Transfer functions

$$G(s) = \frac{q(s)}{r(s)}$$

Compactly gives us all information we need to predict major features of system response

- $y_h(t)$, modes, stability : all from $r(s)$ the denominator polynomial of $G(s)$

$$r(s) = \alpha_n \prod_{k=1}^{n} (s - p_k)$$

- forced response : Evaluate $G(s)$ at specific complex values of s.

Numerator Terms

Can also factor $q(s)$:

$$q(s) = \beta_m (s - z_1)(s - z_2) \cdots (s - z_m)$$

where $q(z_i) = \emptyset$ for $i = 1, \ldots, m$

The values z_i are called the zeros of $G(s)$

since $G(z_i) = \dfrac{\cancelto{\emptyset}{q(z_i)}}{r(z_i)} = \emptyset$

The values p_k are called the poles of $G(s)$

since $G(p_k) = \dfrac{q(p_k)}{\cancelto{\emptyset}{r(p_k)}} = \infty$

Zero/Pole/Gain (ZPK) form

$$G(s) = K\left[\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{k=1}^{n}(s-p_k)}\right]$$

Poles p_k satisfy $r(p_k) = \emptyset$

Zeros z_i satisfy $q(z_i) = \emptyset$

Gain: $K = \frac{\beta_m}{\alpha_n}$ (always real)

Alternate ZPK form:

When $G(s)$ has complex poles and/or zeros, we commonly combine the conjugate roots of $r(s)$ or $q(s)$ into 2^{nd} order polynomials.

For example, if $p = \sigma + j\omega$ and $\bar{p} = \sigma - j\omega$ are complex roots of $r(s)$:

$$(s-p)(s-\bar{p}) = \underline{s^2 - 2\sigma s + (\sigma^2 + \omega^2)}$$

replace with in $G(s)$

<u>Stability and G(s)</u>

→ $G(s)$ is stable if <u>all</u> its poles are in LHP.

→ $G(s)$ is <u>unstable</u> if <u>any</u> of its poles are in <u>RHP</u>.

→ What role do zeros of $G(s)$ have in stability?

⟹ ABSOLUTELY NONE!

→ OK, so what role <u>do</u> zeros play?

Effect of zeros in $G(s)$

→ Certainly zeros influence the coefficients C_K of homogeneous response.

→ They also influence calculation of $y_f(t)$.

→ Special example: Suppose $u(t) = e^{z_i t}$

then:

$$y_f(t) = G(z_i)e^{z_i t} = \emptyset$$

The forced response is exactly zero here!

"Input absorbing" property of zeros

More complicated $u(t)$

$$u(t) = U e^{st} \Rightarrow y_f(t) = G(s) U e^{st}$$

Suppose $u(t) = U_1 e^{s_1 t} + U_2 e^{s_2 t}$

Substitute into DE, can show

$$y_f(t) = G(s_1) U_1 e^{s_1 t} + G(s_2) U_2 e^{s_2 t}$$

More complicated $u(t)$

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Suppose $u(t) = \boxed{U_1 e^{s_1 t}} + \boxed{U_2 e^{s_2 t}}$

Substitute into DE, can show

$$y_f(t) = \boxed{G(s_1) U_1 e^{s_1 t}} + \boxed{G(s_2) U_2 e^{s_2 t}}$$

The sum of the responses to the individual parts of the input.

Linearity of Systems

<u>If</u> $y_1(t)$ is a possible sol'n of DE
with input $u_1(t)$

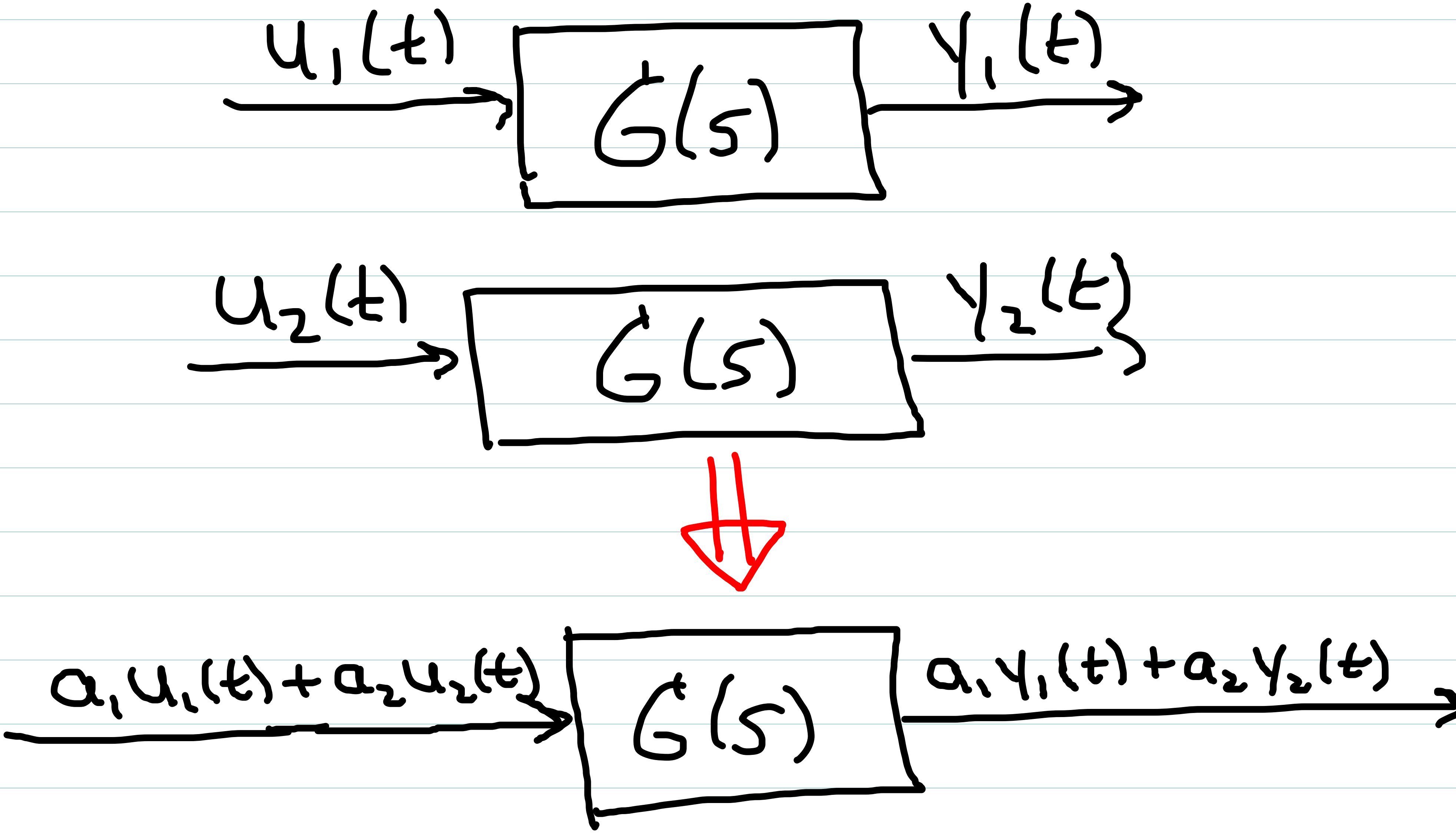
<u>and</u> similarly $y_2(t)$ is a sol'n for input $u_2(t)$

<u>Then</u>:

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

is a sol'n for input $u(t) = a_1 u_1(t) + a_2 u_2(t)$

For <u>any</u> constants a_1, a_2 <u>and</u> any
inputs $u_1(t), u_2(t)$


$u_1(t)$
$G(s)$
$y_1(t)$
$u_2(t)$
$G(s)$
$y_2(t)$
$a_1u_1(t)+a_2u_2(t)$
$G(s)$
$a_1y_1(t)+a_2y_2(t)$

We've already seen an example

$$u_1(t) = e^{st} \longrightarrow y_1(t) = G(s)e^{st}$$

$$u_2(t) = \emptyset \longrightarrow y_2(t) = y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$$

$$u(t) = U e^{st} = U e^{st} + \emptyset$$

$$= U u_1(t) + u_2(t)$$

$$\Rightarrow \quad y(t) = U y_1(t) + y_2(t)$$

$$= U G(s) e^{st} + y_h(t)$$

$$= y_f(t) + y_h(t)$$

Linearity can be used multiple times

$$u(t) = \sum_{i=1}^{N} a_i u_i(t) \implies y(t) = \sum_{i=1}^{N} a_i y_i(t)$$

$y_i(t)$ sol'n for $u_i(t)$

$\Rightarrow$ Holds for any number N

In particular,

$$u(t) = \sum_{i=1}^{N} U_i e^{s_i t} \implies y(t) = \sum_{i=1}^{N} G(s_i) U_i e^{s_i t}$$

Even for infinite sum, $N = \infty$.

Is this enough to make <u>any</u> u(t)?

Not quite, need to go to differential limit

$$\sum_{i=1}^{N} U_i e^{s_i t} \longrightarrow \int U(s) e^{st} ds$$

integral over all complex freqs

i.e. $u(t) = \int U(s) e^{st} ds$

$U(s)$ is the "amount" (complex amplitude) of e^{st} present in $u(t)$, for <u>each</u> $s \in \mathbb{C}$.

Similarly $y(t) = \sum_{i=1}^{N} G(s_i) U_i e^{s_i t} \rightarrow \int G(s) U(s) e^{st} ds$

OR: $y(t) = \int Y(s) e^{st} ds$ with $\boxed{Y(s) = G(s) U(s)}$

Laplace Transform

More formally, for any $f(t)$ define:

(1) $$f(t) = \frac{1}{2\pi j} \int F(s) e^{st} ds$$

Normalizing constant

where:

(2) $$F(s) = \int_0^\infty f(t) e^{-st} dt$$

Notation:

$F(s) = \mathcal{L}\{f(t)\}$ (transform)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$ (inverse transform)

Limitations of Laplace Transform

Only defined for $f(t)$ where the integral (2) converges.

Requires: $e^{-\sigma_0 t}|f(t)| \to \emptyset$

for some finite $\sigma_0 \in \mathbb{R}$

The transform $F(s)$ is then defined for any

$s = \sigma + j\omega$ with $\sigma \geq \sigma_0$

and the integral (1) is over all values of s which satisfy this condition. "region of convergence"

Examples

$f(t) = e^{pt}$ can be transformed for any finite $p \in \mathbb{C}$

However, $f(t) = e^{t^2}$ cannot be transformed since $e^{-\sigma_0 t} f(t) = e^{(t^2 - \sigma_0 t)} \to \infty$

for any finite σ_0.

Note:

When working with Laplace transforms we assume we are using values of s in the region of convergence. (ROC)

By above def'n of ROC,

$$\lim_{t \to \infty} e^{-st} f(t) = \emptyset$$

for these values of s.