Impulse Response

The *impulse response* of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

The <u>impulse response</u> $g(t)$ is the inverse transform of the transfer function $G(s)$

Conversely, Knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Rightarrow$ Foundation of "system identification" theory.

Additional Laplace Property

For any two functions $f_1(t), f_2(t)$ with transforms $F_1(s), F_2(s)$

$$\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_{0^-}^{\infty} f_1(t-\tau)f_2(\tau)\,d\tau$$

"convolution"

Implication: $\mathcal{L}^{-1}\{G(s)U(s)\} = \int_{0^-}^{\infty} g(t-\tau)u(\tau)\,d\tau$

proving generally what we showed specifically for the hovercraft problem.

There we had $\ddot{y}(t) = Ku(t)$

$$\Rightarrow G(s) = \frac{K}{s^2} \Rightarrow g(t) = Kt$$

and thus $g(t-\tau) = K(t-\tau)$.

Note:

Laplace actually let's us "divide out" the effect of any Known input to recover the transfer function (impulse response)

$$Y(s) = G(s)U(s) \qquad \text{(assuming } \emptyset \text{ ICs)}$$

$$\left[Y(s) = G(s)U(s)\right] \times \left(\frac{1}{U(s)}\right)$$

$$\left[\frac{Y(s)}{U(s)}\right] = G(s)\left[\frac{U(s)}{U(s)}\right]$$

$$= G(s) \cdot 1$$

response to ideal impulse.

Structure of Impulse Response

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{q(s)}{r(s)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\sum_{k=1}^{n} \frac{\gamma_k}{(s-p_k)}\right\}$$ p_k poles of $G(s)$

or

$$\boxed{\begin{aligned} g(t) &= \sum_{k=1}^{n} \gamma_k e^{p_k t} \\ \gamma_k &= \left[(s-p_k)G(s)\right]_{s=p_k} \end{aligned}}$$

(assuming non-repeated modes for simplicity)

$$g(t) = \sum_{k=1}^{n} \gamma_k e^{p_k t}$$

Note:

$\Rightarrow$ $g(t)$ is a specific linear combination of the modes.

$\Rightarrow$ Like a special homogeneous response

Alternate characterization of system stability

$$\boxed{\lim_{t \to \infty} |g(t)| \to \emptyset}$$

(if system is stable)

Step Responses

The (unit) step response of a system is the output $y(t)$ when $u(t) = \mathbb{1}(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[\cancelto{\emptyset}{c}(s) - \cancelto{\emptyset}{b}(s)]}{r(s)}$$

$U(s) = \frac{1}{s}$ here, so

$$Y(s) = \left(\frac{1}{s}\right) G(s) = \frac{q(s)}{s\, r(s)}$$

General Thoughts about step responses

① **Every** system has a unit step response:

$$Y(s) = \left[\left(\tfrac{1}{s}\right) G(s)\right]$$

$$y(t) = \mathcal{L}^{-1}\left\{\tfrac{1}{s} G(s)\right\} \triangleq y_{us}(t)$$

Find $y_{us}(t)$ as usual by partial fraction expansion and inverse transform of each term

However, we want to be able to predict main features of $y_{us}(t)$ by inspection for 1^{st} and 2^{nd} order systems

⇒ Very common special cases

⇒ "Building blocks" for more complex systems

② (Use of linearity, I)

$$u(t) = c\,\mathbb{1}(t) \implies y(t) = c\,y_{us}(t)$$

All $y(t)$ values are the unit step values multiplied by c.

Equivalent to "rescaling" vertical axis on plot of $y(t)$, however horizontal (time) axis is unaffected

$\Rightarrow$ Characteristic times (t_s, t_r, t_p) are unaffected

We'll encounter these shortly.

$\Rightarrow$ Corresponding $y(t)$ values scaled by c:

$$y_{ss} = c\,G(0), \quad y_p = c\,G(0)[1 + M_p]$$

$\Rightarrow$ True for any c, positive or negative

③ (Use of Linearity, II)

By definition, unit step response assumes all ICs are zero.

However, can easily "Add on" effects of nonzero ICs.

Nonzero Now

$$Y(s) = \left[\frac{1}{s} G(s)\right] + \left[\frac{c(s)}{r(s)}\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right) G(s)\right\} + \mathcal{L}^{-1}\left\{\frac{c(s)}{r(s)}\right\}$$

$$= y_{us}(t) + \mathcal{L}^{-1}\left\{\frac{c(s)}{r(s)}\right\}$$

Added terms from ICs

Solve for last term by PFE

Effect of added terms on t_s, t_p, y_p etc depends on specific ICs. No simple formulae to quantify their effects.

"1st Order" Responses

$$\dot{y}(t) + \alpha_0 y(t) = \beta_0 u(t) \implies G(s) = \frac{\beta_0}{s+\alpha_0}$$

Single real pole at $P_1 = -\alpha_0$ (stable if $\alpha_0 > 0$)

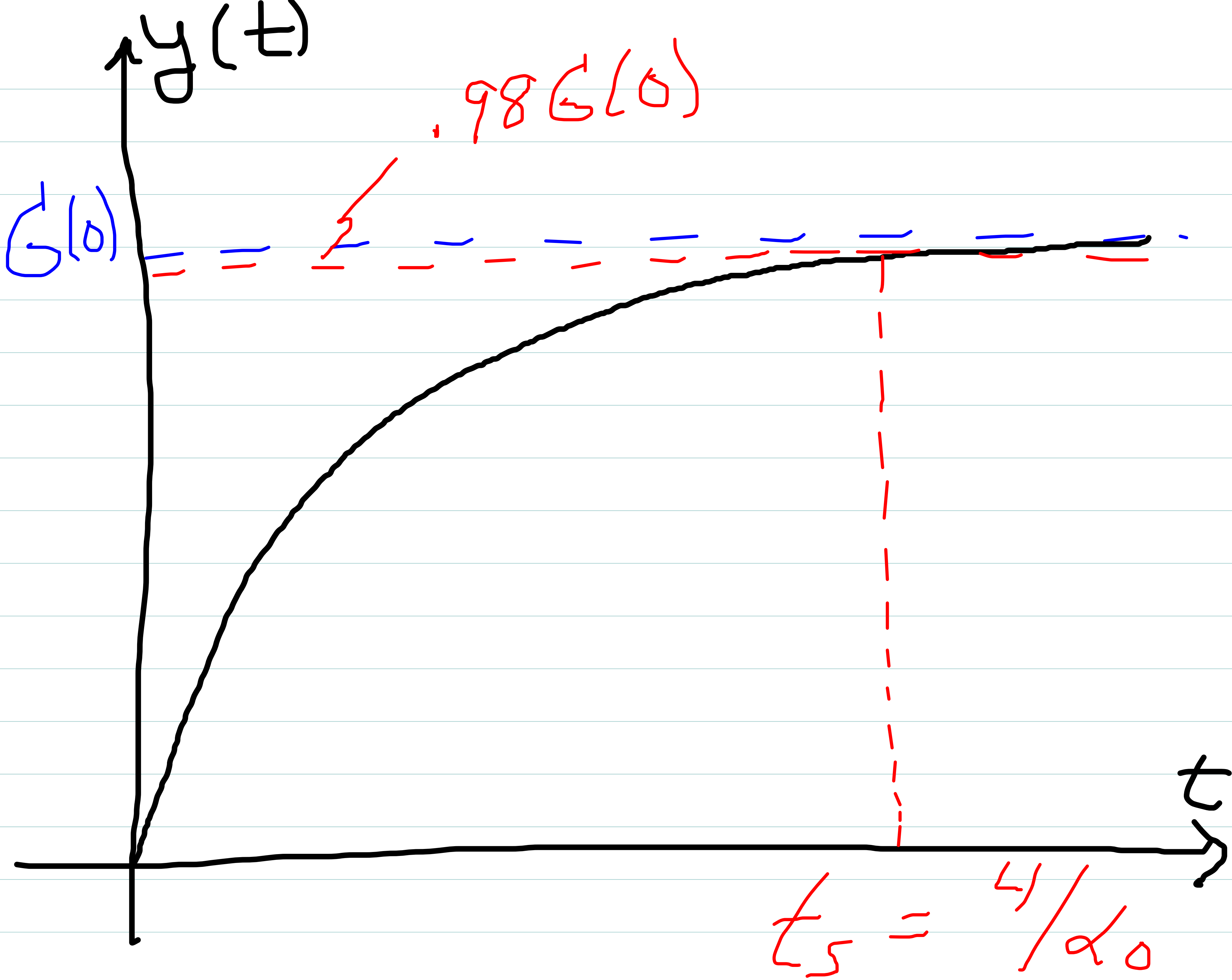
$$Y(s) = \frac{\beta_0}{s(s+\alpha_0)} = \frac{A_1}{s} + \frac{A_2}{s+\alpha_0}$$

$$A_1 = [sY(s)]_{s=0} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = [(s+\alpha_0)Y(s)]_{s=-\alpha_0} = \frac{-\beta_0}{\alpha_0} = -G(0)$$

Thus:

$$\boxed{y(t) = G(0)\left[1 - e^{-\alpha_0 t}\right]}$$



y(t)
G(0)
.98G(0)
t
$t_s = 4/\alpha_0$

Notes

1. Response asymptotically approaches steady-state

$$y_{ss}(t) = G(0)$$ (as expected)

2. Response <u>never</u> crosses its steady-state

3. Response <u>settles</u> within 2% of its steady-state in

$$t_s = \frac{4}{|\mathrm{Re}\{p\}|} = \frac{4}{\alpha_0}$$

4. "Shape" of graph is same for any 1st order system

Responses only differ by:

- steady-state level, $G(0)$
- settling time, t_s

"2^{nd} Order" Step Responses

$$\ddot{y}(t)+\alpha_1\dot{y}(t)+\alpha_0 y(t)=\beta_0 u(t) \implies G(s)=\frac{\beta_0}{s^2+\alpha_1 s+\alpha_0}$$

2 poles, both stable if $\alpha_1>0, \alpha_0>0$.

3 possibilities for poles:

1. $\alpha_1^2<4\alpha_0 \implies p_1, p_2$ complex conjugates
2. $\alpha_1^2=4\alpha_0 \implies p_1=p_2$ repeated real
3. $\alpha_1^2>4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case 1. is most interesting (and complicated) tackle this after the other two

2^{nd} order response, Case 2

$$G(s) = \frac{\beta_0}{s^2+\alpha_1 s+\alpha_0} \qquad \alpha_1^2 = 4\alpha_0 \quad (\zeta = 1)$$

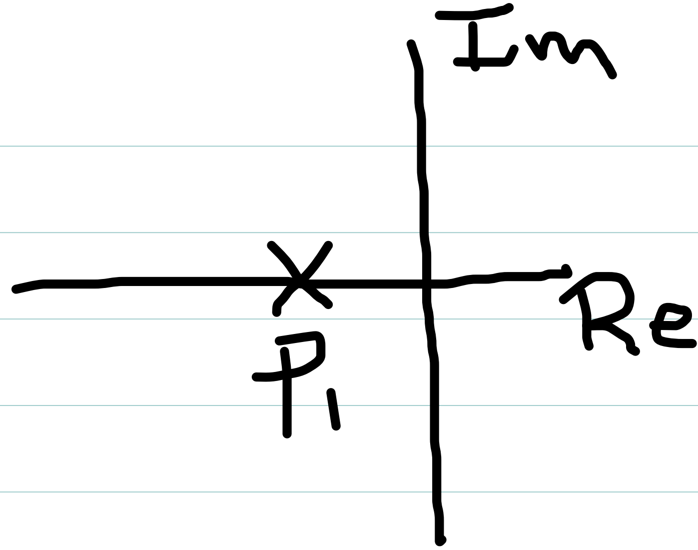
$$= \frac{\beta_0}{(s-p_1)^2} \qquad \text{repeated real pole}$$

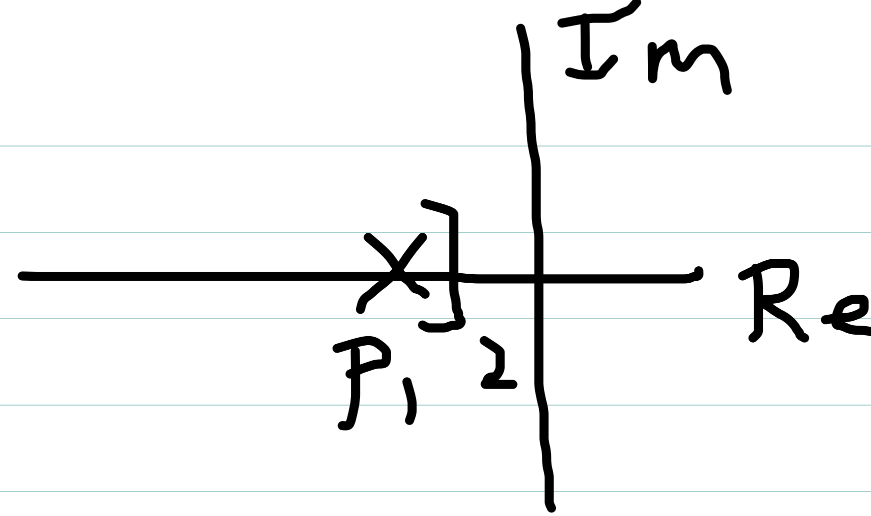
$$Y(s) = \left(\frac{1}{s}\right)G(s) = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{A_3}{(s-p_1)^2}$$

$$y(t) = G(\phi) + [A_2 + A_3 t]\,e^{p_1 t}$$

<u>Non-oscillatory</u>, since poles are <u>real</u>

features resemble 1^{st} order response
(No overshoot, $y_{ss} = G(0)$ approached asymptotically from below), <u>but</u> $\boxed{t_s \text{ 50\% longer } \left(\frac{6}{|p_1|}\right)}$

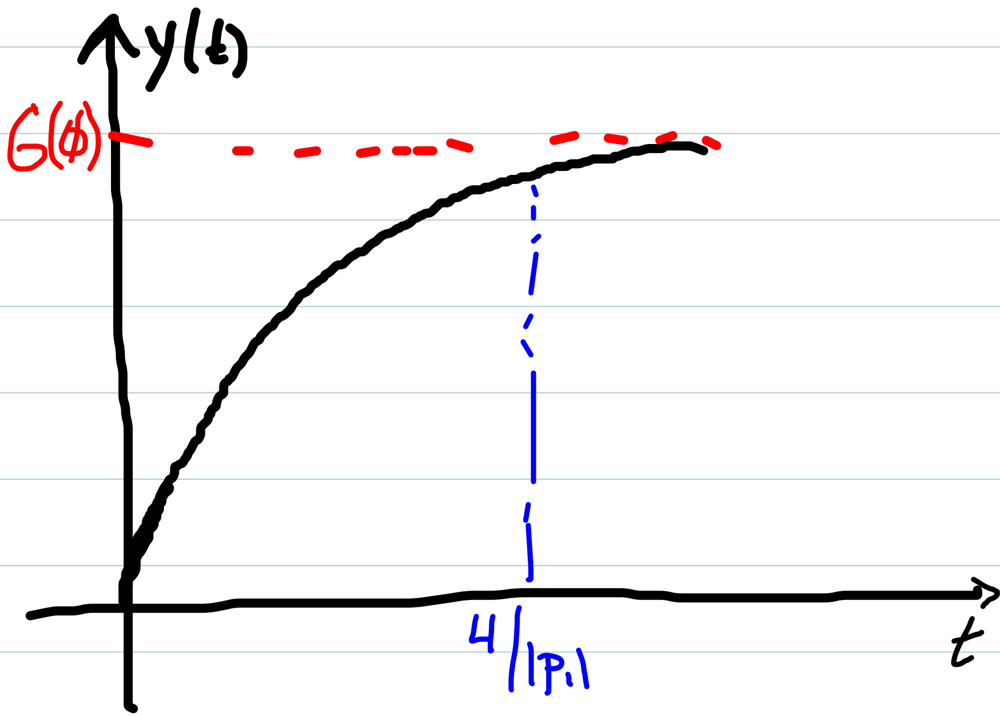




1^{st} order

2^{nd} order, repeated real

Add'l $te^{p_1 t}$ term "slows down" response.



2^{nd} order Response, Case 3

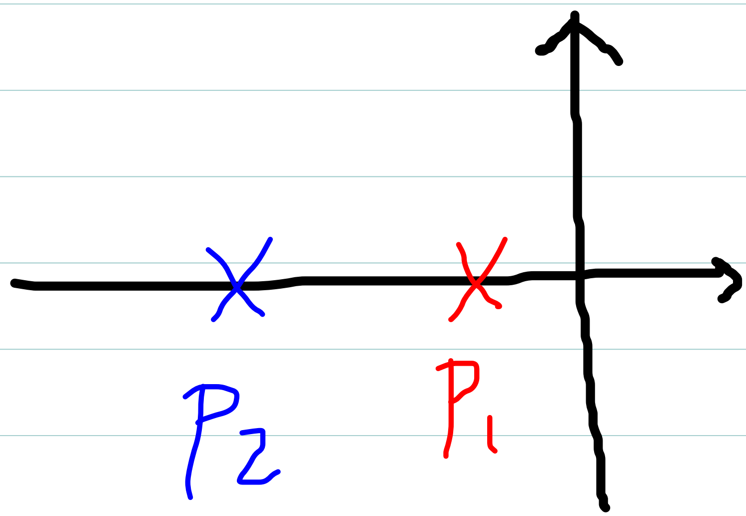
$\alpha_1^2 > 4\alpha_0$

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-p_2)} \qquad p_1 \neq p_2 .$$

$$\Rightarrow y(t) = G(0) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

Assume for notation sake that poles are numbered so that

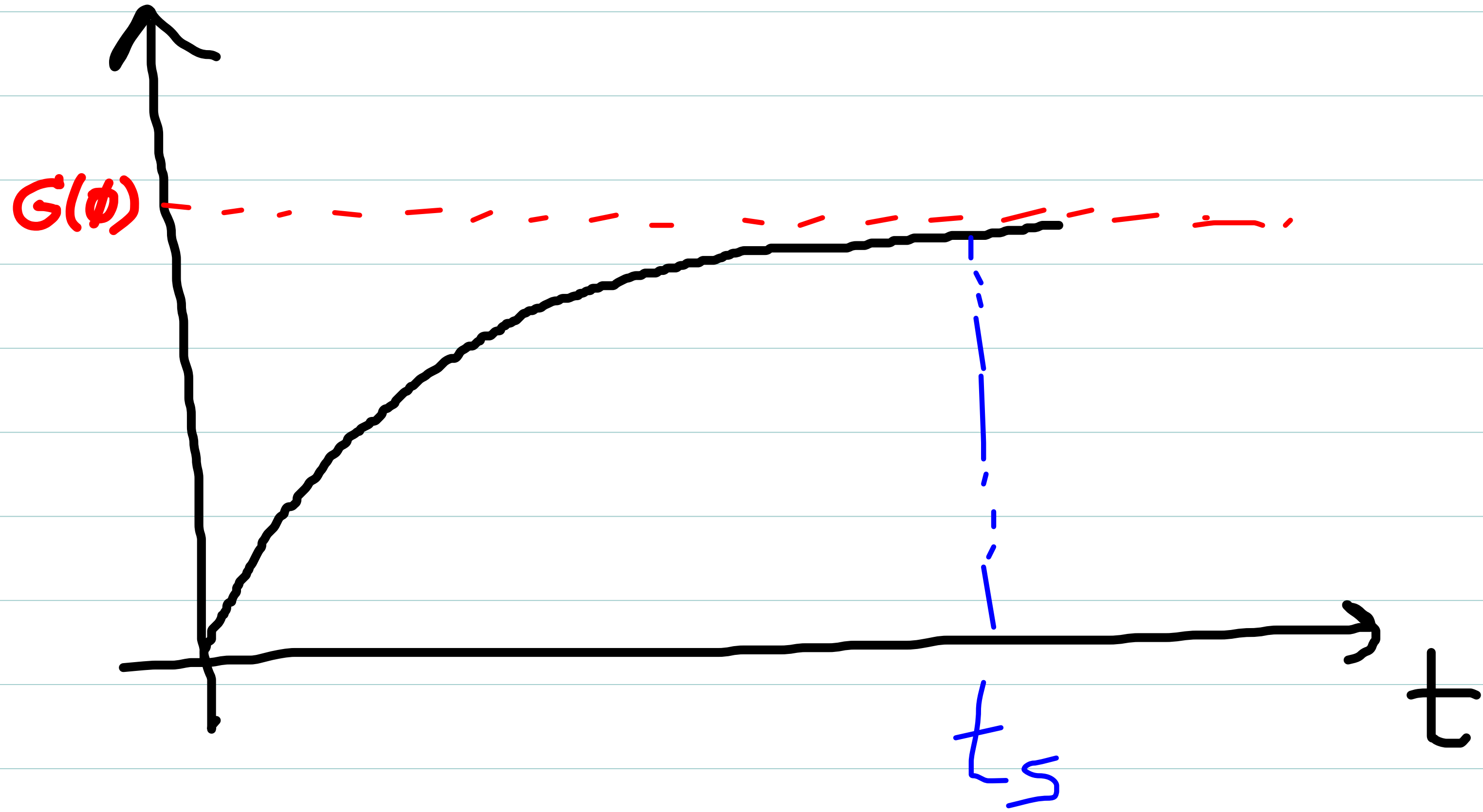
$p_2 < p_1$ $\quad$ ($\Rightarrow |p_2| > |p_1|$ since p_1, p_2 assumed negative)



p_1 is the "slow pole"

p_2 is the "fast pole"

General sol'n again resembles 1^{st} order response



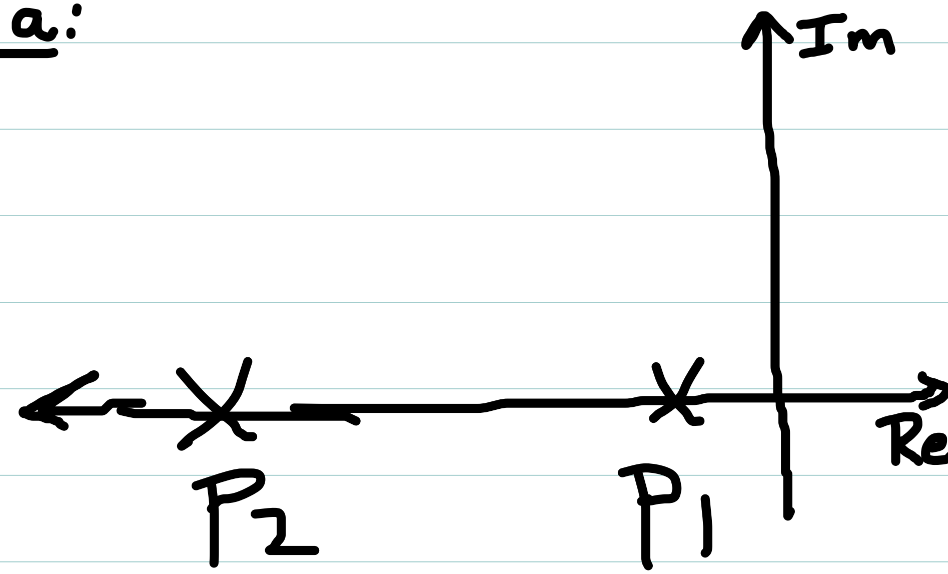
t_s difficult to quantify precisely for arbitrary p_1, p_2

Two limiting cases:

<u>Case 3a:</u> $|p_2| \gg |p_1|$

<u>Case 3b:</u> $|p_2| \approx |p_1|$

Case 3a:



$y(t) = G(\emptyset) + A_1 e^{p_1 t} + A_2 e^{p_2 t}$

$|P_2| >> |p_1|$

$\Rightarrow P_2$ much further into LHP than p_1

$\Rightarrow e^{p_2 t} \to \emptyset$ much faster than $e^{p_1 t}$

$\Rightarrow e^{p_1 t}$ controls settling time ("slow pole")

So $t_s \approx \dfrac{4}{|p_1|}$ in this case

$\Rightarrow$ Corresponds with previous "1st cut" of approximating system settling time with settling time of slowest mode.

Dominant Modes

When $|p_2| \gg |p_1|$ we say that mode $e^{p_1 t}$ "dominates" transient response, or that $e^{p_1 t}$ ("slow mode") is the

Dominant mode

What is a sufficient separation for a mode to be dominant

Generally, if $|p_2| > 5|p_1|$ or $|p_2| > 10|p_1|$

i.e. if p_2 is 5-10 times further into LHP

$\Rightarrow$ setting time of $e^{p_2 t}$ 5-10 times faster than that of $e^{p_1 t}$

(5 is usually sufficient. Some authors use 8 or even 10)

Case 3b

$|p_2| \approx |p_1| \implies p_2 \approx p_1$ Poles are "nearly" repeated

Here it is best to approximate the settling time as though the poles were actually repeated

$$t_s \approx \frac{6}{|p_1|}$$

Simple rule of thumb for this:

$$1 \leq \frac{|p_2|}{|p_1|} \leq 1.1$$

Intermediate Case 3 Situations

If $1.1 < \frac{|P_2|}{|P_1|} < 5$ (or 8 or 10)

$$\frac{4}{|P_1|} < t_s < \frac{6}{|P_1|}$$

Unfortunately, there is no simple formula for interpolating between the two limits based on the exact ratio.