

Robustness

Robustness is range of inaccuracy in our Nominal model $G(s)$ that we can tolerate before feedback loop might become unstable.

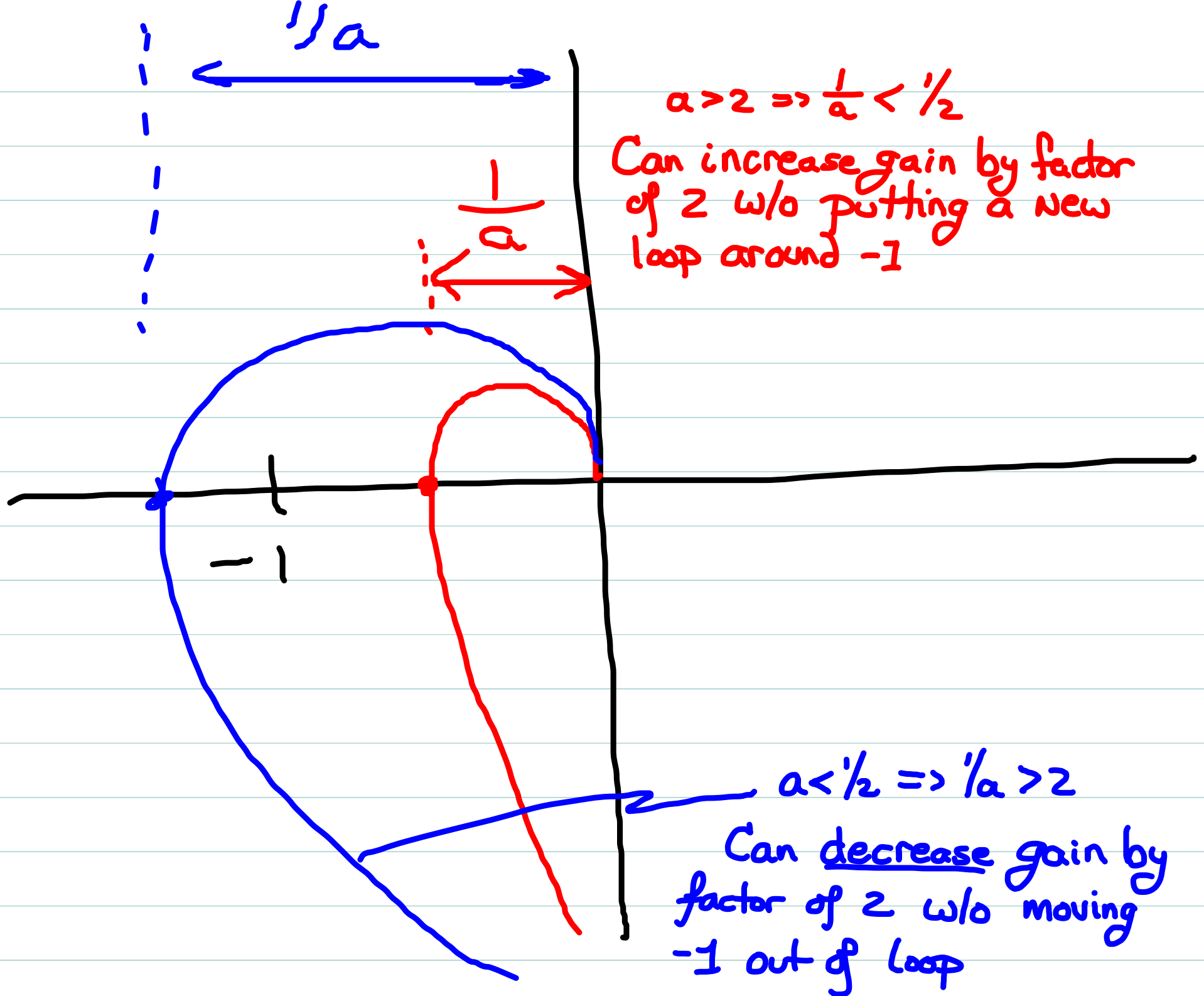
"Perturbations" to Nyquist analysis: how much can polar plot of $L(j\omega)$ be changed without changing the number of -1 encirclements.

Simple measures:

(1) gain margin: Measures tolerance to pure gain uncertainty

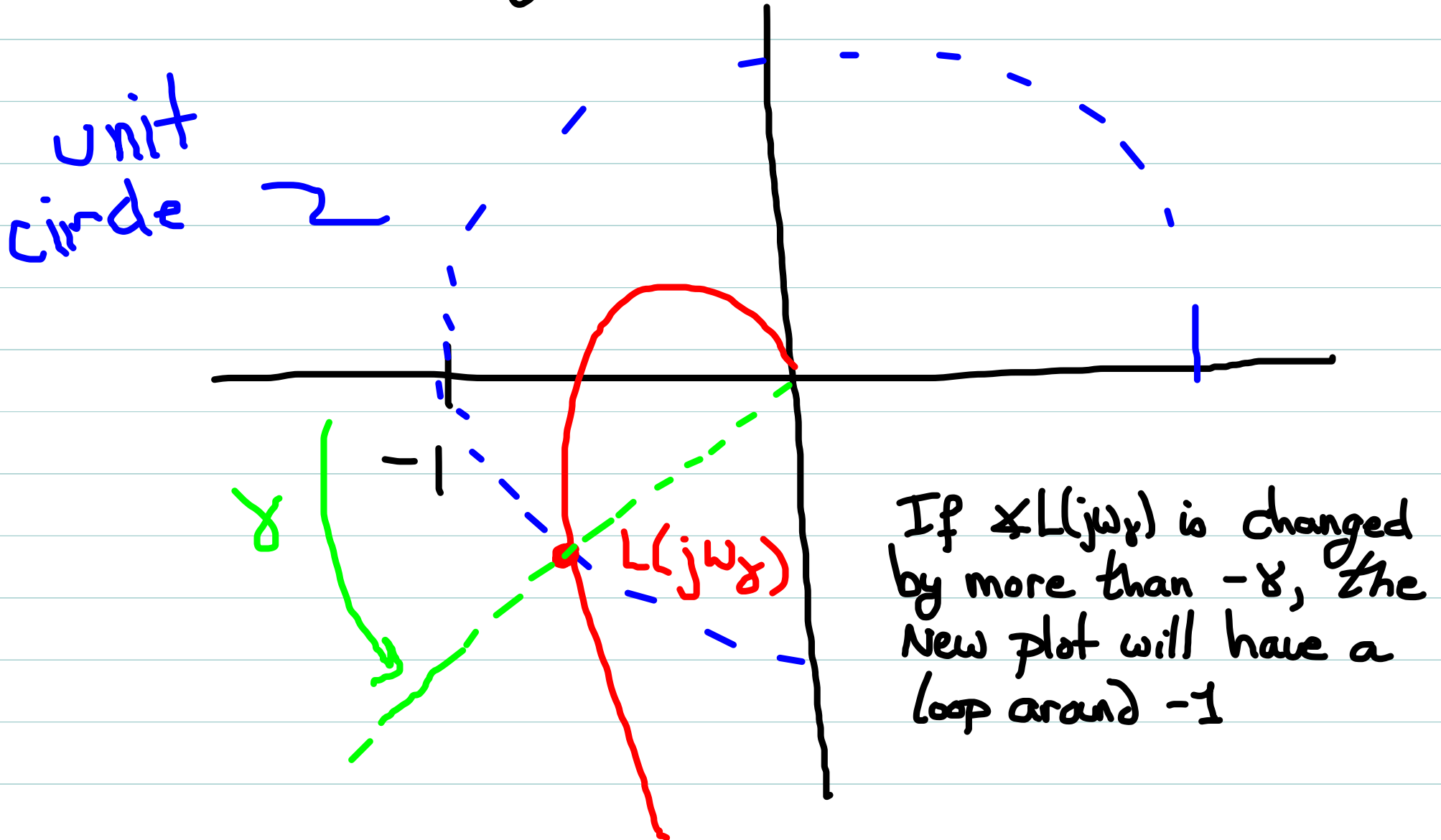
Common requirement: $|a|_{dB} \geq 6 \Rightarrow a \geq 2$ or $a \leq 1/2$

\Rightarrow Plant gain could be a factor of 2 larger or smaller and -1 encirclements will not change.



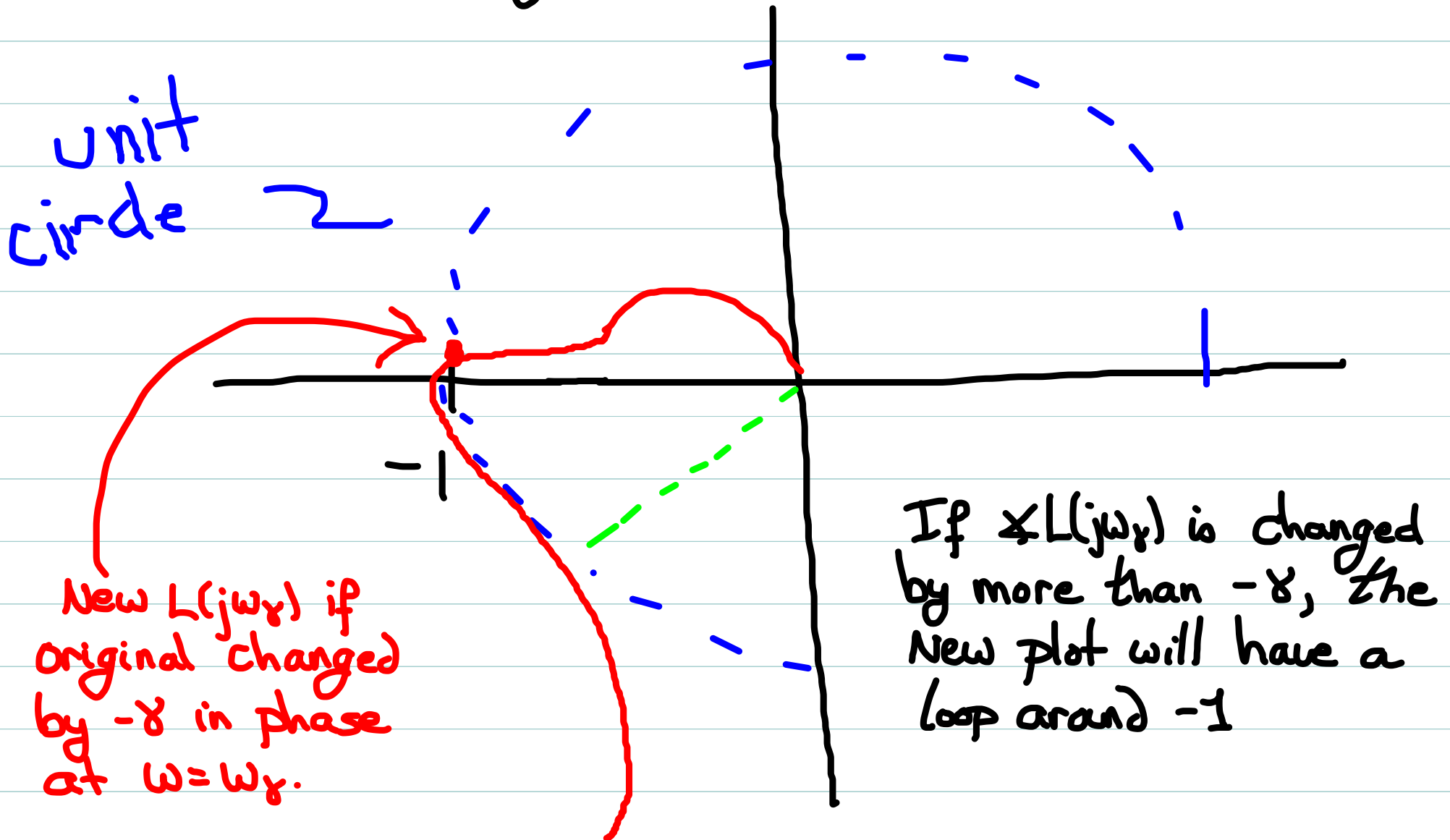
Simple Robustness measure #2: phase margin, δ

Measures pure Phase uncertainty tolerable before -1 encirclements change



Simple Robustness Measure #2: phase margin, γ

Measures pure phase uncertainty tolerable before -1 encirclements change



Physical Sources of Pure Phase Change

Phase margin is an important metric, so there must be an important, common physical mechanism which can introduce pure phase changes. What is it?

Time Delay

We've been modeling our controller as continuously evolving, just like the physical system being controlled.

But the controller is different than a physical system with dynamics governed by continuous diff'l equations.

Models of these differences will create pure phase changes to $\angle(j\omega)$.

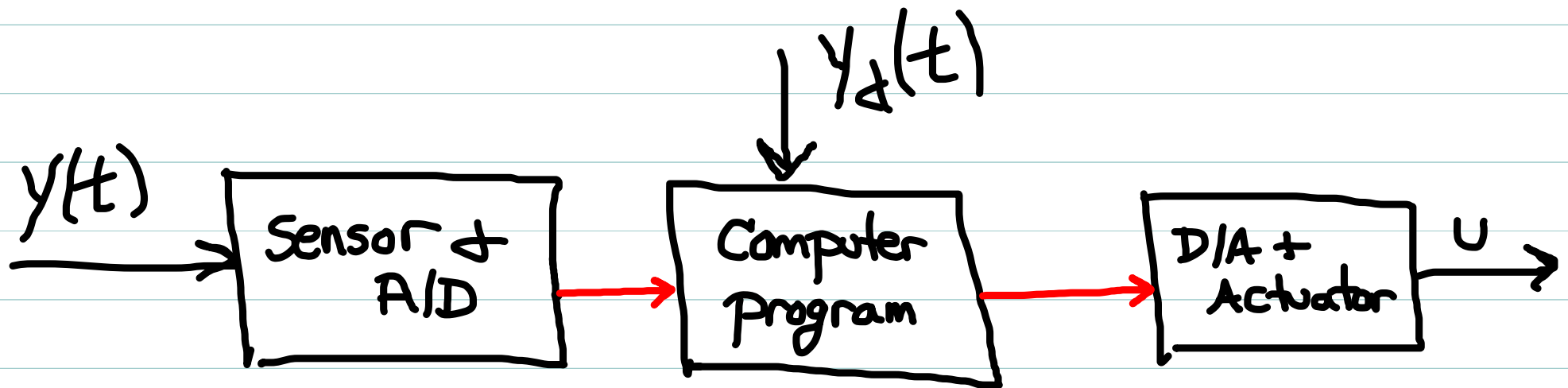
Time Delay

Three typical steps in controller implementation

- ① Measure output $y(t)$, and input to computer
- ② Compute $u(t)$ via computer program
- ③ Output $u(t)$ from computer to physical actuator

Each of these steps requires nonzero amount of time!

- ① A/D conversion and transmission/read time
- ② Time to execute program
- ③ D/A conversion and transmission time



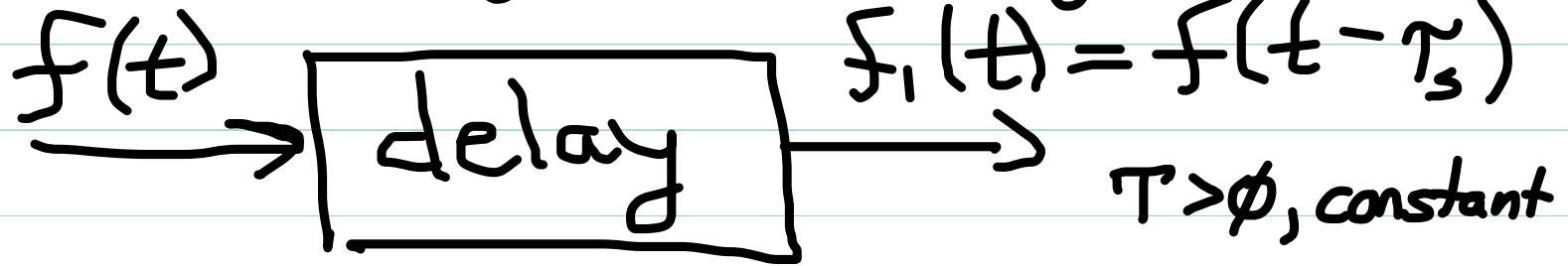
Each block, and each red arrow, requires nonzero time to operate. Call total required time τ_s

τ_s may be small (msec), but is always > 0 !

The implication is that the $u(t)$ which actually gets applied to the plant depends on the measurement taken τ_s seconds ago, i.e. $y(t - \tau_s)$

We haven't modeled this!

Laplace analysis of ideal delay



By def'n:

$$F_1(s) = \int_{0^-}^{\infty} f_1(t) e^{-st} dt = \int_{0^-}^{\infty} f(t - \tau_s) e^{-st} dt$$

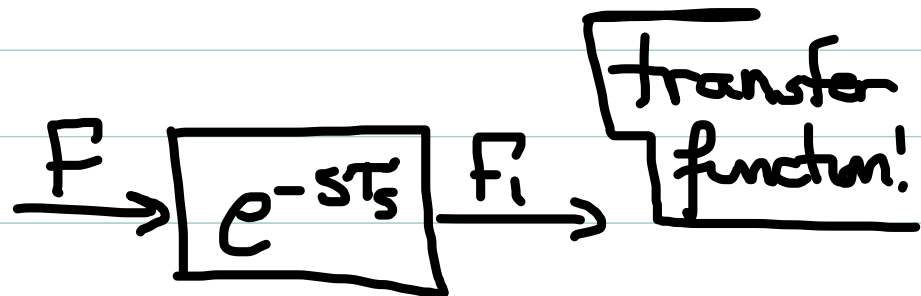
$$\text{let } \sigma = t - \tau_s, \text{ so } d\sigma = dt$$

$$F_1(s) = \int_{-\tau_s}^{\infty} f(\sigma) e^{-s(\sigma + \tau_s)} d\sigma = e^{-s\tau_s} \int_{0^-}^{\infty} f(\sigma) e^{-s\sigma} d\sigma$$

Since τ_s is constant, and Laplace assumes $f(t) = 0$ for $t < 0$

Thus:

$$F_1(s) = e^{-s\tau_s} F(s)$$



So really:

$$L(s) = e^{-s\tau_s} [G(s) H(s)]$$

transfer function of delay

$L_o(s)$: "ideal" (no delay) open-loop TF.

Now, $e^{-s\tau_s}$ is difficult to deal with in standard TF manipulations, because it is not rational. Cannot be described with a finite number of poles and zeros.

It's impact on freq. Domain properties of $L(j\omega)$ are easy to determine, however.

$$|L(j\omega)| = |e^{-j\omega\tau_s}| |L_o(j\omega)|$$

$$\angle L(j\omega) = \angle e^{-j\omega\tau_s} + \angle L_o(j\omega)$$

↳ what are these?

Recall for complex number in polar form:

$$z = r e^{j\theta} \Leftrightarrow |z| = r, \angle z = \theta$$

$$e^{-j\omega\tau_s} = 1 \cdot e^{j(-\omega\tau_s)} \Rightarrow r = 1, \theta = -\omega\tau_s$$

so $|e^{-j\omega\tau_s}| = 1$ for all ω , and

$$\angle e^{-j\omega\tau_s} = -\omega\tau_s \text{ for all } \omega$$

Hence:

$$|L(j\omega)| = |L_0(j\omega)| \text{ (unaffected by delay)}$$

$$\angle L(j\omega) = \angle L_0(j\omega) - \omega\tau_s$$

Effect of delay is pure phase change in $L(j\omega)$!

Delay thus acts to reduce phase margin:

$$\gamma = 180^\circ + \angle L(j\omega) = \underbrace{180^\circ + \angle L_0(j\omega_x)}_{\gamma_0: \text{expected Phase margin w/o delay}} - \underbrace{\omega_x T_s}_{\text{reduction in actual phase margin due to delay}}$$

γ_0 : expected
Phase margin
w/o delay

reduction in
actual phase
margin due to
delay

i.e.

$$\gamma = \gamma_0 - \omega_x T_s$$

Note: ω_x in rad/sec, T_s in sec $\Rightarrow \omega_x T_s$ in rad

γ_0, γ expressed in deg, so must convert $\omega_x T_s$ to deg here

Recall we typically need $\gamma > 0^\circ$ for Nyquist to show Stability

$$\Rightarrow \gamma_0 - \omega_x T_s > 0 \quad \text{or} \quad T_s < \frac{\gamma_0(\text{rad})}{\omega_x}$$

$$T_{\max} = \frac{\gamma_0(\text{rad})}{\omega_x}$$

is the maximum tolerable delay, or the "delay margin"

Now, typically T_s is fixed by available hardware.

Then $\gamma_0 - \omega_x T_s > 0$ Becomes a design constraint

\Rightarrow Cannot have $\omega_x T_s$ "too big" or it will be impossible to design $H(s)$ to provide necessary positive phase for γ_0 .

Typical guideline: Keep $\omega_x T_s < 0.1$ ($\omega_x < \frac{1}{10 T_s}$)

Then $\gamma = \gamma_0 - \omega_x T_s \geq \gamma_0 - 5.7^\circ$ ($0.1 \text{ rad} = 5.7^\circ$)

Can design $H(s)$ to provide additional $+5.7^\circ$ of phase margin in γ_0 to offset (or, just tolerate the small reduction)

\Rightarrow Note this constrains ω_x in a manner which works against guideline for good performance (big ω_x)

\Rightarrow Sample rate fundamentally restricts performance!

Gain and Phase margins are measures of robustness

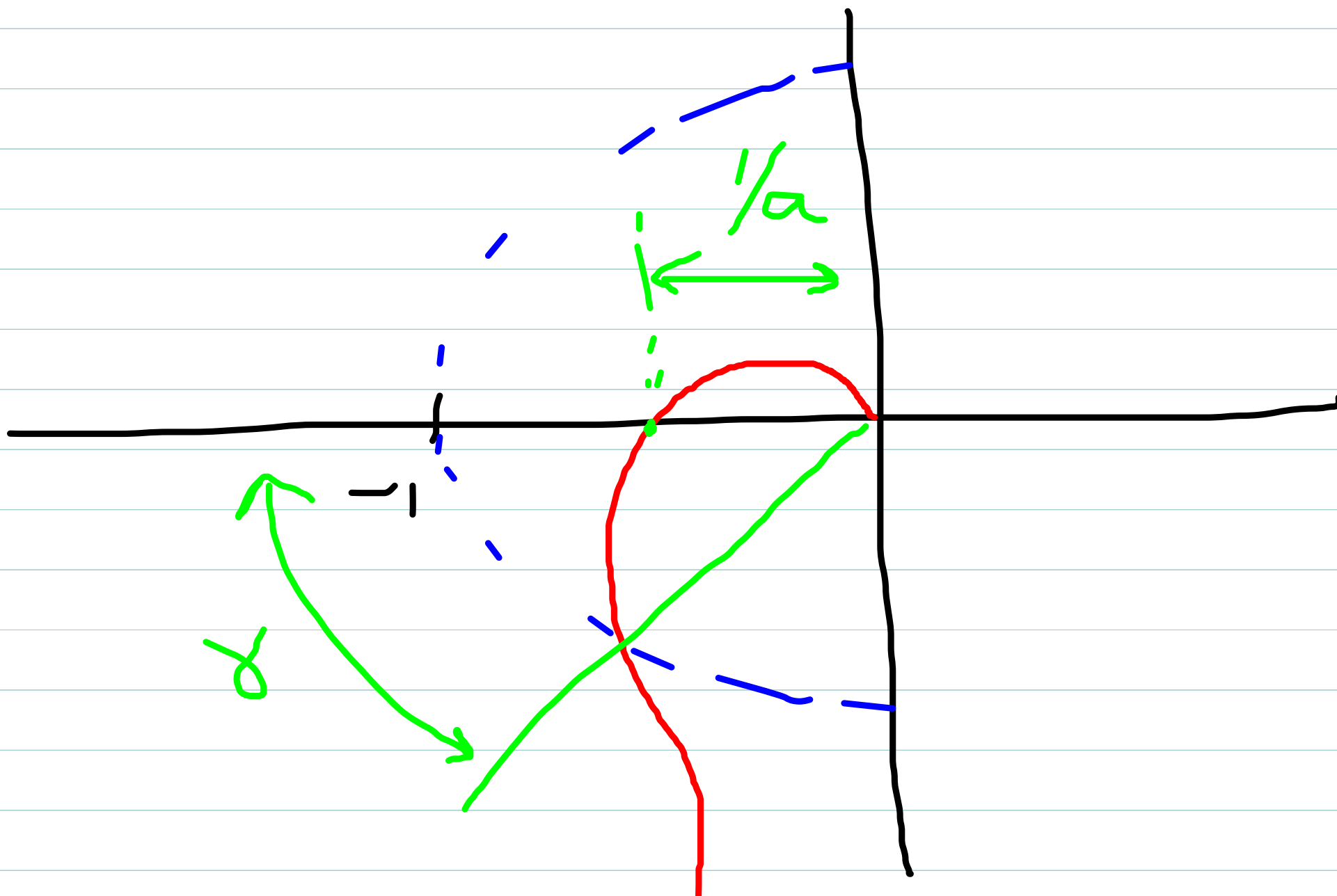
They quantify how close Nyquist diagram comes to -1 in two simple senses

Very common and popular since each corresponds to a physical source of possible model inaccuracy

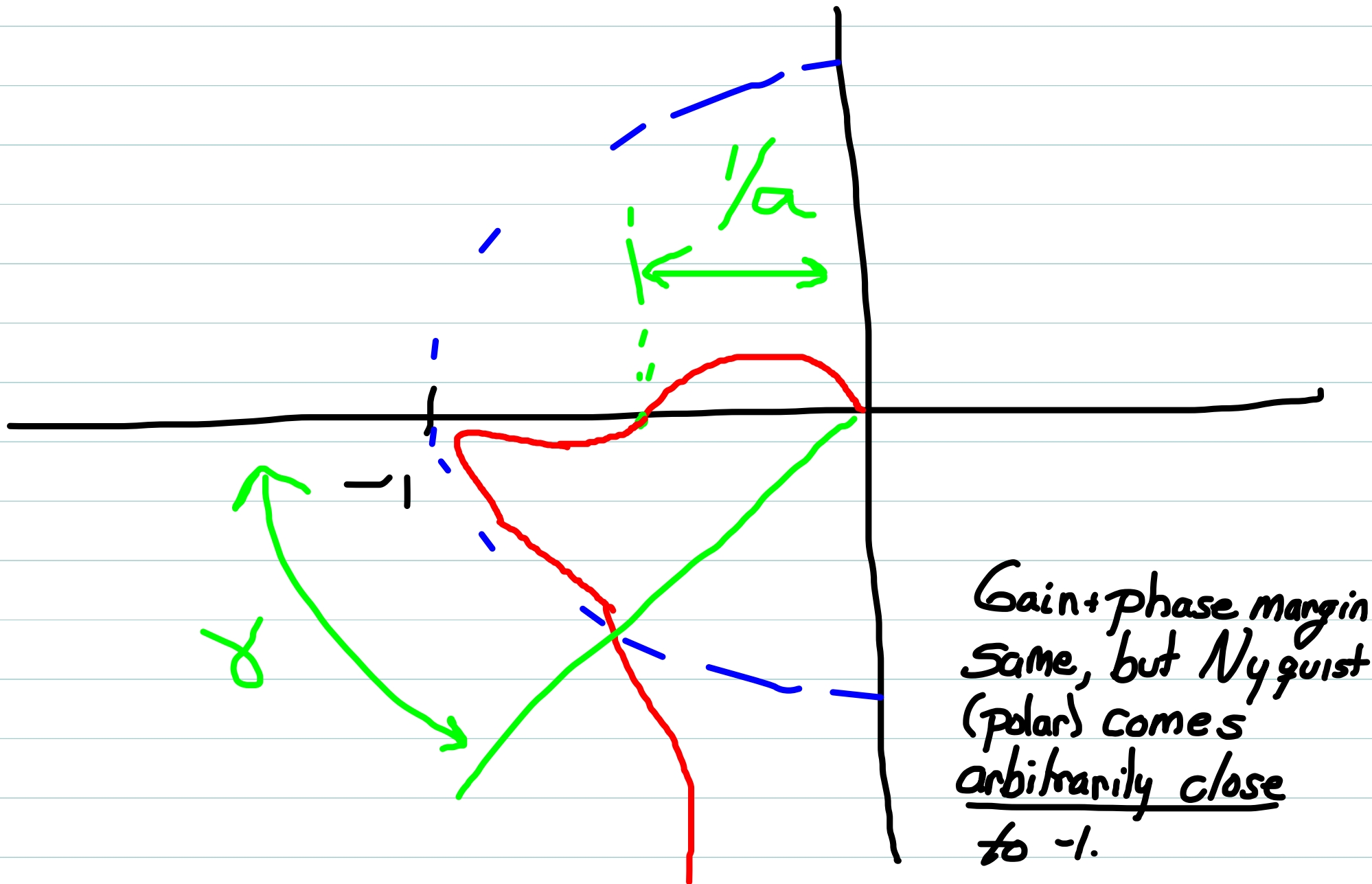
\Rightarrow gain margin: tolerance to variations in overall gain of plant

\Rightarrow phase margin: tolerance to time delays associated with computer implementation of controller

However: mathematically they are poor indicators of the tolerance of the Nyquist diagram to small perturbations



A typical case



But this is possible too!

Gain and phase margin are useful, intuitive measures but cannot capture the effects of Simultaneous gain and phase changes to $L(j\omega)$

Such changes would occur due to:

\Rightarrow mismatching of pole/zero locations in $G(s)$

\Rightarrow Incompleteness of $G(s)$ model, i.e. physics has additional dynamics which are too uncertain, or too difficult, to model accurately

\Rightarrow "real" $G(s)$ has additional poles/zeros which aren't present in model we use for design!

\Rightarrow Want a robustness test which can also handle these!