ENAE 301:

$$\xrightarrow{\text{force}} \boxed{\text{Dynamics}} \xrightarrow{\text{motion}}$$

More generally:

$$\xrightarrow{\text{input}} \boxed{\text{System}} \xrightarrow{\text{output}}$$

ENAE 301:

$$\text{force} \longrightarrow \boxed{\text{Dynamics}} \longrightarrow \text{motion}$$

More generally:

$$\xrightarrow[u(t)]{\text{input}} \boxed{\text{System}} \xrightarrow[y(t)]{\text{output}}$$

A (dynamic) system "transforms" inputs $u(t)$ into outputs $y(t)$.

We must first understand as completely as possible this "transformation".

Simple Hovercraft Example

$$\frac{d}{dt}(mv) = f \quad \text{(dynamics)}$$

$$\frac{d}{dt}(y) = v \quad \text{(Kinematics)}$$

Where:

m = mass (assume constant)
v = velocity
y = position
f = applied force

Thus:

$$\left.\begin{aligned} m\frac{dv}{dt} &= f \\ \frac{dy}{dt} &= v \end{aligned}\right\} \text{Governing DE}$$

Equivalently: $m\dfrac{d^2y}{dt^2} = f$

Driving Force

Force driving system is due to fan:

$$f \approx K_f \, \omega$$

where K_f constant, ω is rotation rate of fan

Similarly: $\omega \approx K_m V_m$

where K_m constant, V_m is voltage applied to motor

Then $\boxed{\ddot{y}(t) = K\,u(t)}$, $K = \left[\frac{K_f K_m}{m}\right]$

treating $V_m(t) = u(t)$ as the input to the system

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Analysis:

Given:

$$\boxed{\begin{aligned}\dot{V}(t) &= K u(t)\\ \dot{y}(t) &= v(t)\end{aligned}}$$

($v(t)$ velocity)

Then:

$$V(t) = V_0 + K\int_{\phi}^{t} u(\tau)\, d\tau$$

$$y(t) = y_0 + \int_{\phi}^{t} V(\sigma)\, d\sigma$$

Take $V_0 = y_0 = \phi$ for simplicity now, then

$$y(t) = K\int_{\phi}^{t}\left[\int_{\phi}^{\sigma} u(\tau)\, d\tau\right] d\sigma$$

So: $$y(t) = K\int_{\phi}^{t}\int_{\phi}^{\sigma} u(\tau)\, d\tau\, d\sigma$$ (Double integral!)

Or: $$y(t) = K\int_{\phi}^{t} (t-\tau)\, u(\tau)\, d\tau$$ (How...?)

Example Control Problem

Find $u(t)$ so that, for a specified t_f, y_f

$$V(t_f) = \emptyset \Rightarrow \boxed{\begin{aligned} \emptyset &= K\int_{\emptyset}^{t_f} u(\tau)\,d\tau \\ y_f &= K\int_{\emptyset}^{t_f} (t_f - \tau)\,u(\tau)\,d\tau \end{aligned}}$$

$$y(t_f) = y_f \Rightarrow$$

Solve for $u(\tau)$

Here, we are assuming vehicle starts at rest ($V(\emptyset) = \emptyset$) on the "start line" ($y(\emptyset) = \emptyset$).

Want the vehicle to move to position y_f in t_f seconds and stop there.

Many sol'ns $u(t)$ possible! Typically would also constrain:

1.) $|u(t)| \leq u_{max}$

2.) Behavior of $y(t)$, $t \in [\emptyset, t_f]$

Issues

1.) m, K_f, K_m Not Known precisely:

Hovercraft will not stop exactly where we want.

2.) Requires an accurate clock:

Must use correct $u(t)$ at exactly right times t.

3.) Cannot handle an external ("disturbance") force:

Headwind or cross-breeze will drive hovercraft off the track.

Mathematically sound, but not practical!

Do you drive like that? I hope not!

<u>Mathematically sound, but not practical!</u>

Do you drive like that? I hope not!

Instead you continually <u>compare</u> where you are ($y(t)$) with where you want to be ($y_d(t)$) and continually adjust actions ($u(t)$) based on difference.

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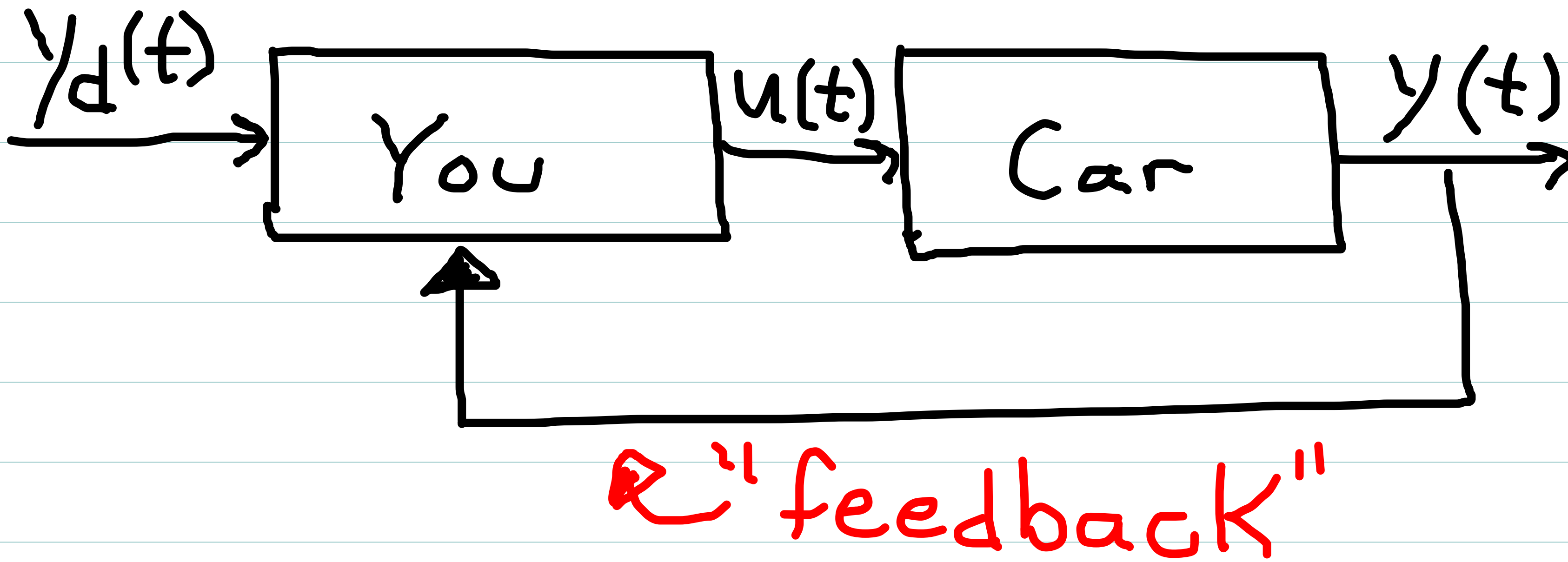
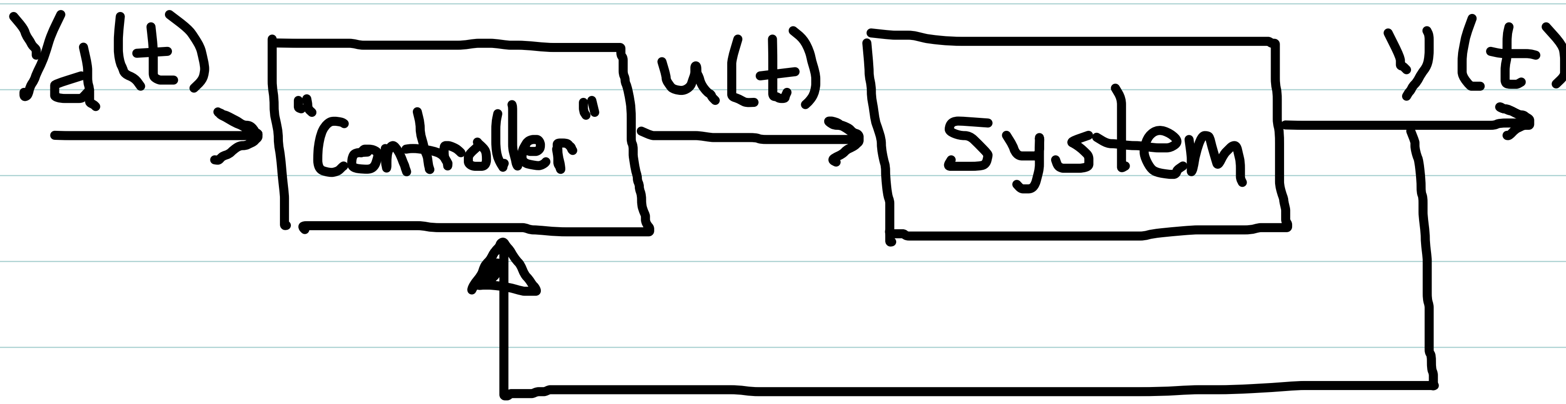


Feedback Control



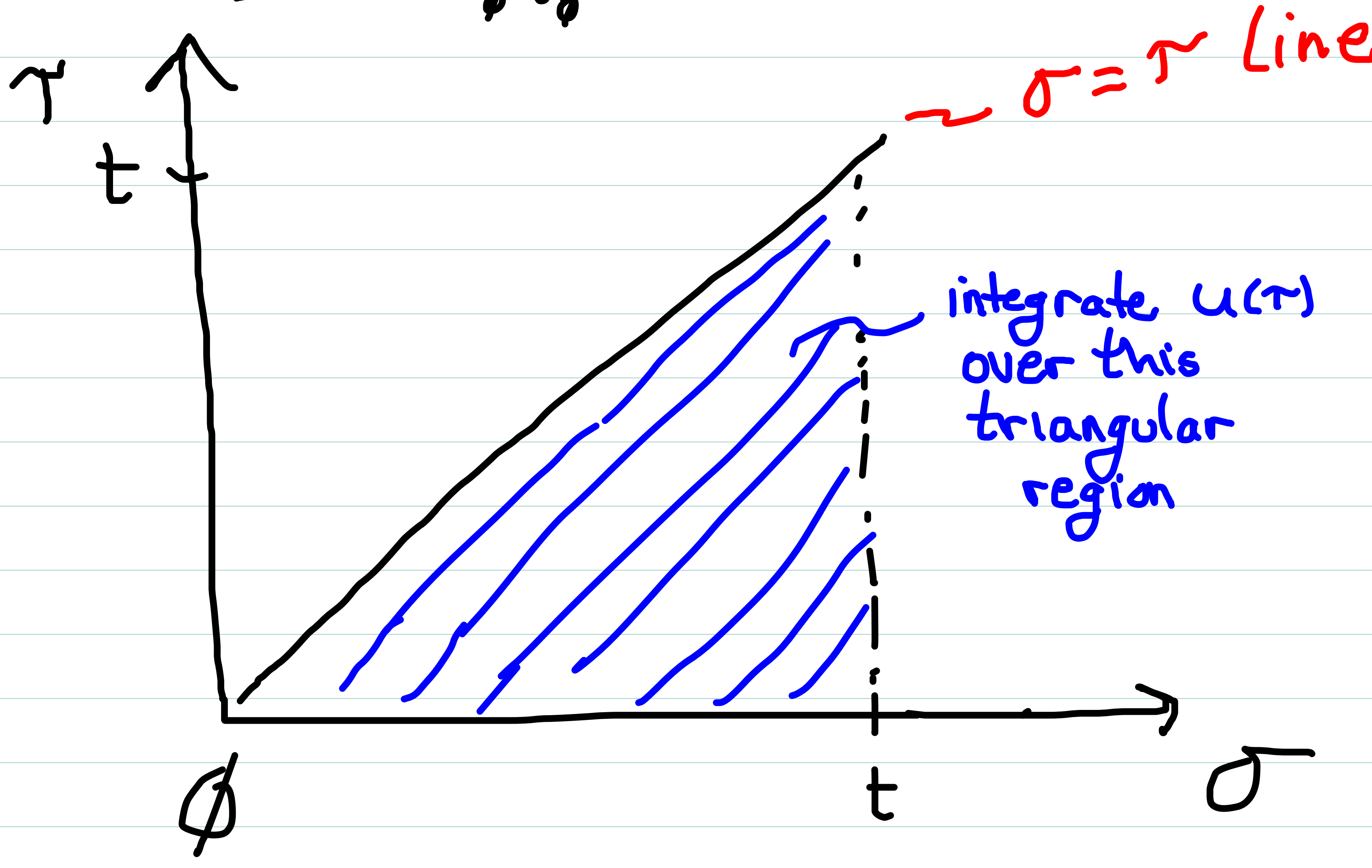
The controller is a mathematical algorithm (implemented as a computer program) which calculates required $u(t)$ from $y(t)$ and $y_d(t)$.

Addresses all 3 issues: uncertainty, disturbance, clocking

This course is about the derivation + implementation of suitable feedback control algorithms based on governing dynamics of system.

Reduction of the double integral

$$y(t) = K \int_0^t \int_0^\sigma u(\tau)\, d\tau\, d\sigma$$



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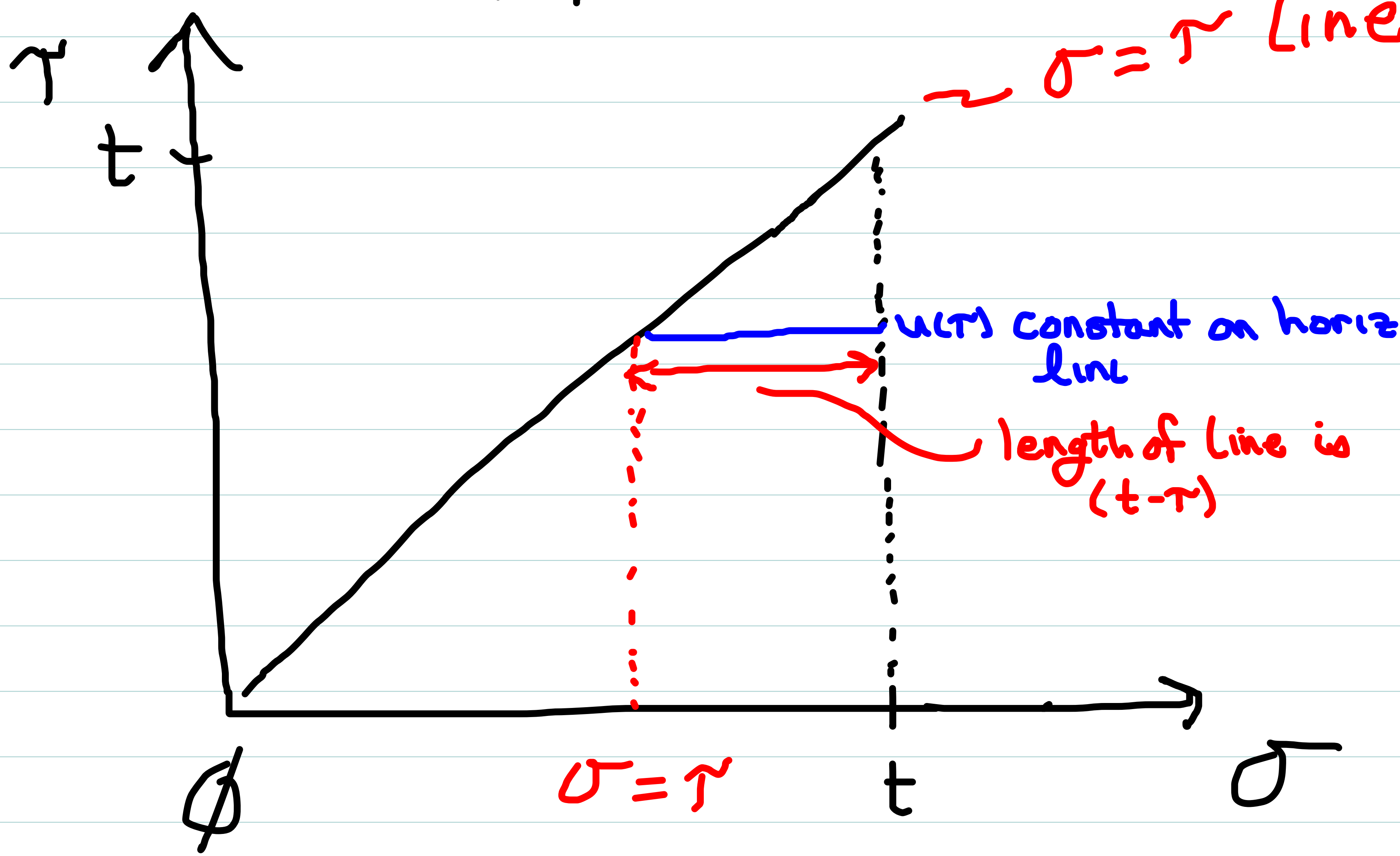

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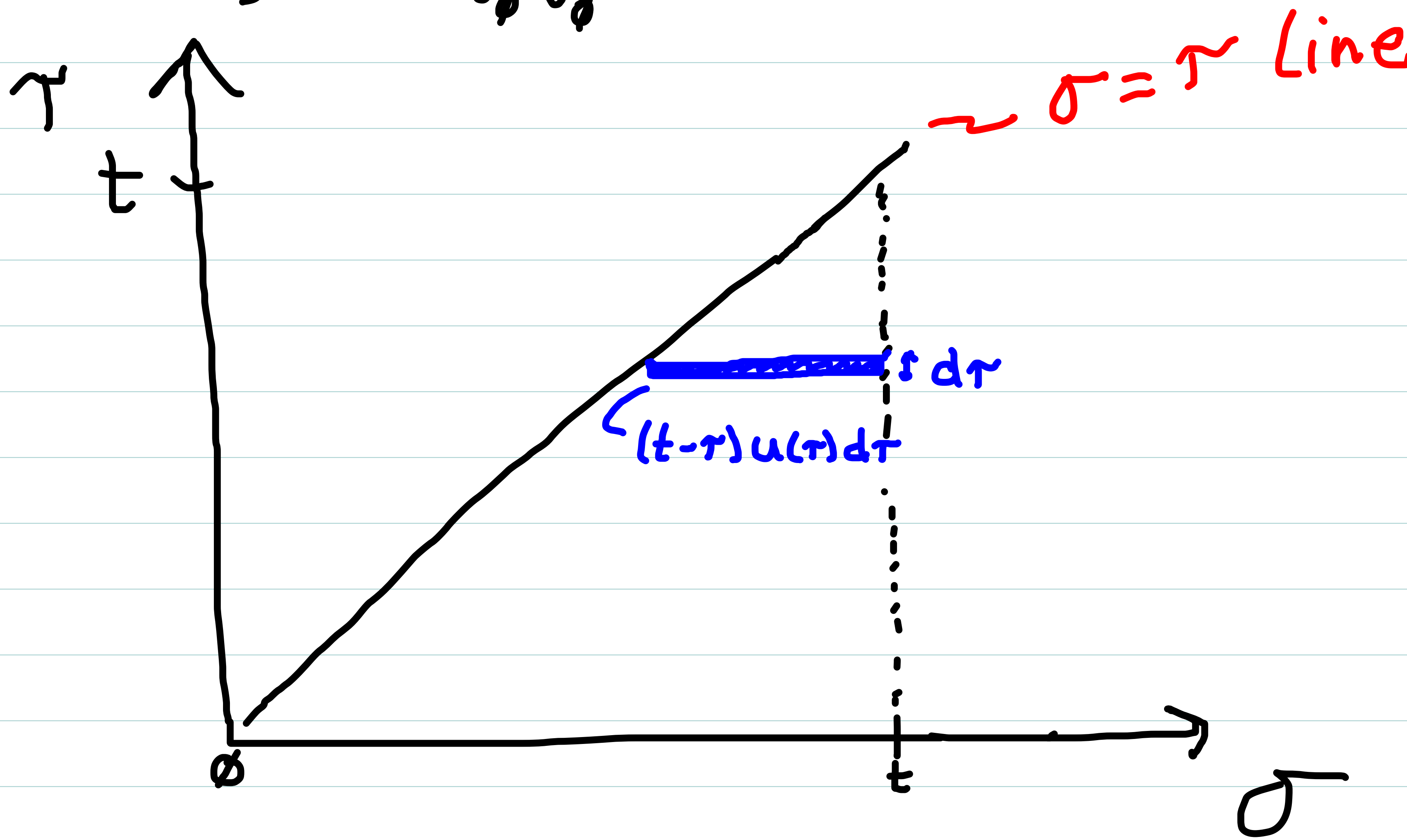

Integrate over all strips + multiply by K: $\Rightarrow y(t) = K\int_{0}^{t}(t-\tau)u(\tau)\,d\tau$

An alternate form

Our sol'n has the general form:

$$\boxed{y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau}$$

where here $g(t) = Kt$ $\quad [\text{so } g(t-\tau) = K(t-\tau)]$

We will (indirectly) show that for any system, no matter how complex the dynamics, this relationship between $u(t)$ and $y(t)$ holds.

Different systems are characterized by different functions $g(t)$.

The characteristic function $g(t)$ is called the

Impulse response

Implication

Suppose:

$$y_1(t) = \int_0^t g(t-\tau)\, u_1(\tau)\, d\tau$$

$$y_2(t) = \int_0^t g(t-\tau)\, u_2(\tau)\, d\tau$$

are two known input-output pairs.

Suppose that $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$; α_1, α_2 constant

Then:

$$y(t) = \int_0^t g(t-\tau) \left[\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)\right] d\tau$$

$$= \alpha_1 \int_0^t g(t-\tau)\, u_1(\tau)\, d\tau + \alpha_2 \int_0^t g(t-\tau)\, u_2(\tau)\, d\tau$$

hence

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$ Principle of Linearity.

This suggests an approach:

1. Identify a "family" of functions $u_k(t)$ for which it is easy to calculate response $y_k(t)$:

$$u_k(t) \mapsto y_k(t) \quad \text{(easy)}$$

2. "Break down" an arbitrarily complicated $u(t)$ into a linear combination of the $u_k(t)$:

$$u(t) = \sum \alpha_k u_k(t) \qquad \text{(easy?)}$$

3. Use Linearity:

$$y(t) = \sum \alpha_k y_k(t) \qquad \text{(easy)}$$

Time varying complex numbers

$$z(t) = a(t) + b(t)\,j$$
$$= r(t)\,e^{j\theta(t)}$$

Important example:

$$z(t) = e^{st} \text{ with } s \in \mathbb{C}$$

"complex exponential functions"

Let $s = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

So $\text{Re}\{s\} = \sigma$, $\text{Im}\{s\} = \omega$

① If $\omega = \emptyset$, then

$$e^{st} = e^{\sigma t} \text{ (\underline{real} exponential)}$$

② If $\sigma = \emptyset$ then

$$e^{st} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Note: $\text{Im}\{s\}$ gives <u>frequency</u> of the oscillations

3. Most general case

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t}[\cos\omega t + j\sin\omega t]$$

$$\left.\begin{aligned} Re\{e^{st}\} &= e^{\sigma t}\cos(\omega t) \\ Im\{e^{st}\} &= e^{\sigma t}\sin(\omega t) \end{aligned}\right]$$

$\sigma \rightarrow$ amplitude envelope

$\omega \rightarrow$ oscillation frequency

$s = \sigma + j\omega$ is the "Complex frequency"