"Phasor" Notation

Observation: Complex number add'n/sub'n follows same rules as 2D (planar) vectors

$z_1 = a_1 + b_1 j, \quad z_2 = a_2 + b_2 j$ $\qquad$ $\underline{V}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad \underline{V}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$

$z_3 = z_1 + z_2$ $\qquad$ $\underline{V}_3 = \underline{V}_1 + \underline{V}_2$

$= (a_1 + a_2) + (b_1 + b_2) j$ $\qquad$ $= \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$

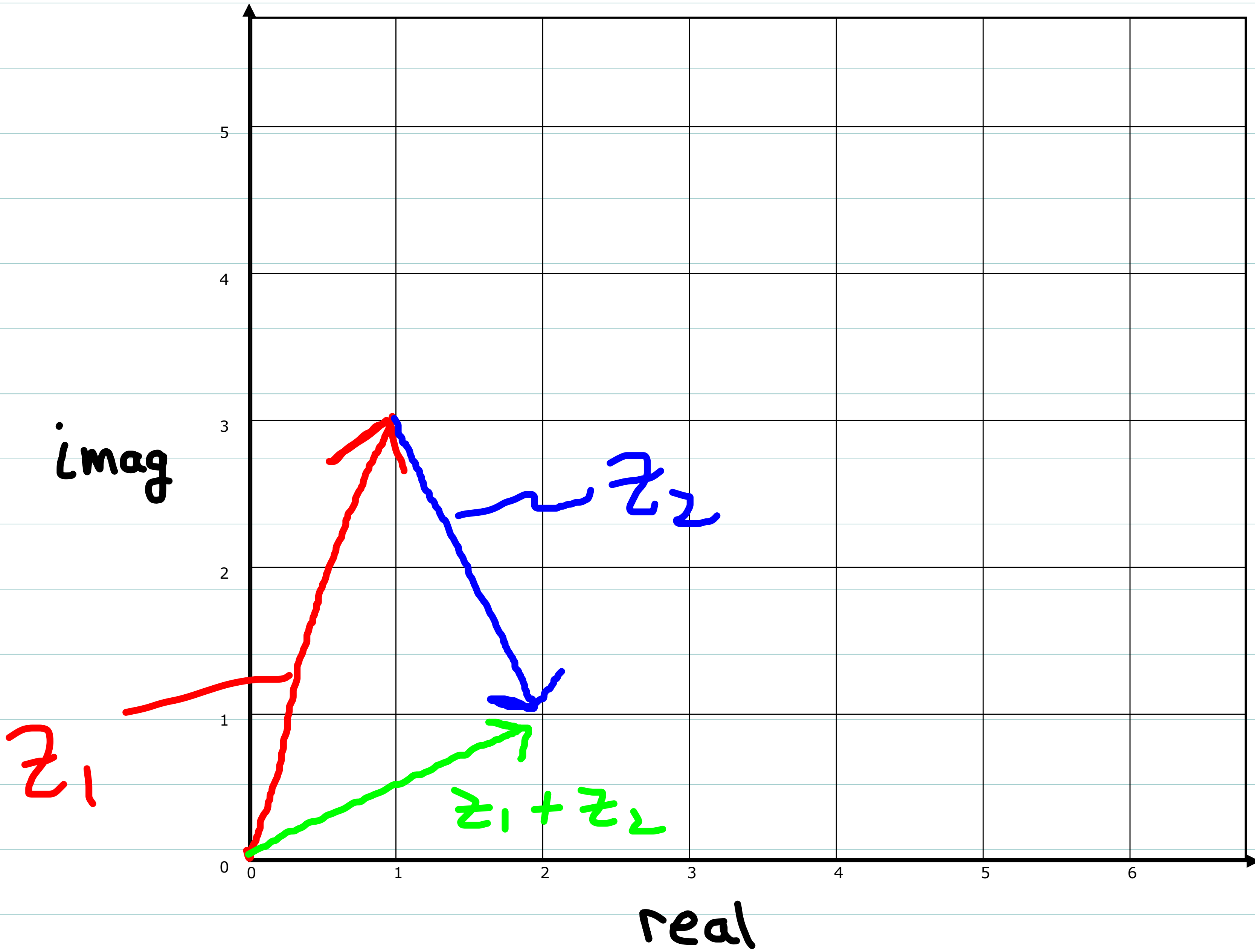
i.e. identify real part with 1st component of 2D vector; imag part with 2nd component.

⟹ Can interpret complex numbers as planar vectors

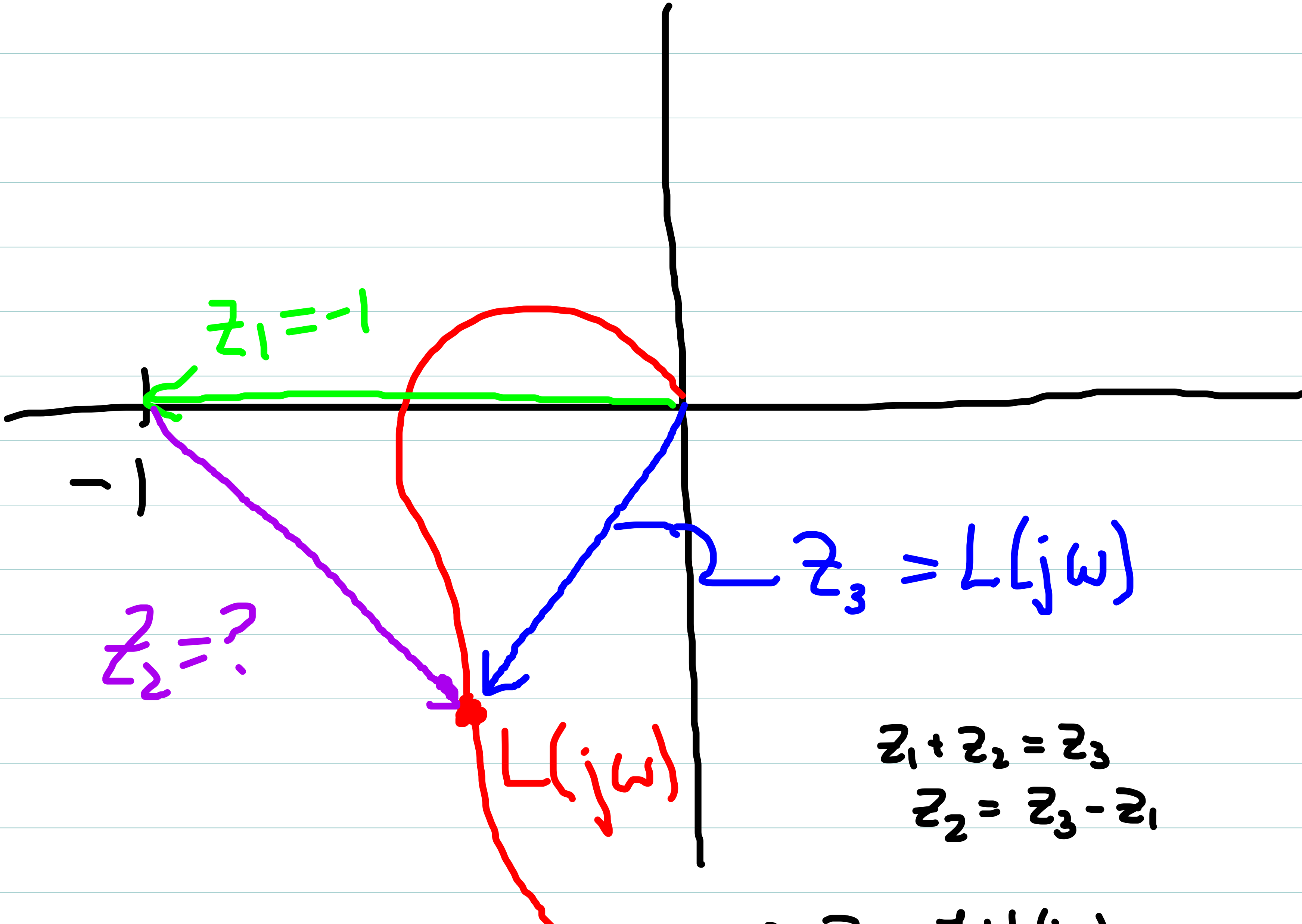
⟹ Can use vector graphical add'n tricks for complex numbers

Example

$z_1 = 1+3j, \; z_2 = 1-2j \Rightarrow z_3 = z_1 + z_2 = 2+j$



Important Application



$$z_1 + z_2 = z_3$$
$$z_2 = z_3 - z_1$$

$$\Rightarrow z_2 = 1 + L(j\omega)$$

Thus:

Complex number $1+L(j\omega)$ can be graphically visualized as the phasor from -1 to $L(j\omega)$ on polar plot.

$\Rightarrow$ $|1+L(j\omega)|$ is the distance from -1 to polar plot at freq ω.

$\Rightarrow$ Good robustness requires this doesn't get too small!

$\Rightarrow$ But note: $|1+L(j\omega)| = |S(j\omega)|^{-1}$

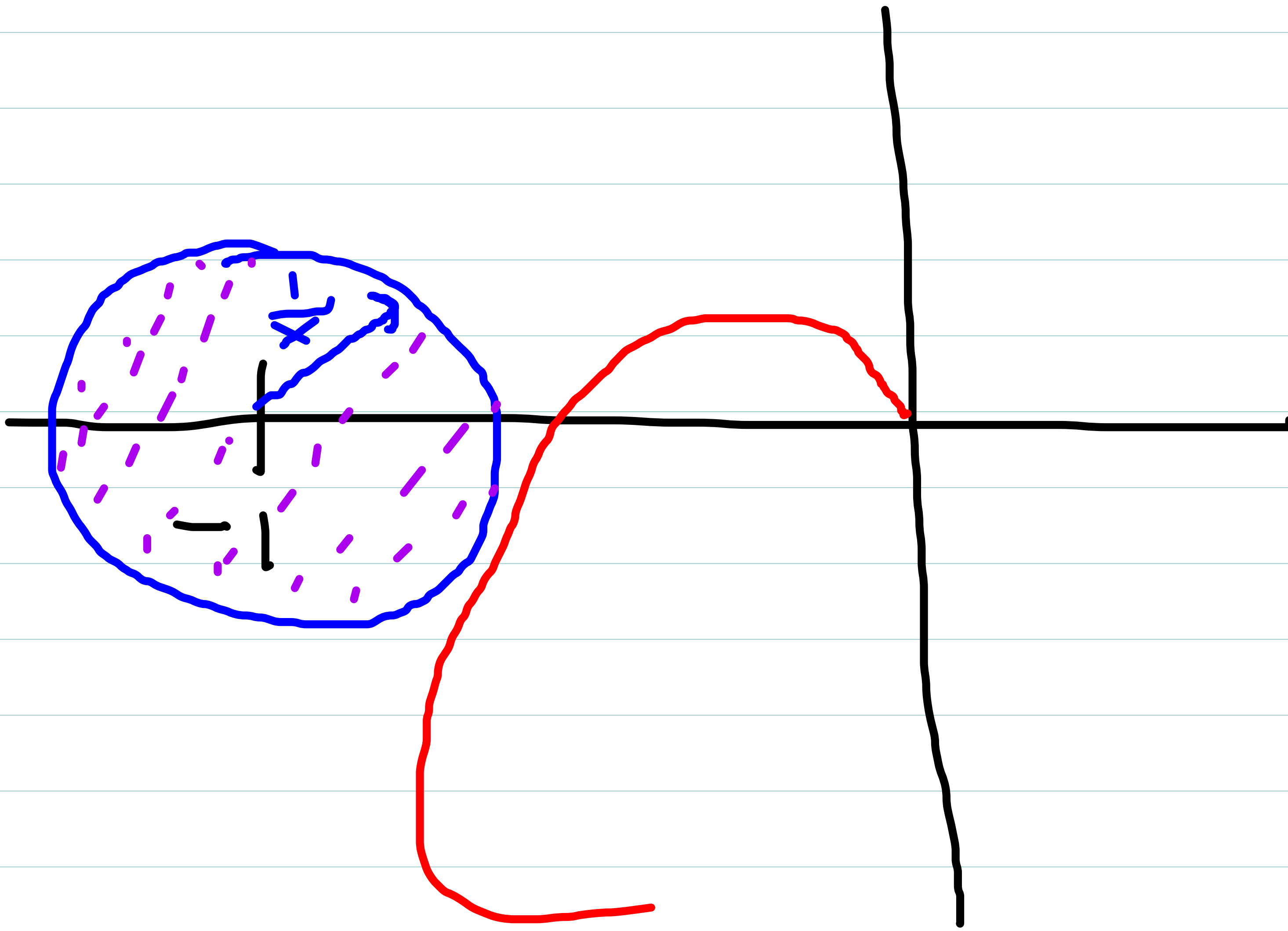
$\Rightarrow$ Thus, good robustness requires $|S(j\omega)|$ not get too big.

$\Rightarrow$ Good designs have $|S(j\omega)|$ which do not exhibit a large <u>peak</u>!

Now:

$$|S(j\omega)|_{max} < X \Rightarrow |1 + L(j\omega)| > \frac{1}{X} \text{ for all } \omega \geq 0$$

$\Rightarrow$ Polar (Nyquist) diagram of $L(j\omega)$ cannot enter a disk of radius $\frac{1}{X}$ centered at -1



This property guarantees certain minimum phase + gain margins

for example, can show: $|S(j\omega)|_{max} < 2$ (+6 dB) $\Rightarrow |1 + L(j\omega)| > 1/2$

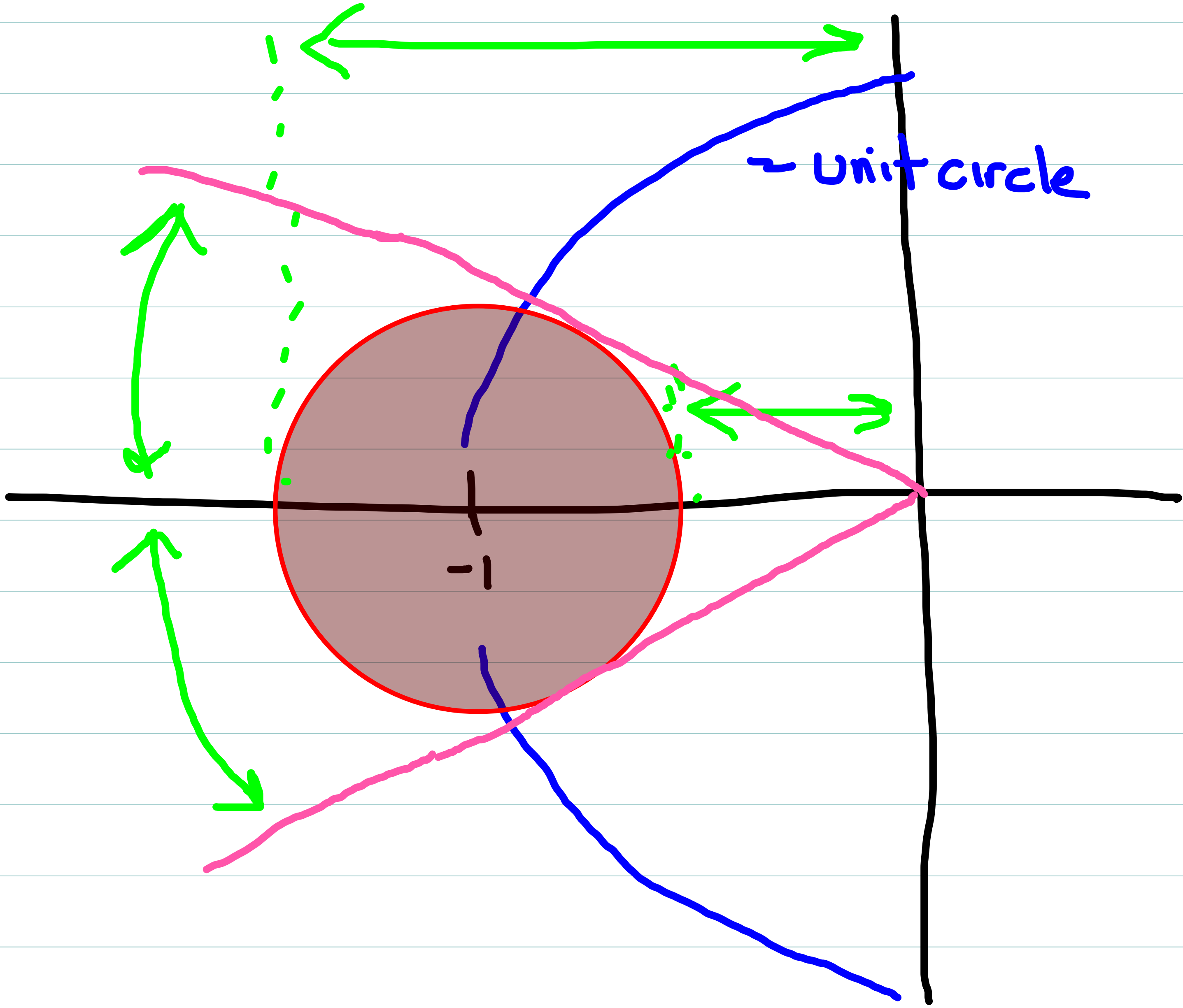
$\Rightarrow a < 2/3$ (−3.5 dB), $a > 2$ (+6 dB)

$\Rightarrow |\gamma| > 29°$

(Note that these are pretty close to the common industry standard req'ts: $|a|_{dB} \geq 6$, $|\gamma| \geq 30°$)

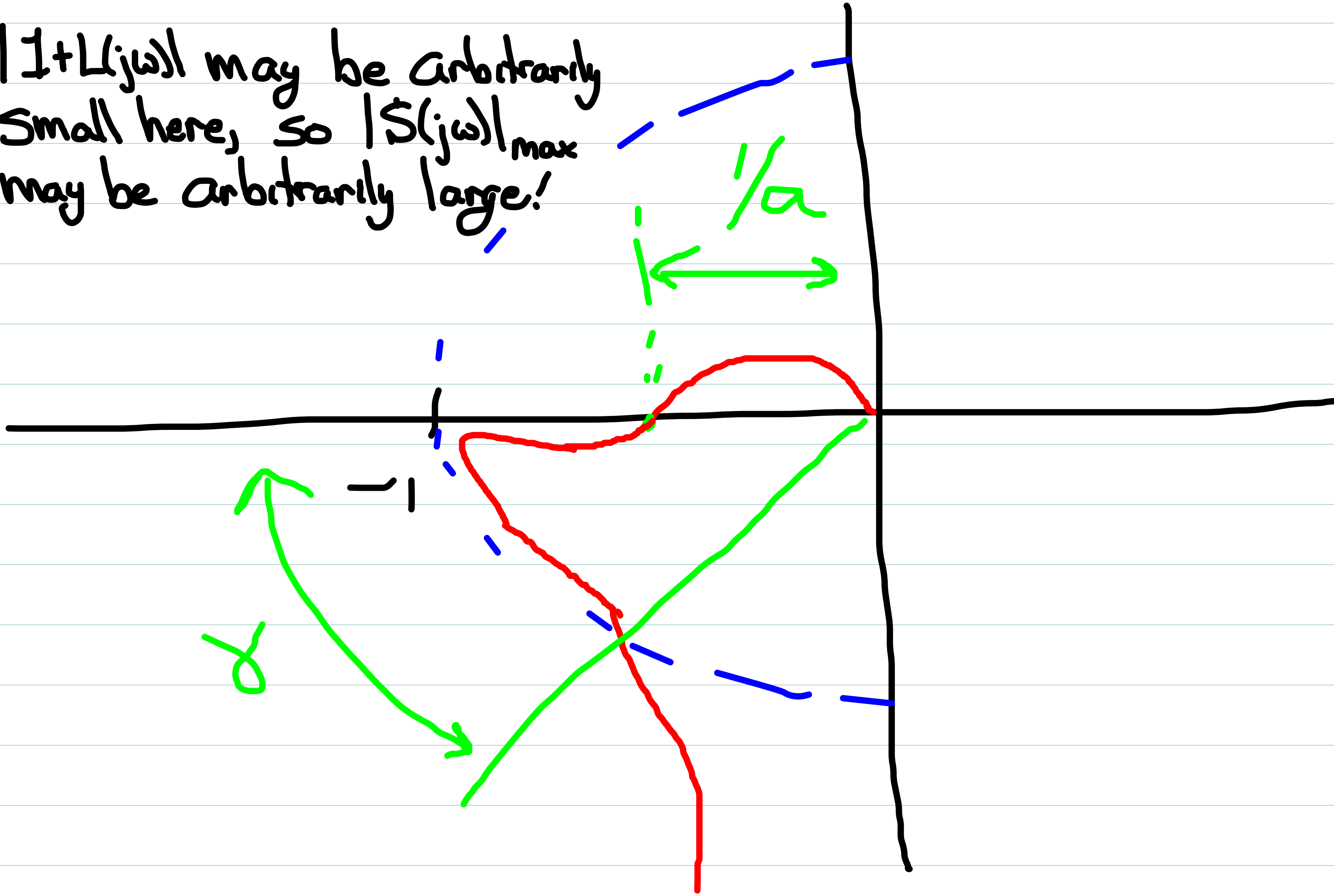
<u>However</u>, a specific set of gain, phase margins <u>does</u> <u>not</u> conversely guarantee a bound on $|S(j\omega)|_{max}$ (as shown in previous example!)

$\Rightarrow |S(j\omega)|_{max}$ (peak of sensitivity diagram) is a superior measure of robustness, and $|S(j\omega)|_{max} \lesssim +6$ dB is a good nominal target.



-1
← unit circle

$|1+L(j\omega)|$ may be arbitrarily small here, so $|S(j\omega)|_{max}$ may be arbitrarily large!



We can do much more with this idea!

Let $G_0(s)$ be our nominal plant model (what we use in Matlab)

Let $G(s)$ be the "true" plant TF (unknown)

Define:
$$\Delta(s) = \left[\frac{G(s) - G_0(s)}{G_0(s)}\right] = \left[\frac{G(s)}{G_0(s)} - 1\right]$$

A normalized measure of error in nominal model

We don't know what $\Delta(s)$ is, but may be able to place bounds on how "big" it can be to still ensure stability of feedback system.

Let:

$L_o(s) = G_o(s)\,H(s)$ Nominal OL TF

$L(s) = G(s)\,H(s)$ True OL TF

The def'n of $\Delta(s)$ implies $G(s) = G_o(s)\left[1+\Delta(s)\right]$

Hence $L(s) = G_o(s)\,H(s)\left[1+\Delta(s)\right]$

$= G_o(s)H(s) + G_o(s)H(s)\Delta(s)$

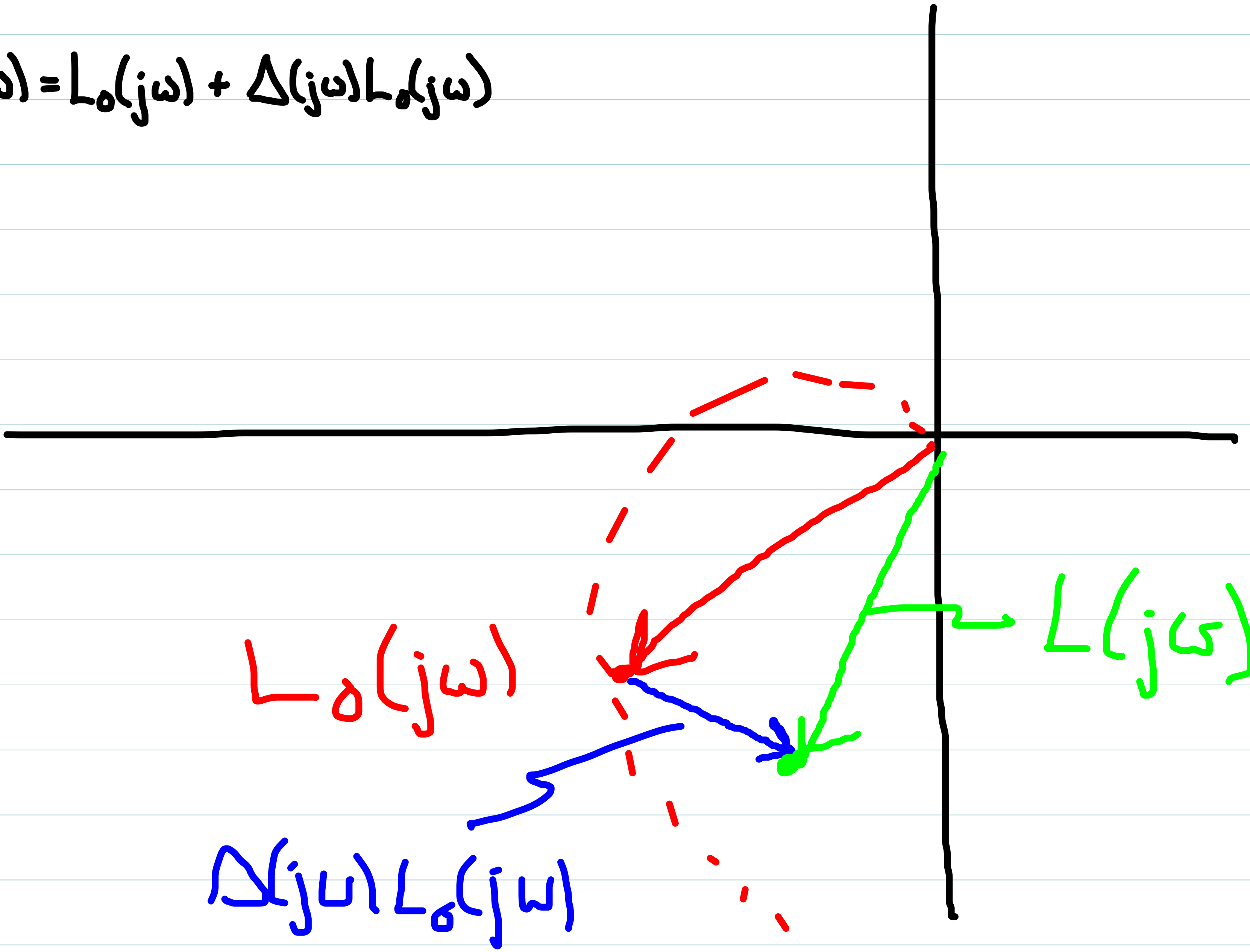
Or: $L(s) = L_o(s) + L_o(s)\Delta(s)$

and for each $\omega \geq 0$:

$$\underbrace{L(j\omega)}_{\text{true polar plot}} = \underbrace{L_o(j\omega)}_{\text{Nominal Polar Plot}} + \overbrace{L_o(j\omega)\Delta(j\omega)}^{\text{effect of model error on polar plot}}$$

Phasor Interpretation

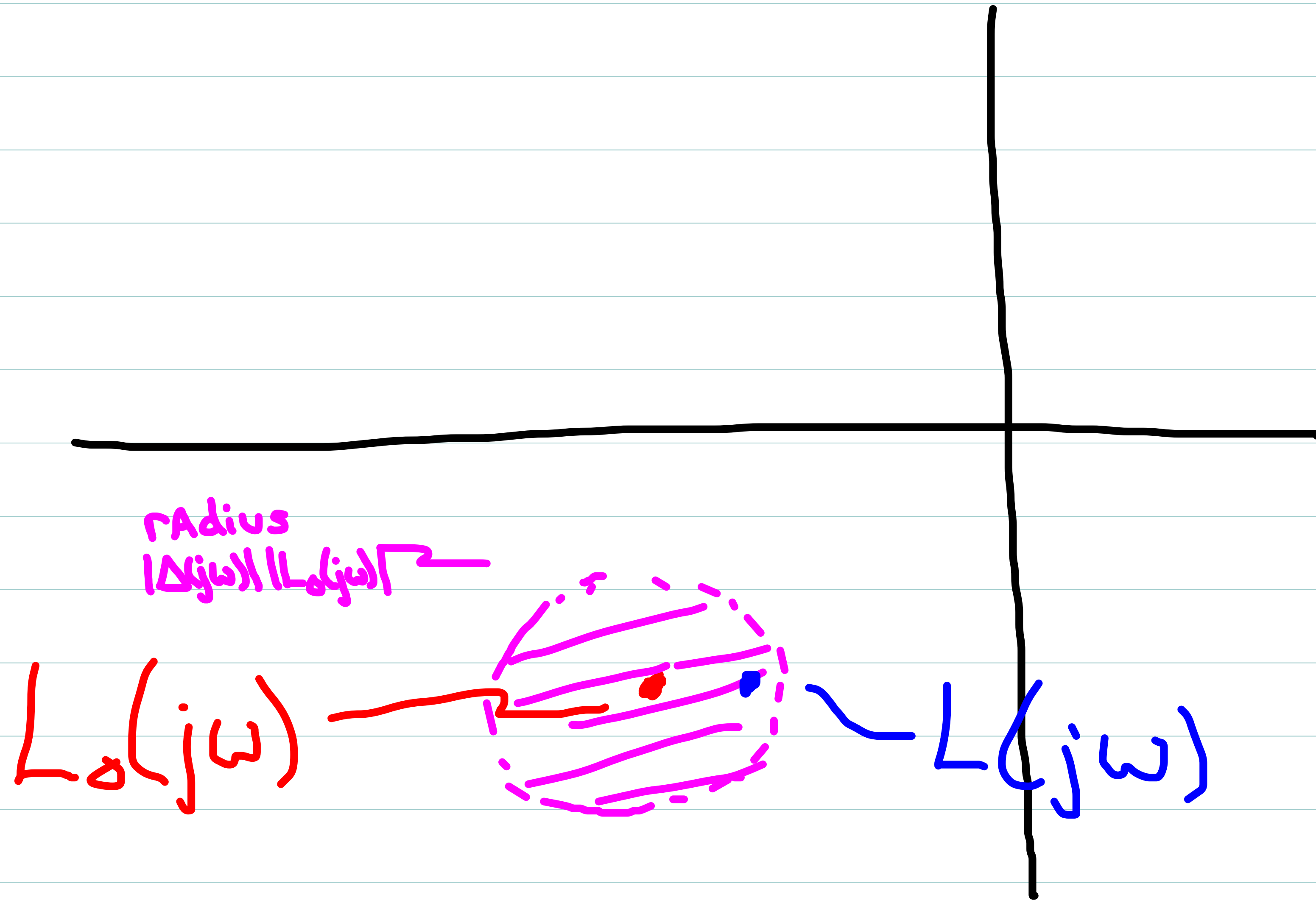
$L(j\omega) = L_o(j\omega) + \Delta(j\omega)L_o(j\omega)$



<u>Note</u>: $\Delta(j\omega)$ has unknown magnitude and direction

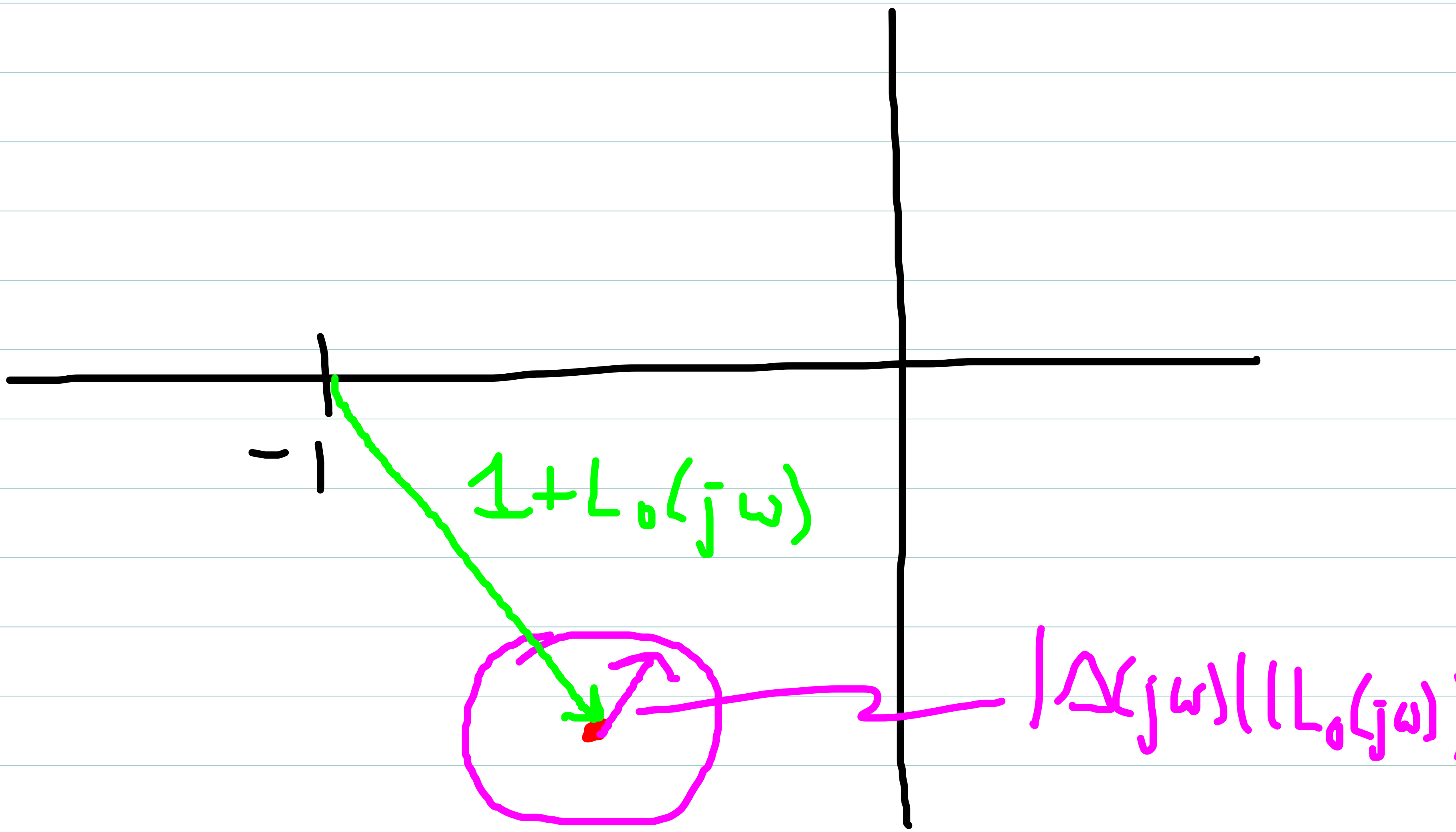
<u>Assume</u>: $\Delta(j\omega)$ can have <u>any</u> direction (worst case).

$\Rightarrow$ $L(j\omega)$ can lie anywhere in a <u>disk</u> of radius $|\Delta(j\omega)||L_0(j\omega)|$ centered at $L_0(j\omega)$



In order to ensure $\Delta(s)$ cannot change number of encirclements:

Each disk of radius $|\Delta(j\omega)||L_0(j\omega)|$ centered at $L_0(j\omega)$ must not extend to -1 point



This can be ensured if:

$$\underbrace{|\Delta(j\omega)||L_o(j\omega)|}_{\text{Radius of Disk}} < \underbrace{|1+L_o(j\omega)|}_{\text{Distance from }-1\text{ to center of disk}} \quad \text{for all } \omega \geq 0$$

Re-arranging:

$$\frac{|L_o(j\omega)|}{|1+L_o(j\omega)|} < |\Delta(j\omega)|^{-1} \quad \text{for all } \omega \geq 0$$

Note that

$$T_o(s) = \frac{L_o(s)}{1+L_o(s)}$$ is the <u>nominal</u> CL TF

So the required condition is:

$$\boxed{|T_o(j\omega)| < |\Delta(j\omega)|^{-1} \quad \text{for all } \omega \geq 0}$$

Uncertainty robustness test

Graphical Interpretation

The Bode magnitude plot $|T_o(j\omega)|$ must lie below the graph of $|\Delta(j\omega)|^{-1}$ at every frequency.

"Multiplicative" Uncertainty Robustness Test

with

$$\Delta(s) = \left[\frac{G(s)}{G_o(s)} - 1\right]$$

test is:

$$|T_o(j\omega)| < |\Delta(j\omega)|^{-1} \text{ for every } \omega$$

Guarantees closed-loop stability only. Performance will generally suffer

Given an assumed bound on magnitude $|\Delta(j\omega)|$

Note: Simultaneous gain/phase uncertainty easily handled in this framework. If plant gain uncertain and time delay present, then

$$\Delta(s) = \left[\frac{K_p}{K_o} e^{-sT_s} - 1\right]$$

Where K_p is true gain of plant, K_o is assumed gain, and T is delay length. Can graph $|\Delta(j\omega)|^{-1}$ given bounds on T and (K_p/K_o).

Note: Test is inherently conservative. If it fails, $T'(s)$ may be unstable, but not necessarily.

For example, with pure time delay uncertainty

$$|\Delta(j\omega)| = |e^{-j\omega T_s} - 1|$$

The above test yields predictions for T_{max} which are about 5-10% shorter than phase margin analysis gives

In this case, the phase margin analysis is exact.

Discrepancy with Δ test is because there exist $\Delta(s)$ with the same magnitude bound as $|e^{-j\omega T_s} - 1|$ which would result in an unstable $T'(s)$. However, these $\Delta(s)$ would include other terms than pure delay.

But only Δ test lets us look at impact of <u>simultaneous</u> gain/phase changes, including effects of

$\Rightarrow$ <u>uncertain</u> pole/zero locations in $G(s)$

$\Rightarrow$ <u>neglected</u> pole/zero locations in $G(s)$

<u>Typically</u>: $|\Delta(j\omega)|$ is small at low frequencies, increases at higher freqs.

$\Rightarrow$ Effects of model errors on freq. response accumulate as freq. increases

<u>Then</u>: Bound on $|T_0(j\omega)|$ is large at low freqs, small at high freqs.

Example: Suppose $G_o(s)$ neglects a pole in $G(s)$, but is otherwise identical:

Then: $\Delta(s) = \left[\frac{1}{\tau s+1} - 1\right] = \frac{-\tau s}{\tau s+1} \Rightarrow \Delta^{-1}(s) = \frac{\tau s+1}{-\tau s}$

Now look at "typical" shapes for $|T_o(j\omega)|$

$$T_o(s) = \frac{L(s)}{1+L(s)}, \quad |T_o(j\omega)| = \frac{|L_o(j\omega)|}{|1+L_o(j\omega)|}$$

Typically, $|L_o(j\omega)| \gg 1$ for small ω (especially if $L_o(s)$ has at least 1 pole at origin)

$\Rightarrow |T_o(j\omega)| \approx 1$ (0 dB) for small ω.

Since relative degree of $L_o(s)$ is positive for any physical system, $|L_o(j\omega)| \to \phi$ as $\omega \to \infty$, and thus

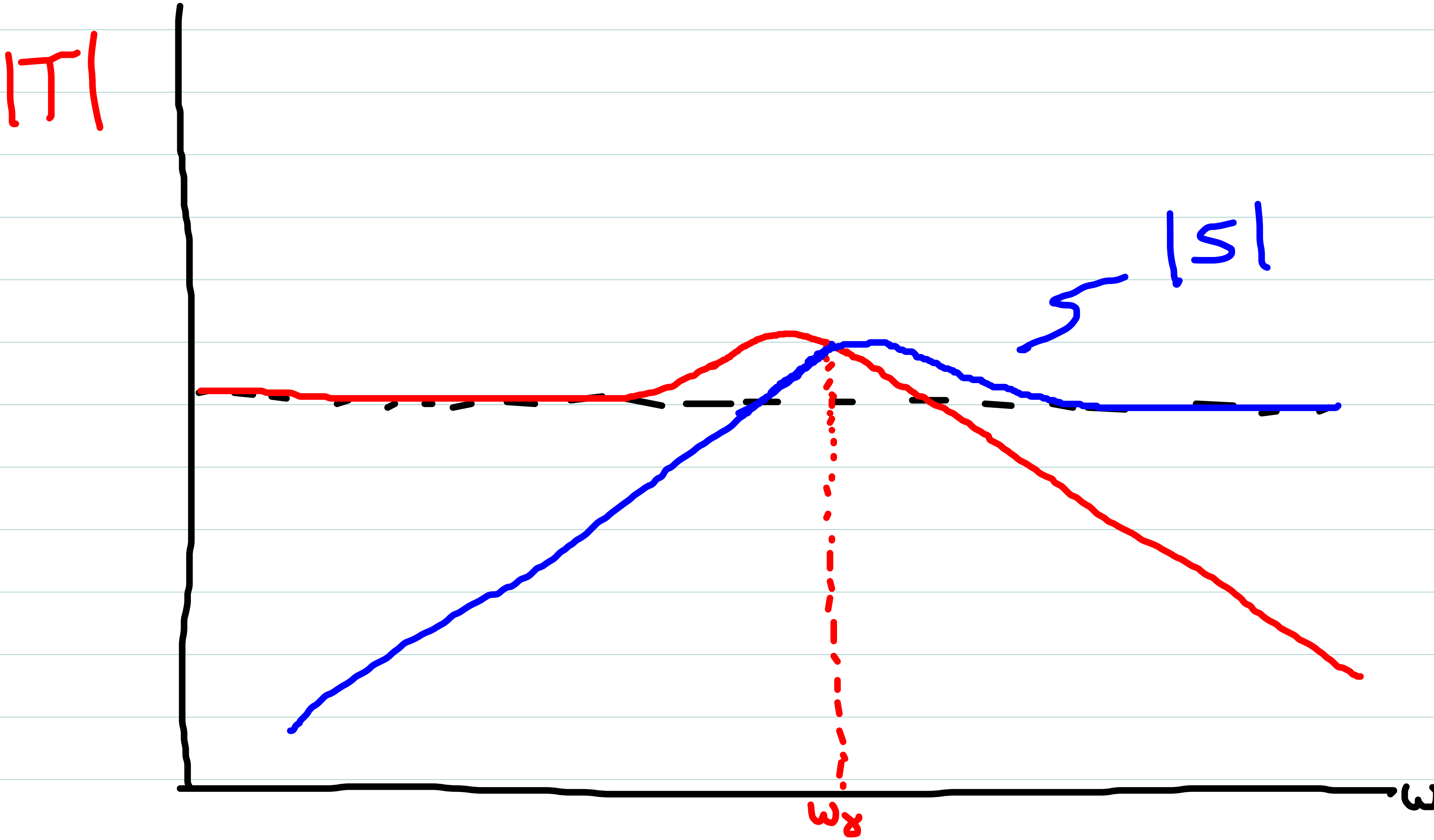
$|T_o(j\omega)| \approx |L_o(j\omega)|$ at high freq. and $|T_o(j\omega)| \to \phi$ also

Finally, note $$|T_o(j\omega_\gamma)| = \frac{|L_o(j\omega_\gamma)|}{|1+L_o(j\omega_\gamma)|} = \frac{1}{|1+L_o(j\omega_\gamma)|}$$

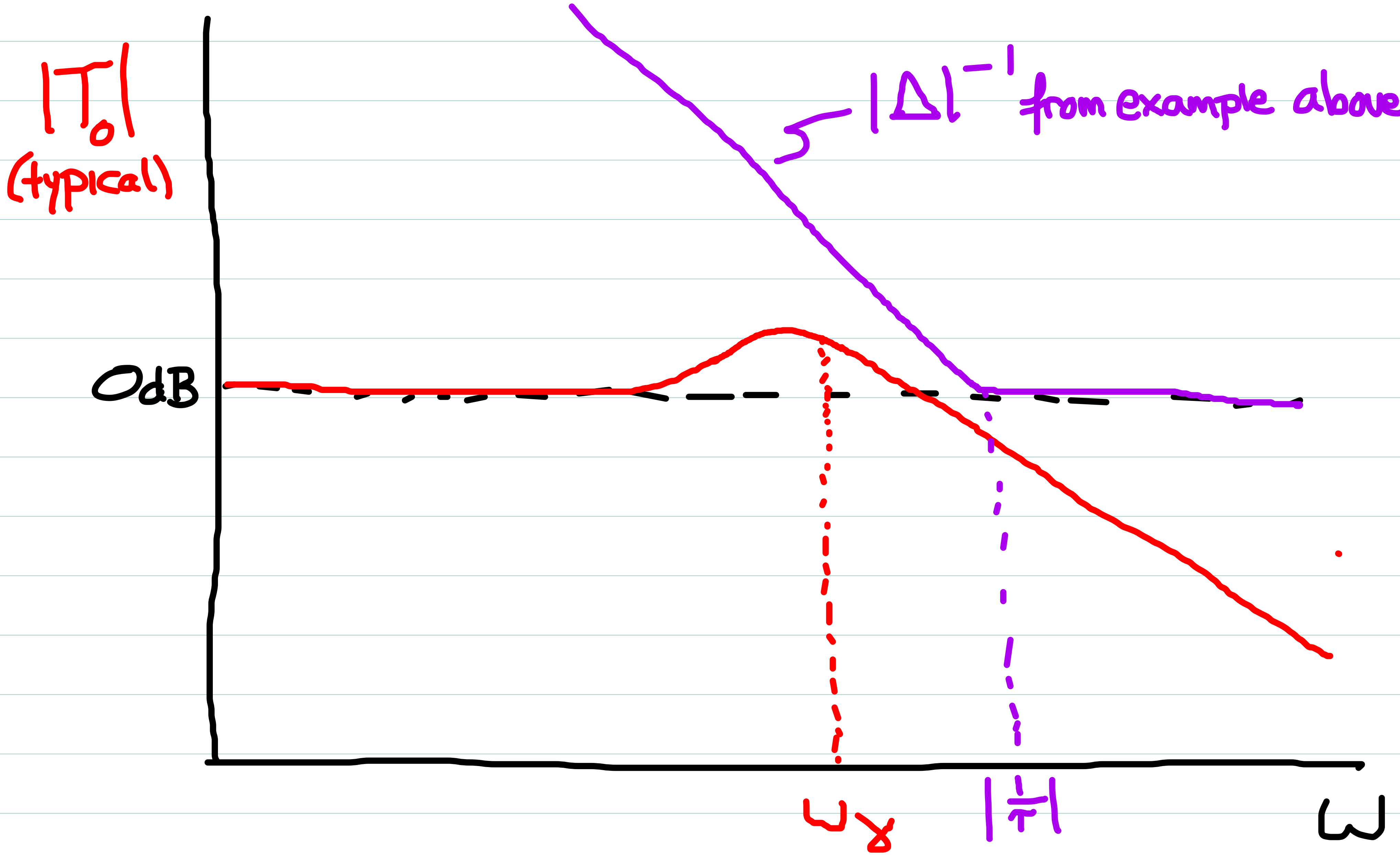
So $$|T_o(j\omega_\gamma)| = |S(j\omega_\gamma)| = \frac{1}{2\sin(\gamma/2)}$$

hence $|T_o|$ is also peaking near ω_γ.


$|T_0|$
(typical)
0dB
$\left|\frac{1}{2\sin(\gamma/2)}\right|_{dB}$
$\approx |L_0|$
ω_γ
ω



Note: $|T_o|$ and $|S_o|$ "complementary" in sense that $|S_o| \approx 0$ when $|T_o| \approx 1$ and vice-versa.
Reflects algebraic identity $S(s) + T(s) = 1$ from def'ns.



Remember: must keep graph of $|T_o(j\omega)|$ <u>below</u> $|\Delta(j\omega)|^{-1}$ at every frequency

Design Implication of robustness

Uncertainty constrains size of ω_γ!

In specific example above, we'd need ω_γ significantly less than freq. $(\frac{1}{\tau})$ of neglected pole.

When $G(s)$ has "unmodeled dynamics" (i.e. poles/zeros neglected in nominal model $G_0(s)$), usually want ω_γ a <u>decade below</u> suspected freq. of neglected poles.

Recall, ω_γ is correlated w/ closed-loop settling time. Above observation means this should be slow compared to neglected poles. We need to avoid control actions so sharp and quick they might "excite" the unmodeled dynamics.