Utility of gain/phase margin

⇒ a, γ measure how close polar comes to -1

⇒ If design is nominally stable (Nyquist shows required number of encirclements of -1), then

a, γ measure how much Nyquist plot can Change in a pure gain or phase fashion, before -1 would enter a different loop, changing the number of encirclements.

Thus: a, γ are measures of the "tolerance" of the system's stability to gain/phase changes in $L(s)$.

⇒ Relative stability measures.

Robustness (classical)

As measures of the tolerance of the control system stability to changes in shape of Nyquist, gain and phase margin are measures of the robustness of the design.

That is, the ability of the design to tolerate model errors which would create pure gain or pure phase errors in $L(s)$

Typically caused by errors in model of $G(s)$, since

$$L(s) = G(s)H(s)$$

and there is no uncertainty in $H(s)$.

Classical Robustness Requirements

A "robustly stable" design thus requires:

$\Rightarrow$ Correct number of Nyquist encirclements

AND $\Rightarrow$ Large $|\alpha|$, $|\gamma|$

Typical professional requirements

$\Rightarrow$ $|\alpha_{dB}| \geq 6$ (i.e. $\alpha > 6\,dB$ or $\alpha < -6\,dB$)

$\Rightarrow$ $|\gamma| \geq 30°$

Requirement on α is physically equivalent to no more than a factor of 2 uncertainty on gain of $G(s)$

Recall: a, γ formally measure only how much Nyquist can change before encirclements change

(Assuming design is nominally stable, such changes would usually be bad!)

<u>By themselves</u> (separate from Nyquist) they are not reliable indicators of stability.

i.e. $a > \emptyset$ dB means Nyquist plot crosses neg. real axis to right of -1; $a < \emptyset$ dB means it crosses left of -1

Which is "better" (necessary for stability) depends on full Nyquist analysis.

However:

For a great many physical systems with:
a) $L(s)$ stable; b.) unique ω_γ; c.) $\gamma(L) > \phi°$, the shape of Nyquist plot ensures $T(s)$ stable.

(True even for many $L(s)$ which satisfy c) but violate a) or b); however need to check actual Nyquist shape carefully here).

Common enough to be a major design guideline:

$\Rightarrow$ Design $H(s)$ to ensure that $L(s)$ has positive phase margin

$$\Rightarrow \measuredangle L(j\omega_\gamma) > -180°$$

Constraints for Stability

For most simple (and common) systems (and many not so simple systems) Nyquist will show stability if phase margin of $\angle L(j\omega)$ is positive.

Design prescription: Add LHP zeros in $H(s)$ to increase phase at magnitude xover.

Indeed, we will show using different techniques that it is rare that such a strategy would fail to stabilize.

$\Rightarrow$ Theoretically interesting counter-example: if $G(s)$ has both a zero and a pole in RHP. Such a system may actually require a RHP pole in $H(s)$ to stabilize!

Always check the Nyquist diagram when using simple Guidelines to design $H(s)$!

How much phase margin is "good"

Again, $\gamma > 30°$ is a typical minimum, and would ensure stability in common cases.

Why $30°$? Is more better? Unfortunately, there is no simple correlation between freq. domain properties of $L(j\omega)$ and the <u>exact</u> location of poles of $T(s)$.

Nyquist tells us only $\mathrm{Re}\{p_K\} < \emptyset$ for each pole p_K of $T(s)$ when the stability condition is satisfied

However, we can develop some useful intuition correlating (γ, ω_γ) with transient properties of $T(s)$ by looking at some typical simple examples.

Simple Example

$$L(s) = \frac{K}{s(s+\alpha)} \Rightarrow T(s) = \frac{K}{s^2+\alpha s+K}$$

$(\alpha > \emptyset)$

If $K=\alpha^2\sqrt{2}$, then $\omega_\gamma = |\alpha|$ and $\gamma = 45°$
(prove this to yourself if not obvious!)

Closed-loop poles are complex since $\alpha^2 - 4K < 0$

$$(\alpha^2 - 4\sqrt{2}\alpha^2) < 0$$

and in fact closed-loop damping ratio is $\zeta = 0.42$ here.

$\Rightarrow$ <u>Increasing</u> K <u>decreases</u> γ here, and also decreases damping ratio of closed-loop poles.

$\Rightarrow$ <u>Decreasing</u> K <u>increases</u> γ here, and also <u>increases</u> DR of closed-loop poles

In fact, for this system we can show

$$\xi_{CL} \approx \frac{\gamma\,(\text{deg})}{100} \qquad (\text{for } 0 \lesssim \gamma \lesssim 70^\circ)$$

i.e. closed-loop damping ratio ξ_{CL} is directly proportional to the phase margin of L.

What about settling times for a step response of $T(s)$?
$\Rightarrow$ controlled by real parts of closed-loop poles.

Here the real parts are at $-\alpha/2 < 0$

$$t_s = \frac{4}{|\alpha/2|} = \frac{8}{|\alpha|} = \frac{8}{\omega_\gamma} \qquad (\text{when } \gamma = 45^\circ \text{ as above})$$

i.e. t_s inversely prop. to ω_γ in this example.

Freq. Domain Constraints for Performance

When simple $\gamma(L) > \emptyset$ constraint works for stabilization, then typically:

$\Rightarrow$ larger γ gives higher damping for poles of $T(s)$

$\Rightarrow$ larger ω_γ gives faster settling time for $T(s)$ transients

Except for very simple systems, there are no direct mathematical connections between the <u>freq. domain</u> properties of $L(j\omega)$ (like γ and ω_γ) and the corresponding <u>time domain</u> properties of $T(s)$.

However, certain general trends have been found to hold:

For more complex systems, above observations do not hold precisely, but general trends do:

Given 2 possible OL TF: $L_1(s)$, $L_2(s)$

a.) If L_1, L_2 have same ω_γ but suppose $\gamma(L_1) > \gamma(L_2)$ then the closed-loop TFs T_1 and T_2 will have comperable settling times, but T_1 will have a higher Damping ratio

b.) If L_1 and L_2 have same phase margin but different ω_γ, $\omega_{\gamma_1} > \omega_{\gamma_2}$, then T_1 and T_2 will have comperable damping, but T_1 will settle faster than T_2

$\Rightarrow$ <u>Design guideline</u>: make γ, ω_γ as large as possible.

Intro. to Controller Design

Stability and healthy margins are just the first of many different constraints for a good design

Often the constraints conflict, and we must use our judgement to achieve an acceptable trade-off

The general design process is typically:

1.) Look at Bode/Nyquist of $G(s)$.

2.) Determine how plots in 1.) must be changed to achieve desired design goals.

3.) From required changes in 2.), determine the ZPK structure $H(s)$ must have.

Controller Implementation, I

But can we actually have any $H(s)$ we want?

Unfortunately no. There are implementation constraints:

i.e., can we actually calculate $u(t)$ from $e(t)$ in real time

Note that we do not calculate $u(t)$ from

$$u(t) = \mathcal{L}^{-1}\{H(s)E(s)\}$$

Why not?

- $y_d(t)$ not always known ahead of time (may come from pilot inputs)

- $y(t)$ cannot be predictedly exactly due to inaccurate model or external "disturbances"

Controller Implementation, II

So how do we implement the controller? By solving in real time the differential equation relating $u(t)$ to $e(t)$.

There are mathematical constraints under which this is possible, and these in turn constrain the "allowable" $H(s)$.

$$U(s) = H(s)E(s) = H(s)[Y_d(s) - Y(s)]$$

Suppose
$$H(s) = \frac{a(s)}{b(s)}$$

$a(s), b(s)$ polynomials in s. Then

$$b(s)U(s) = a(s)E(s)$$

Controller Implementation, II

$$U(s) = H(s)\,E(s)$$

Suppose
$$H(s) = \frac{a(s)}{b(s)}$$

$a(s), b(s)$ polynomials in s. Then

$$\mathcal{L}^{-1}\left\{ b(s)\,U(s) = a(s)\,E(s) \right\}$$

Gives a differential equation relating $u(t)$ ("output") to $e(t)$ ("input")

This diff'l equation must be <u>solvable in real time</u> using <u>only</u> measurements of

$$e(t) = y_d(t) - y(t)$$

Example

$$H(s) = \frac{6(s+1)^2}{(s+3)(s+5)} = \frac{\overbrace{(6s^2+12s+6)}^{a(s)}}{\underbrace{(s^2+8s+15)}_{b(s)}}$$

$$\Rightarrow (s^2+8s+15)U(s) = (6s^2+12s+6)E(s)$$

$$\Rightarrow \ddot{u}(t) + 8\dot{u}(t) + 15u(t) = 6\ddot{e}(t) + 12\dot{e}(t) + 6e(t)$$

DE which must be solved during operation of controller on vehicle.

Must be soluable using <u>only</u> measured $e(t)$.

$\dot{e}(t)$, $\ddot{e}(t)$ terms <u>Not</u> assumed to be available!

Note these terms come from <u>zeros</u> of $H(s)$...

Real-time implementation constraint

Computation of $u(t)$ must require only knowledge of $e(t)$, ($\underline{\text{Not}}$ $\dot{e}(t)$, $\ddot{e}(t)$, etc.)

But note the DE from $\mathcal{L}^{-1}\{b(s)U(s) = a(s)E(s)\}$ will have derivatives of $e(t)$ on RHS.

If you think about it, this would seem to suggest $H(s)$ could never have any zeros [i.e. $a(s)$ must be a constant]

Fortunately this is <u>not</u> *the case, if we think a little more deeply:*

$$H(s)E(s) = \left[\frac{a(s)}{b(s)}\right]E(s) = \left[d(s) + \frac{a'(s)}{b(s)}\right]E(s)$$

where $d(s)$ is the quotient polynomial of $\dfrac{a(s)}{b(s)}$ and $a'(s)$ is the remainder polynomial.

If $\deg\{a(s)\} \geq \deg\{b(s)\}$, then

$$H(s) = \frac{a(s)}{b(s)} = d(s) + \frac{a'(s)}{b(s)}$$

where $\deg\{d(s)\} = \deg\{a(s)\} - \deg\{b(s)\}$

and $\deg\{a'(s)\} = \deg\{b(s)\} - 1$

Since $\deg\{a'(s)\} < \deg\{b(s)\}$ we can expand

$$\frac{a'(s)}{b(s)} = \sum_{k=1}^{M} \frac{c_k}{(s - \ell_k)}$$

(M = # poles of $H(s)$ here!)

where ℓ_k are roots of $b(s)$ (poles of $H(s)$)

and:

$$c_k = \left\{ (s - \ell_k) \left[\frac{a'(s)}{b(s)} \right] \right\}_{s = \ell_k}$$

If instead $\deg\{a(s)\} < \deg\{b(s)\}$ then

$$H(s) = \frac{a(s)}{b(s)} = \sum_{k=1}^{M} \frac{c_k}{(s-\ell_k)} \quad \text{directly}$$

so that $d(s) = \emptyset$ and $a'(s) = a(s)$

in the above.

Thus generally:

PFE of $H(s)$!

$$H(s)E(s) = \boxed{\left[d(s) + \sum_{k=1}^{M} \frac{c_k}{s-\ell_k} \right]} E(s)$$

or $$H(s)E(s) = d(s)E(s) + \sum_{k=1}^{M} c_k \left[\frac{1}{s-\ell_k} \right] E(s)$$

Look at each of the terms individually

$$U(s) = H(s)E(s) = d(s)E(s) + \sum_{k=1}^{M} c_k \left[\frac{1}{s-\ell_k}\right] E(s)$$

Introduce:

$$X_k(s) = \left(\frac{1}{s-\ell_k}\right) E(s)$$

So that

$$U(s) = d(s)E(s) + \sum_{k=1}^{M} c_k X_k(s)$$

and

$$u(t) = \mathcal{L}^{-1}\{d(s)E(s)\} + \sum_{k=1}^{M} c_k x_k(t)$$

with $x_k(t) = \mathcal{L}^{-1}\{X_k(s)\}$

But note from above: $(s-\ell_k)X_k(s) = E(s)$

$$\Rightarrow \dot{x}_k(t) - \ell_k x_k(t) = e(t)$$

DE which only involves $e(t)$!

Thus, generally the control calculations required by $H(s)$ can be implemented using:

$$\boxed{u(t) = \mathcal{L}^{-1}\{d(s)E(s)\} + \sum_{k=1}^{M} c_k x_k(t)}$$

where

$$\boxed{\dot{x}_k(t) = \ell_k x_k(t) + e(t)}$$

$\leftarrow$ M different 1^{st} order DEs. for $x_k(t)$

ℓ_k are poles of $H(s)$, and c_k are the residues:

$$c_k = \left\{(s-\ell_k)\left[\frac{a'(s)}{b(s)}\right]\right\}_{s=\ell_k}$$

What about $\mathcal{L}^{-1}\{d(s)E(s)\}$? Recall $d(s)$ is a polynomial with degree $\deg\{a(s)\} - \deg\{b(s)\}$

If $\deg\{d(s)\} > 1$ $\left(\deg\{a(s)\} > \deg\{b(s)\}\right)$

i.e.

$$d(s) = d_0 + d_1 s + d_2 s^2 + \cdots$$

then $\mathcal{L}^{-1}\{d(s)E(s)\} = d_0 e(t) + \underbrace{d_1 \dot{e}(t) + d_2 \ddot{e}(t) + \cdots}_{}$

Cannot be implemented with assumed measurements.

Thus, these add'l terms can only be implemented if

$$\deg\{d(s)\} = \emptyset$$ (i.e. $d(s)$ is just a constant)

Or equivalently $\deg\{\underline{a(s)}\} \leq \deg\{\underline{b(s)}\}$

numerator of $H(s)$ → Denom $H(s)$