"Root Locus" Method for CL pole prediction

<u>Set up:</u> $L(s) = K\left[\frac{N(s)}{D(s)}\right]$

$\Rightarrow$ $\text{Deg}\{N(s)\} = m$; m zeros z_i such that $N(z_i) = \emptyset$

$$N(s) = (s-z_1)(s-z_2)\cdots(s-z_m) = \prod_{i=1}^{m}(s-z_i)$$

$\Rightarrow$ $\text{Deg}\{D(s)\} = n$; n poles p_k such that $D(p_k) = \emptyset$

$$D(s) = (s-p_1)(s-p_2)\cdots(s-p_n) = \prod_{k=1}^{n}(s-p_k)$$

$\Rightarrow$ $n \geq m$: no more zeros than poles

$\Rightarrow$ Characteristic equation: s is a CL pole if

$$1 + L(s) = \emptyset$$

Basic Observations

$$1+L(s)=0 \Rightarrow 1+K\left[\frac{N(s)}{D(s)}\right]=0$$

$$\Rightarrow D(s)+KN(s)=0$$

This is an n^{th} order polynomial equation to define CL poles:

$\boxed{\Rightarrow \text{There are } n \text{ CL poles, same number as OL poles}}$

Consider limit as $K \to 0$. Then CE becomes: $D(s)=0$

$\Rightarrow$ same eq'n as defines OL poles.

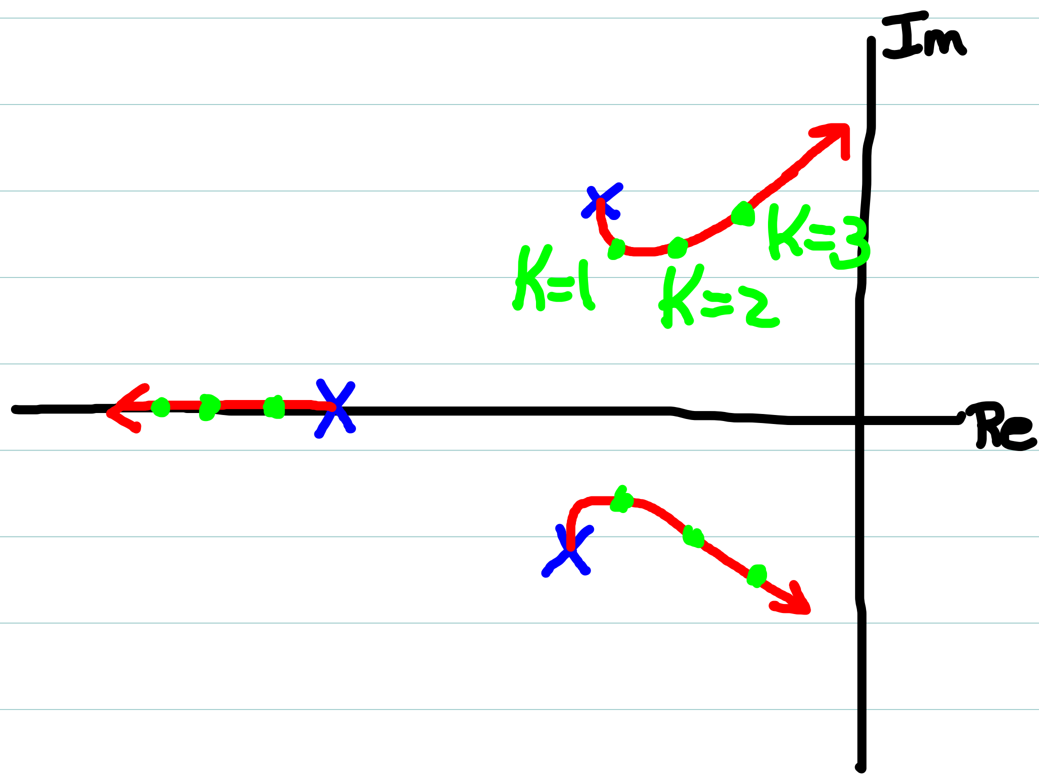
$\boxed{\Rightarrow \text{In low gain limit, } K\to 0\text{, the CL poles are same as OL poles}}$

Varying K

⇒ As K changes, the CL pole locations migrate away from OL poles

⇒ Each CL pole location traces out a continuous curve starting at an OL pole. These curves are called **branches**.

⇒ Since there are n CL poles, there are n branches



X = OL pole

• = CL pole for different K

— = branch

Symmetry

$\Rightarrow$ Recall that complex roots of polynomial equations occur in <u>conjugate pairs</u>.

$\Rightarrow$ If $s \in \mathbb{C}$ satisfies $1 + L(s) = \emptyset$, so also $\bar{s}$ satisfies $1 + L(\bar{s}) = \emptyset$.

$\Rightarrow$ CL pole locations are symmetric about <u>real</u> Axis.

$\Rightarrow$ Branches of CL pole loci are symmetric ("mirror image") about real Axis.

$\Rightarrow$ Can we predict branch behavior as $|K|$ increases?

High gain limit : $|K| \to \infty$

Recall CL poles satisfy $D(s) + KN(s) = \emptyset$

Equivalently if $K \neq \emptyset$:

$$N(s) + \left[\frac{1}{K}\right] D(s) = \emptyset$$

and as $|K| \to \infty$ we have : $N(s) = \emptyset$

$\Rightarrow$ *As $|K| \to \infty$, the CL poles coincide with* OL zeros !

$\Rightarrow$ Branches terminate at OL zeros !

$\Rightarrow$ OL zeros "attract" CL poles to them in *high gain limit*

$\Rightarrow$ RHP zeros in $L(s)$ are dangerous !

High gain limit, cont

$\Rightarrow$ n CL poles (branches), but only $m \leq n$ OL zeros.

$\Rightarrow$ What happens to other $n-m$ CL poles (branches)?

$\Rightarrow$ *The remaining $n-m$ branches asymptote to infinity*

$\Rightarrow$ But how? Depends on sign of K. Suppose for simplicity we take $K > \emptyset$.

$\Rightarrow$ Recall "angle condition" for $K > \emptyset$:

if s is a possible CL pole, then

$$\measuredangle L(s) = (1+2\ell)180^\circ \qquad \text{(odd multiple of } 180^\circ\text{)}.$$

Interpretation of Angle Condition

Just like in Bode, for any $s \in \mathbb{C}$:

$$\measuredangle L(s) = \sum_{i=1}^{m} \measuredangle (s - z_i) - \sum_{k=1}^{n} \measuredangle (s - p_k)$$

More compactly:

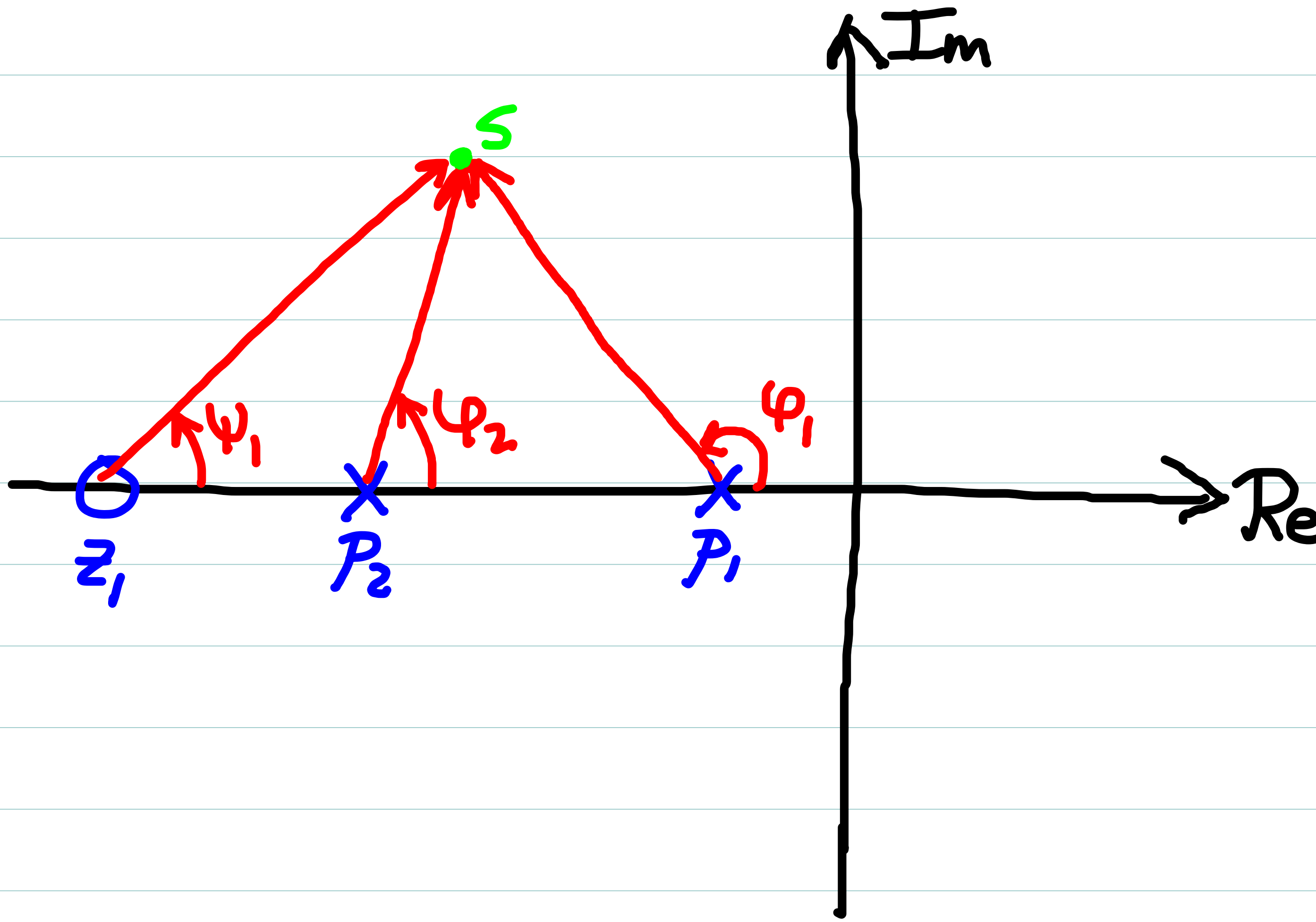
$$\measuredangle L(s) = \sum_{i=1}^{m} \psi_i - \sum_{k=1}^{n} \varphi_k$$

where.

$$\psi_i = \measuredangle (s - z_i) \qquad \text{(Contribution of each zero)}$$

$$\varphi_k = \measuredangle (s - p_k) \qquad \text{(Contribution of each pole)}$$

Graphical (Phasor) Interpretation



$\Rightarrow$ s is a possible CL pole (hence lies on a branch of the locus) if:

$$\psi_1 - \varphi_1 - \varphi_2 = (1+2\ell)180^\circ$$

For high gain limit, look for s with $|s| \gg 1$ which satisfy this

Note that for such s, the phasors become indistinguishable!

$$\psi_1 \approx \varphi_1 \approx \varphi_2 = \alpha$$

Thus, for $|s| \gg 1$, the angle condition becomes:

$$(1+2\ell)180° = \sum_{i=1}^{m} \psi_i - \sum_{k=1}^{n} \varphi_k$$

$$= m\alpha - n\alpha$$

$$= (m-n)\alpha$$

Where α is common phasor angle from z_i or p_k to s.

We then have

$$\boxed{\alpha = \frac{(1+2\ell)180°}{n-m}}$$

is the angular direction in complex plane for s with $|s| \gg 1$ that satisfy angle condition

Asymptotes

$\Rightarrow$ The $n-m$ branches which diverge to infinity do so along asymptotes which make angles of

$$\boxed{\alpha_\ell = \frac{(1+2\ell)180^\circ}{n-m}}$$

with respect to real Axis.

$\Rightarrow$ A slightly messy additional derivation shows these asymptotes intersect at a common point on real Axis, given by

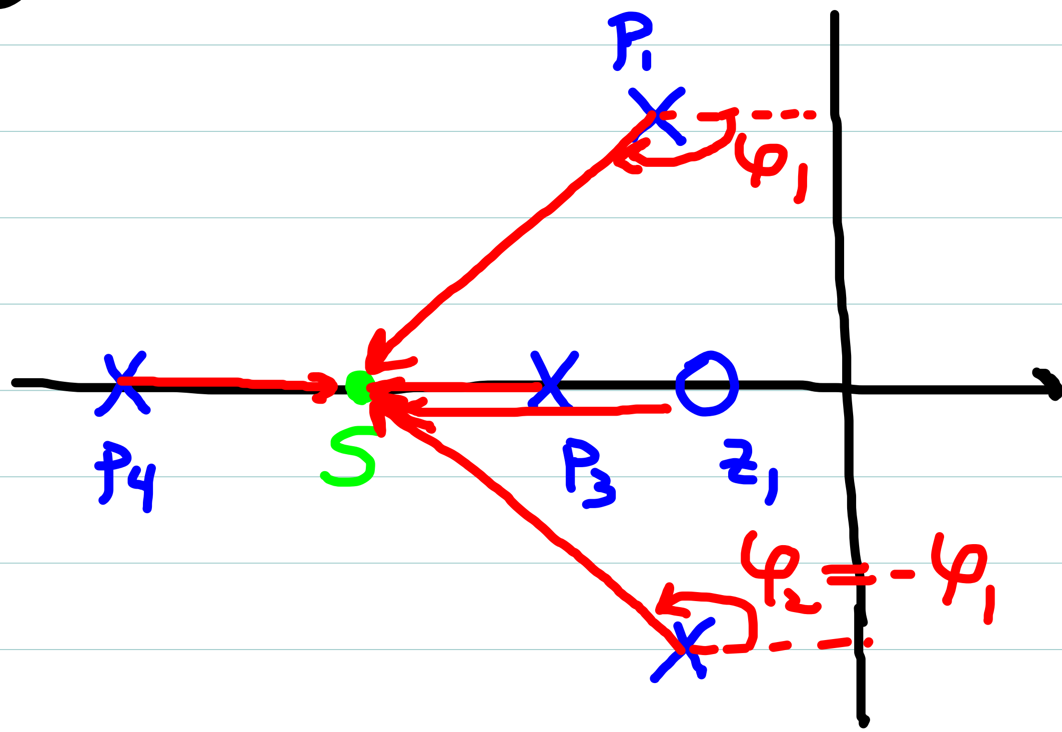
$$\boxed{\sigma_a = \frac{\sum_{k=1}^{n} Re\{p_k\} - \sum_{i=1}^{m} Re\{z_i\}}{n-m}}$$

where again z_i, p_k are zeros and poles of $L(s)$.

"Asymptote rule"

Branches on real Axis

Look at angle condition on real Axis



$\Rightarrow$ Contribution to angle condition from complex conjugate pole or zero pairs will <u>cancel</u>.

$\Rightarrow$ Contribution from any real pole or zero to <u>left</u> of s will be zero

Thus, only the real poles and zeros lying to right of s will contribute to angle condition at a real s.

In particular:

If the total number of real poles and zeros lying to the right of a point s on real Axis is odd, then that point satisfies the angle condition.

Portions of branches of the locus lie on segments of real Axis which satisfy this cond'n:

"real Axis rule"