Question 1

$G(s) = \dfrac{7}{(s/5+1)^3}$ $u(t) = K e(t)$ $K > 0$.

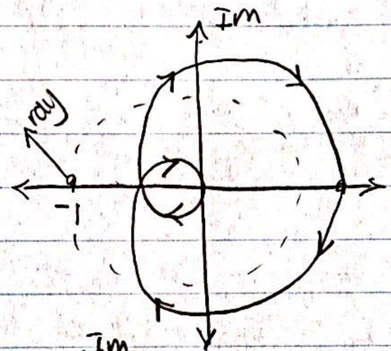
$U(s) = H(s)\, E(s)$

$\therefore H(s) = K$

$L(s) = G(s)H(s) = \dfrac{7K}{(s/5+1)^3}$ $L_N(s) = \dfrac{7}{(s/5+1)^3}$ where $K = K_N = 1$

a.) Nyquist analysis

if $0 < K \leq 1$:

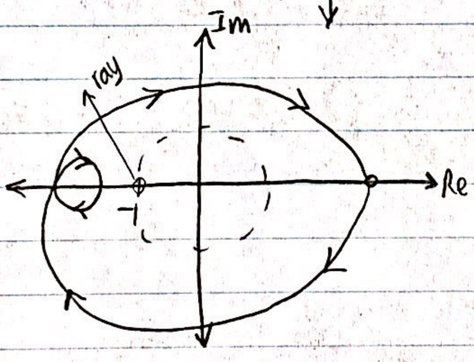


$N_{cw}(L) = -P_R(L)$

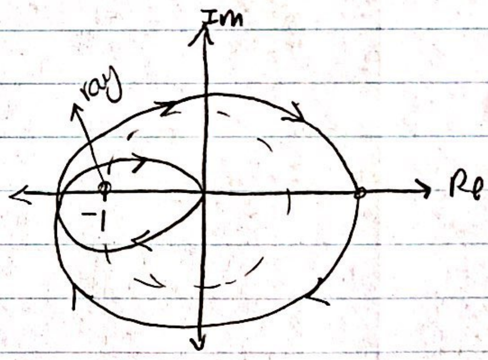
$0 = 0$ ✓

$\boxed{\text{STABLE}}$

if $K > 1$:



OR



$N_{cw}(L) = -P_R(L)$

$1 \neq 0$

$\boxed{\text{UNSTABLE}}$

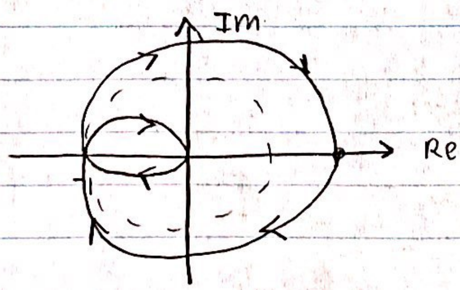
$N_{cw}(L) = -P_R(L)$

$2 \neq 0$

$\boxed{\text{UNSTABLE}}$

determine K_{max} for stability :

if $K = K_{max}$:



Here $L(j\omega)$ passes through the -1 point.

$\begin{cases} \text{If } K > K_{max}, \text{ unstable} \\ \text{If } K \leq K_{max}, \text{ stable.} \end{cases}$

$L(s) = \dfrac{7K_{max}}{(s/5+1)^3}$ @ $w_a = 8.66$ rad/s, $\|L(s)\| = 1$.

$a_{dB} = 1.14$ dB.

$\|L(s)\| = 1 = \left\| \dfrac{7K_{max}}{\left(\frac{8.66j}{5}+1\right)^3} \right\| \Rightarrow \boxed{K_{max} = 1.14}$

→ K_{max} can also be found by realizing if the $L(jw)$ graph is moved upward by 1.14 dB, $L(jw)$ would pass through -1.

$\dfrac{K_{max}}{K_N} = 10^{(1.14/20)}$ where $K_N = 1$.

$\boxed{K_{max} = 1.14}$

b.) $\gamma = 55°$

$\gamma = 180 + \measuredangle L(s) = 55° \Rightarrow \measuredangle L(s) = -125°$ when $\|L(s)\| = 1$.

@ $\measuredangle L_N(s) = -125°$, $\|L_N(s)\| = 9.28$ dB

→ we need to lower the $L(jw)$ graph by 9.28 dB to achieve $\gamma = 55°$.

$\dfrac{K_{new}}{K_N} = 10^{(-9.28/20)}$

$\boxed{K_{new} = 0.3436}$

$T(s) = \dfrac{L(s)}{1+L(s)}$ $L(s) = \dfrac{7(0.3436)}{(s/5+1)^3} = \dfrac{7(.3436)5^3}{(s+5)^3} = \dfrac{300.61}{(s+5)^3}$

$T(s) = \dfrac{n(s)}{n(s)+d(s)} = \dfrac{300.61}{s^3+15s^2+75s+125+300.61}$

$s^3 + 15s^2 + 75s + 125 + 300.61 = 0$

$\boxed{\text{CL poles} = \begin{cases} -11.69 \\ -1.65 \pm 5.80j \end{cases}}$ ← closer to Im axis, so $T(s)$ heavily influenced by this pole expected characteristics:

$$t_s = \frac{4}{|\sigma|} = \frac{4}{|-1.65|} = 2.42$$

$$M_p = \exp\left(\pi\sigma / \omega_d\right) = \exp\left(-1.65\pi / 5.80\right) = 0.4091 = 40.91\%$$

$$y_{ss} = T(0) = 300.6 / 425.6 = 0.7063$$

exact characteristics from step response of $T(s)$:

$t_s = 2.36$

$M_p = 35.5\%$

$y_{ss} = 0.706$

Tabulated values:

PARAMETER	EXPECTED	EXACT
t_s	2.42	2.36
M_p	40.91%	35.5%
y_{ss}	0.7063	0.706

There may be discrepancies in the analysis b/c p_1 was ignored, and in reality, p_2 doesn't fit the definition of a dominant pole. Also since $p_1 \ngtr 10 p_2$, this effect is seen on the step response.

c) u_{max} and u_{ss}.

$$Y_d(s) \longrightarrow \boxed{R(s)} \longrightarrow U(s) \qquad R(s) = \frac{H(s)}{1+L(s)}$$

from the step response of $R(s)$:

$$\boxed{\begin{aligned} U_{max} &= 0.344 \\ U_{ss} &= 0.101 \end{aligned}}$$

d) e_{ss}

$$Y_d(s) \longrightarrow \boxed{S(s)} \longrightarrow E(s) \qquad S(s) = \frac{1}{1+L(s)}$$

from the step response of $S(s)$: $\boxed{e_{ss} = 0.294}$

now $y_d(t) = A\sin(\omega t)$. want $|e_{ss}(t)| < A/2$. Find ω range:

$$|e_{ss}| \leq A\,|S(j\omega)|$$

$$\left|\frac{A}{2}\right| < A\,|S(j\omega)|$$

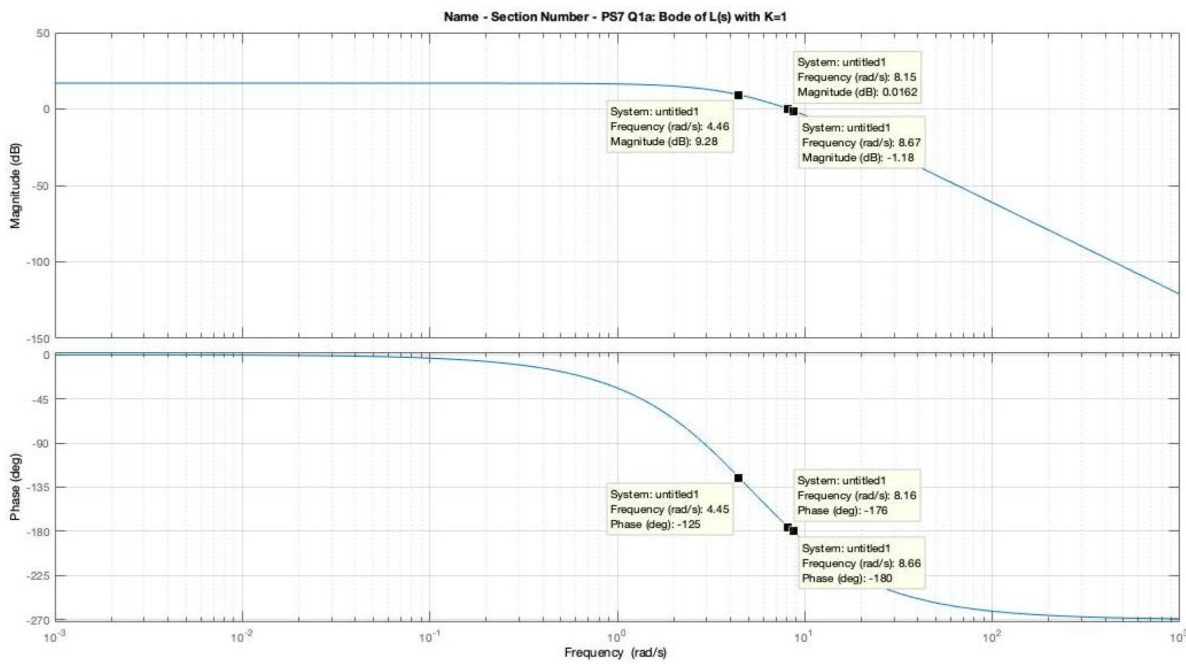
$$1/2 < |S(j\omega)|$$

$$-6.02\text{ dB} = 1/2.$$

→ on bode of $S(s)$, find where $|S(j\omega)| = -6.02$ dB. This occurs at $\omega = 2.68$.

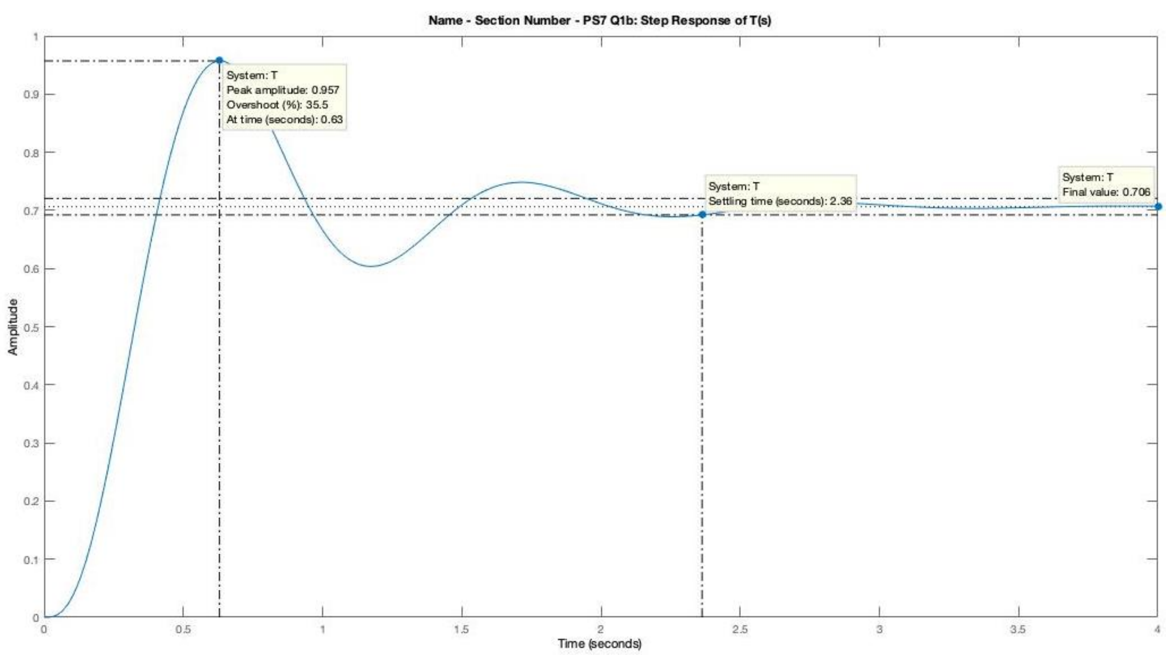
$\therefore$ $\boxed{\omega = [0, 2.68)\ \text{rad/s}}$

```
% ENAE 432, Spring 2019
% TA Solutions
% PS7, Question 1

% Part A
s = tf('s');
w = logspace(-3,3,250000);
G = 7/(s/5+1)^3;
K = 1;
figure(1)
bode(K*G,w); grid on;
title('Name - Section Number - PS7 Q1a: Bode of L(s) with K=1');
[Gm_db, Pm_deg, Wcg, Wcp] = margin(K*G)
Kmax = 10^(Gm_db/20)
```



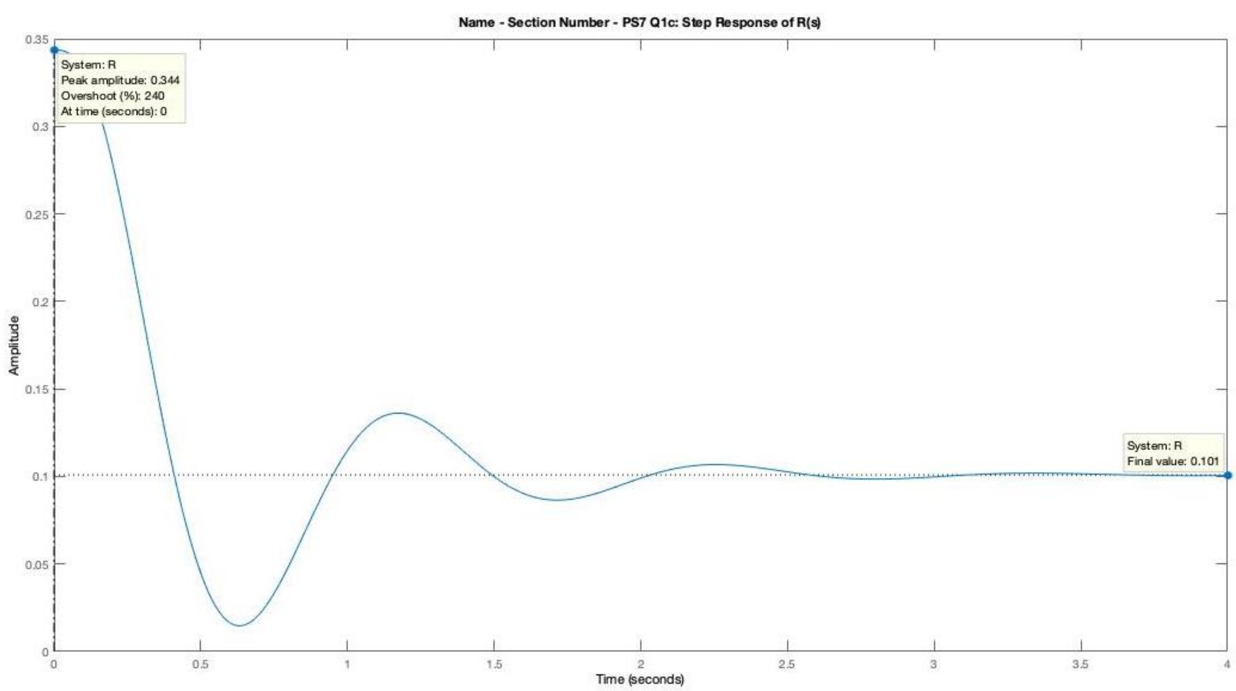

```
Gm_db =

    1.1430


Pm_deg =

    4.5548


Wcg =

    8.6608
```

```
Wcp =

    8.1534


Kmax =

    1.1406


% Part B
Gm_db_b = 9.28;  % value of shift needed (obtained from Bode
diagram)
Knew = 10^(-Gm_db_b/20)
L = Knew*G;
T = feedback(L,1)
CLpoles = pole(T)
step(T); title('Name - Section Number - PS7 Q1b: Step Response
of T(s)');
```

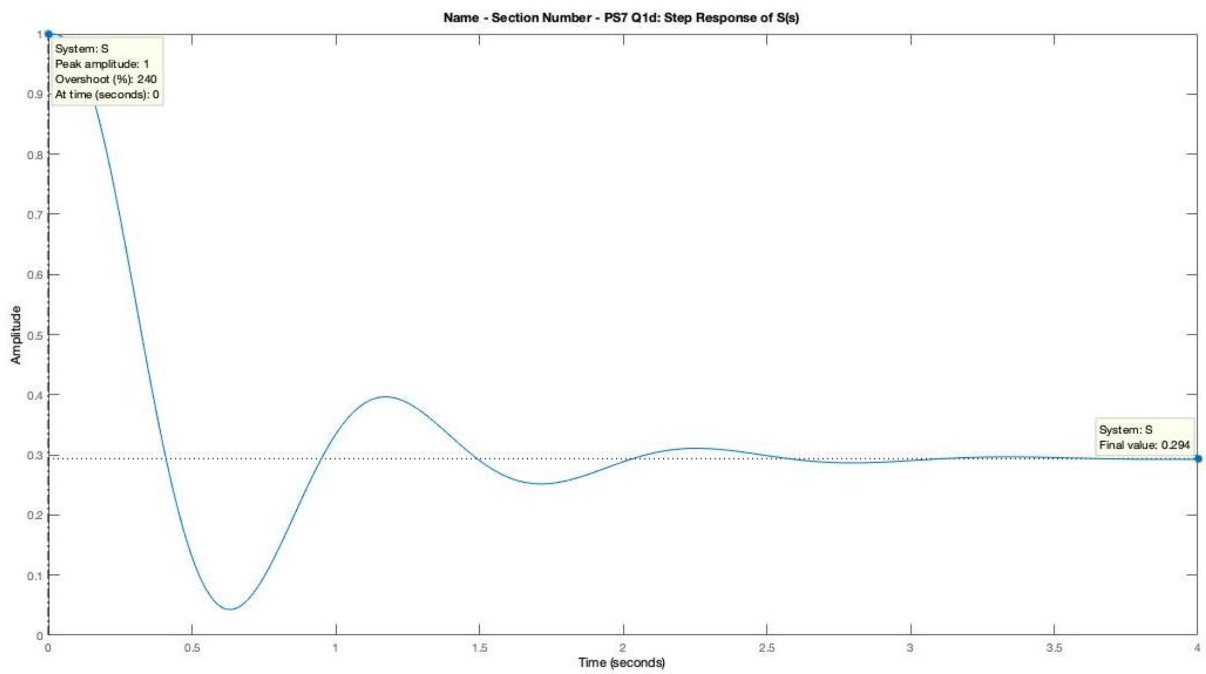


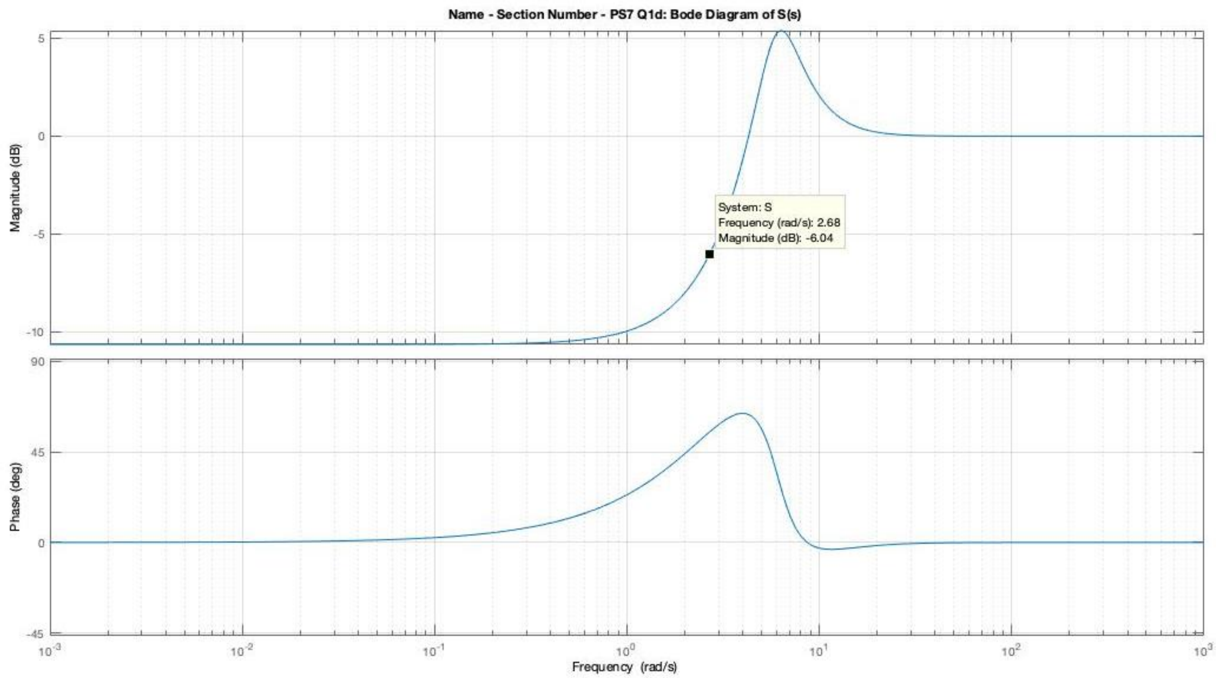

```
Kmax =

    1.1406
```

```
Knew =

    0.3436


T =

              300.6
  ---------------------------
  s^3 + 15 s^2 + 75 s + 425.6

Continuous-time transfer function.


CLpoles =

 -11.6989 + 0.0000i
  -1.6506 + 5.8014i
  -1.6506 - 5.8014i


% Part C
H = Knew;
R = H/(1+L)
step(R); title('Name - Section Number - PS7 Q1c: Step Response of R(s)');
```



```
R =

  0.3436 s^3 + 5.153 s^2 + 25.77 s + 42.94
  ----------------------------------------
        s^3 + 15 s^2 + 75 s + 425.6

Continuous-time transfer function.


% Part D
S = feedback(1,L)
step(S); title('Name - Section Number - PS7 Q1d: Step Response
of S(s)');
bode(S,w); grid on;
title('Name - Section Number - PS7 Q1d: Bode Diagram of S(s)');
```





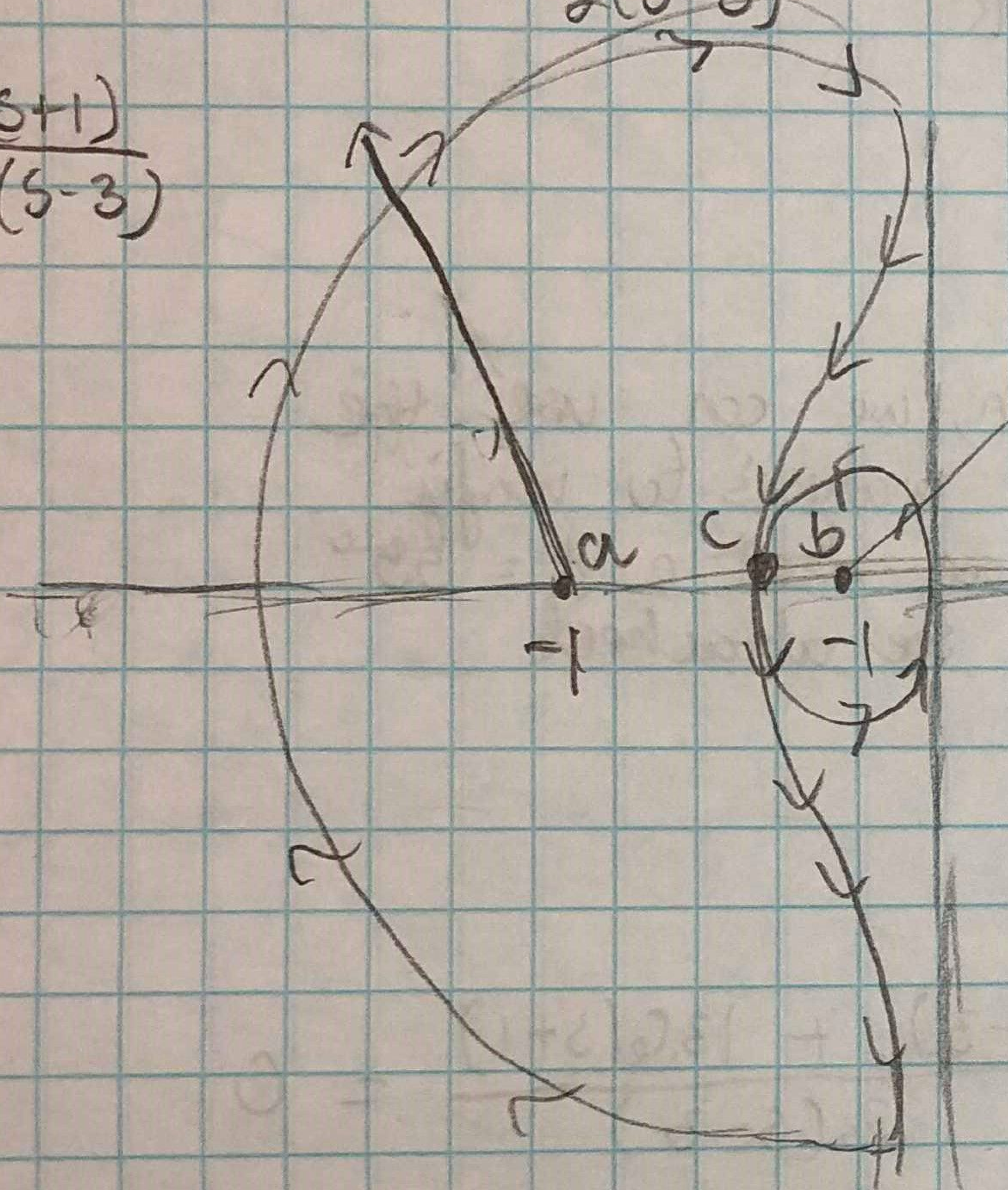

```
S =

    s^3 + 15 s^2 + 75 s + 125
  ---------------------------
  s^3 + 15 s^2 + 75 s + 425.6

Continuous-time transfer function.
```

PST Question #2 Solutions * acceptable to use evalfr functions

Part A) $G(s) = \frac{s+1}{2(s-3)}$ $H(s) = \frac{K}{s}$ $K > 0$

$L(s) = \frac{K(s+1)}{2s(s-3)}$

$P_R(L) = 1$



★ For small K, we have -1 at point a

$N_{cw} = 1$

$N_{cw} \neq -P_R(L)$

unstable

★ for large K we have -1 at point b

$N_{cw} = -1$

$N_{cw} = -P_R(L)$

$-1 = -1$ ✓

stable

- From this analysis, we can find the maximal K-value by finding the K-value that puts the -1 point at point C in the Nyquist diagram

- Observe that the condition that satisfies this is ..

$\omega_a = \omega_\gamma$ (i.e Nyquist/polar crosses the unit circle and the negative real axis at the same time)

- which implies that $\gamma = 0$ and $|L(j\omega_a)| = 1$

$0 = 180 + \angle L(j\omega_\gamma)$ 90°

- From this analysis, we can find the maximal K-value by finding the K-value that puts the -1 point at point C in the Nyquist diagram

- Observe that the condition that satisfies this is...

$$\omega_\alpha = \omega_\gamma$$ (i.e Nyquist/polar crosses the unit circle and the negative real axis at the same time)

- which implies that $\gamma = 0$ and $|L(j\omega_\alpha)| = 1$

$$0 = 180 + \angle L(j\omega_\gamma)$$

$$0 = 180 + \cancel{\angle\left(\frac{K}{2}\right)}^{0} + \angle(j\omega_\gamma + 1) - \cancel{\angle(j\omega)}^{90°} - \angle(j\omega_\gamma - 3)$$

$$0 = 90 + \arctan\left(\frac{\omega_\gamma}{1}\right) - \left[\arctan\left(\frac{\omega_\gamma}{-3}\right) + 180\right]$$

this comes from the fact that we are in the 2nd quadrant

$$\underline{\sqrt{3} = \omega_\gamma = \omega_\alpha}$$

- Finally solve for K

let L_0 be L/K

$$|L(j\sqrt{3})| = 1$$

$$|K||L_0(j\sqrt{3})| = 1$$

$$|K| = |L_0(j\sqrt{3})|^{-1}$$

$$\boxed{K_{max} = 6}$$

* may use evalfr function as well as abs( ) and angle( ) *

See attached Bode Plot of L that verifies this result

b) Find K for which L will have $\gamma = 55°$

Part B

$$55° = 180 + \angle L(j\omega_\gamma)$$
$$1 = |L(j\omega_\gamma)|$$

we use these two conditions to solve for K

$$55° = 90° + \arctan\left(\frac{\omega_\gamma}{1}\right) - \left[\arctan\left(\frac{\omega_\gamma}{-3}\right) + 180\right]$$

$$\boxed{\omega_\gamma = 6.2\,Hz}$$

$$|L(j\omega_\gamma)| = |K||L_0(j\omega_\gamma)| = 1$$

$$|K| = \frac{1}{|L_0(j\omega_\gamma)|} = 1$$

$$\boxed{K = 13.6}$$

again, we can use the bode analysis to verify that we meet a $\gamma = 55°$ see attached

---

Definition of closed loop poles
solutions to

$$1 + L(s) = 0$$

$$1 + \frac{(13.6)(s+1)}{2s(s-3)} = 0 \longrightarrow \frac{2s(s-3) + 13.6(s+1)}{2s(s-3)} = 0$$

$$2s(s-3) + 13.6(s+1) = 0$$

$$2s^2 - 6s + 13.6s + 13.6 = 0$$

solving this equation:
$$P_{1,2} \approx -1.9 \pm 1.786j$$

expected transient metrics

t_s $$t_s = \frac{4}{|\sigma|} = \boxed{2.1s = t_s}$$

%OS $$\%OS = M_p = e^{(\sigma\pi/\omega_d)} \times 100$$

$$\boxed{M_p = 3.59\%}$$

exact transient metrics (see MATLAB)

$$\boxed{t_s = 2.55s}$$

$$\boxed{\%\text{ Overshoot} = 61.8\%}$$

t_s | $t_s = \frac{4}{|\sigma|} = \boxed{2.1s = t_s}$ | $\boxed{t_s = 2.55s}$

$\%OS$ | $\%OS = M_p = e^{(\sigma\pi/w_d)} \times 100$ | $\boxed{\%\text{ Overshoot} = 61.8\%}$

$\boxed{M_p = 3.59\%}$

* the estimates assume the we strictly only have poles; however, T(s) will have zeros present with the poles, which heavily affect overshoot. This is why there are discrepancies in t_s and M_p *

C * See attached MATLAB for plot $|U(t)|_{max} = .46$ $|u(t)|_{ss} = .441$

D) Determine $e_{ss}(t)$......

if $y_d(t) = \mathbb{1}(t)$ or step response

$$e_{ss}(t) = 1 S(0)$$

$$S = \frac{1}{1+L(s)} = \frac{2(\;2s(s-3)(\;)}{2s(s-3)+13.6(s+1)}$$

$S(0) = 0$

$\boxed{e_{ss}(t) = 0 \text{ for step response}}$, see attached step response to verify

this must be found analytically

find the range of w s.t $|e_{ss}(t)| < A/2$

if $y_d(t) = A\sin(wt)$ or frequency respon

$= A\cos(wt + \pi/2)$

$e_{ss}(t) = B\cos(wt + \varphi)$

$B = |S(jw)|A$, where B is the m magnitude of $e_{ss}(t$

we want $|e_{ss}(t)| \leq \frac{A}{2}$

which means

$|e_{ss}(t)|_{max} \leq \frac{A}{2}$

$B \leq \frac{A}{2}$

d) continued
using

$B = |S(j\omega)| A \longrightarrow \frac{1}{2} = |S(j\omega_{max})| \approx -13.46$
$B \leq A/2$

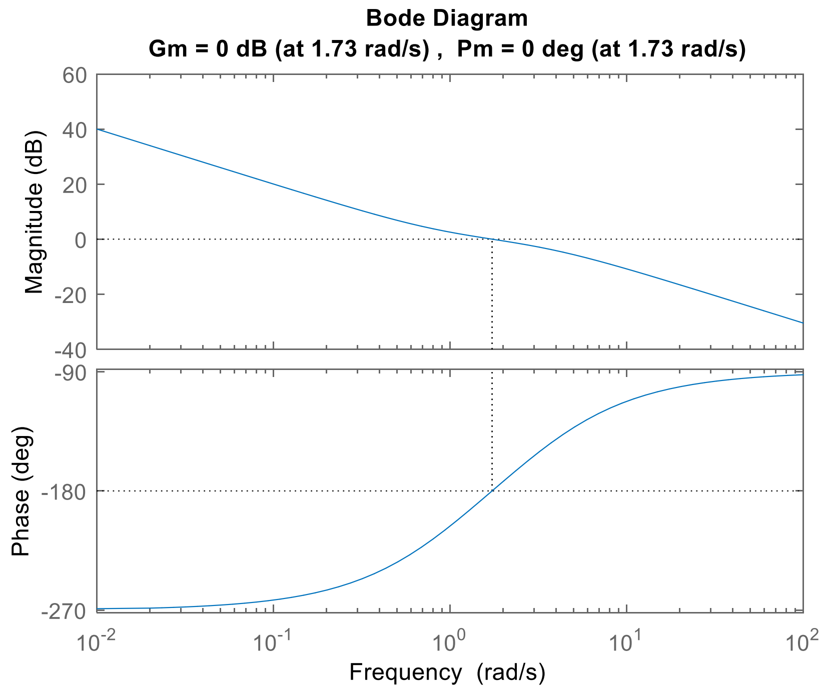
$$|S(j\omega_{max})| = \frac{2|j\omega_{max}| \cdot |j\omega_{max} - 3|}{2|j\omega_{max}| \cdot |j\omega_{max} - 3| + 13.6|j\omega_{max} + 1|} = \frac{1}{2}$$

Solve for ω_{max} (bode analysis is also acceptable)

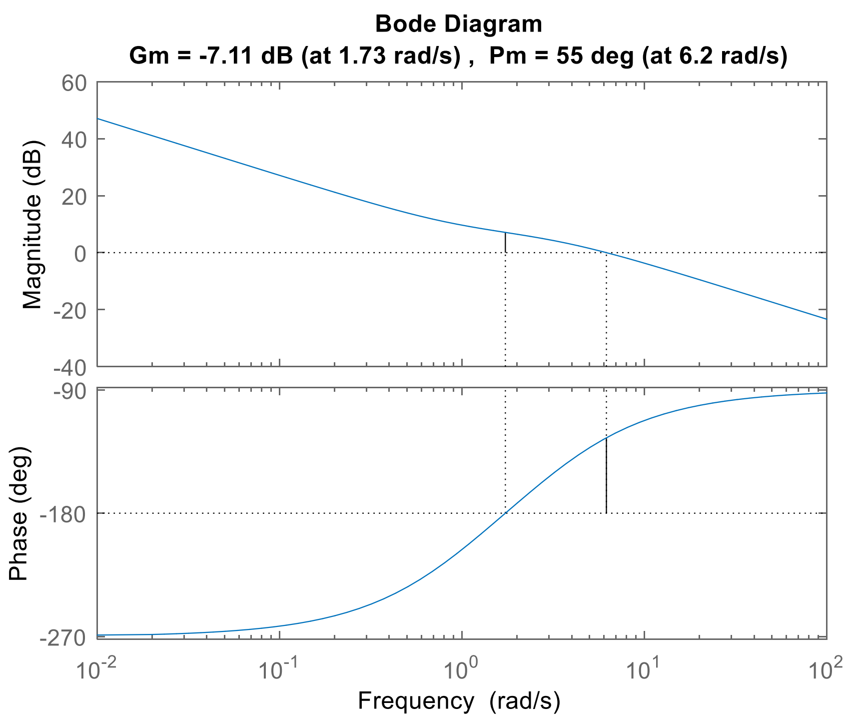
$\omega_{max} \approx .452 \text{ Hz}$
thus the range must be

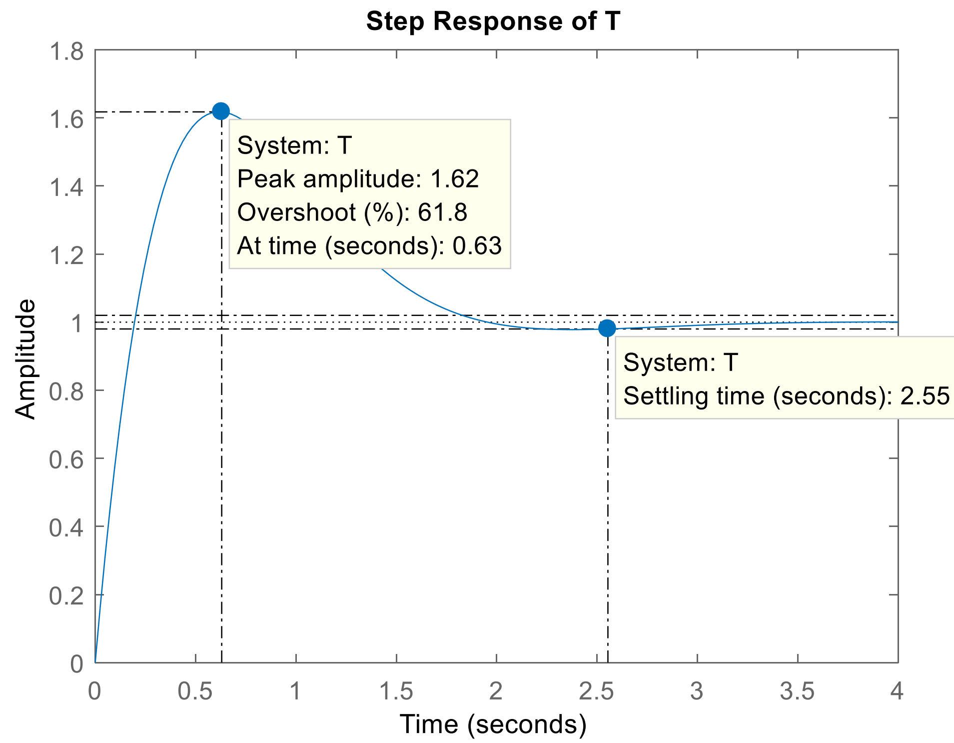
$\boxed{\omega \in [0, .452]}$

Part a

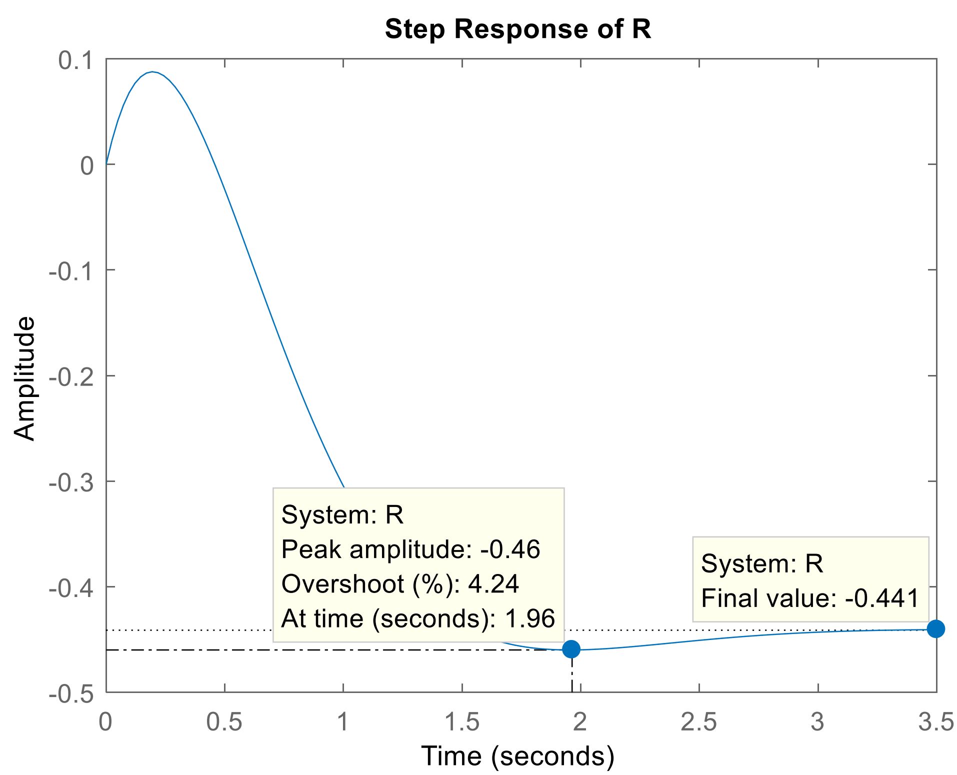

Bode Diagram
Gm = 0 dB (at 1.73 rad/s) , Pm = 0 deg (at 1.73 rad/s)
60
40
20
0
-20
-40
Magnitude (dB)
-90
-180
-270
Phase (deg)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Part b

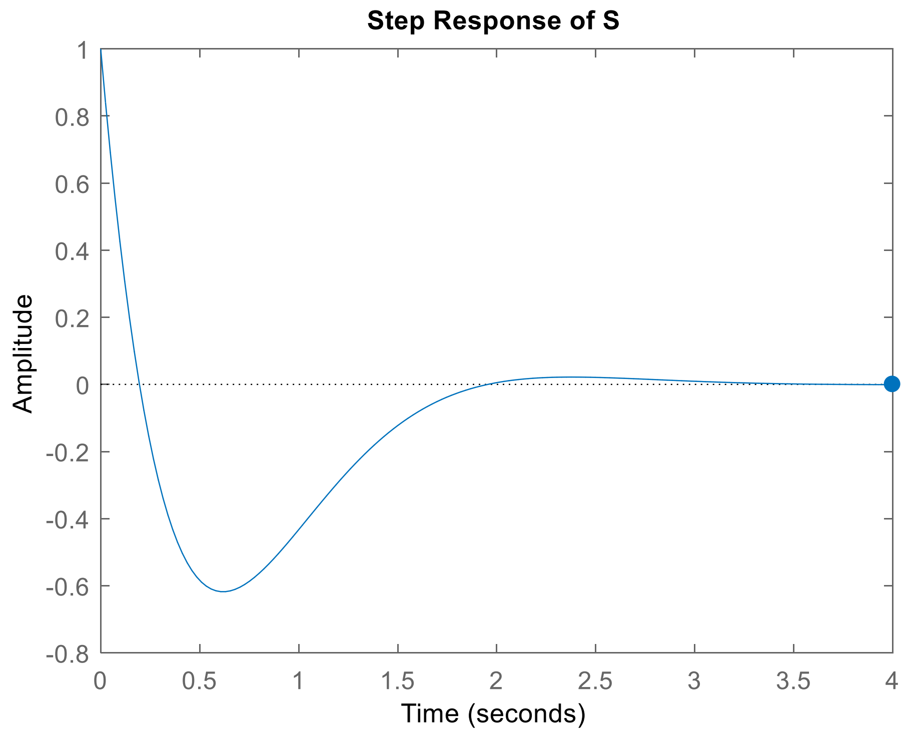

Bode Diagram
Gm = -7.11 dB (at 1.73 rad/s) , Pm = 55 deg (at 6.2 rad/s)
60
40
20
0
-20
-40
Magnitude (dB)
-90
-180
-270
Phase (deg)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)


Step Response of T
System: T
Peak amplitude: 1.62
Overshoot (%): 61.8
At time (seconds): 0.63
System: T
Settling time (seconds): 2.55
Amplitude
Time (seconds)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4

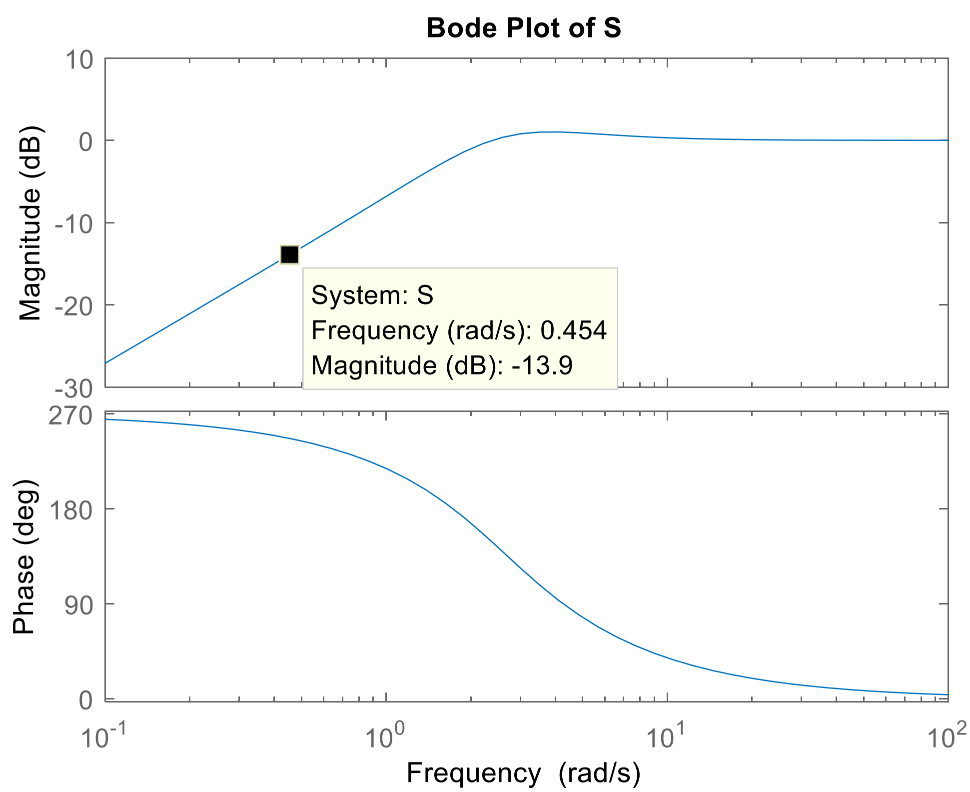
Part c


Step Response of R
System: R
Peak amplitude: -0.46
Overshoot (%): 4.24
At time (seconds): 1.96
System: R
Final value: -0.441
Amplitude
Time (seconds)
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.5
1
1.5
2
2.5
3
3.5

Part d



Step Response of S
Amplitude
Time (seconds)



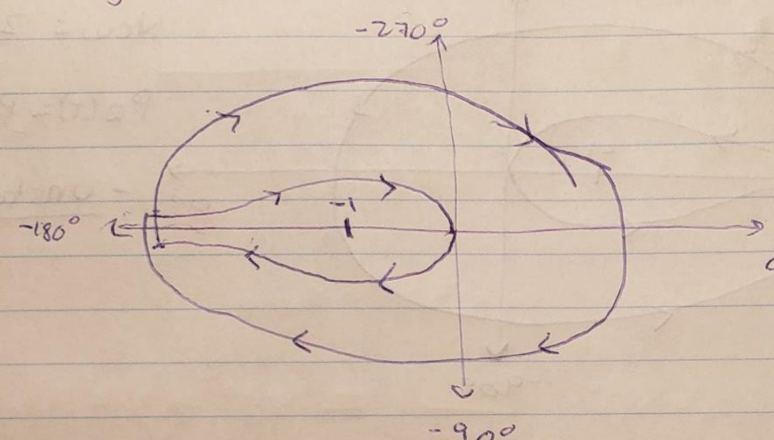
Bode Plot of S
Magnitude (dB)
System: S
Frequency (rad/s): 0.454
Magnitude (dB): -13.9
Phase (deg)
Frequency (rad/s)

PS7 Q3 Solns

$$G(s) = \frac{12}{s^2(\tau s+1)}, \quad \tau > 0$$

a) Proportional control $H = K$ such that $L(s) = \frac{12K}{s^2(\tau s+1)}$

- using Nyquist for $K > 0$:

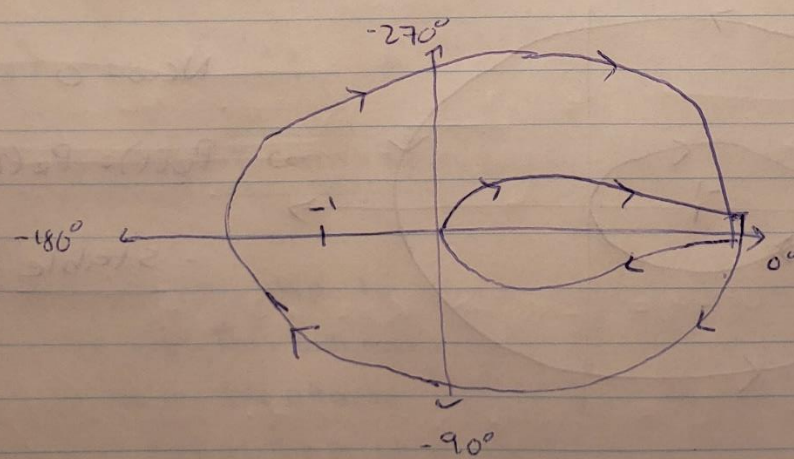


$N_{cw} = 2$

$P_R(T) = P_R(L) + N_{cw} = 2$

$\Rightarrow$ unstable in $T(s)$

- using Nyquist for $K < 0$:



$N_{cw} = 1$

$P_R(T) = P_R(L) + N_{cw} = 1$

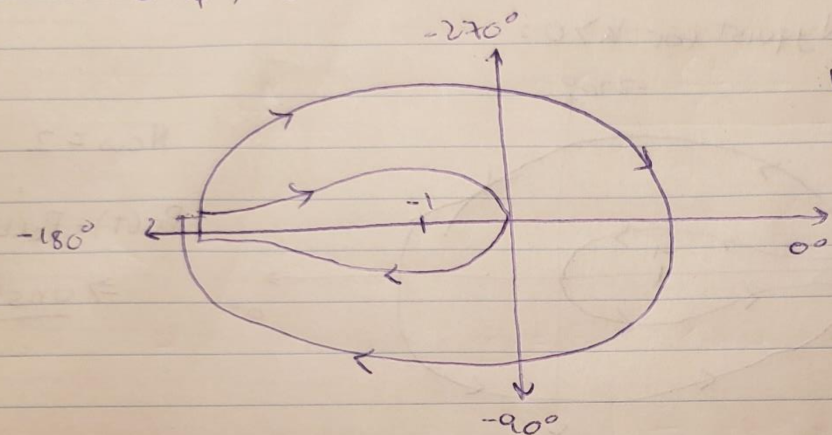
$\Rightarrow$ unstable in $T(s)$

$\therefore$ system cannot be stabilized w/ proportional control

b) $u = K_p e(t) + K_D \dot{e}(t) \rightarrow H(s) = K_D s + K_p$

$$L(s) = \frac{12(K_d s + K_p)}{s^2(\tau s+1)} = \frac{12 K_p \left(\frac{K_d}{K_p} s + 1\right)}{s^2(\tau s+1)}$$

- when $\tau K_p \geq K_d$

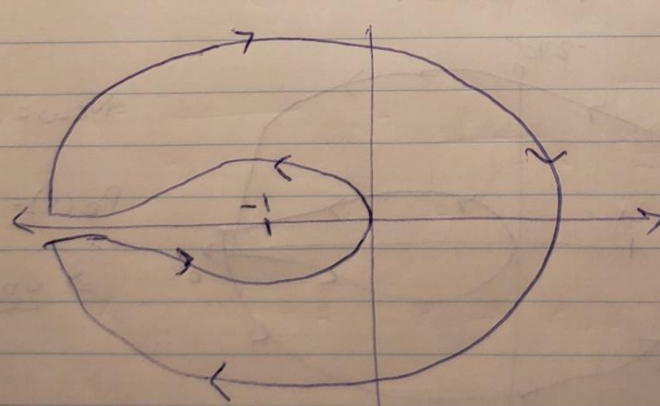


$N_{cw} = 2$

$P_R(T) = P_R(L) + N_{cw} = 2$

- <u>unstable</u>!

- when $\tau K_p < K_d$



$N_{cw} = 0$

$P_R(T) = P_R(L) + N_{cw} = 0$

- <u>stable</u>!

c) $\tau = 0.17$, $K_d = K_p$, such that

$$L(s) = \frac{12 K_p (s+1)}{s^2(0.17s+1)}$$

from Bode, max $(\angle L(j\omega)) = -135°$ @ $\omega = 2.45$ rad/s

- find K_p so that $|L(j(2.45))| = 1$

$$1 = \frac{12 K_p \sqrt{(2.45)^2+1}}{2.45^2 \sqrt{(0.17 \cdot 2.45)^2+1}} \rightarrow \boxed{K_p = .205}$$

If $L(s) = \frac{N(s)}{D(s)}$, CL poles are values which satisfy $N(s)+D(s)=0$

$$\Rightarrow 12K_p(s+1) + s^2(0.17s+1) = 0$$

$p_1 = -2.3474$
$p_{2,3} = -1.7675 \pm 1.742j$ } CL poles are in LHP → system stable

d) $M_p = e^{\sigma\pi/\omega_d} = e^{-1.768\pi/1.742} \approx \boxed{4.12\%}$ increased

overshoot comes from LHP zero | exact %OS and t_s attached

e) Increasing K_p to $10K_p$ will cause phase margin to decrease to 22.4° and increase the ω_γ. This will make the overshoot increase and the settling time decrease. See attached MATLAB for reference. A faster response is achieved at the price of greater overshoot

when K_p increases by a factor of 10:

$\gamma_{new} = 22.4°$
$\omega_\gamma = 11.4$ rad/s
$\%OS = 56.4$
$t_s = 1.48$ s

f) Need $>10^\circ$ phase margin $\Rightarrow \gamma = 55^\circ$ for the system

$$L(s) = \frac{12K_p\left(\frac{K_d}{K_p}s + 1\right)}{s^2(\tau s + 1)}$$

we can make this "look" like a lead compensator problem by breaking L(s) up so that

$$G'(s) = \frac{1}{s^2} \qquad H'(s) = \frac{12K_p\left(\frac{K_d}{K_p}s+1\right)}{\tau s + 1} = k\left[\frac{\beta\tau s + 1}{\tau s + 1}\right]$$

where $\beta = \frac{K_d}{K_p}/\tau$, $k = 12K_p$

$\angle G'(s) = -180^\circ \; \forall \omega$, so $\phi_{req} = 55^\circ = \gamma_{new}$

$$\phi_{req} = \phi_{max} = \sin^{-1}\left[\frac{\beta - 1}{\beta + 1}\right] = 55^\circ \rightarrow \beta = 10.06$$

Them $\omega_{max} = \frac{1}{\tau\sqrt{\beta}} = \frac{1}{0.17\sqrt{10.06}} = 1.855$ rad/s

and $K = \frac{1}{|L_o(j\omega_{max})|} = \left[\frac{1}{1.855^2}\frac{\sqrt{[(10.06 \cdot 0.17) \cdot 1.855]^2 + 1}}{\sqrt{(0.17 \cdot 1.855)^2 + 1}}\right]^{-1} = 1.0844$

so: $K = 1.0844 = 12K_p \rightarrow K_p = .0904$

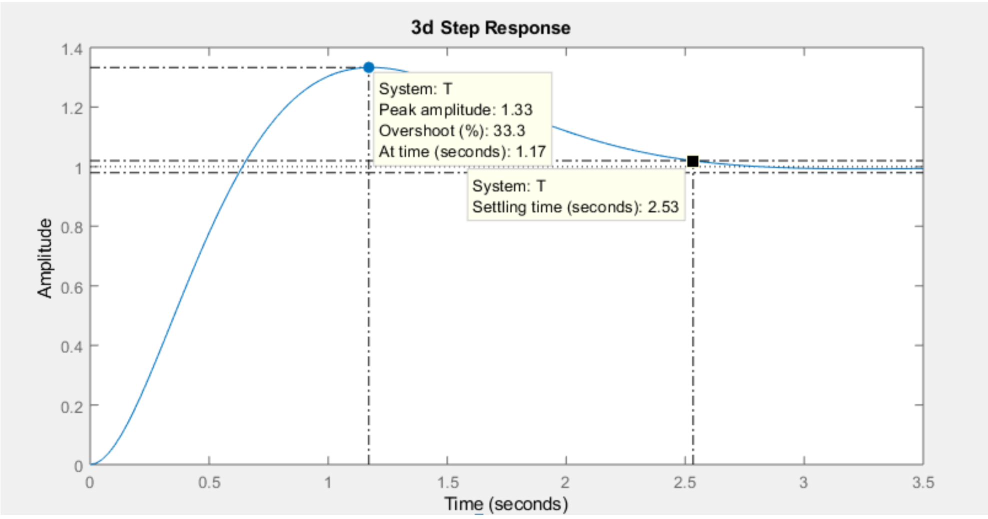
and: $\frac{K_d}{K_p} = \beta\tau \Rightarrow K_d = K_p\beta\tau = 0.094 \cdot 10.06 \cdot 0.17 = 0.1608$

Thus: $K_p = 0.0904$, $K_d = 0.1608$, $\omega_\gamma = 1.855$ rad/s

%OS reduces to 22% but ts inceases to 4.6 sec

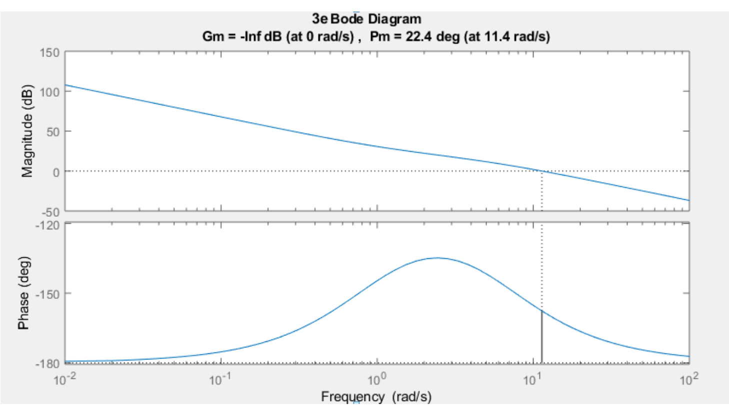
PS8Q3d

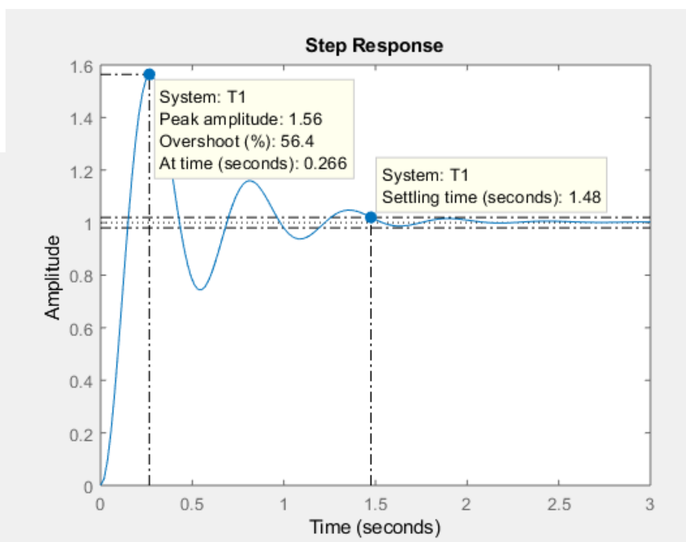
```
% part d
s = tf('s');
L = 12*.205*(s+1)/(s^2*(0.17*s+1));
T = L/(1+L);
step(T);
```



PS8Q3e

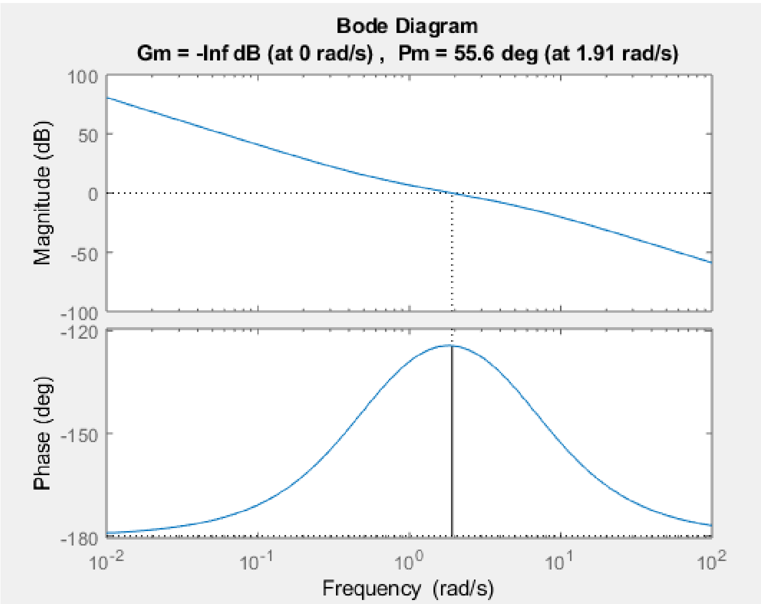
```
% part e
figure
margin(10*L);
figure
T1 = feedback(10*L,1);
step(T1)
```

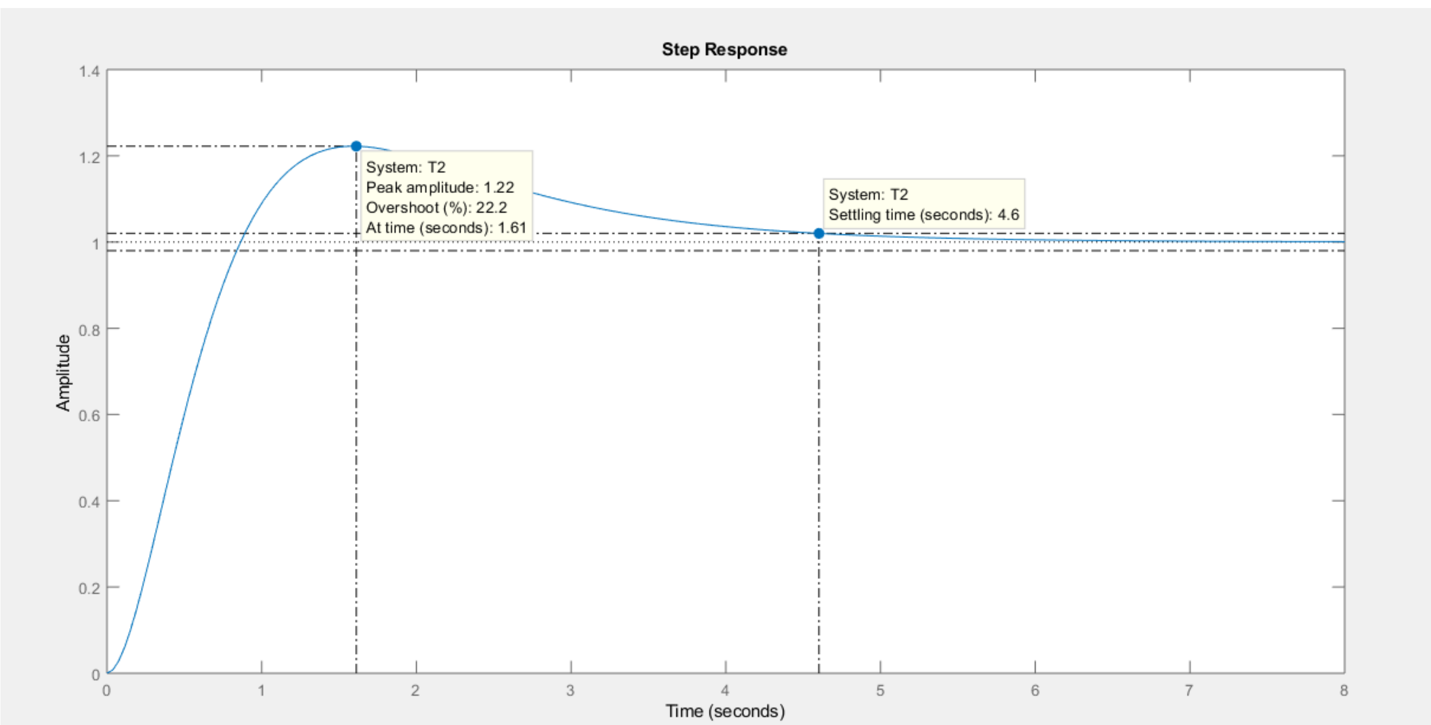




PS8Q3f

```
% part f
figure
L = 12*.0904*((0.1608/0.0904)*s+1)/(s^2*(0.17*s+1));
margin(L)
figure
T2 = feedback(L,1);
step(T2);
```





PS7

Q4. FOR THE COMPENSATOR $H(s) = \dfrac{20(s+2)^3}{s(s+6)(s+12)}$

DETERMINE THE DERIVATIVE FREE IMPLEMENTATION EQUATIONS WHICH WOULD BE USED TO GENERATE $u(t)$ from $e(t)$ IN THE OPERATION OF THE FEEDBACK LOOP.

SOLUTION:

$$H(s) = 20\left[\frac{(s+2)^3}{s(s+6)(s+12)}\right] = 20\left[\frac{s^3+6s^2+12s+8}{s^3+18s^2+72s}\right] = 20H_o(s)$$

Dividing out:

$$\begin{array}{r|l} & 1 \\ s^3+18s^2+72s & s^3+6s^2+12s+8 \\ & -s^3-18s^2-72s \\ \hline & -12s^2-60s+8 \end{array}$$

$$\therefore H_o(s) = 1 + \frac{8-60s-12s^2}{s(s+6)(s+12)} = 1 + \frac{A_1}{s} + \frac{A_2}{s+6} + \frac{A_3}{s+12}$$

$$A_1 = [s\cdot H_o(s)]_{s=0} = \frac{8}{6(12)} = \frac{8}{72} = \frac{1}{9}$$

$$A_2 = [(s+6)H_o(s)]_{s=-6} = \frac{8-60(-6)-12(36)}{-6(-6+12)} = \frac{16}{9}$$

$$A_3 = [(s+8)H_o(s)]_{s=-8} = \frac{8-60(-12)-12(144)}{-12(-12+6)} = \frac{-125}{9}$$

$$H_o(s) = 1 + \frac{1}{9}\cdot\frac{1}{s} + \frac{16}{9}\cdot\frac{1}{(s+6)} - \frac{125}{9}\frac{1}{(s+12)}$$

$$H(s) = 20H_o(s) = 20 + \frac{20}{9}\cdot\frac{1}{(s)} + \frac{320}{9}\cdot\frac{1}{(s+6)} - \frac{2500}{9}\cdot\frac{1}{(s+12)}$$

$$\boxed{\begin{aligned} u(t) &= 20e(t) + \frac{20}{9}x_1(t) + \frac{320}{9}x_2(t) - \frac{2500}{9}x_3(t) \\ \dot{x}_1(t) &= e(t) \\ \dot{x}_2(t) &= -6x_2(t) + e(t) \\ \dot{x}_3(t) &= -12x_3(t) + e(t) \end{aligned}}$$