# Poles/zeros at origin

Poles at origin (type $N > \emptyset$) or zeros at origin ($N < \emptyset$)

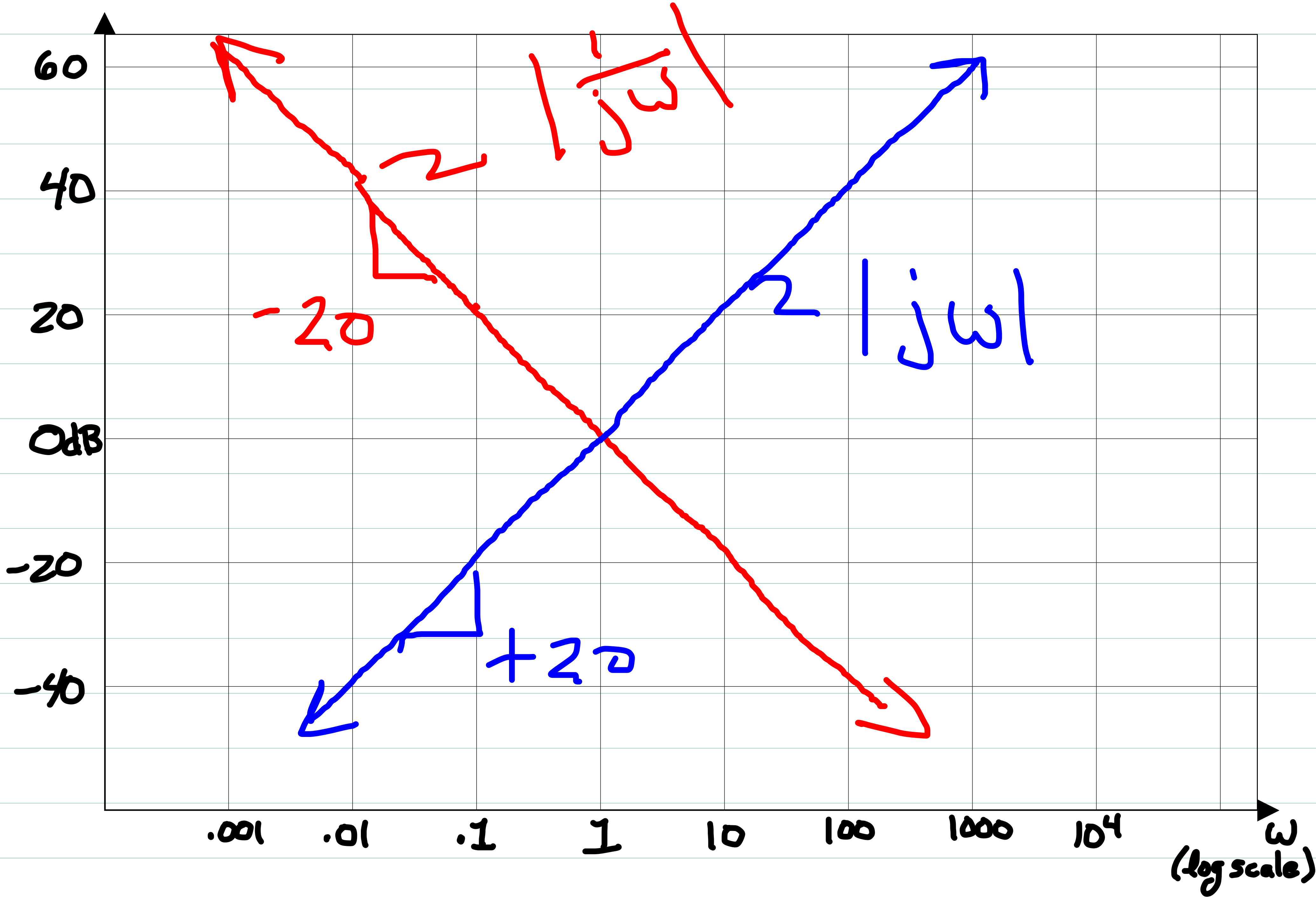
have corner frequencies at $\omega = 0$

$\Rightarrow$ infinitely far to left on horizontal frequency Axis.

These factors do not produce "visible" corners, instead contribute a constant slope of $-20N$ dB/dec for <u>all</u> freqs.

Note also: $|(j\omega)^N| = 1$ at $\omega = 1$ for any $N$

so graph of $|(j\omega)^N|_{dB}$ will pass through 0 dB at $\omega = \underline{1}$



60
40
20
0dB
-20
-40
.001
.01
.1
1
10
100
1000
$10^4$
ω (log scale)
$\angle|j\omega|$
-20
$\angle|j\omega|$
+20

For $G(s)$ with poles/zeros at origin:

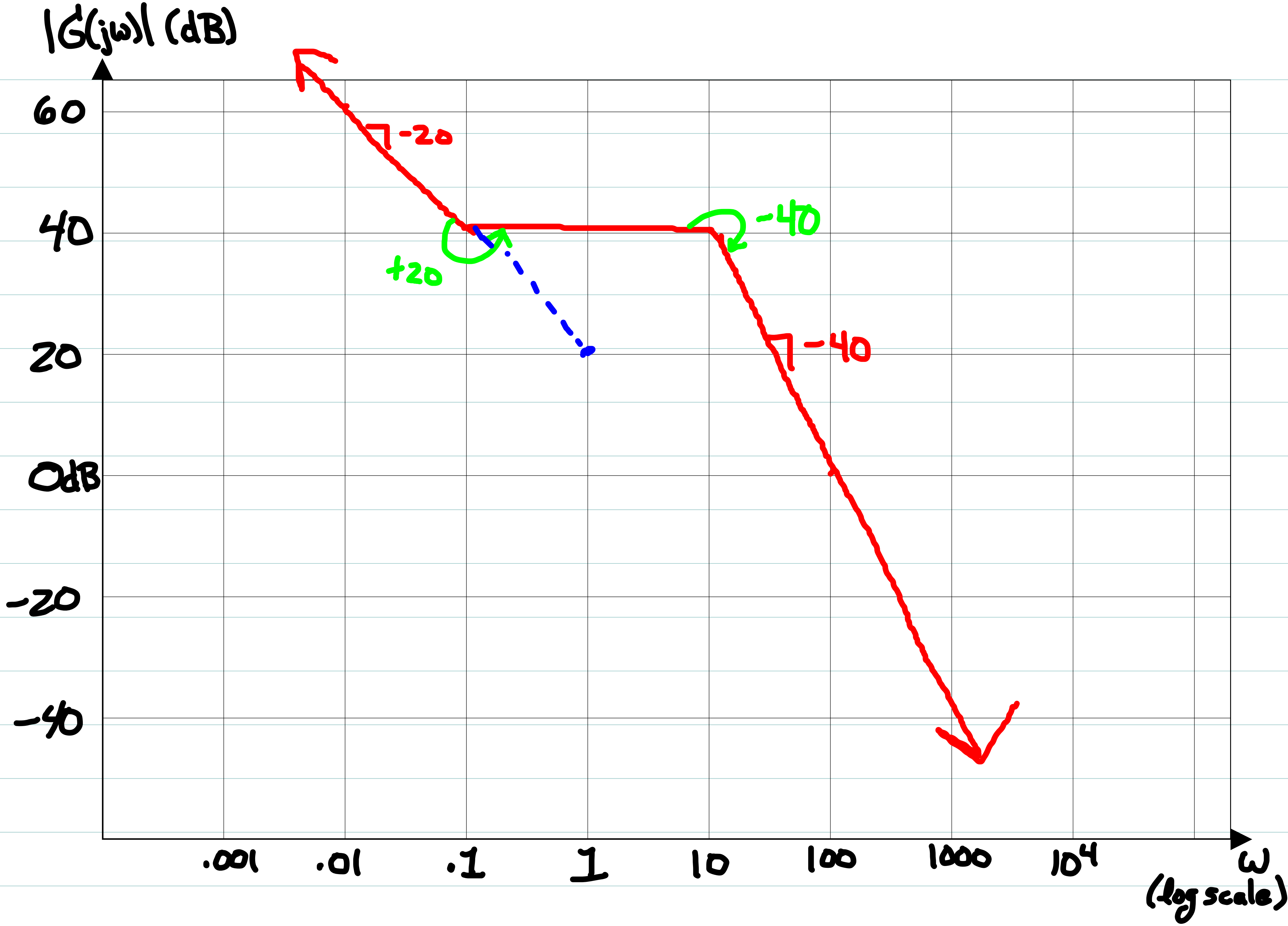
⇒ Start diagram by sketching effect of these poles at low frequencies

⇒ Note if $|K_B| \neq 1$, then this low frequency asymptote will pass through $|K_B|_{dB}$ at $\omega = 1$

⇒ Then add bends due to nonzero $z_i$ and $p_K$ as usual.

<u>Example:</u>

$$G(s) = 10\left[\frac{(10s+1)}{s(s/10+1)^2}\right]$$



|G(jω)| (dB)
60
40
20
0dB
-20
-40
-20
+20
-40
-40
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)

# What about phase?

Recall:

$$\angle G(j\omega) = \angle K_B - N\angle(j\omega) + \sum_{i=1}^{m} \angle\left(1 - \frac{j\omega}{z_i}\right) - \sum_{K=N+1}^{n} \left(1 - \frac{j\omega}{p_K}\right)$$

$$\angle K_B = \begin{cases} 0 & K_B > 0 \\ -180 & K_B < 0 \end{cases} \quad \text{for all } \omega \geq 0$$

$$\angle(j\omega) = 90^\circ \quad \text{for all } \omega \geq 0$$

So, low frequency phase is <u>constant</u> at

$-90N$ if $K_B > 0$

$-180 - 90N$ if $K_B < 0$

Other poles/zeros will cause "bends" at higher freqs.

# Phase response from other poles/zeros

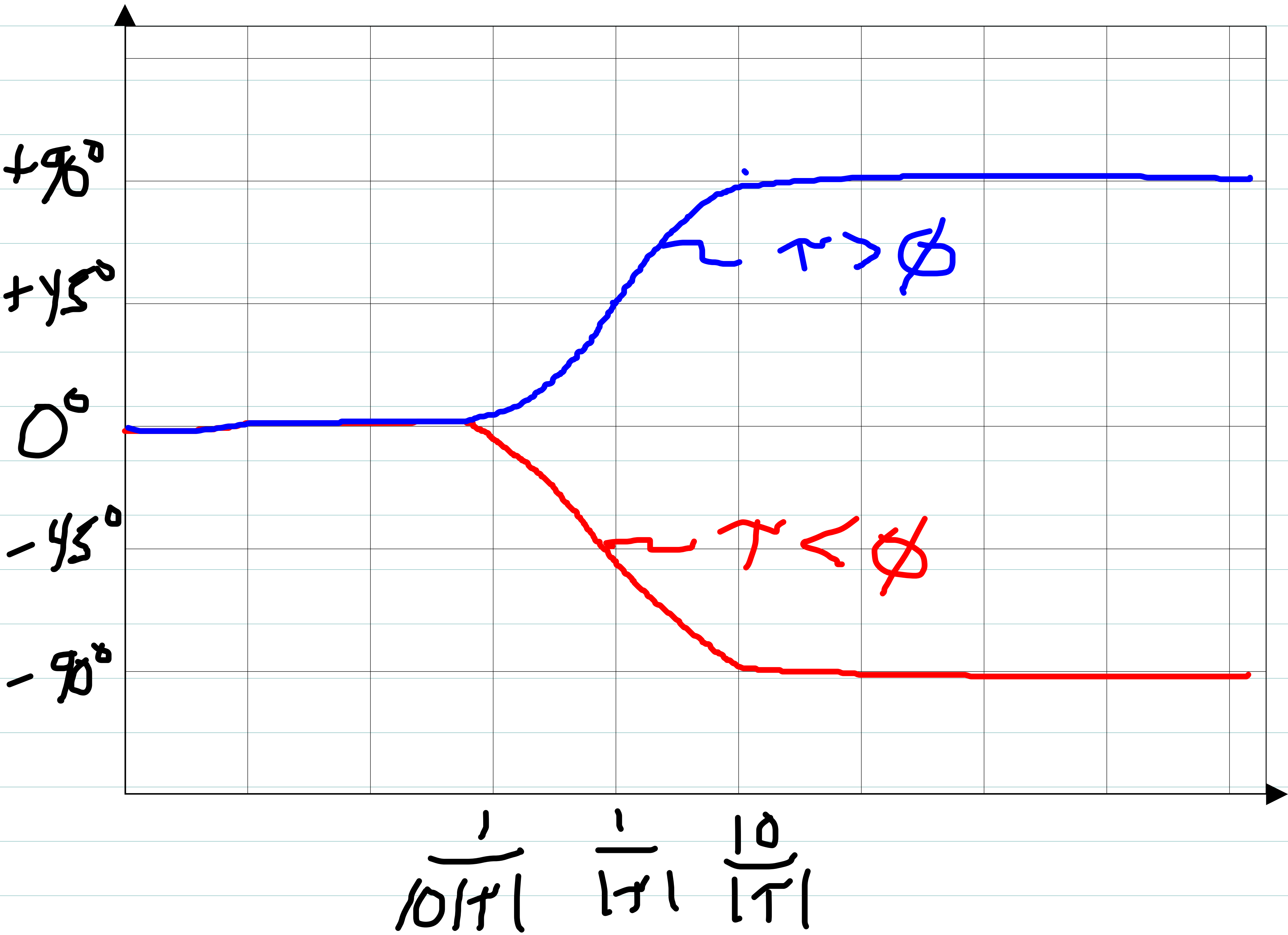
Consider again in generic form $(1+j\omega\tau)$ with $\tau = -1/z_i$ or $\tau = -1/p_k$

$$\measuredangle(1+j\omega\tau) = \tan^{-1}\omega\tau$$

$$= \begin{cases} \phi & \text{if } \omega \ll 1/|\tau| \\ +45^\circ & \text{if } \omega = 1/|\tau| \\ +9\phi^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$

above is for $\tau > 0$. If instead $\tau < 0$

$$\measuredangle(1+j\omega\tau) = -\tan^{-1}\omega|\tau| = \begin{cases} \phi & \text{if } \omega \ll 1/|\tau| \\ -45^\circ & \text{if } \omega = 1/|\tau| \\ -90^\circ & \text{if } \omega \gg 1/|\tau| \end{cases}$$



+90°
+45°
0°
-45°
-90°
τ > 0
τ < 0
1/(10|τ|)
1/|τ|
10/|τ|

# Observations

⇒ Phase change due to a single factor occurs in a 2 decade band of frequencies centered at the magnitude corner frequency $1/|\tau|$

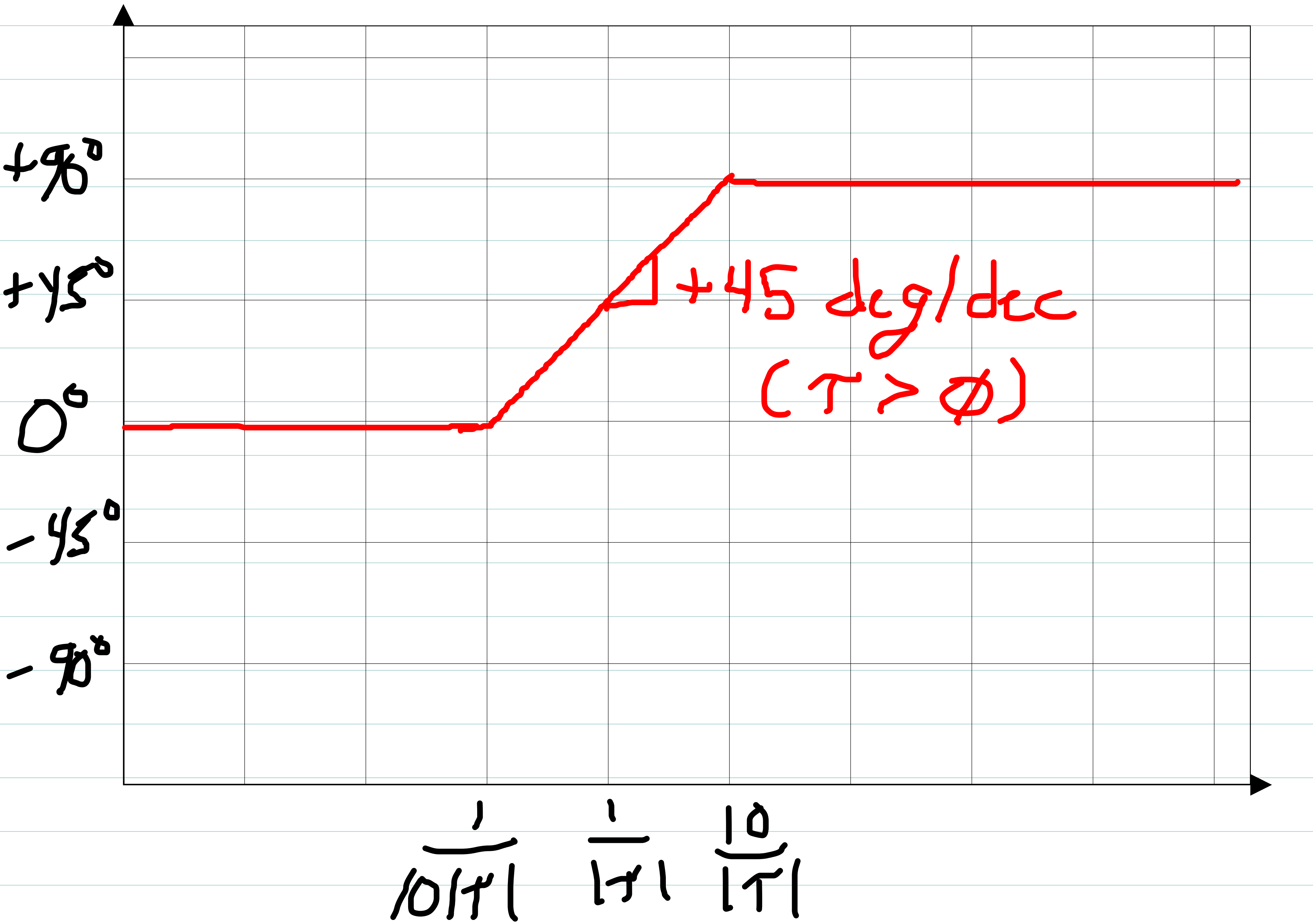
i.e. in band $\frac{1}{10|\tau|} \leq \omega \leq 10/|\tau|$

⇒ Phase is constant outside this band

low freq phase $\approx 0°$

h.f. phase $\approx \pm 90°$ ($+90°$ if $\tau > \phi$, $-90°$ if $\tau < \phi$

⇒ Phase change is approximate linear across band with slope $\pm 45°/\text{dec}$



+90°
+45°
0°
-45°
-90°
+45 deg/dec
(τ > 0)
1/(10|τ|)
1/|τ|
10/|τ|

Sign of phase change depends on:

$\Rightarrow$ whether factor is pole or zero

$\Rightarrow$ whether factor is RHP $(\tau<\emptyset)$ or LHP $(\tau>\emptyset)$

Suppose all factors are LHP, $z_i<\emptyset$ $p_k<\emptyset$

then all $\tau = {}^{-1}/_{z_i}$ or ${}^{-1}/_{p_k}$ are positive.

This is called the "minimum phase" case

Then:

$\Rightarrow$ zeros cause $+90°$ phase change over band $\frac{|z_i|}{10}$ to $10|z_i|$

$\Rightarrow$ poles cause $-90°$ change over $\frac{|p_k|}{10}$ to $10|p_k|$

(Minimum Phase Systems)

<u>Slopes</u> of phase change are $+45°/\text{dec}$ (zeros) or $-45°/\text{dec}$ (poles) in these bands

Note phase changes in minimum phase cases mirror those for magnitude changes:

⇒ zeros cause <u>positive</u> slope changes
⇒ poles cause <u>negative</u> slope changes.

Graphical addition is again straightforward, but requires a little care:

⇒ slopes are nonzero only in a 2 decade band

⇒ bands from different factors may overlap.

# Example:

$$G(s) = \frac{10s+1}{s(s+1)(s/10+1)}$$

low freq. phase $-90°$

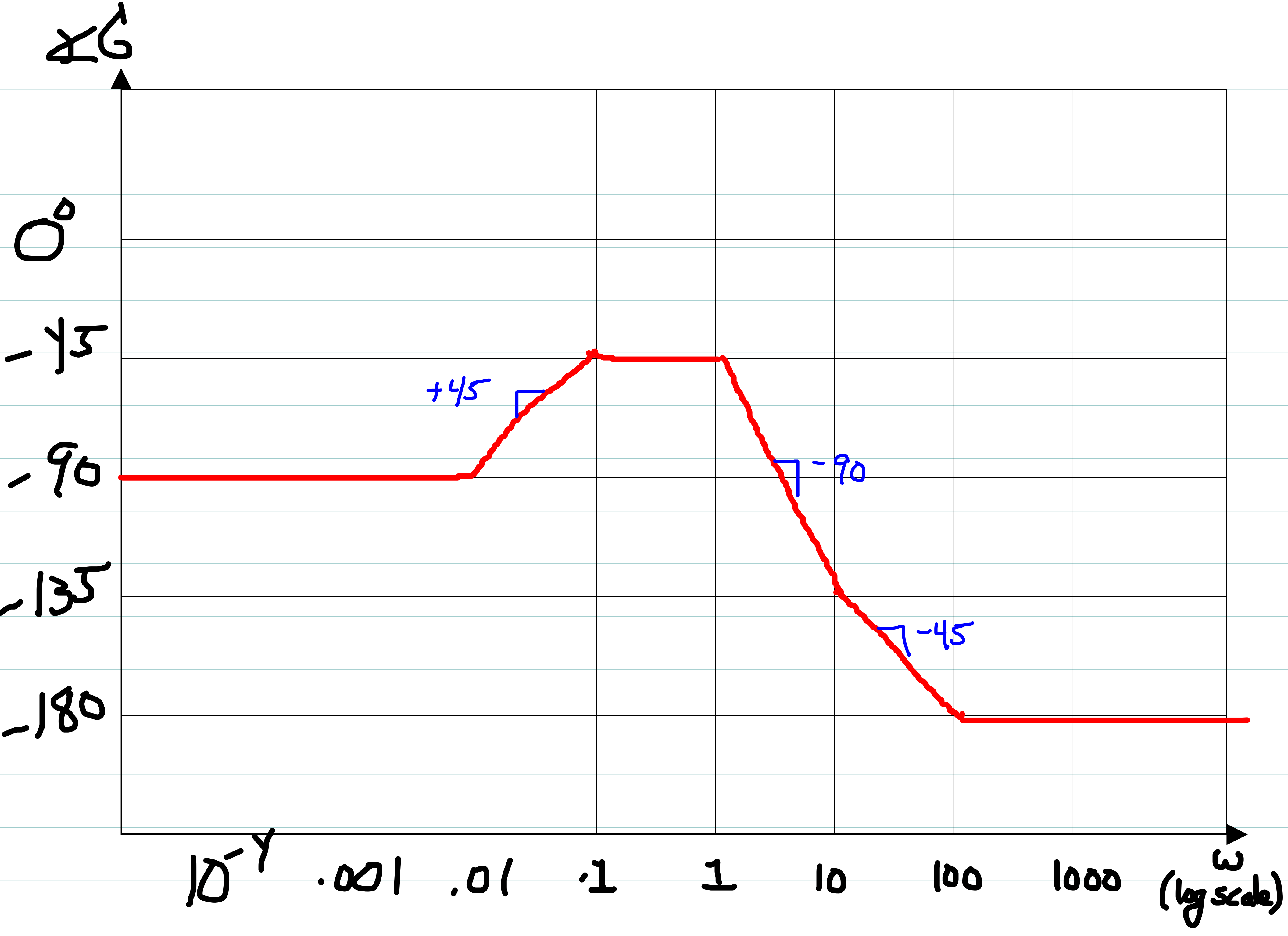
Phase changes:

| | | |
|---|---|---|
| $+45°/\text{dec}$ | in | $.01$ to $1$ |
| $-45°/\text{dec}$ | in | $.1$ to $10$ |
| $-45°/\text{dec}$ | in | $1$ to $100$ |

Net:

| | | |
|---|---|---|
| $+45°/\text{dec}$ | in | $.01$ to $.1$ |
| $0°/\text{dec}$ | in | $.1$ to $1$ |
| $-90°/\text{dec}$ | in | $1$ to $10$ |
| $-45°/\text{dec}$ | in | $10$ to $100$ |

Constant for $\omega > 100$.



∡G
0°
-45
-90
-135
-180
+45
-90
-45
$10^{-4}$
.001
.01
.1
1
10
100
1000
ω
(log scale)

# Repeated factors

Repeated factors $(1+j\omega T)^l$ multiply the phase changes by $l$, just like magnitudes.

**Example:**

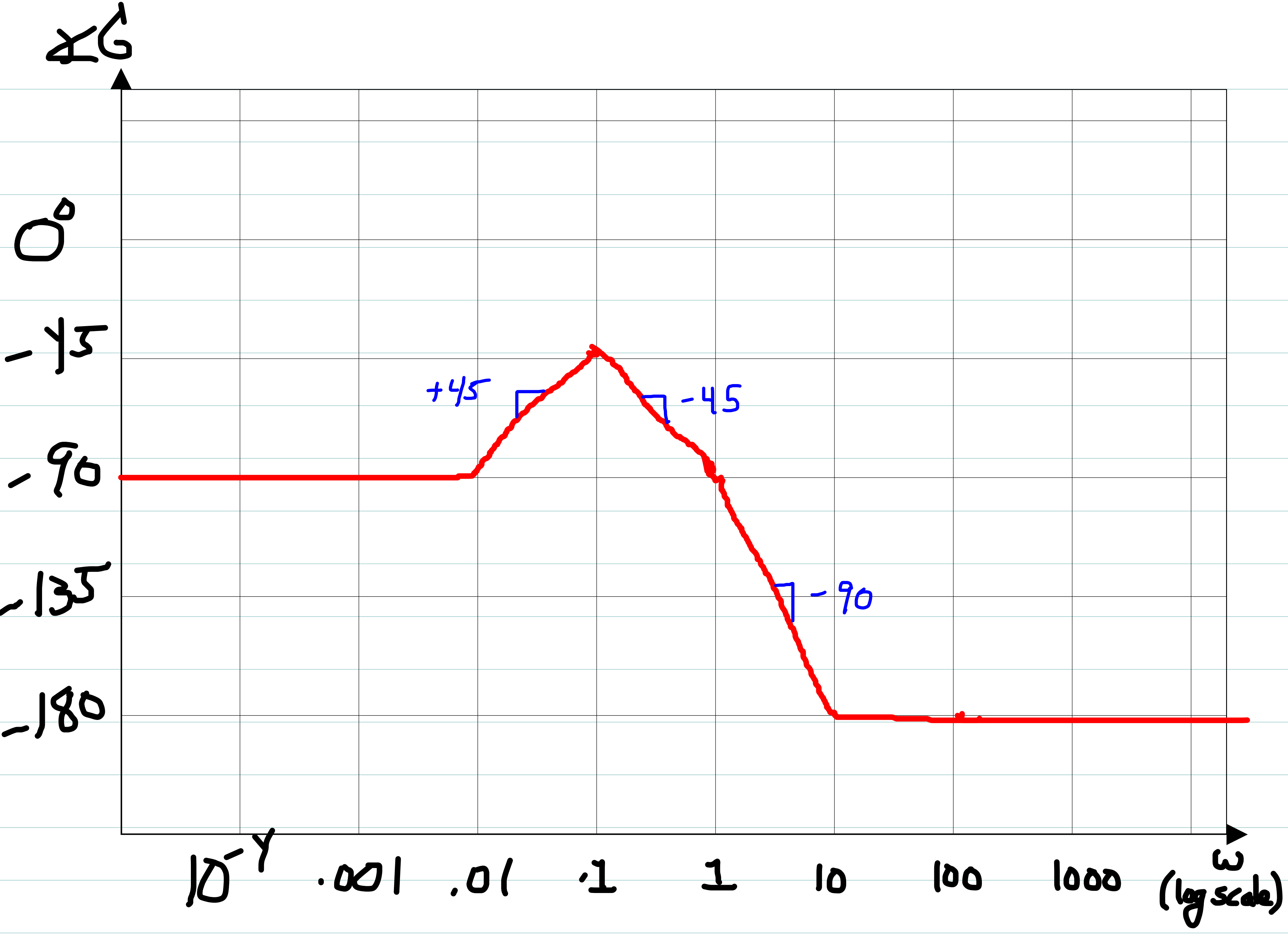
$$\hat{G}(s) = \frac{10s+1}{s(s+1)^2}$$

**Changes:**

+45°/dec in .01 to 1
- 90°/dec in .1 to 10

**Net:**

+45°/dec in .01 to .1
-45°/dec in .1 to 1
-90°/dec in 1 to 10



∡G
0°
-45
-90
-135
-180
+45
-45
-90
$10^{-4}$
.001
.01
.1
1
10
100
1000
ω
(log scale)

# Summary (minimum phase)

=> Low freq. phase is $\angle K_B - N\,90°$

=> high freq. phase is $\angle K_B - 90°(n-m)$

=> Note low and high freq. phases are <u>constant</u> (slope is zero).

=> Recall typically $n>m$ for a physical system so high freq. phase is typically <u>negative</u> for a minimum phase system.

=> zeros cause $+90°$ change at rate of $+45°$/dec in 2 decade band centered at $|z_i|$

=> poles cause $-90°$ change at rate of $-45°$/dec in 2 decade band centered at $|p_k|$.

Can be tricky to accurately sketch phase

=> overlapping change regions for multiple factors

=> No standard formula for phase change of underdamped factors

=> Helps to 1st make a table of slope changes over frequency ranges as above

=> Generally, straight-line phase sketch is less accurate than magnitude sketch.

=> Still sufficiently accurate to give us a good general idea of phase behavior.

=> We'll use Matlab when greater accuracy is required.

# Non-minimum phase systems

If any poles or zeros of $G(s)$ in RHP, the system is "Non-minimum phase"

Corresponds to $\tau < \phi$ in phase analysis and $\angle(1+j\omega\tau) = -\tan^{-1}\omega/|\tau|$.

$\Rightarrow$ Phase response is negative of that seen above

$\Rightarrow$ In particular, zeros cause $-90$ deg phase change in 2 decade band around corner freq.

Poles cause $+90$ deg change

Opposite of minimum phase behavior, but

$\boxed{\text{Corner freqs unchanged}}$ ($|z_i|$ or $|p_k|$)

Example:

$$G(s) = \frac{(10s+1)}{(1-s)}$$

Min phase zero: $+45°$/dec change in .01 to 1

Nonmin phase pole: $+45°$/dec change in .1 to 10

Net: $+45°$/dec in .01 to .1
$+90°$/dec in .1 to 1
$+45°$/dec in 1 to 10.

Note: h.f. phase is $+180°$ here. Above rule for h.f. phase in min phase systems does not apply if $G(s)$ has RHP poles or zeros

# Underdamped factors

$$(s^2+2\zeta\omega_n s+\omega_n^2) \Rightarrow \left[\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]$$ in Bode form.

How do we draw magnitude response when $G(s)$ contains these factors?

$\Rightarrow$ If $\frac{\sqrt{2}}{2} \leq \zeta \leq 1$, we can well approximate the response as a repeated pole at $-\omega_n$ (it isn't really but it's a good approx to sketch this way).

$\Rightarrow$ If $0 \leq \zeta < \frac{\sqrt{2}}{2}$ a more substantial correction is needed...

$\Rightarrow$ To illustrate, suppose

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{\left[(s/\omega_n)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]}$$

Let's find $|G(j\omega)|$ here

$$|G(j\omega)| = \left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\xi\left(\frac{j\omega}{\omega_n}\right) + 1\right|^{-1}$$

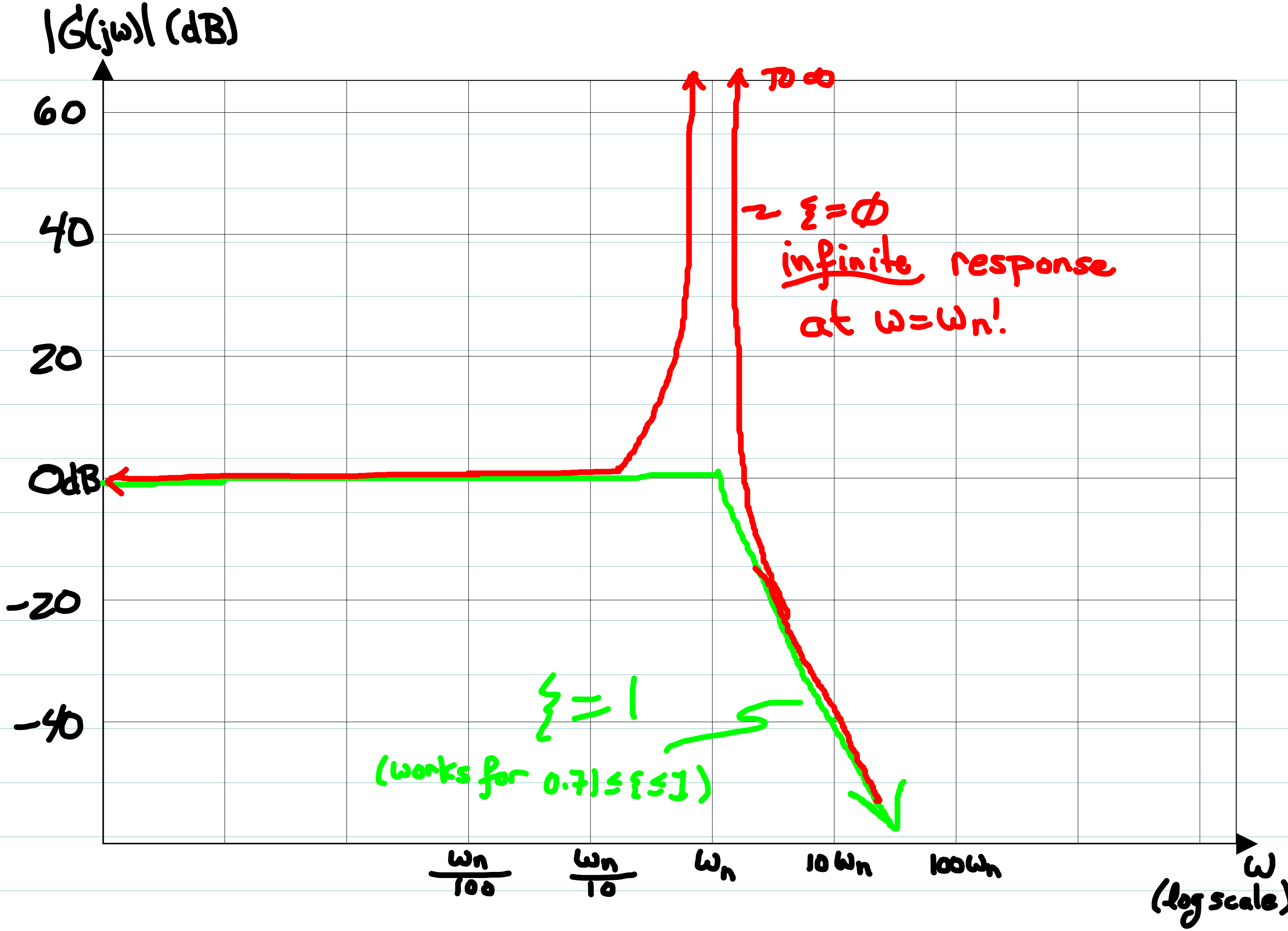
$$= \left[\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}\,\right]^{-1}$$

Which is ugly, so why bother?

Consider if $\xi = \emptyset$, then

$$|G(j\omega)| = \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|} \approx \begin{cases} 1 & \text{if } \omega \ll \omega_n \\ \left|\frac{\omega}{\omega_n}\right|^{-2} & \omega \gg \omega_n \end{cases}$$

so that $|G(j\omega)|_{\omega=\omega_n} = \infty$ !!! Definitely Something Going on!



|G(jw)| (dB)
60
40
20
0dB
-20
-40
TO ∞
← ξ = 0
infinite response
at ω = ωn!
ξ = 1
(works for 0.71 ≤ ξ ≤ 1)
ωn/100
ωn/10
ωn
10ωn
100ωn
ω
(log scale)

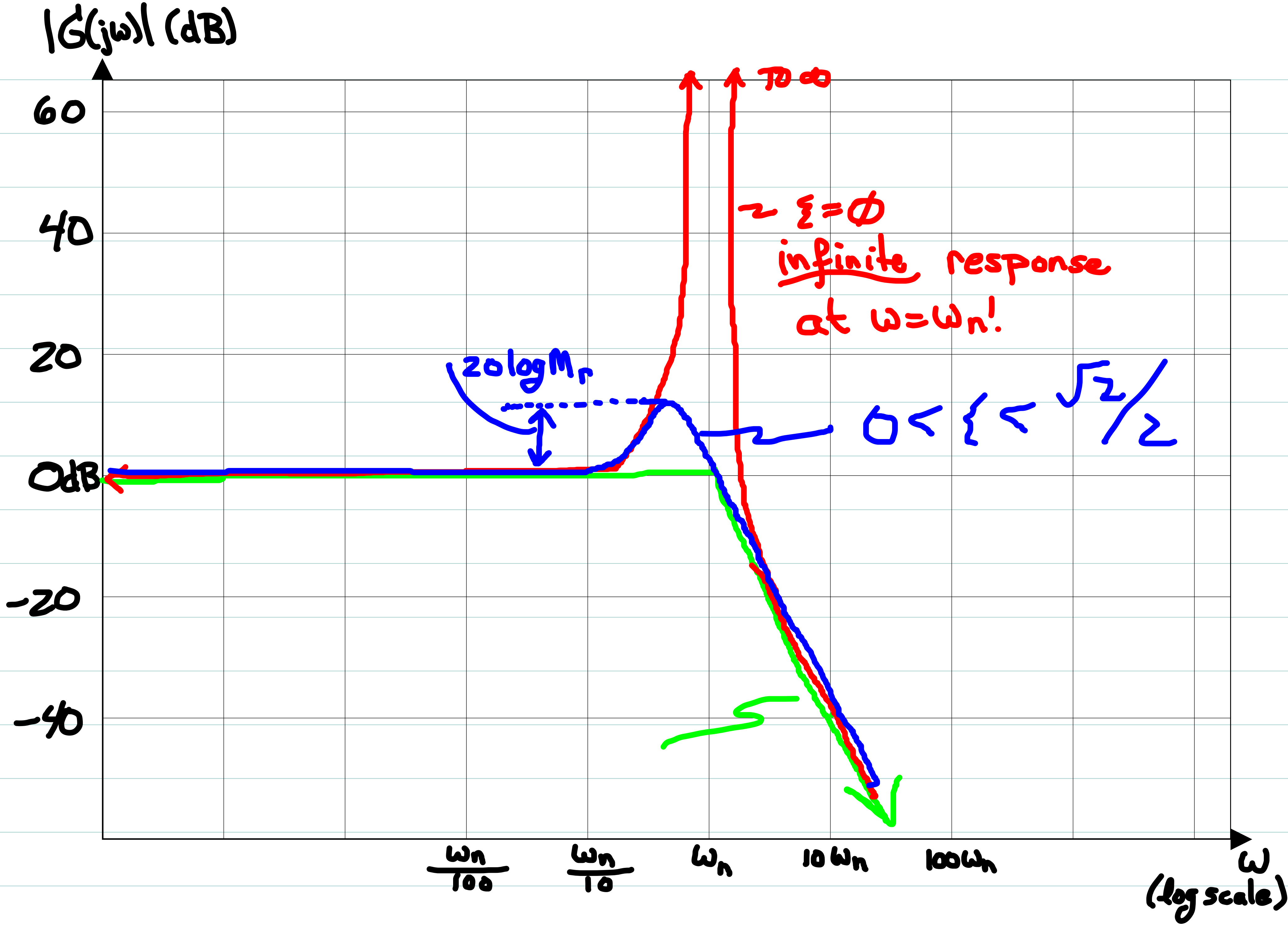
When $0 < \zeta < \sqrt{2}/2$, a similar "peaking" phenomenon occurs, but peak height is finite:

for
$$G(s) = \left[ \left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1 \right]^{-1}$$

$\max\limits_{\omega \geq 0} |G(j\omega)|$ occurs at:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

and $|G(j\omega_r)| = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \triangleq M_r$



|G(jω)| (dB)
60
40
20
0dB
-20
-40
To ∞
ξ=0
infinite response at ω=ωn!
20logMr
0<ξ<√2/2
ωn/100
ωn/10
ωn
10ωn
100ωn
ω (log scale)

This is the phenomenon of resonance

An ideal (no zeros) underdamped $2^{nd}$ order system with $0 \le \xi < \sqrt{2}/2$ will exhibit output amplitudes significantly greater than the input amplitude when input frequency is close to the natural frequency $\omega_n$.

The largest amplitude ratio will occur at the resonant frequency $\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}} < \omega_n$

and the maximal amplitude ratio (maximal resonance) is

$$\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$$

## Notes:

1.) Height of peak on diagram is $M_r$ in dB, i.e. $20\log M_r$

2.) When 2nd order factor in TF with other factors, the peak is $20\log M_r$ above whatever magnitude the plot would otherwise have at $\omega_r$. That is, $M_r$ is a <u>relative</u> offset to plot, not absolute.

3.) For small $\xi$, say $\phi < \xi \lesssim 1/10$

$$\omega_r \approx \omega_n \quad \text{and} \quad M_r \approx 1/(2\xi)$$

So $20\log M_r \approx -[6 + 20\log \xi]$ is a good approximation

i.e. at $\xi = 1/10$, $20\log M_r \approx +14$ dB

## Example

$$G(s) = \left[\frac{(10s+1)}{s\left(\left(\frac{s}{10}\right)^2 + 0.2\left(\frac{s}{10}\right) + 1\right)}\right]$$

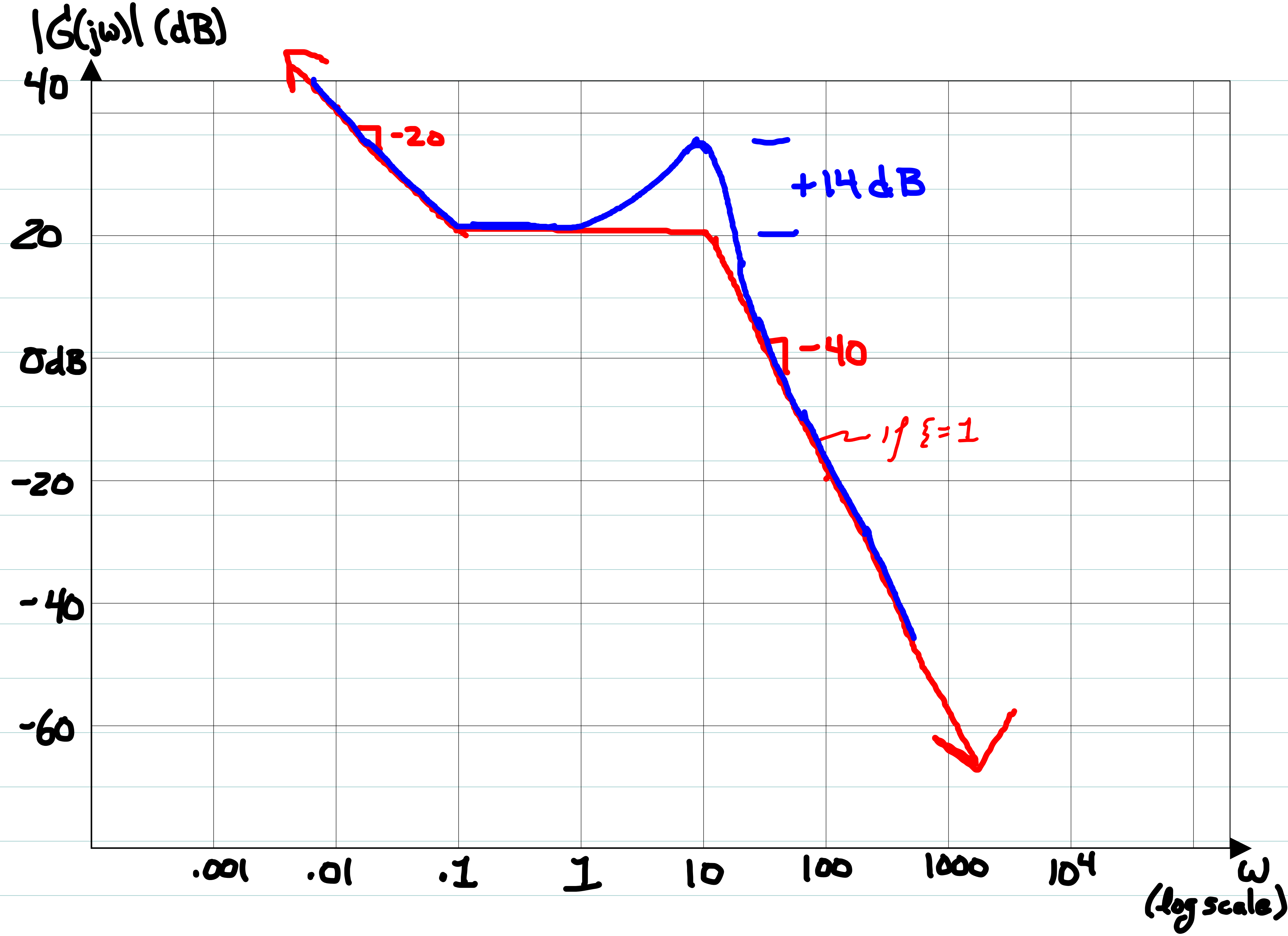
Same as example above, except:

$K_B = 1$ (instead of 10)

$\zeta = 0.1$ (instead of $\zeta = 1$)

$\omega_n = 10$

$\Rightarrow$ Expect resonant peak of height +14dB near $\omega = 10$.



|G(jω)| (dB)
40
20
0dB
-20
-40
-60
-20
+14 dB
-40
if ξ=1
.001
.01
.1
1
10
100
1000
10^4
ω
(log scale)