Linear constant coefficient (time invariant) Diff Eq'n

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \dot{y} + \alpha_0 y$$
$$= \beta_m u^{(m)} + \cdots + \beta_1 \dot{u} + \beta_0 u$$

where $\alpha_n, \ldots \alpha_0$ and $\beta_m \ldots \beta_0$ are <u>real</u> and <u>constant</u>

Suppose $u(t) = U e^{st}$ with $s, U \in \mathbb{C}$

Is $y(t) = Y e^{st}$ a sol'n for some $Y \in \mathbb{C}$?

Substitute into DE

GIVES

$$r(s)Ye^{st} = q(s)Ue^{st}$$

With:

$$r(s) = \alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$$

So Assumption is consistent with

$$Y = \left[\frac{q(s)}{r(s)}\right]U = G(s)U$$

$$Ue^{st} = u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t) = Ye^{st}$$

$$\boxed{\text{If } u(t) = Ue^{st} \text{ for some } U, s \in \mathbb{C} \text{ then } y(t) = Ye^{st}, \text{ with } Y = G(s)U}$$

$$G(s) = \left[\frac{q(s)}{r(s)}\right] \Longleftarrow \text{"transfer function"}$$

$q(s)$ poly from derivs of $u(t)$ in DE

$r(s)$ " " " $y(t)$ " "

- This is one possible sol'n of the DE; the forced sol'n, $\boxed{y_f(t).}$
- Other sol'ns are possible!
- These are also complex exponential functions, at specific characteristic (complex) frequencies, p_k, satisfying

$$\boxed{r(p_k)=0}$$

i.e. p_k are roots of $r(s)$

Other possible sol'ns

Now, suppose $u(t) = \emptyset$. Clearly here $y_f(t) = \emptyset$. But is $y(t) = \emptyset$ necessarily? Or can we still have sol'ns of the form $y(t) = Ce^{st}$? Substitute into DE:

$$r(s) C e^{st} = \emptyset$$

which can be true for any s where

$$\boxed{r(s) = \emptyset}$$

$$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$$

There are n values of s for which $r(s) = 0$.

highest deriv of $y(t)$ in DE
"order" of the system

We denote these roots $p_1, p_2, \ldots p_n$

So $r(s)$ can be factored as

$$r(s) = \alpha_n (s - p_1)(s - p_2) \cdots (s - p_n)$$

$$= \alpha_n \prod_{k=1}^{n} (s - p_k)$$

For any p_K with $r(p_K) = \phi$,
$y(t) = e^{p_K t}$ is a sol'n of the DE
when $u(t) = \phi$. So is $y(t) = C_K e^{p_K t}$
for any constant C_K. So is any
<u>sum</u> of these terms:

$$y(t) = \sum_{K=1}^{n} C_K e^{p_K t} = y_h(t)$$

$\boxed{\text{The "homogeneous" sol'n}}$

Proof:

Substitute $y(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

into diff eq'n:

GIVES:

$$r(p_1)c_1 e^{p_1 t} + r(p_2)c_2 e^{p_2 t} + \cdots + r(p_n)c_n e^{p_n t}$$
$$= 0$$

Which is true if $r(p_1) = r(p_2) = \cdots = r(p_n) = 0$

i.e. the p_k are zeros of polynomial $r(s)$

Since, trivially, we can write $u(t) = u(t) + \phi$

By linearity, the general sol'n of the DE

is $y(t) = \underbrace{y_h(t)}_{\text{homogeneous response independent of } u(t)} + \underbrace{y_f(t)}_{\text{"forced" response from } u(t)}$

homogeneous response <u>independent</u> of $u(t)$

"forced" response from $u(t)$

where $y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

and if $u(t) = U e^{st}$, then

$$y_f(t) = G(s) U e^{st}$$

Both complex! (Generally)

Since any $y_h(t)$ yields $\emptyset$ exactly when substituted into DE, we can add it to any other sol'n and still have a valid sol'n. Generally:

$$y(t) = y_h(t) + y_f(t)$$

where $y_h(t) = \sum_{k=1}^{n} C_k e^{p_k t}$

and if $u(t) = U e^{st}$, then

$$y_f(t) = G(s) U e^{st}$$

Both complex! (Generally)

But $y_f(t)$ is complex generally . . . ?

$\Longrightarrow$ because $u(t)$ is complex here

Suppose $u(t) = B\sin(\omega t + \varphi)$ (real)

$= \operatorname{Im}\{Ue^{st}\}$

Take matching Im part

with $U = Be^{j\varphi}$

and $s = j\omega$

Then $y_f(t) = \operatorname{Im}\{G(s)Ue^{st}\}$

and similarly for cosine inputs, taking real part

What about $y_h(t)$?

Contains terms e^{pt}, where $r(p) = 0$.

If p is complex, $p = \sigma + j\omega$, $\omega \neq 0$

then e^{pt} is complex

However: in this case $r(p) = 0 \Longrightarrow r(\bar{p}) = 0$

i.e. $\bar{p}$ is also a zero of $r(s)$.

$\Longrightarrow$ Complex roots of polynomials occur in "conjugate pairs".

Hence, with complex roots, $y_h(t)$ will contain

$$C_1 e^{pt} + C_2 e^{\bar{p}t}$$

<u>Fact:</u>

$$C_2 = \overline{C_1}$$

i.e. coef of $e^{\bar{p}t}$ will <u>always</u> be the conjugate of the coef of e^{pt}.

Thus, if $r(s)$ has a complex root p, $y_h(t)$ will contain

$$ce^{pt} + \bar{c}e^{\bar{p}t} = ce^{pt} + \overline{ce^{pt}}$$

We've seen this before...

Write: $c = re^{j\varphi}$, $p = \sigma + j\omega$: then

$$ce^{pt} + \overline{ce^{pt}} = 2\,\mathrm{Re}\{ce^{pt}\}$$

$$= 2\,\mathrm{Re}\{re^{j\varphi}e^{(\sigma+j\omega)t}\}$$

$$= 2re^{\sigma t}\,\mathrm{Re}\{e^{j(\omega t+\varphi)}\}$$

$$= 2re^{\sigma t}\cos(\omega t+\varphi)$$

So...

$$ce^{pt} + \bar{c}e^{\bar{p}t} = 2re^{\sigma t}\cos(\omega t+\varphi)$$

with $r = |c|$, $\varphi = \measuredangle c$, $\sigma = \mathrm{Re}\{p\}$, $\omega = \mathrm{Im}\{p\}$

n free parameters in general sol'n

$C_1, C_2, \ldots, C_n \Rightarrow$ coefs in $y_h(t)$.

Determined by n initial cond'ns on DE

$y(0) = y_0, \ \dot{y}(0) = \dot{y}_0, \ldots, y^{(n-1)}(0) = y_0^{(n-1)}$

Can substitute $y(t) = y_h(t) + y_F(t)$

into DE, differentiate, and match IC

Results in system of n eq'ns with n unknown

$\Rightarrow$ We will find much easier methods!

An Example

$$\ddot{y} + 5\dot{y} + 4y = 2\dot{u} + u$$

$$y(0) = 0,\ \dot{y}(0) = 0,\ u(t) = 3\cos(2t - \pi/2)$$

$$y_h(t) = C_1 e^{-t} + C_2 e^{-4t}$$

$$y_f(t) = \mathrm{Re}\{G(2j)\,3e^{(2t-\pi/2)j}\}$$

By inspection

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \frac{3\sqrt{17}}{10}\cos\left(2t - \frac{\pi}{2} - \tan^{-1}\frac{1}{4}\right)$$

(after add'l calculation)

Note here:

$$G(s) =$$

How...?

Here $u(t) = 3\cos(2t - \pi/2) = \text{Re}\{U e^{st}\}$

with $s = 2j$ and $U = 3e^{-\pi/2 j}$

So $y_f(t) = \text{Re}\{G(2j)(3e^{-\pi/2 j})(e^{2jt})\}$

with here: $G(s) = \dfrac{2s+1}{s^2+5s+4}$

$$\Rightarrow G(2j) = \frac{1+4j}{(2j)^2+10j+4} = \frac{1}{10}(4-j) = \frac{\sqrt{17}}{10} \measuredangle -\tan^{-1}\left(\frac{1}{4}\right)$$

Hence:

$$y_f(t) = \frac{3\sqrt{17}}{10}\cos(2t - \pi/2 - \tan^{-1}(1/4))$$

So we know $y_f(t)$ exactly

Homogeneous Sol'n

We have $r(s) = s^2 + 5s + 4$ (denom poly of $G(s)$)

Or: $r(s) = (s+1)(s+4)$

So $p_1 = -1$, $p_2 = -4$ and $y_h(t) = c_1 e^{-t} + c_2 e^{-4t}$

Then $y(t) = y_f(t) + y_h(t)$

$$= \frac{3\sqrt{17}}{10} \cos\left(2t - \pi/2 - \tan^{-1}(1/4)\right) + c_1 e^{-t} + c_2 e^{-4t}$$

So $y(\phi) = c_1 + c_2 - \frac{3}{10} = \phi$ (specified)

and $\dot{y}(\phi) = -c_1 - 4c_2 + 12/5 = \phi$ (specified)

Impose Boundary Cond'ns

Equivalently

$$\begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/10 \\ -12/5 \end{bmatrix}$$

$\Rightarrow$ (linear algebra):

$$c_1 = -4/10, \quad c_2 = 7/10$$

So that:

$$y(t) = \frac{1}{10}\left[7e^{-4t} - 4e^{-t}\right] + \frac{3\sqrt{17}}{10}\cos\left(2t - \frac{\pi}{2} - \tan^{-1}\frac{1}{4}\right)$$

as claimed

Recap

General sol'n of LTI DE is:

$$y(t) = y_h(t) + y_f(t)$$

Forced response $y_f(t)$ depends on $u(t)$

Homogeneous response is <u>independent</u> of $u(t)$:

$$\left. y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t} \quad \text{where } r(p_k) = \emptyset \right\} \text{ for } \underline{\text{any}}\ u(t)$$

Specific coefs c_k depend on initial conditions and $u(t)$.

Repeated roots of $r(s)$

Above formula for $r(s)$ assumes the roots p_K are <u>non-repeated</u>

Suppose instead that there are repeated roots, for example:

$$r(s) = (s-p_1)^{\ell}(s-p_{\ell+1})\cdots(s-p_n)$$

i.e. p_1 is repeated ℓ times. Then:

$$y_h(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_\ell t^{\ell-1})e^{p_1 t} + \sum_{K=\ell+1}^{n} c_K e^{p_K t}$$

(will prove later)

(Natural) Modes

$y_h(t)$ is a linear combination of $e^{p_k t}$ (or $t^i e^{p_k t}$). These describe solutions which are possible <u>without any input</u>

They are "natural" motions which are <u>intrinsic</u> to the dynamics of the system.

We call them the "<u>MODES</u>".

$\boxed{\text{MODES:}}$ Terms in sol'n for $y(t)$ of form e^{pt}, where $r(p) = \emptyset$

Two cases (non-repeated, to start)

① p real : e^{pt} is a real exponential function

"1st order mode"

② <u>p complex</u>: e^{pt} and $e^{\bar{p}t}$ both present in solution, and will combine to form the "2^{nd} order mode"

$$Ae^{\sigma t}\cos(\omega t+\varphi)$$

where $\sigma = \mathrm{Re}\{p\}$, $\omega = \mathrm{Im}\{p\}$

and A, φ depend on the initial conditions