Direct Solution

Our sol'n has the general pattern:

$$\boxed{y(t) = \int_{\phi}^{t} g(t-\tau)\, u(\tau)\, d\tau}$$

where here $g(t) = Kt$ $\left[\text{so } g(t-\tau) = K(t-\tau)\right]$

We will (indirectly) show that for <u>any</u> system, no matter how complex the dynamics, this relationship between $u(t)$ and $y(t)$ holds.

Different systems are characterized by different functions $g(t)$.

The characteristic function $g(t)$ is called the

<u>Impulse response</u>

Implication

Suppose:

$$y_1(t) = \int_0^t g(t-\tau)\,u_1(\tau)\,d\tau$$

$$y_2(t) = \int_0^t g(t-\tau)\,u_2(\tau)\,d\tau$$

are two known input-output pairs.

Suppose that $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$; α_1, α_2 constant

Then:

$$y(t) = \int_0^t g(t-\tau)\left[\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)\right] d\tau$$

$$= \alpha_1 \int_0^t g(t-\tau)\,u_1(\tau)\,d\tau + \alpha_2 \int_0^t g(t-\tau)\,u_2(\tau)\,d\tau$$

hence

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$ Principle of Linearity.

This suggests an approach:

1. Identify a "family" of functions $u_K(t)$ for which it is EASY to calculate response $y_K(t)$:

$$u_K(t) \mapsto y_K(t) \quad \text{(easy)}$$

2. "Break down" an arbitrarily complicated $u(t)$ into a linear combination of the $u_K(t)$:

$$u(t) = \sum \alpha_K u_K(t) \quad \text{(easy?)}$$

3. Use Linearity:

$$y(t) = \sum \alpha_K y_K(t) \quad \text{(easy)}$$

Time varying complex numbers

$$z(t) = a(t) + b(t)\,j$$
$$= r(t)\,e^{j\theta(t)}$$

Important example:

$$z(t) = e^{st} \text{ with } s \in \mathbb{C}$$

"complex exponential functions"

Let $s = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

So $\mathrm{Re}\{s\} = \sigma$, $\mathrm{Im}\{s\} = \omega$

① If $\omega = 0$, then

$$e^{st} = e^{\sigma t} \text{ (real exponential)}$$

② If $\sigma = 0$ then

$$e^{st} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Note: $\mathrm{Im}\{s\}$ gives frequency of the oscillations

3. Most general case

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t}$$

$$= e^{\sigma t}\left[\cos\omega t + j\sin\omega t\right]$$

$$\boxed{\begin{aligned} \mathrm{Re}\{e^{st}\} &= e^{\sigma t}\cos(\omega t) \\ \mathrm{Im}\{e^{st}\} &= e^{\sigma t}\sin(\omega t) \end{aligned}}$$

$\sigma \rightarrow$ amplitude envelope

$\omega \rightarrow$ oscillation frequency

$s = \sigma + j\omega$ is the "complex frequency"

Utility of e^{st}

For different values of s, e^{st} is:

- a constant
- a real exponential
- a pure sine/cosine wave
- an exponentially decaying or increasing sine/cosine

Covers 90% of cases needed to solve linear diff'l eq'ns

Complex Amplitudes

Now consider $z(t) = Ae^{st}$ with both $A, s \in \mathbb{C}$.

$s = \sigma + j\omega$, $A = re^{j\varphi}$ (polar)

$$Ae^{st} = \left(re^{j\varphi}\right)\left(e^{(\sigma+j\omega)t}\right)$$

$$= \left(re^{\sigma t}\right)\left(e^{j(\omega t+\varphi)}\right)$$

$$= re^{\sigma t}\left[\cos(\omega t+\varphi) + j\sin(\omega t+\varphi)\right]$$

so $$\boxed{\begin{aligned} \mathrm{Re}\{Ae^{st}\} &= re^{\sigma t}\cos(\omega t+\varphi) \\ \mathrm{Im}\{Ae^{st}\} &= re^{\sigma t}\sin(\omega t+\varphi) \end{aligned}}$$

$r = |A|$ is <u>initial amplitude</u> of oscillations

$\varphi = \measuredangle A$ is <u>phase shift</u> of oscillations

$\varphi > 0$ called "phase lead"

$\varphi < 0$ called "phase lag"

Property of e^{st}:

Let $f(t) = e^{st}$ for any $s \in \mathbb{C}$

Then $\dot{f}(t) = \frac{d}{dt} f(t) = \frac{d}{dt}(e^{st})$

$= s e^{st}$

or $\dot{f}(t) = s f(t)$

Similarly: $\ddot{f}(t) = s^2 f(t)$

$f^{(3)}(t) = s^3 f(t)$, etc

Linear constant coefficient (time invariant) Diff Eq'n

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \dot{y} + \alpha_0 y$$
$$= \beta_m u^{(m)} + \cdots + \beta_1 \dot{u} + \beta_0 u$$

where $\alpha_n, \ldots \alpha_0$ and $\beta_m \ldots \beta_0$ are real and constant

Suppose $u(t) = U e^{st}$ with $s, U \in \mathbb{C}$

Is $y(t) = Y e^{st}$ a sol'n for some $Y \in \mathbb{C}$?

Substitute into DE

GIVES

$$r(s) Y e^{st} = q(s) U e^{st}$$

With:

$$r(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$$

So Assumption is consistent with

$$Y = \left[\frac{q(s)}{r(s)} \right] U = G(s) U$$

$$Ue^{st} = u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t) = Ye^{st}$$

system

$$Y = G(s)U$$

If $u(t) = Ue^{st}$ for some $U, s \in \mathbb{C}$
then $y(t) = Ye^{st}$, with $Y = G(s)U$

This is one possible sol'n of the DE, the forced sol'n, $\boxed{y_f(t)}$.
Other sol'ns are possible.

Other possible sol'ns

Now, suppose $u(t) = \emptyset$. Clearly here $y_f(t) = \emptyset$. But is $y(t) = \emptyset$ necessarily? Or can we still have sol'ns of the form $y(t) = Ce^{st}$? Substitute into DE:

$$r(s) C e^{st} = \emptyset$$

which can be true for any s where

$$\boxed{r(s) = \emptyset}$$

$$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$$

There are n values of s for which $r(s) = 0$. We denote these

$$p_1, p_2, \cdots p_n$$

So $r(s)$ can be factored As

$$r(s) = \alpha_n (s - p_1)(s - p_2) \cdots (s - p_n)$$

$$= \alpha_n \prod_{k=1}^{n} (s - p_k)$$

For any p_K with $r(p_K) = \phi$,
$y(t) = e^{p_K t}$ is a sol'n of the DE
when $u(t) = \phi$. So is $y(t) = C_K e^{p_K t}$
for any constant C_K. So is any
sum of these terms:

$$y(t) = \sum_{K=1}^{n} C_K e^{p_K t} = y_h(t)$$

The "homogeneous" sol'n.

Proof:

Substitute $y(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

into diff eq'n:

GIVES:

$$r(p_1)c_1 e^{p_1 t} + r(p_2)c_2 e^{p_2 t} + \cdots + r(p_n)c_n e^{p_n t}$$

$$= \emptyset$$

Which is true if $r(p_1) = r(p_2) = \cdots = r(p_n) = \emptyset$

i.e. the p_k are zeros of polynomial $r(s)$

Since any $y_h(t)$ yields $\emptyset$ exactly when substituted into DE, we can add it to any other sol'n and still have a valid sol'n. Generally:

$$y(t) = y_h(t) + y_f(t)$$

where $y_h(t) = \sum_{k=1}^{n} c_k e^{p_k t}$

and if $u(t) = U e^{st}$, then

$$y_f(t) = G(s) U e^{st}$$

Both complex! (Generally)

But $Y_f(t)$ is complex generally . . . ?

$\Rightarrow$ because $u(t)$ is complex here

Suppose $u(t) = B\sin(\omega t + \varphi)$ (real)

$= \text{Im}\{Ue^{st}\}$ Take matching Im part

with $U = Be^{j\varphi}$

and $s = j\omega$

Then $Y_f(t) = \text{Im}\{G(s)Ue^{st}\}$

and similarly for cosine inputs, taking real part

What about $y_h(t)$?

Contains terms e^{pt}, where $r(p) = \emptyset$.

If p is complex, $p = \sigma + j\omega$, $\omega \neq \emptyset$

then e^{pt} is complex

However: in this case $r(p) = \emptyset \Longrightarrow r(\bar{p}) = \emptyset$

i.e. $\bar{p}$ is also a zero of $r(s)$.

$\Longrightarrow$ Complex roots of polynomials occur in "conjugate pairs".

Hence, with complex roots, $y_h(t)$ will contain

$$c_1 e^{pt} + c_2 e^{\bar{p}t}$$

<u>Fact:</u>

$$c_2 = \overline{c_1}$$

i.e. coef of $e^{\bar{p}t}$ will <u>always</u> be the conjugate of the coef of e^{pt}.

Thus, if $r(s)$ has a complex root p, $y_h(t)$ will contain

$$c e^{pt} + \bar{c} e^{\bar{p}t} = c e^{pt} + \overline{c e^{pt}}$$