Now we have an idea of the constraints on $L(s)$ for closed-loop stability and transient performance

$\Rightarrow$ Make $L(j\omega)$ have large positive phase margin γ and large crossover freq ω_γ;
(but check Nyquist in unusual or unfamiliar cases)

Let's examine constraints on $L(s)$ which ensure good tracking, i.e. which ensure $|e_{ss}(t)|$ is small for a variety of $y_d(t)$.

Recall that $e(t)$ for a given $y_d(t)$ is governed by sensitivity transfer function $S(s)$ where

$$E(s) = S(s)Y_d(s) \qquad \text{with} \qquad S(s) = \frac{1}{1+L(s)}$$

Intuitively, we make $e(t)$ small by making $L(s)$ "big"

Simple Relationships

Already seen:

$\Longrightarrow$ $e_{ss}(t) = \phi$ when $y_d(t) = A$ (constant)

if $S(\phi) = \phi$

$\Longrightarrow$ $|e_{ss}(t)| \leq 0.7A$ ($\leq$ 70% error)

if $y_d(t) = A\cos(\omega t + \psi)$

for any ω such that $|S(j\omega)| \leq -3dB$

and we call the range of such ω the "tracking bandwidth" ω_B of the system.

More general observations

$$|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| = \frac{1}{|1+L(j\omega)|}$$

All physical systems with implementable controllers satisfy:

$$\boxed{|L(j\omega)| \to \emptyset \text{ as } \omega \to \infty}$$

i.e. $L(s)$ has relative degree of 1 or more (at least one more pole than zeros).

Since $H(s)$ is constrained to have relative degree zero or greater, and all physical systems have $G(s)$ with relative degree 1 or greater. Thus $L(s) = G(s)H(s)$ has relative degree at least 1

Implication: $\boxed{|S(j\omega)| \to 1 \ (\emptyset \text{ dB}) \text{ as } \omega \to \infty}$

$|S(j\omega)|_{dB} \to \phi$ as $\omega \to \infty$

Thus there is always an <u>upper</u> bound on bandwidth.

Let's see if we can more precisely characterize this bound in terms of properties of $L(s)$.

Looking at lower freqs: $\omega \to \phi$

$$S(\phi) = \frac{1}{1+L(\phi)}$$

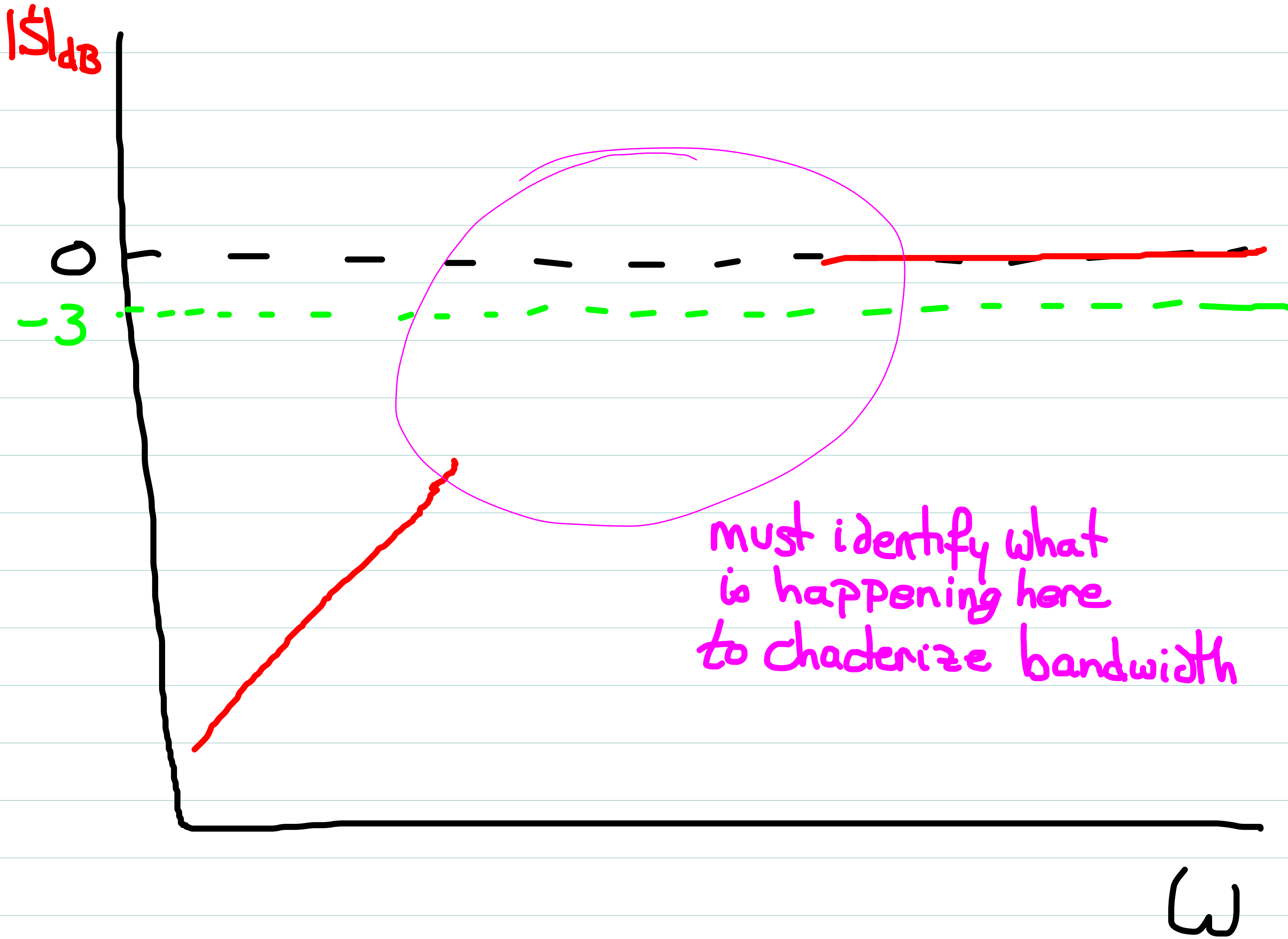
$S(\phi) = \phi \Rightarrow L(\phi) = \infty \Rightarrow L(s)\Big]_{s=\phi} = \infty \Rightarrow$ $\boxed{L(s) \text{ has pole at origin}}$

$\Downarrow$

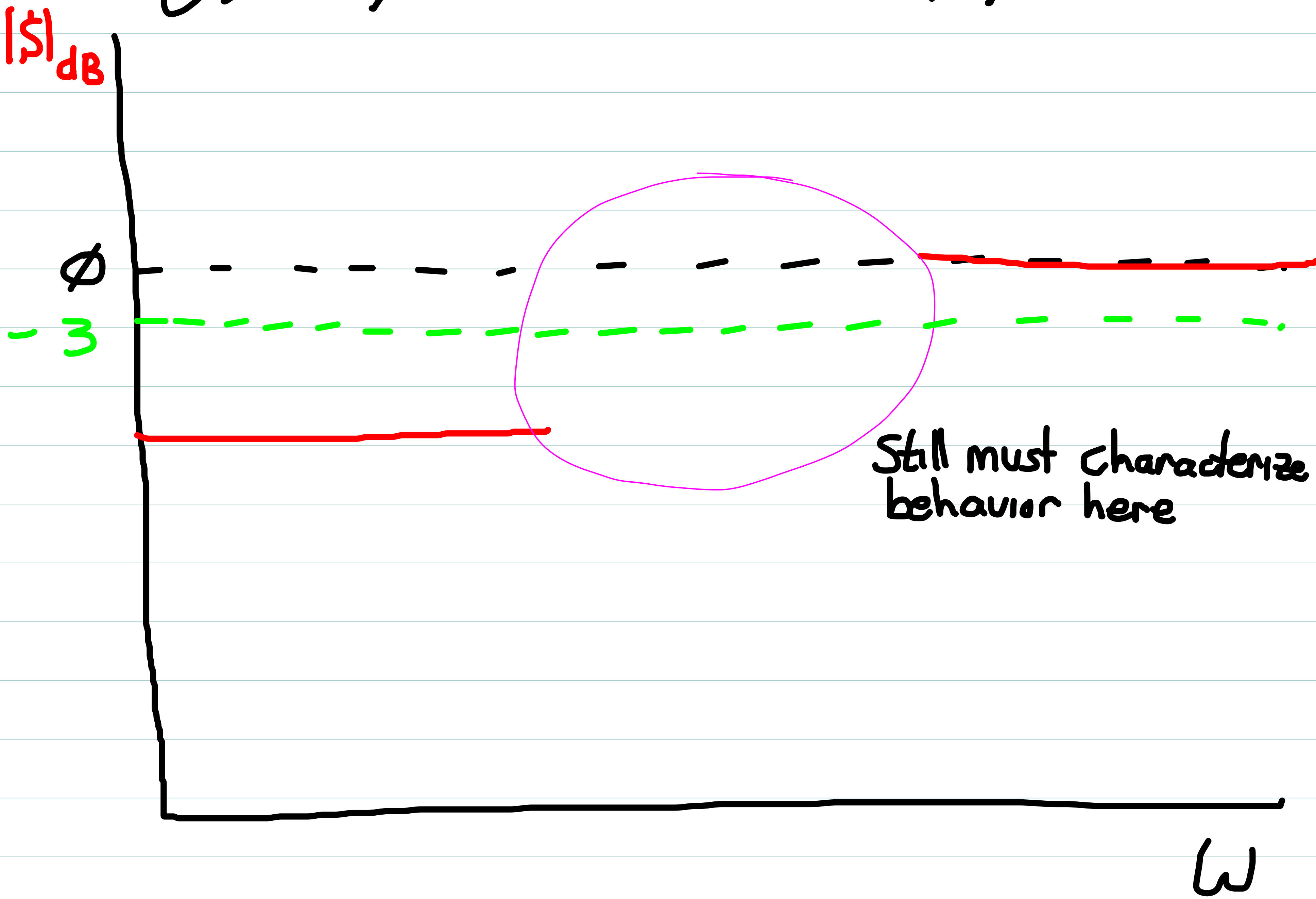
$\boxed{S(s) \text{ has } \underline{\text{zero}} \text{ at origin}}$

$\Rightarrow$ low freq slope of $|S|$ is <u>positive</u>

Remember this correlation! We will see it again!


|S|dB
0
-3
ω
must identify what is happening here to characterize bandwidth

If $L(0) \neq \infty$, then $|S(0)|$ is constant and mag plot of $|S(j\omega)|$ has zero low freq slope



Bandwidth is region for which $|S(j\omega)| \leq -3\,dB$

in actual units $|S(j\omega)| \leq \frac{1}{\sqrt{2}}$

And hence is the region for which $|1+L(j\omega)| \geq \sqrt{2}$

Want to identify constraints on $L(j\omega)$ which guarantee this.

If $|L(j\omega)| > 1$ (ϕ dB), then it is true that

$$|1+L(j\omega)| \geq |L(j\omega)| - 1$$

Hence, if $|L(j\omega)| \geq 1+\sqrt{2}$ (~ 7.7 dB), then

$$|1+L(j\omega)| \geq \sqrt{2} \quad \text{and} \quad |S(j\omega)| \leq -3\,dB$$

So, tracking bandwidth is guaranteed to be <u>at least</u> the range of ω for which $|L(j\omega)| \gtrsim 7.7\,dB$

This is pretty close to ω_γ $(|L(j\omega_\gamma)| = \emptyset\ dB)$
Let's see if we can more precisely relate ω_B to ω_γ:

Assume that $|L(j\omega)|$ is decreasing with slope at least $-20\,dB/dec$ from $+7.7\,dB$ through $\emptyset\,dB$ (typical, but not always).

Then $|L(j\omega)| \geq 7.7\ dB$ starting at frequencies $(7.7/20)$ of a decade <u>below</u> ω_γ

i.e. for $\omega \leq \left(10^{-7.7/20}\right)\omega_\gamma \approx \omega_\gamma/2.5$

Now, let's look more precisely at what is happening at ω_γ

$$|S(j\omega_\gamma)| = \frac{1}{|1+L(j\omega_\gamma)|}$$

$|1+L(j\omega_\gamma)|$ depends on phase $\measuredangle L(j\omega_\gamma)$ and hence on phase margin γ:

Since $|L(j\omega_\gamma)| = 1$ by definition:

$$L(j\omega_\gamma) = e^{j\phi} \quad \text{where } \phi = \measuredangle L(j\omega_\gamma)$$

By definition of $\gamma = 180 + \measuredangle L(j\omega_\gamma)$, $\phi = \gamma - 180°$

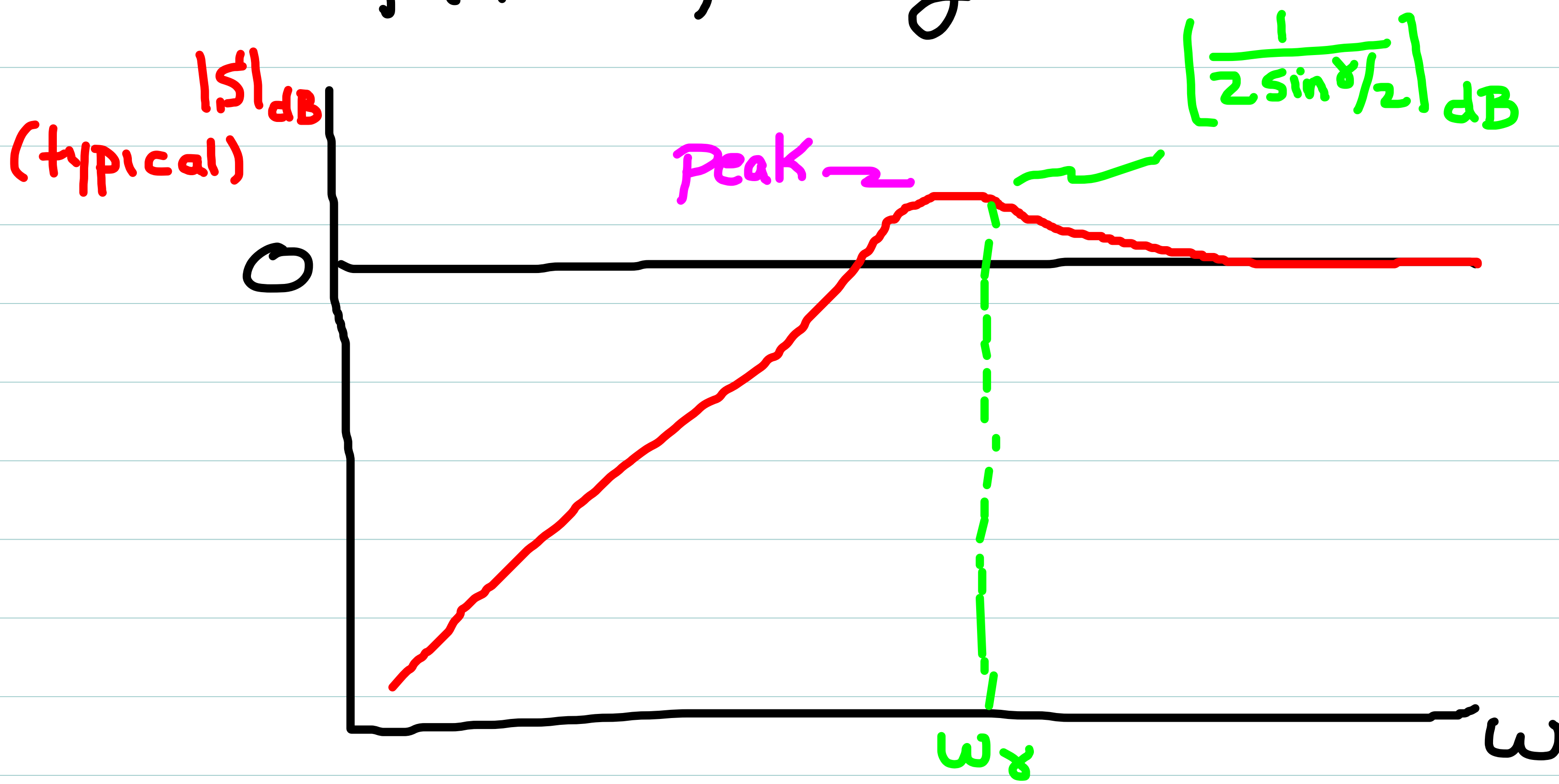
So $1 + L(j\omega_\gamma) = 1 + e^{j(\gamma-\pi)} = (1+\cos(\gamma-\pi)) + j\sin(\gamma-\pi)$

and $|1+L(j\omega_\gamma)| = 2\sin(\gamma/2)$

Hence:

$$\boxed{\left|S(j\omega_\gamma)\right| = \frac{1}{2\sin\gamma/2}}$$

Note $|S(j\omega_\gamma)| > 1$ when $\gamma < 60°$, thus generally $|S(j\omega)|$ will exhibit a <u>peak</u> of height <u>at least</u> as tall as $|S(j\omega_\gamma)|$ (may be higher)



$$|S(j\omega_\gamma)| = \frac{1}{2\sin \gamma/2}$$

Note if $\gamma = 90°$ $|S(j\omega_\gamma)| = \frac{1}{\sqrt{2}}$

Together with previous observations we can conclude that typically for a feedback system with $0 \le \gamma \le 90°$

$$\frac{\omega_\gamma}{2.5} \le \omega_B \le \omega_\gamma$$

And in particular increasing ω_γ increases tracking bandwidth ω_B

Thus in this sense, our design guidelines for performance are aligned with the design guidelines for good tracking

$\Rightarrow$ Larger ω_B means a greater range of sinusoidal $y_d(t)$ which can be tracked with minimal error.

But, this isn't the whole Story!

Many times we require our designs to have $|e_{ss}(t)| = \emptyset$ ("perfect tracking") for specified classes of $y_d(t)$ (even sinusoidal)

When can this be guaranteed?

Let $L(s) = \dfrac{N(s)}{D(s)}$ $N(s), D(s)$ Polynomials

Then $S(s) = \dfrac{1}{1+L(s)} = \dfrac{D(s)}{N(s)+D(s)}$

$\Rightarrow$ Zeros of $S(s)$ are poles of $L(s)$

In particular, perfect tracking of step $y_d(t)$ requires $S(\emptyset) = \emptyset \Rightarrow D(\emptyset) = \emptyset \Rightarrow L(s)$ has at least 1 pole at origin, as we have seen.

More generally, suppose

$$Y_d(s) = \frac{a(s)}{b(s)} \qquad a(s), b(s) \text{ polynomials}$$

Then $E(s) = S(s)\, Y_d(s)$

$$= \left[\frac{D(s)}{N(s)+D(s)}\right]\left[\frac{a(s)}{b(s)}\right]$$

Now, assuming our controller at least stabilizes the feedback loop, the poles of $S(s)$ [same as poles of $T(s)$] are stable

If all poles of $Y_d(s)$ (roots of $b(s)$) are stable, then partial fraction expansion and inverse transform of $E(s)$ will give $e(t)$ as a sum of decaying exponential functions.

$\Rightarrow e_{ss}(t) = \emptyset$ here

Above result makes sense:

for a stable system, $y(t)$ naturally "wants" to converge to ϕ. If $Y_d(s)$ has all stable poles, then $y_d(t)$ is a sum of decaying exponentials and $y_d(t) \to \phi$

So asymptotically, we are requiring the system to do what it already wants to do, and thus we get perfect steady-state tracking.

More interesting is when $y_d(t) \not\to \phi$ As $t \to \infty$. So suppose that poles of $Y_d(s)$ are <u>not</u> stable.

$$E(s) = \left[\frac{D(s)}{N(s)+D(s)} \right] \left[\frac{a(s)}{b(s)} \right]$$

and $e(t)$ will contain same non-stable poles as $Y_d(s)$, unless...

$$E(s) = \left[\frac{D(s)}{N(s)+D(s)}\right]\left[\frac{a(s)}{b(s)}\right]$$

Unless, the non-stable poles of $Y_d(s)$ are cancelled by zeros of $S(s)$

i.e. if $D(s) = D'(s)\,b(s)$, $D'(s)$ polynomial

For $y_d(t) = A \Rightarrow Y_d(s) = \frac{A}{s}$ ($b(s) = s$, root at origin)

Need $D(s) = s D'(s)$ i.e. $D(s)$ also has root at origin

$\Rightarrow L(s)$ has pole at origin (as we have already seen)

But the above result is much more general!

Suppose $y_d(t) = At \Rightarrow Y_d(s) = \frac{A}{s^2}$ $\Rightarrow b(s) = s^2$

If $L(s)$ has 2 poles at origin $D(s) = s^2 D'(s)$, Non-stable terms will cancel.

General Result

If $L(s)$ has the same non-stable poles as $Y_d(s)$ then $e_{ss}(t) = \emptyset$

If true, we say that $L(s)$ has an "internal model" of $Y_d(t)$, and the above fact is known as the "internal model principle" (IMP)

<u>Note</u>: while theoretically this applies even if $Y_d(s)$ has unstable poles, practically we use this only for <u>marginally stable</u> poles of Y_d, i.e. poles on imag axis.

One common special case: "type p" (polynomial) $Y_d(t)$, i.e.

(p <u>integer</u> $\geq \emptyset$) $\quad Y_d(t) = \left(\frac{A_p}{p!}\right) t^p \quad A_p$ constant

$$\Rightarrow Y_d(s) = \frac{A_p}{s^{p+1}}$$

p = power of t in $Y_d(t)$ $\Leftrightarrow$ $p+1$ poles at $\emptyset$

Type p: $\quad Y_d(s) = \frac{A_p}{s^{p+1}}$

Via IMP: perfect tracking ($e_{ss} = \phi$) requires $L(s)$ to have $p+1$ poles at origin

$p = \phi$, $\; y_d(t) = A_0$ (constant) $\Rightarrow L(s)$ needs 1 pole at origin

$p = 1$, $\; y_d(t) = A_1 t$ (linear) $\Rightarrow L(s)$ needs 2 poles at origin

and so on.

Now, suppose $L(s)$ does not have enough poles at origin what happens? Look more closely at

$$E(s) = S(s) Y_d(s) = \left[\frac{D(s)}{D(s) + N(s)}\right]\left(\frac{A_p}{s^{p+1}}\right)$$

when $y_d(t)$ is type p.

$$E(s) = \left[\frac{D(s)}{D(s)+N(s)}\right]\frac{A_p}{s^{p+1}}$$

Pull out any poles $L(s)$ has at origin: Let

$$D(s) = s^N D'(s) \quad (N = \#\text{poles of } L(s) \text{ at origin}$$
$$\text{"type" of } L(s))$$

So $$E(s) = A_p\left[\frac{s^N}{s^{p+1}}\right]\left[\frac{D'(s)}{N(s)+D(s)}\right]$$

If $N \geq p+1$, $E(s)$ will have only stable poles remaining and $e(t) \to \varnothing \Rightarrow e_{ss}(t) = \varnothing$

If $N = p$, however, ($L(s)$ has one less pole at origin than $Y_d(s)$) then

$$E(s) = \left(\frac{A_p}{s}\right)\left[\frac{D'(s)}{D(s)+N(s)}\right] = \frac{c_0}{s} + \underbrace{\frac{c_1}{s-d_1} + \cdots}_{}$$

So $e_{ss}(t) = c_0$ <u>constant</u> here

from stable poles of $S(s)$ (and $T(s)$)

We can compute c_0 in this case using residue formula:

$$C_0 = A_p \left[\frac{D'(s)}{D(s) + N(s)} \right]_{s=\phi}$$

But recall $D'(s) = D(s)/s^p$ (since $p = N$ here)

So $$C_0 = A_p \left[\frac{D(s)}{s^p D(s) + s^p N(s)} \right]_{s=\phi} = \left[\frac{A_p}{s^p + s^p L(s)} \right]_{s=\phi}$$

But note (again since $N = p$ here)

$$\left[s^p L(s) \right]_{s=\phi} = K_{B,L} \quad \text{Bode gain of } L(s)$$

So:

$$C_0 = \left[\frac{A_p}{s^p + K_{B,L}} \right]_{s=\phi} = \begin{cases} \dfrac{A_p}{1 + K_{B,L}} & p = \emptyset \\[2ex] \dfrac{A_p}{K_{B,L}} & p > \emptyset \end{cases}$$

Now suppose $N = P - 1$ (2 less poles at origin in $L(s)$)

Then $E(s) = A_p\left(\frac{s^N}{s^{P+1}}\right)\left[\frac{D'(s)}{D(s)+N(s)}\right] = \frac{A_p}{s^2}\left[\frac{D'(s)}{D(s)+N(s)}\right]$

$$= \frac{c_0}{s} + \frac{c_1}{s^2} + \underbrace{\frac{c_2}{(s-d_1)} + \cdots}_{\text{from stable poles of } S(s)}$$

So $e_{ss}(t) = c_0 + c_1 t \longrightarrow \infty$ as $t \to \infty$

<u>Diverges</u>

Easy to show similar phenomenon for <u>any</u> $N < P$.

i.e. if $N = P - 2$ then

$$e_{ss}(t) = c_0 + c_1 t + c_2 t^2 \longrightarrow \infty$$

etc.

Summary: Tracking for "type-p" $y_d(t)$

$y_d(t) = \left(\frac{A_p}{p!}\right)t^p$, N = # poles at origin in $L(s)$.

$$\boxed{e_{ss}(t) = \begin{cases} \emptyset & N > p \\ c_0 \neq \emptyset & N = p \\ \infty & N < p \end{cases}}$$

Where

$$\boxed{c_0 = \begin{cases} \dfrac{A_p}{1 + K_{B,L}} & p = \emptyset \\ \dfrac{A_p}{K_{B,L}} & p > \emptyset \end{cases}}$$

and $K_{B,L}$ is the Bode gain of $L(s)$.

Very important design constraint!