Use of $\mathcal{L}T$ for Diff'l Eqn

$$\mathcal{L}\{\alpha_n y^{(n)}(t) + \cdots + \alpha_1 \dot{y}(t) + \alpha_0 y(t)\}$$

$$= \mathcal{L}\{\beta_m u^{(m)} + \cdots + \beta_1 \dot{u}(t) + \beta_0 u(t)\}$$

Gives:

$$\alpha_n\left[s^n Y(s) - y^{(n-1)}(0) - s y^{(n-2)}(0) - \cdots - s^{n-1} y(0)\right]$$

$$+ \cdots + \alpha_1\left[sY(s) - y(0)\right] + \alpha_0 Y(s)$$

$$= \beta_m\left[s^m U(s) - u^{(m-1)}(0) - s u^{(m-2)}(0) - \cdots - s^{m-1} u(0)\right]$$

$$+ \cdots + \beta_1\left[sU(s) - u(0)\right] + \beta_0 U(s)$$

Collect Terms

$$\boxed{r(s)Y(s) - c(s) = q(s)U(s) - b(s)}$$

$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$

$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$

$c(s)$ = $n-1$ order polynomial in s from IC terms on $y(t)$

$b(s)$ = $m-1$ order polynomial in s from IC terms on $u(t)$.

Re-arrange for $Y(s)$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

IC terms

Or:

$$Y(s) = G(s)U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

Alternate def'n of TF:

$$G(s) = \left[\frac{Y(s)}{U(s)}\right]_{ICs=0} = \left[\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}\right]_{\substack{\text{zero} \\ \text{ICs}}}$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$(2s^3 + 8s^2 + 14s + 10)Y(s)$$

$$- \left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]$$

$$= (3s^2 + 15s + 18)U(s) - \left[3su_0 + (3\dot{u}_0 + 15u_0)\right]$$

Thus:

$$Y(s) = \underbrace{\boxed{\left[\frac{3s^2+15s+18}{2s^3+8s^2+14s+10}\right]}}_{G(s)} U(s)$$

$$+ \left[\frac{2s^2 y_0 + s(2\dot{y}_0 + 8y_0 - 3u_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0 - 3\dot{u}_0 - 15u_0)}{2s^3+8s^2+14s+10}\right]$$

$\Rightarrow$ We assume all ICs on $y(t)$ known; and $u(t)$ known so $U(s)$ can be computed and ICs on $u(t)$

$\Rightarrow$ All terms on RHS are known, so we know $Y(s)$

$\Rightarrow$ "Simply" invert transform to get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Inverse Transform

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{2\pi j}\int Y(s)e^{st}\,ds$$

$\Rightarrow$ contour integral over ROC in complex plane

$\Rightarrow$ ugly! Math 463

$\Rightarrow$ We can sidestep this in many cases

General form of Y(s)

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

all polynomials

Suppose $U(s)$ is rational in s
(ratio of polynomials)

i.e. $U(s) = \dfrac{a(s)}{h(s)}$, $a(s)$ $h(s)$ polys

Note: ① Not true for every $u(t)$
② True for many "useful" $u(t)$

Then . . .

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]\underbrace{\left(\frac{a(s)}{h(s)}\right)}_{U(s)} + \frac{c(s)-b(s)}{r(s)}$$

$$= \frac{q(s)a(s) + h(s)\left[c(s)-b(s)\right]}{r(s)h(s)}$$

or

$$Y(s) = \frac{N(s)}{D(s)}$$

where both $N(s)$ and $D(s)$ are polynomials

(i.e. $Y(s)$ is rational)

$$Y(s) = \frac{N(s)}{D(s)}$$

Suppose $\deg\{N(s)\} < \deg\{D(s)\} = L$

Let d_ℓ be the roots of $D(s)$: $D(d_\ell) = \emptyset$

Then:

$$Y(s) = \frac{A_1}{s-d_1} + \frac{A_2}{s-d_2} + \cdots + \frac{A_L}{s-d_L}$$

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{s-d_\ell}$$

Partial fraction expansion

and

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

How to find expansion coefficients

"Residue formula":

$$A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$$

(also called "cover up" rule).

Example:

$$Y(s) = \frac{2s+3}{(s+2)(s+3)}$$

$$Y(s) = \frac{A_1}{s+2} + \frac{A_2}{s+3}$$

$$A_1 = \left[\frac{2s+3}{s+3}\right]_{s=-2} = -1, \quad A_2 = \left[\frac{2s+3}{s+2}\right]_{s=-3} = 3$$

$$y(t) = 3e^{-3t} - e^{-2t}$$

Complex d_ℓ

Note if d_ℓ is a complex root of $D(s)$, then its conjugate $\bar{d}_\ell$ will also be a root.

The residue formula then tells us that

for d_ℓ: $A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$

and for $\bar{d}_\ell$ we instead have

$$\left[(s-\bar{d}_\ell)Y(s)\right]_{s=\bar{d}_\ell} = \bar{A}_\ell$$

i.e. the PFE coefficients are also conjugates

Complex d_ℓ (cont)

Thus, the expression for $y(t)$ will contain

$$A_\ell e^{d_\ell t} + \bar{A}_\ell e^{\bar{d}_\ell t}$$

$$= 2|A_\ell| e^{\sigma t} \cos(\omega t + \measuredangle A_\ell)$$

where $\sigma = \text{Re}\{d_\ell\}$ $\omega = \text{Im}\{d_\ell\}$

Example:

$$Y(s) = \frac{4(s^2+2s+6)}{(s+1)(s^2+4s+13)}$$

$$d_1 = -1; \quad d_2 = -2+3j; \quad d_3 = -2-3j = \bar{d}_2$$

Then:

$$A_1 = \left[(s+1)Y(s)\right]_{s=-1} = 2$$

$$A_2 = \left[(s+2-3j)Y(s)\right]_{s=-2+3j} = 1+j = \boxed{\sqrt{2}\,\angle \pi/4 = A_2}$$

$$A_3 = \left[(s+2+3j)Y(s)\right]_{s=-2-3j} = 1-j = \bar{A}_2$$

Hence:

$$y(t) = 2e^{-t} + \boxed{(1+j)e^{(-2+3j)t} + (1-j)e^{(-2-3j)t}}$$

or:

$$y(t) = 2e^{-t} + \boxed{2\sqrt{2}\,e^{-2t}\cos(3t + \pi/4)}$$

Recap

$G(s)$

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

IC terms

If $U(s)$ rational, $U(s) = \dfrac{a(s)}{h(s)}$

Then $\quad Y(s) = \dfrac{N(s)}{D(s)} \quad$ (also rational)

$$= \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)} \quad \text{where} \quad D(d_\ell) = \emptyset$$

and $\quad A_\ell = \left[(s-d_\ell) Y(s)\right]_{s=d_\ell}$

Inverse transform:

$$y(t) = \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

Assumptions

Above assumes:

1. $\deg\{N(s)\} < \deg\{D(s)\}$
2. No repeated roots of $D(s)$

Simplest, most common case

Both can be relaxed:

1. Suppose $\text{Deg}\{N(s)\} = \text{Deg}\{D(s)\}$

Then do polynomial long division:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}, \quad \text{Deg}\{N_1(s)\} < \text{Deg}\{D(s)\}$$

and $\frac{N_1(s)}{D(s)}$ can be expanded using above

So:

$$Y(s) = \frac{N(s)}{D(s)} = A_0 + \frac{N_1(s)}{D(s)}$$

$$= A_0 + \sum_{\ell=1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

PFE

Where:

$$A_\ell = \left[(s-d_\ell)\frac{N_1(s)}{D(s)}\right]_{s=d_\ell}$$

<u>Inverse transforming</u>:

$$y(t) = \mathcal{L}^{-1}\{A_0\} + \sum_{\ell=1}^{L} A_\ell e^{d_\ell t}$$

What is this?? We'll see later...

<u>Note:</u> $\mathrm{Deg}\{N(s)\} > \mathrm{Deg}\{D(s)\}$

nonphysical + won't be seen

Repeated Roots

Now suppose:

$$D(s) = (s-d_1)^K (s-d_{K+1}) \cdots (s-d_L)$$

i.e. d_1 is repeated K times, then:

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

<u>for $\ell = K+1, \ldots, L$</u>:

$$A_\ell = \left[(s-d_\ell)Y(s)\right]_{s=d_\ell}$$ (unchanged)

<u>for $\ell = 1, \ldots, K$</u>:

(ugh!)

$$A_\ell = \frac{1}{(K-\ell)!}\left\{\frac{d^{K-\ell}}{ds^{K-\ell}}\left[(s-d_1)^K Y(s)\right]\right\}_{s=d_1}$$

Inverse Transform (Repeated Roots)

$$Y(s) = \sum_{\ell=1}^{K} \frac{A_\ell}{(s-d_1)^\ell} + \sum_{\ell=K+1}^{L} \frac{A_\ell}{(s-d_\ell)}$$

$$\Rightarrow \quad y(t) = \sum_{\ell=1}^{K} \left(\frac{A_\ell\, t^{\ell-1}}{(\ell-1)!} \right) e^{d_1 t} + \sum_{\ell=K+1}^{L} A_\ell\, e^{d_\ell t}$$

Example:

$$Y(s) = \frac{2s+1}{(s+1)^3(s+2)}$$

$d_1 = -1$, $K=3$

$d_4 = -2$

$$\Rightarrow y(t) = \left[A_1 + A_2 t + \frac{A_3}{2} t^2\right] e^{-t} + A_4 e^{-2t}$$

$$A_3 = \left[(s+1)^3 Y(s)\right]_{s=-1} = -1$$

$$A_2 = \left(\frac{1}{1}\right)\left\{\frac{d}{ds}\left[(s+1)^3 Y(s)\right]\right\}_{s=-1} = \left[\frac{3}{(s+2)^2}\right]_{s=-1} = 3$$

$$A_1 = \left(\frac{1}{2}\right)\left\{\frac{d^2}{ds^2}\left[(s+1)^3 Y(s)\right]\right\}_{s=-1}$$

$$= \left(\frac{1}{2}\right)\left\{\frac{d}{ds}\left[\frac{3}{(s+2)^2}\right]\right\}_{s=-1} = -3$$

And

$$A_4 = \left[(s+2)Y(s)\right]_{s=-2} = 3$$

So finally:

$$y(t) = \left[-3 + 3t - \frac{1}{2}t^2\right]e^{-t} + 3e^{-2t}$$

Note: You aren't responsible for repeated root residue formula. However you should know the general pattern for repeated root solutions.

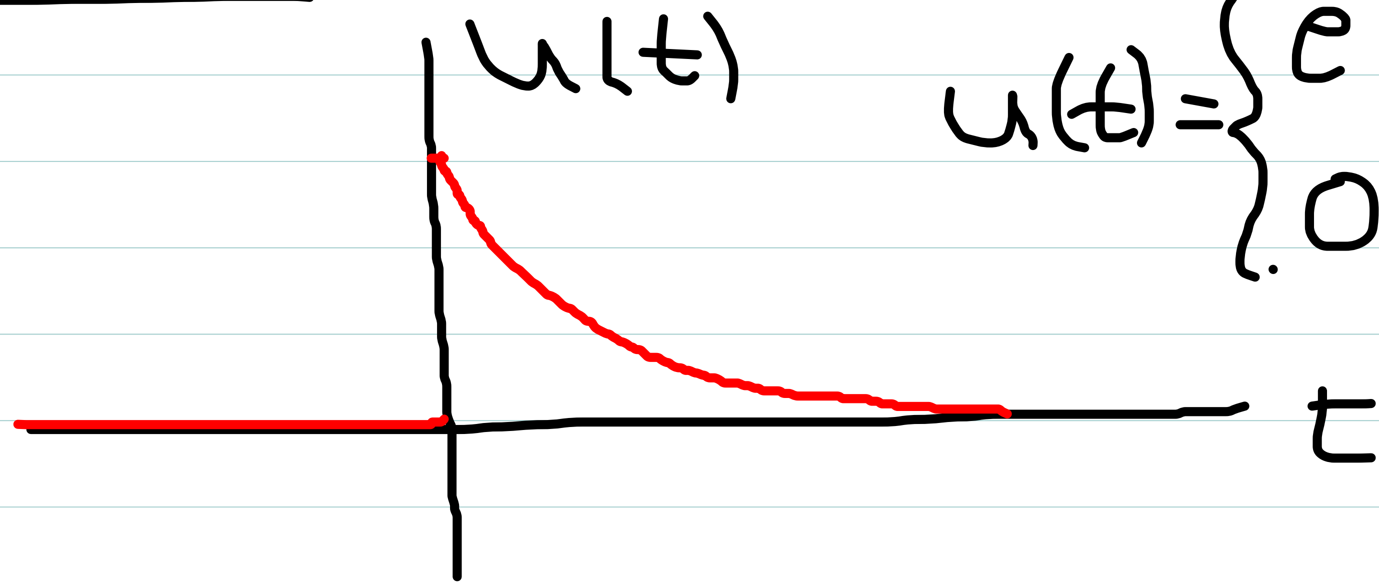
Philosophical Question: What is $t=0$?

$\Rightarrow$ The instant we start acting on the system with external input.

$\Rightarrow$ In control Theory, we assume these inputs are completely "off" for $t<0$.

$\Rightarrow$ $u(t), \dot{u}(t), \ddot{u}(t)$, etc all zero for $t<0$

$\Rightarrow$ Discontinuities exist when $u(0) \neq 0$



$$u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 & t<0 \end{cases}$$

$$u(t) = \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{u(t) = e^{pt}\,\mathbb{1}(t)}$$

where

$$\boxed{\mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

"Unit step function" (Very important!)

Now, Laplace is concerned about behavior of functions only for $t \geq 0$.

For all intents and purposes, functions in Laplace are considered 0 for $t < 0$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\,\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \emptyset & t \neq \emptyset \\ & \end{cases}$$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \quad (???) \end{cases}$$