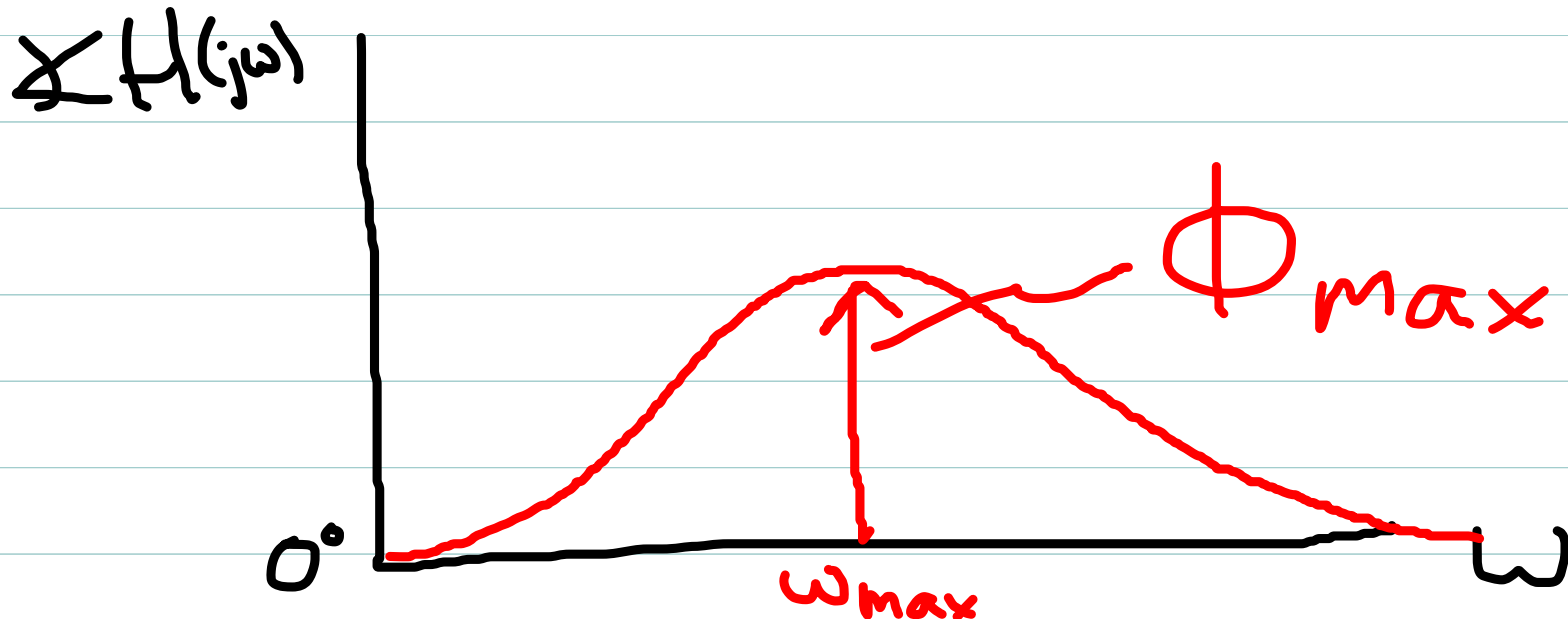


Compensators of general form:

$$H(s) = K \left[ \frac{\beta T s + 1}{T s + 1} \right] \quad \text{with} \quad \begin{cases} T > 0 \\ \beta > 1 \end{cases}$$

$\Rightarrow z_c = -1/\beta T, p_c = -1/T$  so  $|z_c| < |p_c|$  (zero closer to imag axis)  
and  $\beta = p_c/z_c$  ("lead ratio")

Are called "lead compensators", since  $\angle H(j\omega) > 0$   
for all  $\omega > 0$  (positive phase is called "lead").



Note here that:

$$U(s) = H(s)E(s) = K \left[ \frac{\beta \tau s + 1}{\tau s + 1} \right] E(s)$$
$$= K \left[ \beta - \frac{\beta - 1}{\tau s + 1} \right] E(s)$$

$$\Rightarrow u(t) = K\beta e(t) + K(1-\beta)x_1(t)$$
$$\tau \dot{x}_1(t) = -x_1(t) + e(t)$$

Corresponding  
implementation equations

So that  $|u(t)| \propto \beta$  generally.  
Want to find smallest value of  $\beta = P_c/z_c$  which ensures  
 $\Phi_{\max} \geq \Phi_{\text{req}}$ .

We can compute:

$$\Phi_{\max} = \sin^{-1} \left[ \frac{\beta - 1}{\beta + 1} \right]$$

Which is an increasing function of  $\beta > 1$

Thus, the minimum required  $\beta$  value is obtained when

$$\Phi_{\max} = \Phi_{\text{req.}}$$

Selection of  $\tau$  can then be achieved using analytical result

$$\omega_{\max} = \frac{1}{\tau\sqrt{\beta}}$$

and setting  $\omega_{\max} = \omega_{\text{Des.}}$

Revisit previous example:

$$G(s) = \frac{3}{s(s+2)} \quad \text{Want } \gamma_{\text{Des}} = 60^\circ, \omega_{\text{Des}} = 6$$

for which  $\Phi_{\text{req}} = 41.56^\circ$

Previous design used  $H(s) = 93.4 \left[ \frac{s+5.54}{s+60} \right] \quad (\beta = 10.8)$

New design:

$$41.56^\circ = \phi_{\text{req}} = \phi_{\text{max}} = \sin^{-1} \left[ \frac{\beta - 1}{\beta + 1} \right]$$

$$\Rightarrow \beta = 4.94$$

$$\text{Then } \omega = \omega_{\text{des}} = \omega_{\text{max}} = \frac{1}{T\sqrt{\beta}} \Rightarrow T = \frac{1}{\omega_{\text{des}}\sqrt{\beta}}$$

So  $T = 0.075$  here.

$$\text{and thus } H(s) = K \left[ \frac{0.375s + 1}{0.075s + 1} \right] \quad \begin{matrix} (z_c = -2.67 \\ p_c = -13.3) \end{matrix}$$

Choose  $K$  as before: let  $L_0(s) = [L(s)]_{K=1}$

$$\text{Then take } K = \frac{1}{|L_0(j\omega_{\text{des}})|} = 5.63 \text{ here}$$

## Comparison of Designs:

For a unit step  $y_d(t)$ :

Old design:  $t_s \approx 0.96 \text{ sec}$ ,  $M_p \approx 15\%$ ,  $u_{\max} \approx 92$

New design:  $t_s \approx 0.86 \text{ sec}$ ,  $M_p \approx 10\%$ ,  $u_{\max} \approx 27$

New design is essentially the same (a little better) in transient performance, and requires a factor of 3 less maximum control effort.

$\Rightarrow$  minimizing  $\beta$  is very beneficial!

## Another Example

Suppose  $G(s) = \frac{3}{s(s-2)}$

and we again want  $\gamma_{des} = 45^\circ$  with  $\omega_{des} = 6$ , which we know is assured if:

$$L(s) = \frac{6^2 \sqrt{2}}{s(s+6)}$$

You might be tempted to try  $H(s) = \left( \frac{6^2 \sqrt{2}}{3} \right) \left[ \frac{s-2}{s+6} \right]$

Don't do it! Pole-zero cancellation cannot be guaranteed to be exact here, and CL system will have an unstable pole (try it! Use  $(s-2.1)$  in numerator and see what happens).

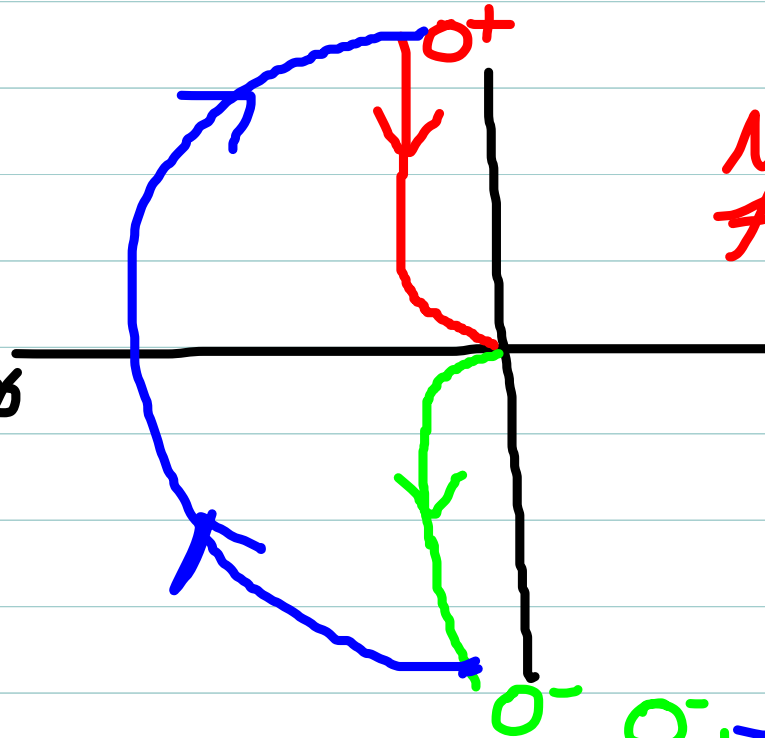
Instead, meet targets using only LHP poles/zeros.

Check Nyquist: Ensure  $\gamma > 0$  still stabilizing

Using  $H(s) = K > 0$

$$P_R(L) = 1$$

$N_{cw}(L) = 1$  for any  $K > 0$   
 $\Rightarrow T(s)$  unstable

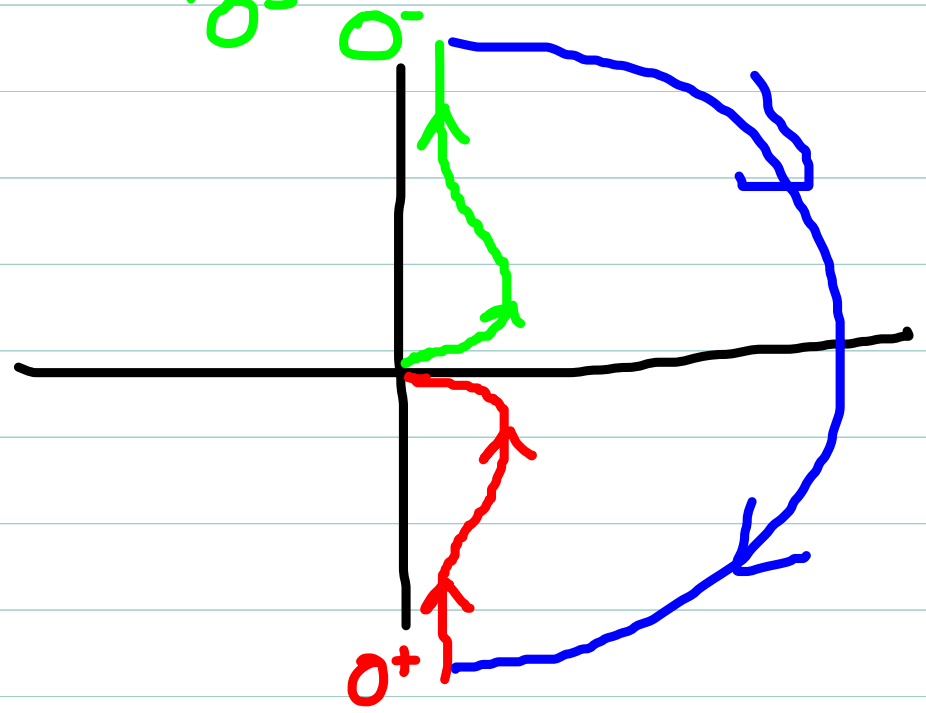


Need  $N_{cw}(L) = -1$   
for stability!

Using  $H(s) = K < 0$

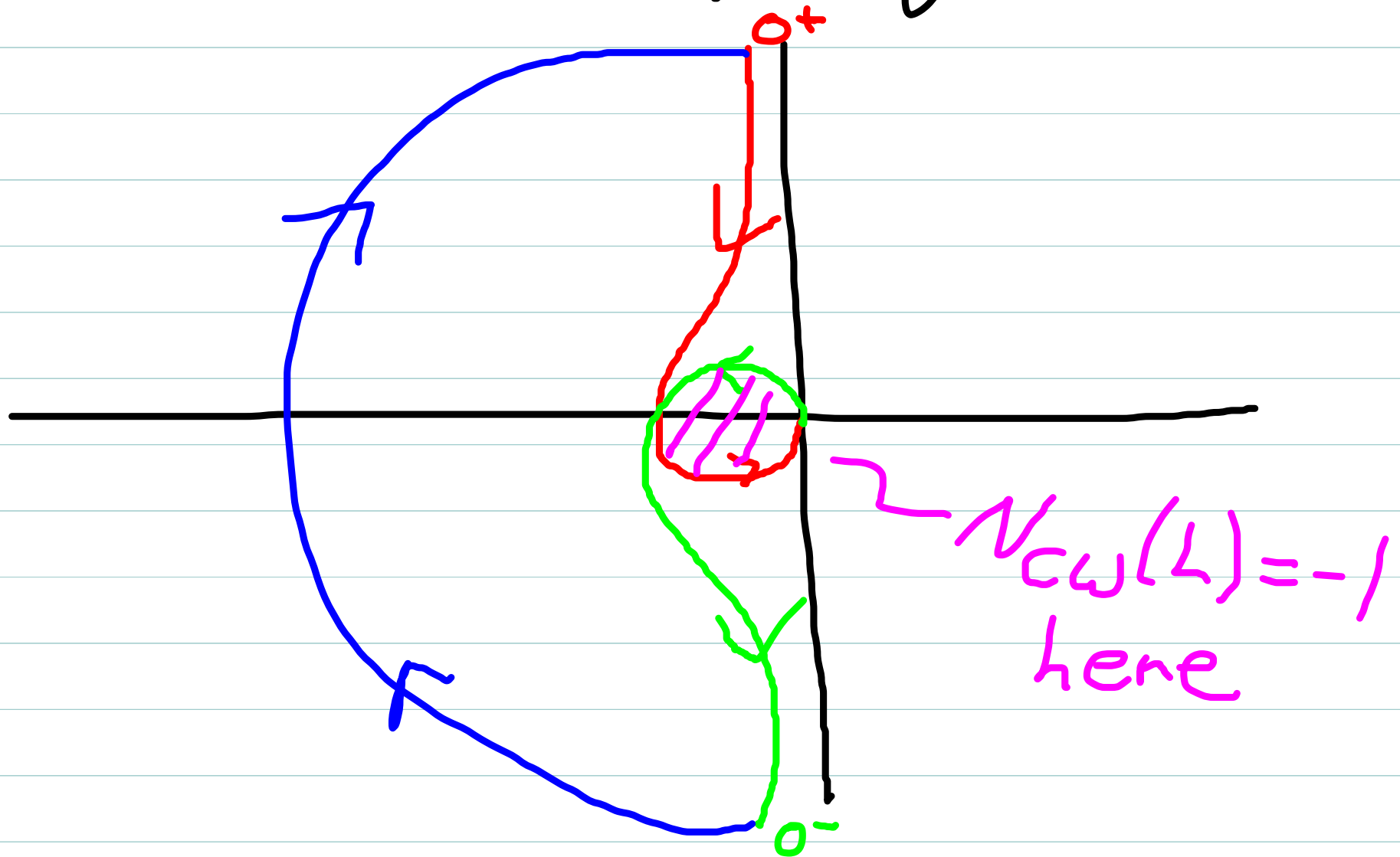
$N_{cw}(L) = 0$  for any  $K < 0$

$T(s)$  unstable



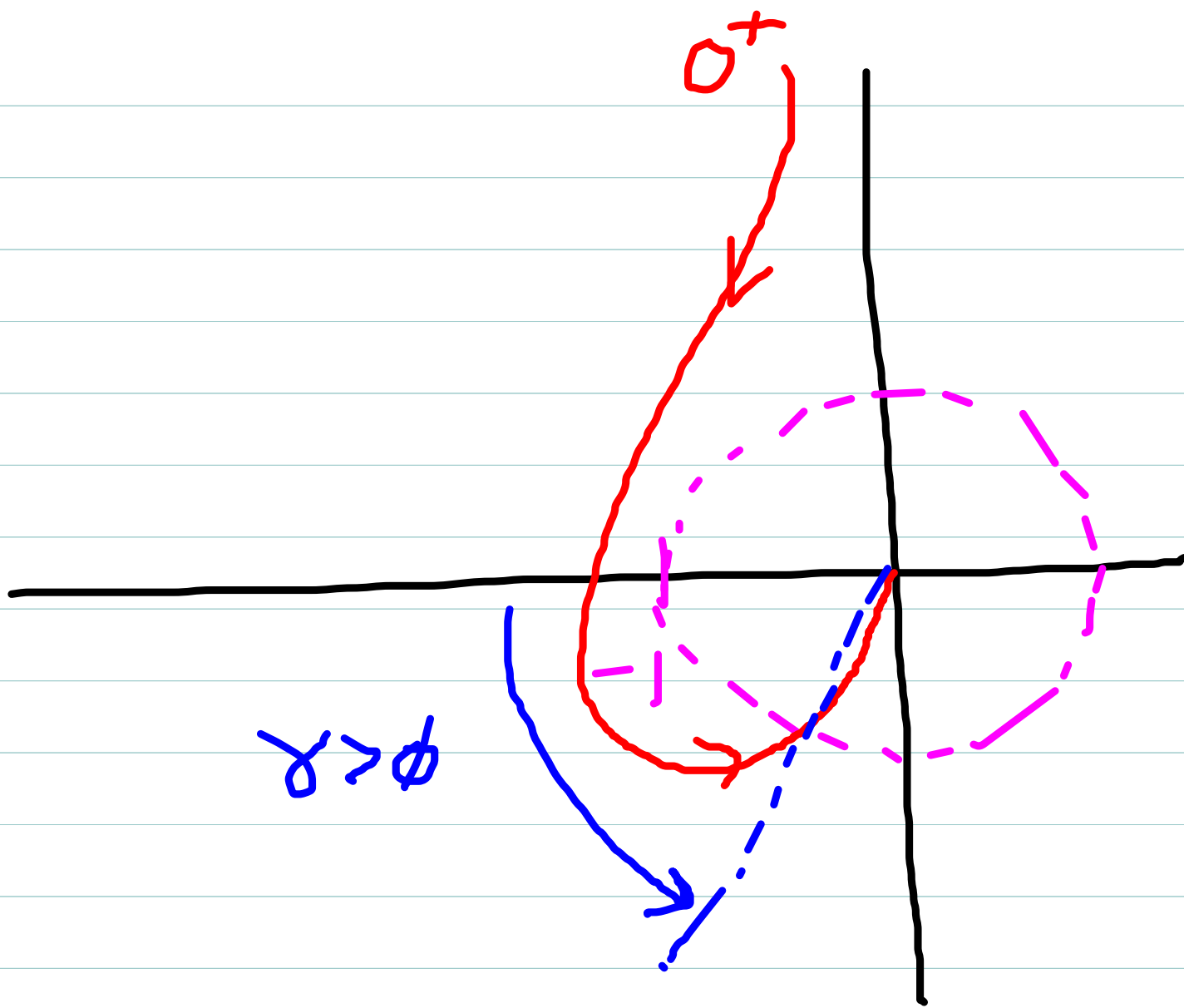
Conclusion: Cannot stabilize  
with  $H(s) = K$

If  $H(s) = K(Ts+1)$ ,  $K > 0$ ,  $T > 0$   
 (LHP zero, positive gain)



Will stabilize if  $-1$  in shaded area!





With  $-1$  in desired region, Phase margin will be positive  $\Rightarrow$  agrees with our basic design guideline  
positive  $\gamma$  is stabilizing here!

For  $G(s) = \frac{3}{s(s-2)}$ ,  $\gamma_{des} = 45^\circ$ ,  $\omega_{des} = 6$ :

$$\begin{aligned}\phi_{req} &= 45 - 180 - \angle G(6j) \\ &= 63.43^\circ\end{aligned}$$

$$\Rightarrow z_c = -\left[\frac{\omega_{des}}{\tan \phi_{req}}\right] = -3$$

$$\Rightarrow K = \frac{1}{|L_o(6j)|} = 1.89. \text{ So } H(s) = 1.89(s+3) \text{ works}$$

For an implementable design, simple approach is again to put pole at  $-\omega_{des}$ , increase  $\phi_{req}$  by  $5.7^\circ$ , giving

$$H(s) = 118.79 \left[ \frac{(s+2.29)}{(s+60)} \right]$$

(Or, we could do a lead comp design, if the above requires excessively large  $u(t)$ .)

$$G(s) = \frac{3}{s(s-2)}$$

Suppose we want instead  $\gamma_{des} = 70^\circ$  at  $\omega_{des} = 6$

$$\phi_{req} = 70 - 180 - \angle G(j\omega) = 88.43^\circ, \text{ Add } +5.7^\circ \text{ for pole, need } \angle(j\omega_{des} - z_c) = 94.13^\circ$$

This condition cannot be satisfied with a single LHP zero  
Need 2 zeros here.

With 2 zeros, can add up to  $+180^\circ$  to  $\angle L(j\omega)$ . Lots of choices for 2 zeros adding up to  $94.13^\circ$  at  $\omega = 6$ . Can simplify design if we assume zeros/poles repeated

$$H(s) = K \left[ \frac{(s - z_c)^2}{(s - p_c)^2} \right]$$

$$\text{So } \angle H(j\omega) = 2\angle(j\omega - z_c) - 2\angle(j\omega - p_c)$$

$$\Rightarrow \angle(j\omega_{des} - z_c) = \frac{\phi_{req}}{2} + \angle(j\omega_{des} - p_c)$$

for example, using  $\mathcal{P}_c = -10\omega_{des} = -60$  again

We get  $z_c = -5.05$ ,  $K = 747.89$

and  $H(s) = 747.89 \left[ \frac{(s+5.05)^2}{(s+60)^2} \right]$  (Again, could instead do a lead comp design - See next page).

Note: Above considerations apply any time we would need  $\angle H(j\omega_{des}) > 90^\circ$ . Not specific to this example.

We can apply similar thinking for more complicated situations:  
i.e. if we would need

$$\angle H(j\omega_{des}) > 180^\circ$$

Then we need  $\geq 3$  LHP zeros in  $H(s)$ , etc.

## Alternate lead comp design for above

Using again  $\gamma_{des} = 70^\circ$ ,  $\omega_{des} = 6$

$$\text{Set } \Phi_{max} = \frac{\Phi_{req}}{2} \Rightarrow \beta = 5.61$$

$$\omega_{max} = 6 \Rightarrow \tau = .07$$

Then:

$$H(s) = K \left[ \frac{(.395s+1)^2}{(.07s+1)^2} \right]$$

and

$$K = \frac{1}{|L_o(6j)|} = 2.255$$

# Comparison of different designs

$$\gamma = 45^\circ:$$

zero/far pole:  $t_s = 1.95 \text{ sec}$ ,  $M_p = 51\%$ ,  $u_{\max} = 119$

Lead:  $t_s = 1.4 \text{ sec}$ ,  $M_p = 53\%$ ,  $u_{\max} = 52$

$$\gamma = 70^\circ$$

zero/far pole:  $t_s = 3.0 \text{ sec}$ ,  $M_p = 40\%$ ,  $u_{\max} = 700$

Lead:  $t_s = 3.9 \text{ sec}$ ,  $M_p = 47\%$ ,  $u_{\max} = 71$

Which is better...? That becomes a judgement call  
Also need to consider tracking, bandwidth, robustness,  
and how "tight" your requirements actually are...

## (Intro:) Effect of Uncertainty

Suppose again we had wanted  $\gamma = 45^\circ$ ,  $\omega_\gamma = 6$   
and we (naively) ignored my caveat and chose

$$H(s) = \left( \frac{6^2 \sqrt{2}}{3} \right) \left( \frac{s-2}{s+6} \right)$$

If the  $G(s)$  really is:

$$G(s) = \frac{3}{s(s-1.9)}$$

[5% uncertainty in  
unstable pole location]

we'd have CL poles at:  $-3.04 \pm 6.5j$ , 1.97 unstable!

With the two designs above, we'd instead have CL poles:

zero/far pole:  $-2.1 \pm 3.3j$ ,  $-53.9$   
Lead:  $-2.8 \pm 2.3j$ ,  $-18$  ] Both still  
stable!