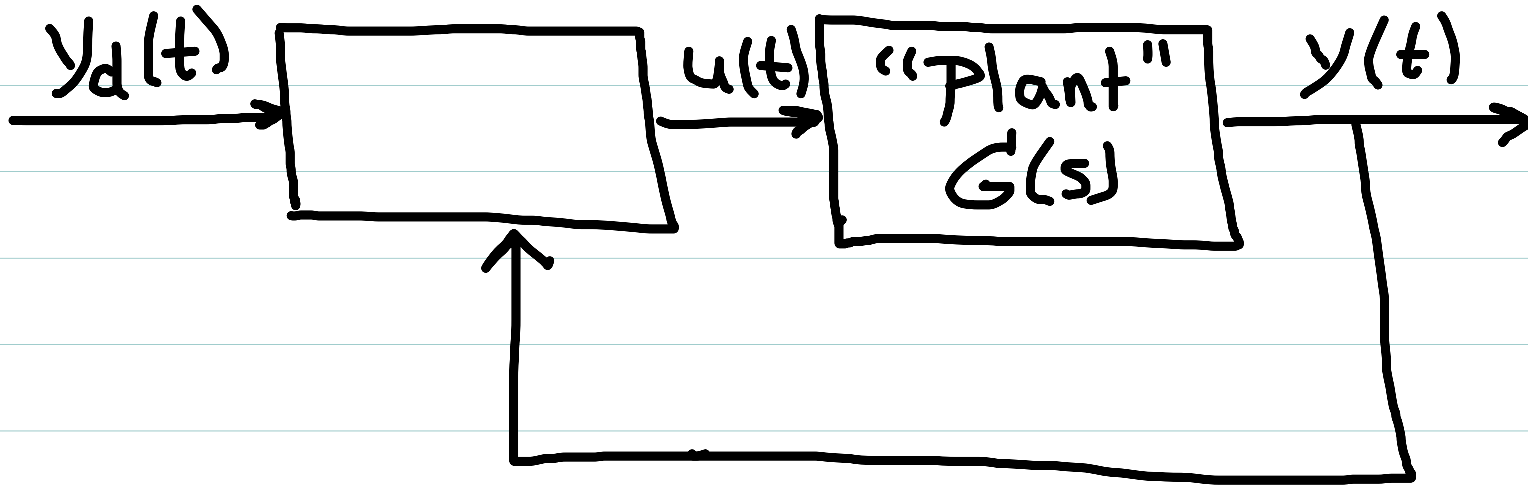
Feedback Control (finally!)

⇒ **Automatically** generate inputs $u(t)$ so that output $y(t)$ tracks "desired output" $y_d(t)$ as closely as possible

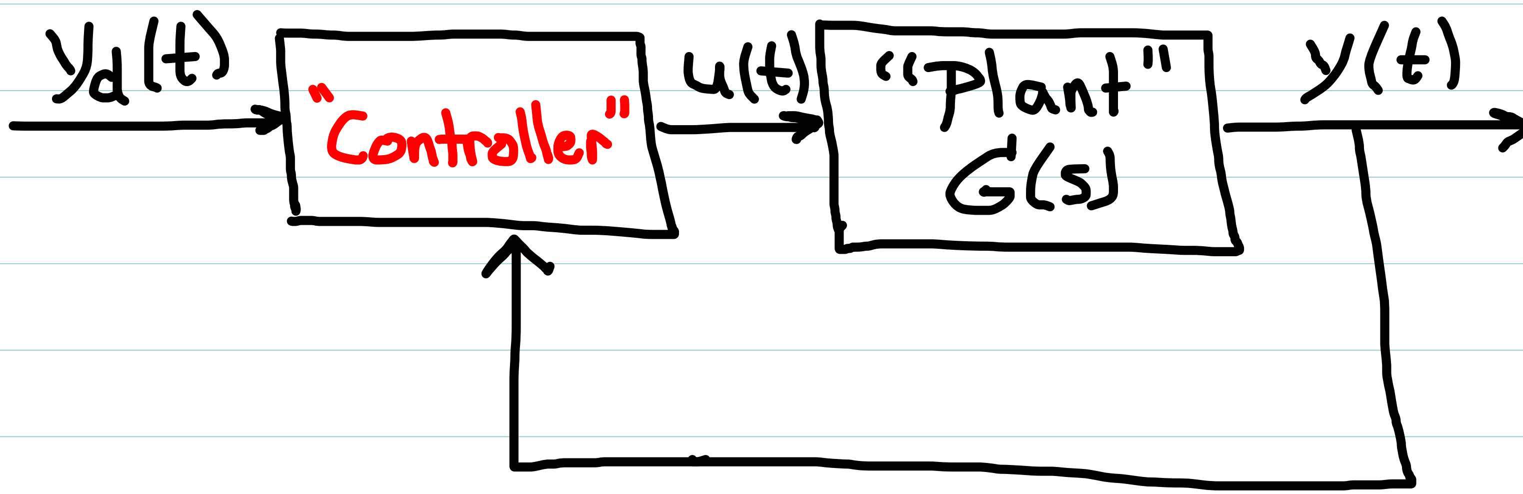
⇒ Input determined in real-time by continually comparing $y(t)$ with $y_d(t)$



Feedback Control (finally!)

⇒ Automatically generate inputs $u(t)$ so that output $y(t)$ tracks "desired output" $y_d(t)$ as closely as possible

⇒ Input determined in real-time by continually comparing $y(t)$ with $y_d(t)$



Feedback Controllers

=> The controller is a device that we design to compute $u(t)$ from $y_d(t)$ and $y(t)$, to satisfy specified constraints.

=> The relationship between $y_d(t)$, $y(t)$ and $u(t)$ is known as the "control law". This is a mathematical algorithm for computing $u(t)$.

=> For example:

$$u(t) = K\left[y_d(t) - y(t)\right]$$

In this control law, $u(t)$ is proportional to the difference between $y_d(t)$ and $y(t)$.

=> Controllers are implemented as programs (usually in C/C++) on a digital computer onboard the vehicle.

Control Laws

⇒ Control laws can be any mathematical function of $y(t)$ and $y_d(t)$, including <u>differential equations</u>

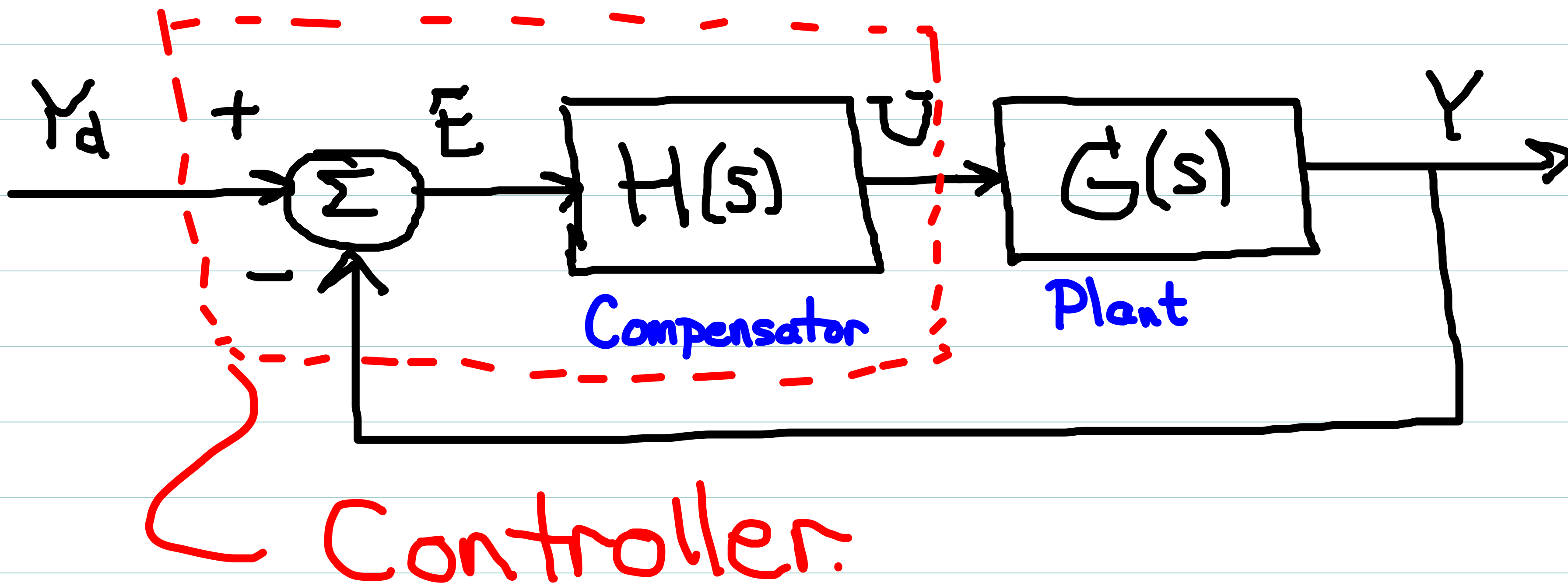
⇒ For example:

$$\dot{u}(t) + \alpha_0 u(t) = \beta_1 \frac{d}{dt}\left[y_d(t) - y(t)\right] + \beta_0 \left[y_d(t) - y(t)\right]$$

⇒ In such cases, we can model the operation of the controller in the same transfer function framework used to model the physical system being controlled.

⇒ The "standard servo loop" is a systematic framework for analyzing these control strategies.

Standard Servo Loop



Action of controller is:

$$U(s) = H(s)\,E(s)$$

"Tracking error"

where $E(s) = Y_d(s) - Y(s) \Rightarrow \boxed{e(t) = y_d(t) - y(t)}$

however finding $u(t)$ from $e(t)$ requires solving differential equation corresponding to $H(s)$.

Controller Design

$$U(s) = H(s)E(s)$$

$H(s)$ is a new transfer function that we design

It has no physical basis, we create it to solve the control problem for a particular physical system $G(s)$.

There is no unique specification of $H(s)$ for a specific $G(s)$. Many different design tradeoffs which do not have a unique sol'n.

Guiding principle: Use the simplest $H(s)$ (fewest poles + zeros) which will provide desired performance.

Servo Loop Analysis

$$U(s) = H(s)[Y_d(s) - Y(s)]$$

$$Y(s) = G(s)\,U(s)$$

Circular! Y depends on U, but U depends on Y.

Very tricky to "untangle" the circularity using the governing diff'l eq'ns for G, H.

Laplace makes it easy!

$$Y = GU = GHE = GH(Y_d - Y)$$

$$\Rightarrow (1+GH)Y = GHY_d$$

$T(s)$ "closed-loop" TF

or

$$Y(s) = \boxed{\left[\frac{G(s)H(s)}{1+G(s)H(s)}\right]}Y_d(s)$$

Loop Transfer Functions

Define

$$L(s) = G(s)H(s)$$ "open-loop" TF

then

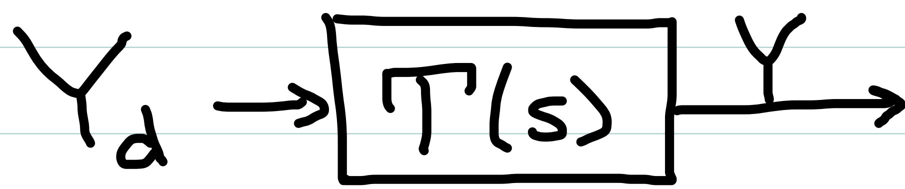
$$T(s) = \frac{L(s)}{1+L(s)}$$ "closed-loop" TF

and

$$Y(s) = L(s)E(s)$$ open-loop dynamics

$$Y(s) = T(s)Y_d(s)$$ closed-loop dynamics

$T(s)$ gives us direct information about system performance



$L(s)$ is an important intermediate quantity in analysis + design

Another Useful relationship

$$E(s) = Y_d(s) - Y(s) = Y_d(s) - T(s)\,Y_d(s)$$

$$= \underbrace{[1 - T(s)]}_{S(s)\text{ : "Sensitivity" TF}}\,Y_d(s)$$

Note that:

$$S(s) = 1 - T(s) = 1 - \frac{L(s)}{1 + L(s)}$$

So:

$$S(s) = \frac{1}{1 + L(s)}$$

Thus:

$$S(s) = 1 - T(s) = \frac{1}{1 + L(s)}$$

are equivalent, although we will primarily work with the second form.

Final Important Relationship

$U(s) = H(s)E(s)$ [TF model of control law]

$$= \underbrace{\left[H(s)S(s)\right]}_{R(s)} Y_d(s)$$

$$R(s) = H(s)S(s) = \frac{H(s)}{1+L(s)}$$

Used to predict control signals which will be generated under ideal conditions

⇒ $y_d(t)$ known perfectly for all $t \geq 0$
⇒ perfect model of system (no errors in model, no disturbances)

R(s) used only theoretically. H(s) is used for actual implementation.

Example:

Suppose $G(s) = \frac{2(s+1)}{s+3}$ $\quad H(s) = \frac{K}{s}$

Then $L = GH = \frac{2K(s+1)}{s(s+3)}$

$$T = \frac{L}{1+L} = \frac{2K(s+1)}{s(s+3)+2K(s+1)} = \frac{2K(s+1)}{s^2+(3+2K)s+2K}$$

$$S = \frac{1}{1+L} = \frac{s(s+3)}{s^2+(3+2K)s+2K}$$

$$R = \frac{H}{1+L} = \frac{K(s+3)}{s^2+(3+2K)s+2K}$$

Three <u>Derived</u> TFs for Feedback Loops

Given $G(s)$ and $H(s)$, we can derive $R(s)$, $S(s)$, $T(s)$ so that:

$$Y_d(s) \longrightarrow \boxed{\;T(s)\;} \longrightarrow Y(s) \qquad T(s) = \frac{L(s)}{1+L(s)}$$

$$Y_d(s) \longrightarrow \boxed{\;S(s)\;} \longrightarrow E(s) \qquad S(s) = \frac{1}{1+L(s)}$$

$$Y_d(s) \longrightarrow \boxed{\;R(s)\;} \longrightarrow U(s) \qquad R(s) = \frac{H(s)}{1+L(s)}$$

$\Rightarrow$ Each of these <u>derived</u> TFs can be analyzed using the same tools developed for $G(s)$.

Example use of loop TF:

Suppose $y_d(t) = A\mathbb{1}(t)$ (step of magnitude A)

Then:

$$y(t) = A \times \{\text{step response of } T(s)$$

$$u(t) = A \times \{\text{ step response of } R(s)\}$$

$$e(t) = A \times \{\text{step response of } S(s)\}$$

Note in particular here that:

$$e_{ss}(t) =$$

Example use of loop TF:

Suppose $y_d(t) = a\mathbb{1}(t)$ (step of magnitude A)

Then:

$$y(t) = a \times \{\text{step response of } T(s)$$

$$u(t) = a \times \{\text{step response of } R(s)\}$$

$$e(t) = a \times \{\text{step response of } \mathcal{S}(s)\}$$

Note in particular here that:

$$e_{ss}(t) = A\,\mathcal{S}(0) \qquad \text{(constant)}$$

Thus generally we'd like to make sure $\mathcal{S}(0) = 0$ (or at least very small).

Closed-loop poles

=> Performance of controlled system (settling time, steady-state, overshoot, etc) depends on poles of $T(s)$

=> ($R(s)$ and $S(s)$ have same poles!!)

=> Where are these poles??

=> Determined by denominator of $T(s)$

=> ($R(s)$ and $S(s)$ have same denominator)

=> Denom of all 3 derived TF is:

$$1+L(s)$$

Charactistic Equation

Poles of $T(s), R(s), S(s)$ are at values of $s \in \mathbb{C}$ such that

(CE) $$\boxed{1 + L(s) = \emptyset}$$ "Chaderistic equation" of feedback system

We need sol'ns of this equation to be in "good" locations of complex plane.

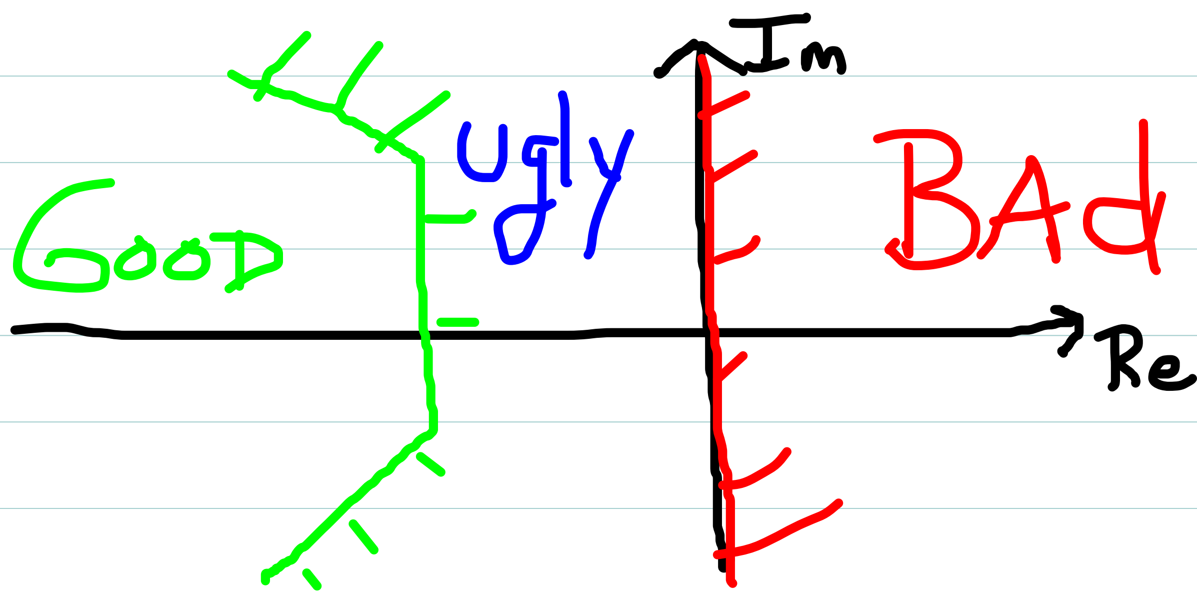
Will identify required properties for $L(s)$ so this is true, then work backwards to determine required properties of $H(s)$.

Fundamental Consideration: Closed-loop Stability

Most basic design consideration: ⟸ !!

Closed-loop poles should be "good", and certainly <u>must be stable</u>.

Thus, sol'ns of $\boxed{CE: \quad 1+L(s)=\emptyset}$ must be in left half of complex plane, preferably in "good region" (far from imag Axis, relatively close to or on the real Axis).



A crucial Observation:

If $L(j\omega) = -1$ for some ω, then

$1 + L(s) = \emptyset$ has a sol'n $s = j\omega$ for some ω

$\Rightarrow$ closed-loop dynamics has poles at $\pm j\omega$, on imag Axis

$\Rightarrow$ Such poles are on the boundary between bad and ugly

$\Rightarrow$ This situation must be avoided!!!

Now if $\boxed{\angle L(j\omega) = -1}$ for some $\omega > \emptyset$, then:

$\Rightarrow$ Polar plot of $L(j\omega)$ passes through -1

$\Rightarrow$ $\omega_a = \omega_\gamma$ (both crossover freqs same)

$\Rightarrow$ $\boxed{a = \emptyset\ dB,\ \gamma = \emptyset^\circ}$ (both margins $\emptyset$)

Any such feedback loop is bad!

Now, suppose $\exists \omega \geq \emptyset \ni$: $L(j\omega) \approx -1$ (i.e. close to, but not exactly -1)

By continuity of $L(s)$, $1 + L(s) = 0$ would have a sol'n very near (but not exactly on) the imag axis.

Some poles of $T(s)$ would be in bad or ugly region
$\Rightarrow$ Also undesireable!

Now, if $L(j\omega) \approx -1$ for some $\omega \geq \emptyset$

$\Rightarrow$ polar plot of $L(j\omega)$ comes very close to -1 but doesn't pass exactly through it

$\Rightarrow$ (typically) $|a_{dB}|$ and $|\gamma|$ very small (small margins)

$\Rightarrow$ $\boxed{\text{This should Also be avoided.}}$

Thus, for $T(s)$ to have only good poles, we need conditions:

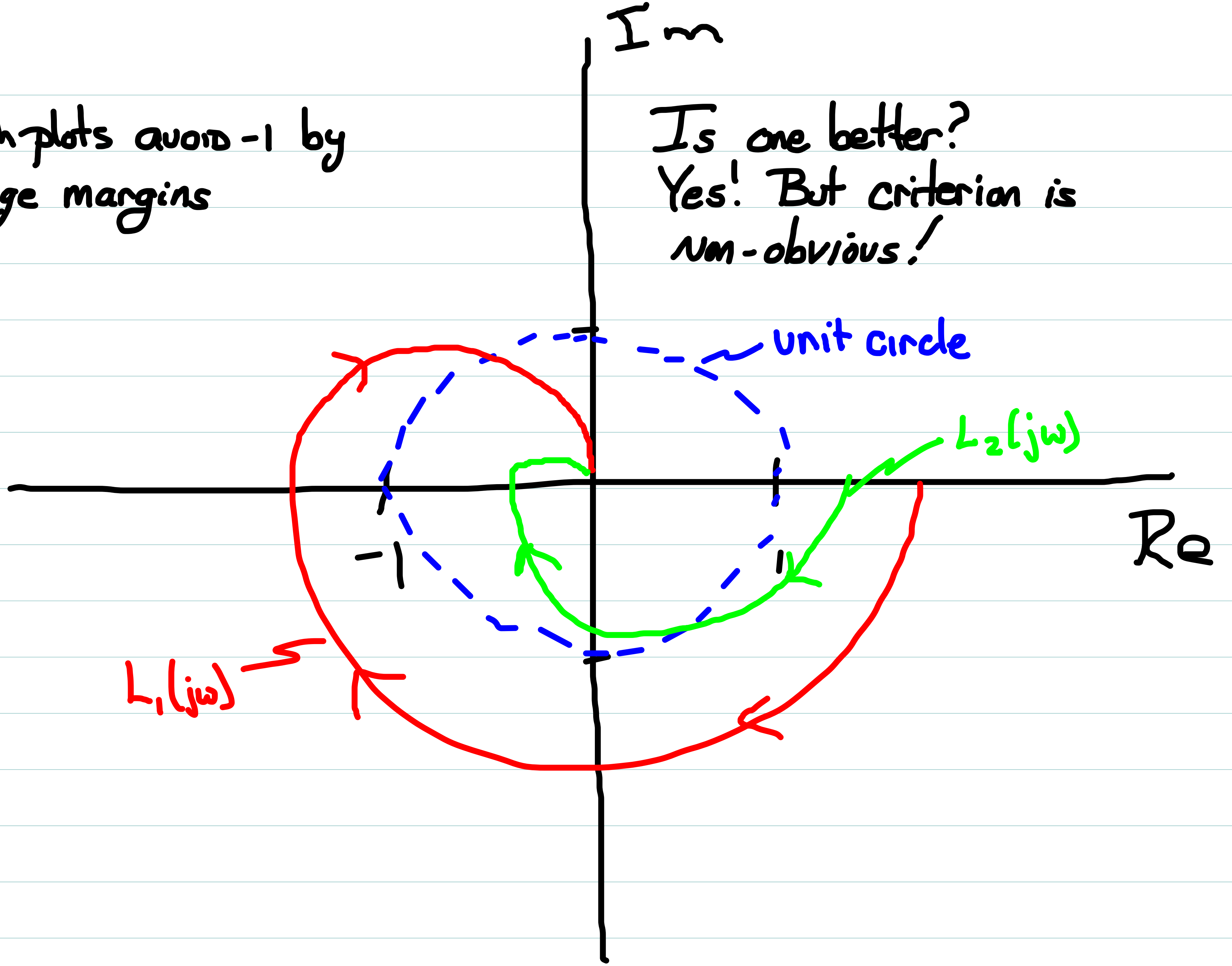
$\Rightarrow$ Gain and phase margins $\boxed{\text{of } L(s)}$ $\Leftarrow$!!! to be Large

$\Rightarrow$ polar plot of $L(j\omega)$ avoids -1 by wide margins

Necessary, but <u>Not sufficient</u>!

Both plots avoid -1 by large margins

Is one better?
Yes! But criterion is non-obvious!



Nyquist Stability Criterion

All roots of $1+L(s)=\emptyset$ are in LHP if

the Nyquist diagram (a modified polar plot) $\boxed{\text{of } L(j\omega)}$

circles the -1 point the correct number of times.

⇒ Major Theoretical result! Used extensively in control theory

⇒ Questions to answer

⇒ How to create diagram from polar?

⇒ How to count encirclements of -1?

⇒ How many encirclements needed?