Exercise 3

wave equation in polar coordinates for surface deformation ϕ

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right]$$

through the boundary conditions is the driving force for a material initially at rest

$\phi(r=R, \theta, t) = A \cos(\omega t) \cos(\theta)$
$\phi(r=0, \theta, t) = \text{finite}$
$\phi(r, \theta + 2\pi, t) = \phi(r, \theta, t)$

and initial conditions

$\phi(r, \theta, t=0) = 0$
$\frac{\partial \phi(r, \theta, t=0)}{\partial t} = 0$

solve for ϕ with eigenfunction expansion methods, determine if there is any resonance value for ω

first use separation of variables to decouple the time component

$$\nabla^2 \psi = \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] = -\lambda \psi$$

with boundary conditions

$\psi(r=R, \theta) = 0$
$\psi(r=0, \theta) = \text{Finite}$
$\psi(r, \theta + 2\pi) = \psi(r, \theta)$

then subtract from the solution to ϕ a function that makes the boundary conditions homogeneous, but may introduce inhomogeneities in the initial condition or equation itself

last solve for ϕ using eigenfunction expansion with the eigenfunctions determined from separation of variables

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with separation of variables

$\phi = \psi(r, \theta)\, T(t)$

$$\frac{\partial^2}{\partial t^2}(\psi T) = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}(\psi T) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}(\psi T) \right]$$

$$\psi T'' = c^2 \left[\frac{T}{r} \frac{\partial}{\partial r}\left[r \frac{\partial \psi}{\partial r} \right] + \frac{T}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] \quad \text{divide by } \psi T c^2$$

$$\frac{1}{\psi} \left[\frac{1}{r} \frac{\partial}{\partial r}\left[r \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] = \frac{1}{c^2} \frac{T''}{T} = -\lambda^2 \quad \text{constant}$$

note for convenience I define λ^2 contrary to the problem statement

Exercise 3 (continued)

first solving the time component with integrating factor

$T + c^2\lambda^2 T = 0 \qquad T \sim e^{zt}$

$z^2 + c^2\lambda^2 = 0 \qquad T' \sim z e^{zt}$

$z = \pm c\lambda i \qquad T'' \sim z^2 e^{zt}$

$T = C_1 e^{c\lambda i} + C_2 e^{-c\lambda i}$

applying Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$

$T(t) = C_1 \cos(\lambda c t) + C_2 \sin(\lambda c t)$

now using separation of variables again on the spatial component

$\psi = R(r)\,X(\theta)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(RX)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(RX) = -\lambda^2 RX$$

$$X\frac{1}{r}\frac{\partial}{\partial r}(rR') + \frac{1}{r^2}RX'' = -\lambda^2 RX$$

$$X\frac{1}{r}(R' + rR'') + \frac{1}{r^2}RX'' = -\lambda^2 RX$$

$$XrR' + Xr^2R'' + \lambda^2 r^2 RX = -RX''$$

$$\frac{r^2R'' + rR' + \lambda^2 r^2 R}{R} = \frac{-X''}{X} = \alpha^2 \text{ constant}$$

solving the radial component

$r^2R'' + rR' + (\lambda^2 r^2 - \alpha^2)R = 0$

this takes the form of the Bessel equation of order α

$R(r) = C_3 J_\alpha(\lambda r) + C_4 Y_\alpha(\lambda r)$, where J_α and Y_α are Bessel function of the first and second kind, respectively, and order α

applying boundary conditions

$R(r=0) = \text{finite} \rightarrow C_4 = 0$

$R(r=R) = 0 \rightarrow 0 = C_3 J_\alpha(\lambda R)$ for $C_3 \neq 0$

a relation between the eigenvalues and the zeroes of the Bessel function

$\lambda_{m,n} R = z_{m,n}$ for $n = 1, 2, 3\ldots$

and m corresponding to Bessel order α_m

Exercise 3 (continued)

solving the angular component with integrating factor

$\chi'' + \alpha^2 \chi = 0$ $\qquad \chi \sim e^{zt}$

$z^2 + \alpha^2 = 0$ $\qquad \chi' \sim z e^{zt}$

$z = \pm \alpha i$ $\qquad \chi'' \sim z^2 e^{zt}$

$\chi = C_5 e^{\alpha i} + C_6 e^{-\alpha i}$

applying Euler's identity $e^{i\delta}$ $\cos\delta + i\sin\delta$

$\chi(\theta) = C_5 \cos(\alpha\theta) + C_6 \sin(\alpha\theta)$

applying boundary conditions

$\chi(\theta + 2\pi) = \chi(\theta)$

$C_5 \cos(\alpha\theta + 2\pi\alpha) + C_6 \sin(\alpha\theta + 2\pi\alpha) = C_5 \cos(\alpha\theta) + C_6 \sin(\alpha\theta)$

α is an integer

$\alpha_m = m$ for $m = 0, 1, 2 \ldots$

combining $\psi(r,\theta) = R(r)\,\chi(\theta)$

$\psi_{m,n}(r,\theta) = J_m(\lambda_{m,n} r)\left[C_5 \cos(m\theta) + C_6 \sin(m\theta)\right]$

with the modes of C_5, C_6 a function of t, to be determined

$\psi_{m,n}(r,\theta) = J_m(\lambda_{m,n} r)\left[C_{m,n}(t)\cos(m\theta) + B_{m,n}\sin(m\theta)\right]$

now determining a function $\tilde{\phi}$ that when subtracted from ϕ makes the boundary conditions homogeneous (but introduces inhomogeneities in the initial conditions and equations)

$\phi - \tilde{\phi} = A(r/R)^2 \cos(\omega t)\cos(\theta)$

$\longrightarrow$ $\tilde{\phi}(r=0,\theta,t) = \text{finite}$

$\tilde{\phi}(r=R,\theta,t) = 0$

similarly $\tilde{\phi}(r,\theta,t) = \tilde{\phi}(r,\theta+2\pi,t)$

$\tilde{\phi}(r,\theta+2\pi,t) + A(r/R)^2\cos(\omega t)\cos(\theta+2\pi) = \tilde{\phi}(r,\theta,t) + A(r/R)^2\cos(\omega t)\cos(\theta)$

$\dfrac{\partial \tilde{\phi}(r,\theta,t=0)}{\partial t} = 0$

$\tilde{\phi}(r,\theta,t=0) = -A(r/R)^2\cos(\theta)$

now substituting into the PDE

$$\frac{\partial^2\tilde{\phi}}{\partial t^2} - \omega^2 A(r/R)^2\cos(\omega t)\cos(\theta) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\tilde{\phi}}{\partial\theta^2}\right]$$

$$= c^2\left(\frac{4A}{R^2}\cos(\omega t)\cos(\theta)\right) + c^2\left(-\frac{A}{R^2}\cos(\omega t)\cos(\theta)\right)$$

Exercise 3 (continued)

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{\phi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \tilde{\phi}}{\partial \theta^2}\right] + \frac{A}{R^2}(r^2 w^2 + 3c^2)\cos(wt)\cos(\theta)$$

defining $S(r,\theta,t) \equiv \frac{A}{R^2}(r^2 w^2 + 3c^2)\cos(wt)\cos(\theta)$

now using eigenfunction expansion

$$\tilde{\phi}(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{m,n} r)\left[C_{m,n}(t)\cos(m\theta) + B_{m,n}(t)\sin(m\theta)\right]$$

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{m,n} r)\left[\ddot{C}_{m,n}(t)\cos(m\theta) + \ddot{B}_{m,n}(t)\sin(m\theta)\right]$$

and $\frac{\partial^2 \tilde{\phi}}{\partial \theta^2}, \frac{\partial \tilde{\phi}}{\partial r}, \frac{\partial^2 \tilde{\phi}}{\partial r^2}$ are already eigenfunctions of spatial operator

$$S(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{m,n} r)\left[D_{m,n}(t)\cos(m\theta) + E_{m,n}(t)\sin(m\theta)\right]$$

here $E_{m,n}(t) = 0$ because $S(r,\theta,t)$ is only a function with cosine and can be represented by the $\cos(m\theta)$ eigenfunction

because $\cos(m\theta)$ are orthogonal functions, only when $m=1$ will $D_{m,n}(t)$ be non-zero, $D_{m,n}(t) = 0$ for $m \neq 1 \rightarrow \int_0^{2\pi}\cos^2\theta \, d\theta = \pi$

substituting into the PDE

$$\longrightarrow \frac{\partial^2 \tilde{\phi}}{\partial \theta^2} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{m,n} r)\left[-m^2 C_{m,n}\cos(m\theta) - m^2 B_{m,n}\sin(m\theta)\right]$$

$$\longrightarrow \frac{\partial \tilde{\phi}}{\partial r} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\lambda_{m,n}}{2}\left(J_{m-1}(\lambda_{m,n} r) - J_{m+1}(\lambda_{m,n} r)\right)\right]\left[C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right]$$

$$\longrightarrow \frac{\partial^2 \tilde{\phi}}{\partial r^2} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{\lambda_{m,n}}{2}\right)^2\left(J_{m-2}(\lambda_{m,n} r) - 2J_m(\lambda_{m,n} r) + J_{m+2}(\lambda_{m,n} r)\right)\right]\left[C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right]$$

$$J_m(\lambda_{m,n} r)\left[\ddot{C}_{m,n}\cos(m\theta) + \ddot{B}_{m,n}\sin(m\theta)\right] = \ldots$$

$$\ldots \frac{c^2}{r}\frac{\lambda_{m,n}}{2}\left[J_{m-1}(\lambda_{m,n} r) - J_{m+1}(\lambda_{m,n} r)\right]\left[C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right] + \ldots$$

$$\ldots c^2(\lambda_{m,n}/2)^2\left[J_{m-2}(\lambda_{m,n} r) - 2J_m(\lambda_{m,n} r) + J_{m+2}(\lambda_{m,n} r)\right]\left[C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right] + \ldots$$

$$\ldots (c^2/r^2)\, J_m(\lambda_{m,n} r)\left[-m^2 C_{m,n}\cos(m\theta) - m^2 B_{m,n}\sin(m\theta)\right] + \ldots$$

$$\ldots J_m(\lambda_{m,n} r)\left[D_{m,n}\cos(m\theta) + E_{m,n}\sin(m\theta)\right]$$

where $J_{m-2} = \frac{2(m-1)}{\lambda r} J_{m-1} - J_m$

$$J_{m+2} = \frac{2(m+1)}{\lambda r} J_{m+1} - J_m$$

$$J_{m+1} + J_{m-1} = \frac{2m}{\lambda r} J_m$$

Exercise 3 (continued)

for convenience define

$G = C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)$

$F = D_{m,n}\cos(m\theta) + E_{m,n}\sin(m\theta)$

and in short-hand

$$J_m G'' = \frac{c^2\lambda}{2r}\left[J_{m-1} - J_{m+1}\right]G + Gc^2\left[\frac{\lambda}{2}\right]^2\left[J_{m-2} - 2J_m + J_{m+2}\right] + \frac{c^2}{r^2}J_m(-m^2)G + J_m(F)$$

$$J_m G'' = \frac{c^2\lambda}{2r}\left[J_{m-1} - J_{m+1}\right]G + G\left[\frac{c\lambda}{2}\right]^2\left[\frac{2(m-1)}{\lambda r}J_{m-1} - 4J_m + \frac{2(m+1)}{\lambda r}J_{m+1}\right] + \dots$$

$$\dots\left[\frac{-c^2m^2}{r^2}\right]J_m(G) + J_m(F)$$

$$J_m G'' = \left[\frac{c^2\lambda}{2r}J_{m-1} + \frac{c^2\lambda^2}{4}\left[\frac{2(m-1)}{\lambda r}\right]J_{m-1}\right]G + \left[\frac{c^2\lambda^2}{4}\left[\frac{2(m-1)}{\lambda r}\right]J_{m+1} - \frac{c^2\lambda}{2r}J_{m-1}\right]G + \dots$$

$$\dots - G\left[\frac{c^2\lambda^2}{\cancel{4}}\right]\cancel{4}J_m - \frac{c^2m^2}{r^2}J_m(G) + J_m(F)$$

$$J_m G'' = \frac{c^2\lambda m}{2r}J_{m-1}G + \frac{c^2\lambda m}{2r}J_{m+1}G - c^2\lambda^2 J_m G - \frac{c^2m^2}{r^2}J_m G + J_m F$$

$$J_m G'' = \frac{c^2\cancel{\lambda} m}{\cancel{2r}}\left[\frac{\cancel{2m}}{\cancel{\lambda r}}J_m\right]G - c^2\lambda^2 J_m G - \cancel{\frac{c^2m^2}{r^2}J_m G}$$

$$\cancel{J_m}G'' = -c^2\lambda^2\cancel{J_m}G + \cancel{J_m}F \rightarrow G'' = -c^2\lambda^2 G + F$$

substitute G, F

$$C_{m,n}''\cos(m\theta) + B_{m,n}''\sin(m\theta) = -c^2\lambda_{m,n}^2\left[C_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right] + \dots$$

$$\dots + \left[D_{m,n}\cos(m\theta) + E_{m,n}\sin(m\theta)\right]$$

$\longrightarrow$ $C_{m,n}'' = -c^2\lambda_{m,n}^2 C_{m,n} + D_{m,n}$

$B_{m,n}'' = -c^2\lambda_{m,n}^2 B_{m,n} + E_{m,n}$

as shown previously $E_{m,n}(t) = 0$ and $D_{m,n}(t) = 0$ for $m \neq 1$

thus

$C_{m,n}'' = -c^2\lambda_{m,n}^2 C_{m,n}$ for $m \neq 1$

$C_{1,n}'' = -c^2\lambda_{1,n}^2 C_{1,n} + D_{1,n}$

$B_{m,n}'' = -c^2\lambda_{m,n}^2 B_{m,n}$

Exercise 3 (continued)

applying initial conditions

since $\cos\theta$ corresponds to $m=1$ and $\cos(m\theta)$ is orthogonal to $\cos(n\theta)$ for $n \neq m$, all terms with $m \neq 1$ are $0 \rightarrow m=1$ and summation over m can be removed

$$\tilde{\phi}(r,\theta,t=0) = -\frac{Ar^2}{R^2}\cos\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{m,n} r)\left[C_{m,n}(0)\cos(m\theta) + B_{m,n}(0)\sin(m\theta)\right]$$

$$-\frac{Ar^2}{R^2}\cos\theta = \sum_{n=1}^{\infty} J_1(\lambda_n r)\left[C_n(0)\cos\theta + B_n(0)\sin\theta\right]$$

thus $B_n(0) = 0$

$$C_n'' = -c^2\lambda_n^2 C_n + D_n$$
$$B_n'' = -c^2\lambda_n^2 B_n$$

since $C_{m,n}'' = -c^2\lambda_{m,n}^2 C_{m,n}$ for $m \neq 1$

$C_{m,n} = a\cos(c\lambda_{m,n}t) + b\sin(c\lambda_{m,n}t) \rightarrow C_{m,n}(0) = C'_{m,n}(0) = 0,\ a = b = 0$

solving B_n using method of integrating factor

$B_n \sim e^{zt}$

$B_n' \sim ze^{zt}$

$B_n'' \sim z^2 e^{zt}$

$z = \pm c\lambda i$ and apply Euler's identity $e^{i\gamma} = \cos\gamma + i\sin\gamma$

$$B_n = C_1\cos(c\lambda_n t) + C_2\sin(c\lambda_n t)$$

for D_n, using orthogonality

$$\frac{A}{R^2}(r^2w^2 + 3c^2)\cos(wt)\cancel{\cos\theta} = \sum_{n=1}^{\infty} J_1(\lambda_n r) D_n(t)\cancel{\cos\theta}$$

$$D_n = \frac{\langle \frac{A}{R^2}(r^2w^2+3c^2)\cos(wt), J_1(\lambda_n r)\rangle_w}{\langle J_1(\lambda_n r), J_1(\lambda_n r)\rangle_w} \quad \text{where } w(r) = r$$

$$D_n = \frac{\int_0^R r\frac{A}{R^2}(r^2w^2+3c^2)\cos(wt)J_1(\lambda_n r)\,dr}{\int_0^R r J_1^2(\lambda_n r)\,dr}$$

$$D_n = \cos(wt)\left[\frac{A\int_0^R r(r^2w^2+3c^2)J_1(\lambda_n r)\,dr}{R^2\int_0^R rJ_1^2(\lambda_n r)\,dr}\right]$$

Exercise 3 (continued)

solving for $C_n(t)$ homogeneous with integrating factor

$C_n \sim e^{zt}$

$C_n' \sim ze^{zt}$

$C_n'' \sim z^2e^{zt}$

$z = \pm c\lambda i$ and apply Euler's identity $e^{\gamma i} = \cos\gamma + i\sin\gamma$

$$C_{n,hom}(t) = C_3\cos(c\lambda_n t) + C_4\sin(c\lambda_n t)$$

solving for $C_n(t)$ particular with undetermined coefficients

$$C_{n,p}(t) = H\cos(wt) + I\sin(wt)$$
$$C_{n,p}'(t) = -wH\sin(wt) + wI\cos(wt)$$
$$C_{n,p}''(t) = -w^2H\cos(wt) - w^2I\sin(wt)$$

$$-w^2H\cos(wt) - w^2I\sin(wt) + c^2\lambda_n^2[H\cos(wt) + I\sin(wt)] = D_n$$

$$-w^2H + c^2\lambda_n^2H = D_n/\cos(wt) \rightarrow H = \frac{D_n}{(c^2\lambda_n^2 - w^2)\cos(wt)}$$
$$-w^2I + c^2\lambda_n^2I = 0 \rightarrow I = 0$$

combining

$$C_n(t) = C_3\cos(c\lambda_n t) + C_4\sin(c\lambda_n t) + D_n/(c^2\lambda_n^2 - w^2)$$

now solving C_1, C_2, C_3, C_4

knowing that $B_n(0) = 0$ and $\dot{\phi}(r,\theta,t=0) = 0 = \sum_{n=1}^{\infty} J_1(\lambda_n r)[C_n'(0)\cos\theta + B_n'(0)\sin\theta]$

thus $C_n'(0) = B_n'(0) = 0$

$$B_n(t) = C_1\cos(c\lambda_n t) + C_2\sin(c\lambda_n t)$$
$$B_n(0) = B_n'(0) = 0 \rightarrow C_1 = C_2 = 0 \text{ and } B_n(t) = 0$$

finding $C_n(t)$ from orthogonality

$$\frac{-Ar^2}{R^2}\cancel{\cos\theta} = \sum_{n=1}^{\infty} J_1(\lambda_n r)[C_n(0)\cancel{\cos\theta}]$$

$$C_n(0) = \frac{\langle \frac{-Ar^2}{R^2}, J_1(\lambda_n r)\rangle_w}{\langle J_1(\lambda_n r), J_1(\lambda_n r)\rangle_w} \quad \text{where } w(r) = r$$

$$C_n(0) = \frac{\int_0^R r\left(\frac{-Ar^2}{R^2}\right)J_1(\lambda_n r)\,dr}{\int_0^R J_1^2(\lambda_n r)\,dr}$$

Exercise 3 (continued)

applying boundary conditions to $C_n(t)$

$$C_n(t) = C_3 \cos(c\lambda_n t) + C_4 \sin(c\lambda_n t) + D_n/(c^2\lambda_n^2 - w^2)$$

$$C_n'(t) = -c\lambda_n C_3 \sin(c\lambda_n t) + c\lambda_n C_4 \cos(c\lambda_n t) + D_n'/(c^2\lambda_n^2 - w^2)$$

where $D_n'(t) = w \sin(wt) \left[\dfrac{A \int_0^R r(r^2w^2 + 3c^2) J_1(\lambda_n r)\, dr}{R^2 \int_0^R r J_1^2(\lambda_n r)\, dr} \right]$

$$C_n'(0) = 0 \rightarrow C_4 = 0$$
$$C_n(0) = C_3 + \frac{D_n}{\cos(wt)(c^2\lambda_n^2 - w^2)} \rightarrow C_3 = C_n(0) - D_n/[\cos(wt)(c^2\lambda_n^2 - w^2)]$$

$$C_n(t) = \left[C_n(0) - \frac{D_n}{\cos(wt)(c^2\lambda_n^2 - w^2)} \right] \cos(c\lambda_n t) + \left[\frac{D_n}{c^2\lambda_n^2 - w^2} \right]$$

$$\tilde{\phi}(r,\theta,t) = \sum_{n=1}^{\infty} \left[\left[C_n(0) - \frac{D_n}{\cos(wt)(c^2\lambda_n^2 - w^2)} \right] \cos(c\lambda_n t) + \frac{D_n}{c^2\lambda_n^2 - w^2} \right] J_1(\lambda_n r) \cos\theta$$

and $\phi = \tilde{\phi} + A\left(\frac{r}{R}\right)^2 \cos(wt) \cos\theta$

and $D_n = \cos(wt) \left[\dfrac{A \int_0^R r(r^2w^2 + 3c^2) J_1(\lambda_n r)\, dr}{R^2 \int_0^R r J_1^2(\lambda_n r)\, dr} \right]$ where λ_n can be replaced by $\sqrt{\lambda_n}$ as defined in the problem statement

by inspection of $\tilde{\phi}(r,\theta,t)$, the coefficient in the summation goes to infinity and "blows up" as the denominator $c^2\lambda_n^2 - w^2 \rightarrow 0$, thus a resonance condition exists when $c^2\lambda_n^2 - w^2 = 0$ and $w^2 = c^2\lambda_n^2$, thus $c\lambda_n$ can be interpreted as an eigenfrequency which when excited, results in resonance