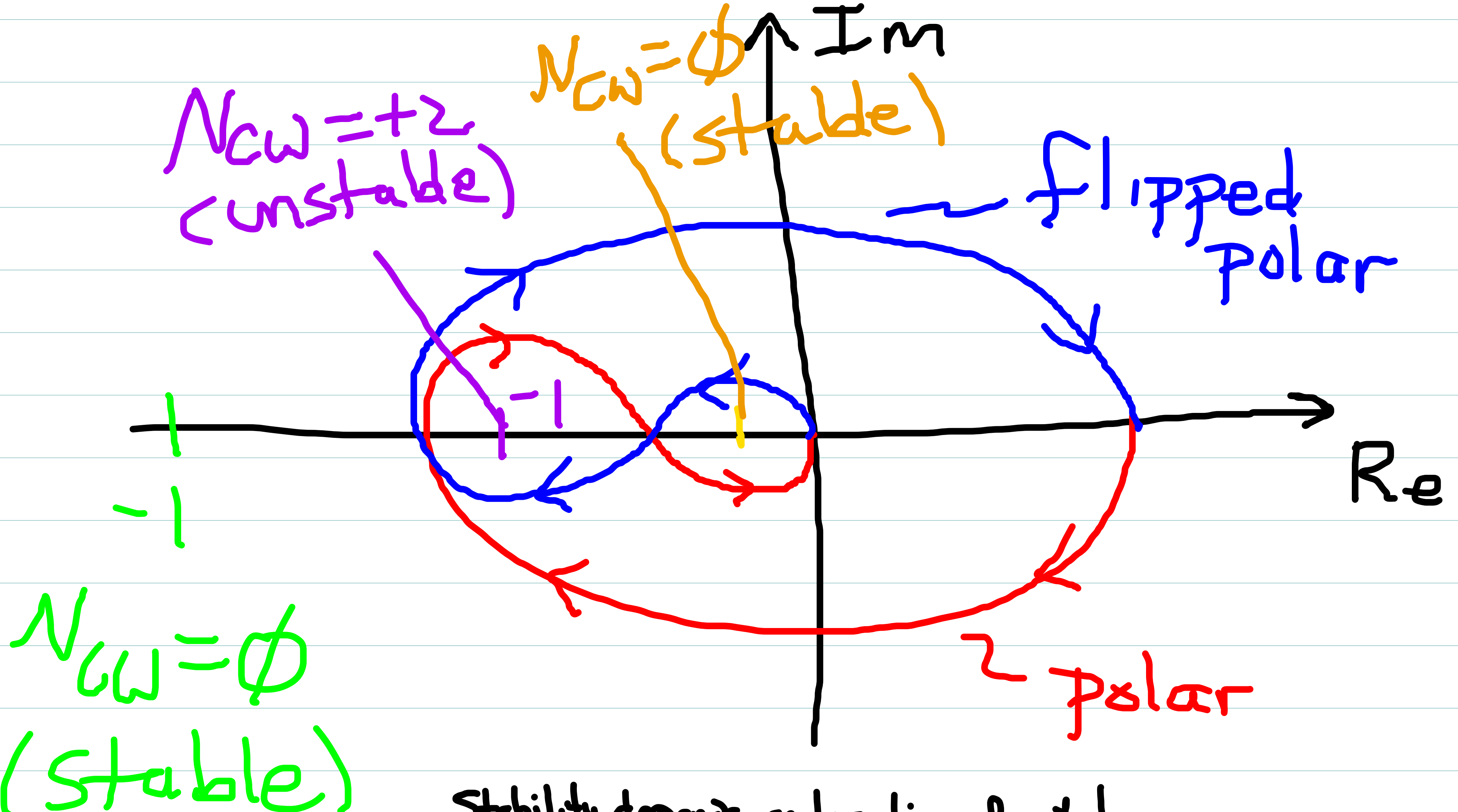
A more complicated Example:

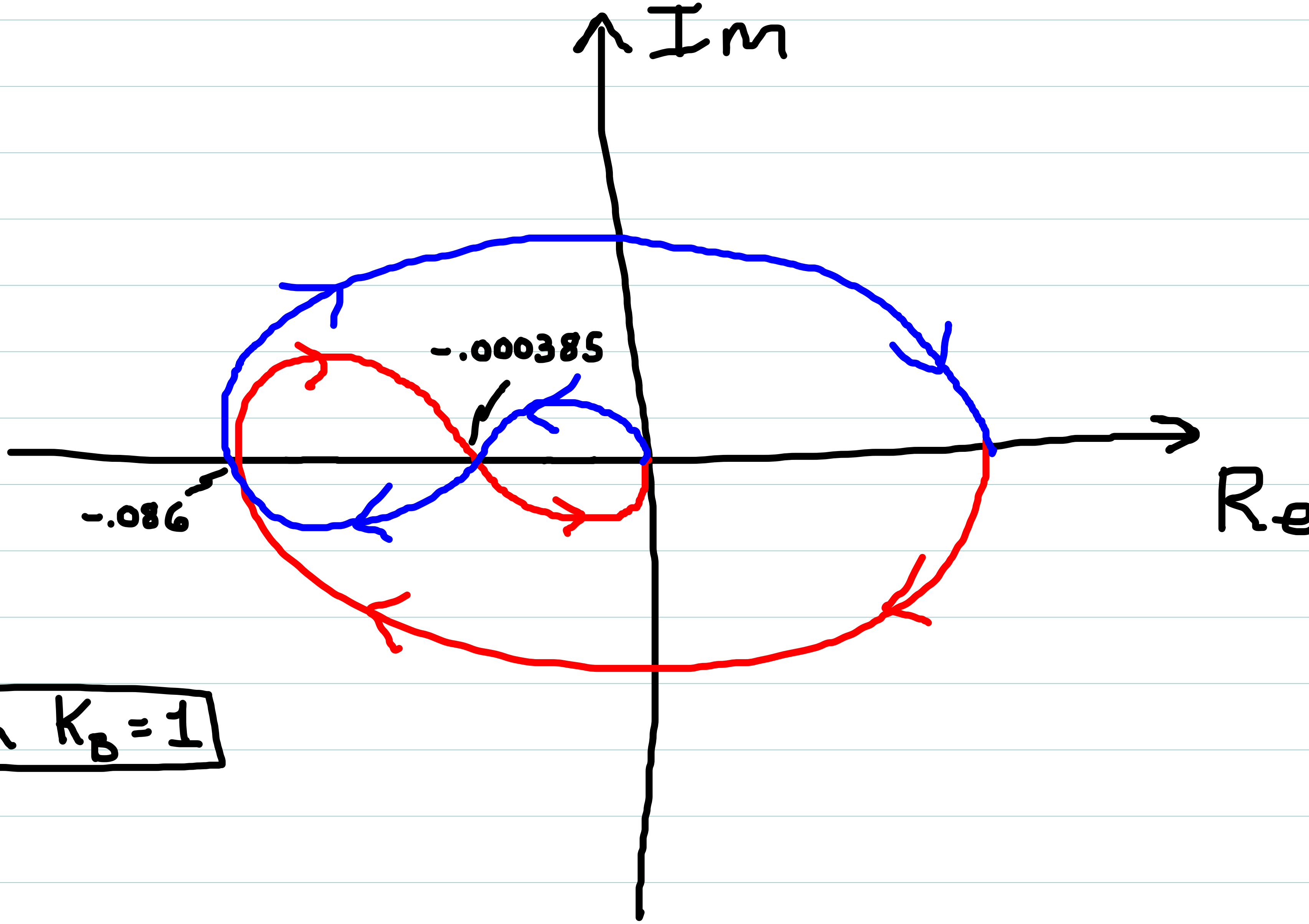
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_2 \gg \tau_1 > \phi \quad \Big] \; P_R(L) = \phi$$



Stability depends on location of -1!

Effect of gain Changes

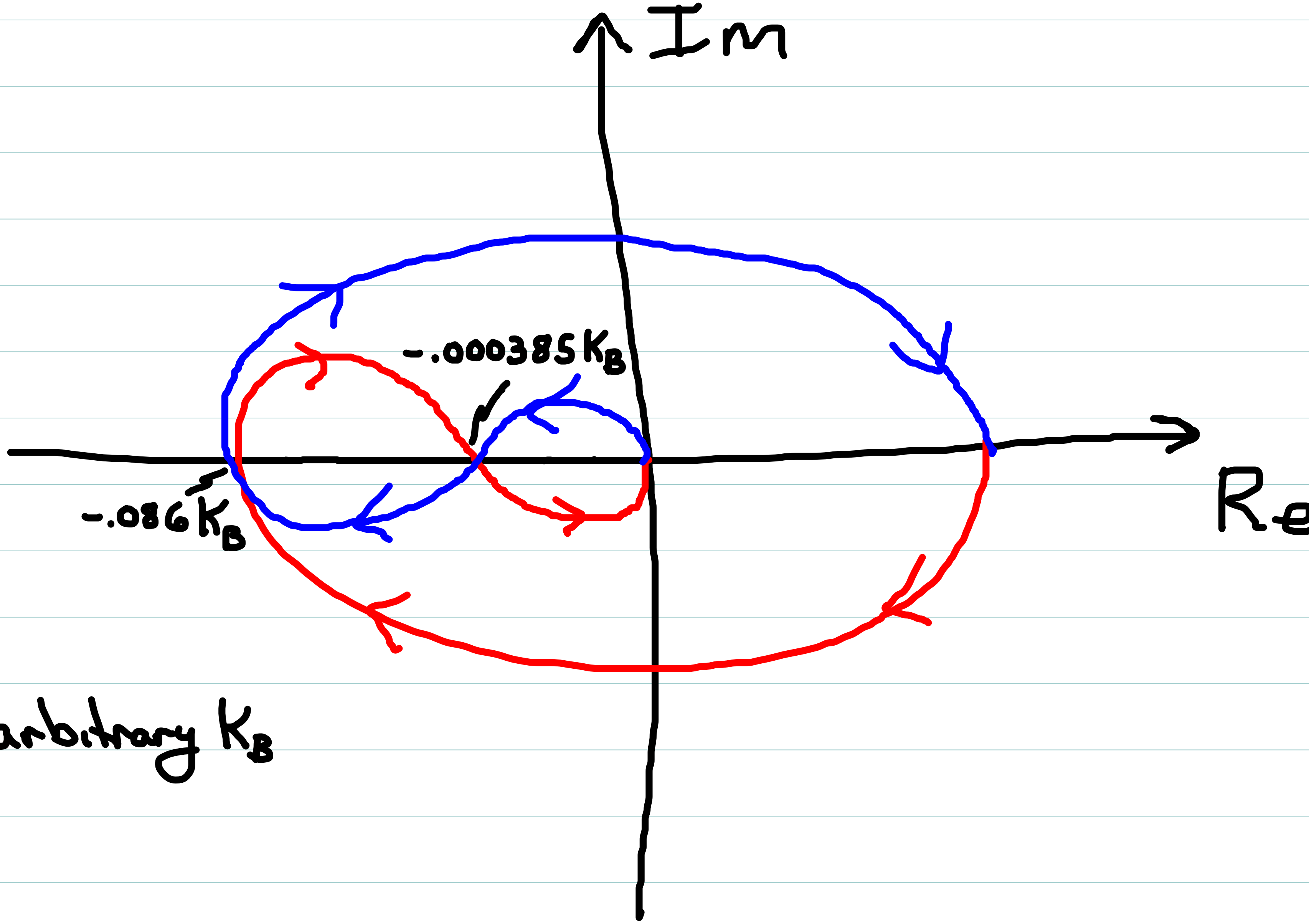
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



$\boxed{\text{When } K_B = 1}$

Effect of gain changes

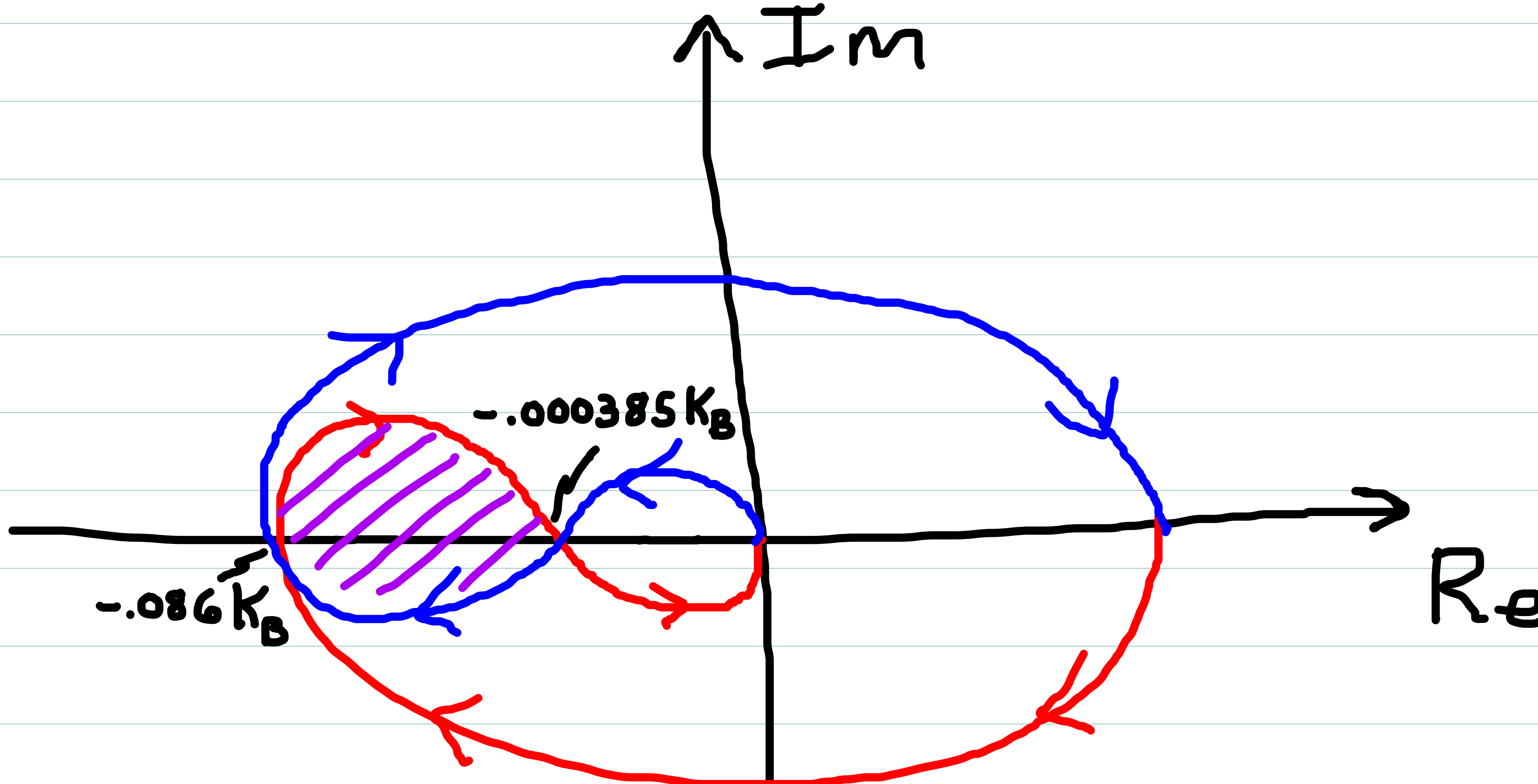
$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



for arbitrary K_B

Effect of gain changes

$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_1 = 10,\ \tau_2 = 1$$



For this system, $T(s)$ stable unless -1 is in hashed area

Need: $-1 < -.086K_B$ or $-.0004K_B < -1$

Thus: stable for $K_B < 1/.086 \cong 11.63$
or $K_B > 1/.000385 \cong 2597$

Note: Gain change is easy to accomplish with compensator:

$$H(s) = K \quad (\Rightarrow u(t) = Ke(t) \text{ "proportional" control})$$

$$L(s) = H(s)G(s) = KG(s) \text{ here}$$

$$\Rightarrow (K_B)_L = K(K_B)_G$$

However, gain change only affects "size" of polar (hence location of -1 relative to loops in Nyquist).

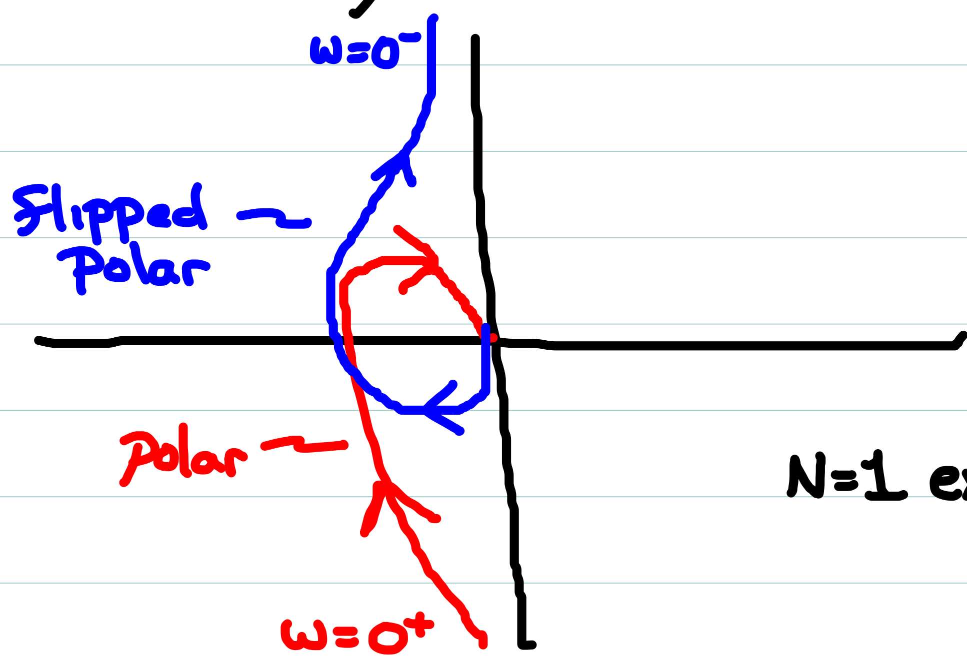
More substantial changes to polar/Nyquist diagram (changes to number and/or location of its loops) require also zeros/poles in $H(s)$.

Nyquist Diagram for $N > \emptyset$ systems

When $L(s)$ has type $N > \emptyset$ (one or more poles at origin) the first step to creating Nyquist diagram is same:

$\Rightarrow$ Draw polar of $L(j\omega)$
$\Rightarrow$ Flip polar about real Axis

However, the resulting diagram is Not connected; both halfs have "tails" parallel to coordinate AXES



N=1 example

Completing the diagram, $N > \emptyset$

$\Rightarrow$ Connect the $\omega = 0^-$ tail of flipped polar to $\omega = 0^+$ tail of original polar with a clockwise circular arc of total rotation $N\pi$

(i.e. ½ circle for every pole at origin in $L(s)$)

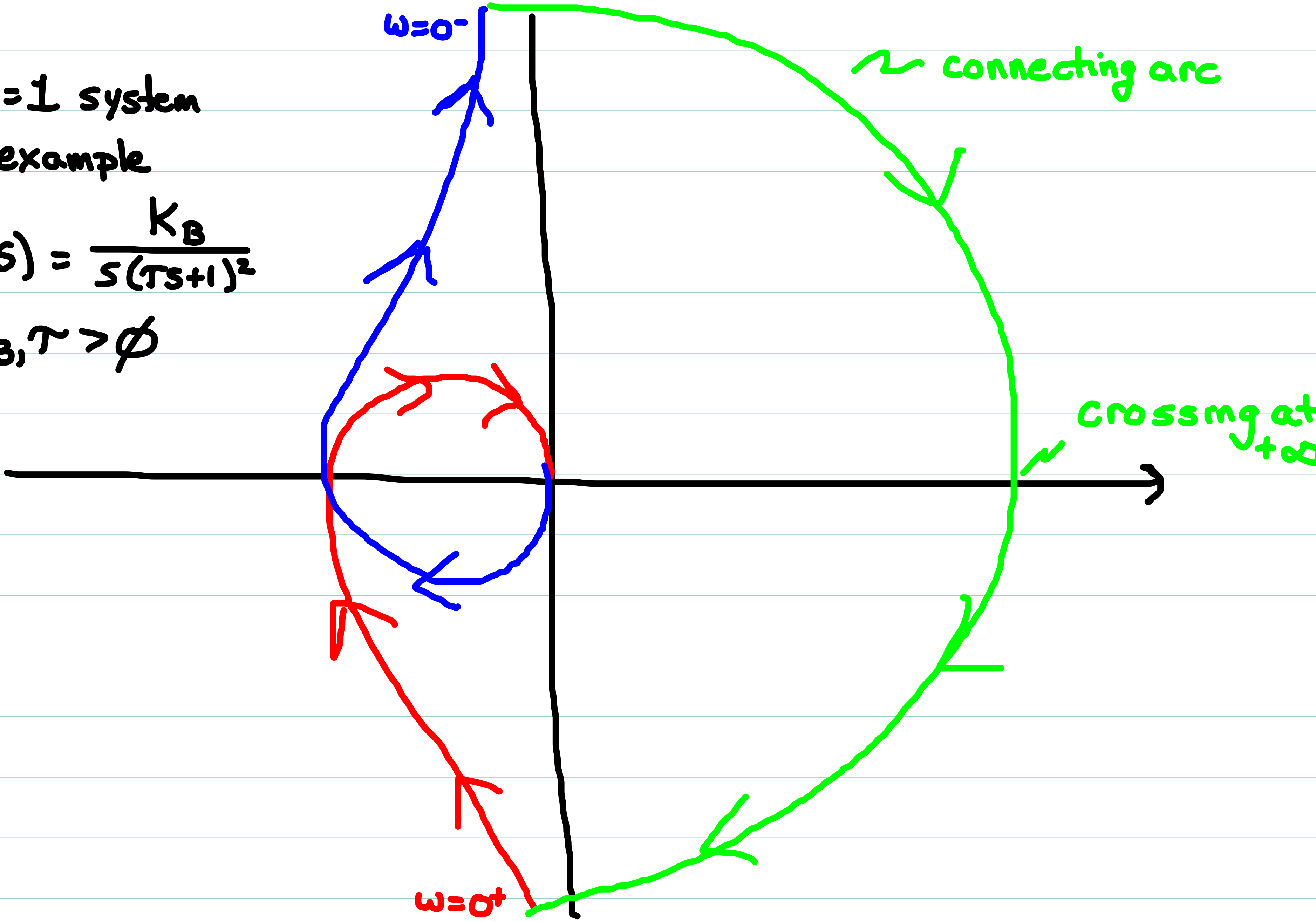
Note: Connecting arc has infinite radius, although we draw it as finite.

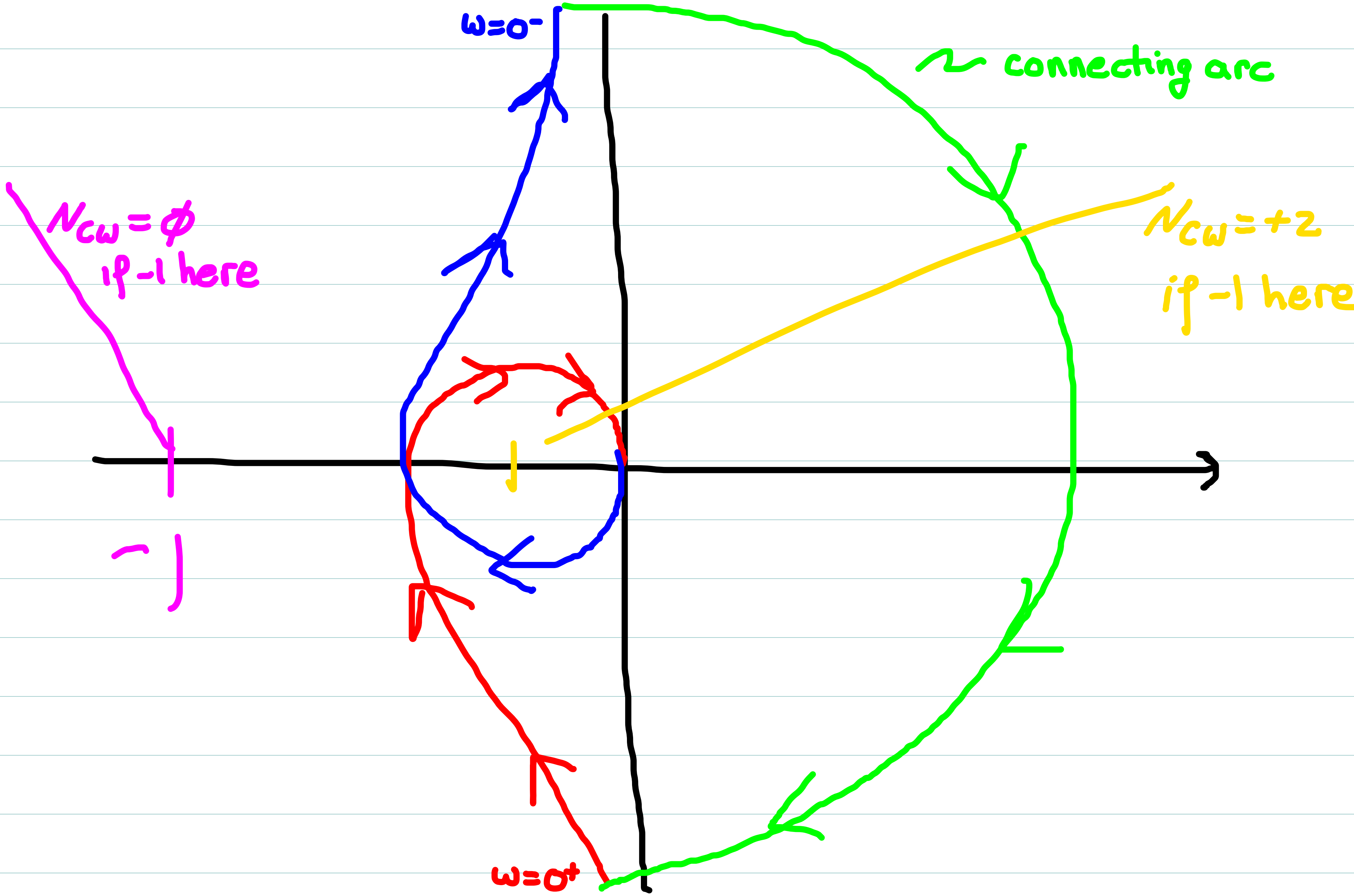
$\Rightarrow$ After connecting tails, compute $N_{cw}(L)$ as before.

N=1 system
for example

$$L(s) = \frac{K_B}{s(\tau s+1)^2}$$

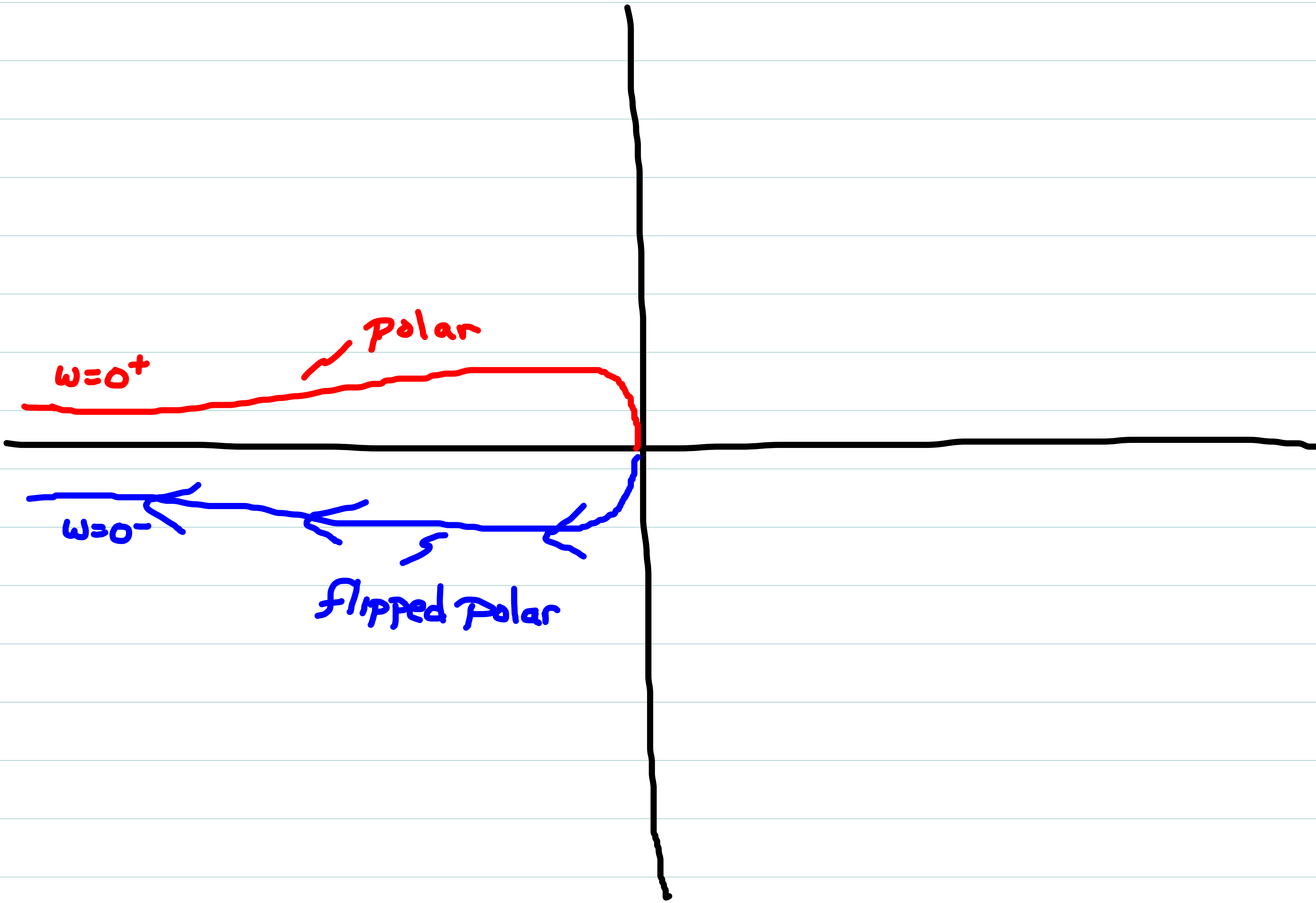
$K_B, \tau > \phi$



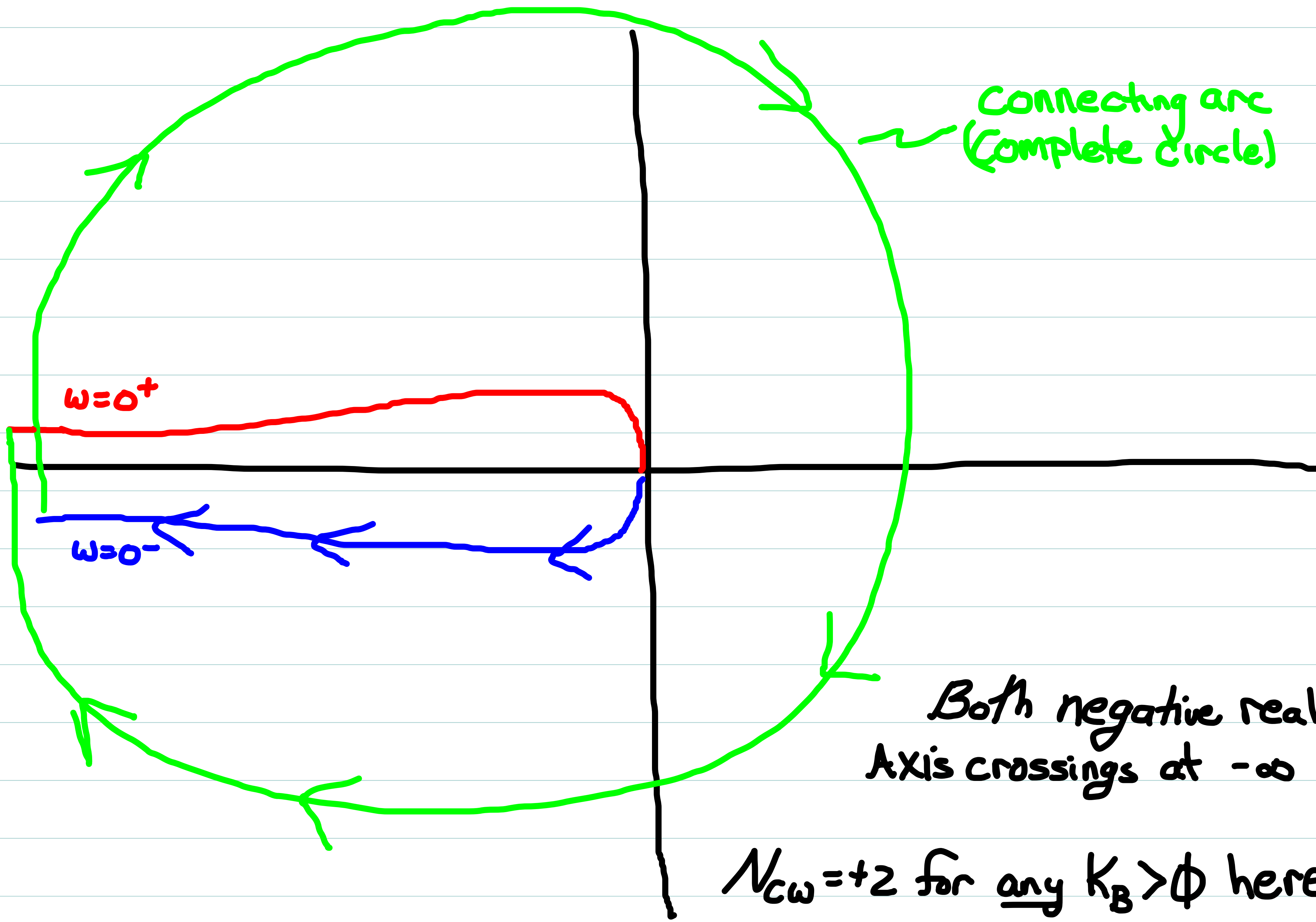


ω=0⁻
connecting arc
N_cw = ∅
if -1 here
N_cw = +2
if -1 here
-1
ω=0⁺

Be careful with N=2 systems!



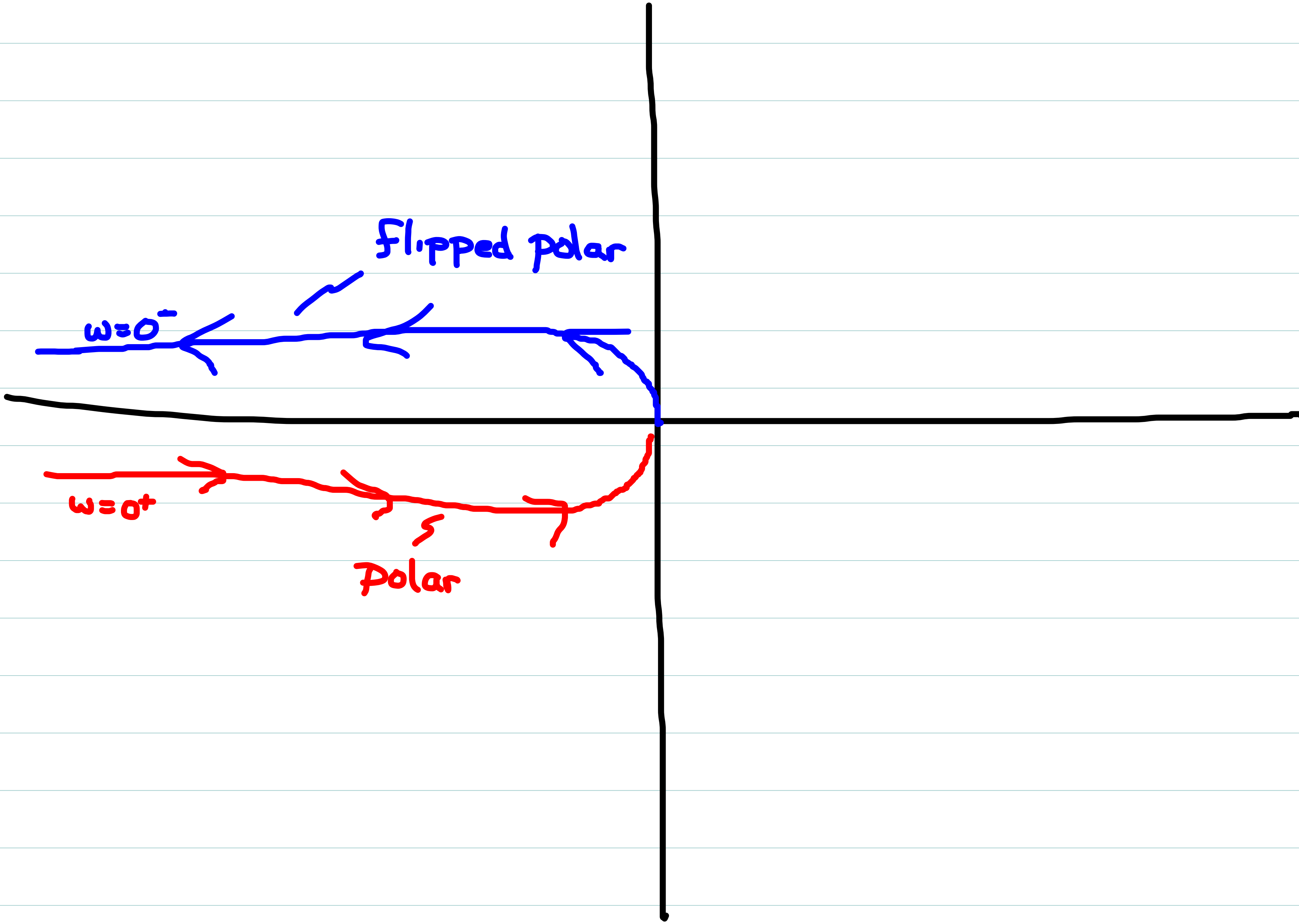
Be careful with N=2 systems!

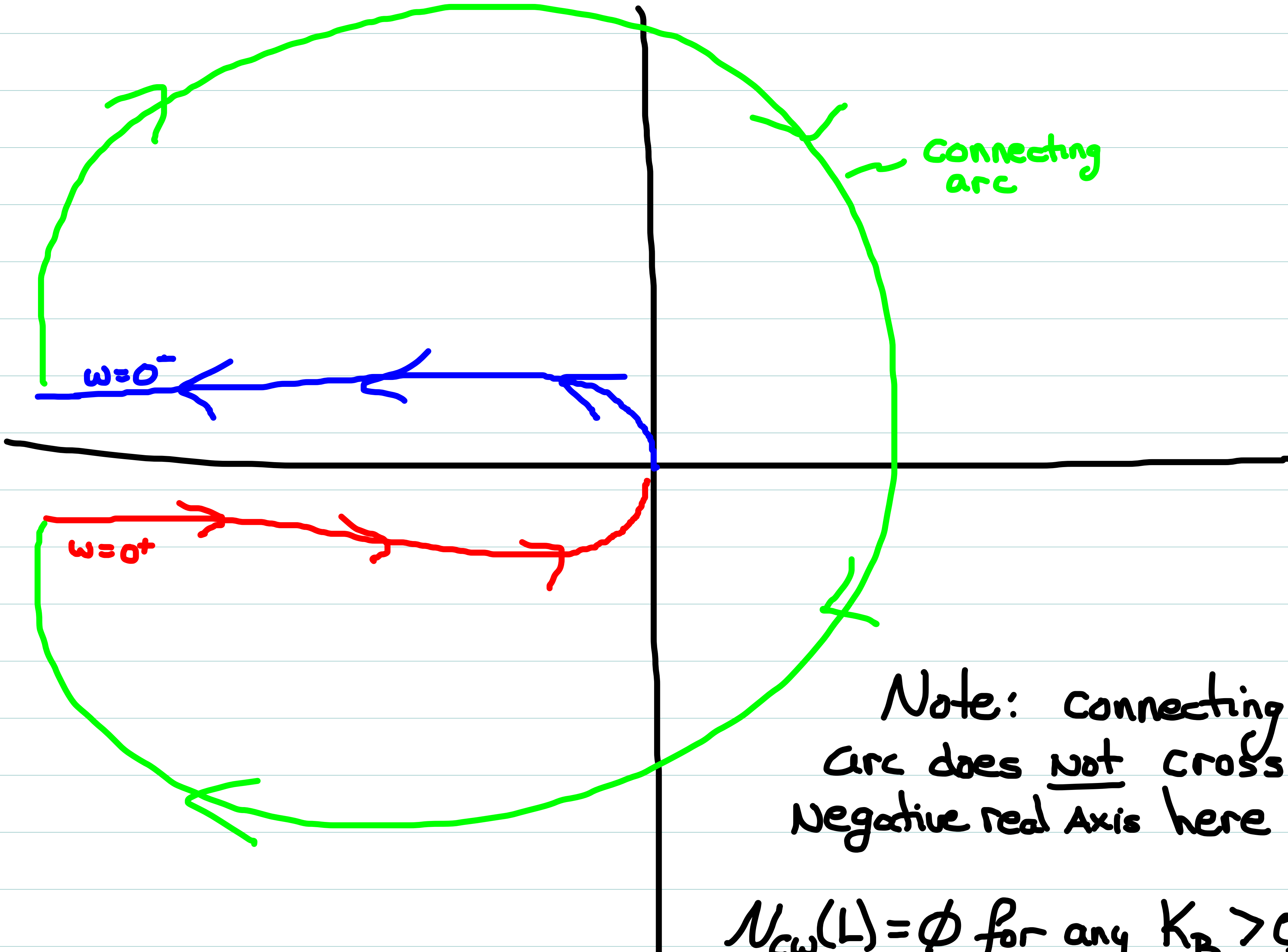


Both negative real Axis crossings at $-\infty$

$N_{cw}=+2$ for <u>any</u> $K_B>\phi$ here

An apparently similar system (N=2 still)





Note: connecting arc does not cross Negative real Axis here

$N_{cw}(L) = \emptyset$ for any $K_B > \emptyset$ here

Utility of gain/phase margin

⇒ a, γ measure how close polar comes to -1

⇒ If design is nominally stable (Nyquist shows required number of encirclements of -1), then

a, γ measure how much Nyquist plot can <u>change</u> in a pure gain or phase fashion, before -1 would enter a different loop, changing the number of encirclements.

Thus: a, γ are measures of the "tolerance" of the system's stability to gain/phase changes in $L(s)$.

⇒ <u>Relative</u> stability measures.

Robustness (classical)

As measures of the <u>tolerance</u> of the control system stability to changes in shape of Nyquist, gain and phase margin are measures of the <u>robustness</u> of the design.

That is, the ability of the design to tolerate model errors which would create pure gain or pure phase errors in $L(s)$

Typically caused by errors in model of $G(s)$, since

$$L(s) = G(s)H(s)$$

and there is no uncertainty in $H(s)$.

Classical Robustness Requirements

A "robustly stable" design thus requires:

$\Rightarrow$ Correct number of Nyquist encirclements

AND $\Rightarrow$ Large $|a|$, $|\gamma|$

Typical professional requirements

$\Rightarrow$ $|a_{dB}| \geq 6$ (i.e. $a > 6\,dB$ or $a < -6\,dB$)

$\Rightarrow$ $|\gamma| \geq 30°$

Requirement on a is physically equivalent to no more than a factor of 2 uncertainty on gain of $G(s)$

Recall: a, γ formally measure only how much Nyquist can change before encirclements change

(Assuming design is nominally stable, such changes would usually be bad!)

<u>By themselves</u> (separate from Nyquist) they are not reliable indicators of stability.

i.e. $a > \emptyset$ dB means Nyquist plot crosses neg. real axis to right of -1; $a < \emptyset$ dB means it crosses left of -1

Which is "better" (necessary for stability) depends on full Nyquist analysis.

However:

For a great many physical systems with:
a.) $L(s)$ stable; b.) unique ω_γ; c.) $\gamma(L) > \phi°$, the shape of Nyquist plot ensures $T(s)$ stable.

(True even for many $L(s)$ which satisfy c) but violate a) or b); however need to check actual Nyquist shape carefully here).

Common enough to be a major design guideline:

$\Rightarrow$ Design $H(s)$ to ensure that $L(s)$ has positive phase margin

$$\Rightarrow \measuredangle L(j\omega_\gamma) > -180°$$

Constraints for Stability

For most simple (and common) systems (and many not so simple systems) Nyquist will show stability if phase margin of $L(j\omega)$ is positive.

Design prescription: Add LHP zeros in $H(s)$ to increase phase at magnitude xover.

Indeed, we will show using different techniques that it is rare that such a strategy would fail to stabilize.

⇒ Theoretically interesting counter-example: if $G(s)$ has both a zero and a pole in RHP. Such a system may actually require a RHP pole in $H(s)$ to stabilize!

Always check the Nyquist diagram when using simple guidelines to design $H(s)$!

How much phase margin is "good"

Again, $\gamma > 30°$ is a typical minimum, and would ensure stability in common cases.

Why 30°? Is more better? Unfortunately, there is no simple correlation between freq. domain properties of $L(j\omega)$ and the <u>exact</u> location of poles of $T(s)$.

Nyquist tells us only $\mathrm{Re}\{p_K\} < 0$ for each pole p_K of $T(s)$ when the stability condition is satisfied

However, we can develop some useful intuition correlating (γ, ω_γ) with transient properties of $T(s)$ by looking at some typical simple examples.

Simple Example

$$L(s) = \frac{K}{s(s+\alpha)} \Rightarrow T(s) = \frac{K}{s^2+\alpha s+K}$$

$(\alpha > \emptyset)$

If $K = \alpha^2\sqrt{2}$, then $\omega_\gamma = |\alpha|$ and $\gamma = 45°$

(prove this to yourself if not obvious!)

Closed-loop poles are complex since $\alpha^2 - 4K < 0$

$(\alpha^2 - 4\sqrt{2}\alpha^2) < 0$

and in fact closed-loop damping ratio is $\zeta = 0.42$ here.

$\Rightarrow$ <u>Increasing</u> K <u>decreases</u> γ here, and also decreases damping ratio of closed-loop poles.

$\Rightarrow$ <u>Decreasing</u> K <u>increases</u> γ here, and also <u>increases</u> DR of closed-loop poles

In fact, for this system we can show

$$\xi_{CL} \approx \frac{\gamma\,(\text{deg})}{100} \quad (\text{for } 0 \lesssim \gamma \lesssim 70^\circ)$$

i.e. closed-loop damping ratio ξ_{CL} is directly proportional to the phase margin of L.

What about settling times for a step response of $T(s)$?
$\Rightarrow$ controlled by real parts of closed-loop poles.

Here the real parts are at $-\alpha/2 < 0$

$$t_s = \frac{4}{|\alpha/2|} = \frac{8}{|\alpha|} = \frac{8}{\omega_\gamma} \quad (\text{when } \gamma = 45^\circ \text{ as above})$$

i.e. t_s inversely prop. to ω_γ in this example.