Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at $z_1 = -\beta_0/\beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both? Yes!

Depending on system

② Transient Suppression

Suppose $s^2+\alpha_1 s+\alpha_0 = (s-p_1)(s-p_2)$ p_1, p_2 real

So
$$G(s) = \frac{\beta_1(s-z_1)}{(s-p_1)(s-p_2)}$$

Suppose $z_1 \approx p_1$, i.e $|z_1-p_1| = \varepsilon << 1$

We know $y(t) = G(\phi) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$

where $A_1 = \left[(s-p_1)Y(s)\right]_{s=p_1} = \dfrac{\beta_1 \underbrace{(p_1-z_1)}_{\sim <<1}}{p_1(p_1-p_2)}$ is small

so, for sufficiently small ε, the $e^{p_1 t}$ term in transient is negligible, and response is equivalent to a <u>1st order</u> system with single pole p_2

Pole-zero Cancellation

Algebraically, if $z_1 \approx p_1$

$$G(s) = \frac{\beta_1 \cancel{(s-z_1)}}{\cancel{(s-p_1)}(s-p_2)} \approx \frac{\beta_1}{(s-p_2)}$$

Usually, if

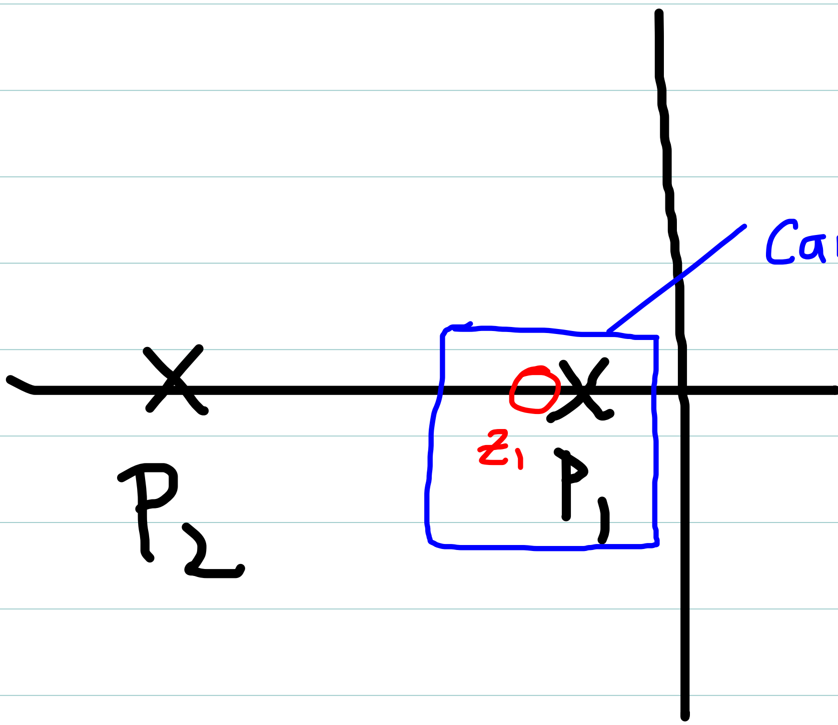
$$0.9 \leq \left| \frac{z_1}{p_1} \right| \leq 1.1$$

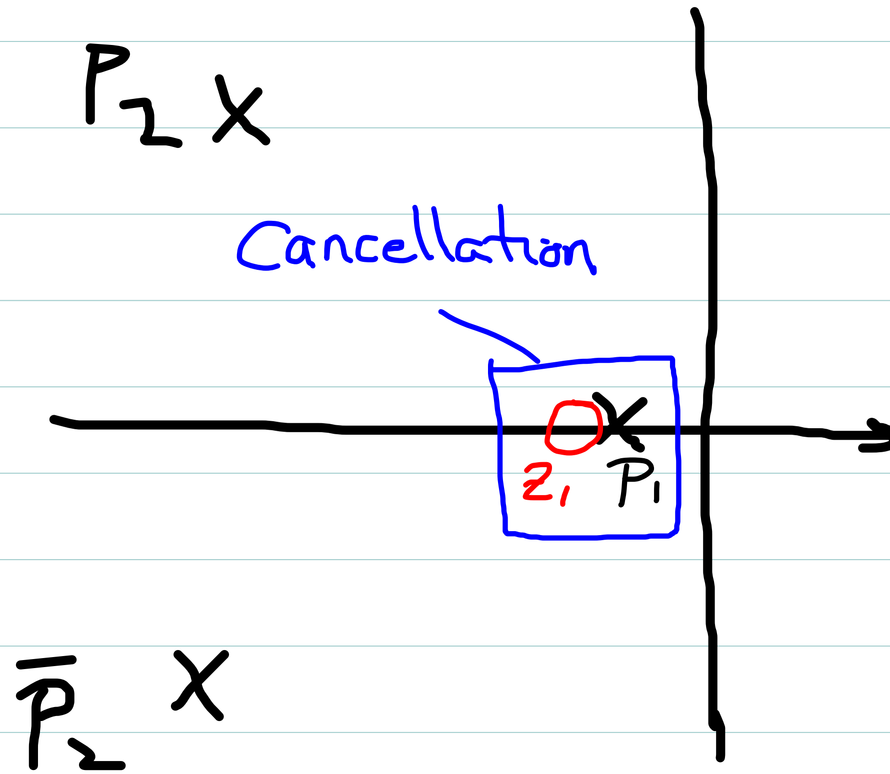
i.e. zero location within 10% of pole location

this is a good approximation

Cancellation and Dominance

Pole-zero cancellations can change dominance calculation





"fast" pole becomes dominant

2nd order poles become dominant

Cancellation is never exact!

$\Rightarrow$ z_1, p_1 come from different coefs. in diff'l eq'n.

$\Rightarrow$ These coefs come from physical properties of system whose values are not known precisely.

$\Rightarrow$ Cancellation should always be considered approx.

$\Rightarrow$ If p_1 is stable, it is a good approximation to cancel it

$$A_1 e^{p_1 t} \approx \varepsilon e^{p_1 t}$$

This term starts small, and gets smaller as t increases

But

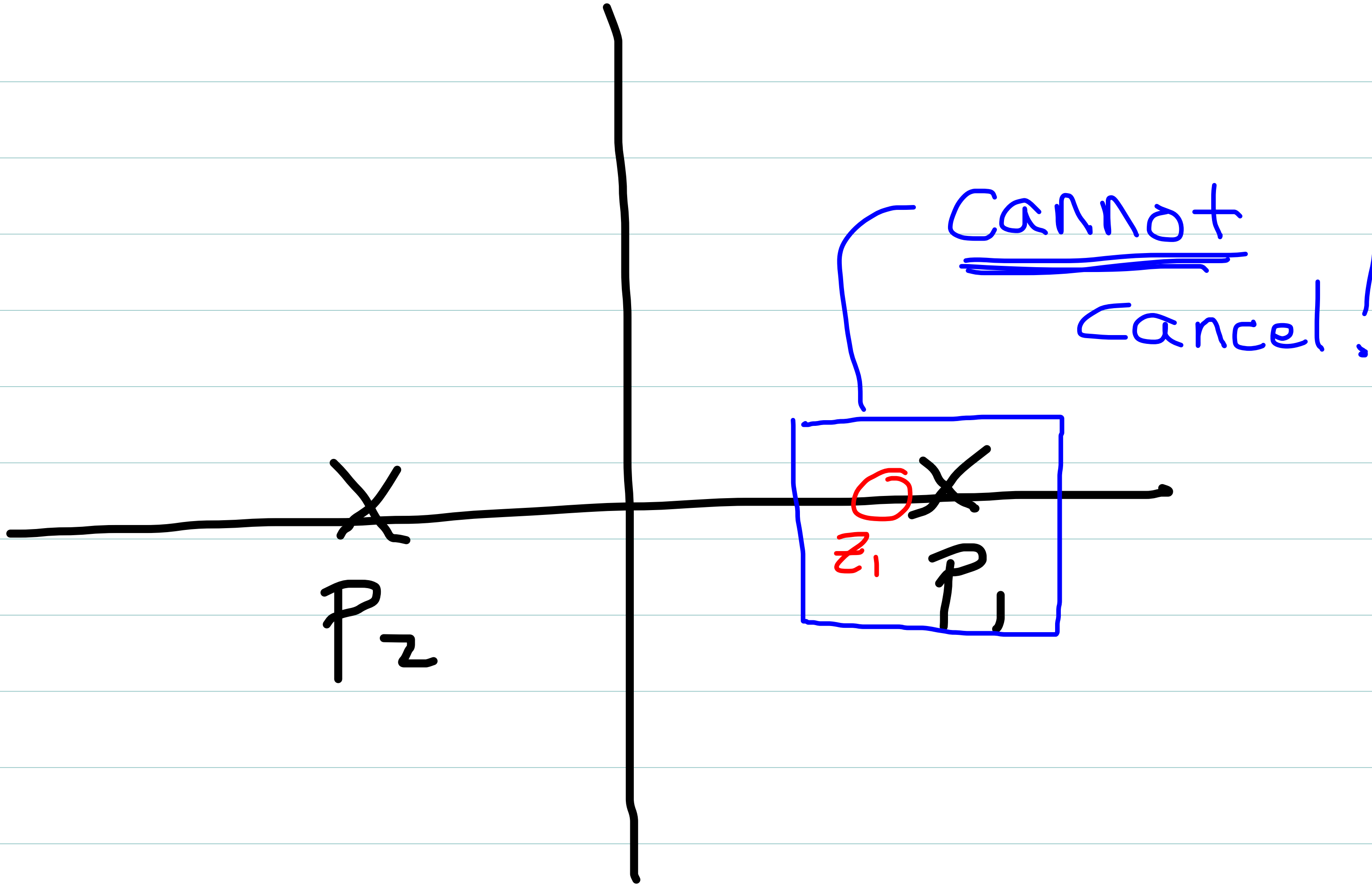
Suppose P_1 Not stable: $P_1 > 0$

then $A_1 e^{P_1 t} \propto \varepsilon e^{P_1 t}$

may start small, but increases w/o bound as t increases

Term will diverge to ∞, regardless how small ε is!

$\boxed{\text{Pole-zero cancellation can \underline{Never} be performed in RHP}}$



Cannot
cancel!
z_1
P_1
P_2

Moreover...

Generally, if ICs on $y(t)$ are not all zero

$$Y(s) = G(s)U(s) + \frac{c(s)}{r(s)}$$

Nonzero

Will contribute terms to $y(t)$ which contain unstable mode <u>even if</u> this mode "cancels" in $G(s)$

<u>Moral</u>: Can <u>Never</u> "cancel" an unstable mode

!!!

Effects of zeros on step response

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}, \quad \text{zero at } z_1 = \frac{-\beta_0}{\beta_1}$$

1. Input absorbtion (if $\beta_0 = \emptyset \Rightarrow z_1 = \emptyset$)

2. Transient suppression via pole-zero cancellation

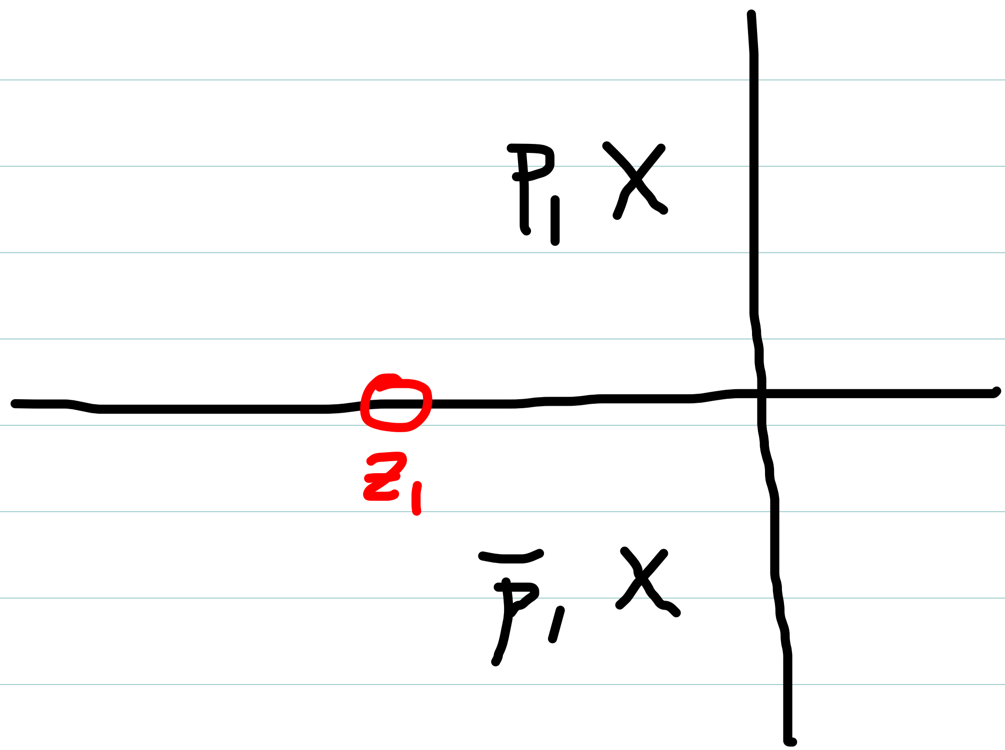
   $\Rightarrow$ if $s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - p_2)$; p_1, p_2 real and $z_1 \approx p_1$ (or p_2)

3. Transient amplification $\Rightarrow$ examine this now.

③ Transient <u>Amplification</u>

Now suppose $s^2+\alpha_1 s+\alpha_0 = (s-p_1)(s-\bar{p}_1)$

$p_1 = \sigma + j\omega_d, \quad \omega_d \neq \emptyset$



Pole-zero cancellation cannot occur here

What <u>is</u> the effect of the zero?

$$Y(s) = \frac{\beta_1 s + \beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{\beta_1 s}{s(s-p_1)(s-\bar{p}_1)} + \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}$$

$$= \left[\left(\frac{\beta_1}{\beta_0}\right)s\right]\left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right] + \left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right]$$

Let

$$Y_1(s) = \left[\frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)}\right]$$

So

$$Y(s) = \left(\frac{\beta_1}{\beta_0}\right)\left[s\,Y_1(s)\right] + Y_1(s)$$

$$\Rightarrow \boxed{y(t) = \left(\frac{\beta_1}{\beta_0}\right)\dot{y}_1(t) + y_1(t)} \quad , \; y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\}$$

<u>Note:</u> $y_1(t)$ is ideal 2^{nd} order step response

$$y(t) = \left(\frac{\beta_1}{\beta_0}\right)\dot{y}_1(t) + y_1(t)$$

or equivalently:

$$\boxed{y(t) = y_1(t) - \left(\frac{1}{z_1}\right)\dot{y}_1(t)} \quad \left(z_1 = -\beta_0/\beta_1\right)$$

Where $y_1(t)$ is the "ideal" (no zero) step response

The total response $y(t)$ is the sum of the ideal response, and a fraction of the <u>derivative</u> of this response.

Suppose 1^{st} $z_1 < 0$ (LHP zero)

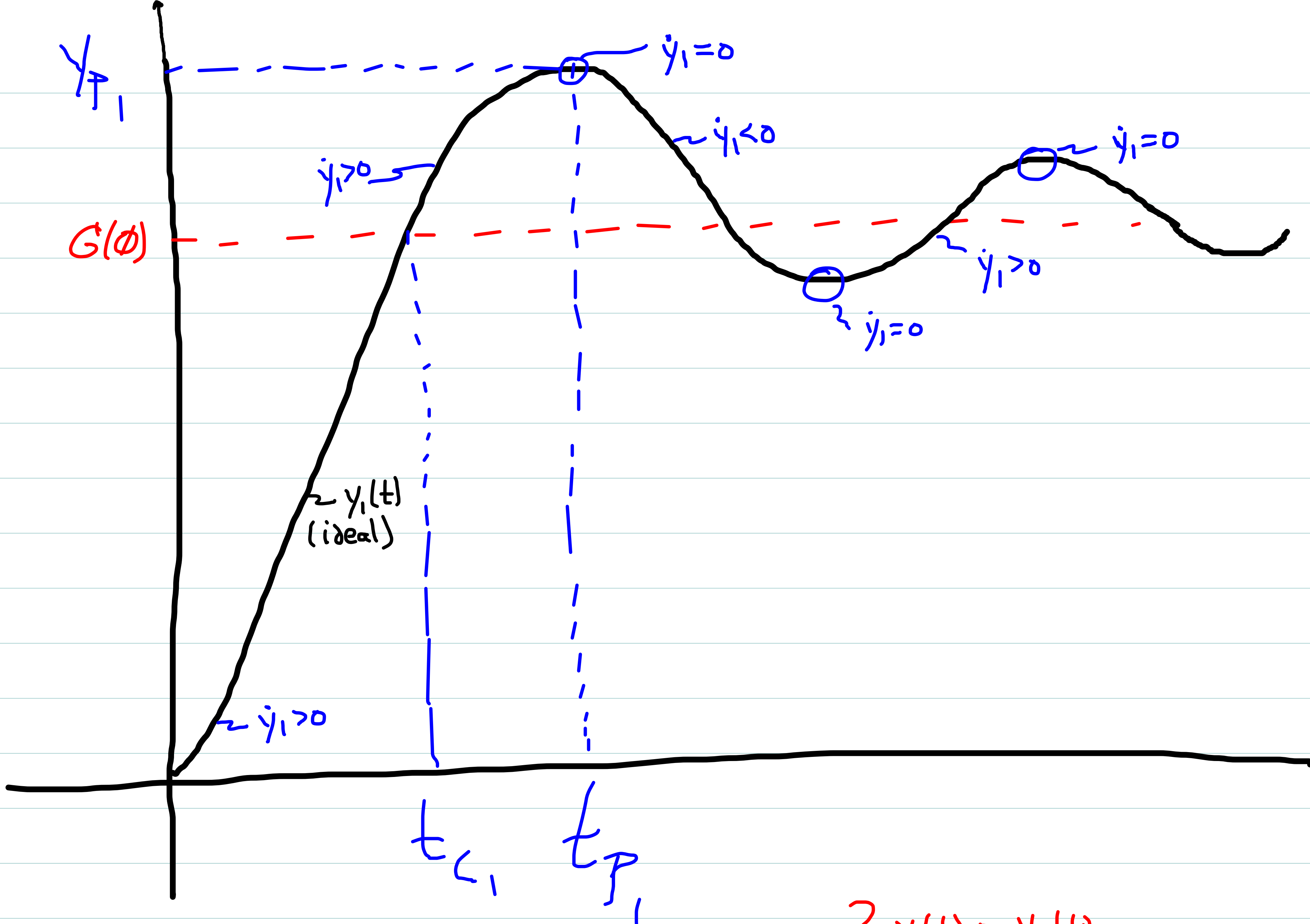
then $z_1 < 0$ and $\left(-\frac{1}{z_1}\right) > 0$ so we can write

$$y(t) = y_1(t) + \left(\frac{1}{|z_1|}\right) \dot{y}_1(t)$$

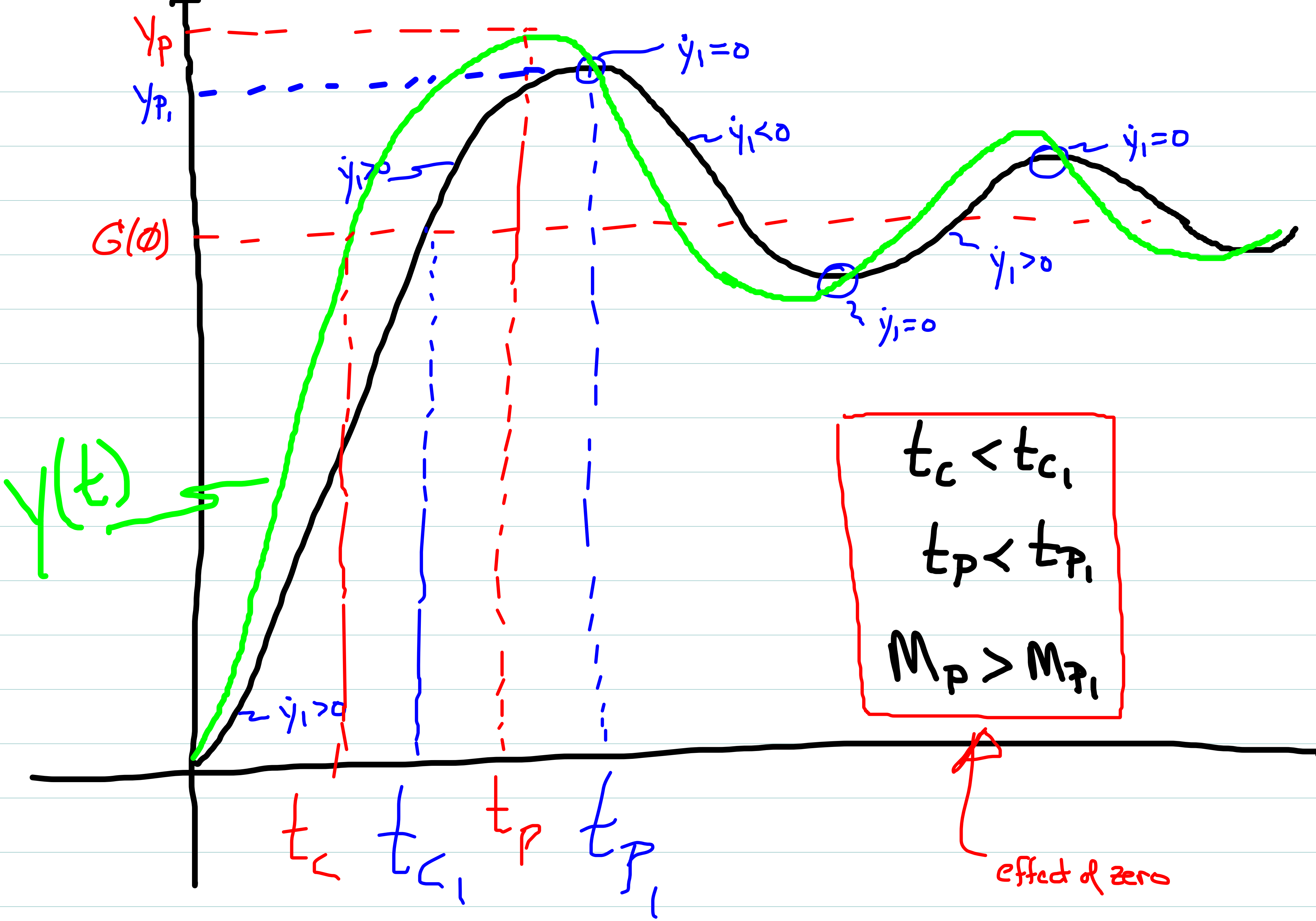
Derivative <u>adds</u> to total response. To understand effect of this, must examine behavior of $\dot{y}_1(t)$

Note that $\dot{y}_1(t) \to \phi$ As $t \to \infty$, so the steady-state of the new response will be the <u>same</u> as the ideal response

$$y_{ss} = G(\phi)$$



<u>Note:</u> $\dot{y}_1(t) > \phi$ for all $\phi \leq t < t_{P_1}$ $\Big\}$ $y(t) > y_1(t)$ in this region



y_p
y_{p1}
G(0)
y(t)
ẏ1=0
ẏ1<0
ẏ1>0
ẏ1=0
ẏ1>0
ẏ1=0
ẏ1>0
t_c < t_{c1}
t_p < t_{p1}
M_p > M_{p1}
t_c
t_{c1}
t_p
t_{p1}
effect of zero

Summary of observations

A LHP zero changes a 2^{nd} order step response by:

⇒ Increasing overshoot y_p and M_p

⇒ decreasing t_c and t_p

In a sense, system "responds" faster (crosses y_{ss} more quickly), but price is greater overshoot.

⇒ Note: tricky to quantify exact changes to t_c, t_p, y_p based on z_1

⇒ However, note change from "ideal" response is proportional to $\frac{1}{|z_1|}$

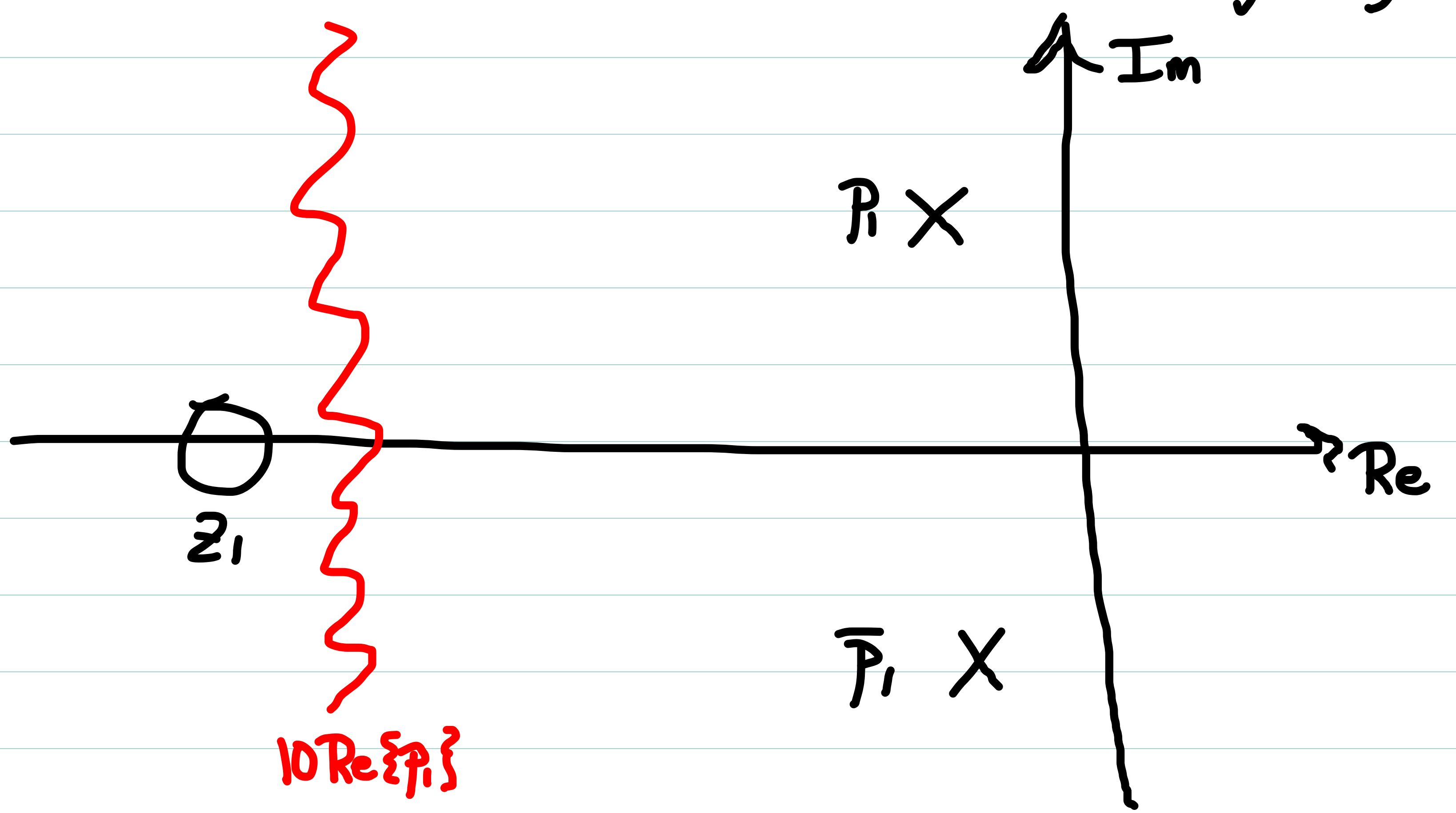
⇒ The further z_1 is from imag axis, the smaller the effect

Rule of Thumb

Effect of zero in this case is negligible if

$$|z_1| > 10\,|\mathrm{Re}\{p_1\}|$$

i.e. zero is 10 times further into LHP than complex poles.



Question

$\Rightarrow$ A zero increases (amplifies) the overshoot of a 2^{nd} order system with $\zeta < 1$ (complex poles).

$\Rightarrow$ Can it actually create overshoot in a system with 2 real poles ($\zeta \geq 1$)?

$\Rightarrow$

Question

$\Rightarrow$ A zero increases (amplifies) the overshoot of a 2^{nd} order system with $\zeta < 1$ (complex poles).

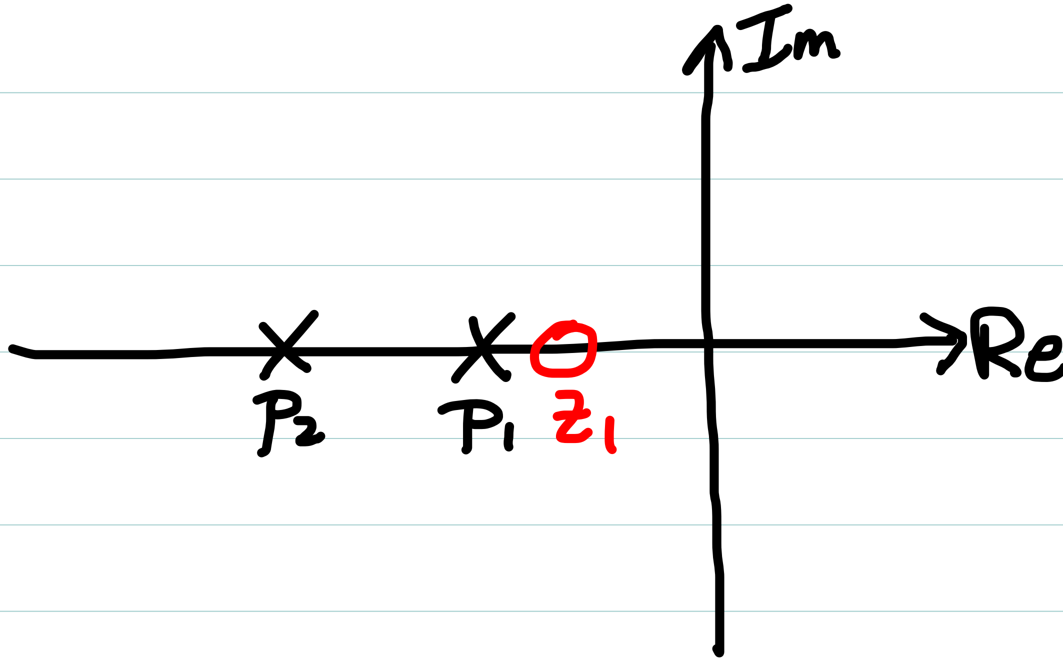
$\Rightarrow$ Can it actually create overshoot in a system with 2 real poles ($\zeta \geq 1$)?

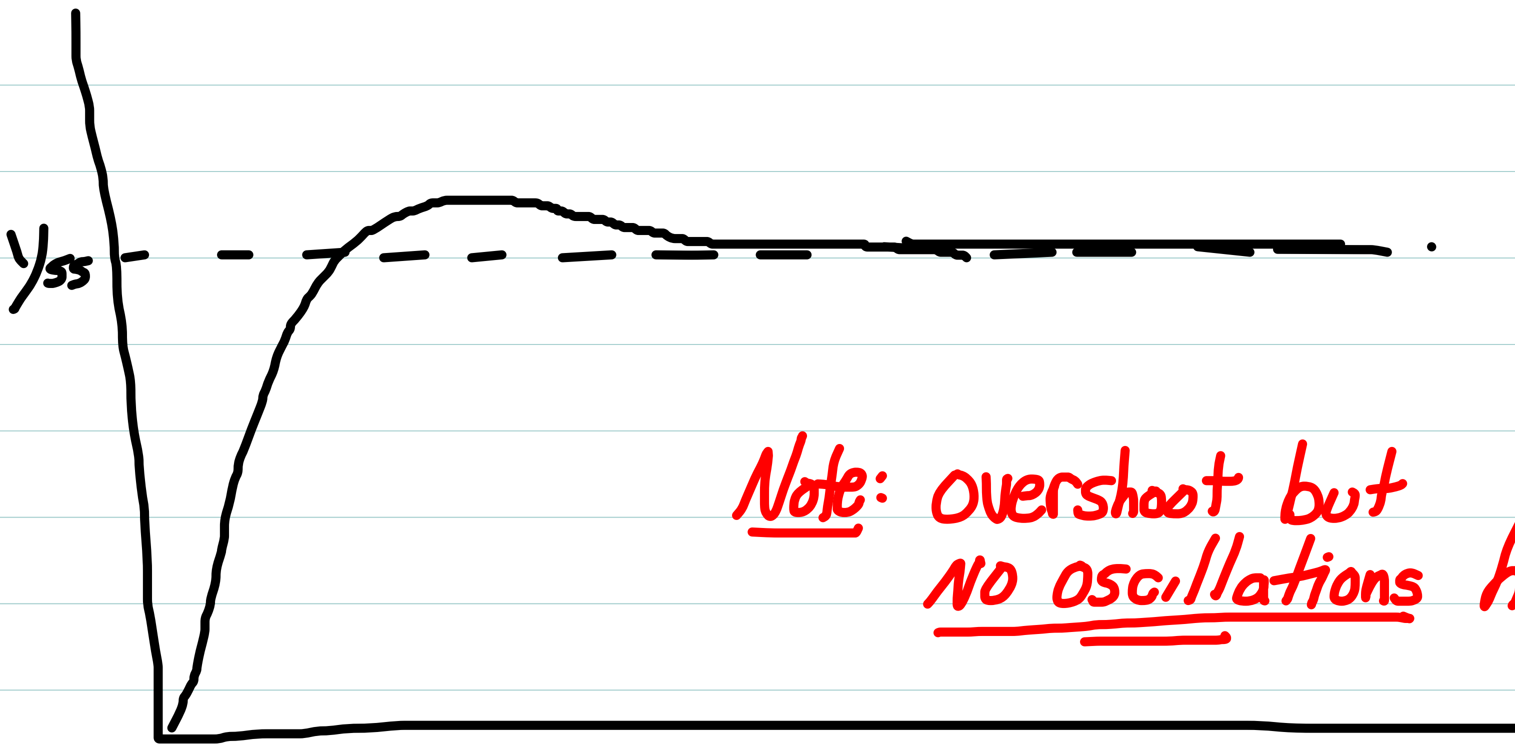
$\Rightarrow$ Yes!

$\Rightarrow$ With 2 real poles p_1 and p_2, $y_p > y_{ss}$ if

$$|z_1| < \min(|p_1|, |p_2|)$$

i.e. if zero is closer to imag axis than either of the two poles.





Note: overshoot but
NO oscillations here

Back to 2nd order ($\zeta < 1$ case)

Suppose $z_1 > \phi$, i.e. z_1 in RHP, then

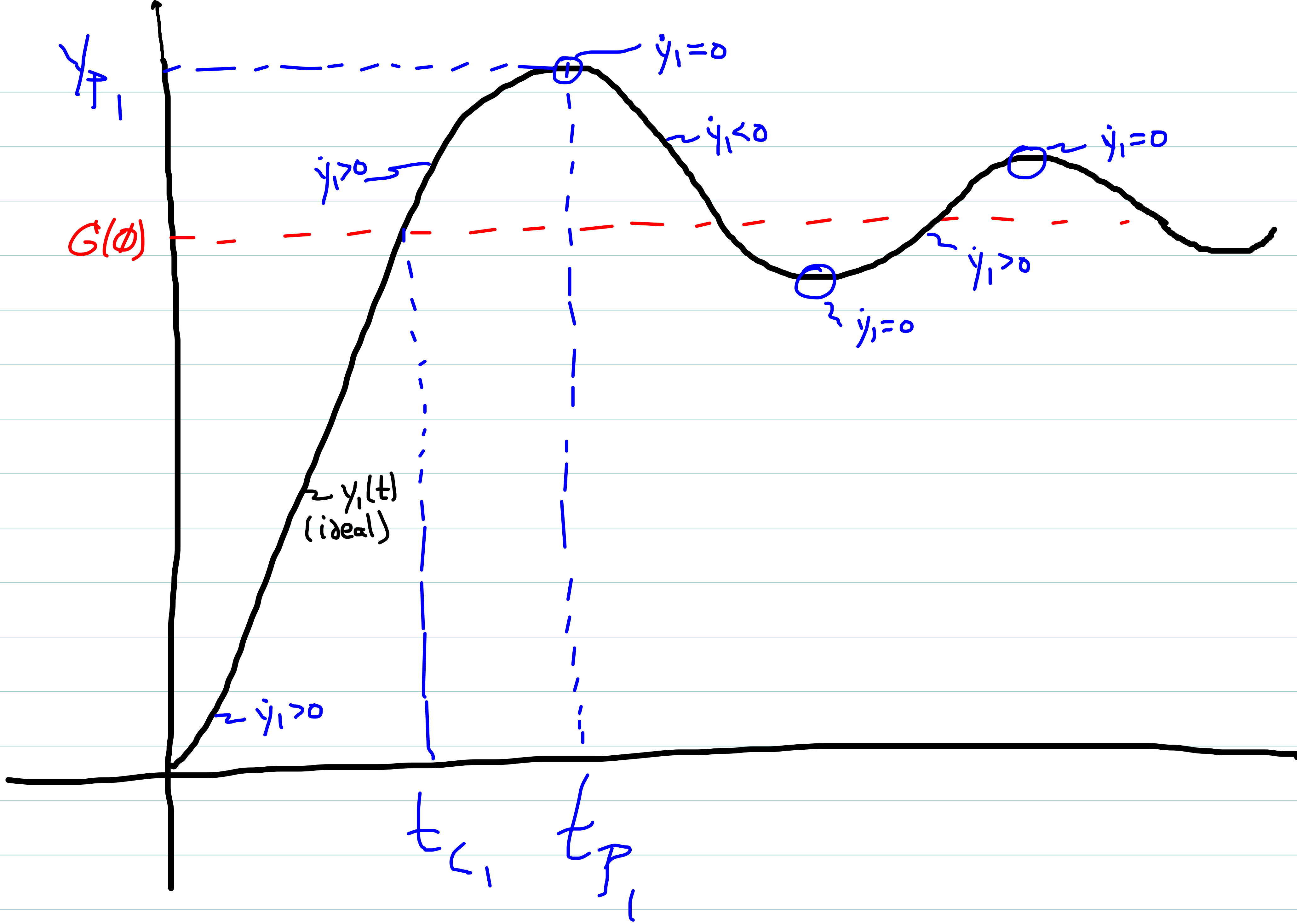
$$y(t) = y_1(t) - \left(\frac{1}{z_1}\right) \dot{y}_1(t)$$

So we are subtracting the derivative from the ideal response.

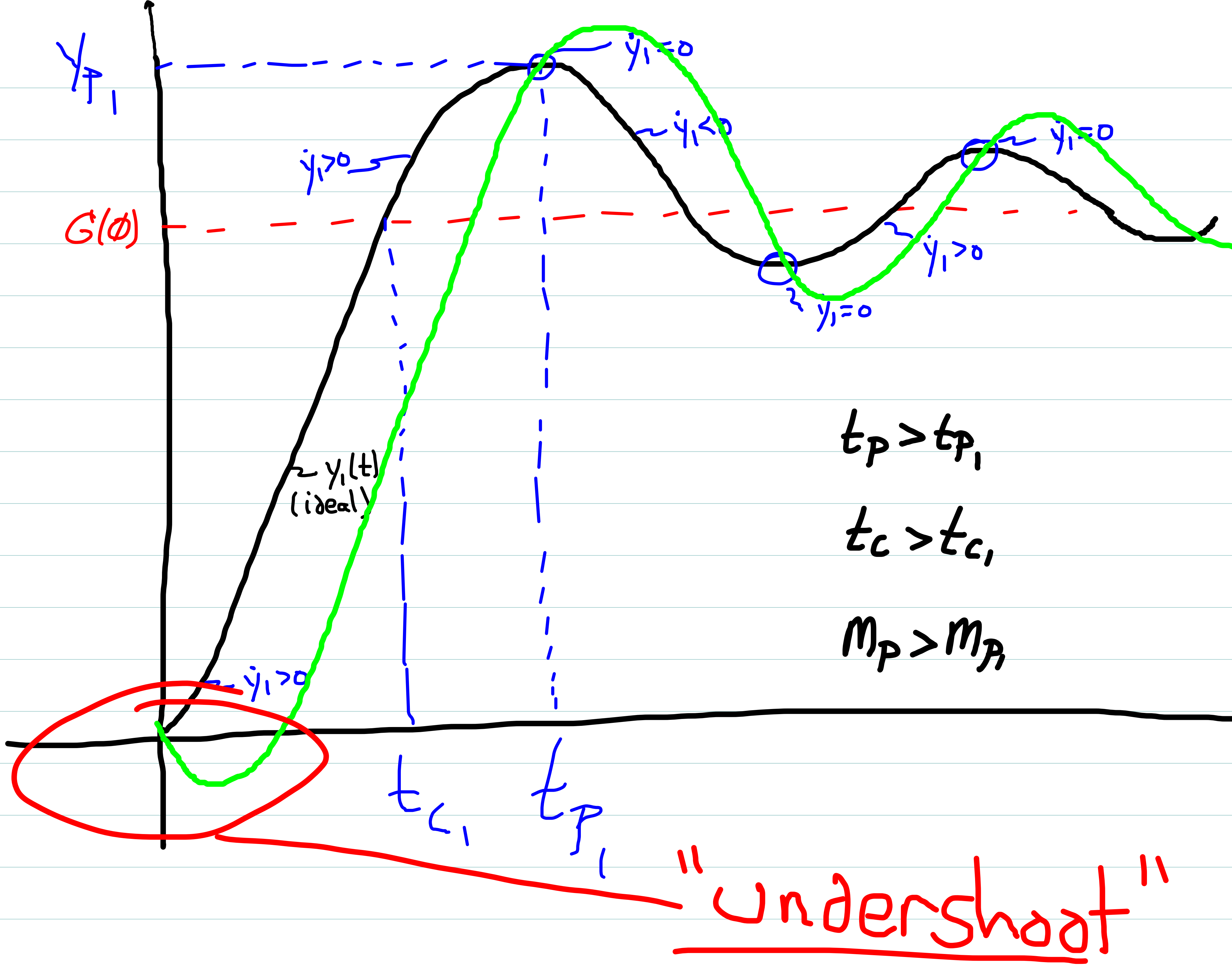
But recall $\dot{y}_1(t) \geq \phi$ for $\phi < t < t_p$,

And $y_1(t) \approx \phi$ for t close to zero

Seems to suggest that $y(t)$ may become negative for times near $t = \phi \ldots$?



y_{p_1}
$\dot{y}_1 = 0$
$\dot{y}_1 < 0$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$G(0)$
$\dot{y}_1 > 0$
$\dot{y}_1 = 0$
$y_1(t)$ (ideal)
$\dot{y}_1 > 0$
t_{c_1}
t_{p_1}



yP1
G(0)
ẏ1>0
ẏ1=0
ẏ1<0
ẏ1=0
ẏ1>0
ẏ1=0
y1(t) (ideal)
ẏ1>0
tc1
tP1
tp > tp1
tc > tc1
Mp > MP1
"undershoot"

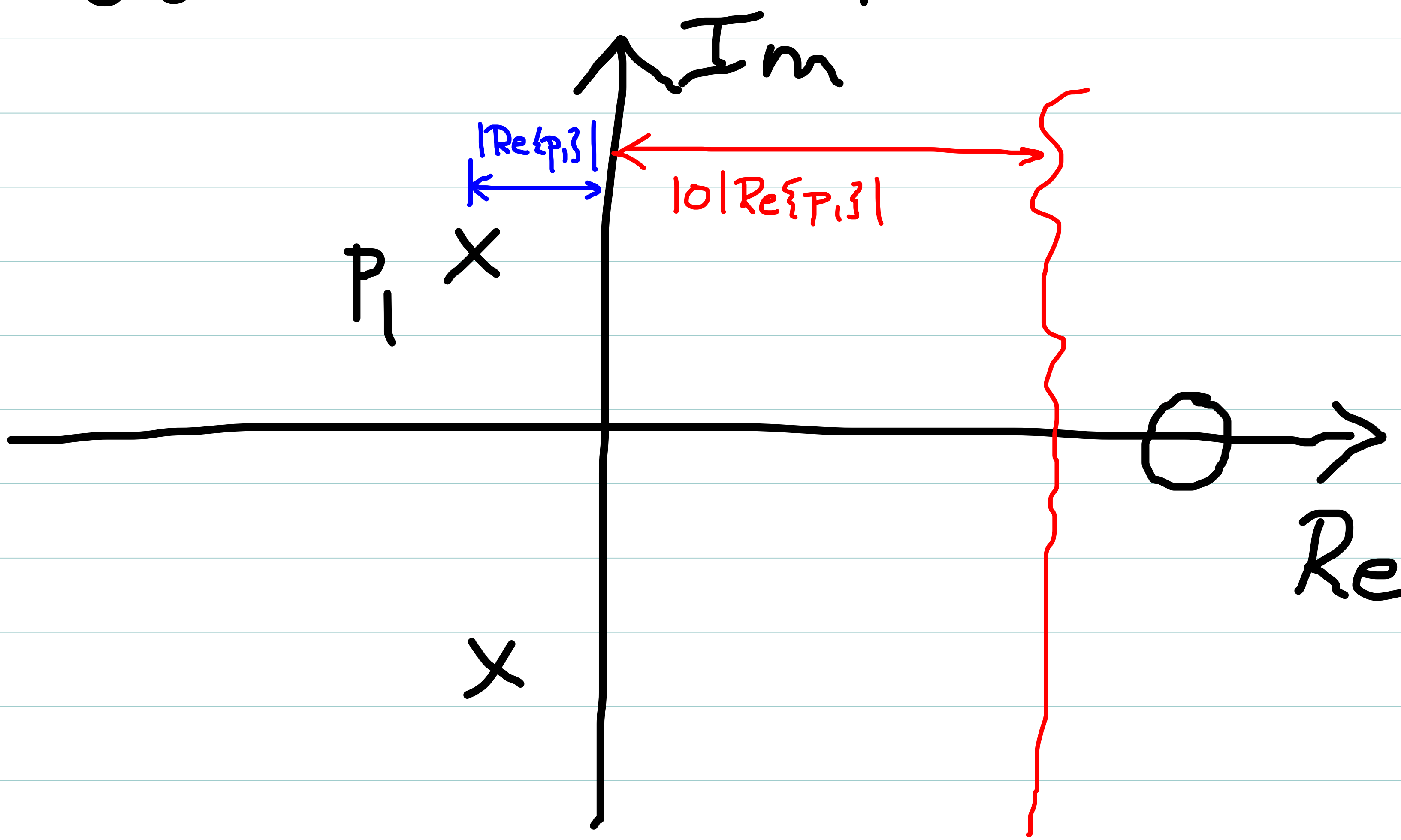
Observations (RHP zero)

- ⇒ Again, the peak response is greater
- ⇒ However, t_c and t_p have increased
- ⇒ Appearence of a new feature : "undershoot"
- ⇒ Response initially heads "in wrong direction" before ultimately returning to the same steady-state
- ⇒ Such behavior is NOT UNstable
- ⇒ It is, however, very tricky to design controllers for such systems.

Effect is still proportional to $\frac{1}{|z_1|}$

hence diminishes as z_1 moves further from Im axis

Again negligible if $|z_1| > 10|Re\{p_1\}|$



Effect on settling time

How a zero, either LHP or RHP, affects t_s is difficult to predict.

$\Rightarrow$ Often, but not always, t_s is longer with zero due to increased amplitude of transient oscillations

$\Rightarrow$ No hard and fast rule here

$\Rightarrow$ Primary effect is increased overshoot and:

- reduction of t_c, t_p (LHP)
- undershoot, with increase of t_c, t_p (RHP)