Michael D. Walker (mw6136), Philip Satterthwaite (ps1639), Michael Schroeder (ms2774)
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Analytical Solution to the Wave Equation in Cylindrical Coordinates

0.1. **Wave Equation Classification.** The wave equation is a second-order linear partial differential equation (PDE) for the description of waves or standing wave fields. The scalar wave equation describes the mechanical wave propagation of any scalar φ as

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) = c^2 \, \nabla^2 \, \varphi \quad ,$$

where c is a fixed non-negative real coefficient. ∇ is the *nabla* operator and $\nabla^2 = \nabla \cdot \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the spatial Laplacian operator (in Cartesian coordinates). Because the wave equation is linear and homogeneous, it can be analyzed as a linear combination (superposition) of simple solutions that are sinusoidal plane waves with various directions of propagation and wavelengths, but all with the same finite propagation speed c.

Projecting the wave equation in the general form of a second-order PDE

$$A \frac{\partial^2 \varphi}{\partial t^2} + B \frac{\partial^2 \varphi}{\partial x \partial t} + C \frac{\partial^2 \varphi}{\partial x^2} + D \frac{\partial \varphi}{\partial t} + E \frac{\partial \varphi}{\partial x} + F \varphi = 0$$

gives $A = 1$, $B = 0$, and $C = -c^2$, and thus the discriminant $B^2 - 4AC > 0$ defines a hyperbolic PDE with a well-posed initial value problem .

0.2. **Algebraic Solution to the 1-D Wave Equation.** The wave equation in one spatial dimension

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2}$$

is unusual for a partial differential equation in that a relatively simple general solution may be found. Using an algebraic change of variable $\xi = x - ct$ and $\eta = x + ct$, and computing the partial derivatives

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \frac{\partial \varphi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \varphi}{\partial \eta}\frac{\partial \eta}{\partial t} \\
\frac{\partial^2 \varphi}{\partial t^2} &= c^2 \frac{\partial^2 \varphi}{\partial \xi^2} - 2c \frac{\partial^2 \varphi}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 \varphi}{\partial \eta^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &= \frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta} \\
\frac{\partial^2 \varphi}{\partial x^2} &= \frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial^2 \varphi}{\partial \xi \partial \eta} + \frac{\partial^2 \varphi}{\partial \eta^2} \quad .
\end{aligned}$$

Substituting into the wave equation for $c \neq 0$, the wave equation becomes

$$\frac{\partial^2 \varphi}{\partial \xi \partial \eta}(x, t) = 0 \quad ,$$

which is separable and can be integrated for a general solution, where F, G, H are arbitrary forcing functions.

$$\begin{aligned}
\int \frac{\partial^2 \varphi}{\partial \xi \partial \eta}\, d\xi = 0 &\rightarrow \frac{\partial \varphi}{\partial \eta} = H(\eta) \\
\varphi(\xi, \eta) = \int \frac{\partial \varphi}{\partial \eta}\, d\eta &= \int H(\eta)\, d\eta + F(\xi) = F(\xi) + G(\eta) \\
\varphi(x, t) &= F(x - ct) + G(x - ct) \quad .
\end{aligned}$$

Thus the solutions of the 1-D wave equation are sums of a right-translating function F and a left-translating function G at the speed c.

0.3. **The Wave Equation in Cylindrical Coordinates.** For circularly propagating waves, it is useful to project the wave equation in cylindrical coordinates.

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) = c^2\,\nabla^2\,\varphi$$

Substituting the transformation from Cartesian variables (x, y, z) to cylindrical (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$.

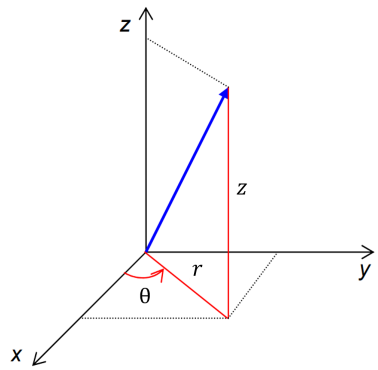

FIGURE 1. Cartesian and Cylindrical coordinates.

The partial derivatives can be evaluated as:

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos(\theta) \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin(\theta)$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r}$$

Applying the chain rule:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial y}$$

Therefore, the Laplacian in cylindrical coordinates is

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \;=\; \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

and the wave equation can be written as

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \cancel{\frac{\partial^2\varphi}{\partial z^2}} \quad .$$

It is clear that it remains a hyperbolic PDE, but now with a singular point at the origin $r = 0$. The study of surface deformation requires only plane-polar coordinates, so the $\partial^2\varphi/\partial z^2$ term may be neglected. We now seek an analytical solution to the 2-D equation, given well-posed boundary and initial conditions.

0.4. **Discretization and Finite Difference Methods.** To discretize the wave equation in polar coordinates, define $r = i\Delta r$, $\theta = j\Delta\theta$, and $t = n\Delta t$. Thus,

$$\varphi(r,\theta,t) = \varphi(i\Delta r, j\Delta\theta, n\Delta t) = \varphi\left(r_i, \theta_j, t^{(n)}\right) \equiv \varphi_{i,j}^{(n)}$$

where r_i and φ_j are the positions and Δr and $\Delta\theta$ are the cell widths.

Considering the Laplacian is divergent at $r = 0$:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

so too will the discretization diverge. This can be managed in several ways.

- Choose a problem such that $r = 0$ does not need to be considered (e.g., $r_{\min} = 0.001$)
- Use cell-centered values for r (e.g., $r_{\min} = 0.5\Delta r$)
- Use two meshes: a Cartesian one for $r \leq 1$ and a polar one for $r \geq 1$, merging them at $r \simeq 1$ (requires at least 1 cell of overlap).

0.4.1. *Explicit (forward difference) methods.* Using the forward difference in r and central differences in θ and t,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r_i}\left(r_{i+1/2}\frac{\varphi_{i+1,j}^{(n)} - \varphi_{i,j}^{(n)}}{\Delta r^2} - r_{i-1/2}\frac{\varphi_{i,j}^{(n)} - \varphi_{i-1,j}^{(n)}}{\Delta r^2}\right)$$

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{1}{r_i{}^2}\left(\frac{\varphi_{i,j+1}^{(n)} - 2\varphi_{i,j}^{(n)} + \varphi_{i,j-1}^{(n)}}{\Delta\theta^2}\right)$$

where $r_{i+1/2} = 0.5\left(r_{i+1} + r_i\right)$. Thus the full wave equation

$$\begin{aligned}
\varphi_{i,j}^{(n+1)} &= 2\varphi_{i,j}^{(n)} - \varphi_{i,j}^{(n-1)} \\
&+ \frac{c^2\Delta t^2}{\Delta r^2}\left[\varphi_{i+2,j}^{(n)} + \frac{1}{i}\varphi_{i+2,j}^{(n)} - 2\varphi_{i+1,j}^{(n)} - \frac{1}{i}\varphi_{i+1,j}^{(n)} + \varphi_{i,j}^{(n)}\right] \\
&+ \frac{c^2\Delta t^2}{i^2\Delta r^2\Delta\theta^2}\left[\varphi_{i,j+1}^{(n)} + \varphi_{i,j-1}^{(n)} - 2\varphi_{i,j}^{(n)}\right]
\end{aligned}$$

The spatial discretization is Δr, $\Delta\theta$ is user defined. The timestep Δt is limited by the Courant-Friedrichs-Levy (CFL) condition. For the wave equation, such that

$$c^2\frac{\Delta t}{\min(\Delta r^2, \Delta\theta^2)} \leq \frac{1}{2}$$

Where formally the constant is limited by the dimensionality of the system $1/N_{\text{dim}}$. For time-stepping purposes, this is the speed limitation of the algorithm. Thus, for performance, it is worthwhile to consider Implicit methods.

0.4.2. *Implicit (backward difference) methods.*

0.4.3. *Crank–Nicolson scheme.* To summarize, usually the Crank–Nicolson scheme is the most accurate scheme for small time steps. For larger time steps, the implicit scheme works better since it is less computationally demanding. The explicit scheme is the least accurate and can be unstable, but is also the easiest to implement and the least numerically intensive.

0.5. **Problem Statement.** Note: this problem was adapted from Problem Set 9, Exercise 3 of Princeton University Course MAE501: Mathematical Methods of Engineering Analysis I given in Fall semester 2023.

For a small deformation of any planar circular elastic material, the surface deformation, φ, satisfies the wave equation in plane-polar coordinates

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right]$$

where the driving force of the wave motion for a material initially at rest can be specified through the boundary and initial conditions. Seeking to model waves on the surface of the fluid generated by the motion of the walls, we choose the following boundary and initial conditions.

Boundary Conditions:
$\varphi(r = R, \theta, t) = A\cos(\omega t)\cos(\theta)$
$\varphi(r = 0, \theta, t) \rightarrow$ finite
$\varphi(r, \theta + 2\pi, t) = \varphi(r, \theta, t)$

Initial Conditions:
$\varphi(r, \theta, t = 0) = 0$
$\partial\varphi/\partial t\,(r, \theta, t = 0) = 0$

0.6. **Method of Solution.** We will solve for φ for all t, θ, and r using the method of separation of variables and eigenfunction expansions. We will also determine the resonance condition for the value of ω.

Steps in the solution:

1) Use Separation of Variables to de-couple the time component and determine the eigenfunctions for the operator with boundary conditions
$$\nabla^2 \psi = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] = -\lambda \psi$$
Boundary Conditions:
$\psi(r = R, \theta) = 0$
$\psi(r = 0, \theta) \rightarrow$ finite
$\psi(r, \theta + 2\pi) = \psi(r, \theta)$

2) Determine a function $\tilde{\varphi}$, that when subtracted from φ makes the boundary conditions homogeneous but perhaps introduces inhomogeneities in the initial conditions and the equation itself.
3) Use the eigenfunctions determined from the separation of variables in the method of eigenfunction expansions and solve for φ.

0.7. **Final Solution.** These conditions lead to the analytical solution

$$\varphi(r,\theta,t) = A\left(\frac{r}{R}\right)^2 \cos(\omega t)\cos(\theta) + \tilde{\varphi}(r,\theta,t)$$

$\tilde{\varphi}$ is composed as follows, where $J_\alpha(\cdot)$ is the Bessel function of the 1st kind and order α, η is a variable of integration, and λ are the eigenfunctions for each of the modes.

$$\tilde{\varphi}(r,\theta,t) = \sum_{m=1}^{\infty}\left(\frac{1}{c^2\lambda_{n,m}-\omega^2}\frac{2A}{R^2}\frac{\int_0^1 (\omega^2\eta^2+3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\cos(\omega t)\ \ldots\right.$$

$$-\frac{2A}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\left[\int_0^1 \eta^3 J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta\ \ldots\right.$$

$$\left.\left.+\frac{1}{c^2\lambda_{n,m}-\omega^2}\frac{1}{R^2}\int_0^1\left(\omega^2\eta^2+3c^2\right)\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta\right]\cos\left(c\sqrt{\lambda_{1,m}}t\right)\right)\cos(\theta)J_1\left(\sqrt{\lambda_{1,m}}r\right).$$

0.8. **Full Derivation.** The method of Separation of Variables can be applied to a partial differential equation (PDE) when the equation is linear and homogeneous, where the boundary conditions are also linear and homogeneous. Thus, considering the homogeneous stationary boundary value problem, using separation of variables, assume $\varphi(r,\theta,t) = \psi(r,\theta)\mathcal{T}(t)$ and substituting in the wave equation

$$\frac{\partial^2(\psi\mathcal{T})}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\psi\mathcal{T})}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2(\psi\mathcal{T})}{\partial\theta^2}\right] \quad\rightarrow\quad \psi\mathcal{T}'' = c^2\left[\frac{\mathcal{T}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{\mathcal{T}}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right] \quad .$$

Dividing by $\psi\mathcal{T}c^2$,

$$\frac{1}{\psi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right] = \frac{1}{c^2}\frac{\mathcal{T}''}{\mathcal{T}} = -\lambda \quad ,$$

with λ a constant that will be shown to correspond to the eigenfunctions for each of the modes. Now using separation of variables again on the spatial component, assume $\psi(r,\theta) = \mathcal{R}(r)\,\Theta(\theta)$ and again substitute into the $\nabla^2\psi$ operator

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\mathcal{R}\Theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2(\mathcal{R}\Theta)}{\partial\theta^2} = -\lambda\mathcal{R}\Theta \quad\rightarrow\quad \frac{\Theta}{r}(\mathcal{R}'+r\mathcal{R}'')+\frac{\mathcal{R}\Theta''}{r^2} = -\lambda\mathcal{R}\Theta \quad .$$

The PDE can then be rewritten as

$$\frac{1}{\mathcal{R}}r\frac{d}{dr}\left(r\mathcal{R}'\right)+\lambda r^2 = -\frac{\Theta''}{\Theta} = \alpha^2$$

with α^2 as some constant. First solving the angular component by applying an integrating factor and solving for the roots of the characteristic polynomial $\Theta \sim \exp(\delta\theta) \rightarrow \delta = \pm\alpha i$, then substituting Euler's identity $\exp(i\alpha) = \cos(\alpha) + i\,\sin(\alpha)$.

$$\Theta'' = -\alpha^2\Theta \quad\rightarrow\quad \Theta(\theta) = a\cos(\alpha\theta)+b\sin(\alpha\theta) \quad .$$

Applying the condition $\psi(r,\theta+2\pi) = \psi(r,\theta)$,

$$a\cos(\alpha\theta+\alpha 2\pi)+b\sin(\alpha\theta+\alpha 2\pi) = a\cos(\alpha\theta)+b\sin(\alpha\theta)$$
$$\rightarrow \cos(\alpha\theta+\alpha 2\pi) = \cos(\alpha\theta) \quad\text{and}\quad \sin(\alpha\theta+\alpha 2\pi) = \sin(\alpha\theta)$$
$$\rightarrow \alpha_n = n \in \mathbb{Z}^{\geq 0} \ \text{ an integer}$$

Therefore a solution to the angular problem is

$$\Theta_n(\theta) = a_n\cos(n\theta)+b_n\sin(n\theta) \quad .$$

Now re-arranging the radial problem (where the substitution $\alpha_n = n$) is made,

$$\frac{1}{\mathcal{R}} r \frac{d}{dr}\Big(r\mathcal{R}'\Big) + \lambda r^2 = n^2 \quad \rightarrow \quad r\frac{d}{dr}\Big(r\mathcal{R}'\Big) + (\lambda r^2 - n^2)\mathcal{R} = 0$$

$$r^2\mathcal{R}'' + r\mathcal{R}' + (\lambda r^2 - n^2)\mathcal{R} = 0$$

This is the form of a Bessel equation of order n and therefore the solutions to the radial problem are the Bessel functions of order n,

$$\mathcal{R}(r) = cJ_n(\sqrt{\lambda}r) + d\cancel{Y_n(\sqrt{\lambda}r)} \quad .$$

Applying boundary conditions:
$\mathcal{R}(r=0)$ finite $\rightarrow$ $d = 0$
$\mathcal{R}(r=R) = 0$ $\rightarrow$ $cJ_n(\sqrt{\lambda_{n,m}}r) = 0$ $\rightarrow$ $\sqrt{\lambda_{n,m}}r = z_{n,m}$ for $c \neq 0$

This gives a relation between the eigenvalues and zeroes of the Bessel function, where $z_{n,m}$ is the m-th zero of the n-th order Bessel function J_n for $m = 1, 2, 3\dots$ Therefore the eigenfunctions are of the form

$$\psi_{n,m}(r,\theta) = \mathcal{R}(r)\,\Theta(\theta) = [a_{n,m}\cos(n\theta) + b_{n,m}\sin(n\theta)]J_n(\sqrt{\lambda_{n,m}}r)$$

With the modes of $a_{n,m}(t)$ and $b_{n,m}(t)$ to be determined as functions of time. Now considering the original problem, the boundary condition is inhomogeneous, so we must rescale the problem to achieve homogeneous boundary conditions. Let

$$\tilde{\varphi} \equiv \varphi - A\left(\frac{r}{R}\right)^2 \cos(\omega t)\cos(\theta)$$

Here the r^2/R^2 term is necessary to resolve the singularity from the ODE. Taking the time and spatial derivatives of $\tilde{\varphi}$ to rewrite the PDE:

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \frac{\partial \tilde{\varphi}}{\partial t} - A\omega\left(\frac{r}{R}\right)^2 \sin(\omega t)\cos(\theta) \\
\frac{\partial \varphi}{\partial r} &= \frac{\partial \tilde{\varphi}}{\partial r} - 2A\left(\frac{r}{R^2}\right)\cos(\omega t)\cos(\theta) \\
\frac{\partial^2 \varphi}{\partial t^2} &= \frac{\partial^2 \tilde{\varphi}}{\partial t^2} - A\omega^2\left(\frac{r}{R}\right)^2 \cos(\omega t)\cos(\theta) \\
\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \varphi}{\partial r}) &= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \tilde{\varphi}}{\partial r}) + 4A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta) \\
\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} &= \frac{1}{r^2}\frac{\partial^2 \tilde{\varphi}}{\partial \theta^2} - A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta)
\end{aligned}$$

Now combining to rewrite the solution in terms of the transformed variable $\tilde{\varphi}$,

$$\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - A\omega^2\left(\frac{r}{R}\right)^2 \cos(\omega t)\cos(\theta) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \tilde{\varphi}}{\partial r}) + \frac{4A}{R^2}\cos(\omega t)\cos(\theta) + \frac{1}{r^2}\frac{\partial^2 \tilde{\varphi}}{\partial \theta^2} - \frac{A}{R^2}\cos(\omega t)\cos(\theta)\right]$$

which simplifies as

$$\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \frac{A}{R^2}\cos(\omega t)\cos(\theta)(\omega^2 r^2 + 3c^2) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \tilde{\varphi}}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \tilde{\varphi}}{\partial \theta^2}\right] \quad .$$

The initial and boundary conditions are similarly re-scaled.

Boundary Conditions:
$\tilde{\varphi}(r = R, \theta, t) = 0$
$\tilde{\varphi}(r = 0, \theta, t) \to \text{finite}$
$\tilde{\varphi}(r, \theta + 2\pi, t) = \tilde{\varphi}(r, \theta, t)$

Initial Conditions:
$\tilde{\varphi}(r, \theta, t = 0) = -A\left(\frac{r^2}{R^2}\right)\cos(\theta)$
$\partial\tilde{\varphi}/\partial t\,(r, \theta, t = 0) = 0$

Note that the boundary conditions are now homogeneous, but one of the initial conditions is inhomogeneous. For convenience, define a function

$$f(r, \theta, t) \equiv A\left(\frac{1}{R^2}\cos(\omega t)\cos(\theta)(\omega^2 r^2 + 3c^2)\right) \quad .$$

Solving the new PDE in $\tilde{\varphi}$ we expand the inhomogeneity $f(r, \theta, t)$ in the eigenfunctions of the stationary problem

$$f(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_{n,m}(t)\cos(n\theta) + d_{n,m}(t)\sin(n\theta)]J_n(\sqrt{\lambda_{n,m}}r)$$

Using orthogonality of eigenfunctions we can solve for $c_{n,m}(t)$ and $d_{n,m}(t)$ by calculating the inner product using an appropriate weight function r

$$c_{n,m}(t) = \frac{\langle f(r, \theta, t), J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\rangle_r}{\langle J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta), J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\rangle_r} = \frac{\int_0^R\int_0^{2\pi} f(r, \theta, t) r J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)dr\,d\theta}{\int_0^R\int_0^{2\pi} r {J_n}^2(\sqrt{\lambda_{n,m}}r)\cos^2(n\theta)dr\,d\theta}$$

$$d_{n,m}(t) = \frac{\langle f(r, \theta, t), J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta)\rangle_r}{\langle J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta), J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta)\rangle_r} = \frac{\int_0^R\int_0^{2\pi} f(r, \theta, t) r J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta)dr\,d\theta}{\int_0^R\int_0^{2\pi} r {J_n}^2(\sqrt{\lambda_{n,m}}r)\sin^2(n\theta)dr\,d\theta}$$

We can rewrite

$$\int_0^R\int_0^{2\pi} f(r, \theta, t)\, r J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\,dr\,d\theta \;=\; \frac{A}{R^2}\cos(\omega t)\left[\int_0^{2\pi}\cos(\theta)\cos(n\theta)\,d\theta\int_0^R(\omega^2 r^2 + 3c^2) r J_n(\sqrt{\lambda_{n,m}}r)\,dr\right]$$

Because $\cos(n\theta)$ are orthogonal functions, only when $n = 1$ is $c_{n,m}$ non-zero.

$$\int_0^{2\pi}\cos(\theta)\cos(n\theta)d\theta = \begin{cases}\pi & \text{if } n = 1\\ 0 & \text{if } n \neq 1\end{cases}$$

Thus $c_{n,m}$ simplifies to

$$c_{n,m} = \begin{cases}0 & \text{if } n \neq 1\\ \frac{2A}{R^4}\cos(\omega t)\frac{\int_0^R(\omega^2 r^2 + 3c^2) r J_1(\sqrt{\lambda_{1,m}}r)\,dr}{{J_2}^2(\sqrt{\lambda_{1,m}}R)} & \text{if } n = 1\end{cases}$$

This also requires the integration property of Bessel functions $\int x^n J_{n-1}(x)dx = x^n J_n(x)$. Similarly, as $f(r, \theta, t)$ is only a function of cosine,

$$\int_0^{2\pi}\cos(\theta)\sin(n\theta)d\theta = 0\,\forall\,n \quad \to \quad d_{n,m} = 0\ .$$

Since $\cos(\theta)$ corresponds to $m=1$ and $\cos(n\theta)$ is orthogonal to $\cos(m\theta)$ for $n \neq m$, all terms $n \neq 1$ are 0 and the summation over n can be removed.

$$f(r,\theta,t) = \frac{2A}{R^4}\cos(\omega t)\sum_{m=1}^{\infty}\frac{\int_0^R(\omega^2\zeta^2+3c^2)\zeta J_1(\sqrt{\lambda_{1,m}}\zeta)d\zeta}{{J_2}^2(\sqrt{\lambda_{1,m}}R)}\cos(\theta)J_1(\sqrt{\lambda_{n,m}}r)$$

Here ζ is the variable of integration. If we non-dimensionalize the variable in the integral with $R\eta = \zeta$,

$$f(r,\theta,t) = \frac{2A}{R^2}\cos(\omega t)\sum_{m=1}^{\infty}\frac{\int_0^1(\omega^2\eta^2+3c^2)\eta J_1(\sqrt{\lambda_{1,m}}R\eta)d\eta}{{J_2}^2(\sqrt{\lambda_{1,m}}R)}\cos(\theta)J_1(\sqrt{\lambda_{n,m}}r) \quad .$$

Similar to the stationary problem, the eigenfunction expansion of $\tilde{\varphi}(r,\theta,t)$ is

$$\tilde{\varphi}(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[a_{n,m}(t)\cos(n\theta)+b_{n,m}(t)\sin(n\theta)]J_n(\sqrt{\lambda_{n,m}}r) \quad .$$

Then we can deduce the following ODE relationships from the PDE:

$$\begin{aligned} n \neq 1 &: a''_{n,m}(t) = -\lambda_{n,m}\, c^2\, a_{n,m}(t) \\ n = 1 &: a''_{1,m}(t) - c_{1,m}(t) = -\lambda_{1,m}\, c^2\, a_{1,m}(t) \\ \forall\, n &: b''_{n,m}(t) = -\lambda_{n,m}\, c^2\, b_{n,m}(t) \end{aligned}$$

Now using the initial conditions for the PDE to get initial conditions for these coefficient ODEs

$$\tilde{\varphi}(r,\theta,t=0) = -A\,\frac{r^2}{R^2}\cos(\theta)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[a_{n,m}(0)\cos(n\theta)+b_{n,m}(0)\sin(n\theta)]J_n(\sqrt{\lambda_{n,m}}r) \quad .$$

The initial condition does not have a $\sin(\theta)$ term and therefore $b_{n,m}(0) = 0$. To solve for $a_{n,m}(0)$, we use orthogonality of eigenfunctions and take inner products of both sides with $\cos(n\theta)J_n(\sqrt{\lambda_{n,m}}r)$ and using an appropriate weight function r

$$\begin{aligned} a_{n,m}(0) &= \frac{\langle -A\,\frac{r^2}{R^2}\cos(\theta), J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\rangle_r}{\langle J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta), J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\rangle_r} \\ &= \frac{\int_0^R\int_0^{2\pi} -A(\frac{r}{R})^2\cos(\theta) r J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\,dr\,d\theta}{\int_0^R\int_0^{2\pi} r\,{J_n}^2(\sqrt{\lambda_{n,m}}r)\cos^2(n\theta)\,dr\,d\theta} \\ &= \frac{\int_0^R -A(\frac{r}{R})^2 r J_n(\sqrt{\lambda_{n,m}}r)\,dr\int_0^{2\pi}\cos(\theta)\cos(n\theta)\,d\theta}{\frac{1}{2}\pi R^2 {J_2}^2(\sqrt{\lambda_{n,m}}R)} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } n \neq 1 \\ \frac{-2A}{R^4 J_{n+1}^2(\sqrt{\lambda_{n,m}}R)}\int_0^R r^3 J_1(\sqrt{\lambda_{1,m}}r)\,dr & \text{if } n = 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \neq 1 \\ \frac{-2A}{{J_2}^2(\sqrt{\lambda_{n,m}}R)}\int_0^1 \eta^3 J_1(\sqrt{\lambda_{1,m}}R\eta)\,d\eta & \text{if } n = 1 \end{cases}$$

This also requires the integration property of Bessel functions $\int x^n J_{n-1}(x)dx = x^n J_n(x)$. Applying the second initial condition

$$\frac{\partial\tilde{\varphi}}{\partial t}(r,\theta,t=0) = 0 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[a'_{n,m}(0)\cos(n\theta)+b'_{n,m}(0)\sin(n\theta)]J_n(\sqrt{\lambda_{n,m}}r)$$

implies that $a'_{n,m}(0) = 0$ and $b'_{n,m}(0) = 0$. Now solving the ODEs for $a_{n,m}$ and $b_{n,m}$. First for the case $n \neq 1$, by applying an integrating factor and solving for the roots of the characteristic polynomial $a \sim \exp(\delta t) \rightarrow \delta = \pm c\sqrt{\lambda_{n,m}} i$, then substituting Euler's identity $\exp(ic\sqrt{\lambda_{n,m}}t) = \cos(c\sqrt{\lambda_{n,m}}t) + i\,\sin(c\sqrt{\lambda_{n,m}}t)$. We use the initial conditions to solve for the constants κ_1, κ_2

$$a_{n,m}(t) = \kappa_1 \cos(c\sqrt{\lambda_{n,m}}t) + \kappa_2 \sin(c\sqrt{\lambda_{n,m}}t) \quad \rightarrow \quad \kappa_1 = 0,\ \kappa_2 = 0 \quad \rightarrow \quad a_{n,m}(t) = 0$$

Similarly for all n, $b_{n,m}(t) = 0$. For the case $n = 1$:

$$a''_{n,m}(t) = \frac{2A}{R^2}\cos(\omega t)\,\frac{\int_0^1 (\omega^2\eta^2 + 3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)} = -c^2\lambda_{n,m}a_{1,m}$$

The homogeneous solution to this ODE is of the form (again using an integrating factor approach)

$$a^{\mathrm{hom}}_{n,m}(t) = \kappa_3\cos(c\sqrt{\lambda_{n,m}}t) + \kappa_4\sin(c\sqrt{\lambda_{n,m}}t)$$

and by inspection we find a particular solution

$$a^{\mathrm{part}}_{n,m}(t) = \kappa_5\cos(\omega t) \quad .$$

Solving for k_5 yields

$$\kappa_5 = \frac{1}{c^2\lambda_{n,m} - \omega^2}\frac{2A}{R^2}\frac{\int_0^1 (\omega^2\eta^2 + 3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)} \quad .$$

The general solution for the coefficients is thus

$$a_{1,m}(t) = \kappa_3\cos(c\sqrt{\lambda_{n,m}}t) + \kappa_4\sin(c\sqrt{\lambda_{n,m}}t) + \frac{1}{c^2\lambda_{n,m} - \omega^2}\frac{2A}{R^2}\frac{\int_0^1 (\omega^2\eta^2 + 3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\cos(\omega t)\ .$$

From the initial conditions we find that $\kappa_4 = 0$ and

$$\kappa_3 = \frac{-2A}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\int_0^1 \eta^3 J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta - \frac{1}{c^2\lambda_{n,m} - \omega^2}\frac{2A}{R^2}\frac{\int_0^1 (\omega^2\eta^2 + 3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}$$

Substituting to write the closed form solution for $\tilde{\varphi}$

$$\begin{aligned}\tilde{\varphi}(r,\theta,t) = \sum_{m=1}^{\infty}\Bigg(&\frac{1}{c^2\lambda_{n,m} - \omega^2}\frac{2A}{R^2}\frac{\int_0^1 (\omega^2\eta^2 + 3c^2)\,\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\cos(\omega t)\ \ldots \\ &-\frac{2A}{{J_2}^2\left(\sqrt{\lambda_{1,m}}R\right)}\Bigg[\int_0^1 \eta^3 J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta\ \ldots \\ &+\frac{1}{c^2\lambda_{n,m} - \omega^2}\frac{1}{R^2}\int_0^1 \left(\omega^2\eta^2 + 3c^2\right)\eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right)d\eta\Bigg]\cos\left(c\sqrt{\lambda_{1,m}}t\right)\Bigg)\cos(\theta)J_1\left(\sqrt{\lambda_{1,m}}r\right).\end{aligned}$$

and the solution in the original variable

$$\varphi(r,\theta,t) = A\left(\frac{r}{R}\right)^2\cos(\omega t)\cos(\theta) + \tilde{\varphi}(r,\theta,t) \quad .$$

By inspection of $\tilde{\varphi}(r,\theta,t)$, the coefficient in the summation goes to infinity and "blows up" as the denominator $c^2\lambda_{n,m} - \omega^2 \rightarrow 0$, thus a resonance condition exists when $c^2\lambda_{n,m} = \omega^2$. Recall that $\lambda_{n,m}$ is the eigenmode for the m-th root of the n-th order Bessel function J_n for $m = 1, 2, 3 \ldots$. Thus, $c\sqrt{\lambda_{n,m}}$ can be interpreted as an eigenfrequency which, when excited, results in resonance.