NOTES FOR MATH 503 BY PROF. M. MAZUR

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Contents

This course is an introduction to group theory: the second course in the graduate algebra sequence.

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Definition 1.1. Let X be a set. A binary operation on X is a function $f : X \times X \to X$. We will denote $f(x, y)$ by $x\Box y$. A binary operation is said to be associative if $(x\Box y)\Box z = x\Box(y\Box z)$.

Definition 1.2. A monoid is a set M with a binary operation $\cdot$ which is associative and such that $\exists e \in M$ s.t. $e \cdot m = m \cdot e = m$ for all $m \in M$.

Proposition 1.3. *e in the previous definition of monoid is unique.*

Proof. Let e_1 be another element so that $e_1 \cdot m = m \cdot e_1 = m$ for all $m \in M$. Then $e = e_1 \cdot e = e_1$. $\Box$

We can thus uniquely define such e to be the identity element or neutral element of M.

Example 1.4. The natural number $\mathbb{N}$ with addition is a monoid, and $e = 0$.

Definition 1.5. A group is a monoid G s.t. $\forall a \in G\ \exists b \in G$ s.t. $a \cdot b = e$.

Example 1.6. The natural number $\mathbb{N}$ with addition and $e = 0$ is not a group. But the integers $\mathbb{Z}$ with addition and $e = 0$ is a group.

Proposition 1.7. *Let G be a group. If $a \cdot b = 0$, then $b \cdot a = e$.*

Proof. We have $c \in G$ s.t. $b \cdot c = e$. Then $a = a \cdot e = a \cdot (b \cdot c) = (a \cdot b) \cdot c = e \cdot c = c$. Hence, $b \cdot a = e$. □

This also shows that b is unique of a. We call it the inverse of a and denote it a^{-1}.

Definition 1.8. We say that a, b commute if $ab = ba$. In a group, this is the same as $aba^{-1}b^{-1}$.

Definition 1.9. The commutator of $a \cdot b$ is $[a, b] = aba^{-1}b^{-1}$.
Note that some books use $[a, b] = a^{-1}b^{-1}ab$ and, in general, they are different.

Definition 1.10. A group G is commutative or abelian if any two elements commute; i.e., $ab = ba$ for all $a, b \in G$.

In abelian group, we often use additive notation; i.e., denote the operation $+$, $e = 0$, and $a^{-1} = -a$.

Example 1.11. These are some examples of groups.

(1) The trivial group: $\{e\}$ where $e \cdot e = e$.
(2) The integers $\mathbb{Z}$ with addition $+$.
(3) The real $\mathbb{R}$ with addition $+$.
(4) If R is a ring, then $(\mathbb{R}, +)$ is an abelian group. Called the additive group of the ring R.
(5) If R is a ring, the units of $\mathbb{R}$ is $\mathbb{R}^\times = \{a \in R : ab = 1 = ba \text{ for some } b \in R\}$. This is a ring with multiplication and is called the multiplicative group of R.
(6) If K is a field, then the $n \times n$ matrices over K, $M_n(K)$, is a ring. Note that $M_n(K)^\times = \mathrm{GL}_n(K)$, the general linear group of degree n over K.
(7) We have $\mathbb{Z}^\times = \{1, -1\}$. So, $\mathrm{GL}_2(\mathbb{Z}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1\}$ as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Definition 1.12. Let X be a set. Then the symmetry group of S, $S(X) = \mathrm{Sym}(X)$ is the set of all bijections $X \to X$ with composition of functions as the binary operation and $e = \mathrm{id} : X \to X$ by $\mathrm{id}(X) = X$. The inverse of f, f^{-1} is just the inverse function of f (whose existence is guaranteed by bijectivity).

Example 1.13. Let $X = V$ be a vector space. Then $\mathrm{GL}(V)$ is the set of all linear bijections of V.

Definition 1.14. Let $X = \{1, 2, ..., n\}$. The symmetry group or permutation group on n letter is just $S_n = S(X)$.

Consider $X = \{a, b\}$, then $S(X) = S_2$ consists of two element, the identity map id, and $f : X \to X$ by $f(a) = b$ and $f(b) = a$.

Example 1.15. Consider a square $ab - cd$. Let r be the action of rotating $90°$ clockwise and s be the action of reflecting along the axis across ab and cd. Then $D_4 = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3\}$.

Multiplication of two actions gives a new rotation or reflecting, for example, $sr(a) = d$, $sr(b) = c$, $sr(c) = d$, and $sr(d) = a$.

Note that we observe $rs = sr^3$, and can thus write the multiplication table as following.

$\cdot$	1	r	r^2	r^3	s	sr	sr^2	sr^3
1	1	r	r^2	r^3	s	sr	sr^2	sr^3
r	r	r^2	r^3	1	sr^3	s	sr	sr^2
r^2	r^2	r^3	1	r	sr^2	sr^3	s	sr
r^3	r^3	1	r	r^2	sr	sr^2	sr^3	s
s	s	sr	sr^2	sr^3	1	r	r^2	r^3
sr	sr	sr^2	sr^3	s	r^3	1	r	r^2
sr^2	sr^2	sr^3	s	sr	r^2	r^3	1	r
sr^3	sr^3	s	sr	sr^2	r	r^2	r^3	1

Definition 1.16. Let G be a group. Then a subgroup of G is a subset $H \subseteq G$ s.t. $e \in H$ and if $a, b \in H$ then $ab \in H$ and $a^{-1} \in H$.

Proposition 1.17. *With the above definition, the subgroup H is also a group under the restriction of the operation on G to H.*

Proof of this is left as an exercise to the reader.

2. Jan. 28

Example 2.1. The following are examples of groups:

(1) Let X be a set. Then $S(X) = \mathrm{Sym}(X) = \{f : X \to X : f \text{ is a bijection}\}$ with function composition is the symmetry group on X.
(2) Take $X = \{1, ..., n\}$. Then $S_n = S(X)$ is the symmetry (permutation) group on n letter.
(3) Let S be a ring. Then $\mathrm{GL}_n(S) = M_n(S)^\times$ is all invertible $n \times n$ matrices with entries in S. Note that $\mathrm{GL}_1(S) = S^\times$.

Definition 2.2. S with two binary operations $+, \cdot$ is a (unitary) ring if

(1) $(S, +)$ is an abelian group
(2) $(S, \cdot)$ is a monoid
(3) $(a + b) \cdot c = a \cdot c + b \cdot c$ and $c \cdot (a + b) = c \cdot a + c \cdot b$.

Definition 2.3. Let G be a group. Then $H \subseteq G$ is a subgroup if $e \in H$ and $\forall a, b \in H$, $ab \in H$ and $a^{-1} \in H$.

Note that $e \in H$ follows from the closure under multiplication and inverse, given H is nonempty.

Example 2.4. Let G be a group. Then $Z(G) = \{a \in G \text{ s.t. } \forall g \in G \; ag = ga\}$ is the center of the group. As an exercise, check it is a subgroup.

It is easy to see that G is abelian iff $G = Z(G)$.

Note 2.5. One objective in group theory is to understand all subgroups of a given group G. Unfortunately, this is, usually, not easy.

Theorem 2.6. *A subset S of $(\mathbb{Z}, +)$ is a subgroup iff $S = d\mathbb{Z}$ for some $d \geq 0$.*

Proof. The "if" direction is obvious: every $S = d\mathbb{Z}$ is a subgroup.

Let S be a subgroup of $\mathbb{Z}$. If $S = \{0\}$, then $d = 0$ has $S = d\mathbb{Z}$. Otherwise, S has positive elements.

Take the smallest positive element $d \in S$. Take $a \in S$, then $a = nd + k$ where $0 \leq k < d$. But $k = a - nd \in S$ which is necessarily 0 as d being the smallest positive element in S and thus $a \in d\mathbb{Z}$;i.e., $S \subseteq d\mathbb{Z}$.

Since $d \in S$, so $d\mathbb{Z} \subseteq S$. Thus, $S = d\mathbb{Z}$. □

As an exercise, proove that $k\mathbb{Z} \cap m\mathbb{Z} = \text{lcm}(k, m)\mathbb{Z}$.

Proposition 2.7. *The intersection of any collection of subgroups of a group G is also a subgroup.*

Proof. Take $\{H_i\}_{i\in I}$ be a collection of subgroups of G. Then $\forall i \in I$, we have $e \in H_i$; i.e., $e \in \cap H_i$.

Take $a, b \in \cap H_i$, then $\forall i \in I$, $a, b \in H_i$. Thus, $ab \in H_i$ and $a^{-1} \in H_i$. Therefore, $ab \in \cap H_i$ and $a^{-1} \in \cap H_i$. □

Definition 2.8. Let X be a subset of G. Then $\langle X\rangle$ is the intersection of all subgroups containing X, called the subgroup generated by X.

Informally, $\langle X\rangle$ is the smallest subgroup that contains X, but subsets might not be comparable under the partial order relation.

Proposition 2.9. *Let X be a subset of group G. Then $g \in \langle X\rangle$ iff $g = e$ or $g = x_1^{\epsilon_1} \cdot ... \cdot x_s^{\epsilon_s}$ for $x_1, ..., x_s \in X$ and $\epsilon_i = \pm 1$ for all i. Note that it is necessary to list the disjunct $g = e$ as X could be $\emptyset$, in which case, $\langle\emptyset\rangle = \{e\}$.*

Proof. Let $T = \{x_1^{\epsilon_1} \cdot ... \cdot x_s^{\epsilon_s} : x_1, ..., x_s \in X, \epsilon_i = \pm 1\}$ for $X \neq \emptyset$. Then, we have

(1) $e = x^1 x^{-1} \in T$.
(2) If $a, b \in T$, then $ab \in T$.
(3) If $a = x_1^{\epsilon_1} \cdot ... \cdot x_s^{\epsilon_s} \in T$, then $a^{-1} = x_s^{-\epsilon_s} \cdot ... \cdot x_1^{-\epsilon_1} \in T$.

Therefore, T is a subgroup. Now, if H is a subgroup of G, then $X \subseteq H$ implies $T \subseteq H$. Therefore, $T = \langle X\rangle$. □

When $X = \{g\}$, then we often denote $\langle X\rangle = \langle g\rangle$, and it is equal to $\{g^i : i \in \mathbb{Z}\}$.

Definition 2.10. Let $g \in G$. Then $g^n = \begin{cases} \overbrace{g \cdot ... \cdot g}^{n} & n > 0 \\ e & n = 0 \\ \underbrace{g^{-1} \cdot ... \cdot g^{-1}}_{-n} & n < 0 \end{cases}$

As an exercise, shoe that $g^m \cdot g^n = g^{m+n}$ and $(g^m)^n = g^{mn}$ for all $m, n \in \mathbb{Z}$.

Definition 2.11. Groups generated by one element are called cyclic groups; i.e., $G = \langle g\rangle$ is cyclic.

For example, $\mathbb{Z} = \langle 1\rangle$ and in D_4, $\langle r\rangle = \{1, r, r^2, r^3\}$.

Note 2.12. (1) If $g^n \neq g^m$ for all $n \neq m$, then $\langle g\rangle$ is infinite.
(2) If $g^n = g^m$ for some $n > m$, then $g^{n-m} = e$.
(3) Let $k > 0$ be the smallest s.t. $g^k = e$, then $e, g, g^2, ..., g^{k-1}$ are all different.

If $l \in \mathbb{Z}$, $l = ak + r$ where $0 \leq r < k$, then $g^l = g^{ak+r} = e \cdot g^r = g^r$. So, $\langle g\rangle = \{e, g, ..., g^{k-1}\}$.

Definition 2.13. G is finite if G has finitely many element; i.e., $|G| < \infty$. Otherwise, it is infinite.

$g \in G$ is of finite order if $|\langle g \rangle| < \infty$.

The order of $g \in G$ is the smallest $k \in \mathbb{N}$ s.t. $g^k = e$.

Example 2.14. In S_n, take f by $f(1) = 2$, $f(2) = 3$,...,$f(n-1) = n$,$f(n) = 1$. Then, f is of order n. We thus have $\langle f \rangle$ is a cyclic group of order n.

Definition 2.15. A group G_1 is isomorphic to group G_2 if there is a bijection $f : G_1 \to G_2$ s.t. $f(ab) = f(a)f(b)$.

Note 2.16. If $e_1 \in G_1$ and $e_2 \in G_2$ are identities. Then $e_2 f(e_1) = f(e_1) = f(e_1 e_1) = f(e_1)f(e_1)$, and so, $f(e_1) = e_2$.

Also, $e_2 = f(aa^{-1}) = f(a)f(a^{-1})$, and so, $f(a^{-1}) = (f(a))^{-1}$.

Example 2.17. Suppose that $\langle g \rangle$ is infinite. Then $f : \mathbb{Z} \to \langle g \rangle$ by $m \mapsto g^m$ is a bijection. Also, $f(a+b) = g^{a+b} = g^a g^b = f(a)f(b)$. So, f is an isomorphism.

Another example is given by $\{1, -1\}$ with multiplication and $\{0, 1\}$ with addition. These are isomorphic and can be shown by their multiplication table.

$$\begin{array}{c|cc} \cdot & 1 & -1 \\ \hline 1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

Example 2.18. Consider $\mathbb{R}_{>0}$ with multiplication and $\mathbb{R}$ with addition. These are groups. Also, $\mathbb{R}_{>0} \subseteq \mathbb{R}^\times = \langle \mathbb{R}_{>0} \cup \{-1\} \rangle$.

$a \mapsto e^a : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$ is an isomorphism.

Definition 2.19. Let G, H be groups. A function $f : G \to H$ is a homomorphism if $f(ab) = f(a)f(b)$.

3. Jan. 31

Definition 3.1. Let G, H be groups. A function $f : G \to H$ is a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Note 3.2.
(1) f is a homomorphism $\implies$ $f(e_G) = e_h$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$.
(2) f is called a monomorphism if f is injective (1-to-1).
(3) f is called an epimorphism if f is surjective (onto).
(4) f is called an isomorphism if f is bijective; and $f^{-1} : H \to G$ is also an isomorphism.

If there is an isomorphism between G and G, we write $G \cong H$ and consider G, H "the same."

Example 3.3. G a group, $g \in G$. Then there is a homomorphism $f : \mathbb{Z} \to G$ s.t. $f(n) = g^n$ for all n. f is injective iff g has finite order.

Example 3.4. If X and Y are sets and $|X| = |Y|$ then $S(X) \cong S(Y)$.

Proof. Suppose $\phi : X \to Y$ is a bijection, then $S(X) \to S(Y)$ by $f \mapsto \phi f \phi^{-1}$ is an isomorphism. $\square$

Note that if $|X| = n$, then $|S(X)| = n!$.

Example 3.5. R a commutative ring. Then $\det : \mathrm{GL}_n(R) \twoheadrightarrow R^\times$ is a homomorphism.

$$\left| \begin{bmatrix} a & & & & \\ & 1 & & \text{\huge 0} & \\ & & \ddots & & \\ & \text{\huge 0} & & 1 & \\ & & & & 1 \end{bmatrix} \right| = a$$

Example 3.6. For all n, for all R a ring. Let $P : S_n \to \mathrm{GL}_N(R)$ be for $f \in S_n$, define $P_f = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & \text{if } i = f(j) \\ 0 & \text{if otherwise} \end{cases}$; i.e., P_f has only one non-zero entry in every row and every column, and all non-zero entries are 1. Such matrices are called permutation matrices.

For example, let $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$. Then $fg = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Note that $P_f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{fg} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

As an exercise, show that $P_{fg} = P_f P_g$.

In S_n, consider $r = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$. Let $D_n = \langle r, s \rangle$. This is the dihedral group on regular n-gon.

r is rotation by $\frac{2\pi}{n}$ clockwise, s is reflection in perpendicular bisector of $\overline{1n}$, and D_n is all rigid motions of regular n-gon.

As an exercise, show $rs = sr^{n-1}$, order of $r = n$, and order of $s = 2$.

Note that $D_n = \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\}$.

When $n = 5$, then $rs^3rs^4 = rsr = sr^{n-1}r = s$. Note that $srs = r^{n-1}$. D_n is called dihedral group of order $2n$.

Example 3.7. Let G be a group. For $g \in G$, define $L_g : G \to G$ by $a \mapsto g \cdot a$ (Left multiplication by g). Then L_g is a bijection as $ga = gb \implies a = b$ and $g(g^{-1}a) = a$.

We have that $L_g \in S(G)$, so we can define $\phi : G \to S(G)$ by $g \mapsto L_G$. Then $L_g \circ L_h(a) = gha = L_{gh}a$, so, this is an injective homomorphism.

Theorem 3.8 (Caley)**.** *Every group is isomorphic to a subgroup of $S(X)$ for some set X.*

If G is a group and $g \in G$. Define $C_g : G \to G$ by $C_g(a) = gag^{-1}$. Then, C_g is a homomorphism as $C_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = C_g(a)C_g(b)$. Also, C_g is a bijection as $gag^{-1} = gbg^{-1} \implies a = b$ and $g(g^{-1}ag)g^{-1} = a$.

These forms a homomorphism $f : G \to \mathrm{Aut}(G)$ by $g \mapsto C_g$ where $Aut(G)$ is the group of all automorphisms of G under compositions.

Definition 3.9. Elements of the form C_g are called <u>inner automorphisms</u> and C_g is called "conjugation by g."

Note: $\mathrm{Aut}(\mathbb{Z}) = \{\mathrm{id}, x \mapsto -x\}$.

If $G = \langle X \rangle$ and $f, h : G \to H$ are two automorphisms. Show as an exercise that if $f(x) = h(x)$ for all $x \in X$, then $f = h$.

$\mathrm{GL}_2(\mathbb{Z})$ is finitely generated, $(\mathbb{Q},+)$ and $(\mathbb{Q}^\times,\cdot)$ are not.

Show as an exercise that if $f: G \to H$ is a homomorphism, then $f(G)$ is a subgroup of H.

Definition 3.10. Let A, B be subsets of G. Then $AB = \{ab : a \in A, b \in B\}$.

Definition 3.11. Let G be a group. A, B are subsets of G. Then

(1) $AB = \{ab \mid a \in A, b \in B\}$.
(2) $A^{-1} = \{a^{-1} \mid a \in A\}$.
(3) $aB = \{a\}B = L_a(B)$

Let $f : G \to G$ be a homomorphism. Then $H = f(G) \leq G$ and we have $f : G \twoheadrightarrow H \hookrightarrow G$.

Definition 3.12. $f^{-1}(e) = \{a \in G : f(a) = e\} = \ker(f)$ is the kernel of f.

Proposition 3.13. *The kernel of f is a subgroup of G.*

Note 3.14. $f(a) = f(b) \iff f(ab^{-1})f(a)f(b)^{-1} = e \iff ab^{-1} \in \ker(f)$. so, $f^{-1}(f(a)) = a\ker(f) = \ker(f)a$.

Definition 3.15. A subgroup N of G is Normal if $aN = Na$ for all $a \in G$; alternatively, $aNa^{-1} = N$ for all $a \in G$.

(N is normal iff N is preserved by all inner automorphism)

As an exercise, show that If $N \leq G$ and $aNa^{-1} \subseteq N$ for all $a \in G$, then $aNa^{-1} = N$ for all $a \in G$.

Note 3.16. We denote N is a subgroup of G by $N \leq G$ and N is a normal subgroup of G by $N \trianglelefteq G$.

Example 3.17. (1) Every subgroup of an abelian group is normal.
(2) $H = \{e, s\} \subseteq D_4$ has $rH = \{r, rs\} = \{r, sr^3\}$ and $Hr = \{r, sr\} \neq rH$, so not normal.
(3) $N = \{e, r^2\}$ is normal in D_4 as $r^kNr^k = N$ and $sNs^{-1} = N$

Show as an exercise that $Z(D_4) = \{e, r^2\}$.

Proposition 3.18. *If $G = \langle X\rangle$, $X \subseteq G$, then N is normal iff $\forall s \in X$ $sNs^{-1} \subseteq N$ and $s^{-1}Ns \subseteq N$.*

Consider $f : G \twoheadrightarrow H \subseteq G$. We observe that elements of H are in bijective correspondence with subsets of the form $a\ker f$ since if $h \in H$ then $f^{-1}(h) = a\ker f$ for some $a \in G$.

Definition 3.19. Let $K \leq G$. A subset of G of the form aK (Ka) is called a left (right) coset of K in G for $a \in G$.

Proposition 3.20. *$c \in aK$ iff $aK = cK$*

Proof. If $cK = aK$, then $c = c \cdot e \in cK = aK$.

If $c \in aK$, then $c = ak$ for some $k \in K$. so, $cK = akK = a(kK) \subseteq aK$. Also, $a = ck^{-1} \in cK$, so $aK \subseteq cK$. Hence,$cK = aK$. □

Corollary 3.21. *Two left (right) cosets either coinside or are disjoints; i.e., the left (right) cosets partition the group.*

Show as an exercise that $(aK)^{-1} = Ka^{-1}$.

Definition 3.22. $[G : K]$ is the index of K in G which is the number of left (right) cosets of K in G.

Proposition 3.23. *Suppose G is finite, so K is finite. For $a \in G$, $|aK| = |K|$, so all cosets have the same number of elements.*

So, $|G| = [G : K]|K|$.

Corollary 3.24. $|K|||G|$ *if* $K \leq G$.

Corollary 3.25. *If $g \in G$, then the order of g divides $|G|$.*

Corollary 3.26. $g^{|G|} = e$.

Theorem 3.27 (Fermat's Last Theorem)**.** *p a prime, if $p \not| a$ then $p|a^{p-1} - a$.*

Note 3.28. $\mathbb{Z}/p\mathbb{Z}$ is a field. $|(\mathbb{Z}/p\mathbb{Z})^{\times}| = p - 1$, and $a \in (\mathbb{Z}/p\mathbb{Z})^{\times} \implies a^{p-1} = e$.

Proposition 3.29. *$N \trianglelefteq G$ iff every left coset of N is also a right coset.*

The proof is left as an exercise.

Consider $f : G \twoheadrightarrow H \subseteq G$. H is in a bijection w/ cosets of $\ker f$; i.e., $h \leftrightarrow f^{-1}(h)$.

Definition 3.30. G/N is the set of all cosets of a nornal group $N \trianglelefteq G$.

Note 3.31. We can consider $f : G \twoheadrightarrow H$. Then $N = \ker f$, $aN = f(a)$, $bN = f(b)$, so, $abN = f(a)f(b) = f(ab)$. Then, G/N is a group isomorphic to H.

Definition 3.32. Multiplication on G/N by $(aN)(bN) = (ab)N$. Need to check that if $aN = a_1N$, $bN = b_1N$, then $abN = a_1b_1N$.

Proof. We have $a_1 = an_1$, $b_1 = bn_2$. Then $a_1b_1 = an_1bn_2$. $Nb = bN \implies n_1b = bn_3 \implies a_1b = abn_3n_2 = abn_4 \in abN$. □

As an exercise, show that $(aN)(bN) = (ab)N$ as sets.

Proposition 3.33. $(G/N, \cdots)$ *is a group.*

Proof. We have $[(aN)(bN)](cN) = (ab)NcN = (ab)cN = a(bc)N = aN[bNcN]$. $e = N$. $aN \cdot N = aN$. $(aN)(a^{-1}N) = aa^{-1}N = N$. □

We have a canonical map called the quotient map. $\phi : G \to G/N$ by $g \mapsto gN$. It is surjective and is a homomorphism. $\ker \phi = N$.

Example 3.34. Let $G = \mathbb{Z}$. Consider $n\mathbb{Z}$ where $n \geq 0$. Then $\mathbb{Z}/n\mathbb{Z} = \{0 + n\mathbb{Z}, 1 + n\mathbb{Z}, ..., (n-1) + n\mathbb{Z}\}$.

$(a+n\mathbb{Z})+(b+n\mathbb{Z}) = ab+n\mathbb{Z} = (a+b \mod n)+n\mathbb{Z}$ and $(a+n\mathbb{Z})(b+n\mathbb{Z}) = ab+n\mathbb{Z}$. So, $\mathbb{Z}/n\mathbb{Z}$ is a ring.

4. Feb. 7

Theorem 4.1. *Let G be a group. $H \leq G$, then the following are euivalent*

(1) $aH = Ha$ *for all* $a \in G$
(2) $aHa^{-1} = H$ *for all* $a \in G$
(3) $aHa^{-1} \subseteq H$ *for all* $a \in G$
(4) *Every left (right) coset of H is also a right (left) coset.*

If H has these properties, then we call H to be normal, denoted $H \trianglelefteq G$.

Proposition 4.2. *Let $H \leq G$. Suppose for any $a, b \in H$, $(aH)(bH)$ is also a left coset. Then $H \trianglelefteq G$ and $(aH)(bH) = (ab)H$.*

The proof is left as an exercise.

Proposition 4.3. *If $f : G \to K$ is a homomorphism, then* $\ker f \trianglelefteq G$.

Definition 4.4. Let $N \trianglelefteq G$, then G/N is the set of all coset of N in G.

With multiplication defined as $(aN)(bN) = (ab)N$, this is well-defined and G/N is a group called the quotient group of G by N.

The map $\phi : G \to G/N$ by $g \mapsto gN$ is a surjective group homomorphism, called the quotient map and $\ker \phi = N$.

Proposition 4.5. *Suppose $f : G \to H$ is a surjective homomorphism and let $K = \ker f$, $\phi : G \to G/K$ the quotient map.*

Then there is a unique homomorphism $\bar{f} : G/K \to H$ s.t. $\bar{f}\phi = f$ and $\bar{f}$ is an isomorphism.

$$\begin{tikzcd} G \arrow[r, two heads, "f"] \arrow[d, two heads, "\phi"'] & H \\ G/H \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

Proof. If $\bar{f}$ exists, then $\bar{f}(aK) = \bar{f}\phi(a) = f(a)$. so, it is unique if exists.

Define $\bar{f}(aK) = f(a)$. If $aK = bK$, then $a = bk$ for $k \in K$, so $f(a) = f(bk) = f(b)f(k) = f(b)$. Therefore, it is well-defined. □

Proposition 4.6. (1) *Intersection of any collection of normal subgroups of G is still normal.*
(2) *If $x \subseteq G$ and $aXa^{-1} \subseteq X$ for all $a \in G$, then $\langle X \rangle$ is normal.*
(3) *If $N \trianglelefteq G$ and $H \leq G$, then $NH = HN$ is a subgroup of G.*
(4) *If $N \trianglelefteq G$ and $H \trianglelefteq G$ then $NH = HN \trianglelefteq G$.*
(5) *If $N \trianglelefteq G$ and $H \leq G$, then $H \cap N \trianglelefteq H$.*

Proof. (1) $N_i \trianglelefteq G$, $i \in I$. Then $a \cap N_i a^{-1} = \cap a N_i a^{-1} = \cap N_i$.

(2) Let $N = \langle X \rangle$. Then $aXa^{-1} \subseteq X \subseteq N$. So, $\langle aXa^{-1} \rangle = a\langle X \rangle a^{-1} = aNa^{-1} \subseteq N$ and $\langle X \rangle_n = \langle \bigcap_{a \in G} aXa^{-1} \rangle$, where $\langle \cdot \rangle$ is the smallest normal subset containing $\cdot$.

(3) Let $nh = h(h^{-1}nh) = hn' \in HN$ so $NH \subseteq HN$. Similarly $HN \subseteq NH$. So, $NH = HN$.

Note that $NH = \langle N \cup H \rangle$. $nh(n_1h_1) = nhn_1h^{-1}hh_1 \in NH$ and $nh = h^{-1}n^{-1} = h^{-1}nhh^{-1} \in NH$.

(4) $a(HN)a^{-1} = (aHa^{-1})(aNa^{-1}) = HN$.

(5) $h(N \cap H)h^{-1} = (hNh^{-1}) \cap (hHh^{-1}) = N \cap H$. □

Theorem 4.7 (First homomorphism theorem)**.** *Let $\phi : G \to K$ be a surjective homomorphism and $f : G \to H$ a homomorphism s.t. $\ker f \subseteq \ker \phi$. Then there is a unique homomorphism $\bar{f} : K \to H$ s.t. $\bar{f}\phi = f$. Also, $f(G) = \bar{f}(K)$ and* $\ker \bar{f} = \phi \ker f$.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, two heads, "\phi"'] & H \\ K \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

Proof. if $\bar{f}$ exists then $\bar{f}(k) = \bar{f}(\phi g) = f(g)$ for $\phi g = k$. So, $\bar{f}$ is unique if exists.

If $\phi(g_1) = \phi(g_2) = k$, then $g_1 g_2^{-1} \in \ker \phi$ and so $g_1 g_2^{-1} \in \ker f$. Therefore, $f(g_1) = f(g_2)$ and thus $\bar{f}$ is well-defined. Define $\bar{f}(k) = f(g)$ for any $g \in G$ s.t. $\phi(g) = k$. □

Corollary 4.8. *If $\phi : G \to G/N$ is a quotient map. and $N \in \ker f$, then* $\ker \bar{f} = \ker f/N$.

Theorem 4.9 (Correspondence Theorem)**.** *Let $f : G \twoheadrightarrow H$ be a surjective homomorphism. Then*

(1) $K \leq G \implies f(K) \leq H$ $(K \trianglelefteq G \implies f(K) \trianglelefteq H)$.
(2) $T \leq H \implies f^{-1}T \leq G$ *and* $\ker f \subseteq f^{-1}(T)$.
(3) *If* $K \leq G$, *then* $f^{-1}(f(K)) = K \ker f$.
(4) *If* $T \leq H$, *then* $f(f^{-1}(T)) = T$.

To summarize: $T \mapsto f^{-1}(T)$: subgroups of $H \to$ subgroups of G containing $\ker f$ *is a bijective correspondence that preserves inclusion and intersection with normal subgroups corresponding to normal subgroups. In particular, if $f : G \to G/N$ is the quotient map, then subgroups of $G/N \leftrightarrow$ subgroups of G containing N; i.e.,*

$$N \subseteq K \subseteq G \leftrightarrow K/N \subseteq G/N$$

Theorem 4.10 (Second homomorphism theorem)**.** *If $K \trianglelefteq G$, $H \leq G$, $A \trianglelefteq H$. Then $KH \lhd G$, $KA \trianglelefteq KH$ and the quotient map $\phi : KH \to KH/KA$ takes H onto KH/KA and the kernel is $(H \cap K)A$.*

Furthermore, $H/(H \cap K)A \cong KH/KA$ in a canonical way by $h((H \cap K)A) \mapsto h(KA)$. If $A = \{e\}$, then we have $H/H \cap K \cong KH/K$.

Theorem 4.11 (Modular Law)**.** *G a group. H, K, L subgroups of G s.t. $K \subseteq L$. Then $(HK) \cap K = (H \cap L)K$.*

5. Feb. 9

Theorem 5.1 (Homomorphism Theorems)**.** *The following are the four homomorphism theorems.*

(1)
$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, two heads, "\phi"'] & H \\ K \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$
If $\phi : G \twoheadrightarrow K$ is a surjective homomorphism and $f : G \to H$ is a homomorphism s.t. $\ker \phi \subseteq \ker f$. *Then there is a unique homomorphism $\bar{f} : K \to H$ s.t. $\bar{f}\phi = f$. Hence $\bar{f}(k) = f(a)$ and* $\ker \bar{f} = \phi(\ker f)$.

(2) *Let $f : G \twoheadrightarrow H$ a surjective homomorphism then the assignment $K \to f(K)$ is a bijective correspondence between subgroups of G that contain* $\ker f$ *and subgroups of H which preserves inclusion, intersection, and normality.*

(3) *Let $N \trianglelefteq G, H \leq G, A \trianglelefteq H$. Then $H/(H \cap N)A \to NH/NA$ by $h(H \cap N)A \mapsto hNA$ is a group isomorphism. In particular, if $A = \{e\}$, then $H/H \cap N \cong NH/N$.*

$$\begin{tikzcd} NH \arrow[r, "\phi"] & NH/NA \\ H \arrow[u, dash, "\subseteq"] \arrow[ur, "\widetilde{\phi}"'] \arrow[dr, two heads, "quotient"'] & \\ & H/(H \cap N)A \arrow[uu] \end{tikzcd} \qquad \ker \widetilde{\phi} = H \cap NA = (H \cap N)A$$

(4) *Let $K \trianglelefteq G, H \trianglelefteq G, K \subseteq H$. Then $G/H \to (G/K)/(H/K)$ by $gH \mapsto (gK)H/K$ is an isomorphism.*

Example 5.2. G a group, $g \in G$. Consider $\phi : \mathbb{Z} \to G$ by $n \mapsto g^n$. Then $\ker \phi = m\mathbb{Z}$ where m is the order of g. So, $\mathbb{Z}/m\mathbb{Z} \cong \langle g \rangle$

Note that in $\mathbb{Z}/m\mathbb{Z}$, take $a \in \mathbb{Z}$. $a + m\mathbb{Z}$ generates $\mathbb{Z}/m\mathbb{Z}$ iff $\gcd(a, m) = 1$.

Example 5.3. $\det : \mathrm{GL}_n(K) \to K^\times$ is a surjective group homomorphism (for any commutative ring). Note that $\ker(\det) = \mathrm{SL}_n(K)$.

Scalar notation: $aI, a \in K^\times$ form a normal subgroup of $\mathrm{GL}_n(K)$. This is the center.

The quotient $\mathrm{GL}_n(K)/\{aI\} = PGL_n(K) \supseteq PSL_n(K)$.

$$SL_n(\mathbb{Z}/p\mathbb{Z}) \supseteq (\mathbb{Z}/pZ)^\times I \subseteq GL_{p-1}(\mathbb{Z}/p\mathbb{Z}) \overset{\det}{\to} (\mathbb{Z}/p\mathbb{Z})^\times$$

Example 5.4. Consider the permutation group on n letters.

$$S_n \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \overset{\det}{\to} \{1, -1\} = \mathbb{Z}^\times$$

This induces $\pi : S_n \to \mathbb{Z}^{1,-1}$.

Note that π is surjective as $\det \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix} = -1.$

Here $\ker \pi = A_n$ is the alternating group. $[S_n : A_n] = 2$ and $S_n/A_n \cong \{1, -1\} = \mathbb{Z}/2\mathbb{Z}$.

Example 5.5. Let $\phi : G \to \operatorname{Aut} G$ by $g \mapsto C_g$ where $C_g : a \mapsto gag^{-1}$.

Then $\ker \phi = Z(G)$ which is the center of G. $\phi(G) = \operatorname{Inn} G$ which are the inner automorphism on G.

As an exercise, show that $\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G$. $\phi C_g \phi^{-1} = C_{\phi g}$.

Definition 5.6. The outer automorphisms $\operatorname{Out} G = \operatorname{Aut} G/InnG$.

G is <u>complete</u> if $G \to \operatorname{Aut} G$ is an isomorphism.

G is <u>simple</u> if $\{e\}$ and G are the only normal subgroups of G.

Example 5.7. p a prime. Then $\mathbb{Z}/p\mathbb{Z}$ are simple. These are the only simple abelian simple groups.

Proposition 5.8. *G a group. $N \trianglelefteq G, K \trianglelefteq G$. If $N \cap K = \{e\}$ then $nk = kn$ for all $n \in N, k \in K$.*

Proof. Consider $nkn^{-1}k^{-1}$. On one hand, $nkn^{-1} \in K$ and $k^{-1} \in K$, so it is in K. On the other hand, $n \in N$ and $kn^{-1}k^{-1} \in N$, so it is in N. Therefore, $nkn^{-1}k^{-1} \in K \cap N = \{e\}$. Therefore, $nk = kn$. □

Therefore, $NK = N \times K$ if $N, K \trianglelefteq G$ and $N \cap K = \{e\}$.

Definition 5.9. Given a collection of groups $(G_i)_{i \in I}$, we define $\prod_{i \in I} G_i$ to be the set of all functions $f : I \to \cup G_i$ s.t. $\forall i \in I f(i) \in G_i$ where $(g \star f)(i) = f(i)g(i)$, this is the groups called the <u>product</u> of G_i.

Note that this definition corresponds to the strings of g_i where $f \leftrightarrow (g_i)$ s.t., $f(i) = g_i$.

Definition 5.10. For all $i \in I$, we have a homomorphism $\alpha_i : G_i \to \prod G_i$ by $g \mapsto f$ where $f(j) = \begin{cases} e & j \neq i \\ g & i = j \end{cases}$

Also, we have $\pi_i : \prod G_i \to G_i$ by $(g_i) \mapsto g_i$.

Given $\phi_i : H \to G_i$, there is a unique $\phi : H \to \prod G_i$ s.t. $\phi_i = \pi_i \phi$. for all i

Inside of $\prod\limits_{i \in I} G_i$, we have subgroups $\bigoplus\limits_{i \in I} G_i$; the direct sums of G_i which consists of all those f s.t. $f(i) \neq e$ for at most finitely many i.

Proposition 5.11. *Given any collection $\phi_i : G_i \to A_i$ where A_i abelian groups. There is a unique $\phi : \bigoplus\limits_{i \in I} G_i \to A$ s.t. $\phi\alpha_i = \phi_i$ by $\phi((g_i)) = \sum \phi_i(g_i)$.*

Example 5.12. Suppose $\gcd(m, n) = 1$, then $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ as we have $\langle m + mn\mathbb{Z}\rangle \cap \langle n + mn\mathbb{Z}\rangle = \{e\}$ where $\langle m + mn\mathbb{Z}\rangle = \mathbb{Z}/n\mathbb{Z}$ and $\langle n + mn\mathbb{Z}\rangle = \mathbb{Z}/m\mathbb{Z}$.

As an exercise, show that

(1) $n = p_1^{k_1} \cdot ... \cdot p_s^{k_s}$, $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{k_1}\mathbb{Z} \times ... \times \mathbb{Z}/p_s^{k_s}\mathbb{Z}$.
(2) $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z} \times \mathbb{Z}/\operatorname{lcm}(m, n)\mathbb{Z}$.

Consider $A = \bigoplus\limits_{i \in I} \mathbb{Z}$, then every element of A can be uniquely written as $\sum m_i e_i = (m_i)$ for $m_i \in \mathbb{Z}$ and finitely many of them are not zero.

Let G be an abelian group (we use additive notation). Then the elements $(g_i)_{i \in I}$ have the property that $G \cong \bigoplus\limits_{i \in I} \rangle g_i \langle$ is an isomorphism iff $\oplus\langle g_i\rangle = G$ ($\{g_i : i \in I\}$ generates G).

If $m_1 g_1 + ... + m_s g_s = 0$ then $m_1 g_1, ..., m_s g_s = 0$.

6. Feb. 11

Definition 6.1. Let G_i, $i \in I$ be groups. Then $\prod_{i \in I} G_i = \{f : I \to \bigcup\limits_{i \in I} G_i : \forall i \in I\ f(i) \in G_i\}$.

A function f is often denoted $(f_i)_{i \in I}$ where $f_i = f(i)$. We have $(f \star g)(i) = f(i)g(i)$.

There are projections: $\pi_i : \prod G_i \to G_i$ by $\pi_i(f) = f(i)$.

There are also embbedings: $e_i : G_i \to \prod G_i$ by $e_i(g)(j) = \begin{cases} e & j \neq i \\ g & j = i \end{cases}$.

Definition 6.2. The direct sum $\bigoplus\limits_{i \in I} G_i \subseteq \prod G_i$ of the groups G_i consists of f s.t. $f(i) = e$ except for finitely many i.

Proposition 6.3 (Universal Property)**.** *Given an abelian group A and homomorphisms $\phi_i : G_i \to A$, there is a unique $\phi : \bigoplus\limits_{i \in I} G_i \to A$ s.t. $\phi e_i = \phi_i$ by $\phi((g_i)) = \sum_i \phi_i(g_i)$.*

Example 6.4.
(1) V is a vector space over a field K then $(V, +) \cong \bigoplus\limits_{i \in I} K$ for some I.
(2) K a field. Then K contains either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ where p is a prime as a subfield (it is called the prime subfield of K). $(K, +) \cong \begin{cases} \bigoplus\limits_{i \in I} \mathbb{Q} & \mathbb{Q} \subseteq K \\ \bigoplus\limits_{i \in I} \mathbb{Z}/p\mathbb{Z} & \mathbb{Z}/p\mathbb{Z} \subseteq K \end{cases}$

Definition 6.5. A abelian group, (a_i), $i \in I$ some elements in A. The natural homomorphism $\phi : \bigoplus_{i\in I} \langle a_i \rangle \to A$ by $(m_i a_i) \mapsto \sum_{i \in I} m_i a_i$.

1. ϕ is onto iff A is generated by $\{a_i\}_{i\in I}$.
2. ϕ is injective iff whenever $\sum_{i\in I} m_i a_i = 0$, we have $m_i a_i = o$ for all $i \in I$.

If (a_i) has property 2, we say that a_i are independent in A. If in addition they have property 1, we say they form a basis of A.

Example 6.6. $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, we have $\mathbb{Z}/6\mathbb{Z} \cong \langle 3 + 6\mathbb{Z}\rangle \oplus \langle 2 + 6\mathbb{Z}\rangle$. So, $\{1 + 6\mathbb{Z}\}$ is a basis of $\mathbb{Z}/6\mathbb{Z}$ and $\{3 + 6\mathbb{Z}, 2 + 6\mathbb{Z}\}$ is also a basis of $\mathbb{Z}/6\mathbb{Z}$.

Definition 6.7. An abelian group F is called free abelian if it has a basis consisting of elements of infinite orders (then every element $\neq e \in F$ has infinite orders).

$$F \text{ is free abelian} \iff F \cong \oplus_{i\in I}\mathbb{Z}$$

Corollary 6.8. *Every abelian group is a quotient of a free abelian group. An abelian group can be generated by n elements iff it is a quotient of $\mathbb{Z}^n$.*

Proof. If a_i, $i \in I$ generates A, then the maps $\phi_i : \mathbb{Z} \to A$ by $i \mapsto a_i$ gives surjective homomorphism $\bigoplus i \in IZ \twoheadrightarrow A$.

If A is generated by n elements then we get $\mathbb{Z}^n \twoheadrightarrow A$. Conversely, if $\mathbb{Z}^n \twoheadrightarrow A$, then since $\mathbb{Z}^n$ is generated by n elements, we have A is generated by their images. $\square$

Idea: in order to understand n-generated abelian groups, we need to understand subgroups of $\mathbb{Z}^n$.

Example 6.9. $n = 1$, subgroups of $\mathbb{Z}$ are $k\mathbb{Z}$ where $k \geq 0$, so they are all cyclic.

Proposition 6.10. *Let $N \trianglelefteq G$, if N cam be generated by s elements and G/N can be generated by t elements, then G can be generated by $s + t$ elements.*

Proof. Let $a_1, ..., a_s$ generates N and $b_1N, ..., b_tN$ generates G/N. Consider $H = \langle a_1, ..., a_s, b_1, ..., b_t\rangle$. Note that $N \subseteq H$

Also, let $\pi : G \to G/N$, then $\pi(H)$ contains $b_1N, ..., b_tN$. So, $\langle g_1N, ..., g_tN\rangle \subseteq \pi(H)$. So, $\pi(H) = G/N$. By correspondence, $H = G$. $\square$

Corollary 6.11. *A subset of $\mathbb{Z}^n$ can be generated by n-elements.*

Proof. Induction on n. If $n = 1$, $d\mathbb{Z}$ can be generated by d.

Define $K \leq \mathbb{Z}^n$, let $e_1, ..., e_n$ be the standard basis.

$\mathbb{Z} \cong \langle e_1\rangle \subseteq \mathbb{Z}^n \overset{\pi}{\twoheadrightarrow} \mathbb{Z}^{n-1}$. Also, $K \cap \langle e_1\rangle \subseteq K \twoheadrightarrow \pi(K)$. Note that $K \cap \langle e_1$ is a subgroup of $\langle e_1\rangle$, so it is cyclic.

By induction, $\pi(k)$ can be generated by $n - 1$ elements, and $\pi(K) \cong K/(K \cap \langle e_1\rangle)$. $\square$

Note 6.12. Let F be a free abelian group with basis $e_1, ..., e_n$ and A be subgroups generated by $w_1, ..., w_m$ (we don't necessarily have $m \leq n$).

Now, $w_i = \sum_{j=1}^{n} m_{i,j} e_j$ where $m_{i,j} \in \mathbb{Z}$. Let $M = (m_{i,j})$ a $m \times n$ matrix.

Pick $i \neq j$, 1. if we replace w_i by $w_i + kw_j$ and keep the rest unchanged, then we get another generating set and the new matrix M which is obtained from m by adding $k \cdot j$th row to the ith row.

2. if we replace e_j by $e_j - k \cdot e_j$ and keep the rest unchanged, then we get a new basis of F and the corresponding M is obtained from M by adding $k \cdot j$th column to ith column of M.

3. Permuting e_i's permutes the column and permuting w_i's permutes the rows.

We start with M. Find the non-zero entry of the smallest absolute value of M and permute, so it is the $1-1$ entry. Replacing e_i by $-e_i$ we may assume that $k_{1,1} > 0$.

Suppose $k_{1,1} \not| k_{i,1}$ for some i. Then $k_{i,1} = pk_{1,1} + r$ for $0 < r < k_{1,1}$. Subtracting $p\cdot$ 1st row from ith and have $k_{i,1} = r < k_{1,1}$.

Repeat the process, then we have the resulting $\bar{e}_1, ..., \bar{e}_n$ is a basis, $\bar{w}_1 = k_{1,1}\bar{e}_1$ and $\{\bar{w}_2, ..., \bar{w}_m\} \subseteq \rangle \bar{e}_2, ..., \bar{e}_n \langle$.

Theorem 6.13. *There is a basis $\{\bar{e}_1, ..., \bar{e}_n\}$ of F and $k_1|k_2|k_3|...|k_r$ s.t. $k_1\bar{e}_1, ..., k_r\bar{e}_4$ generate A.*

Corollary 6.14. *A is free with basis $k_1\bar{e}_1, ..., k_r\bar{e}_4$.*

Corollary 6.15. *$F/A \cong \mathbb{Z}/k_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/k_r\mathbb{Z} \oplus \mathbb{Z}^{n-s}$.*

7. Feb. 14

Theorem 7.1. *Let F be a free abelian group with basis of size n, and let $\{0\} \neq A < F$. Then there is a basis $e_1, ..., e_n$ of F and positive integers $k_1|k_2|...|k_s$ for some $s \leq n$ s.t. $k_1e_1, k_2e_2, ..., k_3e_3$ generate A.*

The idea of the proof is to start with a basis $b_1, ..., b_n$ of F and generating set $w_1, ..., w_v$ of A. Write $w_i = \sum_j m_{ij}b_j$ and consider $M = (m_{ij})$. By a sequence of operations of the form

(1) For $i \neq j$, replace w_i by $w_i + kw_j$ for some $k \in \mathbb{Z}$.
(2) For $i \neq j$, replace e_i by $e_i + kw_j$ for some $k \in \mathbb{Z}$.
(3) Permute the basis basis elements or the generators of A.
(4) Replace a basis element or generator by its inverse.

transform the bases and generating set, so that the corresponding M is $\left[\begin{array}{ccc|c} k_1 & & 0 & \\ 0 & \ddots & & 0 \\ & & k_s & \\ \hline 0 & & 0 & 0 \end{array}\right]$.

We often call the bases in the theorem a compatible choice of bases of F and A.

Corollary 7.2. *A is free abelian. In general, a subgroup of any free abelian group is free abelian.*

Theorem 7.3. *Let G be a finitely generated abelian group. Then $G \cong \mathbb{Z}/k_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/k_r\mathbb{Z} \oplus \mathbb{Z}^t$ for some $1 < k_1|k_2|...|k_r$ and $t \geq 0$.*

Proof. Since G is n-generated, then we have a surjective map $\mathbb{Z}^n \overset{\pi}{\twoheadrightarrow} G$. If $\ker(\pi) = A$, choose compatible basis $\{e_1, ..., e_n\}$ of $\mathbb{Z}^n$ and $l_1e_1, ..., l_se_s$ of A so that $l_1|l_2|...|l_s$. Then we have $\mathbb{Z}/A \cong \mathbb{Z}/l_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/l_s\mathbb{Z} \oplus \mathbb{Z}^{n-s}$ and if we remove all $l_i = 1$, we have the result. □

Proposition 7.4. *$\mathbb{Z}^n$ can not be generated by fewer than n elements.*

Proof. $\mathbb{Z}^n \subseteq \mathbb{Q}^n$ and if $e_1, ..., e_k$ generates $\mathbb{Z}^n$ as abelian group, then $e_1, ..., e_k$ span $\mathbb{Q}^n$ as $\mathbb{Q}$-vector space.

If $v \in \mathbb{Q}^n$ then $N \cdot v = \mathbb{Z}^n$ and thus $N \cdot v = \sum m_ie_i$, $v = \sum \frac{m_i}{N}e_i$. Therefore, $k \geq n$.) □

Corollary 7.5. *If $k \neq n$ then $\mathbb{Z}^k \ncong \mathbb{Z}^n$.*

Proof. If $k < n$, then $\mathbb{Z}^k$ is generated by k elements, but $\mathbb{Z}^n$ cannot be generated by n elements. □

Definition 7.6. The number of basis elements of a finitely generated abelian group F is unique, and is called the rank of F.

Let $G \cong \mathbb{Z}/k_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/k_r\mathbb{Z} \oplus \mathbb{Z}^t$, where $1 < k_1|k_2|...|k_r$. Then

(1) $\mathbb{Z}/k_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/k_r\mathbb{Z}$ are the elements of finite order, we call it the torsion of G, and denote $T(G)$.
(2) $\mathbb{Z}^t \cong G/T(G)$, so t is the rank of $G/T(G)$.
(3) k_r is the exponent of $T(G)$.
(4) Let r be the smallest number of generator of $T(G)$, $T(G) = \mathbb{Z}/k_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/k_r\mathbb{Z}$ can be generated by r elements.

Let $p|k_1$ be a prime. Then $T(G)/pT(G) = (\mathbb{Z}/p\mathbb{Z})^r$. This is a vector space over $\mathbb{Z}/p\mathbb{Z}$. So cannot be spanned by fewer than r elements.

As an exercise show that k_i is the smallest positive integers so that $k_i \cdot T(G)$ can be generated by $r - i$ elements.

Corollary 7.7. *k_i are unique for G, and called the invariant factors of G.*

Show as an exercise that $r + t$ is the smallest number of generators of G.

Definition 7.8. Let A be an abelian group. Then $T(A)$ is all the elements of of finite order in A. This is a subgroup of A.

Definition 7.9. A subgroup N of G is characteristic if for every $\phi(N) = N$.

As an exercise, show

(1) N is characteristic in G implies that N is normal in G.
(2) $T(A)$ is characteristic in A.

Definition 7.10. A is torsion if $A = T(A)$. A is torsion free if $T(A) = \{0\}$.

Proposition 7.11. *$A/T(A)$ is torsion free.*

Definition 7.12. Given $n \in \mathbb{N}$. Then $nA = \{na : a \in A\} \leq A$, and $A[n] = \{a \in A : na = 0\} \leq A$.

Note that there is a natural injection from $A[n]$ into A, and a natural surjection from A onto nA.

Definition 7.13. Let P be a prime, then $A_p = \{a \in A : p^k a = 0 \text{ for some } k \in \mathbb{N}\} = \bigcup_{k=1}^{\infty} A[p^k]$. We call it the p-primary part of A.

Note that $A[p] \subseteq A[p^2] \subseteq ... \subseteq A[p^n] \subseteq ...$

Definition 7.14. Let H_i for $i \in I$ be a family of subgroups of G. It is a chain if for any $i, j \in I$, either $H_i \subseteq H_j$ or $H_j \subseteq H_i$.

Show as an exercise that the union of any chain of subgroups is a subgroup.

Proposition 7.15. *If A is a torsion abelian group, then $A \cong \bigoplus_{p \text{ prime}} A_p$*

Proof. Since A_p are subgroups, we have the natural embeddings $A_p \hookrightarrow A$. Take the induced homomorphism $\bigoplus_p A_p \to A$. Then $(a_p) \mapsto \sum_p a_p$.

Let $a \in A$, and n be the order of a. Then $n = p_1^{k_1} \cdots p_s^{k_s}$ is its prime factorization. Then $\frac{n}{p_i^{k_i}} a \in A_{p_i}$ since $p_i^{k_i} \cdot \frac{n}{p_1^{k_i}} a = na = 0$.

We observe that $\frac{n}{p_1^{k_1}}, ..., \frac{n}{p_s^{k_s}}$ have non trivial common divisors, so $m_1 \frac{n}{p_1^{k_1}} + ... + m_s \frac{n}{p_s^{k_s}} = 1$ for some $m_1, ..., m_s$. So, $a = m_1 \frac{n}{p_1^{k_1}} a + ... + m_s \frac{n}{p_s^{k_s}} a$.

Suppose $a_{p_1} \in A_{p_i}$ and $a_{p_1} + ... + a_{p_k} = 0$. There is N s.t. $p_i^N \cdot a_{p_i} = 0$ for all p_i. Then $p_2^N \cdot ... \cdot p_t^N (a_{p_1} + \cdot + a_{p_t}) = 0 = p_2^N \cdot ... \cdot_t^N a_{p_1}$, so order of $ap_1 | p_2^N \cdot ... \cdot_t^N$ and so order of $ap_t | p_1^s$, therefore, s=0. $\square$

Note 7.16. G a finite abelian group. Then $G = G_{p_1} \oplus ... \oplus G_{p_s}$ for some p_i. Then $G_{p_i} \cong \mathbb{Z}/p_1^{m_{i1}}\mathbb{Z} \oplus ... \oplus \mathbb{Z}/p_i^{m_{tk_i}}\mathbb{Z}$, $m_{i1} \leq ... \leq m_{ik_i}$.

$G_{p_i} = p_i^{m_{i1} + ... + m_{ik_s}} = p_i^{k_i}$ where $|G| = N = P_1^{k_1} \cdot ... \cdot p_s^{k_s}$.

Corollary 7.17. *Every finite abelian group is a direct sum of cyclic groups of prime power orders and the collection of all prime power order is unique for the group. We call the prime powers appearing elementary divisors.*

Example 7.18. $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/36\mathbb{Z}$

8. Feb. 16

Theorem 8.1. *G finitely generated abelian group. Then*

1. *$G \cong T(G) \times \mathbb{Z}^t$ for some t which is unique and called the (torsion free) rank of G.*
2. *$T(G) \cong \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_s\mathbb{Z}$ is finite, where $1 < k_1 | k_2 | ... | k_s$ are unique for G and called the invariant factors of G.*
3. *$T(G) \cong T(G)_{p_1} \times ... \times T(G)_{p_k}$ where $|T(G)| = p_1^{m_1} ... p_k^{m_k}$, and the invariant factors of $T(G)_{p_i}$ together are unique for G and called the elementary divisors of G.*

   So, $T(G)$ is a direct sum of cyclic groups of prime power order in an essentially unique way.

Definition 8.2. G abelian group, $n \in N$. Then

1. $G[n] = \{g \in G : ng = 0\}$ is a subgroup.
2. $nG = \{ng : g \in G\}$ is a subgroup.
3. p a prime. $G_p = \{g \in G : p^k g = o \text{ for some } k\} = \bigcup_k G[p^k]$ is a subgroup called the p-primary component.
4. $T[G] = \{g : ng = 0 \text{ for some } n > 0\} = \bigcup_n G[n!]$ is a subgroup.

Note, we have $G/T(G)$ is torsion-free.

Theorem 8.3. *If G torsion, then* $G \cong \bigoplus_{p \text{ prime}} G_p$.

Show as an exercise that if G is abelian and G/A is free abelian, then $G \cong A \times G/A$.

Warning: $T(G)$ is not always a direct summand to G ($G \ncong T(G) \times G/T(G)$)

Example 8.4. Consider $(\mathbb{Q}, +)$. Every 2 elements of $\mathbb{Q}$ are dependent, for $\frac{p}{q}$, $\frac{m}{n}$, we have $mq\frac{p}{q} - pn\frac{m}{n} = 0$. So, $\mathbb{Q}$ is not free abelian, it is torsion-free, not cyclic.

Example 8.5. Consider $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with $\cdot$.

$T(S^1) = \mu_\infty$ is all roots of unity which is $\{e^{2\pi i \frac{m}{n}} : \frac{m}{n} \in \mathbb{Q}\}$.

$T(S^1)_p = \mu_p^\infty$ is all roots of unity of p-power order.

We have a surjective homomorphis, $E : (\mathbb{R}, +) \to S^1$ by $t \mapsto e^{2\pi i t} = \cos(2\pi t) + i\sin(2\pi t)$. Here, $\ker E = \mathbb{Z}$. So, $S^1 \cong \mathbb{R}/\mathbb{Z}$ with $E^{-1}(T(S^1)) = \mathbb{Q}$.

So, $\mu_\infty \cong \mathbb{Q}/\mathbb{Z}$ and $\mu_p^\infty = \{\text{rational numbers with } p\text{th power denominators}\}/\mathbb{Z}$

$S^1/T(S^1) \cong (\mathbb{R}/\mathbb{Z})/(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Q} \cong \bigoplus \mathbb{Z}$.

As an exercise. show that $S^1 \cong T(S^1) \times \mathbb{R}/\mathbb{Q}$ where $\mathbb{R}/\mathbb{Q} \cong S^1/T(S^1)$.

Note that μ_p^∞ is infinite but every proper subgroup is finite and cyclic.

Definition 8.6. G abelian, $n \in \mathbb{N}$. Then

(1) $a \in G$ is n-divisible if $a = nb$ for some $b \in G$.
(2) G is n-divisible if all elements of G are n-divisible.
(3) G is divisible if it is n-divisible for every n.

Example 8.7. $\mathbb{Q}$ is divisible, $\mathbb{Q}/\mathbb{Z}$ is divisible, μ_p^∞ are divisible, S^1 is divisible.

Show as an exercise that if G is divisible then G/A is divisible for any $A \leq G$. Also, if A is divisible then $A \leq G \implies G \cong A \oplus G/A$.

Definition 8.8. G is abelian. $A \leq G$, then A is called pure in G if for any $a \in A$ and any $n \in \mathbb{Z}$ if $a = ng$ for some $G \in G$ then $a = nb$ for some $b \in A$ (i.e., $A \cap nG = nA$).

Theorem 8.9. *Every divisible group is a direct sum of groups isomorphic to* $\mathbb{Q}$ *or* μ_p^∞ *for some prime* p.

Note 8.10. A torsion and $A[n] = \{0\}$ then $A = nA$. If $|g| = k$, $\gcd(n, k) = 1$ then $\langle g^n \rangle = \langle g \rangle$.

Theorem 8.11. *G abelian, $A < G$ pure, G/A a direct sum of cyclic groups (i.e., G/A has a basis), then $G \cong A \oplus G/A$*

Theorem 8.12. *$G = G_p$ is an abelian p-group of finite exponent ($G = G[p^k]$ for some k) then G is a direct sum of cyclic groups.*

Corollary 8.13. *If G abelian of finite exponent, then G is a direct sum of cyclic groups.*

Show as an exercise that $T(G)$ is always pure in G.

Theorem 8.14. *If $T(G)$ is of finite exponent then $G \cong T(G) \times G/T(G)$.*

9. Feb. 18

Theorem 9.1. *An abelian group of finite exponent is a direct sum of cyclic groups.*

Theorem 9.2. *If $A \leq G$ and A is pure in G and G/A is a direct sum of cyclic group, then $G \cong A \times G/A$.*

Theorem 9.3. *$A \leq G$ pure and of finite exponent, then $G \cong A \oplus G/A$.*

Theorem 9.4. *If $T(A)$ is of finite exponent then $A \cong T(A) \times A/T(A)$.*

Theorem 9.5 (Kulikov)**.** *G torsion abelian then G has a pure subgroup A which is a direct sum of cyclic groups and G/A is divisible.*

$$A \hookrightarrow G \twoheadrightarrow G/A$$

Let G be a group. $X \subseteq G$ s.t. $G = \langle X \rangle$. This means that every elememnt of G is of the form $g_1^{\epsilon_1}...g_k^{\epsilon_k}$ with $g_i \in X$, $\epsilon_i = \pm 1$.

Usually there are many ways a given element can be written like.

Trivial reasons: We can always insert somewhere gg^{-1} or $g^{-1}g$; $g \in X$.

Question: Are there groups G and $X \subseteq G$ where this is the only reason?

Definition 9.6. X a set. A word of length n over X is a sequence of n elements from X (repetition allowed): $a_1a_2...a_n$ where $a_i \in X$. Note, word of length 0 is the empty word.

$W(X)$ is the set of all finite words. Given 2 words, $u, w \in W(X)$, we can concatenate them with $u \star w = uw$. This is an associative binary operation, and it makes $W(X)$ a monoid. It is called the free monoid on X.

Show as an exercise that given any monoid M and any function $f : X \to M$ it extends uniquely to a homomorphism $W(X) \to M$.

Definition 9.7. X a set. Consider $X \times \{1, -1\}$. We write x for $(x, 1)$ and x^{-1} for $(x, -1)$. Consider $W(X \times \{1, -1\})$.

A word $x_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}$ in $W(X \times \{1, -1\})$ is <u>reduced</u> if whenever $x_i = x_{i+1}$ we have $\epsilon_i \neq -\epsilon_{i+1}$.

$R(X)$ is the set of all reduced words in $W(X \times \{1, -1\})$.

Note 9.8. M is a groupa and $f : X \to M$ then it extends to $f : X \times \{-1, 1\} \to M$ by $(x, 1) \mapsto f(x)$ and $(x, -1) \mapsto f(x)^{-1}$ and it extends to monoid homomorphism $W(X \times \{1, -1\}) \to M$. Clearly equivalent words have the same images in M.

$R(X)$ is the set of reduced words in $W(X \times \{1, -1\})$ and it has a binary operation $u \star w = uw$ and reduced.

This operation has inverses as $(x_1^{\epsilon_1}...x_n^{\epsilon_n})^{-1} = x_n^{-\epsilon_n}...x_1^{-\epsilon_1}$. We have $(x_1^{\epsilon_1}...x_n^{\epsilon_n})(x_n^{-\epsilon_n}...x_1^{-\epsilon_1}) = \emptyset$

Problem is that is this operation associative? Yes, but technical complication.

Definition 9.9. G a group. $X \subseteq G$ a subset. We say X generates G freely if the natural map $R(X) \twoheadrightarrow G$ is bijective (So, X generates G).

If this happens than $R(X)$ is a group.

Note that if X generates freely G, Y generates freely H. $f : X \to Y$ is a bijection, then it extends to an isomorphism $G \to H$.

Example 9.10. Let $X = \{1\}$, we have $G = \mathbb{Z}$ and $\{1\}$ generates freely $\mathbb{Z}$.

Show as an exercise that if X generates freely G, $f : X \to H$ any function to a group H, then it extends uniquely to a homomorphism $G \to H$.

10. Feb. 21

Definition 10.1. X a set. $W(X \times \{1, -1\})$ is the free monoid. Then $R(X)$ is all reduced words in $X \cup X^{-1}$ which is a subgroup of $W(X \times \{1, -1\})$. $R(X)$ has a binary operation with every element "invertible," but not yet established that it is surjective.

Given any group G and a function $X : X \to G$, there is a unique monoid homomorphism $f : W(X \times \{1, -1\}) \to G$ by $x \mapsto f(x)$ and $x^{-1} \mapsto f(x^{-1})$ for $x \in X$ and it restricts to a "homomorphism" on $R(X)$.

Definition 10.2. Let G be a group with generating set X. We say that X generates freely G if the natural map $R(X) \to G$ is a bijection.

If such a group exists, then $R(X)$ is a group.

Note 10.3. If $R(X)$ is not a bijection, then there is a non trivial reduced word w which is mapped onto $e \in G$.

Proof. Choose shortest reduced word u s.t. $f(u) = f(v)$ for some $v \neq u$. If $u = \emptyset$, then $w = v$ works.

Otherwise, suppose u starts with x^ϵ, $x \in X$, $\epsilon = \pm 1$ and $u = x^\epsilon u_1$. If $v = x^\epsilon v_1$, then $f(u) = f(x)^\epsilon f(u_1) = f(x)^\epsilon f(v_j)$. So $f(u_1) = f(v_1)$ and u_1 is shorter which is a contradiction. So, $v \neq x^\epsilon v_1$ and therefore $u^{-1}v$ is reduced and $f(u^{-1}v) = f(u)^{-1}f(v) = e$. So G is freely generated by X iff G is generated by X and no non-trivial reduced word in X represents e. $\square$

Definition 10.4. Assume free group on 2 elements exists, $G = \langle a, b\rangle$ is freely generated by a, b.

Notation, for x a letter, $n \in \mathbb{Z}_{\neq 0}$ define $X^n = \begin{cases} \overbrace{x \cdot ... \cdot x}^{n} & n > 0 \\ \underbrace{x^{-1} \cdot ... \cdot x^{-1}}_{-n} & n < 0 \end{cases}$

Note 10.5. Reduced words in a, b are of the form $a^{n_1}b^{n_2}...c^{n_k}$ where $c = a$ or b, or $b^{n_1}a^{n2}...c^{n_k}$ where $c = a$ or b.

Theorem 10.6. *Let* $a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in SL_2(\mathbb{Z})$. *The subgroup* $\langle a, b\rangle$ *of* $SL_2(\mathbb{Z})$ *is freely generated by* $\{a, b\}$.

Proof. Let w be a non-trivial reduced word in $F(\{a, b\})$. We need to show $w \neq e$ in $\langle a, b\rangle$.

First, assume that w starts with b or b^{-1}; i.e., $w = b^i...c^\epsilon$ where $i, \epsilon \in \{1, -1\}$ and $c \in \{a, b\}$. Take $\delta = \begin{cases} 1 & \text{if } c^\epsilon = a, b, b^{-1} \\ -1 & \text{if} c^\epsilon = a^{-1} \end{cases}$, and $u = a^{-\delta}wa^\delta$. Since $a^{-\delta}$ and a^δ does not cancel with b^i and c^ϵ respectively, u is also a reduced word. If $w = e$, then $u = a^{-\delta}ea^\delta = e$; and if $u = e$, then $w = a^\delta e a^{-\delta}$. So, $w = e$ iff. $u = e$. So, it suffices to show that $w = a^{d_1}b^{d_2}...c^{d_k}$ where $c \in \{a, b\}$, $d_1, ..., d_k \in \mathbb{Z}_{\neq 0}$ is not e.

We will first show by induction that $a^d = \begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix}$ and $b^d = \begin{bmatrix} 1 & 0 \\ dz & 1 \end{bmatrix}$ for $d \in \mathbb{Z}$. By definition, $a^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $a^d = \begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix}$ and $b^d = \begin{bmatrix} 1 & 0 \\ dz & 1 \end{bmatrix}$, then $a^{d+1} = \begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (d+1)z \\ 0 & 1 \end{bmatrix}$ and similarly, $b^{d+1} = \begin{bmatrix} 1 & 0 \\ dz & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d+1)z & 1 \end{bmatrix}$. So, by PMI, this is true for $d \in \mathbb{N}$. Now, since $\begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -dz \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have $a^{-dz} = \begin{bmatrix} 1 & -dz \\ 0 & 1 \end{bmatrix}$; and similarly, we have $b^{-dz} = \begin{bmatrix} 1 & 0 \\ -dz & 1 \end{bmatrix}$. Therefore, $\forall d \in \mathbb{Z}$, $a^d = \begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix}$ and $b^d = \begin{bmatrix} 1 & 0 \\ dz & 1 \end{bmatrix}$.

Now, define (α_i) recursively by $\alpha_0 = 1$, $\alpha_1 = d_1 z$, and for $n \geq 2$, $\alpha_n = \alpha_{n-2} + d_n z \alpha_{n-1}$ where d_n are such powers that are defined in $w = a^{d_1} b^{d_2} ... c^{d_k}$. We will now induct on k to show that $w = \begin{cases} \begin{bmatrix} \alpha_k & \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix} & \text{if } k \text{ is even} \\ \begin{bmatrix} \alpha_{k-1} & \alpha_k \\ \cdot & \cdot \end{bmatrix} & \text{if } k \text{ is odd} \end{cases}$ for $k \in \mathbb{N}$.

If $k = 1$, then $w = a^{d_1} = \begin{bmatrix} 1 & d_1 z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \cdot & \cdot \end{bmatrix}$.

If $k = 2$, then $w = a^{d_1} b^{d_2} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_2 z & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 d_2 z & \alpha_1 \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \alpha_2 & \alpha_1 \\ \cdot & \cdot \end{bmatrix}$.

Now, assume for some odd $k > 2$, we have $a^{d_1} b^{d_2} ... b^{k-1} = \begin{bmatrix} \alpha_{k-1} & \alpha_{k-2} \\ \cdot & \cdot \end{bmatrix}$. Then $a^{d_1} b^{d_2} ... b^{d_{k-1}} a^{d_k} = \begin{bmatrix} \alpha_{k-1} & \alpha_{k-2} \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & d_k z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_{k-1} & \alpha_{k-2} + d_k z \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \alpha_{k-1} & \alpha_k \\ \cdot & \cdot \end{bmatrix}$.

Similarly, assume for some even $k > 2$, we have $a^{d_1} b^{d_2} ... a^{k-1} = \begin{bmatrix} \alpha_{k-2} & \alpha_{d_{k-1}} \\ \cdot & \cdot \end{bmatrix}$ Then $a^{d_1} b^{d_2} ... a^{k-1} b^k = \begin{bmatrix} \alpha_{k-2} & \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_k z & 1 \end{bmatrix} = \begin{bmatrix} \alpha_{k-2} + d_k z \alpha_{k-1} & \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \alpha_k & \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix}$.

Therefore, by PMI, $w = \begin{cases} \begin{bmatrix} \alpha_k & \alpha_{k-1} \\ \cdot & \cdot \end{bmatrix} & \text{if } k \text{ is even} \\ \begin{bmatrix} \alpha_{k-1} & \alpha_k \\ \cdot & \cdot \end{bmatrix} & \text{if } k \text{ is odd} \end{cases}$.

Consider $|\alpha_i|$, we will show that $|\alpha_i|$ is an increasing sequence and thus never $= 0$.

Since $|z| \geq 2$, $\alpha_1 = |d_1 z| = |d_1||z| \geq 2 > |\alpha_0|$ as $d_1 \neq 0$. If $|\alpha_{k-1}| > |\alpha_{k-2}|$, then $|\alpha_k| = |\alpha_{k-2} + d_k z \alpha_{k-1} > |d_k z||\alpha_{k-1}| - |\alpha_{k-2}| > (|d_k z| - 1)|\alpha_{k-1}| > (2-1)|\alpha_{k-1}| = |\alpha_{k-1}|$. Therefore, $|\alpha_i|$ is an increasing sequence by PMI. So, $\forall k$, $|a_k| \neq 0$ and thus $w \neq e$.

Therefore, $\langle a, b \rangle$ is free. □

Proposition 10.7. *Let $x_n = a^n b a^n$ where $n = 1, 2, 3, ...$ Then $H = \langle x_1, x_2, ... \rangle$ is freely generated by $x_1, x_2, ...$*

Proof. x_n^{-1} is represented in G by $a^{-n} b^{-1} a^{-n}$. Elements of $X \cup X^{-1}$ are of the form, $a^m b^{\epsilon_m} a^m$ where $m \in \mathbb{Z}$ and $m \neq 0$. Now reduced words in $R(x_1, ...)$ look like $a^{m_1} b^{\epsilon_1} a^{m_2} b^{\epsilon_2} a^{m_2} ... a^{m_k} b^{\epsilon_k} a^{m_k}$; $\epsilon_i = \text{sign } m_i$ and $m_i + m_{i+1} \neq 0$. So these are also non-trivial reduced words of a, b and hence non-zero. □

Corollary 10.8. *For any finite set X, $R(X)$ is a group (i.e., the operation is associative).*

Corollary 10.9. *For every X, $R(X)$ is a group.*

Proof. Take 3 reduced words, u, v, w. We need $(uv)w = u(vw)$. But $u, v, w \in R(Y)$ for some finite subset Y of X which we know is a group. □

Definition 10.10. A a group. It is free if it is freely generate by a subset X. Then $A = R(X) = \text{Free}(X)$.

Theorem 10.11. *Every group is isomorphic to a quotient of a free group.*

Proof. We have a surjective homomorphism $\text{Free}(G) \twoheadrightarrow G$, so $G \cong \text{Free}(G)/\ker$. □

Definition 10.12. Let $(w_i)_{i\in I}$ be words of $\text{Free}(X)$. Let H be the smallest normal subgroup of $\text{Free}(X)$ generated by $\{w_i : i \in I\}$. Then $\langle X|w_i, i \in I\rangle$ is the group $\text{Free}(X)/H$.

Example 10.13. $\langle\{a\}|a^n\} = \mathbb{Z}/n\mathbb{Z}$, for $n > 0$

Theorem 10.14. *A subgroup of a free group is free.*

11. Feb. 23

Theorem 11.1. *Let* $a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ *and* $H = \langle a, b\rangle$. *Then this is freely generated by* $\{a, b\}$.

Corollary 11.2. *For any set X, the structure $R(X)$ is a group, denoted* $\text{Free}(X)$ *and called the free group on X.*

Theorem 11.3. *Any group is isomorphic to a quotient of a free group.*

Definition 11.4. Given a set X and a collection of reduced words w_i, $i \in I$ in $\text{Free}(X)$. Then $\langle X|w_i, i \in I\rangle = \text{Free}(X)/N$ with N is the smallest normal subgroups of $\text{Free}(X)$ which contains all $w_i, i \in I$. If a group G is isomorphic to $\langle X|w_i, i \in I\rangle$. Then any isomorphism $\langle X|w_i, i \in I\rangle \to G$ is called a presentation of G.

Example 11.5. $D_\infty = \rangle a, b|a^2, b^2\langle = \rangle c, d|d^2, dcd^{-1}c\rangle$.

Definition 11.6. G is called finitely presented if it has a presentation of finitely generators and finitely many relations.

Theorem 11.7. *Any finite group is finitely presented.*

Proof. G finite. $G \cong \text{Free}(X)/N$ where X is finite. So N is of finite index in $\text{Free}(X)$. □

Theorem 11.8. *A subgroup of finite index in a finitely generated group is finitely generated.*

Example 11.9. $\mathbb{Z}^2 \cong \langle a, b|a^{-1}b^{-1}ab\rangle = \text{Free}(\{a, b\})/N$ but N is not finitely generated as $N = [\text{Free}(a, b), \text{Free}(a, b)]$

Goal: To prove the Nielsen-Schreier theorme. A subgroup of any free group is free.

G a group. X a generating set, $H \leq G$. Let S be the set of choice of left coset representatives for H in G s.t. $e \in S$.

For any $g \in G$, there is a unique $\bar{g} \in S$ s.t. $gH = \bar{g}H$.

Note 11.10. $\bar{\bar{g}} = \bar{g}$, $\overline{g_1\bar{g}_2} = \overline{g_1 g_2}$. For $s \in S$, $\bar{s} = s$ and $\forall g$, $\bar{g}^{-1}g \in H$ and for all $h \in H$, $\bar{h} = e$,

Given $g \in G$, $s \in S$, there is unique $t \in S$ s.t. $t^{-1}gs \in H$ with $t = \bar{gs}$. We denote $t^{-1}gs$ by $h(g,s) = (\bar{gs})^{-1}(gs)$; i.e., $h(g,s) = t^{-1}gs$.

Here, $h(g,s)^{-1} = s^{-1}g^{-1}t = h(g^{-1}, t)$.

Proposition 11.11. *Let $Y = \{h(x,s) : x \in X, s \in S\}$, then $Y' = \{h(x^{-1}, s) : x \in X, s \in S\}$. Thus, $H = \langle Y \rangle$.*

Definition 11.12. Let $G = \mathrm{Free}(X)$, $H \leq G$. A set S is called a Schreier set for H if it is a set of left coset representatives for H in G and if a reduced word $x_1^{\epsilon_1}\mu \in S$, then also $\mu \in S$. (with any reduced word in S, all its final sequences are in S).

12. Feb. 25

Theorem 12.1 (Nielson-Schrier)**.** *A subgroup of a free group is free.*

Outline of Proof. $G = \mathrm{Free}(X)$ a free group. $H \leq G$. S a Schrier set for H (S is a left coset representative for H s.t. if a reduced word is in S, then all its final segments are in S).

Given $x \in X$, $s \in S$, there is one unique $t \in S$ s.t. $h(x,s) \in H$ s.t. $t^{-1}xs = h(xs)$. Let $Y = \{h(x,s) : x \in X, s \in S, h(x,s) \neq e\}$ We look at a reduced word $h(x_1, a_1)^{\epsilon_1} ... h(x_n, a_n)^{\epsilon_n} =$

$$t_1^{-1}x_1^{\epsilon_1}s_1t_2^{-1}x_2^{\epsilon_2}s_2...t_n^{-1}x_n^{\epsilon_n}s_n$$

Here, each $t_i^{-1}x_i^{\epsilon_i}s_i$ is a reduced word and study possible collections in $t_i^{-1}x_i^{\epsilon_i}s_i$ show that all the letters $x_i^{\epsilon_i}$ will survive, so this element is not $\emptyset$.

Note that $h(x,s) = e \iff xs \in S$. □

As an exercise, show that if $|X| = k$ and $|S| = [G : H] < \infty$, then $h(x,s) = e$ for exactly $[G : H] - 1$ pairs (x,s), so $|Y| = k[G : H] - [G : H] + 1 = (k-1)[G : H] + 1$.

Theorem 12.2. *A subgroup of index n in a free group of rank k is free of rank $(k-1)n + 1$.*

As an exercise, find a Schrerier set for the commutator subgroup of $\mathrm{Free}(\{a,b\})$. Show that if $N \trianglelefteq \mathrm{Free}(X)$ and $[\mathrm{Free}(X) : N] = \infty$ and $N \neq \{e\}$, then N is not finitely generated. Also, let F_1, F_2 be free subgroups of G s.t. $[G : F_1] = [G : F_2] < \infty$, show that they have the same rank.

Definition 12.3. G a group.

(1) The center of G, $Z(G) = \{a \in G : [g,a] = e \text{ for all } g \in G\}$. Note that $[h,g] = hgh^{-1}g^{-1}$. We always have $Z(G) \trianglelefteq G$.
(2) $[G,G] = G' = \langle\{[h,g] : h,g \in G\}\rangle$ is the derived group of G, also called the commutator subgroup of G.

Theorem 12.4. *Let $f : G \to A$ be a homomorphism to an abelian group. Then $f([h,g]) = e$ for any $h, g \in G$.*

Theorem 12.5. *$[G,G]$ is normal in G.*

Proof. We have $a[h,g]a^{-1} = [aha^{-1}, aga^{-1}] \in [G,g]$. □

Definition 12.6. The abelianization of G is $G^{ab} = G/[G,G]$. This is an abelian group.

Corollary 12.7. *G a group, A abelian, f a homomorphism as shown in this diagram.*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & A \\ G^{ab} \arrow[ur, dotted] & \end{tikzcd}$$

Then, $[G, G] \subseteq \ker f$.

Definition 12.8. G is <u>perfect</u> if $G = [G : G]$;i.e., $G^{ab} = \{e\}$.

Corollary 12.9. *G is simple (has only trivial normal proper subgroup) iff G is abelian or perfect.*

Definition 12.10. A, b subsets of G. Then $[a, b] = \langle [a, b] : a \in A, b \in B \rangle$.

Proposition 12.11. *If $G = KH$ where $K \trianglelefteq G$ and $H \trianglelefteq G$ with $K \cap H = \{e\}$, then $G \cong K \times H$.*

In particular, $[K, H] = \{e\}$

We will now study when $G = KH$, $K \trianglelefteq G$ and $K \cap H = \{e\}$ with no assumptions about the normality of H.

Note 12.12. If $K \trianglelefteq G$, we get a homomorphism $G \to \mathrm{Aut}(K)$ with $g \mapsto C_g : h \mapsto ghg^{-1}$.

The kernel is denoted the <u>centralizer</u> of K in G, $C_G(k)$.

Now, restricting this to H, we get $\phi : H \to \mathrm{Aut}(K)$. Since $G = KH$ and $K \cap H = \{e\}$, we have every $g \in G$ is uniquely expressed as $g = k \cdot h$ where $k \in K$ and $h \in H$ since $kh = k_1 h_1$ implies thta $kk_1^{-1} = h_1 h^{-1} = e$.

Hence, we get a bijection $G \to K \times H$ where $(kh)(k_1 h_1) = k(hk_1h^{-1}) = kC_h(k_1)h$.

Definition 12.13. Given that K, H groups, homomorphism $\phi : H \to \mathrm{Aut}(K)$, define the <u>semidirect prodcut</u> of H by K, $K \rtimes_\phi H = K \times H$ with $(k, h) \star (k_1, h_1) = (k\phi_h(k_1), hh_1)$.

As an exercise, show that this is a group operation on $K \times H$.

Note 12.14. $K \cong K \times \{e\}$, $H \cong \{e\} \times H$, and $hkh^{-1} = \phi_h(k)$.

Example 12.15. A a cyclic group ($A \cong \mathbb{Z}/n\mathbb{Z}$ or $A \cong \mathbb{Z}$), then A always has the following automorphism.

1. $\mathrm{id} : A \to A$.
2. $\phi : a \mapsto a^{-1}$

We note that $\{\mathrm{id}, \phi\} \cong \mathbb{Z}/2\mathbb{Z}$ and if we take $\eta : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$, we have $A \rtimes_\eta \mathbb{Z}/2\mathbb{Z}$ is a dihedral group.

13. Feb. 28

Definition 13.1. K, H groups, $\phi : H \to \mathrm{Aut}(K)$ a homomorphism (denote $\phi_k = \phi(k)$). Then $K \rtimes_\phi H = K \times H$ as a set, with $(k, h) \cdot (k_1, h_1) = (k\phi_h(k_1), hh_1)$.

As an exercise, show that this is a group structure on $K \rtimes_\phi H$ which is called the <u>semi-direct</u> product of H by K, we correspond $(k, 0)$ with K and $(0, h)$ with H.

Theorem 13.2. *We have $K \trianglelefteq K \rtimes_\phi H$, $H \leq K \rtimes_\phi H$, $K \cap H = \{e\}$, $K \rtimes_\phi H = KH$ and $hkh^{-1} = \phi_h(k)$.*

Conversely, if $K \trianglelefteq G$, $H \leq G$, $K \cap H = \{e\}$, $G = KH$, then $G \cong K \rtimes_\phi H$ where $\phi : H \to \mathrm{Aut}(K)$ by $h \mapsto C_h$.

Example 13.3. A abelian. Then $\operatorname{Aut}(A)$ contains id and $\eta : a \mapsto a^{-1}$. So we have a homomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut} A$by $0 \mapsto \mathrm{id}$ and $1 \mapsto \eta$. We thus construct $A \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$.

In particular, if $A = \mathbb{Z}/n\mathbb{Z}$, we have $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong D_n$ and if $A = \mathbb{Z}$, $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong D_\infty$.

Example 13.4. Take $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ where p is a prime. Then $\operatorname{Aut}(N) = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

Take $\eta : N \to N$ by $\eta(a, b) = (a, a + b)$; i.e., $\eta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Note that $\eta^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, so that $\eta^p = \mathrm{id}$.

We thus have a homomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \to \operatorname{Aut}(N)$ by $1 \mapsto \eta$ and then , we have $P = N \rtimes_\phi \mathbb{Z}/p\mathbb{Z}$.

Note that $|P| = p^3$, $\exp(P) = p$, and P is non-abelian.

Show as an exercise that $P \cong \langle a, b, c \mid a^p, b^p, c^p, cbc^{-1}a^{-1}b^{-1}, [a, b], [a, c]\rangle$.

Example 13.5. Let $N = \mathbb{Z}/p^2\mathbb{Z}$. The map $\eta : N \to N$ by $a \mapsto (1 + p)a$ is an automorphism of order p since $\eta^p(a) = (1+p)^p a$ where $(1+p)^a = 1+\binom{p}{1}p+\binom{p}{2}P^2 + ... = a + p^A \equiv 1(\mod p^2)$, so $\eta^p = \mathrm{id}$.

We get a homomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/p^2\mathbb{Z})$ by $1 \mapsto \eta$ and get $Q = (\mathbb{Z}/p^2\mathbb{Z}) \rtimes_\phi \mathbb{Z}/p\mathbb{Z}$.

Note that $|Q| = p^3$, $\exp(Q) = p^2$, and Q is non-abelian.

Show as an exercise that $Q \cong \langle a, b \mid a^{p^2}, b^p, (bab^{-1}a^{-1})^{-p}$.

Theorem 13.6. *If p is an odd prime, then*

1. *Every group of order p is cyclic.*
2. *Every group of order p^2 is abelian (either $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$).*
3. *A non abelian group of order p^3 is isomorphic to either P or Q.*

Also, we have

1. *Every group of exponent 2 is abelian.*
2. *Every group of order 4 is abelian.*
3. *A non-abelian group of order 8 is isomorphic to D_4 or Q_8.*

Example 13.7. R a commutative ring. $R^n = N$, $\operatorname{Aut}(R^n) \supseteq \mathrm{GL}_N(R)$.

Then $\operatorname{Aff}(n, R) = \{f : R^n \to R^n : f(v) = Av+w$ for all $v \in R^n$ and some $A \in \mathrm{GL}_n(R), w \in R^n\}$.

We have $\operatorname{Aff}(n, R) \cong R^n \rtimes \mathrm{GL}_n(R)$.

Given a group G, we want to understand $\operatorname{Aut}(G)$.

Definition 13.8. $K \leq G$ is called <u>characteristic</u> if $\phi(K) = K$ for all $\phi \in \operatorname{Aut}(G)$

As an exercise show that if K is characteristic, the it is normal in G.

Example 13.9. First we have that $Z(G)$ and $[G, G]$ are characteristic in G.

If A abelian, then $A[n]$, nA are characteristic in A for all $n \in \mathbb{N}$. The p-primary component A_p is also characteristic for all p prime. Therefore, $T(A)$ is characteristic.

Recall that $\operatorname{Inn}(G)$ is all inner automorphism of $G \subseteq \operatorname{Aut}(G)$. $\operatorname{Inn}(G) \cong G/Z(G)$. Show as an exercise that $Inn(G) \trianglelefteq \operatorname{Aut}(G)$, so we can define the outer automorphisms of G, $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$.

Definition 13.10. G is called complete if $Z(G) = \{1\}$ and $\mathrm{Aut}(G) = \mathrm{Inn}(G) = G$.

Example 13.11. $\mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Consider $\mathrm{Aut}(D_\infty)$.

Recall, $D_\infty = \langle T, S \mid S^2 = e, STST\rangle$. Given any group G with a, b s.t. $b^2 = e$ and $baba = e$, there is a unique homomorphism $D_\infty \to G$ by $T \mapsto a$ and $S \mapsto b$.

So, we have $D_\infty \to D_\infty$ by $T \mapsto T^\epsilon$ and $S \to ST^2$, to be surjective, $\epsilon = \pm 1$.

For every ϵ, L, there is one such automorphism.

Take $\alpha : D_\infty \to D_\infty$ by $\alpha(T) = T^{-1}$ and $\alpha(S) = S$ and $\beta : D_\infty \to D_\infty$ by $\beta(T) = T$ and $\beta(S) = ST$. Then, $\mathrm{Aut}(D_\infty) = \langle \alpha, \beta\rangle \cong D_\infty$.

We note that $Z(D_\infty) = \{1\}$ and $D_\infty \cong \mathrm{Inn}(D_\infty) \subset \mathrm{Aut}(D_\infty)$ and $\mathrm{Out}(D_\infty) = \mathbb{Z}/2\mathbb{Z}$.

Show as an exercise that $\mathrm{Aut}(D_n) \cong \mathrm{Aff}(1, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$. Note here ,$\mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z}) \subseteq \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$.

14. Mar. 2

We constructs 2 non-abelian group of order p^3, where p is an odd prime. One is of exponent p, the other is of exponent p^2

$\mathrm{Aut}(D_\infty) \cong D_\infty$, $\mathrm{Out}(D_\infty) = \mathbb{Z}/2/Z$, and $\mathrm{Inn}(D_\infty) \cong D_\infty$.

Note 14.1. If H and K are characteristic in $H \times K$, then $\mathrm{Aut}(H \times K) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ as $\eta(h, k) = \phi(h)\psi(k)$.

Example 14.2 (Non-example)**.** $G = (\mathbb{Z}/p\mathbb{Z})^k$, $\mathrm{Aut}(G) = \mathrm{GL}_k(\mathbb{Z}/p\mathbb{Z}) \supset \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \times ... \times \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times \times ... \times (\mathbb{Z}/p\mathbb{Z})^\times$.

$\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times = \{a + n\mathbb{Z} : \gcd(a, n) = 1\}$ where $\phi_a(k) = ak$.

Example 14.3. If $n = p_1^{k_1} ... p_s^{k_s}$, then $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{k_1}\mathbb{Z} \times ... \times \mathbb{Z}/p_s^{k_s}\mathbb{Z}$ and each factor is characteristic, so $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}/p_1^{k_1}\mathbb{Z}) \times ... \times \mathrm{Aut}(\mathbb{Z}/p_s^{k_s}\mathbb{Z})$.

Note 14.4. What is $\mathrm{Aut}(\mathbb{Z}/p^k\mathbb{Z})$? $|(\mathbb{Z}/p^k\mathbb{Z})^\infty| = p^k - p^{k-1}$.

Definition 14.5. The Euler's function $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\infty|$. We have $\phi(p_1^{k_1} ... p_s^{k_s}) = \phi(p_1^{k_1}) ... \phi(p_s^{k_s})$.

If $\gcd(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$.

Lemma 14.6. *1. If $k \geq 2$, then $\bar{5} \in (\mathbb{Z}/2^k\mathbb{Z})^\times$ has order 2^{k-2}.*
2. If $k \geq 1$, then $p + 1 \in (\mathbb{Z}/p^k\mathbb{Z})^\times$ has order p^{k-1}.

Proof. 2. if $K = 1$, then $p + 1 = 1$ has order p^{k-1} in $(\mathbb{Z}/p^k\mathbb{Z})^\times$

Assume $p + 1$ has order p^{k-1} in $(\mathbb{Z}/p^k\mathbb{Z})^\times$. Then $(p + 1)^{pk-1} = 1 + Ap^k$ and assume $p \not| A$.

Look at $(p-1)^{p^k} = [(p+1)^{p^{k-1}}]^p = (1 + Ap^k)^p = 1 + \binom{p}{1}Ap^k + \binom{p}{2}A^2p^{2k} + ... = 1 + p^{k+1}B$ for some $p \not| B$.

From this, we have $(1 + p)^{p^{k-1}} \equiv 1(\mod p^k)$ and $(1 + p)^{p^{k-2}} = 1 + Ap^{k-1} \not\equiv 1(\mod p^k)$ since $p \not| A$. □

Corollary 14.7. $(\mathbb{Z}/2^k\mathbb{Z})^\times = \begin{cases} 1 & k = 1 \\ \mathbb{Z}/2\mathbb{Z} & k = 2 \\ \mathbb{Z}/2^{k-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle\bar{5}\rangle \times \langle\bar{-1} & k \geq 3 \end{cases}$

What about $(\mathbb{Z}/p\mathbb{Z})^\times$?

Theorem 14.8. *If F is a field and $A \subseteq F^\times$ is a finite subgroup then A is cclic.*

Proof. Let N be the exponent of A. So, $a^N = 1$ for all $a \in A$.

Recall that a polynomial of degree k has at most k roots in a field $x^N - 1$ is of degree N so $|A| \leq N$.

A abelian of exponent N, so A has an element a of order N so $|A| \geq |\langle a \rangle| = N$. So, $a = \langle a \rangle$. $\square$

Corollary 14.9. *$(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$; i.e., there is a $a \in \mathbb{Z}$ s.t. $a, a^2, ..., a^{p-1}$ are all distinct $\mod p$. Any such a is called a primitive root module p.*

Theorem 14.10. *$(\mathbb{Z}/p^n\mathbb{Z})^\infty$ is cyclic for odd primes p, $n \geq 1$.*

Proof. $(\mathbb{Z}/p\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ and any b which maps to a generator has order divisible by $p-1$ so some power of b has order $(p-1)$. Here, $(\mathbb{Z}/p^n\mathbb{Z})^\times$ has an element u of power $p-1$ and an element $w = 1 + p$ of order p^{n-1}.

So, uw has order $p^{n-1}(p-1) = \phi(p^n)$. So, $(\mathbb{Z}/p^n\mathbb{Z})^\times = \langle uw \rangle$. $\square$

Theorem 14.11 (Euler)**.** *If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1(\mod n)$. Here, $|(\mathbb{Z}/n\mathbb{Z})^\times| = \phi(n)$.*

Example 14.12. $(\mathbb{Z}/20\mathbb{Z})^\times \cong (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Notice $\phi(20) = 8$.

Definition 14.13. A representation of a group G is a homomorphism $\phi : G \to \mathrm{Aut}(M)$ where $\mathrm{Aut}(M)$ are "symmetries" (or "automorphism") of some sort of object. A presentation is faithful if ϕ is injective.

Example 14.14. M a vector space over a field.

$\phi : G \to GL(M)$ where $GL(M)$ is the group of all invertible linear maps $M \to M$ are linear representations.

Example 14.15. M is a metric space, then $\mathrm{Aut}(M)$ are isometries of M.

Example 14.16. Permutation representations is $G \to \mathrm{Sym}(X) = S(X)$ where $S(X)$ is the group of all permutations of X.

15. Mar. 4

Definition 15.1. A <u>permutation representation</u> of a group G on a set X is a homomorphism $\pi : G \to \mathrm{Sym}(X)$. $\mathrm{Sym}(X) = S(X)$ is the permutation of X.

We call a representation faithful if it is injective.

Definition 15.2. Given a representation $\pi : G \to S(X)$, we define a function $\star : G \times X \to X$ $((g, x) \to g \star x)$ by $g \star x = \pi(g)(x)$.

It has 2 properties:

(1) $g \star (h \star x) = (gh) \star x$
(2) $e \star x = x$

Proof of property 1. We have

$$g \star (h \star x) = g \star (\pi(h)(x)) = \pi(g)(\pi(h))x = \pi(gh)(x) = (gh) \star x$$

$\square$

Definition 15.3. Any function $\star : G \times X \to X$ with properties 1 and 2 is called a <u>left group action</u> of G on X.

Conversely, let $\star : G \times X \to X$ be an action of G on X.

For $g \in G$, define $L_g : X \to X$ by $x \mapsto g \star x$.

Then, by 1, we have $L_g \circ L_h = L_{gh}$ and by 2, we have $L_e = id$; in particular, $L_g \circ L_{g^{-1}} = L_{gg^{-1}} = L_e = id = L_{g^{-1}} = L_g$.

So, each L_g is a bijection. Therefore, $\pi : G \to S(X)$ by $g \to L_g$ is a homomorphism and we get a permutation representation.

We thus conclude that permutation representation and actions are essentially the same thing.

Note 15.4. Let G act on X. We write gx instead of $g \star x$ whenever there are no confusions.

Definition 15.5. For $s \in X$, the orbit of s is the set $O(s) = \{gs : g \in G\}$.

Proposition 15.6. *If $s, t \in X$ then either $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.*

Proof. If $v \in O(s) \cap O(t)$, then $v = as = bt$ for some $a, b \in G$. $O(g) \ni gs = (ga^{-1})(as) = (ga^{-1})(bt) = (ga^{-1}b)t \in O(t)$.

Similarly, we have $O(t) \subseteq O(s)$. So, $O(s) = O(t)$. □

Corollary 15.7. *The orbits of an action on X partition the set X.*

Definition 15.8. The stabilizer of $s \in X$ is the set $\mathrm{St}(s) = \{g \in G : gs = s\}$.

Proposition 15.9. (1) *$\mathrm{St}(s)$ is a subgroup of G.*
(2) *$\mathrm{St}(gs) = g\,\mathrm{St}(s)g^{-1}$.*

Proof. If $h \in \mathrm{St}(s)$, then $(ghg^{-1})(gs) = gh(s) = gs$. The converse is easy to see. □

Definition 15.10. Let G act on X. For $Y \subseteq X$, define:

(1) Stabilizer of Y, $\mathrm{St}(Y) = \{g \in G : gY = Y\} = \{g \in G : gY \in Y, g^{-1}y \in Y$ for all $y \in Y\}$ As an exercise show that $\mathrm{St}(Y)$ is a subgroup of G.
(2) the point-wise stabilizer of Y, $G_Y = \{g \in G : gy = y$ for all $y \in Y\} = \bigcap_{y \in Y} \mathrm{St}(Y)$.

Note that $\mathrm{St}(s) = \mathrm{St}(\{s\}) = G_{\{s\}}$ and we sometimes denote it as G_s.

Also, note that $\mathrm{St}(Y)$ acts on Y.

Definition 15.11. Y is G-stable if $\mathrm{St}(Y) = G$.

Note 15.12. (1) Every orbit is G-stable
(2) Y is G-stable iff it is a union of some collection of orbits.

Definition 15.13. The action is transitive if it has only one orbit.

Definition 15.14. Two actions of G on X and Y are equivalent if there is a bijection $f : X \to Y$ s.t. $f(g(x)) = g(f(x))$ for all $x \in X$.

Question: What does it mean in terms of representations?

Show as an exercise that for a subgroup $H \leq G$, we define G/H to be the set of all left cosets of H in G. Then we have an action of G on G/H by $g(aH) = (ga)H$ and this action is transitive with $\mathrm{St}(eH) = H$.

Theorem 15.15. *Given an action of G on X and $s \in X$, the action of G on $O(s)$ is equivalent to the action of G on the left cosets of $\mathrm{St}(s)$.*

Proof. Consider a map $G/\operatorname{St}(s) \to O(s)$ by $g\operatorname{St}(s) \to gs$.

This map is well defined as if $g\operatorname{St}(s) = g_1\operatorname{St}(s)$, then $g_1 s = gh$ for some $h \in \operatorname{St}(s)$ and so $g_1 s = (gh)s = g(hs) = gs$.

It is also clear that this map is surjective.

Now, suppose that $gs = g_1 s$, then $(g_1^{-1}g)s = s$, so $g_1^{-1}g \in \operatorname{St}(s)$. Therefore, $g_1\operatorname{St}(s) = g\operatorname{St}(s)$. So, it is bijective.

Notice that $\phi(g(a\operatorname{St}(s))) = \phi(ga\operatorname{St}(s)) = (ga)s = g(as) = g(\phi(a\operatorname{St}(s)))$. $\square$

Corollary 15.16. *$|O(s)| = [G : \operatorname{St}(s)]$, and if G is finite, then $|O(s) = \frac{|G|}{|\operatorname{St}(s)|}|$.*

16. Mar. 7

Theorem 16.1. *Let a group G act on a set X. For any $s \in X$, the action of G on the orbit $o(s)$ is equivalent to the action of G on the left cosets of $\operatorname{St}(s)$; i.e., $G/\operatorname{St}(s)$.*

In particular, $|O(s)| = [G : \operatorname{St}(s)]$. If G is finite then $|O(s)| = \frac{|G|}{|\operatorname{St}(s)|}$.

Corollary 16.2. *Any translation action is equivalent to the action of G on G/H, with left multiplcaition for some $H \subseteq G$.*

Note 16.3. In the action of G on G/H, we have

(1) $\operatorname{St}(eH) = H$
(2) the kernel of the action, $\bigcap_{g\in G} gHg^{-1}$ is the largest normal subgroup of G contained in H.

Show as an exercise that the action of G on G/H and G/K are equivalent iff H and K are conjugate in G.

Definition 16.4. A point $s \in X$ is called a fixed point if $O(s) = \{s\}$; i.e., $\operatorname{St}(s) = G_s = G$.

Definition 16.5. For any subset $Y \subseteq G$, the fixed pints of Y, $\operatorname{Fix}(Y) = \{s \in X : gs = s \text{ for all } g \in Y\}$

As an exercise, show that $\operatorname{Fix}(Y) = \operatorname{Fix}(\langle Y\rangle)$.

Note 16.6. For G acting on X, we have

(1) if $H \le G$ then H acts on X.
(2) If $Y \subseteq X$ is G-stable ($\operatorname{St}(Y) = G$), then G acts on Y.
(3) this action induces an action on the power set of X, $P(X)$ (the set of all subsets of X) by $g \cdot Y = \{gy : y \in Y\}$ (Note that $g\emptyset = \emptyset$).
(4) For each $k \le |X|$, the set of all subsets of size k, $P_k(X)$ is G-stable, so G acts on $P_k(X)$

Example 16.7. $G = S_n$ acts on $X = \{1, 2, ..., n\}$, so it acts on $P_k(X)$. This action is transitive so it extends to a partition of X.

We note that $\operatorname{St}(\{1, 2..., k\}) \cong S_k \times S_{n-k}$. So, $|P_k(X)| = \frac{|S_n|}{|S_k \times S_{n-k}|} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$.

Let $n = p^k m$, p a prime, $k > 9$. Let $\pi \in S_n$ be

$$\begin{pmatrix} 1 & \dots & p^k & p^k+1 & \dots & 2p^k & \dots & (m-1)p^k+1 & \dots & mp^k \\ 2 & \dots & 1 & p^k+2 & \dots & p^k+1 & \dots & (m-1)p^k+2 & \dots & (m-1)p^k+1 \end{pmatrix}$$

We note that $|\pi| = p^k$ and $\pi^m(1) = m+1$ for $m < p^k$. So, $\langle\pi\rangle \subseteq S_n$ has order p^k and acts on $P_{p^k}(X)$. What are the fixed points of this action? THe fixed points are exactly the fixed points of π and are $\{1, 2, ..., p^k\}, \{p^{k+1}, ..., 2p^k\}, \{(m-1)p^k + 1, ..., mp^k\}$.

Every orbit of $\langle\pi\rangle$ other than the fixed points on $P_{p^k}(X)$ will have size a positive power of p, hence divisible by p. so $|P_{p^k}(X)| = m + Ap$?

Thus, $\binom{p^k m}{p^k} \equiv m (\mod p)$.

Give a "direct" proof of this thm as an exercise.

Corollary 16.8. *If $p \not| m$, then $p \not| \binom{p^k m}{p^k}$.*

Note 16.9. Let $|G| = p^k m$, $p \not| m$, p a prime, $k > 0$. Then G acts on itself by left multiplication $X = G = G/\{e\}$, so it acts on $P_{p^k}(G)$. But $p \not| \binom{p^k m}{p^k} = |P_{p^k}(G)|$.

So at least one orbit $O(A)$ has size not divisible by p. If $p \not| O(A)$, then $|\operatorname{St}(A)| = \frac{G}{O(A)} \geq p^k$, but $|\operatorname{St}(A)| \leq |A| = p^k$ as if $a \in A$, then $\operatorname{St}(A) \cdot a \subseteq A$.

Thus, $|\operatorname{St}(A)| = p^k$.

Theorem 16.10 (Sylow). *If $|G| = p^k m$ and $p \not| m$, then G has a subgroup of order p^k.*

Any such subgroup is called a Sylow p-subgroup of G.

We have three basic rules for a finite group G acting on a finite set X.

Theorem 16.11. *We have three basic rules for a finite group G acting on a finite set X.*

1. *If G acts transitively on X, then $|X| = \frac{|G|}{|\operatorname{St}(s)|}$ for any $s \in X$.*
2. *If p is a prime and $p \not| |X|$, then $p \not| |O(s)|$ for some $s \in X$.*
3. *If $|G| = p^r$, p a prime and $r > 0$ amd $|\operatorname{Fix}(G)| = f$, then $|X| \equiv f (\mod p)$.*

In particular, if $p \not| |X|$, then $f > 0$, so there is a fixed point. and if $p||X|$ and $f > 0$, then $f > p$ so we have at least p fixed points.

Theorem 16.12 (Cauchy). *If G is a finite group $p||G|$ with p a prime. Then G has an element of order P.*

Proof. Let $|G| = p^k m$, $k > 0$ and $p \not| m$. Then G has a subgroup P of size p^k. Take $1 \neq a \in P$. Then $O(a)$ is a power of p. So some power of a has order p. $\square$

Definition 16.13. A group P is a p-group if every element of P is of finite order $=$ power of p

Corollary 16.14. *A finite group is a p-group iff $|G|$ is a power of p.*

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Theorem 17.1. *Let G be a finite group acting on a finite set X.*

1. *If G acts transitively on X, then $|X| = \frac{|G|}{|\operatorname{St}(s)|}$ for any $s \in X$.*
2. *If p is a prime and $p \not| |X|$, then $p \not| |O(s)|$ for some $s \in X$.*
3. *If $|G| = p^r$, p a prime and $r > 0$ amd $|\operatorname{Fix}(G)| = t$, then $|X| \equiv t (\mod p)$.*
   In particular, if $p \not| |X|$, then $t > 0$, so there is a fixed point. and if $p||X|$ and $t > 0$, then $t \geq p$ so we have at least p fixed points.
4. $|X| = \sum_{orbits\ O} |O| = \sum_{orbits} \frac{|G|}{|\operatorname{St}(s)|}$.

Theorem 17.2. *If $|G| = p^k m$, p a prime, $k > 0$ and $p \not| m$, then G has a subgroup of order p^k.*

Theorem 17.3 (Cauchy)**.** *If G is a finite group, $p||G|$ with p a prime. Then G has an element of order P.*

Corollary 17.4. *A finite group is a p-group iff the size of P, $|P|$ is a power of p.*

Note 17.5. We consider the following "key" action.

Any group G acts on itself by conjugation: $g \star s = gsg^{-1}$ ($G \to \mathrm{Aut}(G) \subseteq S(G)$).

G acts on $P_k(G)$ for all k. The set of fixed points are normal subgroups.

Orbits of this action on G are called conjugacy classes and fixed points are exactly those elements in the center of G.

Definition 17.6. For $X \in G$, the normalizer of X in G, $N_G(X) = \mathrm{St}(X)$ under the conjugation action.

The centralizer of X in G, $C_G(X) = G_X$.

If $H \leq G$, then H is normal in $N_G H$.

In particular, G acts on the set $\mathrm{Syl}_p(G)$ of all sylow p-subgroups of G.

Note 17.7. Let $|G| = p^k m$ where p a prime, $k > 0$, $p \not| m$. Then G acts on $\mathrm{Syl}_p(G)$ by conjugation.

Take $P \in \mathrm{Syl}_p(G)$ and $Q \leq G$ where Q is some power of p.

Q acts on the orbit $O(P)$.

Take a G-orbit of P, $O(P)$. We have

1. $\mathrm{St}(P) = N_G(P) \supseteq P$, so $p^k||N_G(P)|$ and since $|O(p)| = \frac{|G|}{|N_G(P)|}$, so we have $p \not| |O(P)|$
2. consider the action of Q on $O(P)$. $|Q|$ is some power of p and $p \not| |O(P)|$, so there exists a fixed point.

Then, we can take $P_1 \in O(P)$ s.t. Q fixes P_1; i.e., $Q \leq N_G(P_1)$ so $Q \subseteq P_1$.

Corollary 17.8. *If $Q \in \mathrm{Syl}_p(G)$, then $Q = P_1$; i.e., $Q \in O(P)$.*

So, G acts transitively on $\mathrm{Syl}_p(G)$.

Also, Q has only 1 fixed point: Q itself. So, $|\mathrm{Syl}_p(G)| \equiv 1 (\mod p)$.

Theorem 17.9 (Sylow)**.** *Let $|G| = p^k m$ where p a prime, $k > 0$, $p \not| m$.*

1. *G has at least one subgroups of order p^k (Sylow p-subgroup).*
2. *All Sylow p-subgroups are conjugate.*
3. *Let $t_p = |\mathrm{Syl}_p(G)|$. Then $t_p \equiv 1 (\mod p)$ and $t_p|m$.*
4. *Any p-subgroup of G is contained in a Sylow p-subgroup.*

Note that $t_p = 1$ iff G has a normal Sylow p-subgroup.

$$P \hookrightarrow G \twoheadrightarrow G/P$$

and $G \cong P \rtimes G/P$

Example 17.10 (Groups of order pq, $p < q$ primes)**.** Let $|G| = pq$. Let's look at $\mathrm{Syl}_q(G)$. Since $t_q|p$ and $t_q \equiv 1 (\mod q)$, we have $t_q = 1$ as $p < q$.

So, G has a normal Sylow p-subgroup Q. Q is cyclic of order q.

By Cauchy, G has an element a of order p and $H = \langle a \rangle \in \mathrm{Syl}_p(G)$ is a cyclic subgroup of order p in G. Look at t_p, if $t_p = 1$, then $H \trianglelefteq G$, so $G \cong Q \times H$.

If $t_p = q$, then $t_p = q \equiv 1(\mod p)$.

So if $q \equiv 1(\mod p)$, then the only finite group of order pq is $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$.