Three <u>Derived</u> TFs for Feedback Loops

Given $G(s)$ and $H(s)$, we can derive $R(s)$, $S(s)$, $T(s)$ so that:

$$Y_d(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s) \qquad T(s) = \frac{L(s)}{1+L(s)}$$

$$Y_d(s) \longrightarrow \boxed{S(s)} \longrightarrow E(s) \qquad S(s) = \frac{1}{1+L(s)}$$

$$Y_d(s) \longrightarrow \boxed{R(s)} \longrightarrow U(s) \qquad R(s) = \frac{H(s)}{1+L(s)}$$

$\Rightarrow$ Each of these <u>derived</u> TFs can be analyzed using the same tools developed for $G(s)$.

Uses of derived TF:

$\Rightarrow$ $T(s)$ tells us about actual response of controlled system for specific $y_d(t)$

$$\boxed{Y(s) = T(s)\,Y_d(s)}$$

$\Rightarrow$ $S(s)$ tells us about tracking accuracy for specific $y_d(t)$

$$\boxed{E(s) = S(s)\,Y_d(s)}$$

$\Rightarrow$ $R(s)$ tells us about required input for specific $y_d(t)$:

$$\boxed{U(s) = R(s)\,Y_d(s)}$$

Note: all 3 of these TF have the same denominator, hence same poles!!!

Example use of loop TF:

Suppose $y_d(t) = A\mathbb{1}(t)$ (step of magnitude A)

Then:

$$y(t) = A \times \{\text{step response of } T(s)\}$$

$$u(t) = A \times \{\text{step response of } R(s)\}$$

$$e(t) = A \times \{\text{step response of } S(s)\}$$

Note in particular here that:

$$e_{ss}(t) =$$

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Note in particular here that:

$$e_{ss}(t) = A\,S(0) \qquad \text{(constant)}$$

Thus generally we'd like to make sure $S(0) = 0$ (or at least very small).

Example Application: Tracking Ability

A good feedback loop needs to ensure $|e_{ss}(t)|$ small for a wide variety of $y_d(t)$.

Suppose $y_d(t) = A$ (constant)

Then (assuming all poles of $\mathcal{S}(s)$ at least stable)

$$e_{ss}(t) = A\,\mathcal{S}(\emptyset)$$

So good tracking requires $|\mathcal{S}(\emptyset)|$ small.

Ideally, $\mathcal{S}(\emptyset) = \emptyset \Rightarrow e_{ss}(t) = \emptyset$ "perfect tracking"

and this is often a basic design <u>requirement</u>.

Tracking (cont)

Suppose more generally $y_d(t) = A\cos\omega t$

then $e_{ss}(t) = A|S(j\omega)|\cos(\omega t + \angle S(j\omega))$

and in particular $|e_{ss}(t)| \leq A|S(j\omega)|$

So we want $|S(j\omega)| \ll 1$ for a wide range of frequencies ω (including $\omega = 0$)

$\Rightarrow$ Want Bode magnitude diagram $|S(j\omega)| \ll 0\,dB$ for a large range of ω (including 0).

$\Rightarrow$ We will show feedback loops with good tracking properties place <u>constraints</u> on design process, which often conflict with other requirements (stability + performance).

Bandwidth

Define ω_B to be largest ω for which

$$|S(j\omega)| \leq -3\,dB \quad \text{for all } \omega \in [0, \omega_B]$$

this is the (tracking) <u>bandwidth</u> of the system.

$\Rightarrow$ We want designs with <u>high bandwidth.</u>

<u>Note</u>: $-3\,dB$ is an arbitrary boundary between acceptable and poor tracking. Realistic performance constraints are typically much tighter:

$$|S(j\omega)| \leq -20\,dB \quad (\leq 10\% \text{ worst case error})$$

or

$$|S(j\omega)| \leq -40\,dB \quad (\leq 1\% \text{ worst case error})$$

Example Application: Utility of $R(s)$

$\Rightarrow$ $R(s)$ lets us theoretically predict the $u(t)$ which will be generated under ideal circumstances given a specified $y_d(t)$.

$$u(t) = \mathcal{L}^{-1}\{R(s)Y_d(s)\}$$

$\Rightarrow$ Primary quantity of interest is $\max\limits_{t \geq 0} |u(t)|$

$\Rightarrow$ Quantifies maximum control effort required.

$\Rightarrow$ Real actuators have limits $|u(t)| \leq u_{max}$

$\Rightarrow$ Must ensure our control strategy does not "saturate" the actuators, i.e. $\max\limits_{t} |u(t)| \leq u_{max}$

Saturation

Saturation of actuators, i.e. $|u(t)| > u_{max}$ for some $t \geq 0$, may produce performance degradation or even <u>instability</u> even when the poles of $R(s)$ are "good."

Unfortunately, no simple design guidelines for $H(s)$ which ensure saturation does not occur.

Some degree of design iteration typically required

Advanced (graduate level) techniques do exist to incorporate actuator limits into the design process.

Closed-loop poles

=> Performance of controlled system (settling time, steady-state, overshoot, etc) depends on poles of $T(s)$

=> ($R(s)$ and $S(s)$ have same poles!!)

=> Where are these poles??

=> Determined by denominator of $T(s)$

=> ($R(s)$ and $S(s)$ have same denominator)

=> Denom of all 3 derived TF is:

$$1 + L(s)$$

Charactistic Equation

Poles of $T(s)$, $R(s)$, $S(s)$ are at values of $s \in \mathbb{C}$ such that

(CE) $$\boxed{1 + L(s) = 0}$$ "Chacteristic equation" of feedback system

We need sol'ns of this equation to be in "good" locations of complex plane.

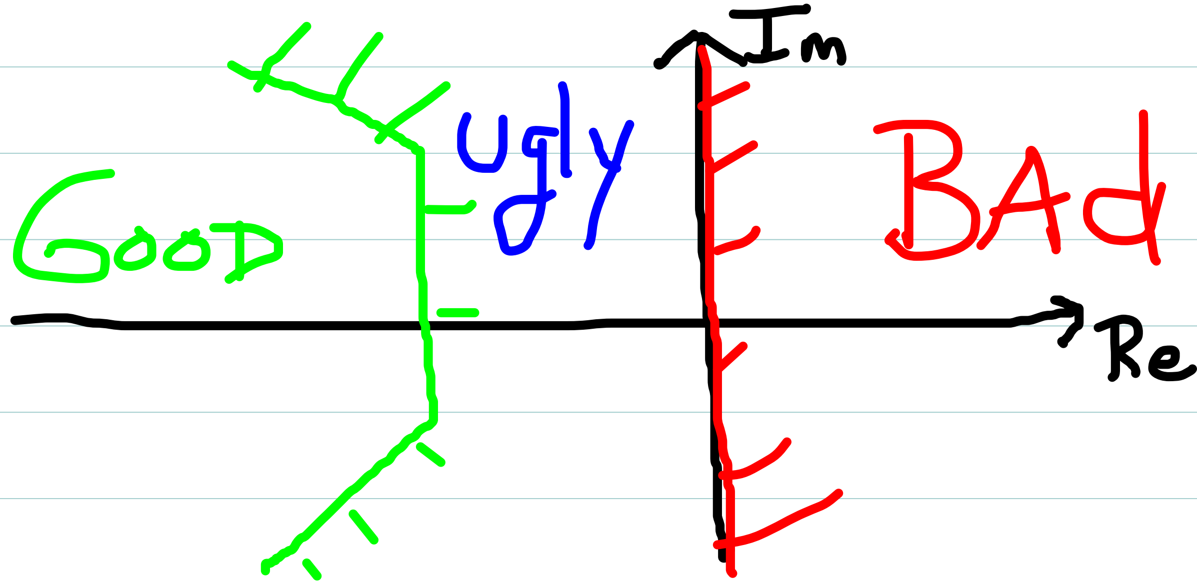
Will identify required properties for $L(s)$ so this is true, then work backwards to determine required properties of $H(s)$.

Fundamental Consideration: Closed-loop Stability

$\boxed{\text{Most basic design consideration:}}$!!

Closed-loop poles should be "good", and certainly <u>must be stable</u>.

Thus, sol'ns of $\boxed{CE: \quad 1+L(s)=\emptyset}$ must be in left half of complex plane, preferably in "good region" (far from imag Axis, relatively close to or on the real Axis).



A Crucial Observation:

If $L(j\omega) = -1$ for some ω, then

$1 + L(s) = \emptyset$ has a sol'n $s = j\omega$ for some ω

$\Rightarrow$ closed-loop dynamics has poles at $\pm j\omega$, on imag Axis

$\Rightarrow$ Such poles are on the boundary between bad and ugly

$\Rightarrow$ This situation must be avoided!!!

Now if $\boxed{L(j\omega) = -1}$ for some $\omega > 0$, then:

$\Rightarrow$ Polar plot of $L(j\omega)$ passes through -1

$\Rightarrow$ $\omega_a = \omega_\gamma$ (both crossover freqs same)

$\Rightarrow$ $\boxed{a = 0 \text{ dB}, \ \gamma = 0^\circ}$ (both margins 0)

$\boxed{\text{Any such feedback loop is bad!}}$

Now, suppose $\exists \omega \geq 0 \ni$: $L(j\omega) \approx -1$ (i.e. close to, but not exactly -1)

By continuity of $L(s)$, $1 + L(s) = 0$ would have a sol'n very near (but not exactly on) the imag axis.

Some poles of $T(s)$ would be in bad or ugly region
$\Rightarrow$ Also undesireable!

Now, if $L(j\omega) \approx -1$ for some $\omega \geq \emptyset$

$\Rightarrow$ Polar plot of $L(j\omega)$ comes very close to -1 but doesn't pass exactly through it

$\Rightarrow$ (typically) $|a_{dB}|$ and $|\gamma|$ very small (small margins)

$\Rightarrow$ $\boxed{\text{This should Also be avoided.}}$

Thus, for $T(s)$ to have only good poles, we need conditions:

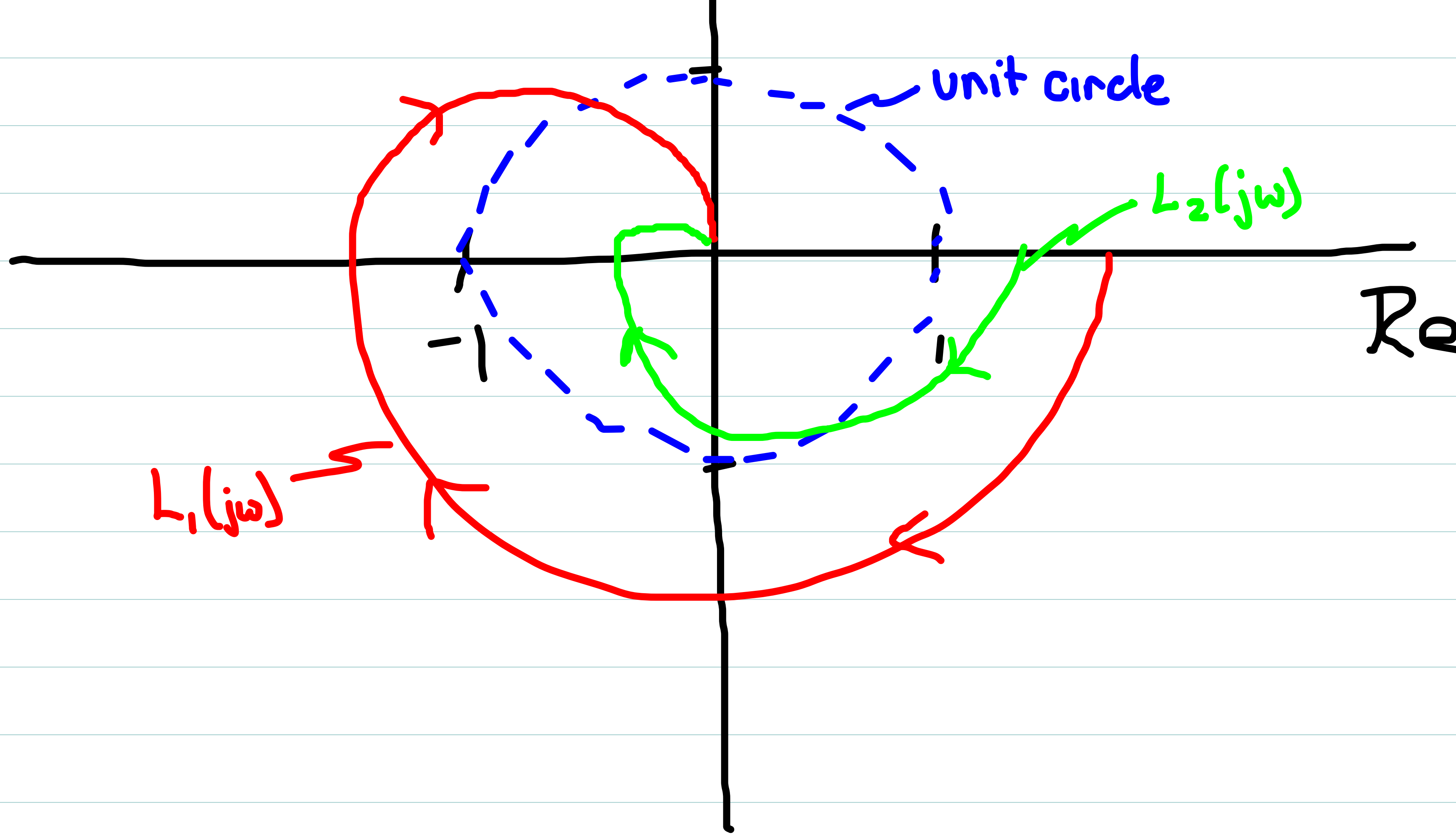
$\Rightarrow$ Gain and phase margins $\boxed{\text{of } L(s)}$ $\Leftarrow$!!! to be Large

$\Rightarrow$ polar plot of $L(j\omega)$ avoids -1 by wide margins

Necessary, but <u>not sufficient</u>!

Both plots avoid -1 by large margins

Is one better?
Yes! But criterion is non-obvious!



Nyquist Stability Criterion

All roots of $1+L(s)=0$ are in LHP if

the Nyquist diagram (a modified polar plot) $\boxed{\text{of } L(j\omega)}$

circles the -1 point the correct number of times.

$\Rightarrow$ Major Theoretical result! Used extensively in control theory

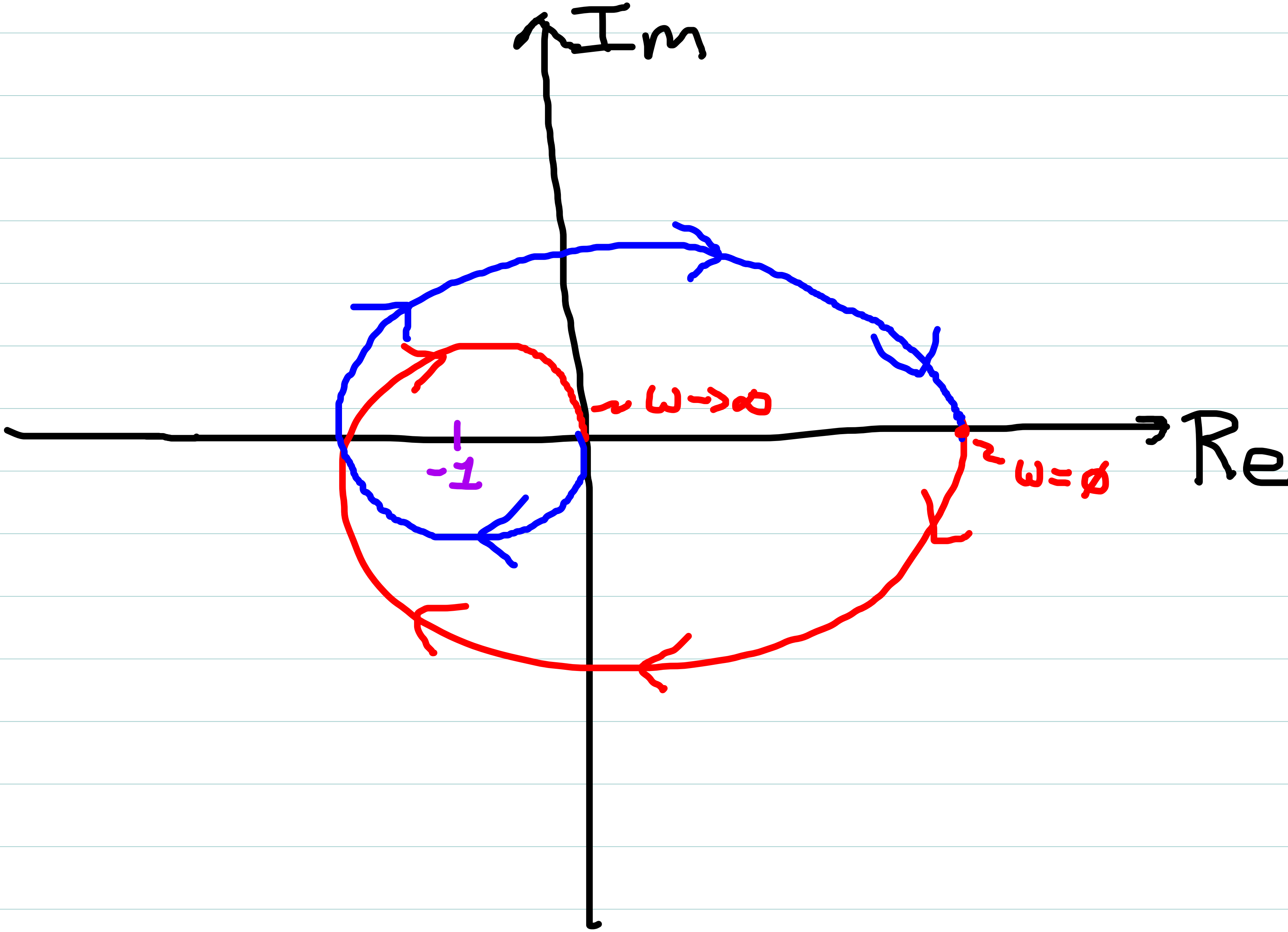
$\Rightarrow$ Questions to answer

$\Rightarrow$ How to create diagram from polar?

$\Rightarrow$ How to count encirclements of -1?

$\Rightarrow$ How many encirclements needed?

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $K_B, \tau > 0$



Nyquist Diagram

When $L(s)$ is type $N \leq \emptyset$ (no poles at origin)

$\Rightarrow$ Draw polar of $L(j\omega)$

$\Rightarrow$ "Flip" polar of L about <u>real</u> axis
(this is the polar of $L(-j\omega)$, i.e. for negative frequencies)

$\Rightarrow$ Put arrows on flipped plot whose direction is consistent with direction of arrows on original polar plot
(i.e. arrows show direction of increasing frequency, from $\omega = -\infty$, through $\omega = \emptyset$, to $\omega = \infty$).

(We will modify for $N > \emptyset$ after we examine complete stability condition.

Counting Encirclements

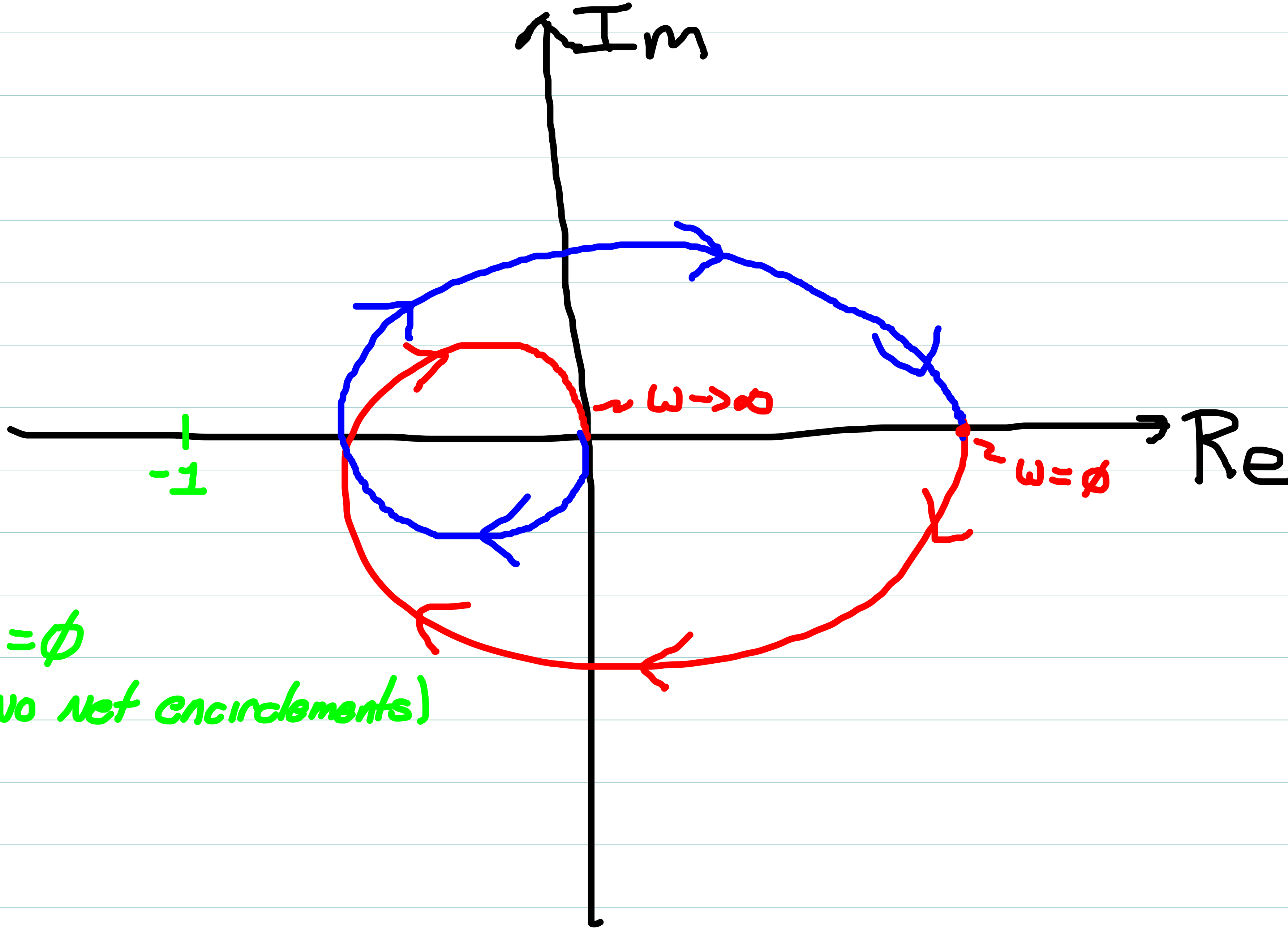
⇒ Count the number of complete loops the diagram makes around -1.

⇒ A clockwise loop counts as $+1$ encirclement
A counter-clockwise loop counts as -1 encirclement

⇒ Diagrams may have both CW or CCW loops around -1

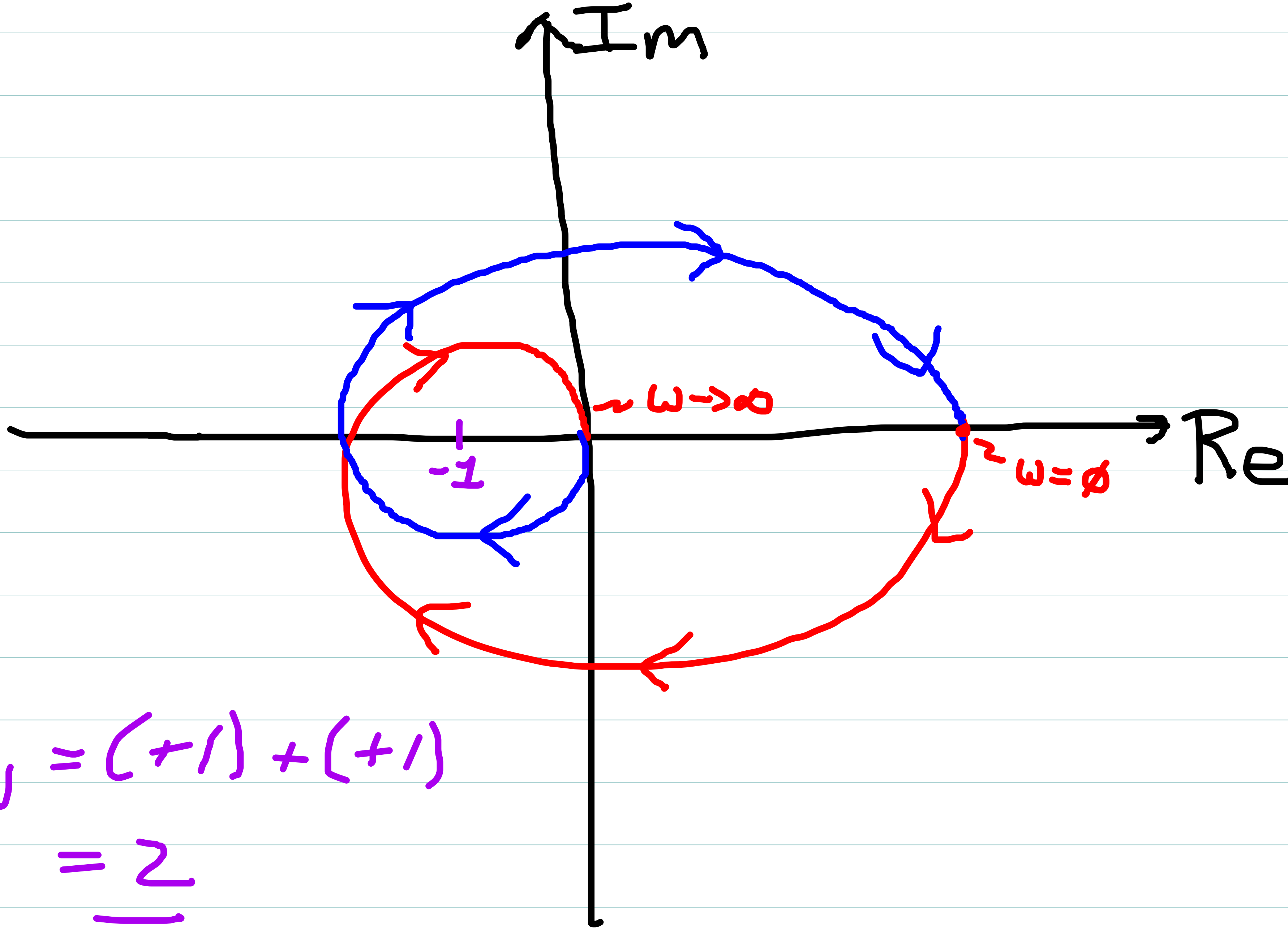
⇒ Let $N_{cw}(L)$ be the net number of CW encirclements for Nyquist diagram of L
(i.e. result of adding contribution of each loop using the ± 1 convention above).

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $K_B, \tau > 0$



$N_{CW} = 0$
(no net encirclements)

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $\quad K_B, \tau > 0$



$N_{CW} = (+1) + (+1)$

$= \underline{2}$