In[87]:= **Clear[cMatch, cTune, zParallel, zSeries, Z, L, r, ω, f, zGood, reZ, imZ, z0];**

In[95]:= **zParallel = (1 / (I * ω * L + r) + 1 / (-I / (ω * cTune)))^(-1)**

Out[95]= $\dfrac{1}{\mathrm{i\,cTune}\,\omega + \dfrac{1}{r + \mathrm{i\,L}\,\omega}}$

In[96]:= **Z = zParallel + -I / (ω * cMatch)**

Out[96]= $-\dfrac{\mathrm{i}}{\mathrm{cMatch}\,\omega} + \dfrac{1}{\mathrm{i\,cTune}\,\omega + \dfrac{1}{r + \mathrm{i\,L}\,\omega}}$

In[97]:= **zGood = ComplexExpand[Z]**

Out[97]= $r \Big/ \left(\left(r^2 + L^2\,\omega^2\right) \left(\dfrac{r^2}{\left(r^2 + L^2\,\omega^2\right)^2} + \left(\mathrm{cTune}\,\omega - \dfrac{L\,\omega}{r^2 + L^2\,\omega^2}\right)^2 \right) \right) +$

$\mathrm{i}\left(-\dfrac{1}{\mathrm{cMatch}\,\omega} - \dfrac{\mathrm{cTune}\,\omega}{\dfrac{r^2}{\left(r^2 + L^2\,\omega^2\right)^2} + \left(\mathrm{cTune}\,\omega - \dfrac{L\,\omega}{r^2 + L^2\,\omega^2}\right)^2} + \right.$

$\left. (L\,\omega) \Big/ \left(\left(r^2 + L^2\,\omega^2\right) \left(\dfrac{r^2}{\left(r^2 + L^2\,\omega^2\right)^2} + \left(\mathrm{cTune}\,\omega - \dfrac{L\,\omega}{r^2 + L^2\,\omega^2}\right)^2 \right) \right) \right)$

$$\text{reZ} = r \Big/ \left(\left(r^2 + L^2\, \omega^2 \right) \left(\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2 \right) \right)$$

$$\text{imZ} = \left(-\frac{1}{\text{cMatch}\, \omega} - \frac{\text{cTune}\, \omega}{\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2} + \right.$$

$$\left. (L\, \omega) \Big/ \left(\left(r^2 + L^2\, \omega^2 \right) \left(\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2 \right) \right) \right)$$

```
(* but hey - since we have to enforce imZ == 0,
it is fruitful to express imZ as a rational function with a common denominator.
*)
```

Out[108]= $r \Big/ \left(\left(r^2 + L^2\, \omega^2 \right) \left(\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2 \right) \right)$

In[122]:= $-\frac{1}{\text{cMatch}\, \omega} - \frac{\text{cTune}\, \omega}{\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2} +$

$(L\, \omega) \Big/ \left(\left(r^2 + L^2\, \omega^2 \right) \left(\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2 \right) \right)$

```
Solve[imZ == 0, cMatch]
```

Out[122]= $-\frac{1}{\text{cMatch}\, \omega} - \frac{\text{cTune}\, \omega}{\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2} +$

$(L\, \omega) \Big/ \left(\left(r^2 + L^2\, \omega^2 \right) \left(\frac{r^2}{\left(r^2 + L^2\, \omega^2 \right)^2} + \left(\text{cTune}\, \omega - \frac{L\, \omega}{r^2 + L^2\, \omega^2} \right)^2 \right) \right)$

$$\left\{ \left\{ \text{cMatch} \to \left(-1 + 2\, \text{cTune}\, L\, \omega^2 - \text{cTune}^2\, r^2\, \omega^2 - \text{cTune}^2\, L^2\, \omega^4 \right) \Big/ \left(\omega^2 \left(-L + \text{cTune}\, r^2 + \text{cTune}\, L^2\, \omega^2 \right) \right) \right\} \right\}$$

```
(* this expression: true when tuned and matched *)
```

In[124]:=

```
(*use built in Together[] to obtain an alternative expression for imZ ...*)


otherImZ = Together[imZ]



(*
== imZ all the time.  when imZ == 0 , the numeator has to also vanish ...
*)
```

Out[125]=
$$\left(-1 + \text{cMatch L } \omega^2 + 2 \text{ cTune L } \omega^2 - \text{cMatch cTune } r^2 \omega^2 - \text{cTune}^2 r^2 \omega^2 - \text{cMatch cTune L}^2 \omega^4 - \right.$$
$$\left. \text{cTune}^2 L^2 \omega^4\right) \Big/ \left(\text{cMatch } \omega \left(1 - 2 \text{ cTune L } \omega^2 + \text{cTune}^2 r^2 \omega^2 + \text{cTune}^2 L^2 \omega^4\right)\right)$$

```
imZNumerator = -1 + cMatch L ω² + 2 cTune L ω² -
    cMatch cTune r² ω² - cTune² r² ω² - cMatch cTune L² ω⁴ - cTune² L² ω⁴;
imZDenominator = cMatch ω (1 - 2 cTune L ω² + cTune² r² ω² + cTune² L² ω⁴);
```

In[130]:=

```
(* try
*)
Solve[imZNumerator == 0, cTune]
```

Out[130]=
$$\left\{\left\{\text{cTune} \rightarrow \frac{1}{2\left(r^2 \omega^2 + L^2 \omega^4\right)} \left(2 \text{ L } \omega^2 - \text{cMatch } r^2 \omega^2 - \text{cMatch L}^2 \omega^4 - \right.\right.\right.$$
$$\left.\left.\sqrt{-4 r^2 \omega^2 + \text{cMatch}^2 r^4 \omega^4 + 2 \text{ cMatch}^2 L^2 r^2 \omega^6 + \text{cMatch}^2 L^4 \omega^8}\right)\right\},$$
$$\left\{\text{cTune} \rightarrow \frac{1}{2\left(r^2 \omega^2 + L^2 \omega^4\right)} \left(2 \text{ L } \omega^2 - \text{cMatch } r^2 \omega^2 - \text{cMatch L}^2 \omega^4 + \right.\right.$$
$$\left.\left.\left.\sqrt{-4 r^2 \omega^2 + \text{cMatch}^2 r^4 \omega^4 + 2 \text{ cMatch}^2 L^2 r^2 \omega^6 + \text{cMatch}^2 L^4 \omega^8}\right)\right\}\right\}$$

In[102]:= `(*`

  `equation 1: enforce real part of impedance is the`
  `characteristic impedance of the transmission line z0 = 50 ohms`
  `*)`

  `Solve[reZ == z0, cTune]`

  `(*`
  `r amnd L are measured properties of the coil,`
  `ω is known by experimenter, and z0 is 50 ohms in our case.`
  `*)`

Out[102]= $\left\{\left\{\text{cTune} \to \left(L\ z0\ \omega^2 - \sqrt{\left(r^3\ z0\ \omega^2 - r^2\ z0^2\ \omega^2 + L^2\ r\ z0\ \omega^4\right)}\right) \Big/ \left(r^2\ z0\ \omega^2 + L^2\ z0\ \omega^4\right)\right\},\right.$
$\left.\left\{\text{cTune} \to \left(L\ z0\ \omega^2 + \sqrt{\left(r^3\ z0\ \omega^2 - r^2\ z0^2\ \omega^2 + L^2\ r\ z0\ \omega^4\right)}\right) \Big/ \left(r^2\ z0\ \omega^2 + L^2\ z0\ \omega^4\right)\right\}\right\}$

  `(*`
  `the quadratic yields two possible values for cTune,`
  `one of which will result in a contradiction`
  `*)`

  `cTunePlus =` $\left(L\ z0\ \omega^2 + \sqrt{r^3\ z0\ \omega^2 - r^2\ z0^2\ \omega^2 + L^2\ r\ z0\ \omega^4}\right) \Big/ \left(r^2\ z0\ \omega^2 + L^2\ z0\ \omega^4\right)$

  `cTuneMinus =` $\left(L\ z0\ \omega^2 - \sqrt{r^3\ z0\ \omega^2 - r^2\ z0^2\ \omega^2 + L^2\ r\ z0\ \omega^4}\right) \Big/ \left(r^2\ z0\ \omega^2 + L^2\ z0\ \omega^4\right)$

  `(* equation 2: no imaginary part.`
  `*)`
  `Solve[imZ == 0, cMatch]`
  `(* its okay to stop at this point technically,`
  `because cTune is determined by equation 1.`
  `*)`

In[84]:= $\left\{\left\{\text{cMatch} \to \left(-1 + 2\ \text{cTune}\ L\ \omega^2 - \text{cTune}^2\ r^2\ \omega^2 - \text{cTune}^2\ L^2\ \omega^4\right) \Big/ \left(\omega^2 \left(-L + \text{cTune}\ r^2 + \text{cTune}\ L^2\ \omega^2\right)\right)\right\}\right\}$

Out[84]= $\left\{\left\{\text{cMatch} \to \left(-1 + 2\ \text{cTune}\ L\ \omega^2 - \text{cTune}^2\ r^2\ \omega^2 - \text{cTune}^2\ L^2\ \omega^4\right) \Big/ \left(\omega^2 \left(-L + \text{cTune}\ r^2 + \text{cTune}\ L^2\ \omega^2\right)\right)\right\}\right\}$

```
(* none of the following is actually necessary to
 solve our problem in a numerical, straight forward way.
*)

CMatch[ctune_] := (-1 + 2 ctune L ω^2 - ctune^2 r^2 ω^2 - ctune^2 L^2 ω^4) /
    (ω^2 (-L + ctune r^2 + ctune L^2 ω^2))
```