"2nd Order" Step Responses

$$\ddot{y}(t)+\alpha_1\dot{y}(t)+\alpha_0 y(t)=\beta_0 u(t) \implies G(s)=\frac{\beta_0}{s^2+\alpha_1 s+\alpha_0}$$

2 poles, both stable if $\alpha_1>0, \alpha_0>0$.

3 possibilities for poles:

1. $\alpha_1^2<4\alpha_0 \implies p_1, p_2$ complex conjugates
2. $\alpha_1^2=4\alpha_0 \implies p_1=p_2$ repeated real
3. $\alpha_1^2>4\alpha_0 \implies p_1, p_2$ real, non-repeated

Case 1. is most interesting (and complicated) tackle this after the other two

Useful Observation (Case 1)

$$p_1 = \sigma + j\omega_d \qquad \omega_d = \mathrm{Im}\{p_1\}$$

Note slight change of notation! $\omega \rightarrow \omega_d$

$$s^2 + \alpha_1 s + \alpha_0 = (s - p_1)(s - \bar{p}_1)$$

$$= s^2 - (p_1 + \bar{p}_1)s + p_1\bar{p}_1$$

$$= s^2 - 2\,\mathrm{Re}\{p_1\}s + |p_1|^2$$

$$= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)$$

Hence:

$$\boxed{\begin{aligned} \alpha_1 &= -2\sigma = -2\,\mathrm{Re}\{p_1\} \\ \alpha_0 &= \sigma^2 + \omega_d^2 = |p_1|^2 \end{aligned}}$$

Rapidly identify pole location from coefs.

2^{nd} Order Response, Case 1:

$$Y(s) = \frac{\beta_0}{s(s-p_1)(s-\bar{p}_1)} = \frac{A_1}{s} + \frac{A_2}{(s-p_1)} + \frac{\bar{A}_2}{(s-\bar{p}_1)}$$

$$A_1 = \left[sY(s)\right]_{s=0} = \frac{\beta_0}{p_1\bar{p}_1} = \frac{\beta_0}{\alpha_0} = G(0)$$

$$A_2 = \left[(s-p_1)Y(s)\right]_{s=p_1} = \frac{\beta_0}{p_1(p_1-\bar{p}_1)} = \frac{\beta_0}{(\sigma+j\omega_d)(2j\omega_d)}$$

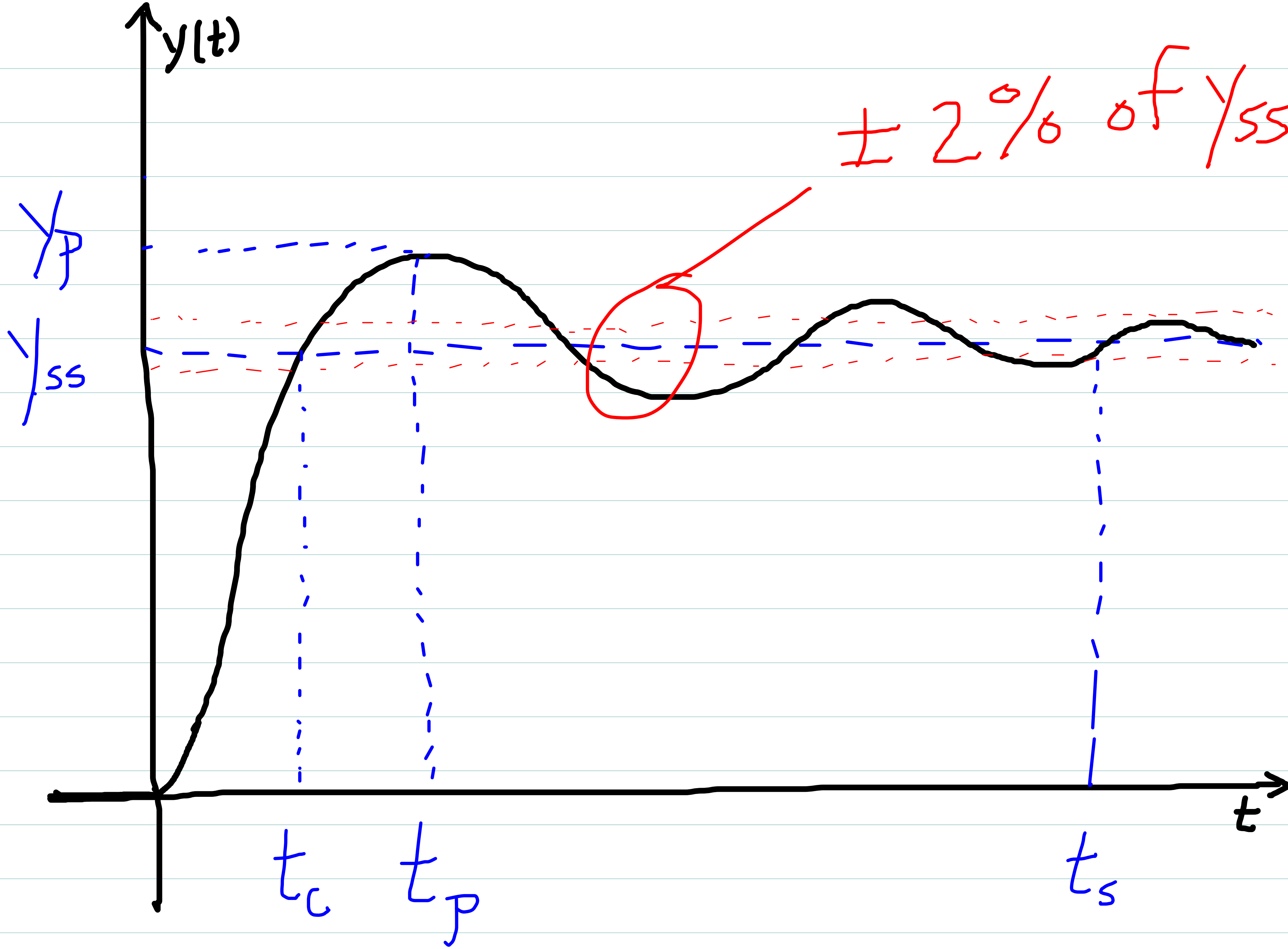
$$= \underbrace{\left(\frac{\beta_0}{2\alpha_0}\right)}_{\frac{1}{2}G(0)} \underbrace{\left(\frac{\alpha_0}{(\sigma+j\omega_d)(j\omega_d)}\right)}_{B}$$

SO:

$$y(t) = G(0) + 2|A_2|e^{\sigma t}\cos(\omega_d t + \angle A_2)$$

OR:

$$\boxed{y(t) = G(0)\left[1 + |B|e^{\sigma t}\cos(\omega_d t + \angle B)\right]}$$



y(t)
y_p
y_{ss}
$\pm 2\%$ of y_{ss}
t_c
t_p
t_s
t

General Observations

1. $y(t)$ continually oscillates about its steady-state value

$y_{ss} = G(\phi)$

2. t_c = time steady-state is first crossed

3. 1^{st} oscillation is largest, and creates an initial overshoot past the steady-state.

4. This initial overshoot has peak value y_p, and occurs at time t_p

5. Settling time t_s defined where response enters ±2% tolerance band and remains within it for times thereafter

Must learn to rapidly quantify these!!

$$y(t) = G(0)\left[1 + |B|\, e^{\sigma t}\cos(\omega_d t + \angle B)\right]$$

where:

$$B = \frac{\alpha_0}{(j\omega_d)(\sigma + j\omega_d)} = \frac{|p_1|^2}{(j\omega_d)\, p_1}$$

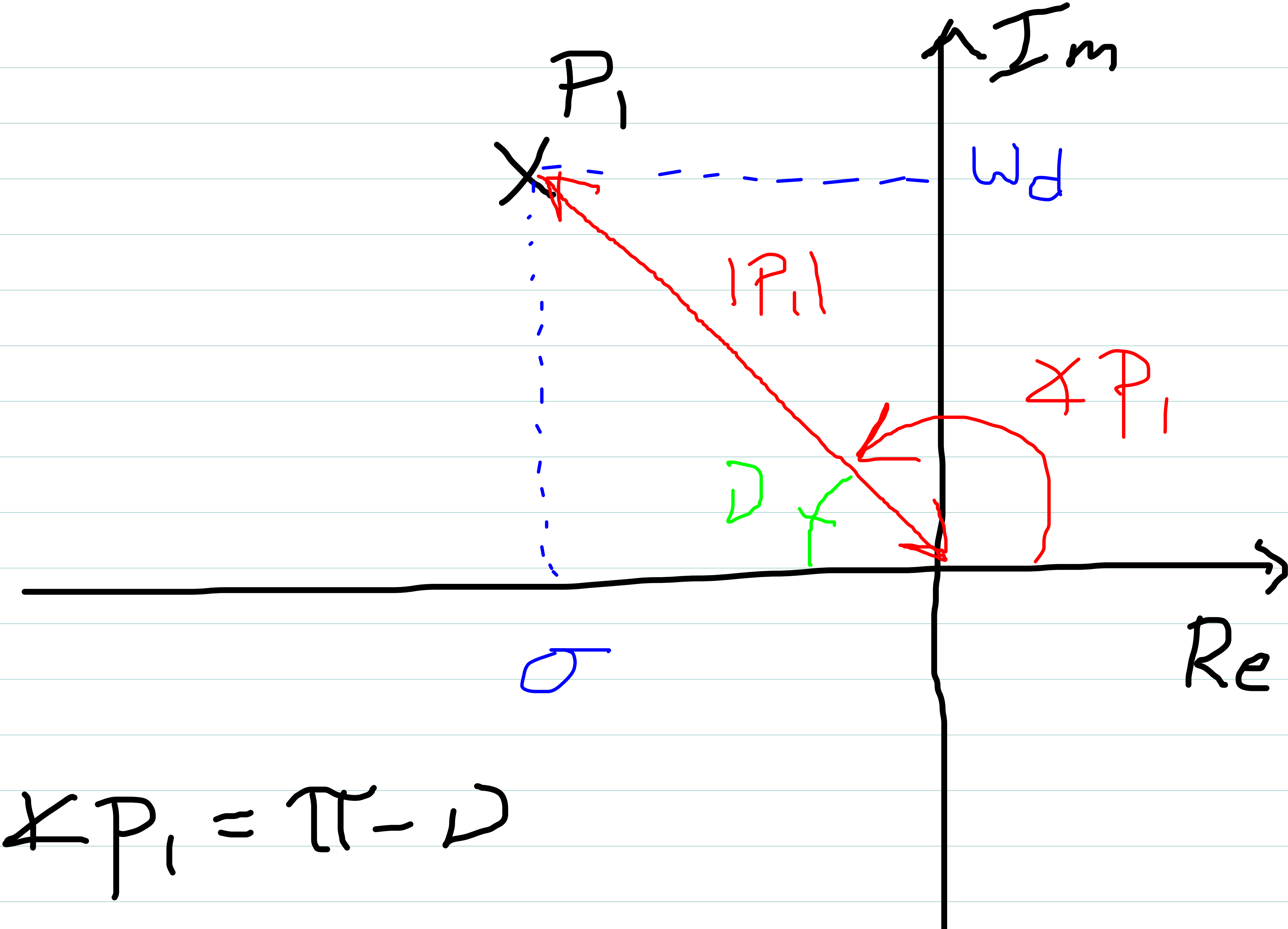
$\Rightarrow$ Transient features completely determined by location of pole $p_1 = \sigma + j\omega_d$ in complex plane

$$|B| = \frac{|p_1|^2}{|j\omega_d| \cdot |p_1|} = \frac{|p_1|}{\omega_d}$$

$$\angle B = \cancelto{\emptyset}{\angle |p_1|^2} - \left(\angle (j\omega_d) + \angle p_1\right)$$

$$= -\left(\frac{\pi}{2} + \angle p_1\right)$$

must quantify this!



$\angle P_1 = \pi - \nu$

<u>Note</u>: $\nu > 0$ is supplement of $\angle P_1$

So:

$$\measuredangle B = -\left(\frac{\pi}{2} + \measuredangle p_1\right) = -\left(\frac{\pi}{2} + (\pi - \nu)\right)$$

$$= -\frac{3\pi}{2} + \nu$$

and thus:

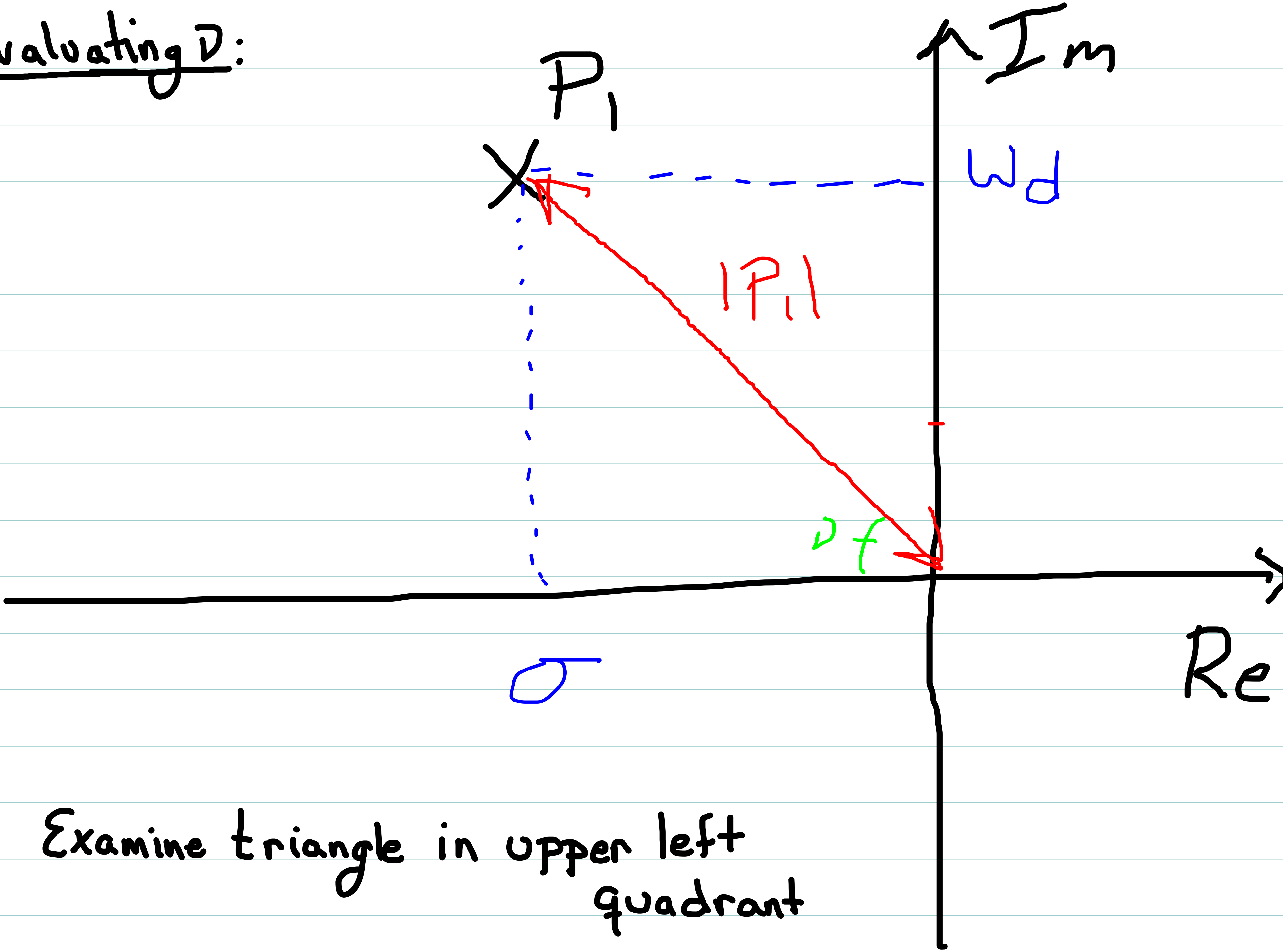
$$y(t) = G(0)\left[1 + \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \cos\left(\omega_d t - \frac{3\pi}{2} + \nu\right)\right]$$

so:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{|p_1|}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \nu)\right]}$$

Need to understand how ν depends on p_1

Evaluating $\mathcal{D}$:



Examine triangle in upper left quadrant

Two Useful Parameters

Define: $\omega_n = |p_1| = \sqrt{\sigma^2 + \omega_d^2}$ "natural" frequency

$\Rightarrow$ purely theoretical! ω_d is physical frequency of transient oscillations

Define: $\xi = \dfrac{|\sigma|}{\omega_n} = \dfrac{|\sigma|}{\sqrt{\sigma^2 + \omega_d^2}}$

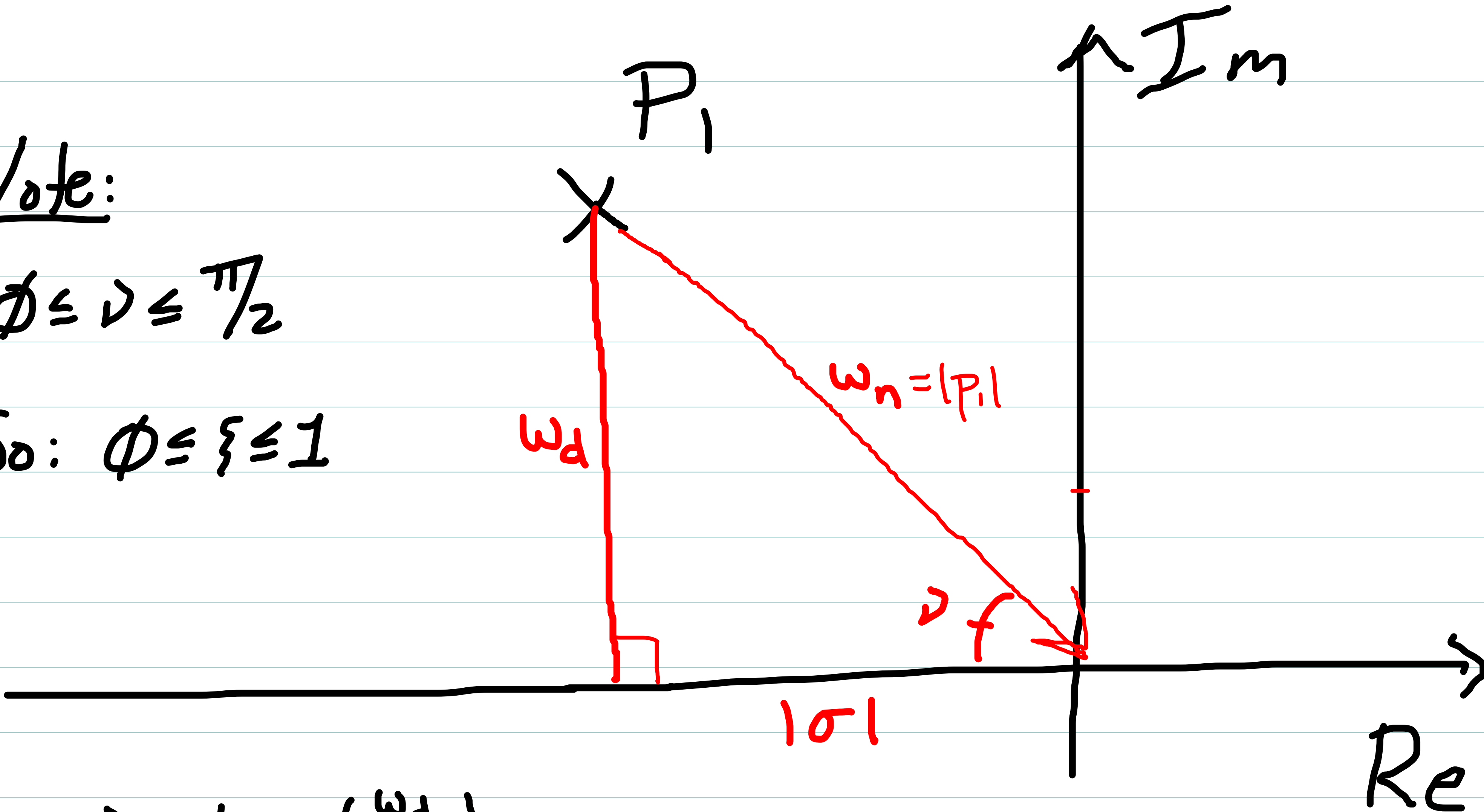
"Damping ratio"

$\Rightarrow$ A normalized measure of the number of transient oscillations observed before amplitude becomes negligible

Note:

$0 \le \nu \le \pi/2$

So: $0 \le \zeta \le 1$



$$\nu = \tan^{-1}\left(\frac{\omega_d}{|\sigma|}\right)$$

$$\nu = \sin^{-1}\left(\frac{\omega_d}{\omega_n}\right)$$

$$\nu = \cos^{-1}\left(\frac{|\sigma|}{\omega_n}\right) = \cos^{-1}\zeta$$ $\Leftarrow$ very useful!

A few more observations

$$\zeta = \frac{|\sigma|}{\omega_n} \Rightarrow \boxed{\sigma = -\zeta\omega_n}$$

(Stable system Assumed)

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

$$\Rightarrow \omega_d^2 = \omega_n^2 - \sigma^2 = \omega_n^2 - (-\zeta\omega_n)^2 = \omega_n^2(1-\zeta^2)$$

So: $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

Then note:

$$\boxed{\begin{aligned} s^2 + \alpha_1 s + \alpha_0 &= (s-p)(s-\bar{p}) \\ &= s^2 - 2\sigma s + (\sigma^2 + \omega_d^2) \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}}$$

all equivalent $\Longleftarrow$

Note:

The three possible cases for 2nd order step responses can be categorized by ξ:

Case 1 (complex poles): $0 \leq \xi < 1$

$$\alpha_1^2 < 4\alpha_0 \Rightarrow (2\xi\omega_n)^2 < 4\omega_n^2$$

Case 2 (repeated real poles): $\xi = 1$

$$\alpha_1^2 = 4\alpha_0 \Rightarrow (2\cancelto{1}{\xi}\omega_n)^2 = 4\omega_n^2$$

Case 3 (distinct real poles): $\xi > 1$

$$\alpha_1^2 > 4\alpha_0 \Rightarrow 4\xi^2\omega_n^2 > 4\omega_n^2$$

Thus finally, the Case 1 step response is:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{\omega_n}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \cos^{-1}\zeta)\right]}$$

We can now solve for important transient parameters

$\Rightarrow$ t_c: Solve for first $t > 0$ such that

$$y(t) = y_{ss}(t) = G(0)$$

$$\Rightarrow \sin(\omega_d t + \cos^{-1}\zeta) = 0$$

$$\Rightarrow t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

or:

$$\boxed{t_c = \frac{\pi - \rho}{\omega_d}}$$

$\Rightarrow$ For t_p, y_p

Solve for first $t>0$ such that

$$\dot{y}(t) = 0$$

$$\Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Substituting:

$$y_p = y(t_p) = G(0)\left[1 + e^{(\sigma\pi/\omega_d)}\right]$$

<u>Define:</u>

$$\boxed{\begin{aligned} M_p &= e^{(\sigma\pi/\omega_d)} \\ \text{then: } \quad y_p &= G(0)\left[1 + M_p\right] \end{aligned}}$$

Peak Overshoot

⇒ M_p is the Normalized peak overshoot

$$y_p = G(0)[1+M_p] \Longrightarrow M_p = \frac{y_p - G(0)}{G(0)} = \frac{y_p - y_{ss}}{y_{ss}}$$

⇒ M_p is entirely determined by damping ratio ξ

$$M_p = \exp\left[\frac{\sigma\pi}{\omega_d}\right]$$

$$= \exp\left[\frac{(-\xi\omega_n)\pi}{\omega_n\sqrt{1-\xi^2}}\right]$$

OR

$$\boxed{M_p = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]}$$

$$\%OS = 100 \times M_p$$

Settling Time

As usual, we can use the approximation

$$\boxed{t_s \approx \frac{4}{|Re\{p_i\}|} = \frac{4}{|\sigma|}}$$

But t_s is actually a function of ξ also here:

$$t_s = \frac{C(\xi)}{|\sigma|}$$

with $3 \leq C(\xi) \leq 5$ for most $0 \leq \xi < 0.9$

so 4 is an "average" value for $C(\xi)$

However for $0.95 \leq \xi \leq 1$ a better approximation is: $t_s \approx \dfrac{6}{|\sigma|}$

Summary: Case I step response; $p_1 = \sigma + j\omega_d$

"Natural" frequency: $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = |p_1|$

Damping ratio: $\xi = \frac{|\sigma|}{\omega_n}$

1st crossing: $t_c = \frac{\pi - \cos^{-1}\xi}{\omega_d} = \frac{\pi - \nu}{\omega_d}$, $\xi = \cos\nu$

1st peak: $t_p = \frac{\pi}{\omega_d}$

Normalized overshoot: $M_p = \exp\left[\frac{\sigma\pi}{\omega_d}\right] = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$$M_p = \left[\frac{y_p - y_{ss}}{y_{ss}}\right]$$

Peak response: $y_p = y_{ss}[1 + M_p]$

$y_{ss} = G(\phi)$ for unit step

Limiting case: $\xi \to 0$

$\xi \to 0 \Rightarrow \sigma = -\xi\omega_n \to 0 \Rightarrow P_1 = j\omega_d$ (pure imaginary)

Overshoot $M_p = e^{(\sigma\pi/\omega_d)} \to 1$ (100% OS)

Peak: $y_p = G(0)[1 + M_p]$

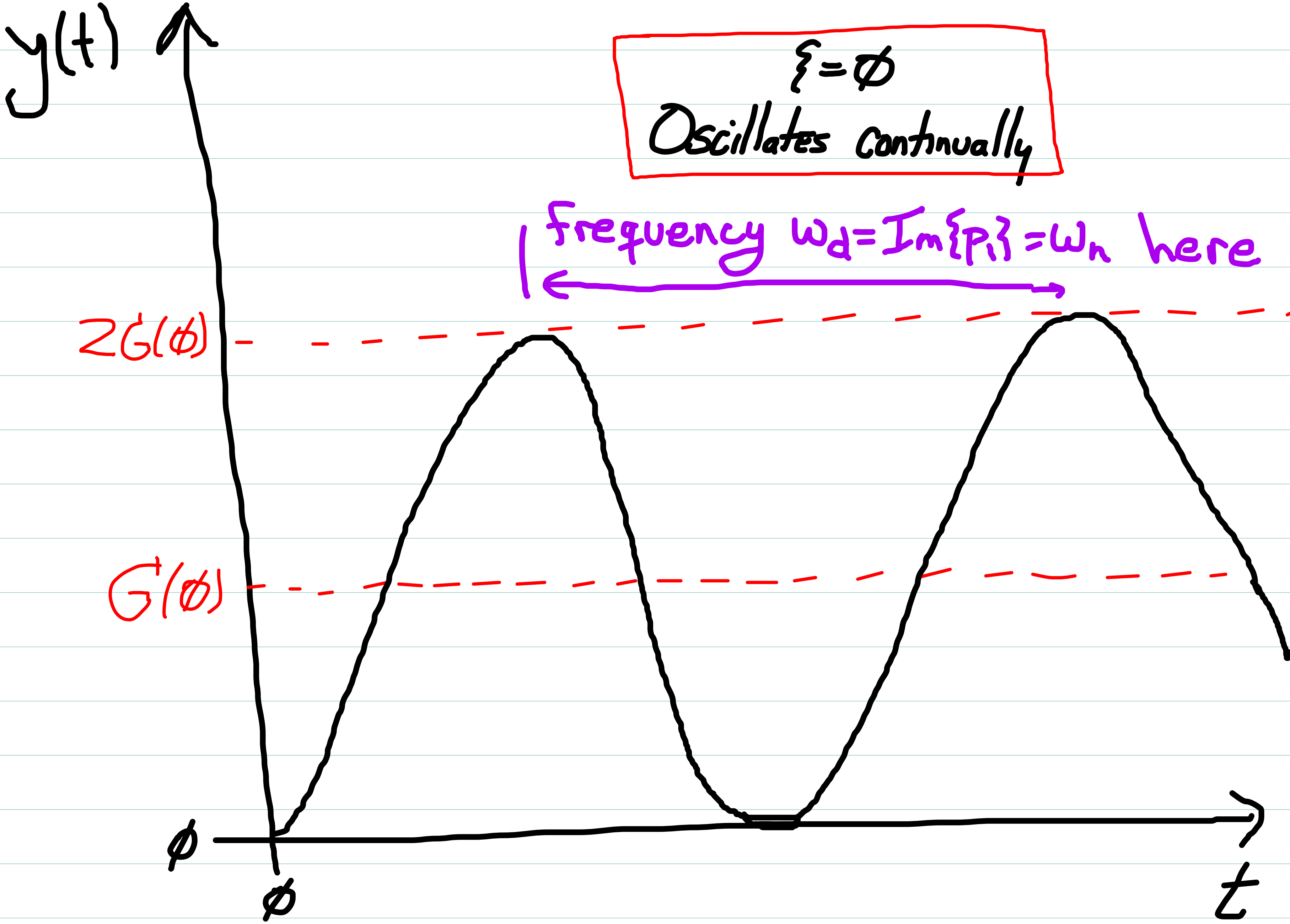
or $y_p = 2y_{ss}$

Settling time: $t_s \approx \dfrac{4}{|\sigma|} = \infty$

Never settles!

Response oscillates infinitely between 0 and $2G(0)$ with frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$

"Undamped"


y(t)
ξ=0
Oscillates continually
frequency ωd=Im{pi}=ωn here
2G(0)
G(0)
0
0
t

Limiting Case, $\zeta \to 1$

$$\zeta \to 1 \Rightarrow \sigma = -\zeta\omega_n \to -\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \to \emptyset$$

Response does not oscillate!

Overshoot:

$$M_p = e^{(\sigma\pi/\omega_d)} = e^{-\omega_n\pi/\emptyset} = \emptyset$$

No overshoot

1st crossing:

$$t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d} = \pi/2 / \emptyset = \infty$$

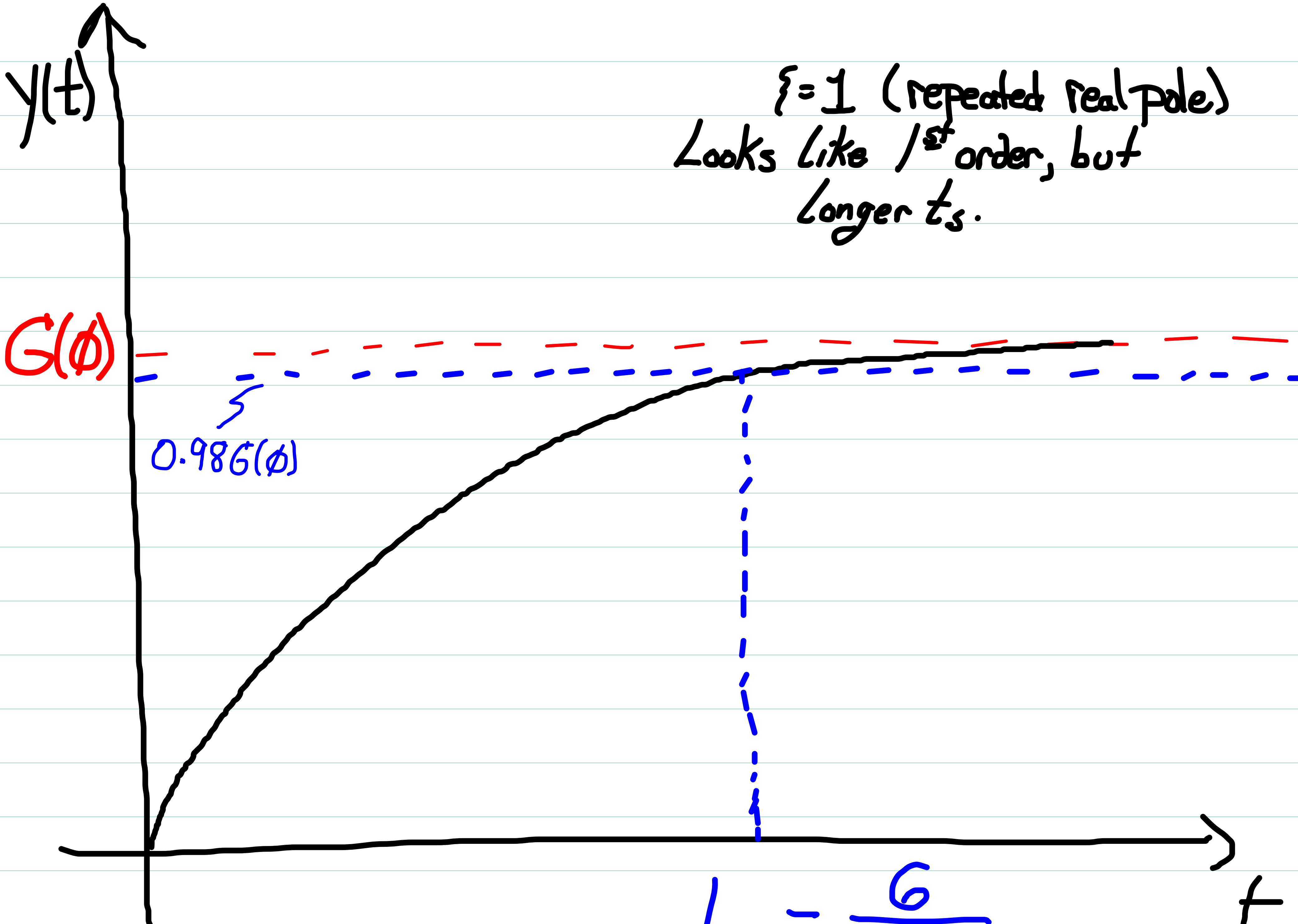
$\Rightarrow$ response asymptotes to y_{ss} from below

Settling:

$$t_s \approx \frac{6}{|\sigma|}$$

use 6 here

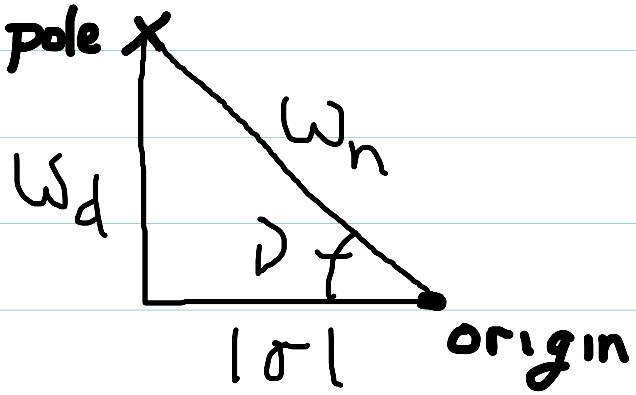
$\zeta = 1$ (repeated real pole)
Looks like 1^{st} order, but longer t_s.



$$t_s = \frac{6}{|\sigma|}$$

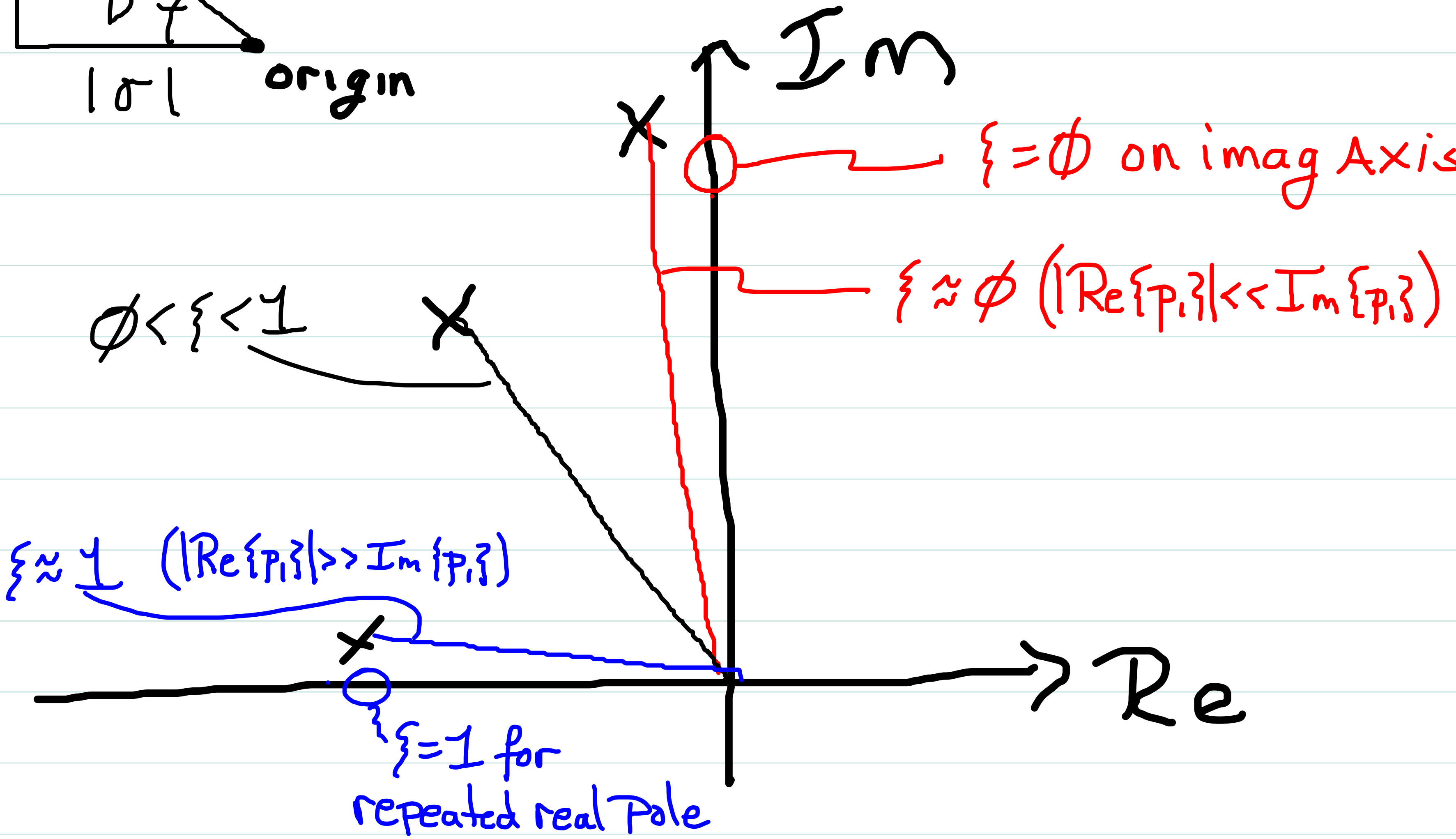
Graphical Interpretation of ζ:

$\boxed{\zeta = \cos\theta :}$

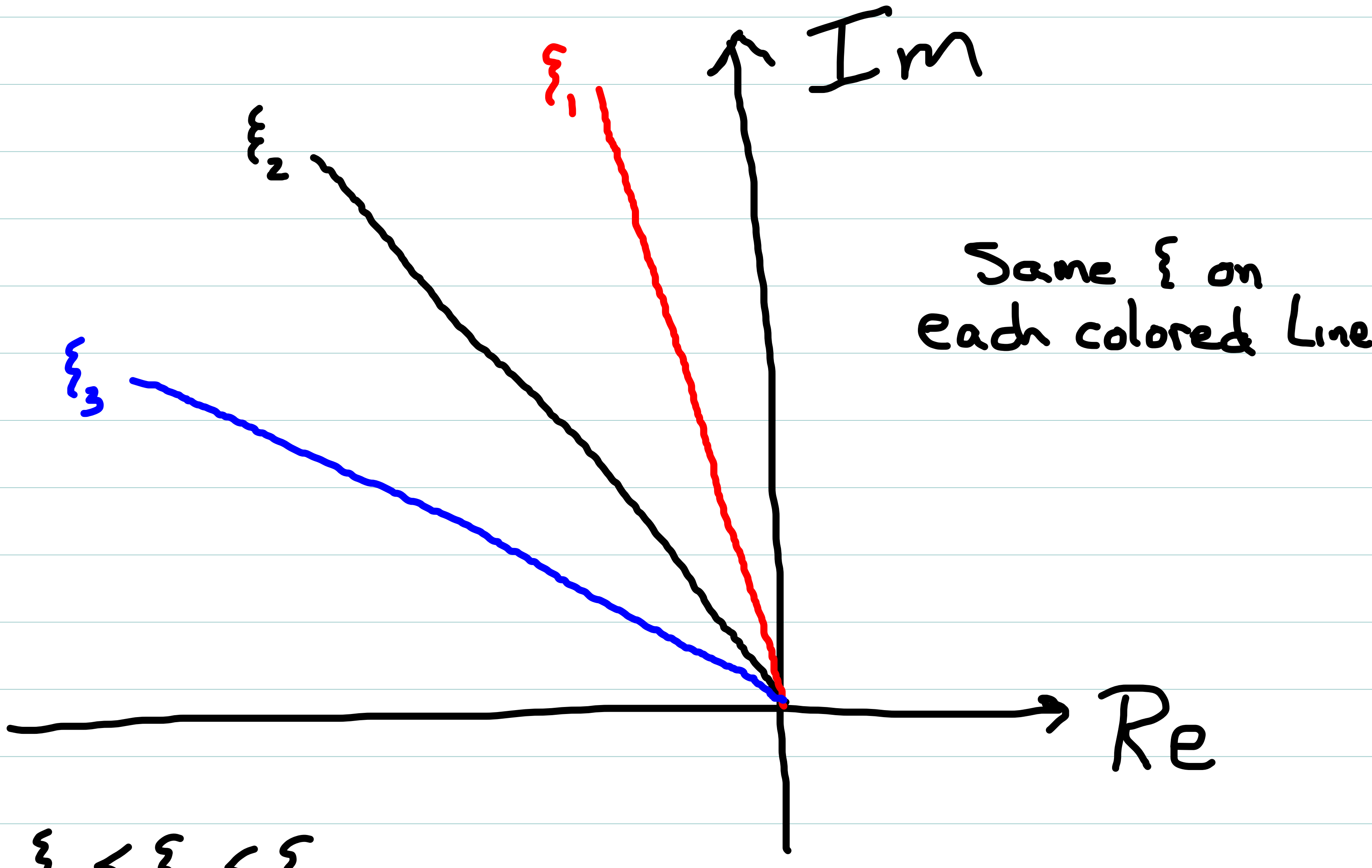


$\zeta \to 0 \Rightarrow \theta \to \pi/2$

$\zeta \to 1 \Rightarrow \theta \to 0$



Lines of constant ξ lie on rays in upper left quadrant of complex plane:



$\xi_1 < \xi_2 < \xi_3$