Homework 9 Solutions   Question #4

i) Show that the implementation equation results in $H(s) = \dfrac{20(s+2)^3}{s(s+6)(s+12)}$

$$u(t) = 20e(t) + 2x_1(t) - 15x_2(t) - 3x_3(t)$$
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\dot{x}_3(t) = -72x_2(t) - 18x_3(t) + 80e(t)$$

* take the Laplace of the implementation equations to get rid of the derivatives

$$U(s) = 20E(s) + 2X_1(s) - 15X_2(s) - 3X_3(s) \quad (1)$$
$$X_1(s) = X_2(s)/s \quad (2)$$
$$X_2(s) = X_3(s)/s \quad (3)$$
$$sX_3(s) = -72X_2(s) - 18X_3(s) + 80E(s) \quad (4)$$

rewrite (4) as

$$X_3(s)(s+18) = -72X_2(s) + 80E(s)$$
$$= -\frac{72}{s}X_3(s) + 80E(s)$$
$$X_3(s)\left[s+18+\frac{72}{s}\right] = 80E(s)$$
$$X_3(s)\left[\frac{s^2+18s+72}{s}\right] = \cancel{80E(s)}$$
$$X_3(s) = \frac{80s}{s^2+18s+72}E(s)$$

rewrite (3) as

$$X_2(s) = \frac{80}{s^2+18s+72}E(s)$$

rewrite (2) as

$$X_1(s) = \frac{80}{s(s^2+18s+72)}E(s)$$

Summary of Steps:

- Laplace
- Systems of Equations
- Extract $H(s)$
- simplify $H(s)$

Substituting into (1)

$$U(s) = 20E(s) + \frac{160}{s(s^2+18s+72)}E(s)$$
$$- \frac{1200}{s^2+18s+72}E(s) - \frac{240s}{s^2+18s+72}E(s)$$
$$U(s) = E(s)\left[20 + \frac{160}{s(s^2+18s+72)} - \frac{1200}{(s^2+18s+72)} - \frac{240s}{(s^2+18s+72)}\right]$$

- from this, we can extract $H(s)$

$$H(s) = 20 + \frac{160}{s(s^2+18s+72)} - \frac{1200}{(s^2+18s+72)} - \frac{240s}{(s^2+18s+72)}$$

- simplify

$$H(s) = \frac{20s(s^2+18s+72) + 160 - 1200s - 240s^2}{s(s+6)(s+12)}$$
$$= \frac{20(s^3+18s^2+72s+8-60s-12s^2)}{s(s+6)(s+12)}$$
$$= \frac{20(s^3+6s^2+12s+8)}{(s)(s+6)(s+12)}$$

$$H(s) = \frac{20(s+2)^3}{s(s+6)(s+2)}$$

∴ ✓

* Use of Linear Algebra Techniques to solve the system is acceptable!

ii) $u(t) = 20e(t) + 2.5x_1(t) - 15.68x_2(t) + 1.92x_3(t)$
$\dot{x}_1(t) = 8e(t)/9$
$\dot{x}_2(t) = -13.61x_2(t) + 3.5x_3(t) + 15.68e(t)$
$\dot{x}_3(t) = -3.5x_2(t) - 4.39x_3(t) + 1.92e(t)$

Take the Laplace of the implementation Eqs.

$U(s) = 20E(s) + 2.5X_1(s) - 15.68X_2(s) + 1.92X_3(s)$ (1)
$sX_1(s) = 8E(s)/9$ (2)
$sX_2(s) = -13.61X_2(s) + 3.5X_3(s) + 15.68E(s)$ (3)
$sX_3(s) = -3.5X_2(s) - 4.39X_3(s) + 1.92E(s)$ (4)

rearrange (4):

$X_3(s)[s+4.39] = -3.5X_2(s) + 1.92E(s)$

$X_3(s) = \dfrac{-3.5X_2(s) + 1.92E(s)}{[s+4.39]}$ (6)

rearrange /substitute into (3)

$sX_2(s) = -13.61X_2(s) + \dfrac{(-12.25X_2(s) + 6.72E(s))}{(s+4.39)} + 15.68E(s)$

$sX_2(s) = \dfrac{-13.61X_2(s)(s+4.39) - 12.25X_2(s) + 6.72E(s) + 15.68(s+4.39)E(s)}{(s+4.39)}$

Simplify

$[s(s+4.39) + 13.61(s+4.39) + 12.25]X_2(s) = (15.68(s+4.39) + 6.72)E(s)$
$[s^2 + 4.39s + 13.61s + 72]X_2(s) = (15.68(s+4.39) + 6.72]E(s)$
$X_2(s) = \dfrac{15.68(s+4.39)E(s)}{(s^2+18s+72)}$ (7)

Substitute (7) into (6)

$[s+4.39]X_3(s) = -3.5\left[\dfrac{(15.68s+68.835)E(s)}{(s^2+18s+72)}\right] + 1.92E(s)$

$X_3(s) = \dfrac{(-54.88s - 240.92)E(s) + 1.92(s^2+18s+72)E(s)}{(s^2+18s+72)(s+4.39)}$

$X_3(s) = \dfrac{E(s)(1.92s^2 - 20.32s - 168.92)}{(s+4.39)(s^2+18s+72)}$

final substiti

substitute into U(s) equation and we get

$$U(s) = E(s)\underbrace{\left[20 + \frac{2.5(9)}{8s} - \frac{15.68(224(700s + 3373))}{10000s^2 + 180000s + 719979} + \frac{1.92(256(75s - 1123))}{10000s^2 + 180000s + 719979}\right]}_{H(s)}$$

simplify $H(s)$

$$H(s) = \left[\frac{20(8)(10000)s(s^2+18s+72) + 2.5(9)(10{,}000)(s^2+18s+72) - (15.68)(8)s(224(700s-3373))}{(8)(10000)s(s^2+18s+72)}\right.$$

$$\ldots \quad \frac{1.92(256(75s-1123))8s}{}$$

$$H(s) = \left[\frac{20(s+2)^3}{s(s+2)(s+6)}\right] \quad \checkmark$$