Laplace Transform

More formally, for any $f(t)$ define:

(1) $$f(t) = \frac{1}{2\pi j} \int F(s) e^{st}\, ds$$

Normalizing constant

where:

(2) $$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Notation:

$F(s) = \mathcal{L}\{f(t)\}$ (transform)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$ (inverse transform)

Limitations of Laplace Transform

Only defined for $f(t)$ where the integral (2) converges.

Requires: $e^{-\sigma_0 t}|f(t)| \to \emptyset$

for some finite $\sigma_0 \in \mathbb{R}$

The transform $F(s)$ is then defined for any

$s = \sigma + j\omega$ with $\sigma \geq \sigma_0$

and the integral (1) is over all values of s which satisfy this condition. "region of convergence"

Examples

$f(t) = e^{pt}$ can be transformed for any finite $p \in \mathbb{C}$

However, $f(t) = e^{t^2}$ cannot be transformed since $e^{-\sigma_0 t} f(t) = e^{(t^2 - \sigma_0 t)} \to \infty$

for any finite σ_0.

Note:

When working with Laplace transforms we assume we are using values of s in the region of convergence. (ROC)

By above def'n of ROC,

$$\lim_{t \to \infty} e^{-st} f(t) = \emptyset$$

for these values of s.

Fundamental Transform

(only one you need!)

$$\boxed{\mathcal{L}\{e^{pt}\} = \frac{1}{s-p}} \qquad \forall p \in \mathbb{C}$$

$$\mathcal{L}\{e^{pt}\} = \int_0^\infty e^{pt}e^{-st}\,dt = \int_0^\infty e^{(p-s)t}\,dt$$

$$= \left[\left(\frac{1}{p-s}\right)e^{(p-s)t}\right]_{t=0}^{t=\infty}$$

$$= \left(\frac{1}{p-s}\right)\left[\cancel{e^{(p-s)\infty}} - 1\right]$$

$\emptyset$

for any s in ROC

Property #1: Linearity

$$\boxed{\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)}$$

for any transformable functions $f_1(t), f_2(t)$
any (complex) constants a_1, a_2

$$\int_0^\infty \{a_1 f_1(t) + a_2 f_2(t)\} e^{-st} dt$$

$$= a_1 \underbrace{\boxed{\int_0^\infty f_1(t) e^{-st} dt}}_{F_1(s)} + a_2 \underbrace{\boxed{\int_0^\infty f_2(t) e^{-st} dt}}_{F_2(s)}$$

And generally:

$$\mathcal{L}\left\{\sum^N a_i f_i(t)\right\} = \sum^N a_i F_i(s)$$

Linearity lets us build more complex transforms:

Consider: $f(t) = Ae^{at}\cos(bt+\psi)$

$$= ce^{pt} + \bar{c}e^{\bar{p}t}$$

with $p = a+bj$, $c = \left(\frac{A}{2}\right)e^{j\psi}$ (polar form)

Then by linearity

$$\boxed{\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{(s-\bar{p})}}$$

We can combine the two terms:

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

$$= \frac{A[(s-a)\cos\psi - b\sin\psi]}{s^2-2as+(a^2+b^2)}$$

so

$$\boxed{\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{A[(s-a)\cos\psi - b\sin\psi]}{(s-a)^2+b^2}}$$

But we will see it is often easier to keep the two terms separate when solving problems.

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\varphi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a + bj$ and $c = \left(\frac{A}{2}\right)e^{j\varphi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a + bj$ and $c = \left(\frac{A}{2}\right)e^{j\psi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

$$f(t) = c = c\,e^{0t} \Longrightarrow F(s) = \left.\frac{c}{s-p}\right]_{p=0}$$

Fundamental Transforms

$$\mathcal{L}\{e^{pt}\} = \frac{1}{s-p} \quad \text{for any } p \in \mathbb{C}$$

$$\mathcal{L}\{Ae^{at}\cos(bt+\psi)\} = \frac{c}{s-p} + \frac{\bar{c}}{s-\bar{p}}$$

with $p = a+bj$ and $c = \left(\frac{A}{2}\right)e^{j\psi}$

What is $\mathcal{L}\{c\}$ for an arbitrary constant c?

i.e. $\mathcal{L}\{f(t)\}$ with $f(t) = c$ for all $t \geq 0$

$$f(t) = c = c\,e^{\emptyset t} \implies F(s) = \left.\frac{c}{s-p}\right]_{p=\emptyset}$$

Hence $\boxed{\mathcal{L}\{c\} = \frac{c}{s}}$ for any $c \in \mathbb{C}$

Common Mistakes

$$\mathcal{L}\{c\} \neq c \qquad \left(\mathcal{L}\{c\} = \frac{c}{s}\right)$$

$$\mathcal{L}\{f_1(t)f_2(t)\} \neq F_1(s)F_2(s)$$

$\mathcal{L}\{f_1(t)f_2(t)\} =$ <unspeakably ugly>

Property #2: Diff'n rule

$$\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$$

$$= \int_0^\infty \frac{df}{dt} e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,df$$

$$(\text{by parts}) \quad = \left[e^{-st} f(t)\right]_{t=0}^{t=\infty} + s\,\underbrace{\boxed{\int_0^\infty f(t)e^{-st}\,dt}}_{F(s)}$$

$$= sF(s) - f(0)$$

Higher Derivatives

$$\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\{\dot{f}_1(t)\} \quad \text{with } f_1(t) = \dot{f}(t)$$

$$= sF_1(s) - f_1(0)$$

but $F_1(s) = \mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$

So $\boxed{\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - \dot{f}(0) - sf(0)}$

and generally

$$\boxed{\mathcal{L}\{f^{(k)}(t)\} = s^kF(s) - f^{(k-1)}(0) - sf^{(k-2)}(0) - \cdots - s^{k-1}f(0)}$$

<u>Note:</u> Laplace will allow us to <u>directly</u> account for IC effects (No Linear algebra!)

Property #3: "t-mult" rule

$$\boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds} F(s)}$$

$$\begin{aligned}
\mathcal{L}\{t f(t)\} &= \int_0^\infty t f(t) e^{-st}\, dt \\
&= \int_0^\infty f(t) \left[t e^{-st} \right] dt \\
&= \int_0^\infty f(t) \left[-\frac{d}{ds} e^{-st} \right] dt \\
&= \int_0^\infty -\frac{d}{ds} \left[f(t) e^{-st} \right] dt \\
&= -\frac{d}{ds} \int_0^\infty f(t) e^{-st}\, dt
\end{aligned}$$

Use of t-mult rule

$$\mathcal{L}\{te^{pt}\} = -\frac{d}{ds}\left[\frac{1}{s-p}\right]$$

$$= \frac{1}{(s-p)^2}$$

Similarly $\mathcal{L}\{t^2e^{pt}\} = \mathcal{L}\{t f_1(t)\}$, $f_1(t) = te^{pt}$

$$= -\frac{d}{ds}F_1(s) = -\frac{d}{ds}\left[\frac{1}{(s-p)^2}\right]$$

So $\mathcal{L}\{t^2e^{pt}\} = \dfrac{2}{(s-p)^3}$

Generally:

$$\boxed{\mathcal{L}\{t^k e^{pt}\} = \frac{k!}{(s-p)^{k+1}}}$$

Recap: Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int F(s)e^{st}\,ds$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

Properties:

1.) Linearity: $\mathcal{L}\left\{\sum_{i=1}^{N} a_i f_i(t)\right\} = \sum_{i=1}^{N} a_i F_i(s)$

2.) Diff. rule: $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$

$$\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - \dot{f}(0) - sf(0)$$

$$\vdots$$

$$\mathcal{L}\{f^{(k)}(t)\} = s^kF(s) - f^{(k-1)}(0) - \cdots - s^{k-1}f(0)$$

Use of $\mathcal{L}T$ for Diff'l Eqn

$$\mathcal{L}\{\alpha_n y^{(n)}(t) + \cdots + \alpha_1 \dot{y}(t) + \alpha_0 y(t)\}$$

$$= \mathcal{L}\{\beta_m u^{(m)} + \cdots + \beta_1 \dot{u}(t) + \beta_0 u(t)\}$$

Gives:

$$\alpha_n\left[s^n Y(s) - y^{(n-1)}(0) - s y^{(n-2)}(0) - \cdots - s^{n-1} y(0)\right]$$

$$+ \cdots + \alpha_1\left[sY(s) - y(0)\right] + \alpha_0 Y(s)$$

$$= \beta_m\left[s^m U(s) - u^{(m-1)}(0) - s u^{(m-2)}(0) - \cdots - s^{m-1} u(0)\right]$$

$$+ \cdots + \beta_1\left[sU(s) - u(0)\right] + \beta_0 U(s)$$

Collect Terms

$$\boxed{r(s)Y(s) - c(s) = q(s)U(s) - b(s)}$$

$r(s) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0$

$q(s) = \beta_m s^m + \cdots + \beta_1 s + \beta_0$

$c(s)$ = $n-1$ order polynomial in s from IC terms on $y(t)$

$b(s)$ = $m-1$ order polynomial in s from IC terms on $u(t)$.

Re-arrange for Y(s)

$$Y(s) = \left[\frac{q(s)}{r(s)}\right] U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

IC terms

Or:

$$Y(s) = G(s)U(s) + \left[\frac{c(s) - b(s)}{r(s)}\right]$$

Alternate def'n of TF:

$$G(s) = \left[\frac{Y(s)}{U(s)}\right]_{ICs=0} = \left[\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}\right]_{\substack{\text{zero}\\ \text{ICs}}}$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$(2s^3 + 8s^2 + 14s + 10)Y(s)$$
$$- \left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]$$
$$= (3s^2 + 15s + 18)U(s) - \left[3su_0 + (3\dot{u}_0 + 15u_0)\right]$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2\left[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0\right] + 8\left[s^2Y(s) - \dot{y}_0 - sy_0\right]$$

$$+ 14\left[sY(s) - y_0\right] + 10Y(s)$$

$$= 3\left[s^2U(s) - \dot{u}_0 - su_0\right] + 15\left[sU(s) - u_0\right] + 18U(s)$$

OR:

$$\underbrace{(2s^3 + 8s^2 + 14s + 10)}_{r(s)}Y(s) - \left[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)\right]$$

$$= \underbrace{(3s^2 + 15s + 18)}_{q(s)}U(s) - \left[3su_0 + (3\dot{u}_0 + 15u_0)\right]$$

Example

$$2y^{(3)} + 8\ddot{y} + 14\dot{y} + 10y = 3\ddot{u} + 15\dot{u} + 18u$$

$$2[s^3Y(s) - \ddot{y}_0 - s\dot{y}_0 - s^2y_0] + 8[s^2Y(s) - \dot{y}_0 - sy_0]$$

$$+ 14[sY(s) - y_0] + 10Y(s)$$

$$= 3[s^2U(s) - \dot{u}_0 - su_0] + 15[sU(s) - u_0] + 18U(s)$$

OR:

$$(2s^3 + 8s^2 + 14s + 10)Y(s)$$

$$- \overbrace{[2s^2y_0 + s(2\dot{y}_0 + 8y_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0)]}^{c(s)}$$

$$= (3s^2 + 15s + 18)U(s) - \overbrace{[3su_0 + (3\dot{u}_0 + 15u_0)]}^{b(s)}$$

Thus:

$$Y(s) = \boxed{\left[\frac{3s^2+15s+18}{2s^3+8s^2+14s+10}\right]} U(s)$$

G(s)

$$+ \left[\frac{2s^2 y_0 + s(2\dot{y}_0 + 8y_0 - 3u_0) + (2\ddot{y}_0 + 8\dot{y}_0 + 14y_0 - 3\dot{u}_0 - 15u_0)}{2s^3+8s^2+14s+10}\right]$$

=> We assume all ICs on $y(t)$ Known; and $u(t)$ Known so $U(s)$ can be computed and ICs on $u(t)$

=> All terms on RHS are Known, so we Know $Y(s)$

=> "Simply" invert transform to get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$