Recap: Frequency Response Analysis

$$u(t) = B\sin(\omega t + \psi) \Rightarrow y_{ss}(t) = A\sin(\omega t + \varphi)$$

$$A = B|G(j\omega)|, \quad \varphi = \angle G(j\omega) + \psi$$

Bode diagrams: show

$|G(j\omega)|$ (dB) vs. ω (log scale) "Magnitude diagram"

$\angle G(j\omega)$ (deg) vs. ω (log scale) "Phase diagram"

Want to learn to rapidly predict the shapes of these diagrams from the ZPK structure of transfer function $G(s)$

How?

Will Show:

(1.) Effect of each pole p_K and zero z_i is concentrated in a narrow band of frequencies near $\omega = |p_K|$ (or $|z_i|$, as appropriate)

$\Rightarrow$ remember: $\omega \geq 0$ on Bode diagrams. There are no negative frequencies shown!

(2.) Effect of individual poles/zeros on *total* Bode diagrams are <u>additive</u>

"Bode form" of transfer function

ZPK form:

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{k=1}^{n}(s-p_k)}\right]$$

Bode form:

$$G(s) = K_B \frac{\prod_{i=1}^{m}(1-s/z_i)}{s^N \prod_{k=N+1}^{n}(1-s/p_k)}$$

N = # of poles at origin "Type" of system

K_B = "Bode gain"; note $N=\emptyset \Rightarrow K_B = G(\emptyset)$

Bode and ZPK forms are two different ways of writing the <u>same</u> transfer function

<u>Example</u>:

$$G(s) = \frac{5(s+2)}{s(s+3)(s+4)} \qquad \text{(ZPK)}$$

(Bode) $$= \left(\frac{5}{6}\right)\left[\frac{(1+s/2)}{s(1+s/3)(1+s/4)}\right]$$

Here $N = 1$ and $K_B = 5/6$

<u>Algebraically equivalent</u> to ZPK form.

i.e. both are the <u>same</u> TF

So:

$$G(j\omega) = K_B \left[\frac{\prod_{i=1}^{m} (1 - j\omega/z_i)}{(j\omega)^N \prod_{k=N+1} (1 - j\omega/p_k)} \right]$$

for any real $\omega \geq 0$, $G(j\omega)$ is complex and so are each individual factor (except K_B, which is real)

recall for any $s_1, s_2 \in \mathbb{C}$

$$\boxed{\begin{aligned} \measuredangle(s_1 s_2) &= \measuredangle s_1 + \measuredangle s_2 \\ \measuredangle\left(\frac{s_1}{s_2}\right) &= \measuredangle s_1 - \measuredangle s_2 \\ \measuredangle s_1^N &= N \measuredangle s_1 \end{aligned}}$$

Thus:

$$\measuredangle G(j\omega) = \measuredangle K_B + \sum_{i=1}^{m} \measuredangle(1 - {j\omega}/{z_i}) - N\measuredangle(j\omega) - \sum_{K=N+1}^{n} \measuredangle(1 - {j\omega}/{p_K})$$

Note: (1.) Each factor contributes additively

(2.) Zeros add to angle, poles subtract

(3) $\measuredangle K_B$ same for any ω:

$$\measuredangle K_B = \emptyset \ (K_B > \emptyset), \quad \measuredangle K_B = \pm 180^\circ \ (K_B < \emptyset)$$

(3.) $\measuredangle(j\omega)$ is same for any $\omega \geq \emptyset$

$$\measuredangle(j\omega) = 90^\circ$$

(4.) Changes to $\measuredangle G(j\omega)$ as ω varies depends on specific z_i and nonzero p_K.

What about Magnitudes?

Recall: for $s_1, s_2 \in \mathbb{C}$

$$|s_1 s_2| = |s_1|\,|s_2|$$

$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$$

$$|s_1^N| = |s_1|^N$$

So:

$$|G(j\omega)| = |K_B| \frac{\prod_{i=1}^{m} |1 - j\omega/z_i|}{|j\omega|^N \prod_{k=N+1}^{n} |1 - j\omega/p_k|}$$

UGLY...

But Bode shows $|G(j\omega)|$ in dB

i.e. $20\log|G(j\omega)|$

Now recall: $\log(xy) = \log x + \log y$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log(x^N) = N\log x$$

Hence in dB:

$$|G(j\omega)|_{dB} = |K_B|_{dB} + \sum_{i=1}^{m} |1 - {}^{j\omega}/z_i|_{dB} - N|j\omega|_{dB} - \sum_{K=N+1}^{n} \left|1 - \frac{j\omega}{P_K}\right|_{dB}$$

Notes:

1. Magnitudes in dB are additive for each factor
2. zeros add to magnitude, poles subtract
3. $|K_B|$ is constant for all ω, like with phase
4. $|j\omega|$ is Not constant, unlike phase.

So, we see effect of individual parts of $G(s)$ contribute additively to

$$\angle G(j\omega) \quad \text{and} \quad |G(j\omega)|_{dB}$$

Look at effect of individual factors

Look at how each $(1 - j\omega/z_i)$ or $(1 - j\omega/p_k)$

changes with ω.

To simplify notation, we'll look at $(1 + j\omega\tau)$, where

$\tau = -1/z_i$ or $\tau = -1/p_k$ as appropriate

Then:

$$|1 + j\omega\tau| = \sqrt{1 + \omega^2\tau^2}$$

and

$$\measuredangle(1 + j\omega\tau) = \tan^{-1}\omega\tau$$

Study how these vary with ω

Consider first magnitude

$$|1+j\omega\tau| = \sqrt{1+(\omega\tau)^2} = \begin{cases} 1 & \text{if } \omega \ll 1/|\tau| \\ \sqrt{2} & \text{if } \omega = 1/|\tau| \\ \omega|\tau| & \text{if } \omega \gg 1/|\tau| \end{cases}$$

and thus:

$$|1+j\omega\tau|_{dB} = \begin{cases} \emptyset & \omega \ll 1/|\tau| \quad \text{"Low freq. Limit"} \\ 3 & \omega = 1/|\tau| \\ 20\log\omega|\tau| & \omega \gg 1/|\tau| \quad \text{"high freq Limit"} \end{cases}$$

Look at 3rd case:

$$20\log\omega|\tau| = 20\left[\log\omega + \log|\tau|\right]$$

Note when $\omega = 1/|\tau|$, $\log\omega = -\log|\tau|$ & 3rd case evaluates to $\emptyset$.

<u>Also</u>:

in high freq limit $\omega \gg 1/|\tau|$

$$|1+j\omega\tau|_{dB} = 20\left[\log\omega + \log|\tau|\right]$$

Suppose we have two freqs, ω_1, ω_2 both $\gg 1/|\tau|$

with $\omega_2 = 10\omega_1$, then:

$$|1+j\omega_2\tau|_{dB} = |1+j(10\omega_1)\tau|_{dB}$$

$$= 20\left[\log(10\omega_1) + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log 10 + \log|\tau|\right]$$

$$= 20\left[\log\omega_1 + \log|\tau|\right] + 20$$

So

$$|1+j\omega_2\tau|_{dB} = |1+j\omega_1\tau|_{dB} + 20 \Leftarrow \text{+20 dB increase}$$

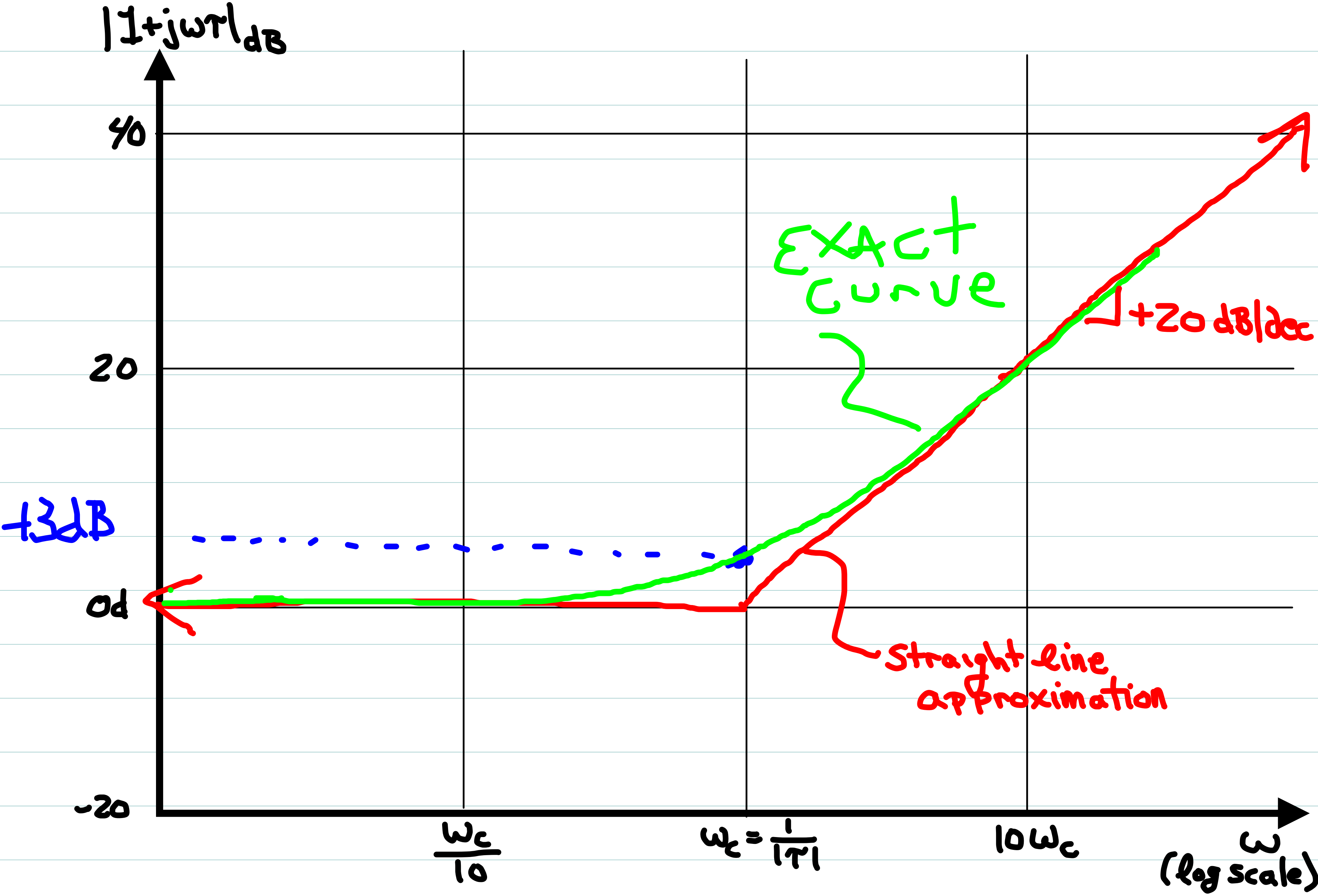
<u>Hence</u>:

in high frequency region $|1+j\omega\tau|_{dB}$ increases by 20dB for every factor of 10 increase in frequency (decade)

$\Rightarrow$ graph has a <u>slope</u> of 20dB/decade in high freq. region

$\Rightarrow$ Recall graph is constant at $\emptyset$dB in low freq. region

$\Rightarrow$ The two limiting cases come together at the "corner frequency", $\omega_c = \dfrac{1}{|\tau|}$.



$|1+j\omega\tau|_{dB}$
40
20
0d
-20
+3dB
EXACT curve
+20 dB/dec
straight line approximation
$\frac{\omega_c}{10}$
$\omega_c = \frac{1}{|\tau|}$
$10\omega_c$
ω (log scale)

Things to note:

- Graph <u>changes</u> slope by +20 dB/dec
- Think in terms of this slope change, not the total shape
- Recall $(1+j\omega\tau)$ is a generic representation of a factor of $G(s)$, either

$$\left(1-\frac{j\omega}{z_i}\right) \quad \text{or} \quad \left(1-\frac{j\omega}{p_K}\right)$$

i.e. $\tau = -1/z_i$ or $\tau = -1/p_K$

Thus the corner freq. $\omega_c = 1/|\tau| = |z_i|$ or $|p_K|$

$\boxed{\text{Corner freq is the absolute VALUE of a pole or zero of } G(s)}$

$\Rightarrow$ Because $|G(j\omega)|_{dB}$ is the sum of the effects of the individual terms $|1 - j\omega/z_i|_{dB}$ $|1 - j\omega/p_k|_{dB}$

each pole or zero will create a "corner" on the complete graph

$\Rightarrow$ The total graph will have corners at every freq. corresponding to $|z_i|$ and $|p_k|$.

$\Rightarrow$ zeros add to overall $|G(j\omega)|_{dB}$ $\Rightarrow$ slope changes of $+20$ dB/dec at $\omega = |z_i|$, $i = 1 \ldots m$

$\Rightarrow$ Poles subtract from overall $|G(j\omega)|_{dB}$ $\Rightarrow$ Slope changes of -20 dB/dec at $\omega = |p_k|$.

Example #1

$$G(s) = (10s+1)(s/10+1)$$

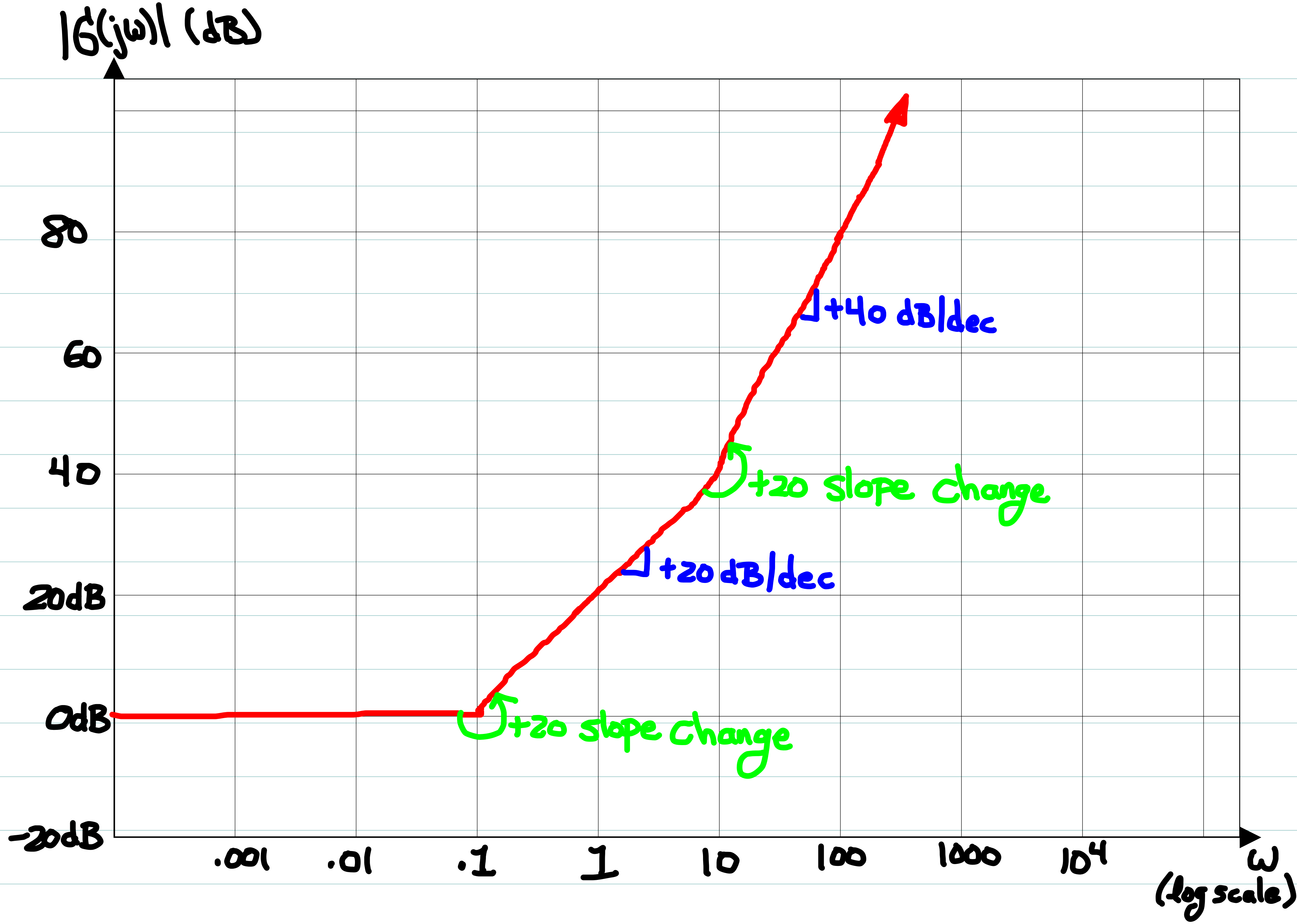
No poles; zeros at $z_1 = -10$, $z_2 = 1/10$

$|G(j\omega)|_{dB}$ will show +20 dB/dec changes at

$\omega = 1/10$ and $\omega = 10$

Below $\omega = 1/10$ the graph will be constant at 0 dB.

Graph bends up by +20 dB/dec at $\omega = 1/10$, and again at $\omega = 10$.



|G(jω)| (dB)
80
60
40
20dB
0dB
-20dB
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)
+20 slope change
+20 dB/dec
+20 slope change
+40 dB/dec

Example #2

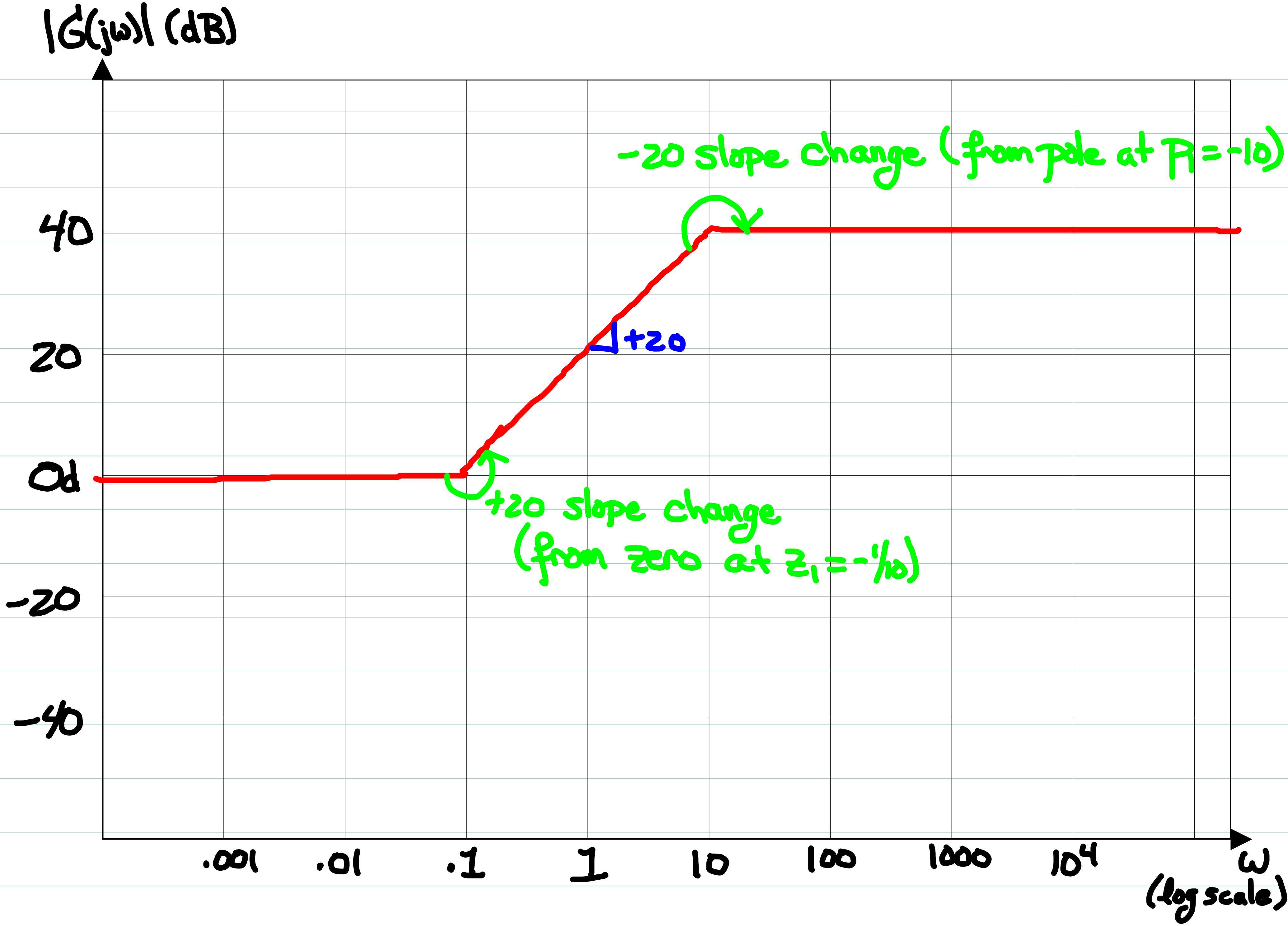
$$G(s) = \frac{(10s+1)}{(s/10+1)}$$

Zero at $z_1 = -1/10$, pole at $p_1 = -10$

Corners at $\omega = 1/10$ and $\omega = 10$ again

But now: at $\omega = 1/10$ slope <u>increases</u> by +20 dB/dec

at $\omega = 10$ slope <u>decreases</u> by -20 dB/dec



|G(jw)| (dB)
40
20
0d
-20
-40
-20 slope change (from pole at $P_1 = -10$)
+20
+20 slope change
(from zero at $z_1 = -1/10$)
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

Gain effect is additive also, and <u>constant</u> for all ω:

$$|K_B(1+j\omega\tau)|_{dB} = |K_B|_{dB} + |1+j\omega\tau|_{dB}$$

$\Rightarrow$ entire graph shifts <u>up or down</u> by $|K_B|_{dB} = 20\log|K_B|$

shifts <u>up</u> if $|K_B|_{dB} > \emptyset$

shifts <u>down</u> if $|K_B|_{dB} < \emptyset$

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$$|K_B(1+j\omega\tau)|_{dB} = |K_B|_{dB} + |1+j\omega\tau|_{dB}$$

$\Rightarrow$ entire graph shifts <u>up or down</u> by $|K_B|_{dB} = 20\log|K_B|$

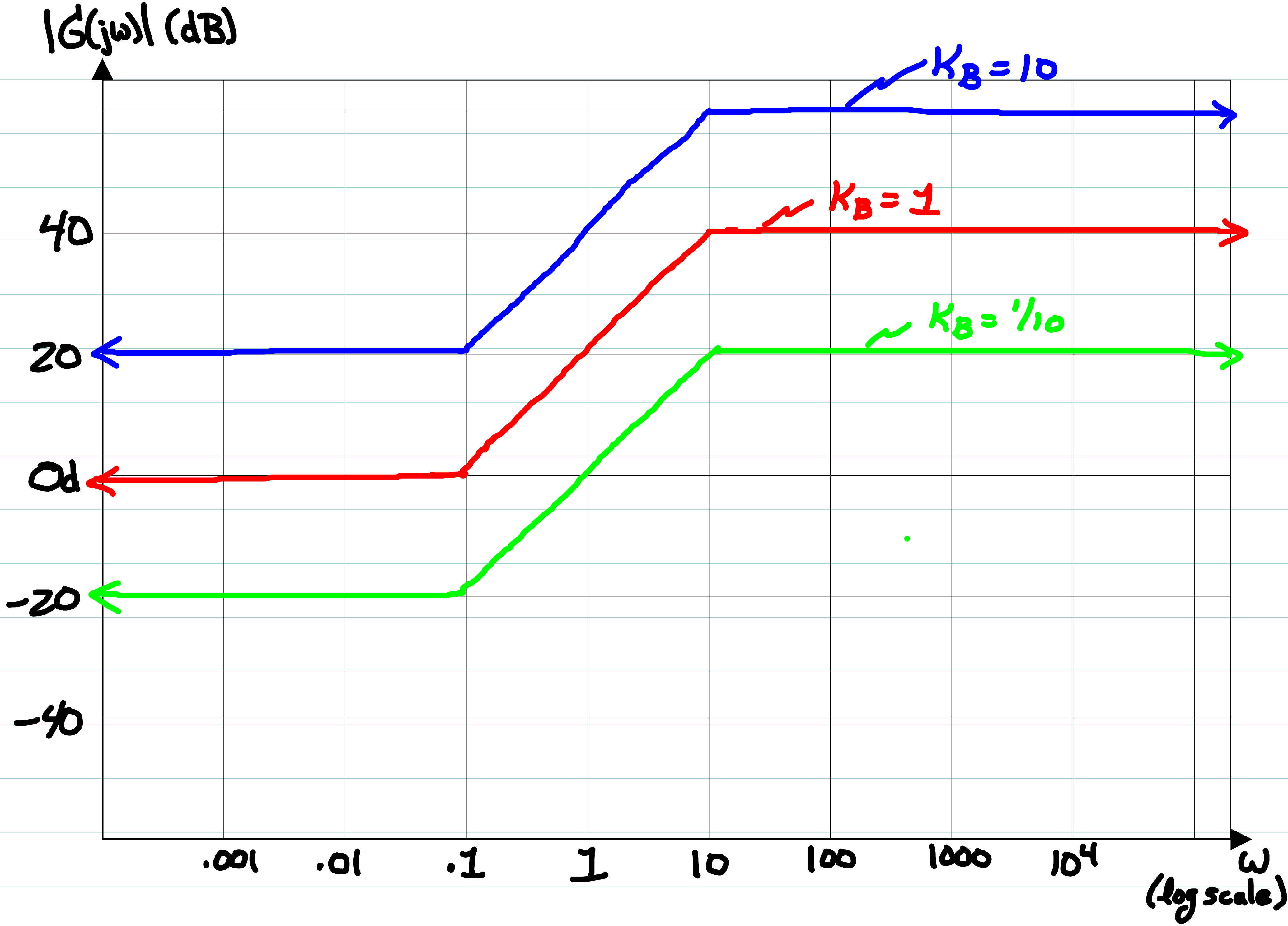
shifts <u>up</u> if $|K_B|_{dB} > \emptyset \Rightarrow |K_B| > 1$

shifts <u>down</u> if $|K_B|_{dB} < \emptyset \Rightarrow |K_B| < 1$

Remember the <u>sign</u> of K_B has <u>no effect</u> on the magnitude diagram!

<u>Example #3:</u>

$$G(s) = K_B\left[\frac{(10s+1)}{(s/10+1)}\right]$$



|G(jω)| (dB)
40
20
0d
-20
-40
$K_B = 10$
$K_B = 1$
$K_B = 1/10$
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

Repeated factors

$(1+j\omega\tau)^{\ell}$, ℓ integer ≥ 1

$$|(1+j\omega\tau)^{\ell}|_{dB} = 20 \log |1+j\omega\tau|^{\ell}$$

$$= (20\ell) \log |1+j\omega\tau|$$

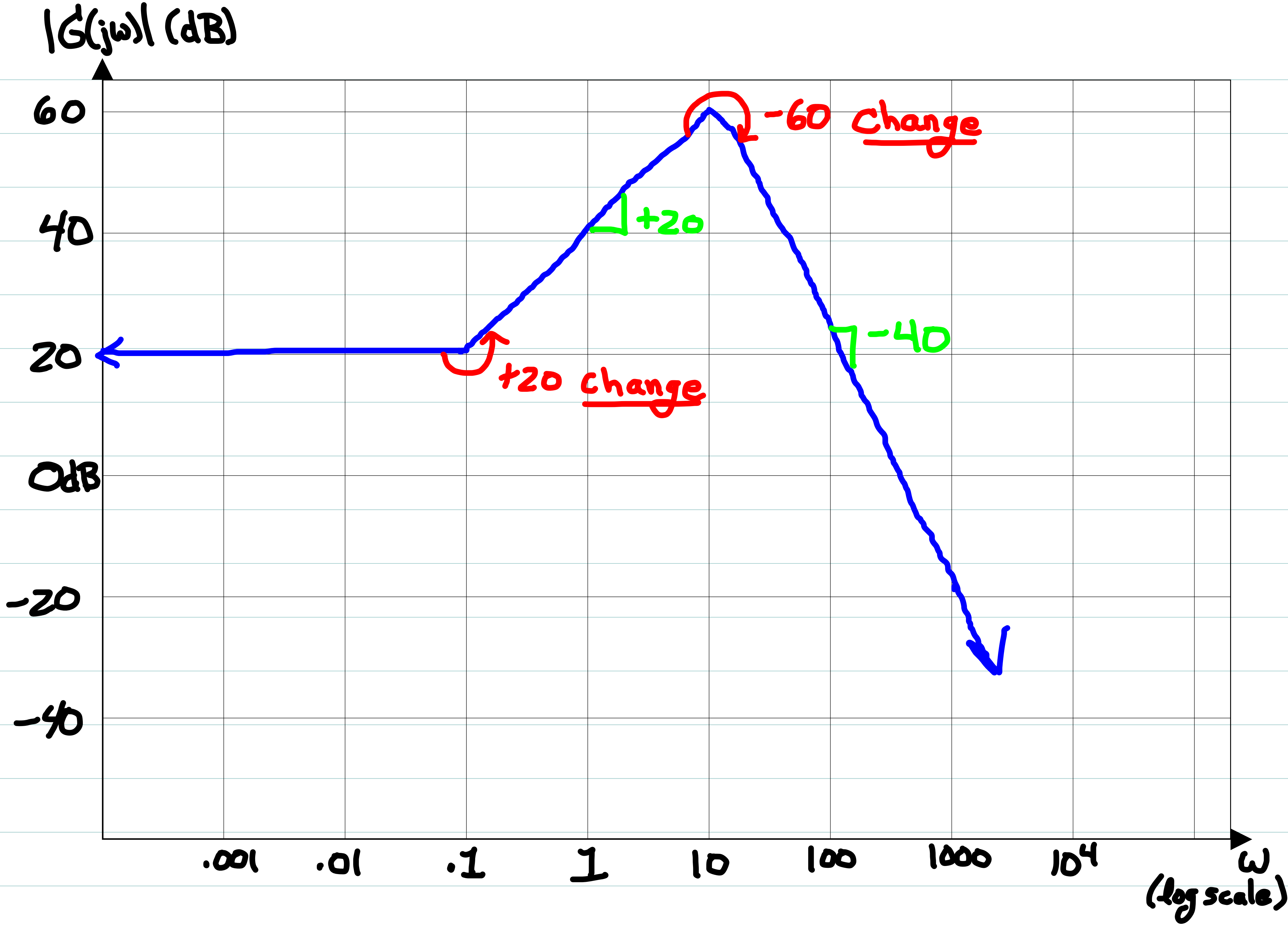
$\Rightarrow$ slope change is $\pm 20\ell$ at $\omega = |1/\tau|$

(positive for zero, negative for pole)

Example #4:

$$G(s) = 10 \left[\frac{(10s+1)}{(s/10+1)^3} \right]$$

$+20$ slope change at $\omega = 1/10$, $\underline{-60}$ change at $\omega = 10$.



|G(jω)| (dB)
60
40
20
0dB
-20
-40
.001
.01
.1
1
10
100
1000
10^4
ω (log scale)
+20 change
+20
-60 change
-40

Summary (so far)

⇒ Poles p_k and zeros z_i cause changes in $|G(j\omega)|_{dB}$ graph at corner frequencies $|p_k|$ and $|z_i|$

⇒ Slope of graph changes at these corners

⇒ Zero corners "bend up", i.e. change slope by +20 dB/dec

⇒ Pole corners "bend down", i.e. change slope by −20 dB/dec

⇒ If $|K_B| \neq 1$, entire graph is raised or lowered by $|K_B|_{dB}$