0}$ and $F_n(\rho) = F(\rho)\delta_{n0}$. So the full 2D Fourier transform pair is given by

$$\frac{1}{2\pi r}\delta(r - r_0) \quad \Leftrightarrow \quad J_0(\rho r_0). \tag{38}$$

5. THE COMPLEX EXPONENTIAL AND ITS TRANSFORM

From Eq. (8), the 2D complex exponential function can be written in polar coordinates as

$$f(\vec{r}) = e^{i\vec{\omega}_0\cdot\vec{r}} = \sum_{n=-\infty}^{\infty} i^n J_n(\rho_0 r)\, e^{-in\psi_0}\, e^{in\theta}, \tag{39}$$

so that the Fourier coefficients can be directly seen from the previous equation or can be found from the formula by

$$\begin{aligned} f_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=-\infty}^{\infty} i^m J_m(\rho_0 r) e^{-im\psi_0}\, e^{im\theta} e^{-in\theta}\, d\theta \\ &= i^n J_n(\rho_0 r)\, e^{-in\psi_0}. \end{aligned} \tag{40}$$

The full 2D Fourier transform is given by

$$\begin{aligned} F(\vec{\omega}) &= \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi} \int_0^{\infty} f_n(r) J_n(\rho r)\, r dr \\ &= \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi} \int_0^{\infty} \left\{ i^n J_n(\rho_0 r)\, e^{-in\psi_0} \right\} J_n(\rho r)\, r dr \\ &= \sum_{n=-\infty}^{\infty} 2\pi\, e^{-in\psi_0} \frac{1}{\rho} \delta(\rho - \rho_0)\, e^{in\psi}, \end{aligned} \tag{41}$$

where the last line follows from the orthogonality of the Bessel functions (Arfken and Weber, 2005). Thus the Fourier coefficients that we seek are given by

$$F_n(\rho) = 2\pi\, e^{-in\psi_0} \frac{1}{\rho} \delta(\rho - \rho_0). \tag{42}$$

Of note, the closure of the complex exponentials (or equivalently, finding the Fourier series of the 1D Dirac delta function) gives

$$\delta(\psi - \psi_0) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\, e^{-in\psi_0}\, e^{in\psi}, \tag{43}$$

so that Eq. (41) actually gives the traditional Fourier transform of the complex exponential as it should:

$$F(\vec{\omega}) = (2\pi)^2 \frac{1}{\rho} \delta(\rho - \rho_0)\, \delta(\psi - \psi_0) = (2\pi)^2 \delta(\vec{\omega} - \vec{\omega}_0). \tag{44}$$

Equation (44) is the correct full 2D Fourier transform of the complex exponential, but the Fourier pair we seek are the coefficients of the function and coefficients of its transform, which are given, respectively, by Eqs. (40) and (42) as

$$f_n(r) = i^n J_n(\rho_0 r)\, e^{-in\psi_0} \quad \Leftrightarrow \quad F_n(\rho) = 2\pi\, e^{-in\psi_0} \frac{1}{\rho} \delta(\rho - \rho_0). \tag{45}$$

This Fourier pair is also included in Table 1.

5.1. Special Case

As a special case of the complex exponential, the Fourier transform of $f(\vec{r}) = 1$ can be computed by substituting $\omega_0 = 0$ into the above formulas. This gives

$$F(\vec{\omega}) = (2\pi)^2 \frac{1}{\rho} \delta(\rho) \delta(\psi) = (2\pi)^2 \delta(\vec{\omega}), \tag{46}$$

or alternatively, in series form the Fourier transform of $f(\vec{r}) = 1$ is given by

$$F(\vec{\omega}) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{\rho} \delta(\rho) e^{in\psi}. \tag{47}$$

The Fourier pair coefficients are given by

$$f_n(r) = i^n J_n(0) = \delta_{n0} \quad \Leftrightarrow \quad F_n(\rho) = 2\pi\, \frac{1}{\rho} \delta(\rho), \tag{48}$$

where δ_{n0} is the standard Kronecker-delta—that is, δ_{n0} is 1 for $n = 0$ and 0 otherwise.

6. MULTIPLICATION

We consider the product of two functions $h(\vec{r}) = f(\vec{r}) g(\vec{r})$, where $f(\vec{r}) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}$ and $g(\vec{r}) = \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta}$; the Fourier series coefficients $f_n(r)$ and $g_n(r)$ are given by Eq. (15); and we seek to find the equivalent coefficients $h_n(r)$ of $h(\vec{r}) = f(\vec{r})g(\vec{r}) = \sum_{n=-\infty}^{\infty} h_n(r)e^{in\theta}$. This is accomplished by finding the Fourier transform of $h(\vec{r})$ and using the expansions $f(\vec{r}) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}$ and $g(\vec{r}) = \sum_{n=-\infty}^{\infty} g_n(r)e^{in\theta}$, along with Eq. (9) for the polar form of the 2D complex exponential, as follows:

$$\begin{aligned} H(\vec{\omega}) &= \int_{-\infty}^{\infty} f(\vec{r})g(\vec{r})\, e^{-i\vec{\omega}\cdot\vec{r}} d\vec{r} \\ &= \int_0^{\infty}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta} \sum_{m=-\infty}^{\infty} g_m(r)e^{im\theta} \sum_{k=-\infty}^{\infty} i^{-k} J_k(\rho r)\, e^{-ik\theta} e^{ik\psi}\, d\theta\, r dr. \end{aligned} \tag{49}$$

Performing the integration over the angular variable yields

$$\begin{aligned} H(\vec{\omega}) &= \sum_{k=-\infty}^{\infty} 2\pi\, i^{-k} e^{ik\psi} \int_0^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r) J_k(\rho r)\, r dr \\ &= \sum_{k=-\infty}^{\infty} H_k(\rho) e^{ik\psi}, \end{aligned} \tag{50}$$

where

$$H_k(\rho) = 2\pi\, i^{-k} \int_0^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r) J_k(\rho r)\, r dr. \tag{51}$$

However, it is known from Eq. (20) that

$$H_k(\rho) = 2\pi\, i^{-k} \int_0^{\infty} h_k(r) J_k(\rho r)\, r dr; \tag{52}$$

hence it follows that

$$h_k(r) = \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r), \tag{53}$$

which is, in fact, the convolution of the Fourier series of $f(\vec{r})$ and $g(\vec{r})$. In other words, the coefficients of the products of the two series is the convolution of the coefficients so that

$$(fg)_k = f_k * g_k, \tag{54}$$

with the convolution of the two series defined as

$$\left(f_k * g_k\right)(r) \equiv \sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r). \tag{55}$$

The definition of the discrete convolution of two series given in Eq. (55) is the same as the standard definition given in the literature (Oppenheim and Schafer, 1989). The Fourier pair is then given by

$$(fg)_k = f_k * g_k \quad \Leftrightarrow \quad 2\pi\, i^{-k} \int_0^{\infty} \left(f_k * g_k\right) J_k(\rho r)\, r dr. \tag{56}$$

7. SPATIAL SHIFT

The correct expression for a Fourier series shifted in space is derived by finding the inverse Fourier transform of the complex exponential-weighted transform. In other words, $F(\vec{r} - \vec{r}_0)$ is found from

$$f(\vec{r} - \vec{r}_0) = \mathbb{F}^{-1}\left\{e^{-i\vec{\omega}\cdot\vec{r}_0} F(\vec{\omega})\right\}. \tag{57}$$

The reason for defining the shifted function according to Eq. (57) is that we have already found the expansion for the complex exponential, in addition to the rules for finding the product of two expansions. It is not sufficient to find any expression for the spatial shift but rather the expression that is sought is one that is (1) in the form of a Fourier series and (2) in terms of the unshifted coefficients of the original function as this builds the rule for what must be done to the coefficients if a shift is desired. Thus, by building on the previous results, the relevant spatial shift result can be found in the desired form.

Using the definition of the inverse Fourier transform given in Eq. (21), along with the expansions in Eqs. (8), (9), and (16), then the desired quantity is given by

$$\begin{aligned} f(\vec{r} - \vec{r}_0) = \frac{1}{(2\pi)^2} \int_0^{\infty} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} i^{-m} J_m(\rho r_0)\, e^{-im\theta_0} e^{im\psi} \sum_{n=-\infty}^{\infty} F_n(\rho) e^{jn\psi} \\ \times \sum_{k=-\infty}^{\infty} i^k J_k(\rho r)\, e^{ik\theta} e^{-ik\psi}\, d\psi \rho d\rho. \end{aligned} \tag{58}$$

Performing the integration over ψ yields a nonzero value only if $m+n-k=0$, so that the preceding equation simplifies to

$$f(\vec{r}-\vec{r}_0)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} i^n e^{-i(k-n)\theta_0} e^{ik\theta}\int_0^\infty F_n(\rho)J_{k-n}(\rho r_0)J_k(\rho r)\rho d\rho. \quad (59)$$

It can be observed from Eq. (59) that the kth Fourier coefficient of the shifted function is

$$\left[f(\vec{r}-\vec{r}_0)\right]_k(r)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} i^n e^{-i(k-n)\theta_0}\int_0^\infty F_n(\rho)J_{k-n}(\rho r_0)J_k(\rho r)\rho d\rho. \quad (60)$$

For reasons that will become apparent later, we may rewrite the indices in the previous equation as

$$\left[f(\vec{r}-\vec{r}_0)\right]_k(r)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} i^{k-n} e^{-in\theta_0}\int_0^\infty F_{k-n}(\rho)J_n(\rho r_0)J_k(\rho r)\rho d\rho. \quad (61)$$

This gives the Fourier coefficients of the shifted function in r space in terms of the unshifted frequency-space coefficients $F_n(\rho)$.

If the values of the shifted coefficients are desired in terms of the original unshifted coefficients $f_n(r)$, where $f(\vec{r})=\sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}$, then the definition of the Fourier-space coefficients from Eq. (20) can be used:

$$F_n(\rho)=2\pi\, i^{-n}\int_0^\infty f_n(r)J_n(\rho r)\, rdr=2\pi\ i^{-n}\mathbb{H}_n\left\{f_n(r)\right\} \quad (62)$$

and substituted into Eq. (60), along with a change in the order of integration so that

$$f(\vec{r}-\vec{r}_0)=\sum_{k=-\infty}^{\infty} e^{ik\theta}\sum_{n=-\infty}^{\infty} e^{-i(k-n)\theta_0}\int_0^\infty f_n(u)S_n^k(u,r,r_0)udu, \quad (63)$$

where $S_n^k(u,r,r_0)$ is defined as a shift-type operator given by the integral of the triple-Bessel product

$$S_n^k(u,r,r_0)=\int_0^\infty J_n(\rho u)J_{k-n}(\rho r_0)J_k(\rho r)\rho d\rho. \quad (64)$$

Thus, the Fourier coefficients of the shifted function in terms of the unshifted function coefficients are given by

$$\left[f(\vec{r}-\vec{r}_0)\right]_k = e^{-ik\theta_0}\sum_{n=-\infty}^{\infty} e^{in\theta_0}\int_0^\infty f_n(u)S_n^k(u,r,r_0)\,udu. \tag{65}$$

The previous equation describes the shift operation in terms of Fourier polar coordinates. In other words, Eq. (65) provides the rule for finding the Fourier coefficients for the shifted function $\left[f(\vec{r}-\vec{r}_0)\right]_k$, if the original unshifted coefficients $f_n(r)$ are known. It can be seen from Eq. (65) that without going into the Fourier domain, this is given by multiplication of the original coefficients $f_n(r)$ by the shift operator $S_n^k(u,r,r_0)$, integrating and then summing over all possible values of n. Alternatively, the rule as given by Eq. (61) is to multiply the Fourier coefficients $F_{k-n}(\rho)$ by $i^{-n}e^{-in\theta_0}J_n(\rho r_0)$ (the shift), sum over all n, and then transform back to spatial coordinates. Interestingly, we need to multiply by all the possible $i^{-n}e^{-in\theta_0}J_n(\rho r_0)$ (all values of n) and integrate to get a proper shift. Recalling that a shift in polar coordinates really consists of a translation by r_0 and then a rotation by θ_0, we can interpret Eq. (61) as

$$\begin{aligned}&\left[f(\vec{r}-\vec{r}_0)\right]_k(r)\\ &= \frac{1}{2\pi}\underbrace{\sum_{n=-\infty}^{\infty}}_{\text{sum over all } n} i^{k-n}\underbrace{e^{-in\theta_0}}_{\text{rotation by } \theta_0}\underbrace{\int_0^\infty F_{k-n}(\rho)\underbrace{J_n(\rho r_0)}_{\text{translation by } r_0} J_k(\rho r)\,\rho d\rho}_{\text{transform back to spatial coordinates}}.\end{aligned} \tag{66}$$

Hence $i^{-n}J_n(\rho r_0)\,e^{-in\theta_0}$ is the kernel of the shift operation (translation + rotation) and, in fact, Eq. (66) is a series convolution between $i^{-n}J_n(\rho r_0)\,e^{-in\theta_0}$ and $F_{k-n}(\rho)$, which is shown below.

In the special case that $\vec{r}_0 = 0$, then the definition of the shift operator, combined with the fact that $J_n(0) = \delta_{n0}$, implies that in this case the shift operator becomes

$$S_n^k(u,r,0) = \int_0^\infty \delta_{nk}J_n(\rho u)J_n(\rho r)\rho d\rho = \frac{1}{u}\delta(u-r)\delta_{nk}, \tag{67}$$

so that Eq. (65) returns the correct unshifted value of f_n, as it should.

7.1. Fourier Domain Coefficients of the Shifted Function

The corresponding coefficients for the 2D Fourier transform can be found. If we define $h(\vec{r}) = f\left(\vec{r} - \vec{r}_0\right)$, then $H(\vec{\omega}) = e^{-i\vec{\omega}\cdot\vec{r}_0}F(\vec{\omega})$, and the Fourier coefficients $H_n(\rho)$ are sought. Thus $H(\vec{\omega})$ is defined as a product of the complex exponential and $F(\vec{\omega})$, and it was previously shown that multiplication in the Fourier series domain implies convolution of the coefficients of the respective series, with the convolution operation defined in Eq. (53). Since the coefficients for the complex exponential are given in Eq. (9), it follows that

$$H_k(\rho) = \left[i^{-k}J_k(\rho r_0)\, e^{-ik\theta_0}\right] * F_k(\rho), \tag{68}$$

or more explicitly,

$$H_k(\rho) = \sum_{n=-\infty}^{\infty} i^{-n}J_n(\rho r_0)\, e^{-in\theta_0}F_{k-n}(\rho), \tag{69}$$

as per the definition of the convolution operation for two Fourier-series coefficients. Converting the coefficients $H_k(\rho)$ into the spatial domain to get $h_k(r) = \left[f\left(\vec{r} - \vec{r}_0\right)\right]_k(r)$ yields Eq. (61) exactly, confirming the previous derivation.

Substituting $n = k - m$ into Eq. (65) so that the indices more closely resemble Eq. (69), we can write the Fourier pair for the shift rule conceptually as

$$\left[f\left(\vec{r} - \vec{r}_0\right)\right]_k \quad \Leftrightarrow \quad \left[e^{-i\vec{\omega}\cdot\vec{r}_0}F(\vec{\omega})\right]_k = \left[e^{-i\vec{\omega}\cdot\vec{r}_0}\right]_k * F_k(\rho), \tag{70}$$

and more explicitly as

$$\left.\begin{array}{l} \frac{i^k}{2\pi}\int\limits_0^\infty \left\{\left[i^{-k}J_k(\rho r_0)\, e^{-ik\theta_0}\right] * F_k(\rho)\right\} J_k(\rho r)\, \rho d\rho \\ \quad = \sum\limits_{n=-\infty}^{\infty} e^{-in\theta_0} \int\limits_0^\infty f_{k-n}(u) S_{k-n}^k(u, r, r_0)\, u du \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \sum\limits_{n=-\infty}^{\infty} i^{-n}J_n(\rho r_0)\, e^{-in\theta_0}F_{k-n}(\rho) \\ \quad = \left[i^{-k}J_k(\rho r_0)\, e^{-ik\theta_0}\right] * F_k(\rho) \end{array}\right., \tag{71}$$

where

$$S_{k-n}^k(u, r, r_0) = \int\limits_0^\infty J_{k-n}(\rho u) J_n(\rho r_0)\, J_k(\rho r) \rho d\rho. \tag{72}$$

7.2. Rule Summary

The shift rule is summarized mathematically in Eq. (71); this rule was actually derived in two different ways. The procedure can be explained as follows: To find the coefficients of the shifted function in spatial coordinates $\left[f\left(\vec{r}-\vec{r}_0\right)\right]_k(r)$, the steps are as follows:

1. Transform the unshifted function to the Fourier domain to find $F(\vec{\omega})$, meaning find its coefficients $F_n(\rho)$.
2. Multiply the Fourier transform $F(\vec{\omega})$ by the complex exponential $e^{-i\vec{\omega}\cdot\vec{r}_0}$. In coefficients, this means convolve the coefficients $F_n(\rho)$ with the coefficients of $e^{-i\vec{\omega}\cdot\vec{r}_0}$ or $\left[i^{-n}J_n(\rho r_0)\, e^{-in\theta_0}\right] * F_n(\rho)$.
3. Transform these new coefficients back to the regular spatial domain to get $\left[f\left(\vec{r}-\vec{r}_0\right)\right]_k(r)$.

In summary, the rule is to transform to the Fourier domain, "multiply" (convolve) by $e^{-i\vec{\omega}\cdot\vec{r}_0}$, and then transform back. The shift operator can be used to avoid the transformation to the Fourier domain and remain in the spatial domain to perform all calculations.

7.3. The Shift Operator

The shift operator can be evaluated in closed form using results derived in Jackson and Maximon (1972). In their work, closed-form results are found for the triple-product integral given by

$$\int_0^\infty J_{n_1}(k_1\rho)J_{n_2}(k_2\rho)J_{n_3}(k_3\rho)\rho d\rho. \tag{73}$$

Jackson and Maximon (1972) state that the integral in expression (73) is nonzero only if $n_1+n_2+n_3=0$. This last condition is a result of integrating over angular variables and the requirement of a nonzero result. The condition $n_1+n_2+n_3=0$ is, in fact, far *too restrictive*, primarily because $J_{-n}(x)=(-1)^nJ_n(x)$. In fact, our shift operator arises after an integration over the angular variables in Eq. (58), where only the relationship $m+n-k=0$ gives a nonzero result. Therefore, the triple-product integral of interest—namely, the shift operator

$$\begin{aligned} S_n^k(u,r,r_0) &= \int_0^\infty J_n(\rho u)J_{k-n}(\rho r_0)\,J_k(\rho r)\rho d\rho \\ &= (-1)^k\int_0^\infty J_n(\rho u)J_{k-n}(\rho r_0)\,J_{-k}(\rho r)\rho d\rho, \end{aligned} \tag{74}$$

requires no additional restrictions on the indices since the requirement for a nonzero angular integral has already been taken into account.

Alternatively, replacing $J_k(x) = (-1)^k J_{-k}(x)$ in Eq. (74) guarantees that the indices add up to zero.

Closed-form results are given in Jackson and Maximon (1972), to which the interested reader is referred. However, since the ultimate purpose of the shift operator is to multiply with the original Fourier coefficients and then to integrate, presumably this is not a computationally efficient way to compute the shifted coefficients. It is probably easier to compute the convolution in the Fourier domain and then inverse transform to the spatial domain than to attempt a direct calculation and subsequent integration with the shift operator. In other words, it is probably easier to compute

$$\frac{i^k}{2\pi}\int_0^\infty \left\{\left[i^{-k}J_k(\rho r_0)\, e^{-ik\theta_0}\right] * F_k(\rho)\right\} J_k(\rho r)\, \rho d\rho \tag{75}$$

instead of

$$\sum_{n=-\infty}^{\infty} e^{-in\theta_0} \int_0^\infty f_{k-n}(u) S_{k-n}^k(u, r, r_0) u du. \tag{76}$$

8. FULL TWO-DIMENSIONAL CONVOLUTION

The 2D convolution of two functions is defined by

$$h(\vec{r}) = f(\vec{r}) ** g(\vec{r}) = \int_{-\infty}^{\infty} g(\vec{r}_0) f(\vec{r} - \vec{r}_0) d\vec{r}_0. \tag{77}$$

The double-star notation $**$ is used to emphasize that this is a 2D convolution and to distinguish it from a 1D convolution.

Using the Fourier expansions for g and the shifted version of f given by Eq. (63), the previous equation becomes

$$h(\vec{r}) = \int_0^\infty \int_0^{2\pi} \sum_{m=-\infty}^{\infty} g_m(r_0) e^{im\theta_0} \sum_{k=-\infty}^{\infty} e^{ik\theta} \sum_{n=-\infty}^{\infty} e^{-i(k-n)\theta_0}$$
$$\int_0^\infty f_n(u) S_n^k(u, r, r_0)\ u du\ d\theta_0 r_0 dr_0. \tag{78}$$

The preceding equation can be simplified by performing the integration over the angular variable θ_0 so that $m = k - n$:

$$h(\vec{r}) = 2\pi \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_0^{\infty} \sum_{n=-\infty}^{\infty} \left(\int_0^{\infty} f_n(u) S_n^k(u, r, r_0)\, udu \right) g_{k-n}(r_0)\; r_0 dr_0. \quad (79)$$

Equation (79) says that

$$\left(f ** g\right)_k = 2\pi \underbrace{\int_0^{\infty} \overbrace{\sum_{n=-\infty}^{\infty} \underbrace{\left(\int_0^{\infty} f_n(u) S_n^k(u, r, r_0)\, udu \right)}_{\substack{\text{kernel of linear "shift" for } f \\ \text{roughly} = f_n(r - r_0)}} g_{k-n}(r_0)}^{\text{convolution over coefficients, } n}\; r_0 dr_0}_{\text{convolution over } r}, \quad (80)$$

which we can loosely interpret as a combination of (1) "shift" for f, (2) convolution over the coefficients—that is, series convolution, and then (3) another convolution over the r dimension. This is a nice insight but serves little to aid in Fourier calculations. To produce a computationally relevant result, we proceed by simplifying Eq. (80) by using the definition of $S_n^k(u, r, r_0)$ and rearranging to give

$$h(\vec{r}) = \sum_{k=-\infty}^{\infty} 2\pi e^{ik\theta} \int_0^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \underbrace{\int_0^{\infty} g_{k-n}(r_0) J_{k-n}(\rho r_0) r_0 dr_0}_{\frac{i^{k-n}}{2\pi} G_{k-n}(\rho)} \underbrace{\int_0^{\infty} f_n(u) J_n(\rho u) udu}_{\frac{i^n}{2\pi} F_n(\rho)} \right\} J_k(\rho r) \rho d\rho. \quad (81)$$

With the simplifications in the brackets as shown in Eq. (81), this becomes

$$h(\vec{r}) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} i^k e^{ik\theta} \int_0^{\infty} \left\{ \sum_{n=-\infty}^{\infty} G_{k-n}(\rho) F_n(\rho) \right\} J_k(\rho r) \rho d\rho. \quad (82)$$

Clearly this can be written in the form

$$h(\vec{r}) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} i^k e^{ik\theta} \int_0^{\infty} H_k(\rho) J_k(\rho r) \rho d\rho = \sum_{k=-\infty}^{\infty} h_k(r)\; e^{ik\theta}, \quad (83)$$

where the coefficients $H_k(\rho)$ are the convolution of the two series given by

$$H_k(\rho) = \sum_{n=-\infty}^{\infty} G_{k-n}(\rho)F_n(\rho) = G_k * F_k. \tag{84}$$

The results of the last section showed that the convolution of two sets of Fourier coefficients is equivalent to multiplication of the functions so that $G_k * F_k = (GF)_k$; hence it follows from Eq. (84) that

$$H(\vec{\omega}) = G(\vec{\omega})F(\vec{\omega}), \tag{85}$$

as would be expected from the standard results of Fourier theory and which serves to confirm the accuracy of the development. However, the main result we seek is Eq. (82), which gives the values of $h_k(r)$ in terms of $g_k(r)$ and $f_k(r)$ (or rather the Hankel transform of those) and essentially defines the convolution operation for functions given in Fourier-series forms. This equation defines the convolution operation; therefore, to find the convolution of two functions given in Fourier-series form, one must first find the kth coefficient of the Fourier *transform* of each function—namely, $G_k(\rho)$ and $F_k(\rho)$—and then subsequently convolve the resulting series as per Eq. (84) to get $H_k(\rho)$. The final step is then to inverse Hankel transform the result to finally obtain $h_k(r)$. It is important to note that convolving the two functions $f(\vec{r})$ and $g(\vec{r})$ is *not* equivalent to convolving their series; this, in fact, was shown previously to be equivalent to the multiplication of the functions themselves.

This preceding result can be conceptually summarized as

$$\left(g ** f\right)_k \quad \Leftrightarrow \quad G_k * F_k = (GF)_k, \tag{86}$$

where

$$\left(G_k * F_k\right)(\rho) = \sum_{n=-\infty}^{\infty} G_{k-n}(\rho)F_n(\rho). \tag{87}$$

8.1. Multiplication Revisited

Equations (56) and (55) gave us the multiplication rule—that is, how to find the Fourier coefficients of the products of two functions. We can now show that the Fourier transform of a product is indeed a 2D convolution of their full transforms, $F(\vec{\omega}) ** G(\vec{\omega})$. For $h(\vec{r}) = f(\vec{r})g(\vec{r})$, we restate Eq. (56) and insert Eq. (55) to get

$$H_k(\rho) = 2\pi\, i^{-k} \int_0^{\infty} \left(\sum_{m=-\infty}^{\infty} f_{k-m}(r) g_m(r) \right) J_k(\rho r)\, r dr. \tag{88}$$

Now each Fourier coefficient can be written in the form of

$$f_n(r) = \frac{i^n}{2\pi}\int_0^\infty F_n(\rho)J_n(\rho r)\,\rho d\rho \tag{89}$$

so that Eq. (88) becomes

$$H_k(\rho) = 2\pi\; i^{-k}\int_0^\infty\left(\sum_{m=-\infty}^{\infty}\frac{i^{k-m}}{2\pi}\int_0^\infty F_{k-m}(u)J_{k-m}(ur)\,udu\;\frac{i^m}{2\pi}\int_0^\infty G_m(v)J_m(vr)\,vdv\right)J_k(\rho r)\,rdr. \tag{90}$$

Recognizing the presence of the shift operator after a change of the order of integration, the preceding equation becomes

$$H_k(\rho) = \frac{1}{2\pi}\int_0^\infty\sum_{m=-\infty}^{\infty}\int_0^\infty F_{k-m}(u)S^m_{k-m}(u,\rho,v)udu\int_0^\infty G_m(v)\,vdv. \tag{91}$$

Comparing Eq. (91) with the form of the Fourier coefficients of a convolution as given in Eq. (80), we see that Eq. (91) states that

$$H_k(\rho) = \frac{1}{(2\pi)^2}\left[F(\vec{\omega})\; {*}{*}\, G(\vec{\omega})\right]_k, \tag{92}$$

or simply

$$H(\vec{\omega}) = \frac{1}{(2\pi)^2}F(\vec{\omega})\,{*}{*}\,G(\vec{\omega}). \tag{93}$$

This is, of course, the result we should obtain and the mathematics show that this result does indeed hold.

9. SPECIAL CASE: SPATIAL SHIFT OF RADIALLY SYMMETRIC FUNCTIONS

We now turn our attention to the special case of radially symmetric functions, those that are only a function of r. For a radially symmetric function, we recall that the 2D forward and inverse transforms are essentially the Hankel transform of order zero. So from Eq. (12) the forward transform is given by

$$F(\rho) = \mathbb{F}_{2D}\left\{f(r)\right\} = 2\pi\int_0^\infty f(r)J_0(\rho r)rdr, \tag{94}$$

and the inverse transform is given by

$$f(\vec{r}) = f(r) = \mathbb{F}_{2D}^{-1}\left\{f(\rho)\right\} = \frac{1}{2\pi}\int_0^\infty F(\rho)J_0(\rho r)\rho d\rho. \tag{95}$$

To obtain the correct expression for the shift of a radially symmetric function, we define the shifted function as before from Eq. (57) as

$$\begin{aligned} f\left(\vec{r}-\vec{r}_0\right) &= \mathbb{F}^{-1}\left\{e^{-i\vec{\omega}\cdot\vec{r}_0}F(\rho)\right\} \\ &= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\sum_{m=-\infty}^{\infty} i^{-m}J_m(\rho r_0)\, e^{-im\theta_0}e^{im\psi}F(\rho) \\ &\quad\times\sum_{k=-\infty}^{\infty} i^k J_k(\rho r)\, e^{ik\theta}e^{-ik\psi}\; d\psi\;\rho d\rho \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta_0)}\int_0^\infty F(\rho)J_k(\rho r_0)J_k(\rho r)\rho d\rho. \end{aligned} \tag{96}$$

Thus, the Fourier coefficients of the shifted radially symmetric function are given by

$$\left[f\left(\vec{r}-\vec{r}_0\right)\right]_k = \frac{1}{2\pi}e^{-ik\theta_0}\int_0^\infty F(\rho)J_k(\rho r_0)J_k(\rho r)\rho d\rho. \tag{97}$$

Although $f(r)$ is radially symmetric and has only the $n = 0$ term in its Fourier series, a properly shifted $f\left(\vec{r}-\vec{r}_0\right)$ is *not* radially symmetric and so needs the full complement of entries in its Fourier series. By a "proper shift," we mean the shift to imply the linear translation in r and angular rotation in θ. It is equally important to note that this applies even if the new "center" $\vec{r}_0$ to which the function f has been shifted is located on the radial axis so that $\theta_0 = 0$. Equation (96) is the proper full shift (translation and rotation) and demonstrates that a function of r becomes a function of r and θ once shifted away from the origin. Hence $f(\vec{r}-\vec{r}_0)$ is the shift so that the radially symmetric $f(r)$ is now centered at $\vec{r}_0$. This is *not* the same as $f(r - r_0)$, which *is* a radially symmetric function. To illustrate these points, consider the Gaussian "dome" of $f(r) = e^{-\left(\frac{r^2}{4}\right)}$, $f(r-4) = e^{-\left(\frac{(r-4)^2}{4}\right)}$, and $f\left(\vec{r}-\vec{r}_0\right)$ as $f(r) = e^{-\left(\frac{r^2}{4}\right)}$ shifted so that is it centered at $\vec{r}_0 = (r = 4, \theta = 0)$. Clearly, the first two of these will be radially symmetric while the third is not.

The interpretation of Eq. (97) is as follows. We want to go from the original radially symmetric $f(r)$ to the kth coefficient $\left[f\left(\vec{r}-\vec{r}_0\right)\right]_k$ of the shifted function. A "shift" in 2D polar space is really a combination of a linear shift in the radial variable and a rotation by the angular variable. The rotation by the angle θ_0 is handled by a multiplication by $e^{-ik\theta_0}$, as is typical in polar coordinates. Completing the linear translation portion of the shift now remains. The "rule" here is to take the Fourier transform of $f(r)$—that is, $F(\rho)$—and multiply it by $J_k(\rho r_0)$, which takes care of the linear shift by r_0 of the kth coefficient. The shifted result is now transformed back into the usual r space by an inverse Hankel transform of order k. In essence, the linear shift for functions that start with radial symmetry is a relatively simple operation as the translation from r to $r-r_0$ is equivalent to a multiplication by $J_k(\rho r_0)$ in Fourier space, much like a shift in Cartesian regular space is the equivalent of a multiplication by $e^{-i\omega\cdot r_0}$ in Cartesian Fourier space. When the function to be shifted is not initially radially symmetric, this multiplication operation to obtain the linear shift in the radial variable becomes a more complicated series-convolution operation, as previously demonstrated in Eq. (69).

9.1. Fourier Transform of the Shifted Radially Symmetric Function

The corresponding coefficients of the 2D Fourier transform can also be found. If we define $h(\vec{r}) = f\left(r-\vec{r}_0\right)$, then from the definition of a Fourier expansion, we know that the form of the expansion must be

$$h(\vec{r}) = f\left(\vec{r}-\vec{r}_0\right) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\, i^k\, e^{ik\theta} \int_0^{\infty} H_k(\rho) J_k(\rho r)\, \rho d\rho. \tag{98}$$

Comparison of Eqs. (98) and (96) gives immediately that

$$H_k(\rho) = i^{-k} e^{-ik\theta_0} F(\rho) J_k\left(\rho r_0\right), \tag{99}$$

and thus for a radially symmetric function, the shift property is to multiply the radially symmetric Fourier transform by the kernel for $e^{-i\vec{\omega}\cdot\vec{r}_0}$. In fact, Eq. (99) implies that

$$\begin{aligned} H\left(\vec{\omega}\right) &= \sum_{k=-\infty}^{\infty} H_k(\rho) e^{ik\theta} \\ &= F(\rho) \sum_{k=-\infty}^{\infty} i^{-k} e^{-ik\theta_0} J_k\left(\rho r_0\right) e^{ik\theta} = F(\rho) e^{-i\vec{\omega}\cdot\vec{r}_0}, \end{aligned} \tag{100}$$

which is exactly the result we should obtain, verifying the development.

As previously mentioned, it is intuitively obvious from a 2D polar-coordinate view of the world that a radially symmetric function is no longer radially symmetric once shifted away from the origin. Thus, since the function is no longer radially symmetric, the 2D Fourier transform is no longer equivalent to a zeroth-order Hankel transform as was the case for the radially symmetric function. In fact, Eq. (100) clearly shows that it is the $e^{-i\vec{\omega}\cdot\vec{r}_0}$ term that destroys the radial symmetry of the 2D Fourier transform. In particular, $e^{-i\vec{\omega}\cdot\vec{r}_0} = e^{-ir_0\rho\cos(\psi-\theta_0)}$ is a function of magnitude 1 but with a non–radially symmetric phase. Continuing with this argument, if Hankel transforms are considered to be merely transforming a function of a variable r to another function of a variable ρ, without consideration of its role in the 2D perspective, it would be reasonable to look for a Hankel analog of the shift and convolution theorems of standard Fourier theory. Clearly, a 1D shift/convolution rule for Hankel transforms does not exist and it is only in considering the matter from the 2D perspective that the reason one cannot exist becomes obvious.

It is interesting to note that since $J_k(0) = 0$ for $k \neq 0$ and $J_0(0) = 1$, Eq. (96) for $r_0 = 0$ becomes

$$\begin{aligned} f(r) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta_0)}\int_0^\infty F(\rho)J_k(0)J_k(\rho r)\rho d\rho \\ &= \frac{1}{2\pi}\int_0^\infty F(\rho)J_0(\rho r)\rho d\rho, \end{aligned} \tag{101}$$

as it should, verifying the development given herein.

9.2. Rule Summary

To summarize the preceding discussion, if $f(r)$ is a radially symmetric function, the shifted function $f\left(r-\vec{r}_0\right)$ is not radially symmetric and its Fourier coefficients are given by

$$\left[f\left(\vec{r}-\vec{r}_0\right)\right]_k = \frac{1}{2\pi}e^{-ik\theta_0}\int_0^\infty F(\rho)J_k(\rho r_0)J_k(\rho r)\,\rho d\rho. \tag{102}$$

Writing $h(\vec{r}) = f\left(r-\vec{r}_0\right)$, then the Fourier coefficients in Fourier space are given by

$$H_k(\rho) = i^{-k}e^{-ik\theta_0}F(\rho)J_k\left(\rho r_0\right), \tag{103}$$

which establishes the Fourier pair of coefficients

$$\left[f(\vec{r}-\vec{r}_0)\right]_k = \frac{1}{2\pi}e^{-ik\theta_0}\int\limits_0^\infty F(\rho)J_k(\rho r_0)J_k(\rho r)\rho d\rho \quad \Leftrightarrow \quad i^{-k}e^{-ik\theta_0}F(\rho)J_k(\rho r_0). \tag{104}$$

9.3. Shift of a Radially Symmetric Function in Terms of the Shift Operator

As previously done for the shift of general function, the coefficients of the shifted function in terms of the original function can be found without reference to its transform. We use the definition of the forward transform and switch the order of integration, so that Eq. (96) becomes

$$f(\vec{r}-\vec{r}_0) = \sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta_0)}\int\limits_0^\infty f(u)\int\limits_0^\infty J_0(\rho u)\, J_k(\rho r_0)J_k(\rho r)\, \rho d\rho\, u du. \tag{105}$$

For a slightly more compact notation, Eq. (105) can be written using Eq. (64) as

$$f(\vec{r}-\vec{r}_0) = \sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta_0)}\int\limits_0^\infty f(u)\, S_0^k(u,r,r_0)\, u du, \tag{106}$$

which gives the shifted coefficients in terms of the original unshifted coefficients:

$$\left[f(\vec{r}-\vec{r}_0)\right]_k = e^{-ik\theta_0}\int\limits_0^\infty f(u)\, S_0^k(u,r,r_0)\, u du. \tag{107}$$

10. SPECIAL CASE: 2D CONVOLUTION OF TWO RADIALLY SYMMETRIC FUNCTIONS

Two radially symmetric functions are functions for which both Fourier series expansions include only the $n = 0$ term. The 2D convolution of the two radially symmetric functions is defined as

$$h(\vec{r}) = f(\vec{r}) ** g(\vec{r}) = \int\limits_{-\infty}^{\infty} g(\vec{r}_0) f(\vec{r}-\vec{r}_0)\, d\vec{r}_0, \tag{108}$$

where it is emphasized that *the integration is over all* $\vec{r}_0$, which includes all possible values of *radial and angular* variables. In other words, Eq. (108) must be properly interpreted as a 2D convolution along with Eq. (109) as

$$\begin{aligned} h(\vec{r}) &= f(\vec{r}) ** g(\vec{r}) \\ &= \int_0^\infty \int_0^{2\pi} g(r_0) \sum_{k=-\infty}^{\infty} e^{-ik\theta_0} e^{ik\theta} \int_0^\infty f(u)\ S_0^k(u,r,r_0)\ u du\, d\theta_0 r_0 dr_0 \end{aligned} \tag{109}$$

and *not* interpreted as

$$\int_0^\infty g(r_0) f(r - r_0) dr_0, \tag{110}$$

which would be a 1D convolution. Equation (109) can be simplified by performing the integration over the angular variable, which is nonzero only if $k = 0$, so that

$$f(\vec{r}) ** g(\vec{r}) = f(r) ** g(r) = 2\pi \int_0^\infty g(r_0) \int_0^\infty f(u)\ S_0^0\,(u,r,r_0) u du\, r_0 dr_0 \tag{111}$$

with

$$S_0^0\,(u,r,r_0) = \int_0^\infty J_0(\rho u) J_0\,(\rho r_0)\, J_0(\rho r) \rho d\rho, \tag{112}$$

so that Eq. (111) is barely recognizable as a convolution operation. The double-star $**$ notation has been used to highlight the fact that a 2D convolution is being taken. In fact, the proper, correct definition of the convolution, Eq. (109) can be interpreted from Eq. (111) as

$$f(\vec{r}) ** g(\vec{r}) = f(r) ** g(r) = \int_0^\infty g(r_0)\ \Phi(r - r_0)\ r_0 dr_0, \tag{113}$$

where

$$\begin{aligned} \Phi(r - r_0) &= 2\pi \int_0^\infty f(u)\ S_0^0\,(u,r,r_0) u du = \int_0^\infty F(\rho) J_0\,(\rho r_0)\ J_0(\rho r) \rho d\rho \\ &= \int_0^{2\pi} f\left(\vec{r} - \vec{r}_0\right) d\theta_0. \end{aligned} \tag{114}$$

Thus, we observe that the definition of convolution that we are tempted to write, as given by Eq. (110), is *almost* correct; what was necessary was to shift the function f from $\vec{r}$ to $\vec{r}_0$ (thus destroying radial symmetry) and *then to integrate the resulting shifted function over all angular variables.* The unshifted function g is still radially symmetric and thus is not affected by the integration over the angular variable; the final result is the form given in Eq. (113).

From Eqs. (83) and (84) it follows that

$$h(r) = f(r) ** g(r) = \frac{1}{2\pi} \int_{0}^{\infty} G_0(\rho)F_0(\rho)J_0(\rho r)\rho d\rho \tag{115}$$

since the interpretation of Eq. (84) in this case is as

$$H_k(\rho) = \sum_{n=-\infty}^{\infty} G_{k-n}(\rho)F_n(\rho) = \begin{cases} G_0(\rho)F_0(\rho) & k = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{116}$$

In the previous two equations, $G_0(\rho)$ and $F_0(\rho)$ are the zeroth-order coefficients in the expansions for $F(\vec{\omega})$ and $G(\vec{\omega})$ themselves. Equation (116) can also be obtained by observing that the integration over the angular variable in Eq. (109) forces k to be zero. Thus, the 2D convolution of two radially symmetric functions yields another radially symmetric function, as can be seen from Eq. (115). Moreover, by using the proper definition of a 2D convolution instead of using the tempting definition of a 1D convolution, the well-known relationship between convolutions in one domain leading to multiplication in the other domain is preserved—namely, that

$$f(\vec{r}) ** g(\vec{r}) = f(r) ** g(r) \Leftrightarrow F(\rho)G(\rho). \tag{117}$$

The key point to this relationship is the *proper definition of the convolution as a 2D convolution.*

11. SPECIAL CASE: CONVOLUTION OF A RADIALLY SYMMETRIC FUNCTION WITH A NONSYMMETRIC FUNCTION

We consider the special case of the convolution of two functions where only one of the functions is radially symmetric. This case holds particular importance for certain physical problems involving nonhomogeneous partial differential equations that can be solved by a convolution of the forcing function with the system's Green's functions. For a large class of problems, the Green's function is radially symmetric while the forcing function is typically not. This is discussed further in a later sections; for now, we consider the mathematics of the convolution without considering any applications, even though this special case has many applications. So, to set up the problem, we are seeking the 2D convolution of two functions defined as

$$h(\vec{r}) = f(\vec{r}) ** g(\vec{r}) = f(r) ** g(\vec{r}) = \int_{-\infty}^{\infty} g(\vec{r}_0) f(\vec{r} - \vec{r}_0)\, d\vec{r}_0, \tag{118}$$

where we assume that only $f(\vec{r}) = f(r)$ is radially symmetric and the notation in Eq. (118) is to remind the reader that a full radial and angular shift is required before integration over *all* possible shifts. Using the usual Fourier expansion for g and the shifted version of f given by Eq. (96), the previous equation becomes

$$h(\vec{r}) = \int_0^{\infty}\int_0^{2\pi} \sum_{m=-\infty}^{\infty} g_m(r_0) e^{im\theta_0} \left\{ \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta_0)} \int_0^{\infty} F(\rho) J_k(\rho r_0) J_k(\rho r) \rho d\rho \right\} d\theta_0 r_0 dr_0. \tag{119}$$

Integrating over θ_0 gives $k = m$ for nonzero results so that

$$h(\vec{r}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_0^{\infty} g_k(r_0) \left\{ \int_0^{\infty} F(\rho) J_k(\rho r_0) J_k(\rho r) \rho d\rho \right\} r_0\, dr_0. \tag{120}$$

This can be written as

$$h(\vec{r}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_0^{\infty} g_k(r_0) \phi_k(r - r_0) r_0\, dr_0, \tag{121}$$

where we use the definition

$$\phi_k(r - r_0) = \begin{cases} \int_0^{\infty} F(\rho) J_k(\rho r_0) J_k(\rho r) \rho d\rho & r_0 \neq 0 \\ \int_0^{\infty} F(\rho) J_k(\rho r) \rho d\rho & r_0 = 0 \end{cases}. \tag{122}$$

The definition of Eq. (122) with a specific version given for the case $r_0 = 0$ is necessary as otherwise $J_k(0) = 0$ when $k \neq 0$. The notation of $\phi_k(r - r_0)$ was used in the definition in Eq. (122) instead of $\phi_k(r, r_0)$ or $f_k(r - r_0)$ since this function essentially arises as a result of shifting $f(r)$ to $f(\vec{r} - \vec{r}_0)$ and then integrating the result over all possible angles θ_0.

The preceding result bears some discussion as it is extremely important. First, the similarity between the $\phi_k(r - r_0)$ of Eq. (122) and the $\Phi\,(r - r_0)$ of Eq. (114) is emphasized. Those two functions have quite similar definitions, and both arise as a result of the shift in a radially symmetric function and then integrating over all possible values of the rotation shift, θ_0. The principal difference between them is that $\Phi\,(r - r_0)$ involves only the zeroth-order term. This is because $\Phi\,(r - r_0)$ is the application of the shift rule as part of a convolution of a radially symmetric function with another radially symmetric function and thus no higher-order terms are required to compute the full convolution between them. In contrast, $\phi_k(r - r_0)$ appears as the application of the shift rule to the radially symmetric function, which is then used in a convolution with a non–radially symmetric function. Since the function that we are convolving is now *not* radially symmetric, it will have higher-order terms in its expansion and the function with radial symmetry must now produce those corresponding terms for a convolution to make sense. If all the functions involved have radial symmetry, then all Fourier transforms become equivalent to zeroth-order Hankel transforms and a kth-order Hankel transform is never required. As soon as one function loses radial symmetry, a kth-order Hankel transform becomes necessary to define a full Fourier transform, even if one of the functions being convolved is radially symmetric.

Second, we are essentially defining a new function (or rather the Fourier coefficients of this new function) as the kth-order inverse Hankel transform of $F(\rho)$:

$$\phi_k(r) = \int_0^\infty F(\rho) J_k(\rho r)\rho\, d\rho, \tag{123}$$

and the only role of this new set of coefficients is to allow a radially symmetric function $F(\rho)$ to convolve with a non–radially symmetric function. In essence, the function's coefficients are as defined in Eq. (123) and then if a shift in the function is required, we use the Hankel shift rule of multiplying by $J_k(\rho r_0)$ before taking the inverse Hankel transform so that the shifted coefficients are

$$\phi_k(r - r_0) = \int_0^\infty F(\rho) J_k\,(\rho r_0)\, J_k(\rho r)\rho\, d\rho \quad r_0 \neq 0. \tag{124}$$

As previously mentioned, the set of functions is defined as in Eq. (123), with the Hankel shift rule given as in Eq. (124) because $r_0 = 0$ in Eq. (124) would yield only the zeroth-order term since $J_k(0) = 0$ for $k \neq 0$ and we would be back to using only the zeroth-order term.

If a radially symmetric function appears by itself, a kth-order Hankel transform is never required. However, once implicated with an operation involving nonsymmetric functions, it becomes necessary. Interestingly enough, with this new, powerful version of the radially symmetric function, Eq. (120) can be interpreted as

$$h(\vec{r}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_{0}^{\infty} g_k(r_0)\phi_k\left(r - r_0\right) \; r_0 dr_0 = \sum_{k=-\infty}^{\infty} e^{ik\theta} \left(g_k *_{1D} \phi_k\right)(r) \quad (125)$$

so that the coefficients of $h(\vec{r})$ are the simple 1D convolution of the g_k and ϕ_k:

$$h_k(r) = \left(g_k *_{1D} \phi_k\right)(r) = \int_{0}^{\infty} g_k(r_0)\phi_k\left(r - r_0\right) r_0 dr_0. \quad (126)$$

It is emphasized that the relationship is the usual 1D convolution, hence the use of the notation $*_{1D}$ instead of writing $g_k * \phi_k$, which would actually imply a *series* convolution.

11.1. In Terms of the Fourier Transforms

The preceding results can be interpreted completely in the Fourier domain without resorting to the spatial domain. Following the procedure for the functions without radial symmetry and using the fact that radially symmetric functions have only a zeroth-order term in their expansions, we can proceed to write the 2D convolution of a radially symmetric function with one that has no radial symmetry from Eq. (82) as

$$h(\vec{r}) = f(r) ** g(\vec{r}) = \sum_{k=-\infty}^{\infty} \frac{i^k}{2\pi} e^{ik\theta} \int_{0}^{\infty} G_k(\rho)F(\rho)J_k(\rho r)\rho d\rho. \quad (127)$$

Clearly this can be written in the form

$$h(\vec{r}) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} i^k e^{ik\theta} \int_{0}^{\infty} H_k(\rho)J_k(\rho r) \, \rho d\rho = \sum_{k=-\infty}^{\infty} h_k(r) \, e^{ik\theta}, \quad (128)$$

where

$$H_k(\rho) = G_k(\rho)F(\rho). \quad (129)$$

Comparing this with Eq. (84) when the two functions were not radially symmetric, we see that the step of convolving the two series has been eliminated—or rather, the convolution of two series when one of them belongs to a spherically symmetric function (and thus consists of only the first term) becomes a simple matter of multiplication.

As a point of interest, note the connection between Eqs. (126) and (129):

$$h_k(r) = \left(g_k *_{1D} \phi_k\right)(r) \Leftrightarrow H_k(\rho) = G_k(\rho)F(\rho). \tag{130}$$

This is yet another example of the convolution/multiplication rule in effect and it holds because (1) a proper 2D convolution was defined and (2) the $\phi_k(r)$ ("shift") version of the inverse Hankel transform of $F(\rho)$ was used instead of naïvely using $f(r)$.

12. CIRCULAR (ANGULAR) CONVOLUTION

For two functions $f(\vec{r}) = f(r,\theta)$ and $g(\vec{r}) = g(r,\theta)$, the notion of a circular or angular convolution can be defined. This is *not* the 2D convolution as previously discussed but rather a convolution over the angular variable only, so that it may be defined as

$$f(\vec{r}) *_\theta g(\vec{r}) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta_0)\, g(r,\theta-\theta_0)\, d\theta_0. \tag{131}$$

Note the notation $*_\theta$ is used to denote the angular convolution. A Fourier relationship can be defined for this operation of angular convolution:

$$\begin{aligned} h(r,\theta) = f(\vec{r}) *_\theta g(\vec{r}) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty} f_n(r)\, e^{jn\theta_0}\right)\left(\sum_{m=-\infty}^{\infty} g_m(r)\, e^{jm(\theta-\theta_0)}\right) d\theta_0 \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f_n(r) g_m(r)\, e^{jm\theta}\frac{1}{2\pi}\int_0^{2\pi} e^{jn\theta_0} e^{-jm\theta_0}\, d\theta_0. \end{aligned} \tag{132}$$

The value of the integral in the preceding equation is $2\pi\ \delta_{mn}$, which results in the following simplification:

$$f(\vec{r}) *_\theta g(\vec{r}) = \sum_{m=-\infty}^{\infty} f_n(r) g_n(r)\, e^{jn\theta}. \tag{133}$$

In other words, the nth coefficient of the *angular* convolution of two functions is simply the product of the nth Fourier coefficient of each of the two functions. Mathematically, if $h(r,\theta) = f(\vec{r}) *_\theta g(\vec{r})$, then $h_n(r) = f_n(r)g_n(r)$. This is also the same result as obtained with the Fourier series of 1D periodic functions.

13. RADIAL CONVOLUTION

Similar to the definition of angular or circular convolution, we define the notion of a radial convolution as

$$h(\vec{r}) = f(\vec{r}) *_r g(\vec{r}) = \int_0^\infty g(r_0,\theta)f(r-r_0,\theta)r_0 dr_0. \tag{134}$$

Such convolutions are less often seen than their 2D or angular counterparts; however, they are included here for the sake of completeness. Using Eq. (63) for only the radially shifted function yields

$$h(r,\theta) = \int_0^\infty \left(\sum_{m=-\infty}^{\infty} g_m(r_0)\, e^{im\theta}\right)\left(\sum_{k=-\infty}^{\infty} e^{ik\theta} \sum_{n=-\infty}^{\infty} \int_0^\infty f_n(u)S_n^k(u,r,r_0)\; udu\right) r_0 dr_0. \tag{135}$$

Using the result for the product of two series so that the resulting Fourier coefficients are convolved, $f_k * g_k \equiv \sum_{m=-\infty}^{\infty} f_m(r)g_{k-m}(r)$, gives

$$\begin{aligned} h(r,\theta) &= \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_0^\infty \sum_{m=-\infty}^{\infty} g_{k-m}(r_0) \sum_{n=-\infty}^{\infty} \int_0^\infty f_n(u)S_n^m\,(u,r,r_0)\, udu\; r_0 dr_0 \\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_0^\infty \sum_{m=-\infty}^{\infty} g_{k-m}(r_0) \sum_{n=-\infty}^{\infty} \frac{i^n}{2\pi} \int_0^\infty F_n(\rho)J_{m-n}(\rho r_0)J_m(\rho r)\rho d\rho\; r_0 dr_0. \end{aligned} \tag{136}$$

Let us define

$$G_{k-m}^{m-n}(\rho) = \int_0^\infty g_{k-m}(r_0)J_{m-n}(\rho r_0)\; r_0 dr_0, \tag{137}$$

which is the $(m-n)$th-order Hankel transform of the $(k-m)$th Fourier coefficient. With this definition, Eq. (136) becomes

$$h(r,\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{i^n}{2\pi} \int_0^\infty G_{k-m}^{m-n}(\rho)F_n(\rho)J_m(\rho r)\rho d\rho. \tag{138}$$

This is the closest we can come to a convolution theorem for the radial convolution. Without the shift and integration over the angular portion of the function, the result is awkward because the order of the Hankel transform of one of the functions does not always correspond to the order of the Fourier coefficient as seen by the definition in Eq. (137). The angular shift and integration portion of the full 2D convolution effectively eliminates all these cross terms to yield the nice result of the full 2D convolution, as previously discussed.

14. PARSEVAL RELATIONSHIPS

A Parseval relationship is important as it deals with the "power" of a signal or function in the spatial and frequency domains. As previously mentioned, it is known that we can write

$$f(\vec{r}) = f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta}. \tag{139}$$

It is noted that in polar coordinates if $\vec{r} = (r,\theta)$, then $-\vec{r} = (r,\theta + \pi)$. Hence

$$\begin{aligned}\overline{f(-\vec{r})} = \overline{f(r,\theta+\pi)} &= \sum_{n=-\infty}^{\infty} \overline{f_n(r)e^{jn(\theta+\pi)}} \\ &= \sum_{n=-\infty}^{\infty} \overline{f_n(r)}\,(-1)^n e^{-jn\theta} = \sum_{n=-\infty}^{\infty} \overline{f_{-n}(r)}\,(-1)^n e^{jn\theta},\end{aligned} \tag{140}$$

where the overbar indicates a complex conjugate. Thus, if we define a function $g(\vec{r}) = \overline{f(-\vec{r})}$, its coefficients are given by $g_n(r) = \overline{f_{-n}(r)}\,(-1)^n$. To derive the Parseval relationship, we evaluate that convolution of two functions $f(\vec{r})$ and $g(\vec{r})$ with $g(\vec{r}) = \overline{f(-\vec{r})}$ and where the convolution is evaluated at $\vec{r} = \vec{0}$, implying $r = 0,\ \theta = 0$. In other words, we evaluate $(f * g)$ as given by Eq. (77) at $\vec{r} = \vec{0}$. This is done by using Eq. (81), along with $J_{-n}(x) = (-1)^n J_n(x)$ and $J_n(0) = \delta_{n0}$ so that

$$\begin{aligned}&\int_{-\infty}^{\infty} g(-\vec{r}_0) f(\vec{r}_0) d\vec{r}_0 \\ &= 2\pi \int_0^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \int_0^{\infty} g_{-n}(r_0) J_{-n}(\rho r_0) r_0 dr_0 \int_0^{\infty} f_n(u) J_n(\rho u) u du \right\} \rho d\rho,\end{aligned} \tag{141}$$

which with the given choice of function for g becomes

$$\int_{-\infty}^{\infty} \overline{f(\vec{r}_0)} f(\vec{r}_0)\, d\vec{r}_0 = 2\pi \int_0^{\infty} \sum_{n=-\infty}^{\infty} \underbrace{\int_0^{\infty} \overline{f_n(r_0)}\, J_n(\rho r_0) r_0 dr_0}_{\frac{i^{-n}}{2\pi}\overline{F_n(\rho)}} \underbrace{\int_0^{\infty} f_n(u) J_n(\rho u) u du}_{\frac{i^n}{2\pi} F_n(\rho)} \rho d\rho. \tag{142}$$

The preceding equation furnishes the desired Parseval relationship as

$$\int_{-\infty}^{\infty} \left|f(\vec{r})\right|^2 d\vec{r} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} |F_n(\rho)|^2 \rho d\rho. \tag{143}$$

Following this same procedure and evaluating the convolution of any two well-behaved functions $f(\vec{r})$ and $\overline{g(-\vec{r})}$ at $\vec{0}$ yields the generalized Parseval relationship:

$$\int_{-\infty}^{\infty} \overline{g(\vec{r})} f(\vec{r}) d\vec{r} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \overline{G_n(\rho)} F_n(\rho) \rho d\rho. \tag{144}$$

Furthermore, from the result of multiplication and observing from Eq. (140) that the coefficients of $\overline{g(\vec{r})}$ are $\overline{g_{-n}(r)}$, we have that

$$f(\vec{r})\overline{g(\vec{r})} = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_m(r) \overline{g_{-(k-m)}(r)}\, e^{ik\theta}. \tag{145}$$

Integrating both sides of the previous equation over all space gives

$$\int_0^{\infty} f(\vec{r})\overline{g(\vec{r})}\, d\vec{r} = 2\pi \sum_{m=-\infty}^{\infty} \int_0^{\infty} f_m(r)\overline{g_m(r)}\, r dr, \tag{146}$$

which gives another version of the Parseval relationship in Eq. (144) as

$$\sum_{n=-\infty}^{\infty} \int_0^{\infty} f_n(r)\overline{g_n(r)}\, r dr = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} F_n(\rho)\overline{G_n(\rho)} \rho d\rho. \tag{147}$$

Clearly, it then follows from the preceding equation that

$$\sum_{n=-\infty}^{\infty} \int_0^{\infty} \left| f_n(r) \right|^2 r dr = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} |F_n(\rho)|^2 \rho d\rho. \tag{148}$$

This last equation is the second Parseval relationship.

15. THE LAPLACIAN

One of the most powerful applications of Fourier transforms is solving partial differential equations. Indeed, the preceding developments can be applied to simplify any partial differential equation involving the Laplacian. In polar coordinates, the 2D Laplacian takes the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{149}$$

Consider a typical function written in standard 2D polar form:

$$f(\vec{r}) = f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{jn\theta}. \tag{150}$$

Taking the Laplacian of $f(\vec{r})$ gives

$$\begin{aligned} \nabla^2 f(\vec{r}) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \sum_{n=-\infty}^{\infty} f_n(r)\, e^{jn\theta} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} - \frac{n^2 f_n}{r^2} \right) e^{jn\theta}. \end{aligned} \tag{151}$$

Hence for a function written in the form of Eq. (150), the required form of the Laplacian is denoted with ∇_n^2, where this operator is defined by

$$\nabla_n^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{152}$$

To obtain the full 2D Fourier transform, we seek $\mathbb{F}_{2D}\left\{\nabla^2 f(\vec{r})\right\}$, which is given by the series

$$\begin{aligned} &\mathbb{F}_{2D}\left\{\nabla^2 f(\vec{r})\right\} \\ &= \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi} \int_0^{\infty} \left(\frac{d^2 f_n(r)}{dr^2} + \frac{1}{r}\frac{df_n(r)}{dr} - \frac{n^2 f_n(r)}{r^2} \right) J_n(\rho r)\, r dr. \end{aligned} \tag{153}$$

A simple application of integration by parts along with the definition of a Bessel function gives

$$\int_0^\infty \nabla_n^2 f_n(r) J_n(\rho r)\, r dr = -\rho^2 \int_0^\infty f_n(r) J_n(\rho r)\, r dr, \tag{154}$$

so that Eq. (153) becomes

$$\begin{aligned}\mathbb{F}_{2D}\left\{\nabla^2 f(\vec{r})\right\} &= -\rho^2 \sum_{n=-\infty}^{\infty} 2\pi\, i^{-n}\, e^{in\psi} \int_0^\infty f_n(r) J_n(\rho r)\, r dr \\ &= -\rho^2 \sum_{n=-\infty}^{\infty} F_n(\rho)\, e^{in\psi} = -\rho^2 F(\rho,\psi).\end{aligned} \tag{155}$$

Thus in polar coordinates, taking the full 2D Fourier transform of the Laplacian—that is, a Fourier series followed by an nth-order Hankel transform of the Laplacian—yields the original function multiplied by $-\rho^2$. It is emphasized that an nth-order Hankel transform is necessary, in keeping with the definition of the full 2D Fourier transform and to eliminate the $\frac{n^2 f_n(r)}{r^2}$ term that results from taking the $\frac{\partial^2}{\partial\theta^2}$ derivative as part of the Laplacian. Thus, although the original Laplacian operator is not a radially symmetric operator due to the presence of the $\frac{\partial^2}{\partial\theta^2}$ derivative, its Fourier space equivalent is a multiplication by $-\rho^2$, which *is* in fact a radially symmetric operation. This seemingly innocent observation has many potential applications and is the primary motivation behind the section on convolutions between a radially symmetric function and one that is not radially symmetric. In summary, the Fourier pair is given by

$$\nabla_n^2 f_n = \frac{d^2 f_n(r)}{dr^2} + \frac{1}{r}\frac{df_n(r)}{dr} - \frac{n^2 f_n(r)}{r^2} \quad \Leftrightarrow \quad -\rho^2 F_n(\rho). \tag{156}$$

16. APPLICATION TO THE HELMHOLTZ EQUATION

All wave fields governed by the wave equation (such as acoustic waves) lead to the Helmholtz equation once a temporal Fourier transform is used to transform from the time domain to the frequency domain:

$$\nabla^2 u(\vec{r},\omega) + k^2 u(\vec{r},\omega) = -s(\vec{r},\omega). \tag{157}$$

Here, the wave number is $k^2 = \frac{\omega^2}{c_s^2}$, where ω is the (temporal) Fourier frequency variable, that is time, t, Fourier transforms to ω. The temporal (time-to-frequency) Fourier transform is assumed to use the same sign convention as given in Eqs. (5) and (6). Here, $s(\vec{r},\omega)$ is the temporal Fourier transform of the inhomogeneous time- and space-dependent source term for the wave equation. The variable $u(\vec{r},\omega)$ represents a physical variable governed by the wave equation—for example, acoustic pressure. Both $s(\vec{r},\omega)$ and $u(\vec{r},\omega)$are functions of position, $\vec{r}$, and (temporal) frequency, ω. The variable c_s represents the speed of the wave, which for an acoustic wave would be the speed of sound and for an electromagnetic wave would be the speed of light. For wave fields governed by the wave equation, k^2 is a real (and positive) quantity.

Equation (157) can also be used to represent other physical phenomena governed by diffusive waves. (Mandelis, 2001). For example, the equation for a diffuse photon density wave describes the photon density $u(\vec{r},t)$ in a solid due to incident energy intensity $s(\vec{r},t)$ (optical source function). The standard heat equation can also be cast in the form of Eq. (157) once a temporal Fourier transform is taken. In these latter two cases, the wave number is complex, indicating a damped or diffusive wave. For example, the temporal Fourier transform of the heat equation gives a complex wave number $k^2 = -\frac{i\omega}{\alpha}$, where α is the thermal diffusivity of a material, $u(\vec{r},t)$ describes the temperature in the material as a function of time and space, and $s(\vec{r},t)$ is a time- and space-dependent heat source.

Thus, the general Helmholtz form of Eq. (157) can be used to describe several different physical phenomena ranging from the propagation of light or acoustic waves to the heavily damped nature of photonic or thermal waves. The exact form of the wave number in each case indicates the propagation characteristics of a wave with a real k^2 indicating a propagating wave and a complex k^2 indicating a damped wave. Generally, the term *Helmholtz equation* refers to Eq. (157) with a real wave number and a complex k^2 can be referred to as *pseudo-Helmholtz equation*. However, this terminology is rather cumbersome and we prefer to refer to Eq. (157) as a *Helmholtz equation,* regardless of whether k^2 is real or complex. Much of the mathematics can be made to yield the same results for a real or complex wave number if a proper choice of sign convention for k is chosen. For example, $e^{\pm ikx}$ can be propagating or damped waves, and the sign convention for a complex k will determine if the wave remains bounded at positive or negative infinity.

The quantity of interest is usually the wave number itself—namely k, which is the square root of the given squared wave number in the Helmholtz equation.Each k can be considered the sum of a real and an imaginary part, so that $k = k_r + ik_i$, with k_r denoting the real part of k and k_i denoting the imaginary part. A proper choice of sign convention for k means defining the complex k as the square root of the corresponding k^2 such that k has a (choice of) a positive or negative imaginary part. This can be done so that the wave(s) gives a physically reasonable result (boundedness) over the domain of interest of the problem. Choices of sign convention can also make the bookkeeping simpler in the sense that by choice of sign convention, results can be written the same way whether for real or complex k. The choice of sign convention for k also determines which expressions are considered outwardly radiating. This is important in the sense that mathematically the Helmholtz equation can allow solutions that are inwardly and outwardly propagating waves (damped or propagating), whereas the choice of the most physically meaningful solution means that only one of these will usually be the right choice. The Sommerfeld radiation condition can be used to choose a radiating solution.

16.1. The Helmholtz Transfer Function

For a Helmholtz equation that is 2D in space, taking the Fourier-series transform of Eq. (157) implies writing $u(\vec{r},\omega) = \sum_{n=-\infty}^{\infty} u_n(r,\omega)e^{in\theta}$ and similarly for $s(\vec{r},\omega)$ and ∇^2, so that $u(\vec{r},\omega) \Leftrightarrow u_n(r,\omega)$, $s(\vec{r},\omega) \Leftrightarrow s_n(r,\omega)$, and $\nabla^2 \Leftrightarrow \nabla_n^2$ and Eq. (157) becomes

$$\nabla_n^2 u_n(r,\omega) + k^2 u_n(r,\omega) = -s_n(r,\omega). \tag{158}$$

Taking an nth-order Hankel transform of Eq. (158) turns this into a full 2D Fourier transform so that $u_n(r,\omega) \Leftrightarrow U_n(\rho,\omega)$, $s_n(r,\omega) \Leftrightarrow S_n(\rho,\omega)$, and $\nabla_n^2 \Leftrightarrow -\rho^2$. This and then rearranging gives

$$U_n(\rho,\omega) = \frac{1}{\rho^2 - k^2} S_n(\rho,\omega) = G(\rho,\omega) S_n(\rho,\omega), \tag{159}$$

where $G(\rho,\omega)$ is given by

$$G(\rho,\omega) = \frac{1}{\rho^2 - k^2} \tag{160}$$

and is referred to as the *Helmholtz transfer function*. The notation $G(\rho,\omega)$ is used as a reminder that the transfer function is (1) only a function of the frequency radial variable ρ in space and (2) a function of temporal frequency ω via the wave number k. Solving the Helmholtz equation for a given source $S(\vec{\omega},\omega)$ is a matter of inverting Eq. (159). Note that Eq. (159) has the exact form as Eq. (129), which means that this simple expression is the frequency domain equivalent of $g(r,\omega) ** s(\vec{r},\omega)$, where $g(r,\omega) = \mathbb{F}_{2D}^{-1}\{G(\rho,\omega)\}$ is the Green's function for the Helmholtz equation. In fact, it is *this* example that was the primary motivator behind the section on convolutions of radially symmetric functions with non–radially symmetric functions.

To invert Eq. (159) back into the spatial domain implies finding

$$u_n(r,\omega) = \frac{i^n}{2\pi} \int_0^\infty \frac{S_n(\rho,\omega)}{\rho^2 - k^2} J_n(\rho r)\, \rho d\rho. \tag{161}$$

Results involving integrals of the type in Eq. (161) are considered elsewhere (Baddour, 2009). Often several solutions to Eq. (161) are mathematically possible, particularly due to its nature as an improper integral. To aid in selecting the most physically meaningful result among several mathematical possibilities, the Sommerfeld radiation condition must be used.

16.2. Green's Function Coefficients

The Green's function is defined as the solution to the Helmholtz equation for a delta function source at $\vec{r} = \vec{r}_0$ for real or complex k:

$$\nabla^2 g\left(\vec{r},\vec{r}_0,\omega\right) + k^2 g\left(\vec{r},\vec{r}_0,\omega\right) = -\delta\left(\vec{r} - \vec{r}_0\right), \tag{162}$$

where we use $g\left(\vec{r},\vec{r}_0,\omega\right)$ to denote the Green's function. Taking the 2D Fourier transform of Eq. (162) gives

$$\left(-\rho^2 + k^2\right) G\left(\vec{\omega},\vec{r}_0,\omega\right) = -e^{-i\vec{\omega}\cdot\vec{r}_0}. \tag{163}$$

For $\vec{r}_0 = 0$, the definition of G in Eq. (163) is the same as that in Eq. (160). We convert to the polar-coordinate form so that the complex exponential is given by Eq. (9) and the Green's function is written as

$$G\left(\vec{\omega},\vec{r}_0,\omega\right) = \sum_{n=-\infty}^{\infty} G_n\left(\rho,\vec{r}_0,\omega\right) e^{in\psi} \tag{164}$$

so that Eq. (163) becomes

$$G_n\left(\rho,\vec{r}_0,\omega\right) = \frac{1}{\rho^2 - k^2} \left[e^{-i\vec{\omega}\cdot\vec{r}_0}\right]_n = \frac{1}{\rho^2 - k^2} i^{-n} J_n(\rho r_0)\, e^{-in\theta_0}. \tag{165}$$

Although we arrived at Eq. (165) directly from the classical definition of a Green's function as the response to a delta function, we see from Eq. (99) that these are, in fact, the Fourier coefficients of a shifted radially symmetric function, the Helmholtz transfer function $G(\rho,\omega) = \frac{1}{\rho^2 - k^2}$, where multiplication by the coefficients $\left[e^{-i\vec{\omega}\cdot\vec{r}_0}\right]_n$ provides the shift.

With the interpretation of these coefficients as those of the shifted function we can write $g_n(r, \vec{r}_0, \omega)$ as $g_n(r - \vec{r}_0, \omega)$, interpreting the shift. The coefficients of the Green's function in spatial (polar) coordinates are

$$g_n(r, \vec{r}_0, \omega) = g_n(r - \vec{r}_0, \omega) = \frac{i^n}{2\pi} \int_0^\infty G_n(\rho, \vec{r}_0, \omega) J_n(\rho r) \rho d\rho$$

$$= \frac{e^{-in\theta_0}}{2\pi} \int_0^\infty \frac{1}{\rho^2 - k^2} J_n(\rho r_0) J_n(\rho r) \rho d\rho, \tag{166}$$

where the notation $g_n(r, \vec{r}_0, \omega) = g_n(r - \vec{r}_0, \omega)$ has been used to indicate that what we have found is actually a shifted version of $\mathbb{F}_{2D}^{-1}[G(\rho, \omega)]$. Of course, the fact that any Green's function is a shift of the response to a delta function at the original is well known, but the point here is that this same interpretation also followed from the rules of Fourier transforms in polar coordinates developed herein.

17. SUMMARY AND CONCLUSIONS

In summary, this article has considered the polar-coordinate version of the standard 2D Fourier transform and derived the operational toolset required for standard Fourier operations. As previously noted, the polar-coordinate version of the 2D Fourier transform is most useful for functions that are naturally described in terms of polar coordinates. Additionally, Parseval relationships were also derived. The results are concisely collected in Table 1. Of particular interest are the results on convolution and spatial shift. Notably, standard convolution/multiplication rules do apply for 2D convolution and 1D circular convolution but not for 1D radial convolution.

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