This can be ensured if:

$$\underbrace{|\Delta(j\omega)||L_o(j\omega)|}_{\text{Radius of Disk}} < \underbrace{|1+L_o(j\omega)|}_{\text{Distance from }-1\text{ to center of disk}} \quad \text{for all } \omega \geq 0$$

Re-arranging:

$$\frac{|L_o(j\omega)|}{|1+L_o(j\omega)|} < |\Delta(j\omega)|^{-1} \quad \text{for all } \omega \geq 0$$

Note that

$$T_o(s) = \frac{L_o(s)}{1+L_o(s)}$$ is the nominal CL TF

So the required condition is:

$$\boxed{|T_o(j\omega)| < |\Delta(j\omega)|^{-1} \quad \text{for all } \omega \geq 0}$$

Uncertainty robustness test

Graphical Interpretation

The Bode magnitude plot $|T_o(j\omega)|$ must lie below the graph of $|\Delta(j\omega)|^{-1}$ at every frequency.

Example: Suppose $G_o(s)$ neglects a pole in $G(s)$, but is otherwise identical:

Then: $\Delta(s) = \left[\frac{1}{\tau s+1} - 1\right] = \frac{-\tau s}{\tau s+1} \Rightarrow \Delta^{-1}(s) = \frac{\tau s+1}{-\tau s}$

$(\tau s+1)$ = neglected pole

Now look at "typical" shapes for $|T_o(j\omega)|$

$$T_o(s) = \frac{L(s)}{1+L(s)}, \quad |T_o(j\omega)| = \frac{|L_o(j\omega)|}{|1+L_o(j\omega)|}$$

Typically, $|L_o(j\omega)| \gg 1$ for small ω (especially if $L_o(s)$ has at least 1 pole at origin)

$\Rightarrow |T_o(j\omega)| \approx 1$ (0 dB) for small ω.

Since relative degree of $L_o(s)$ is positive for any physical system, $|L_o(j\omega)| \to \phi$ as $\omega \to \infty$, and thus

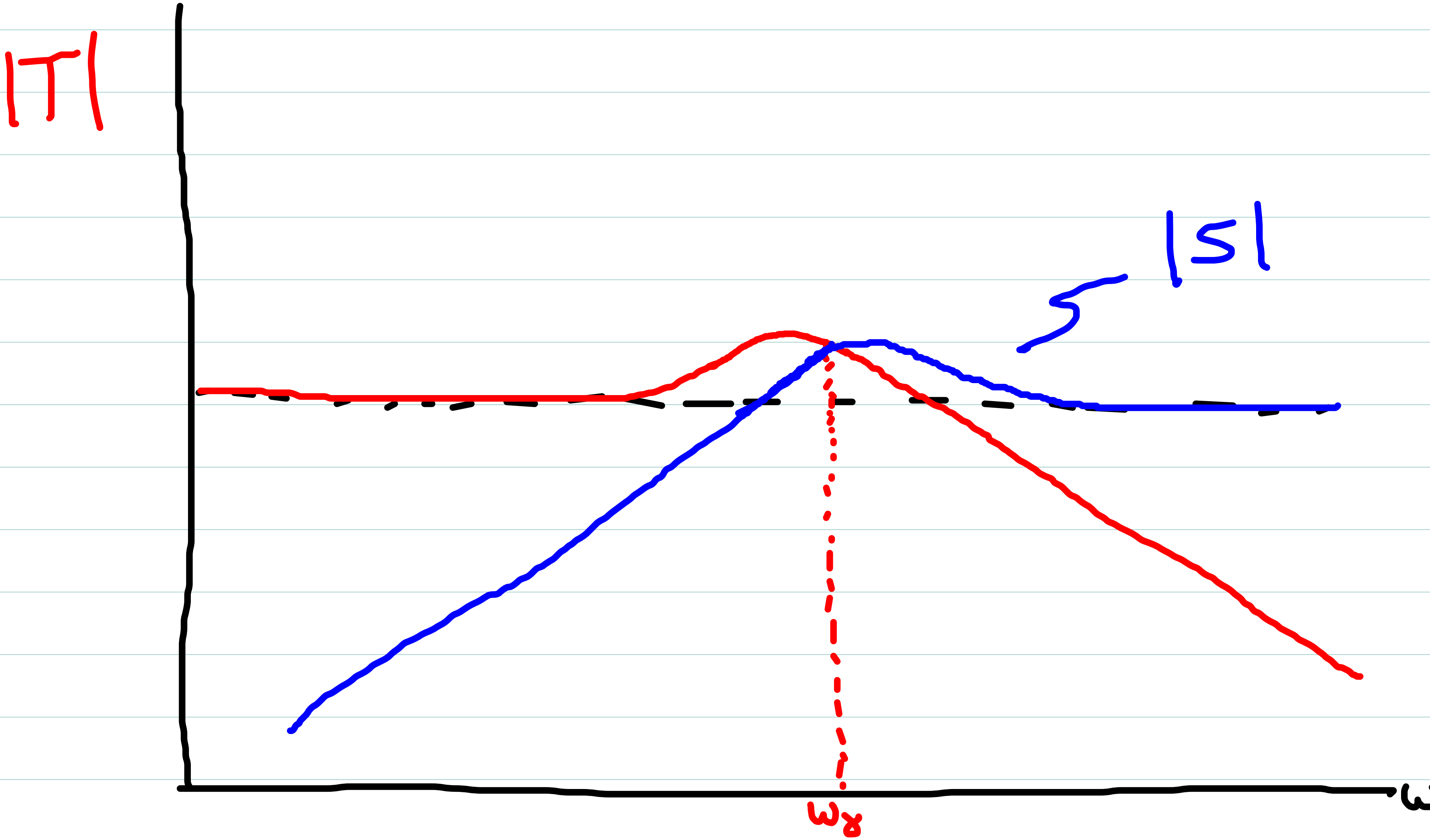
$$|T_o(j\omega)| \approx |L_o(j\omega)| \text{ at high freq. and } |T_o(j\omega)| \to \phi \text{ also}$$

Finally, note $|T_o(j\omega_\gamma)| = \dfrac{|L_o(j\omega_\gamma)|}{|1+L_o(j\omega_\gamma)|} = \dfrac{1}{|1+L_o(j\omega_\gamma)|}$

So $|T_o(j\omega_\gamma)| = |S(j\omega_\gamma)| = \dfrac{1}{2\sin(\gamma/2)}$

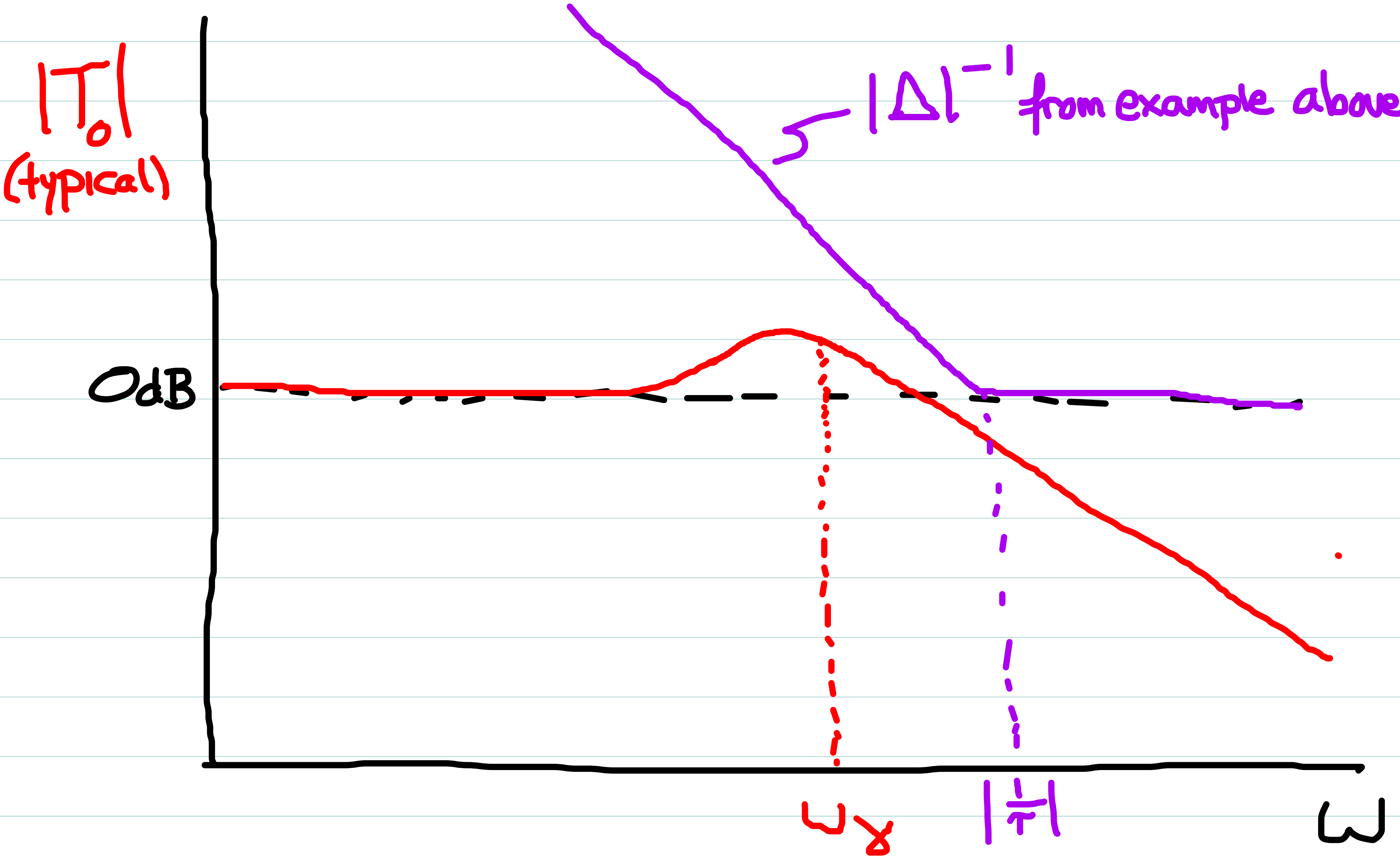
hence $|T_o|$ is also peaking near ω_γ.

$|T_o|$ (typical)
0dB
$\left|\frac{1}{2\sin(\gamma/2)}\right|_{dB}$
$\approx |L_o|$
ω_γ
ω



<u>Note</u>: $|T_0|$ and $|S_0|$ "complementary" in sense that $|S_0| \approx 0$ when $|T_0| \approx 1$ and vice-versa.

Reflects algebraic <u>identity</u> $S(s) + T(s) = 1$ from def'ns.



Remember: must keep graph of $|T_o(j\omega)|$ <u>below</u> $|\Delta(j\omega)|^{-1}$ at every frequency

Design Implication of robustness

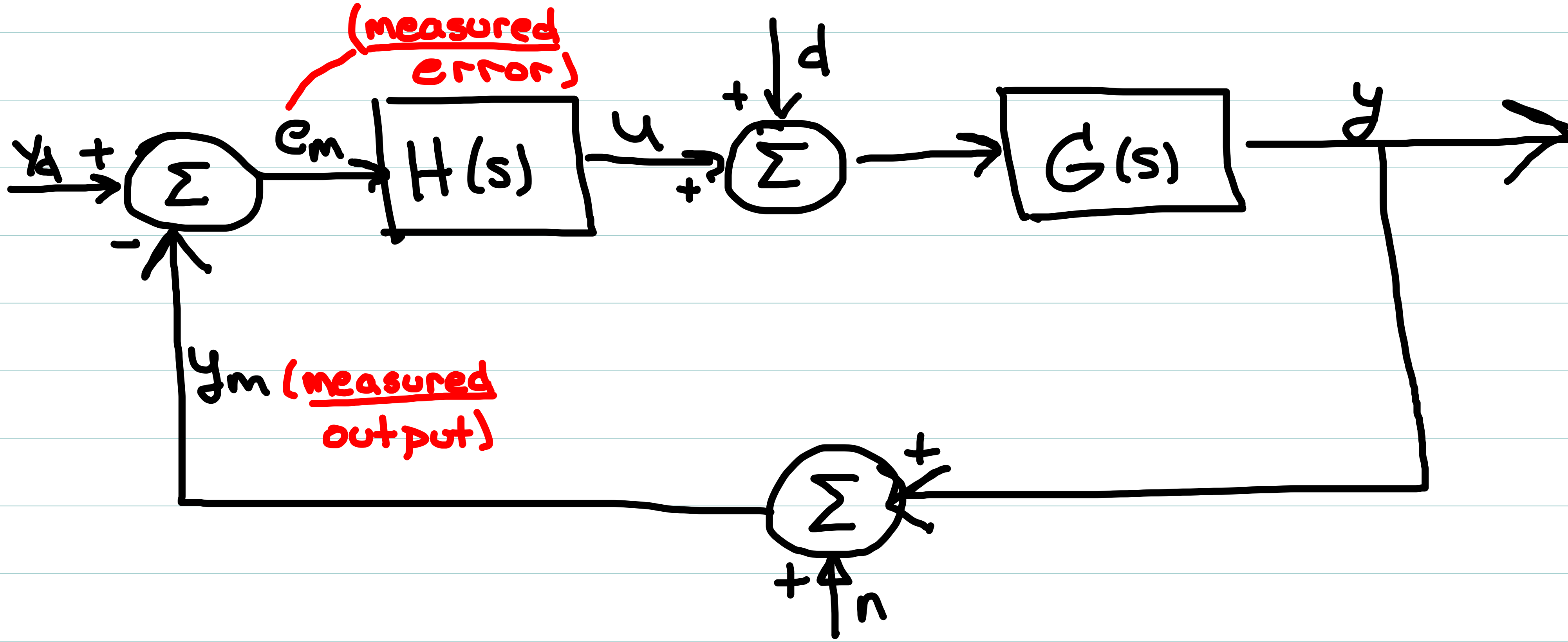
Uncertainty constrains size of ω_γ!

In specific example above, we'd need ω_γ significantly less than freq. $(\frac{1}{\tau})$ of neglected pole.

When $G(s)$ has "unmodeled dynamics" (i.e. poles/zeros neglected in nominal model $G_0(s)$), usually want ω_γ a <u>decade below</u> suspected freq. of neglected poles.

Recall, ω_γ is correlated w/ closed-loop settling time. Above observation means this should be slow compared to neglected poles. We need to avoid control actions so sharp and quick they might "excite" the unmodeled dynamics.

Effect of sensor noise



Now: $Y = G[U+D]$, $U = HE_m = H[Y_d - (Y+N)]$

So: $Y = GHY_d - GHY + GD - GHN$

Or: $\boxed{Y = TY_d - S_i D - TN}$

and hence: $E = Y_d - Y$ satisfies:

$$\boxed{E = (1-T)Y_d - S_i D + TN}$$

New term!

or: $$E = \boxed{SY_d} - \boxed{S_i D} + TN$$

Tracking error

error due to disturbance

Add'l error due to noise

Note: TF from noise to Y is same as TF from Y_d to Y (both are $T(s)$)

Implication: $\Rightarrow$ feedback loop tries to "track the noise"

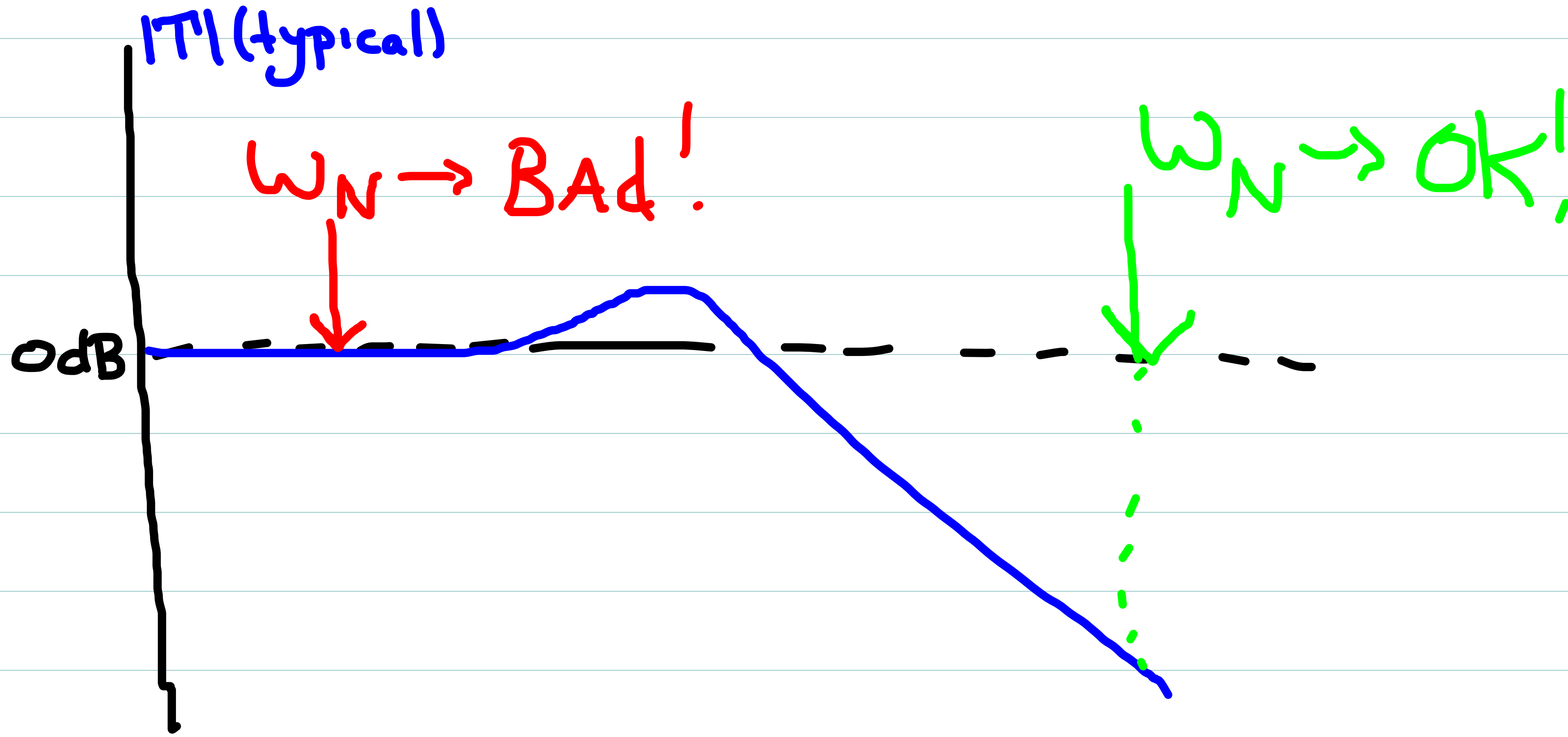
Equivalently: $\Rightarrow$ noise is indistinguishable from "signal" $y(t)$ loop is trying to control!

Impact of Noise

Assume for simplicity noise is "tonal": $n(t) = N\sin(\omega_N t)$
(it isn't really, but useful starting point!)

Then Added error is upper banded by $N|T(j\omega_N)|$

$\Rightarrow$ Need $|T(j\omega)|$ small at noise frequency ω_N!

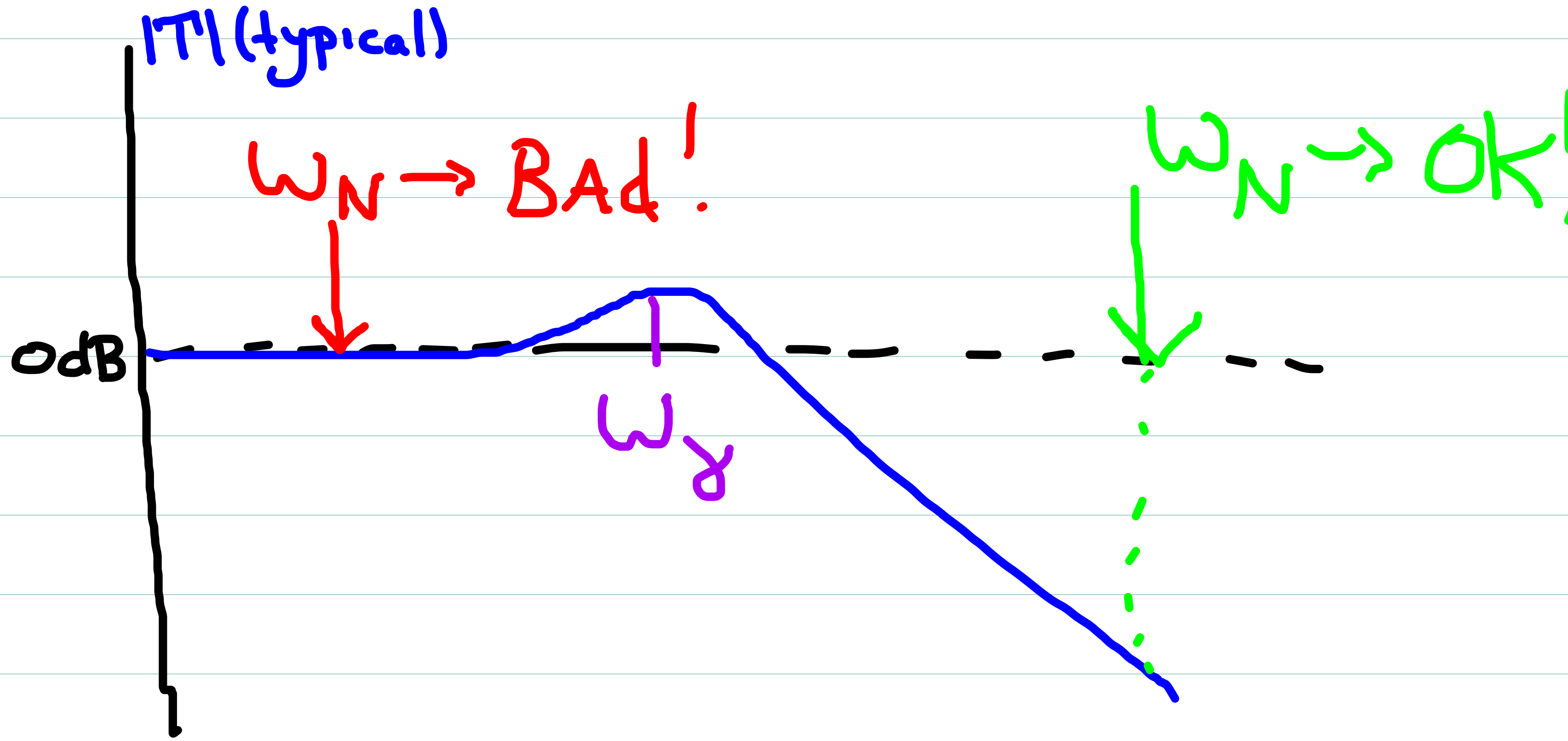


Impact of Noise

Assume for simplicity noise is "tonal": $n(t) = N\sin(\omega_N t)$
(it isn't really, but useful starting point!)

Then Added error is upper bounded by $N|T(j\omega_N)|$

$\Rightarrow$ Need $|T(j\omega)|$ small at noise frequencies!



Design Implications, I

$\Rightarrow$ Need $\omega_\gamma << \omega_N$

$\Rightarrow$ Constrains ω_γ / bandwidth

$\Rightarrow$ Conversely, designs with larger ω_γ will show worse performance due to increased noise impact!

Essentially, we need to make sure there is adequate separation between the frequencies we are trying to track (bandwidth), and the frequency of the noise.

$\Rightarrow$ Works against our desire for large ω_γ (fast settling)

Another perspective:

With noise, controller implementation equation is:

$$u(t) = C_0 \underline{e_m(t)} + \sum c_K X_K(t)$$

$$\dot{X}_K(t) = a_K X_K(t) + \underline{e_m(t)} \qquad [a_K \text{ poles of } H(s)]$$

Noise impacts $u(t)$:

$\Rightarrow$ directly if $C_0 \neq \phi$

$\Rightarrow$ indirectly through $X_K(t)$

$X_K(t)$ diff'l eq'ns have a "filtering" property (reduce magnitude of noise effects)

$\Rightarrow$ Designs with $C_0 = \phi$ have superior noise resistance

Design Implications, II

$C_0 = \emptyset \iff H(s)$ has <u>more</u> poles than zeros

$\Rightarrow$ Designs with this property have better noise resistance!

$\Rightarrow$ Works against our need to increase phase margin

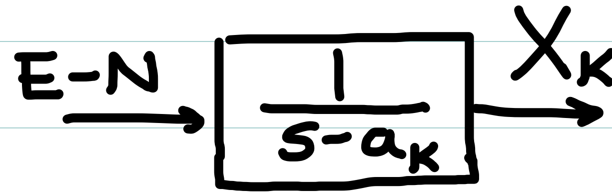
Most "Advanced" controller designs have 1 more pole than zeros to ensure good noise filtering.

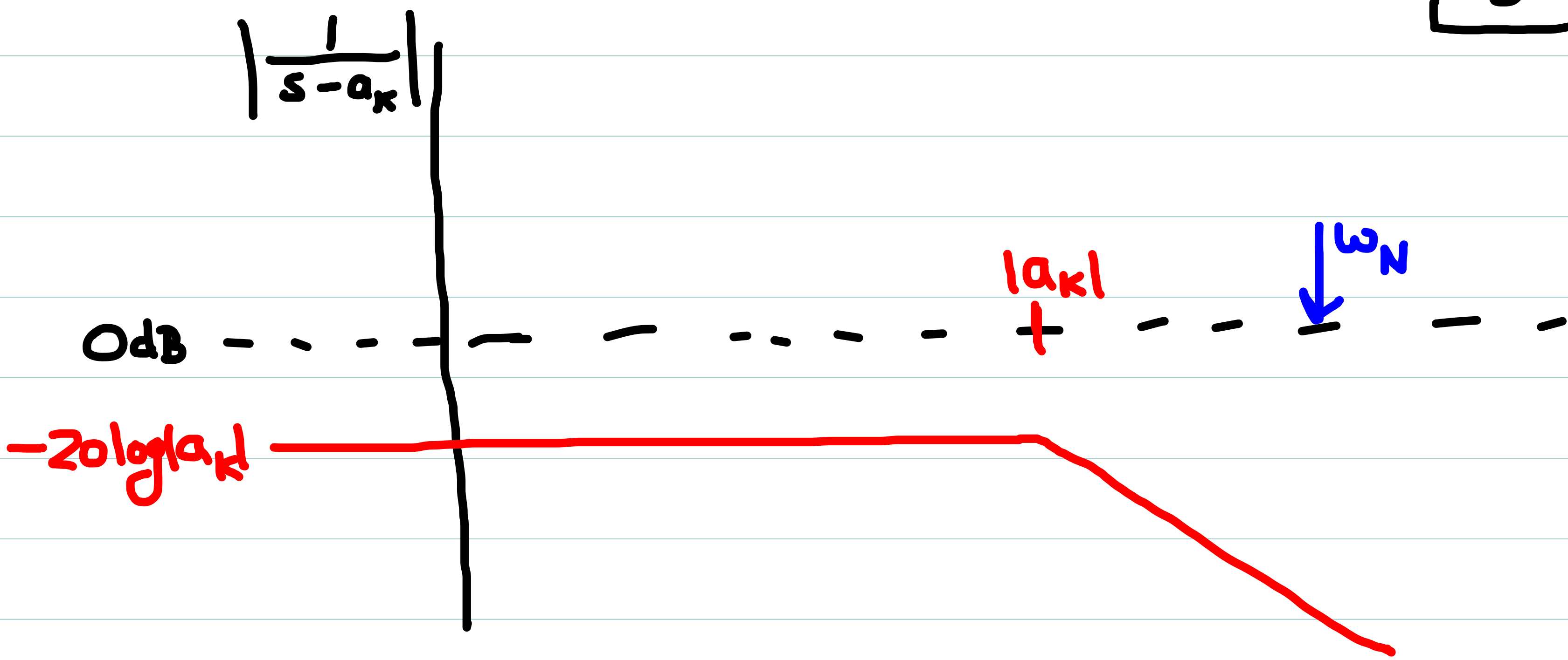
However, superior transient performance is achievable with $C_0 \neq \emptyset$ provided noise is not a significant issue.

"Filtering" by $x_K(t)$ States

$$\dot{x}_K(t) = a_K x_K(t) + e_m = a_K x_K(t) + \underbrace{e(t)}_{\text{true error}} - \underbrace{n(t)}_{\text{sensor noise}}$$

$$\Rightarrow X_K(s) = \left[\frac{1}{s+a_K}\right]\left[E(s) - N(s)\right]$$





Noise is attenuated in $x_K(t)$ if $|a_K| << \omega_N$.

Design implication, III

For good noise rejection, compensator poles should be significantly lower frequency than the noise

$\Rightarrow$ Avoid excessively high frequency poles in $H(s)$ (ie. poles very far from imag axis).

$\Rightarrow$ Another advantage of "minimum β" lead comp design:

By minimizing β (ratio of pole location to zero location in $H(s)$), we are bringing the pole as close to imag axis as possible while still providing necessary φ_{req} at desired ω_x.

Why it's bad to differentiate $y(t)$.

One is tempted to implement a $H(s)$ with only a zero (or more generally with 1 more zero than pole) by numerically differentiating $y(t)$

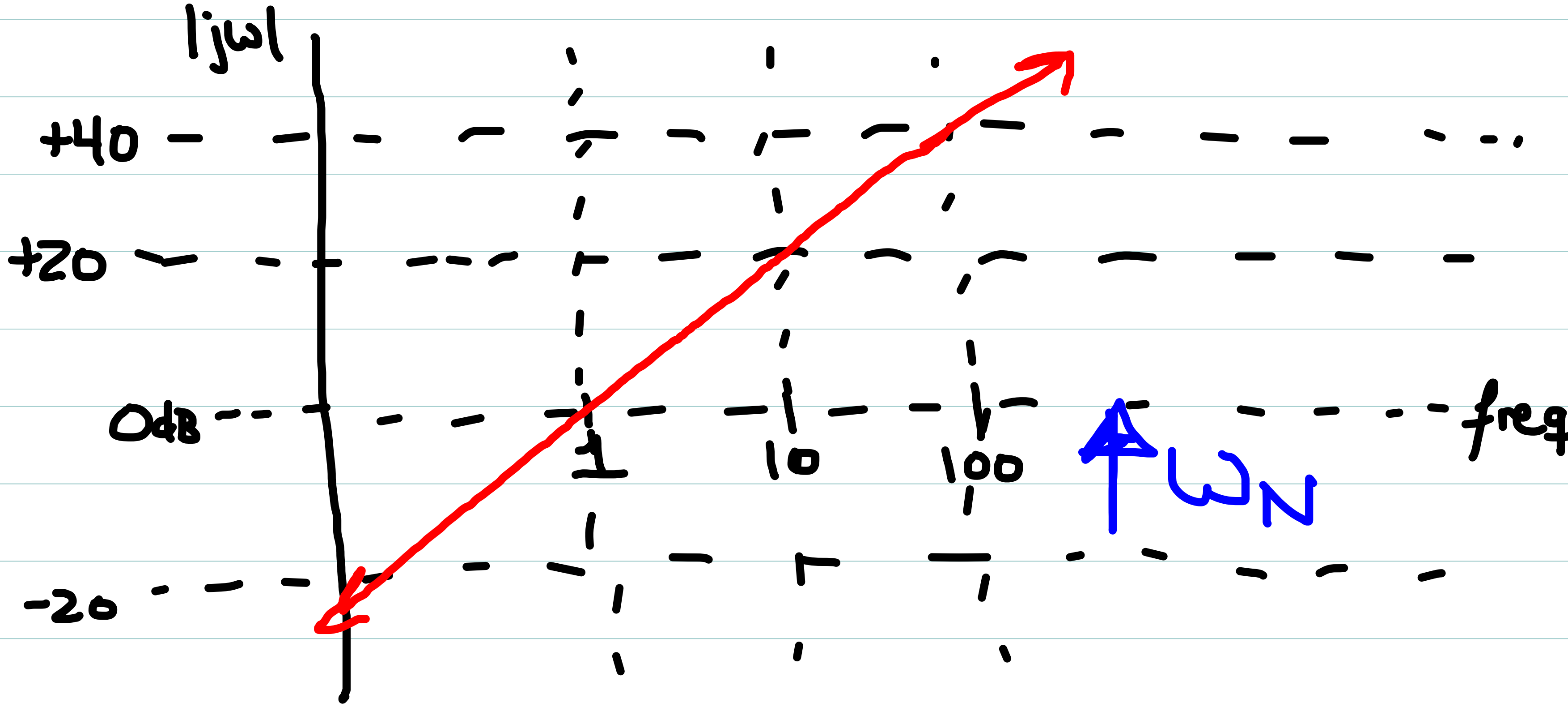
This would be needed since, as we've seen, such compensators will result in $u(t)$ having a term proportional to $\dot{e}(t)$ [hence $\dot{y}(t)$]

But with noise, we're really diffing $y_m(t) = y(t) + n(t)$.

Let $z(t) = \frac{d}{dt} y_m(t)$ be an estimate of $\dot{y}(t)$

$$\Rightarrow Z(s) = s\left[Y(s) + N(s)\right] \qquad \xrightarrow{Y+N} \boxed{s} \xrightarrow{Z}$$

Impact of noise depends on freq. response of s.



Differentiation <u>amplifies</u> the effect of noise

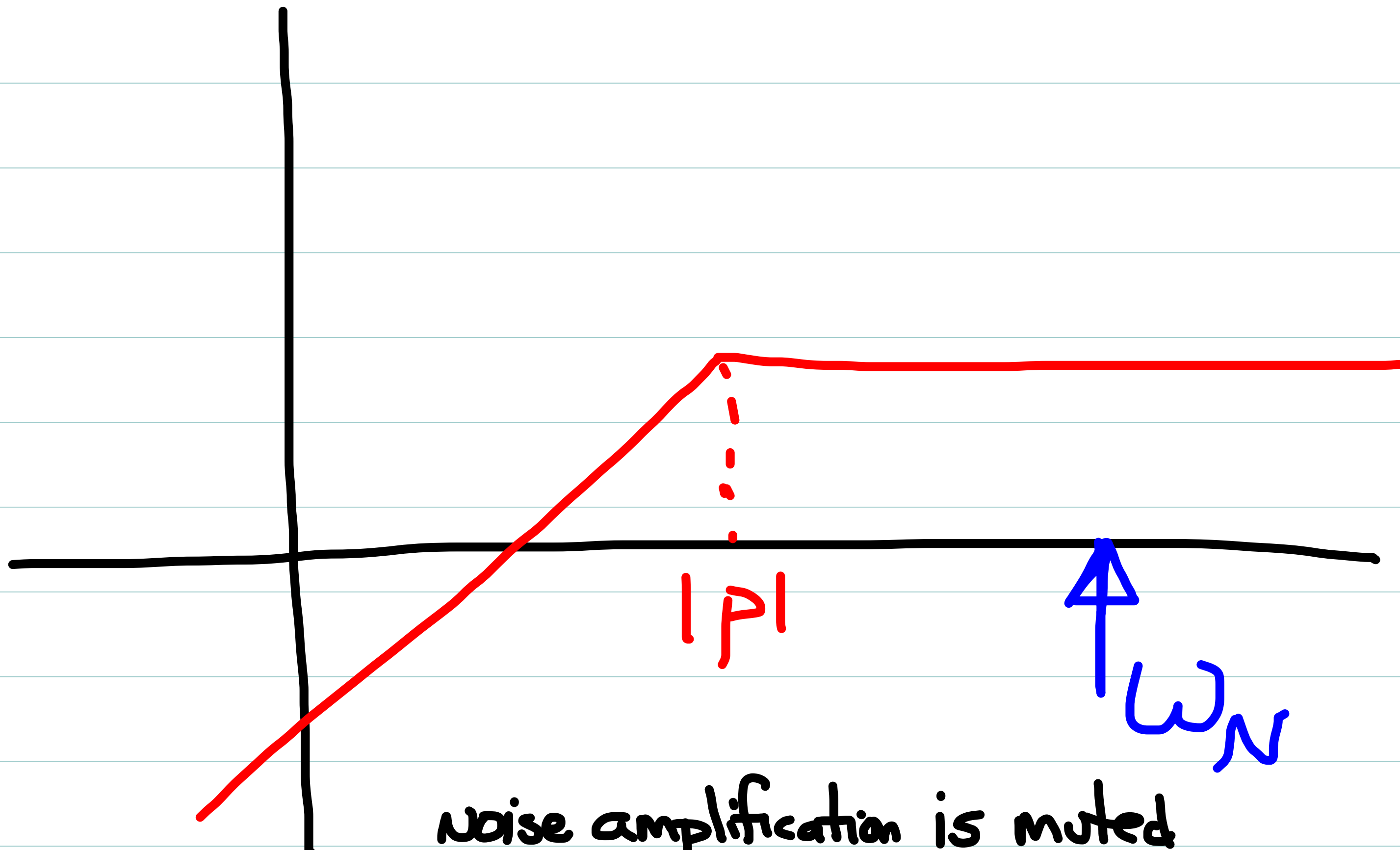
explicitly: if again $n(t) = \varepsilon \sin(\omega_N t)$, $\omega_N \gg 1$
then

$$z(t) = \frac{d}{dt}\left[y(t) + n(t)\right] = \dot{y}(t) + \underline{\varepsilon \omega_N} \cos(\omega_N t)$$

Not small!
(potentially larger than $\dot{y}$)

Note that if we added a pole to our derivative estimation scheme

$$Z(s) = \left[\frac{s}{s-p}\right] Y_m(s)$$



Noise amplification is muted and may be tolerable.

If we used this strategy to replace the derivative information needed for implementation an ideal zero:

$$H(s) = K(s-z) \Longrightarrow H(s) = K\left[\frac{s}{s-p} - z\right]$$

Then:

$$H(s) = K\left[\frac{(1-z)s + pz}{s-p}\right]$$

which is a <u>lead compensator</u> (for typical case $p<z$).

So really, a lead compensator is effectively a "practical" implementation of an ideal zero, which acknowledges the imperfect nature of the measurement process.