Exercise 3. Swirling your coffee.

When ones swirls their coffee, waves on the surface of the fluid are generated by the motion of the walls. For the small deformation of any planar circular elastic material, the surface deformation, ϕ, satisfies the wave equation in plane polar coordinate:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] \tag{22}$$

where we choose to define the driving force of the wave motion for a material initially at rest through the boundary and initial conditions:

$$\phi(r=R,\theta,t) = A\cos(\omega t)\cos(\theta) \tag{23}$$
$$\phi(r=0,\theta,t) \text{ finite} \tag{24}$$
$$\phi(r,\theta+2\pi,t) = \phi(r,\theta,t) \tag{25}$$
$$\phi(r,\theta,0) = 0 \tag{26}$$
$$\phi_t(r,\theta,0) = 0 \tag{27}$$

Solve for ϕ for all t, θ and r using the method of eigenfunction expansions. Also determine whether there is any resonance for any value of ω and if so what value.

Steps in the solution:

a) Complete separation of variables to determine the eigenfunctions for the operator and boundary conditions:

$$\nabla^2\psi = \left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right] = -\lambda\psi \tag{28}$$
$$\psi(r=R,\theta) = 0 \tag{29}$$
$$\psi(r=0,\theta) \text{ finite} \tag{30}$$
$$\psi(r,\theta+2\pi) = \psi(r,\theta) \tag{31}$$

b) Substract from the solution for ϕ a function which makes the boundary conditions homogeneous but perhaps introduces inhomogeneities in the initial conditions and the equation itself.

c) Use the eigenfunctions determined from the separation of variables in the method of eigenfunction expansions and solve for ϕ.

Answer. a) Consider the homogeneous stationary boundary value problem in equations (28)-(31). Using separation of variables, assume

$$\psi(r,\theta) = \mathcal{R}(r)\Theta(\theta). \tag{32}$$

Then we can rewrite the PDE as

$$\Theta\frac{1}{r}\frac{d}{dr}\left(r\mathcal{R}'\right) + \frac{1}{r^2}\mathcal{R}\Theta'' = -\lambda\Theta\mathcal{R} \implies \frac{1}{\mathcal{R}}r\frac{d}{dr}\left(r\mathcal{R}'\right) + \lambda r^2 = -\frac{1}{\Theta}\Theta'' = \alpha^2 \tag{33}$$

with α^2 some constant. First we solve the angular problem:

$$\Theta'' = -\alpha^2\Theta \implies \Theta(\theta) = a\cos(\alpha\theta) + b\sin(\alpha\theta). \tag{34}$$

Applying the condition $\psi(r, \theta + 2\pi) = \psi(r, \theta)$:

$$a\cos(\alpha\theta + \alpha 2\pi) + b\sin(\alpha\theta + \alpha 2\pi) = a\cos(\alpha\theta) + b\sin(\alpha\theta)$$
$$\implies \cos(\alpha\theta + \alpha 2\pi) = \cos(\alpha\theta) \quad \text{and} \quad \sin(\alpha\theta + \alpha 2\pi) = \sin(\alpha\theta) \implies \alpha_n = n \in \mathbb{Z}^{\geq 0} \tag{35}$$

therefore a solution to the angular problem is

$$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta). \tag{36}$$

Now we solve the radial problem

$$\frac{1}{\mathcal{R}} r \frac{d}{dr}(r\mathcal{R}') + \lambda r^2 = n^2 \implies r\frac{d}{dr}(r\mathcal{R}') + (\lambda r^2 - n^2)\mathcal{R} = 0. \tag{37}$$

The above is in the form of a Bessel equation and therefore the solutions to the radial problem are in the form of Bessel functions of order n:

$$\mathcal{R}(r) = cJ_n(\sqrt{\lambda} r) + dY_n(\sqrt{\lambda} r). \tag{38}$$

Applying boundary conditions:

$$\mathcal{R}(0) \text{ finite} \implies d = 0 \tag{39}$$
$$\mathcal{R}(R) = 0 \implies cJ_n(\sqrt{\lambda_{n,m}} r) = 0 \implies \sqrt{\lambda_{n,m}} R = z_{n,m} \tag{40}$$

where $z_{n,m}$ is the mth zero of the nth order Bessel function J_n. Therefore the eigenfunctions of (28) are of the form

$$\psi_{n,m}(r, \theta) = (a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta))J_n(\sqrt{\lambda_{n,m}} r). \tag{41}$$

b) Now consider the original problem (22). The boundary condition (23) is inhomogeneous so we want to rescale the problem to have homogeneous boundary conditions. Let

$$\tilde{\phi} = \phi - A\left(\frac{r^2}{R^2}\right)\cos(\omega t)\cos(\theta) \tag{42}$$

- note that the r^2/R^2 term is necessary to resolve the singularity from the ODE, you may have tried the transformation without it to see that the term is necessary. Taking time and spatial derivatives of $\tilde{\phi}$ to rewrite the PDE:

$$\phi_t = \tilde{\phi}_t - A\omega\left(\frac{r^2}{R^2}\right)\sin(\omega t)\cos(\theta) \tag{43}$$
$$\phi_{tt} = \tilde{\phi}_{tt} - A\omega^2\left(\frac{r^2}{R^2}\right)\cos(\omega t)\cos(\theta) \tag{44}$$
$$\phi_r = \tilde{\phi}_r + 2A\left(\frac{r}{R^2}\right)\cos(\omega t)\cos(\theta) \tag{45}$$
$$\frac{1}{r}\frac{\partial}{\partial r}(r\phi_r) = \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{\phi}_r) + 4A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta) \tag{46}$$
$$\frac{1}{r^2}\phi_{\theta\theta} = \frac{1}{r^2}\tilde{\phi}_{\theta\theta} - A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta) \tag{47}$$

Now we can put it all together to rewrite the problem in terms of the transformed variable $\tilde{\phi}$:

$$\tilde{\phi}_{tt} - A\omega^2\left(\frac{r^2}{R^2}\right)\cos(\omega t)\cos(\theta) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{\phi}_r) + 4A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta) + \frac{1}{r^2}\tilde{\phi}_{\theta\theta} - A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta)\right] \quad (48)$$

which simplifies to

$$\tilde{\phi}_{tt} - A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta)(\omega^2 r^2 + 3c^2) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{\phi}_r) + \frac{1}{r^2}\tilde{\phi}_{\theta\theta}\right] \quad (49)$$

with the initial and boundary conditions

$$\tilde{\phi}(r = R, \theta, t) = 0 \quad (50)$$

$$\tilde{\phi}(r = 0, \theta, t) \ \text{ finite} \quad (51)$$

$$\tilde{\phi}(r, \theta + 2\pi, t) = \tilde{\phi}(r, \theta, t) \quad (52)$$

$$\tilde{\phi}(r, \theta, 0) = -A\left(\frac{r^2}{R^2}\right)\cos(\theta) \quad (53)$$

$$\tilde{\phi}_t(r, \theta, 0) = 0 \quad (54)$$

Note that the boundary conditions are now homogeneous, but one of the initial condition is inhomogeneous. For compactness, define

$$f(r, \theta, t) = A\left(\frac{1}{R^2}\right)\cos(\omega t)\cos(\theta)(\omega^2 r^2 + 3c^2). \quad (55)$$

c) To solve the new PDE in $\tilde{\phi}$ we expand the inhomogeneity $f(r, \theta, t)$ in the eigenfunctions of the stationary problem:

$$f(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(c_{nm}(t)\cos(n\theta) + d_{nm}(t)\sin(n\theta))J_n(\sqrt{\lambda_{n,m}}r). \quad (56)$$

Using orthogonality of eigenfunctions we can solve for $c_{nm}(t), d_{nm}(t)$:

$$c_{nm}(t) = \frac{\int_0^R\int_0^{2\pi} f(r, \theta, t) r J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta)\ dr\ d\theta}{\int_0^R\int_0^{2\pi} r J_n^2(\sqrt{\lambda_{n,m}}r)\cos^2(n\theta)\ dr\ d\theta} \quad (57)$$

$$d_{nm}(t) = \frac{\int_0^R\int_0^{2\pi} f(r, \theta, t) r J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta)\ dr\ d\theta}{\int_0^R\int_0^{2\pi} r J_n^2(\sqrt{\lambda_{n,m}}r)\sin^2(n\theta)\ dr\ d\theta}. \quad (58)$$

Computing some of the integrals:

$$\int_0^R\int_0^{2\pi} r J_n^2(\sqrt{\lambda_{n,m}}r)\cos^2(n\theta)\ dr\ d\theta = \frac{1}{2}\pi R^2 J_{n+1}^2(\sqrt{\lambda_{n,m}}R) \quad (59)$$

We can rewrite

$$
\begin{aligned}
&\int_0^R \int_0^{2\pi} f(r,\theta,t) r J_n(\sqrt{\lambda_{n,m}} r) \cos(n\theta)\ dr\ d\theta \\
&\qquad = \frac{A}{R^2}\cos(\omega t)\left[\int_0^{2\pi}\cos(\theta)\cos(n\theta)d\theta \int_0^R \left(\omega^2 r^2 + 3c^2\right) r J_n\left(\sqrt{\lambda_{n,m}} r\right) dr\right] \qquad (60)
\end{aligned}
$$

Notice that

$$\int_0^{2\pi}\cos(\theta)\cos(n\theta)d\theta = \begin{cases}\pi \text{ if } n=1\\ 0 \text{ if } n\neq 1\end{cases} \tag{61}$$

and so $c_{nm}(t)$ simplifies to

$$c_{nm}(t) = \begin{cases}0 & \text{if } n\neq 1\\ \frac{2A}{R^4}\cos(\omega t)\frac{\int_0^R\left(w^2r^2+3c^2\right)rJ_1\left(\sqrt{\lambda_{1m}}r\right)dr}{J_2^2\left(\sqrt{\lambda_{1m}}R\right)} & \text{if } n=1\end{cases} \tag{62}$$

Note that

$$\int_0^{2\pi}\sin(n\theta)\cos(\theta)d\theta = 0 \ \ \forall n \tag{63}$$

which means that

$$d_{nm}(t) = 0. \tag{64}$$

So we find that

$$f(r,\theta,t) = \frac{2A}{R^4}\cos(\omega t)\sum_{m=1}^{\infty}\frac{\int_0^R\left(w^2\zeta^2+3c^2\right)\zeta J_1\left(\sqrt{\lambda_{1m}}\zeta\right)d\zeta}{J_2^2\left(\sqrt{\lambda_{1m}}R\right)}\cos(\theta)J_1(\sqrt{\lambda_{n,m}}r) \tag{65}$$

or if we non-dimensionalize the variable in the integral with $R\eta = \zeta$

$$f(r,\theta,t) = \frac{2A}{R^2}\cos(\omega t)\sum_{m=1}^{\infty}\frac{\int_0^1\left(w^2\eta^2+3c^2\right)\eta J_1\left(\sqrt{\lambda_{1m}}R\eta\right)d\eta}{J_2^2\left(\sqrt{\lambda_{1m}}R\right)}\cos(\theta)J_1(\sqrt{\lambda_{n,m}}r). \tag{66}$$

Now we can write down the eigenfunction expansion of $\tilde{\phi}(r,\theta,t)$:

$$\tilde{\phi}(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}(t)\cos(n\theta) + b_{nm}(t)\sin(n\theta)\right)J_n(\sqrt{\lambda_{n,m}}r) \tag{67}$$

Then we can deduce the following relationships from the PDE:

$$n\neq 1: a''_{nm}(t) = -\lambda_{n,m}c^2 a_{nm}(t) \tag{68}$$
$$n = 1: a''_{1m}(t) - c_{1m}(t) = -\lambda_{1,m}c^2 a_{1m}(t) \tag{69}$$
$$\forall\ n: \quad b''_{nm}(t) = -\lambda_{n,m}c^2 b_{nm}(t) \tag{70}$$

Now we can use the initial conditions for the PDE to get initial conditions for these coefficient ODEs:

$$\tilde{\phi}(r,\theta,0) = -A\left(\frac{r^2}{R^2}\right)\cos(\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}(0)\cos(n\theta) + b_{nm}(0)\sin(n\theta)\right)J_n(\sqrt{\lambda_{n,m}}r) \tag{71}$$

We note that the initial condition does not have a $\sin(\theta)$ term and therefore

$$b_{nm}(0) = 0. \tag{72}$$

To solve for $a_{nm}(0)$ we use orthogonality of eigenfunctions and take inner products of both sides with $\cos(n\theta) J_n(\sqrt{\lambda_{n,m}} r)$:

$$\begin{aligned}
a_{nm}(0) = & \frac{\int_0^R \int_0^{2\pi} -A\left(\frac{r}{R}\right)^2 \cos(\theta) r J_n(\sqrt{\lambda_{n,m}} r) \cos(n\theta)\ dr\ d\theta}{\int_0^R \int_0^{2\pi} r J_n^2(\sqrt{\lambda_{n,m}} r) \cos^2(n\theta)\ dr\ d\theta} \\
&= \frac{\int_0^R -A\left(\frac{r}{R}\right)^2 r J_n(\sqrt{\lambda_{n,m}} r) dr \int_0^{2\pi} \cos(\theta)\cos(n\theta)\ d\theta}{\pi R^2 \frac{1}{2} J_2^2(\sqrt{\lambda_{n,m}} R)} \\
&= \begin{cases} 0 & \text{if } n \neq 1 \\ -\frac{2A}{R^4 J_{n+1}^2(\sqrt{\lambda_{1,m}} R)} \int_0^R r^3 J_1(\sqrt{\lambda_{1,m}} r) dr & \text{if } n = 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } n \neq 1 \\ -\frac{2A}{J_2^2(\sqrt{\lambda_{1,m}} R)} \int_0^1 \eta^3 J_1(\sqrt{\lambda_{1,m}} R\eta) d\eta & \text{if } n = 1 \end{cases}
\end{aligned} \tag{73}$$

The second initial condition tells us that

$$\tilde{\phi}_t(r, \theta, 0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (a'_{nm}(0)\cos(n\theta) + b'_{nm}(0)\sin(n\theta)) J_n(\sqrt{\lambda_{n,m}} r) = 0 \tag{74}$$

which implies that

$$a'_{nm}(0) = 0, \quad b'_{nm}(0) = 0. \tag{75}$$

Now we can solve the ODEs for a_{nm}, b_{nm}. First we consider $n \neq 1$ with the initial conditions:

$$a_{nm}(t) = K_1 \cos(c\sqrt{\lambda_{m,n}} t) + K_2 \sin(c\sqrt{\lambda_{m,n}} t) \implies K_1 = 0, k_2 = 0 \implies a_{nm}(t) = 0 \tag{76}$$

Similarly, we find for all n

$$b_{nm}(t) = 0. \tag{77}$$

Now we consider the case $n = 1$:

$$a''_{nm} - \frac{2A}{R^2} \cos(\omega t) \frac{\int_0^1 (w^2\eta^2 + 3c^2)\, \eta J_1\left(\sqrt{\lambda_{1m}} R\eta\right) d\eta}{J_2^2\left(\sqrt{\lambda_{1m}} R\right)} = -c^2 \lambda_{n,m} a_{1m} \tag{78}$$

The homogeneous solution to this ODE is of the form

$$a_{1m}^h(t) = K_3 \cos(c\sqrt{\lambda_{m,n}} t) + K_4 \sin(c\sqrt{\lambda_{m,n}} t) \tag{79}$$

and by inspection we find a particular solution

$$a_{1m}^p(t) = K_5 \cos(\omega t) \tag{80}$$

where

$$K_5 = \frac{1}{c^2 \lambda_{n,m} - \omega^2} \frac{2A}{R^2} \frac{\int_0^1 (w^2\eta^2 + 3c^2)\, \eta J_1\left(\sqrt{\lambda_{1,m}} R\eta\right) d\eta}{J_2^2\left(\sqrt{\lambda_{1,m}} R\right)} \tag{81}$$

and so putting it all together, the general solution for the coefficients is

$$a_{1m}(t) = K_3 \cos(c\sqrt{\lambda_{m,n}}t) + K_4 \sin(c\sqrt{\lambda_{m,n}}t) + \frac{1}{c^2\lambda_{n,m} - \omega^2} \frac{2A}{R^2} \frac{\int_0^1 \left(w^2\eta^2 + 3c^2\right) \eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right) d\eta}{J_2^2\left(\sqrt{\lambda_{1,m}}R\right)} \cos(\omega t) \quad (82)$$

Furthermore from the initial conditions we find that

$$K_4 = 0 \quad (83)$$

$$K_3 = -\frac{2A}{J_2^2(\sqrt{\lambda_{1,m}}R)} \int_0^1 \eta^3 J_1(\sqrt{\lambda_{1,m}}R\eta)d\eta - \frac{1}{c^2\lambda_{n,m} - \omega^2} \frac{2A}{R^2} \frac{\int_0^1 \left(w^2\eta^2 + 3c^2\right) \eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right) d\eta}{J_2^2\left(\sqrt{\lambda_{1,m}}R\right)} \quad (84)$$

So finally we have a solution:

$$\tilde{\phi}(r,\theta,t) = \sum_{m=1}^{\infty} \left(\frac{1}{c^2\lambda_{n,m} - \omega^2} \frac{2A}{R^2} \frac{\int_0^1 \left(w^2\eta^2 + 3c^2\right) \eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right) d\eta}{J_2^2\left(\sqrt{\lambda_{1,m}}R\right)} \cos(\omega t) - \frac{2A}{J_2^2(\sqrt{\lambda_{1,m}}R)} \left(\int_0^1 \eta^3 J_1(\sqrt{\lambda_{1,m}}R\eta)d\eta + \frac{1}{c^2\lambda_{n,m} - \omega^2} \frac{1}{R^2} \int_0^1 \left(w^2\eta^2 + 3c^2\right) \eta J_1\left(\sqrt{\lambda_{1,m}}R\eta\right) d\eta \right) \cos(c\sqrt{\lambda_{1,m}}t) \right) \cos(\theta) J_1(\sqrt{\lambda_{1,m}}r) \quad (85)$$

and the solution in the original variable

$$\phi(r,\theta,t) = A\left(\frac{r}{R}\right)^2 \cos(\omega t)\cos(\theta) + \tilde{\phi}(r,\theta,t) \quad (86)$$