

Question 1

$$G(s) = \frac{7}{(s/5 + 1)^3}$$

$$u(t) = K e(t)$$

$$K > 0.$$

$$U(s) = H(s) F(s)$$

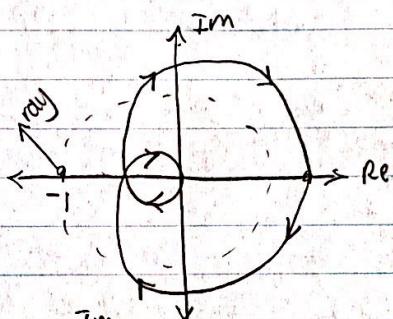
$$\therefore H(s) = K$$

$$L(s) = G(s)H(s) = \frac{7K}{(s/5 + 1)^3}$$

$$L_N(s) = \frac{7}{(s/5 + 1)^3} \quad \text{where } K = K_N = 1$$

a) Nyquist analysis

if $0 < K \leq 1$:

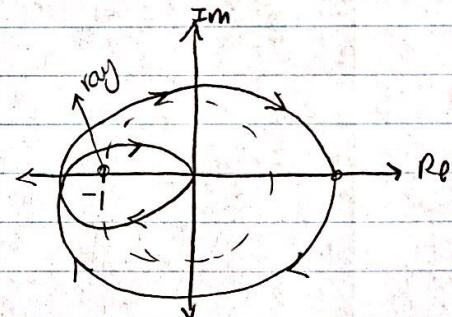
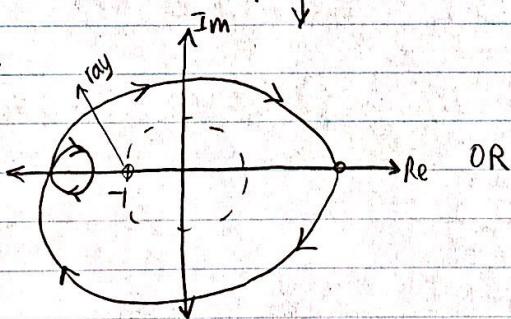


$$N_{CW}(L) = -P_R(L)$$

$$0 = 0 \checkmark$$

STABLE

if $K > 1$:



$$N_{CW}(L) = -P_R(L)$$

$$1 \neq 0$$

UNSTABLE

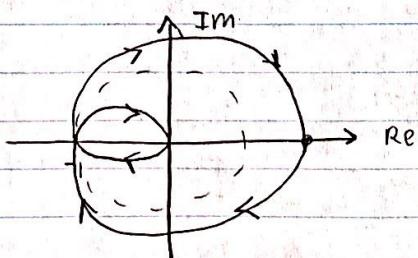
$$N_{CW}(L) = -P_R(L)$$

$$2 \neq 0$$

UNSTABLE

determine K_{max} for stability:

if $K = K_{max}$:



Here $L(j\omega)$ passes through the -1 point.

$\left\{ \begin{array}{l} \text{If } K > K_{max}, \text{ unstable} \\ \text{If } K \leq K_{max}, \text{ stable.} \end{array} \right.$

$$L(s) = \frac{7K_{max}}{(s/5 + 1)^3} \quad @ \omega_a = 8.66 \text{ rad/s}, \|L(s)\| = 1.$$

(dB = 1.14 dB)

$$\|L(s)\| = 1 = \left\| \frac{7K_{max}}{\left(\frac{8.66}{5} + 1\right)^3} \right\| \Rightarrow K_{max} = 1.14$$

$\rightarrow K_{max}$ can also be found by realizing if the $L(j\omega)$ graph is moved upward by 1.14 dB, $L(j\omega)$ would pass through -1.

$$\frac{K_{max}}{K_N} = 10^{\frac{(1.14/20)}{}} \quad \text{where } K_N = 1.$$

$$K_{max} = 1.14$$

b) $\gamma = 55^\circ$

$$\gamma = 180 + \angle L(s) = 55^\circ \Rightarrow \angle L(s) = -125^\circ \text{ when } \|L(s)\| = 1.$$

$$@ \angle L_N(s) = -125^\circ, \|L_N(s)\| = 9.28 \text{ dB}$$

\rightarrow We need to lower the $L(j\omega)$ graph by 9.28 dB to achieve $\gamma = 55^\circ$.

$$\frac{K_{new}}{K_N} = 10^{\frac{(-9.28/20)}{}}$$

$$K_{new} = 0.3436$$

$$T(s) = \frac{L(s)}{1 + L(s)} \quad L(s) = \frac{7(0.3436)}{(s/5 + 1)^3} = \frac{7(0.3436)5^3}{(s+5)^3} = \frac{300.61}{(s+5)^3}$$

$$T(s) = \frac{n(s)}{n(s) + d(s)} = \frac{300.61}{s^3 + 15s^2 + 75s + 125 + 300.61}$$

$$s^3 + 15s^2 + 75s + 125 + 300.61 = 0$$

$$\text{CL poles} = \begin{cases} -11.69 \\ -1.65 \pm 5.80j \end{cases}$$

\leftarrow Closer to Im axis, so $T(s)$ heavily influenced by this pole

expected characteristics:

$$t_s = \frac{4}{|T'|} = \frac{4}{|-1.65|} = 2.42$$

$$M_p = \exp\left(\frac{\pi T}{\omega_d}\right) = \exp\left(\frac{-1.65\pi}{5.80}\right) = 0.4091 = 40.91\%$$

$$y_{ss} = T(0) = 300.6 / 425.6 = 0.7063$$

exact characteristics from step response of $T(s)$:

$$t_s = 2.36$$

$$M_p = 35.5\%$$

$$y_{ss} = 0.706$$

Tabulated values:

PARAMETER	EXPECTED	EXACT
t_s	2.42	2.36
M_p	40.91%	35.5%
y_{ss}	0.7063	0.706

There may be discrepancies in the analysis b/c P_1 was ignored, and in reality, P_2 doesn't fit the definition of a dominant pole.

Also since $P_1 \neq 10P_2$, this effect is seen on the step response.

c) U_{max} and U_{ss} .

$$Y_d(s) \rightarrow [R(s)] \rightarrow U(s) \quad R(s) = \frac{H(s)}{1+L(s)}$$

From the Step response of $R(s)$:

$U_{max} = 0.344$
$U_{ss} = 0.101$

d) e_{ss}

$$Y_d(s) \rightarrow [S(s)] \rightarrow E(s) \quad S(s) = \frac{1}{1+L(s)}$$

From the Step response of $S(s)$:

$e_{ss} = 0.294$

now $y_d(t) = A \sin(\omega t)$. want $|ess(t)| \leq A/2$. Find ω range:

$$|ess| \leq A |S(j\omega)|$$

$$\left| \frac{A}{2} \right| \leq A |S(j\omega)|$$

$$\frac{1}{2} \leq |S(j\omega)|$$

$$-6.02 \text{ dB} = \frac{1}{2}$$

→ on bode of $S(s)$, find where $|S(j\omega)| = -6.02 \text{ dB}$. This occurs

at $\omega = 2.68$.

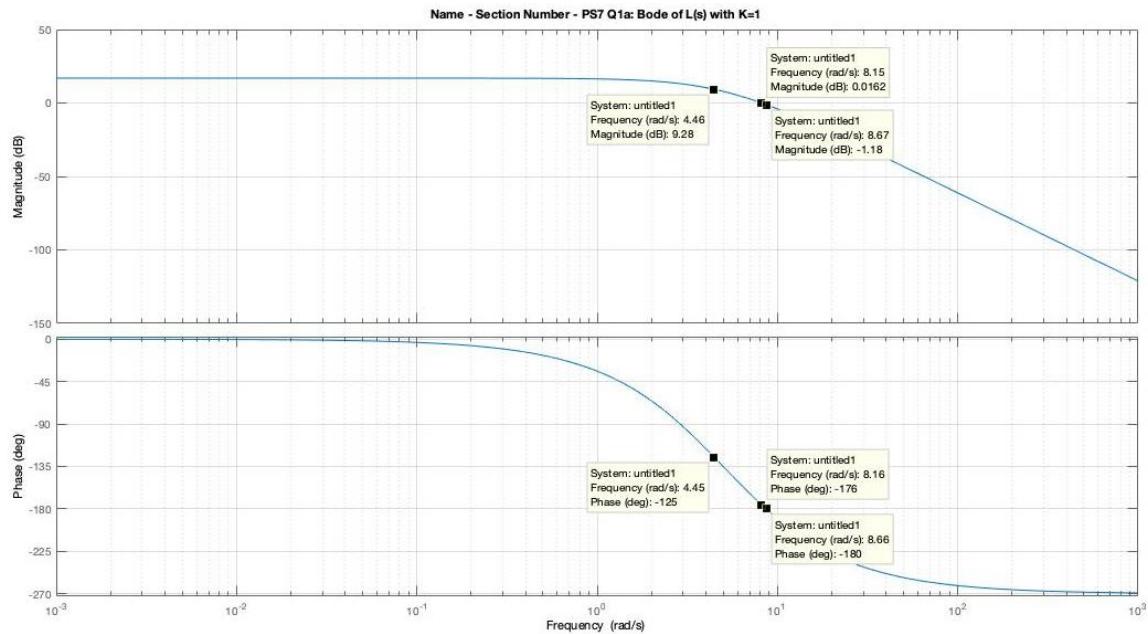
$$\therefore \boxed{\omega = [0, 2.68) \text{ rad/s}}$$

```

% ENAE 432, Spring 2019
% TA Solutions
% PS7, Question 1

% Part A
s = tf('s');
w = logspace(-3,3,250000);
G = 7/(s/5+1)^3;
K = 1;
figure(1)
bode(K*G,w); grid on;
title('Name - Section Number - PS7 Q1a: Bode of L(s) with K=1');
[Gm_db, Pm_deg, Wcg, Wcp] = margin(K*G)
Kmax = 10^(Gm_db/20)

```



$$Gm_db =$$

$$1.1430$$

$$Pm_deg =$$

$$4.5548$$

$$Wcg =$$

$$8.6608$$

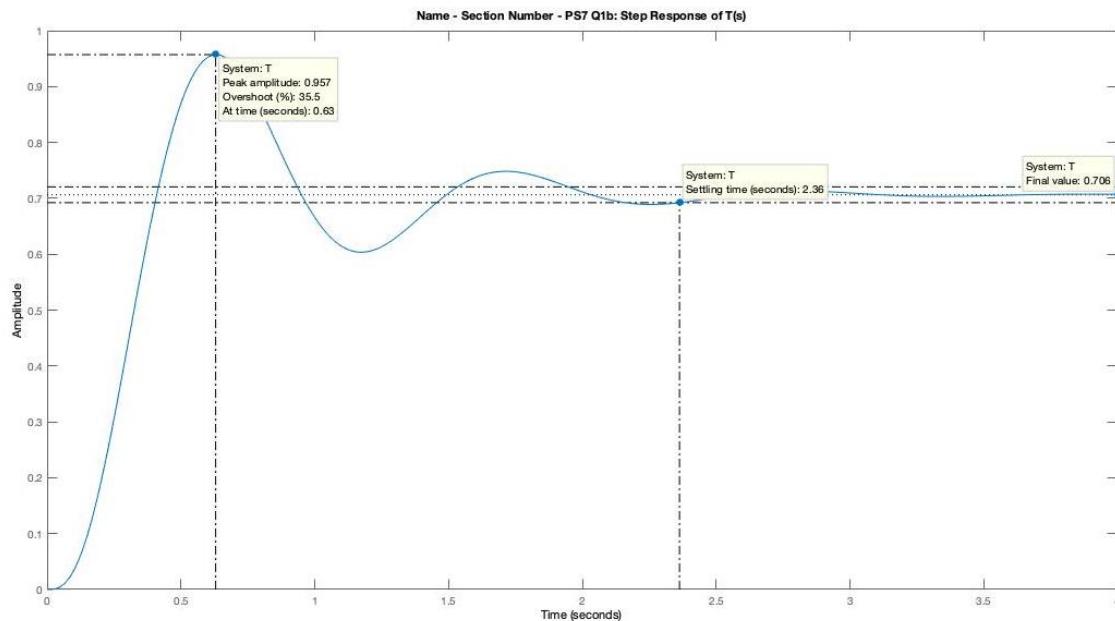
$W_{CP} =$

8.1534

$K_{max} =$

1.1406

```
% Part B
Gm_db_b = 9.28; % value of shift needed (obtained from Bode
diagram)
Knew = 10^(-Gm_db_b/20)
L = Knew*G;
T = feedback(L,1)
CLpoles = pole(T)
step(T); title('Name - Section Number - PS7 Q1b: Step Response
of T(s)');
```



$K_{max} =$

1.1406

Knew =

0.3436

T =

$$\frac{300.6}{s^3 + 15 s^2 + 75 s + 425.6}$$

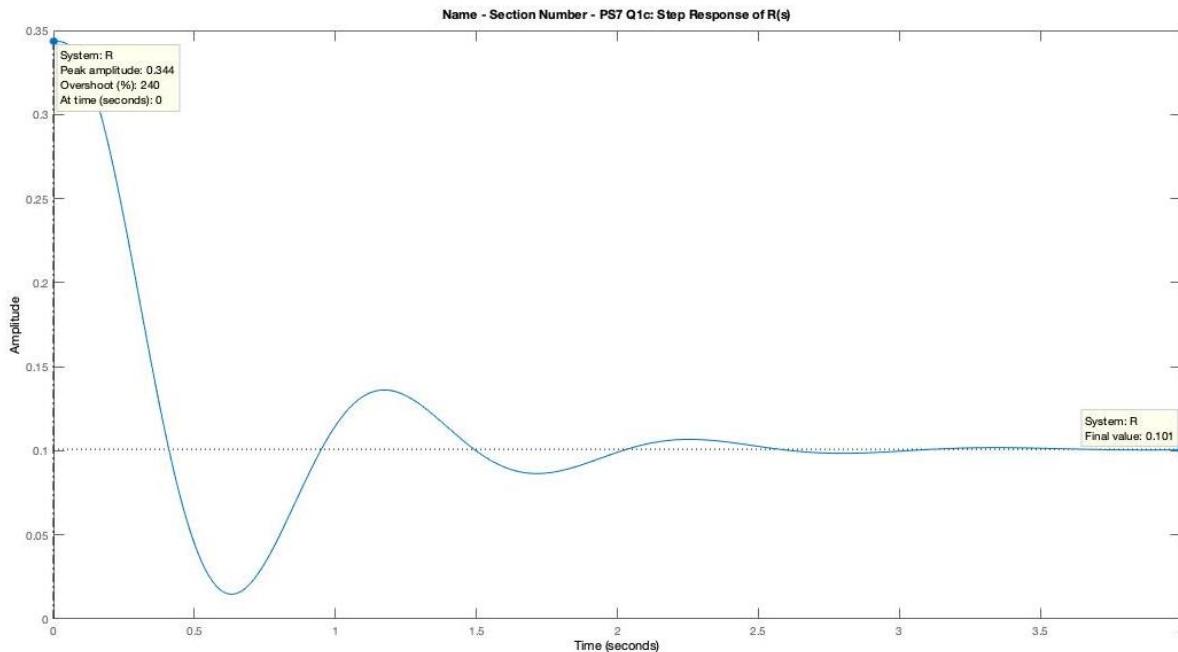
Continuous-time transfer function.

CLpoles =

-11.6989 + 0.0000i
-1.6506 + 5.8014i
-1.6506 - 5.8014i

% Part C

```
H = Knew;  
R = H/(1+L);  
step(R); title('Name - Section Number - PS7 Q1c: Step Response  
of R(s)');
```

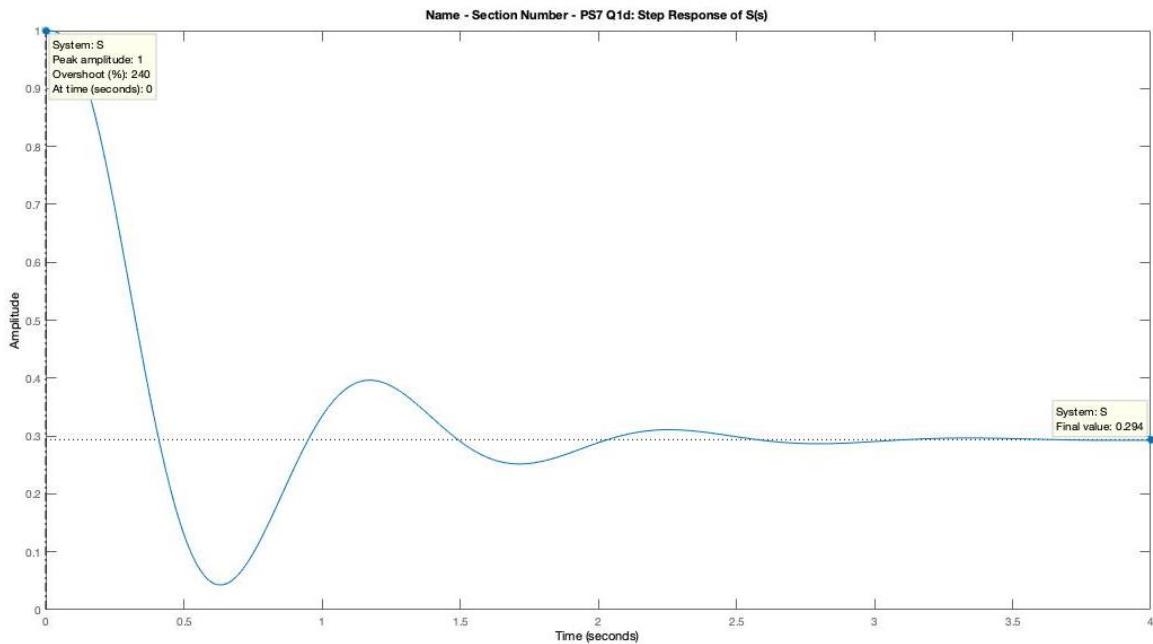


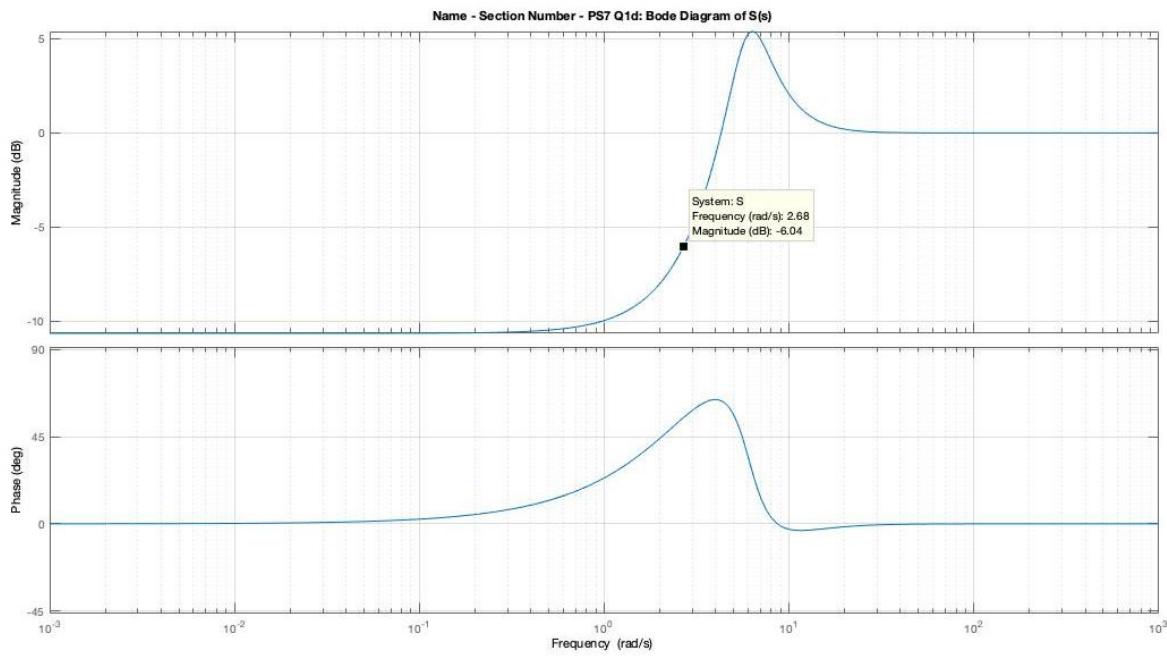
R =

$$\frac{0.3436 s^3 + 5.153 s^2 + 25.77 s + 42.94}{s^3 + 15 s^2 + 75 s + 425.6}$$

Continuous-time transfer function.

```
% Part D
S = feedback(1,L)
step(S); title('Name - Section Number - PS7 Q1d: Step Response
of S(s)');
bode(S,w); grid on;
title('Name - Section Number - PS7 Q1d: Bode Diagram of S(s)');
```





$S =$

$$\frac{s^3 + 15 s^2 + 75 s + 125}{s^3 + 15 s^2 + 75 s + 425.6}$$

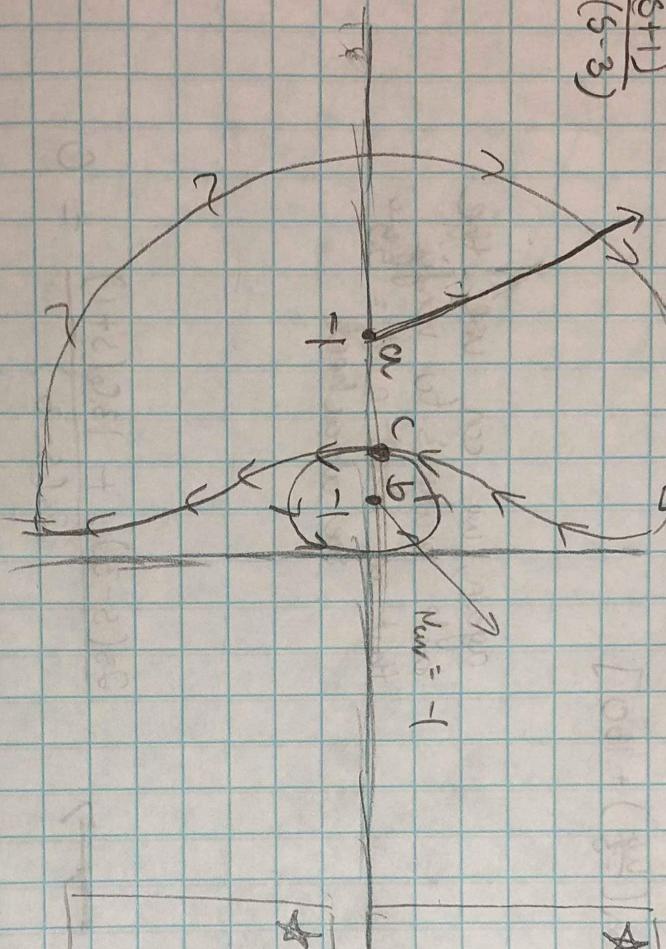
Continuous-time transfer function.

PST Question #2 Solutions

* acceptable to use euler functions

Part A)

$$G(s) = \frac{s+1}{2(s-3)} \quad H(s) = \frac{K}{s} \quad K > 0$$



* for small K we have
at point c

$$Nyq = 1$$

$$P_R(\omega) = 1$$

Unstable

* for large K we have $+1$ at point b

$$Nyq = -1$$

$$Nyq = -P_R(\omega)$$

$$-1 = -1 \checkmark$$

Stable

- From this analysis, we can find the nominal K -value by finding the K -value that puts the -1 point at point C in the Nyquist diagram
- Observe that the condition that satisfies this is ...

$W_R = W_I$ (i.e. Nyquist/polar crosses the unit circle on the negative real axis at the same time)

- which implies that $\gamma = 0$ and $|L(j\omega)| = 1$

$$C_1 = K_C + X_1(j\omega_L)$$

90°

- From this analysis, we can find the maximal k -value by finding the k -value that puts the -1 point at point L in the Nyquist diagram
- Observe that the condition that satisfies this is ...

$$w_x = w_y \quad (\text{i.e. Nyquist/polar crosses the unit circle and the negative real axis at the same time})$$

- which implies that $\gamma = 0$ and $|L(j\omega)| = 1$

$$C = 180 + \gamma L(j\omega)$$

$$\gamma = 180 + \cancel{4(\frac{\sqrt{3}}{2})} + \gamma(j\omega)^{-1} - \gamma(j\omega) - \gamma(j\omega - 3)$$

$$C = 90 + \arctan\left(\frac{w_x}{w_y}\right) - \left[\arctan\left(\frac{w_x}{w_y}\right) + 180^\circ\right]$$

$$\sqrt{3} = w_x = w_y$$

- Finally solve for k

let ω be j/κ

this comes from the fact that w_x are in the 2nd quadrant

$$\left| \frac{L(j\sqrt{3})}{L(j\sqrt{3})} \right| = 1$$

$$|k| |L(j\sqrt{3})| = 1$$

$$|k| = |L(j\sqrt{3})|^{-1}$$

* may use evalf function as well as abs() and angle() *

$$\boxed{|k_{nn}| = 6}$$

See attached Bode Plot of L that verifies this result

b) Find K for which L will have $\theta = 55^\circ$

$$55^\circ = 180 + \arg(j\omega_r)$$

$$1 = |L(j\omega_r)|$$

we use these two conditions to solve for K

$$\frac{55^\circ}{j\omega_r} = 90^\circ + \arctan\left(\frac{4\omega_r}{1}\right) - \left[\arctan\left(\frac{4\omega_r}{3}\right) + 180^\circ\right]$$

$$|L(j\omega_r)| = |K| |L_0(j\omega_r)| = 1$$

$$|K| = \frac{1}{|L_0(j\omega_r)|} = 1$$

$$K = 13.6$$

again, we can use the bode analysis to verify that we meet a $\theta = 55^\circ$
see attached

Definition of closed loop poles
solutions to

$$1 + L(s) = 0$$

$$1 + \frac{(3.6)(s+1)}{2s(s-3)} = 0$$



$$\frac{2s(s-3) + 13.6(s+1)}{2s(s-3)} = 0$$

$$2s(s-3) + (3.6s(s+1)) = 0$$

$$2s^2 - 6s + 13.6s + 13.6 = 0$$

solving this equation:

$$P_{1,2} \approx -1.9 \pm 1.78j$$

expected transient metrics

$$t_5 = \frac{4}{10} = 0.1s = t_5$$

$$k \cos \frac{\% \text{OS}}{\% \text{OOS}} = 10 = e^{\frac{(0.1s)/\omega_d}{100}}$$

$$10 = 3.59$$

Part B

$$t_s = \frac{4}{10} = [2.1s - t_s]$$

$$\%OS = M_p = e^{(0.8\pi/\omega) \times 100}$$

$$M_p = 3.59$$

$$[t_s = 2.55s]$$

$$[\% overshoot = 61.8\%]$$

* The estimates above are we strictly only have poles; however, $H(s)$ will have zeros present with the poles, which heavily affect overshoot. This is why there are discrepancies in t_s and M_p .

C) See attached MATLAB for plot $|U(t)|_{max} = .4t_0$ $|U(t)|_{ss} = .44$

D) Determine $e_{ss}(t)_{max}$

If $y_d(t) = 1(t)$ or step response

$$e_{ss}(t) = 1.5(t)$$

$$S = \frac{1}{1+1(s)} = \frac{s(s-3)}{2s(s-3)+13.6(s+1)}$$

$$S(0) = 0$$

$E_{ss}(t) = 0$
See step response, see attached step response to verify

↑ this must be solved analytically

Find the range of ω s.t $|e_{ss}(t)| < A/2$

$$\text{if } y_d(t) = A \sin(\omega t) \text{ or frequency response} \\ = A \cos(\omega t + \pi/2)$$

$$e_{ss}(t) = [B \cos(\omega t + \varphi)]$$

$$B = 1.5(\omega)/A$$

$$B = 1.5(\omega)/A$$

we want $|e_{ss}(t)| < \frac{A}{2}$
magnitude of $e_{ss}(t)$
which means

$$|e_{ss}(t)|_{max} \leq \frac{A}{2} \Rightarrow$$

$$B \leq \frac{A}{2}$$

d) continued

using

$$B = |S(y_{\text{new}})|A$$

$$B \leq \frac{A}{2}$$

$$\frac{1}{2} = |S(y_{\text{new}})| \approx 13.46$$

$$|S(y_{\text{new}})| = \frac{2|y_{\text{new}}| \cdot |y_{\text{new}} - c|}{|c - y_{\text{new}}| \cdot |y_{\text{new}}|}$$

$$= \frac{1}{2}$$

$$2|y_{\text{new}}| \cdot |y_{\text{new}} - c| = |c - y_{\text{new}}| \cdot |y_{\text{new}}$$

Solve for y_{new}

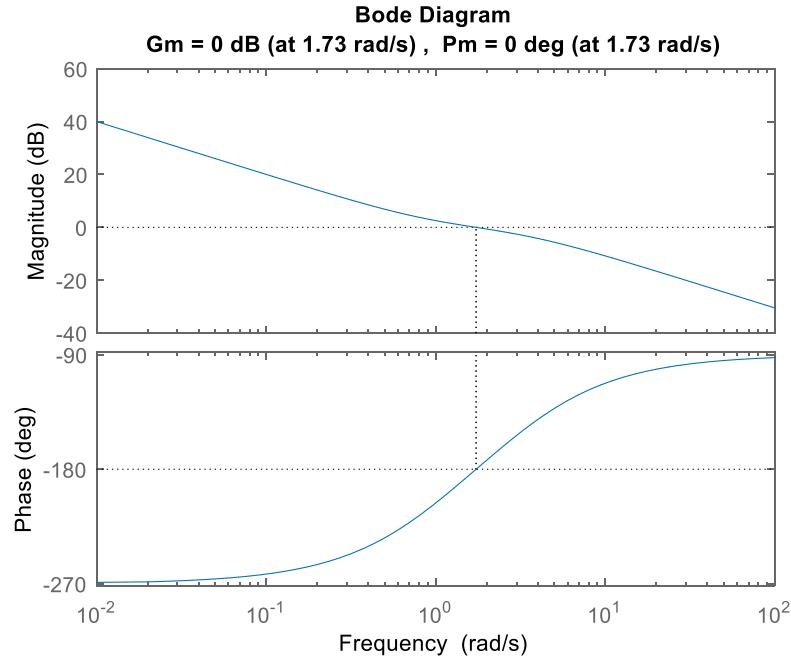
(both solutions is also acceptable)

$$W_{\text{new}} \propto \frac{1}{45.04^2}$$

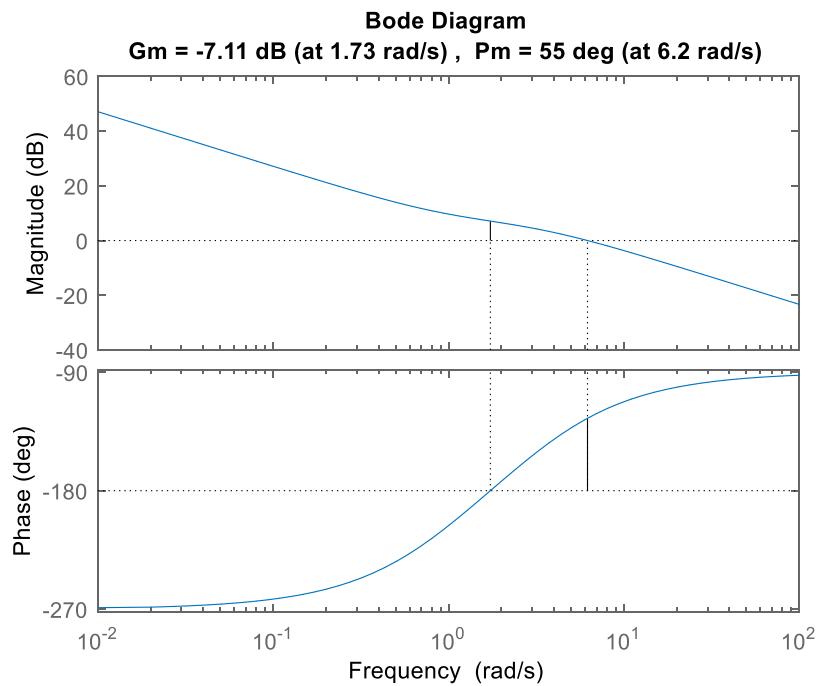
thus the value must be

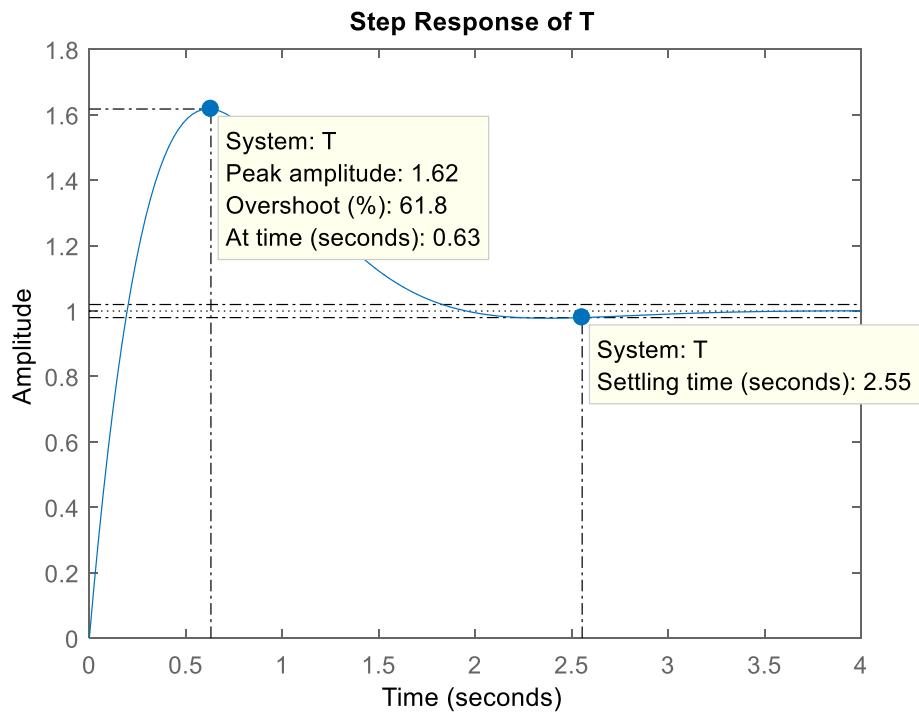
$$W \in [0, 45.04]$$

Part a

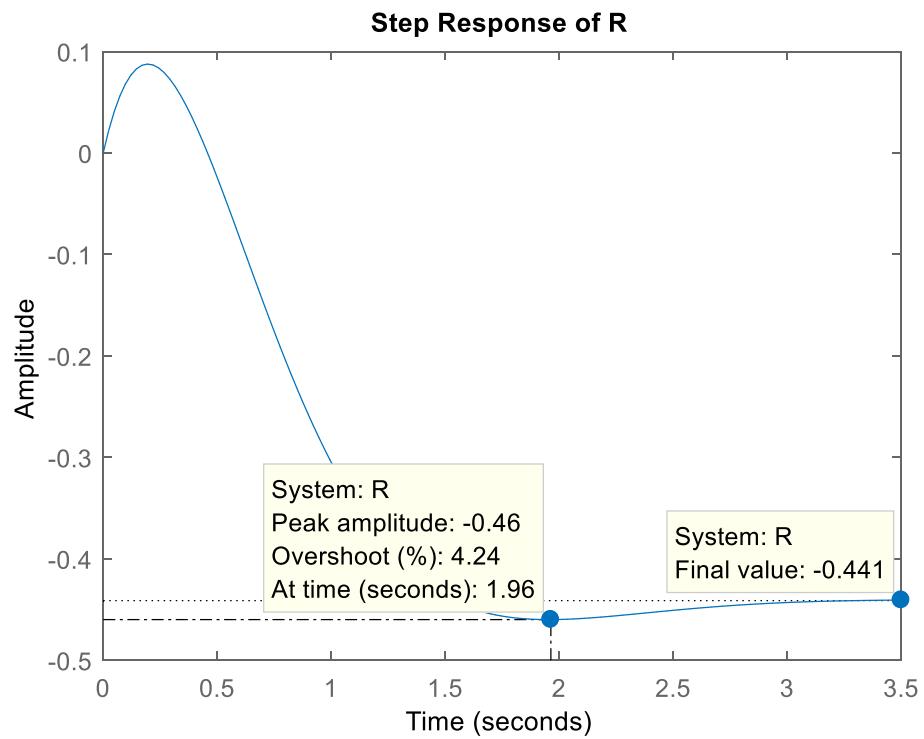


Part b

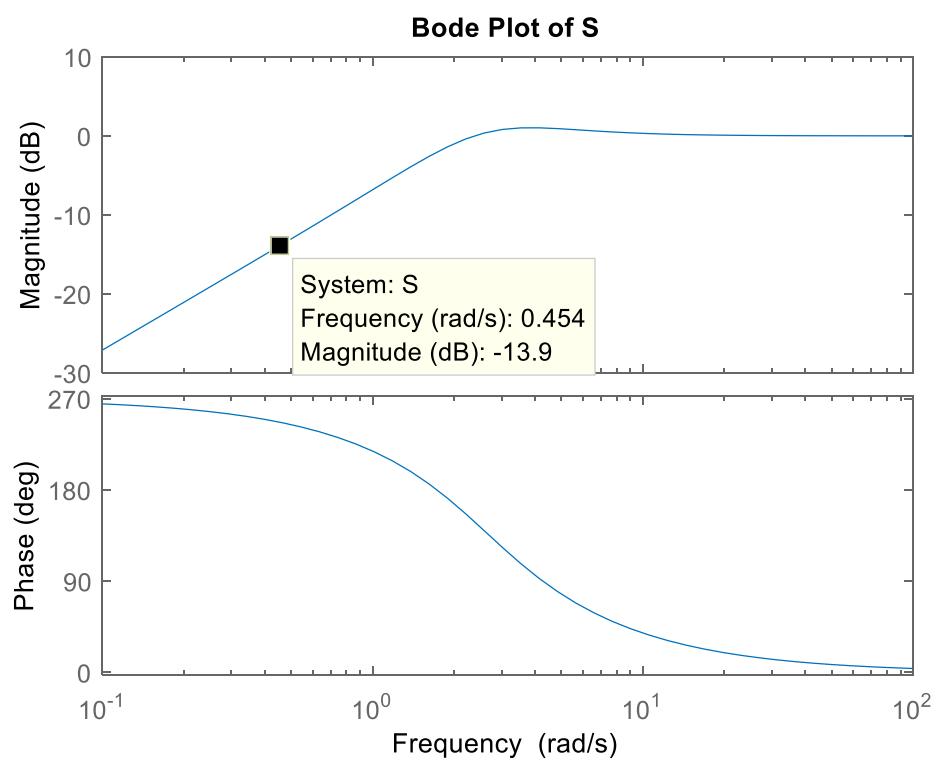
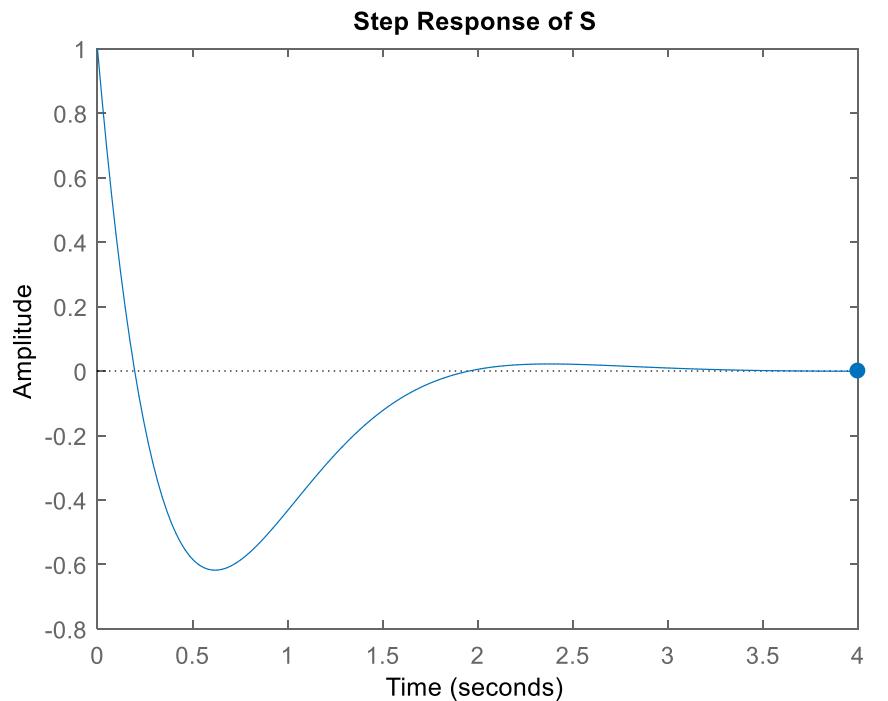




Part c



Part d

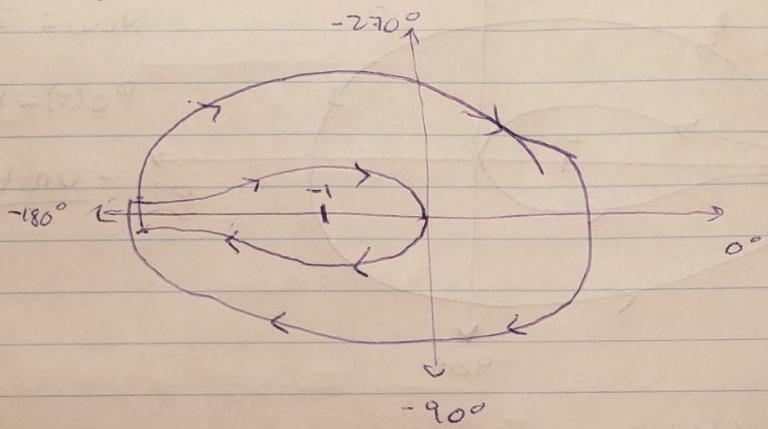


PS7 Q3 Solns

$$G(s) = \frac{12}{s^2(\tau s + 1)}, \tau > 0$$

a) Proportional control $H = K$ such that $L(s) = \frac{12K}{s^2(\tau s + 1)}$

- using Nyquist for $K > 0$:

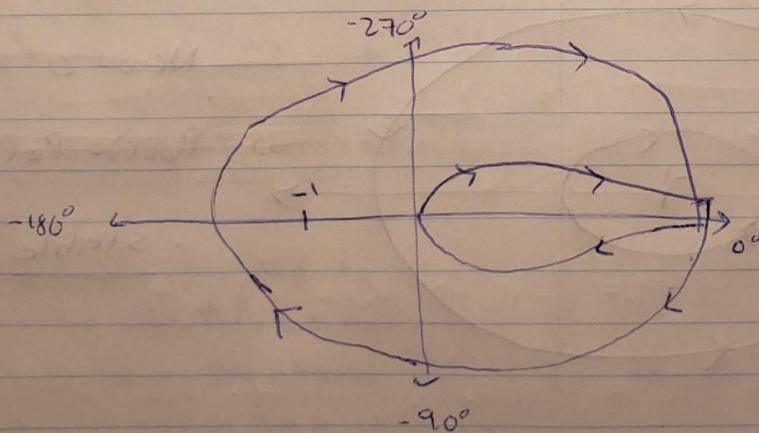


$$N_{cw} = 2$$

$$P_R(\tau) = P_R(L) + N_{cw} = 2$$

\Rightarrow unstable in $T(s)$

- using Nyquist for $K < 0$:



$$N_{cw} = 1$$

$$P_R(\tau) = P_R(L) + N_{cw} = 1$$

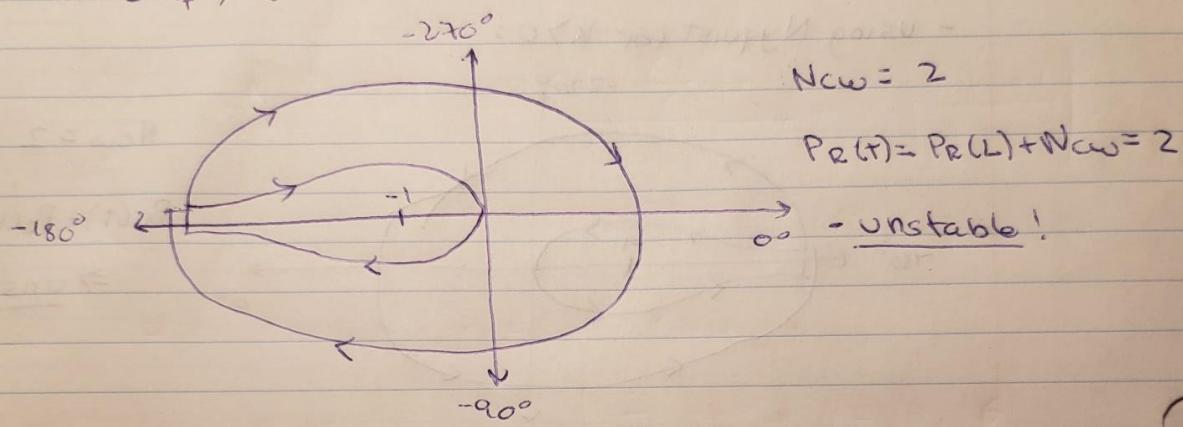
\Rightarrow unstable in $T(s)$

\therefore system cannot be stabilized w/ proportional control

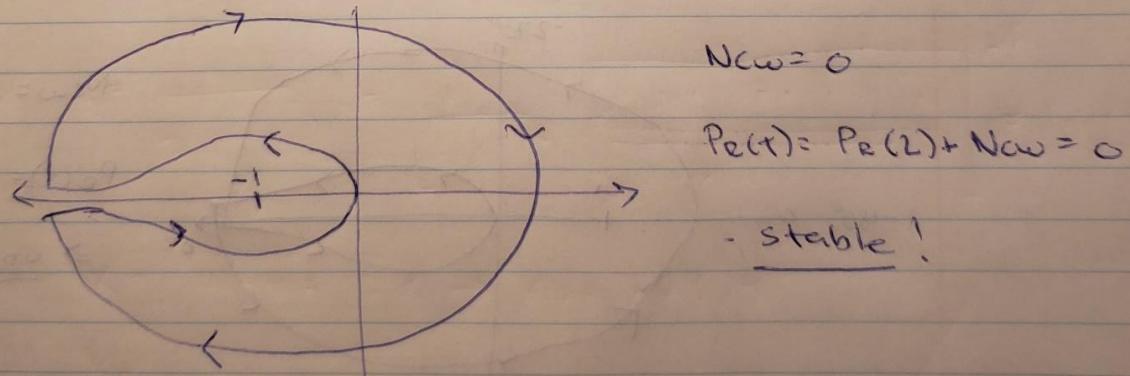
$$b) U = K_p e(t) + K_d \dot{e}(t) \rightarrow H(s) = K_d s + K_p$$

$$L(s) = \frac{12(K_d s + K_p)}{s^2(\tau s + 1)} = \frac{12 K_p \left(\frac{K_d}{K_p} s + 1 \right)}{s^2(\tau s + 1)}$$

- when $\tau K_p > K_d$



- when $\tau K_p < K_d$



c) $\zeta = 0.17$, $K_d = K_p$, such that

$$L(s) = \frac{12 K_p (s+1)}{s^2 (0.17 s + 1)}$$

from Bode, $\max(\angle L(j\omega)) = -135^\circ$ @ $\omega = 2.45 \text{ rad/s}$

- find K_p so that $|L(j(2.45))| = 1$

$$1 = \frac{12 K_p \sqrt{(2.45)^2 + 1}}{2.45^2 \sqrt{(0.17 \cdot 2.45)^2 + 1}} \rightarrow K_p = 205$$

If $L(s) = \frac{N(s)}{D(s)}$, CL poles are values which satisfy $N(s) + D(s) = 0$

$$\Rightarrow 12 K_p (s+1) + s^2 (0.17 s + 1) = 0$$

$P_1 = -2.3474$	CL poles are in LHP
$P_{2,3} = -1.7675 \pm 1.742j$	\rightarrow system stable

d) $M_p = e^{5\pi/\omega_d} = e^{-1.768\pi/1.742} \approx 4.12\%$ increased

overshoot comes from LHP zero | exact cos and ts attached

e) Increasing K_p to 10 K_p will cause phase margin increase to decrease to 22.4°
 the ω_d . This will make the overshoot increase and the settling time decrease. See attached MATLAB for reference. A faster response is achieved at the price of greater overshoot

when K_p increases by a factor of 10:

$\theta_{\text{new}} = 22.4^\circ$
$\omega_d > 11.4 \text{ rad/s}$
$\gamma_{OS} > 56.4$
$t_s = 1.48 \text{ s}$

f) Need $+10^\circ$ phase margin $\Rightarrow \gamma = 55^\circ$ for the system

$$L(s) = \frac{12K_p \left(\frac{K_d}{K_p} s + 1 \right)}{s^2 (\tau s + 1)}$$

we can make this "look" like a lead compensator problem by breaking $L(s)$ up so that

$$G'(s) = \frac{1}{s^2} \quad H'(s) = \frac{12K_p \left(\frac{K_d}{K_p} s + 1 \right)}{\tau s + 1} = k \left[\frac{\beta \tau s + 1}{\tau s + 1} \right]$$

where $\beta = \frac{K_d}{K_p} / \tau$, $k = 12K_p$

$$\angle G'(s) = -180^\circ + \omega, \text{ so } \phi_{req} = 55^\circ = \gamma_{new}$$

$$\phi_{req} = \phi_{max} = \sin^{-1} \left[\frac{\beta - 1}{\beta + 1} \right] = 55^\circ \Rightarrow \beta = 10.06$$

$$\text{Then } \omega_{max} = \frac{1}{\tau \sqrt{\beta}} = \frac{1}{0.17 \sqrt{10.06}} = 1.855 \text{ rad/s}$$

$$\text{and } K = \frac{1}{|L_0(j\omega_{max})|} = \left[\frac{1}{1.855^2} \cdot \frac{\sqrt{[(10.06 \cdot 0.17) + 1.855]^2 + 1}}{\sqrt{(0.17 + 1.855)^2 + 1}} \right]^{-1} = 1.0844$$

$$\rightarrow \text{so: } K = 1.0844 = 12K_p \Rightarrow K_p = 0.0904$$

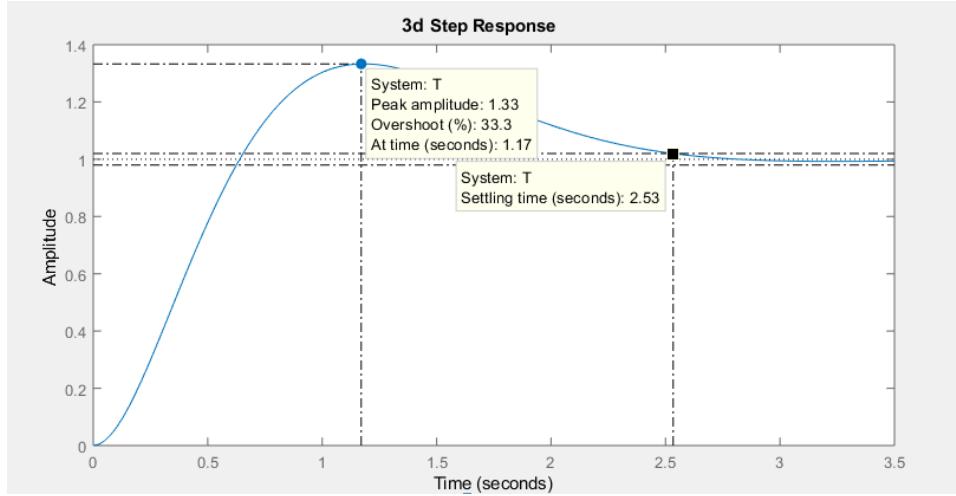
$$\rightarrow \text{and: } \frac{K_d}{K_p} = \beta \tau \Rightarrow K_d = K_p \beta \tau = 0.0904 \cdot 10.06 \cdot 0.17 = 0.1608$$

Thus: $K_p = 0.0904$, $K_d = 0.1608$, $\omega_d = 1.855 \text{ rad/s}$

γ_{OS} reduces to 22%, but T_s increases to 4.6 sec

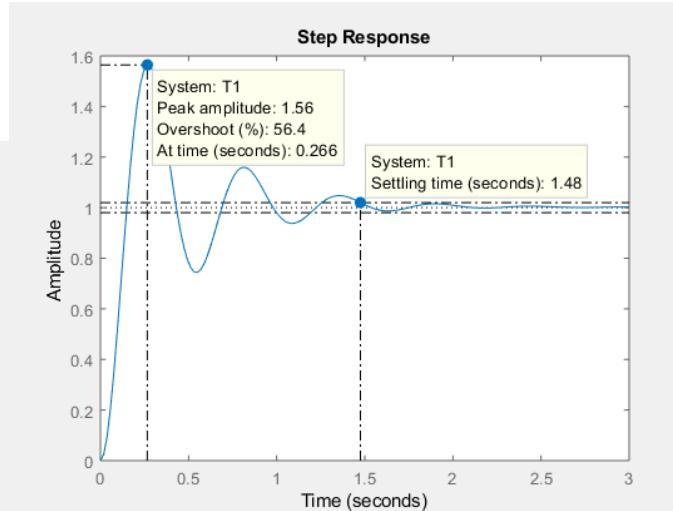
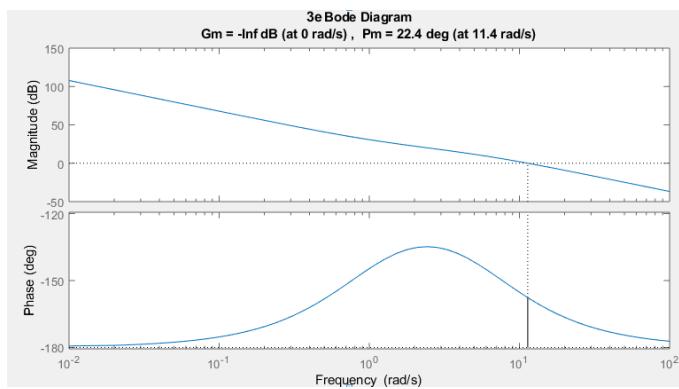
PS8Q3d

```
% part d
s = tf('s');
L = 12*.205*(s+1) / (s^2*(0.17*s+1));
T = L/(1+L);
step(T);
```



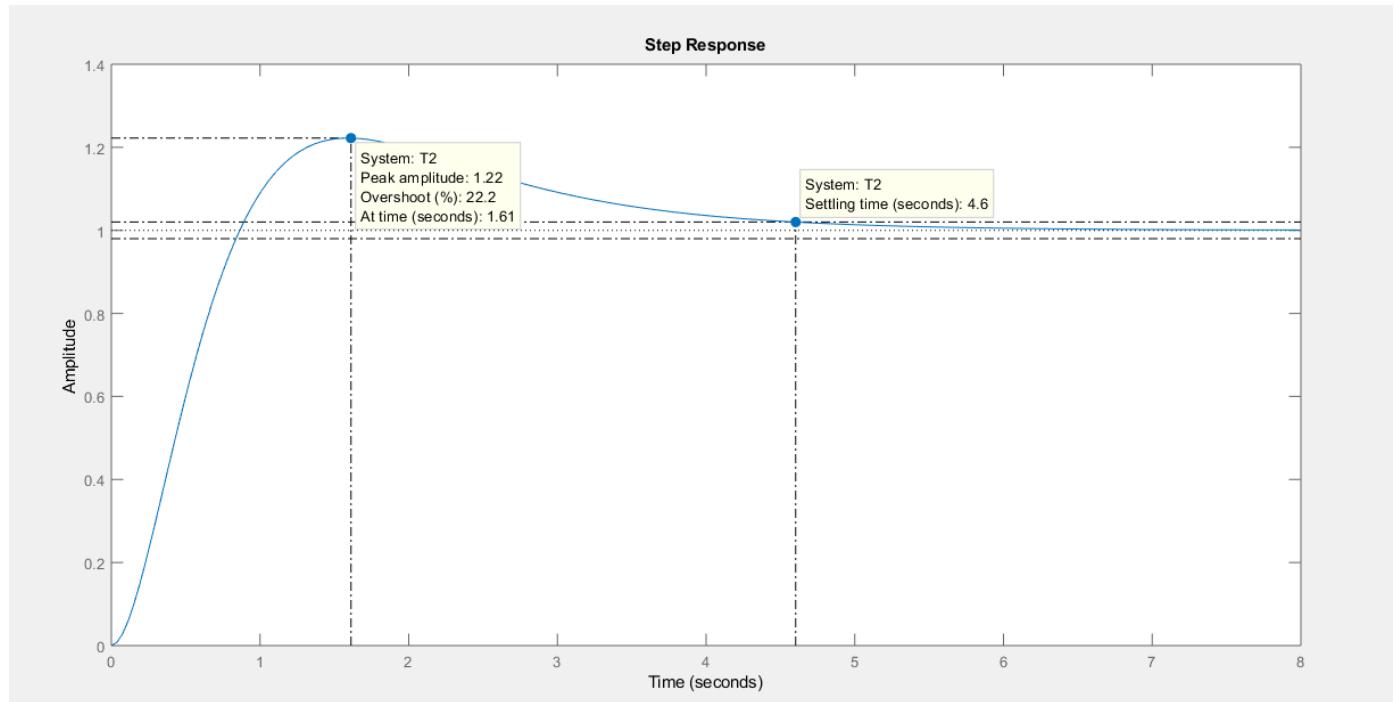
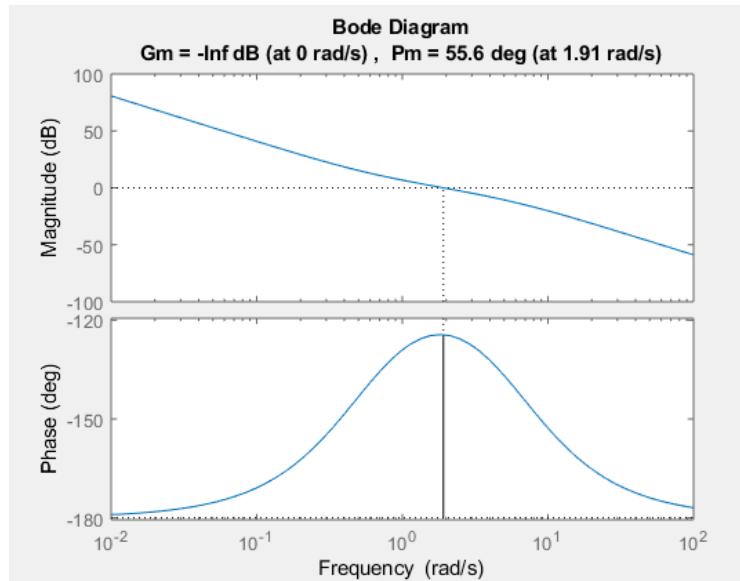
PS8Q3e

```
% part e
figure
margin(10*L);
figure
T1 = feedback(10*L,1);
step(T1)
```



PS8Q3f

```
% part f
figure
L = 12*.0904*((0.1608/0.0904)*s+1)/(s^2*(0.17*s+1));
margin(L)
figure
T2 = feedback(L,1);
step(T2);
```



PS7

Q4. FOR THE COMPENSATOR $H(s) = \frac{20(s+2)^3}{s(s+6)(s+12)}$

DETERMINE THE DERIVATIVE FREE IMPLEMENTATION EQUATIONS WHICH WOULD BE USED TO GENERATE $u(t)$ FROM $e(t)$ IN THE OPERATION OF THE FEEDBACK LOOP.

SOLUTION:

$$H(s) = 20 \left[\frac{(s+2)^3}{s(s+6)(s+12)} \right] = 20 \left[\frac{s^3 + 6s^2 + 12s + 8}{s^3 + 18s^2 + 72s} \right] = 20 H_0(s)$$

Dividing out, $s^3 + 18s^2 + 72s$

$$\begin{array}{r} 1 \\ \hline s^3 + 6s^2 + 12s + 8 \\ -s^3 - 18s^2 - 72s \\ \hline -12s^2 - 60s + 8 \end{array}$$

$$\therefore H_0(s) = 1 + \frac{8 - 60s - 12s^2}{s(s+6)(s+12)} = 1 + \frac{A_1}{s} + \frac{A_2}{s+6} + \frac{A_3}{s+12}$$

$$A_1 = [s \cdot H_0(s)]_{s=0} = \frac{8}{6(12)} = \frac{8}{72} = \frac{1}{9}$$

$$A_2 = [(s+6)H_0(s)]_{s=-6} = \frac{8 - 60(-6) - 12(36)}{-6(-6+12)} = \frac{16}{9}$$

$$A_3 = [(s+12)H_0(s)]_{s=-12} = \frac{8 - 60(-12) - 12(144)}{-12(-12+6)} = -\frac{125}{9}$$

$$H_0(s) = 1 + \frac{1}{9} \cdot \frac{1}{s} + \frac{16}{9} \cdot \frac{1}{s+6} - \frac{125}{9} \cdot \frac{1}{s+12}$$

$$H(s) = 20 H_0(s) = 20 + \frac{20}{9} \cdot \frac{1}{s} + \frac{320}{9} \cdot \frac{1}{s+6} - \frac{2500}{9} \cdot \frac{1}{s+12}$$

$$u(t) = 20 e(t) + \frac{20}{9} x_1(t) + \frac{320}{9} x_2(t) - \frac{2500}{9} x_3(t)$$

$$\dot{x}_1(t) = e(t)$$

$$\dot{x}_2(t) = -6x_2(t) + e(t)$$

$$\dot{x}_3(t) = -12x_3(t) + e(t)$$