Stability

A mode e^{pt} is <u>stable</u> if

$$|e^{pt}| \to 0 \text{ as } t \to \infty$$

A <u>system</u> is stable if

$$|e^{p_k t}| \to 0 \text{ for all } k = 1, \ldots, n$$

i.e. if every mode is stable

<u>Note</u>: if true then $y_h(t) \to 0$ for <u>any</u> set of initial conditions.

Stability Condition

As usual, let $p = \sigma + j\omega$. Then:

$$|e^{pt}| = |e^{(\sigma + j\omega)t}|$$

$$= |e^{\sigma t} e^{j\omega t}| = |e^{\sigma t}|\,|e^{j\omega t}|$$

$$= |e^{\sigma t}|$$

So $|e^{pt}| \to 0$ only if $\sigma < 0$. Hence:

$$\boxed{\text{A mode is stable if } \sigma = \mathrm{Re}\{p\} < 0}$$

System Stability

The system is stable if:

$$Re\{p_k\} < 0 \text{ for all } k = 1, \ldots, n$$

$\Rightarrow$ all roots of $r(s)$ have negative real parts

$\Rightarrow$ all roots of $r(s)$ lie to the left of imaginary axis in the complex plane.

$\Rightarrow$ all roots of $r(s)$ lie in left half of complex plane (LHP)

Instability

A **mode** e^{pt} is **unstable** if $\mathrm{Re}\{p\} > 0$

$\Rightarrow$ root p lies to right of imag axis

$\Rightarrow$ p is in "right half plane" (RHP)

A **system** is unstable if:

$$\mathrm{Re}\{p_k\} > 0 \text{ for } \underline{\text{any}} \quad k = 1, \ldots, n$$

i.e. if **any** roots of $r(s)$ are in RHP.

What about repeated modes?

Repeated real modes have terms like:

$$t^i e^{pt} \quad \text{(powers of } t \text{ multiplying } e^{pt}\text{)}$$

<u>Fact</u>: for <u>any</u> $i > 0$, if $Re\{p\} < \emptyset$ then

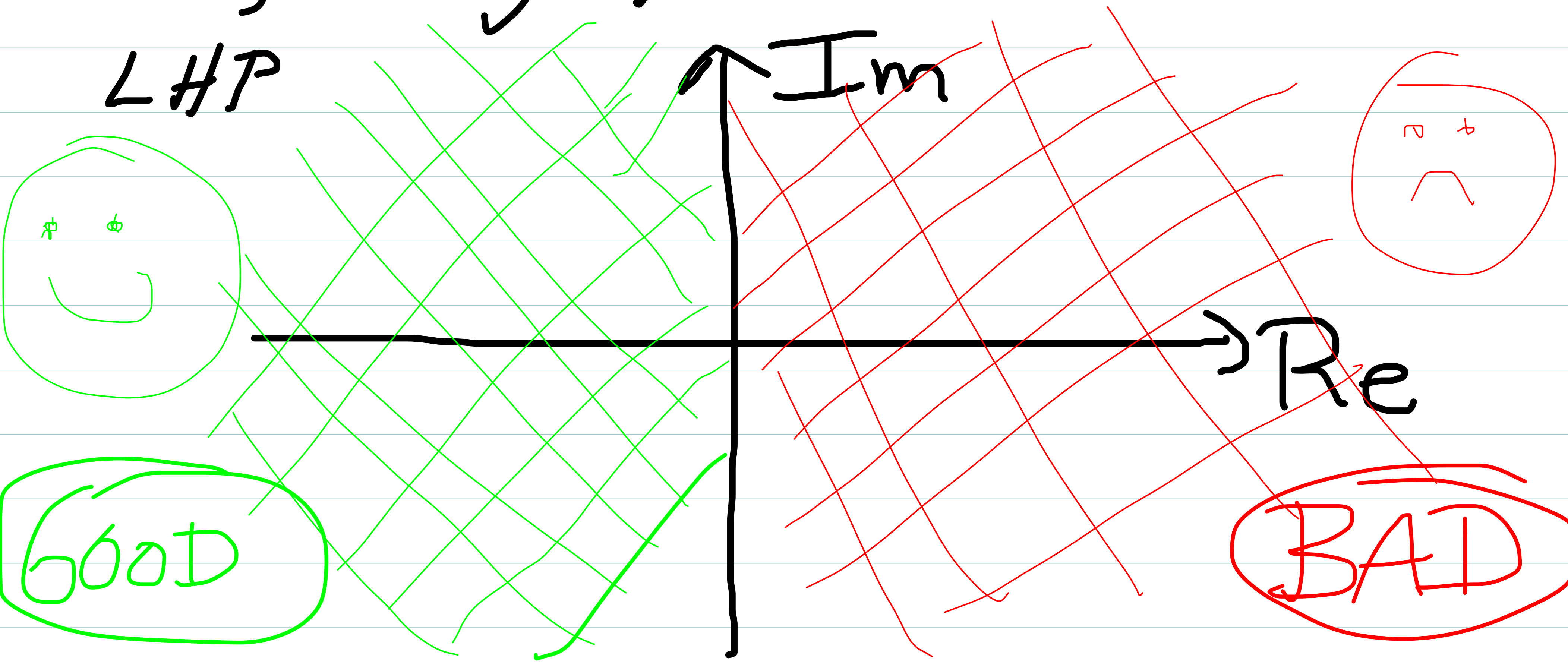
$$\lim_{t \to \infty} |t^i e^{pt}| \to \emptyset$$

Thus a repeated mode is stable as long as the repeated roots are in LHP.

Conversely, a repeated mode is <u>unstable</u> if repeated roots in RHP.

Hence:

A system is stable if <u>all</u> roots of $r(s)$, including repeated roots, lie in LHP



Recap

Stable mode: $|e^{pt}| \to 0$ as $t \to \infty$

$$\iff \mathrm{Re}\{p\} < 0$$

Unstable mode: $|e^{pt}| \to \infty$ as $t \to \infty$

$$\iff \mathrm{Re}\{p\} > 0$$

Stable system: $\mathrm{Re}\{p_k\} < 0$ for all $k = 1, \ldots, n$

Unstable system: $\mathrm{Re}\{p_k\} > 0$ for any $k = 1, \ldots, n$

What happens if $\mathrm{Re}\{p\} = 0$?

Marginally Stable MODES

$$Re\{p\} = \emptyset \Rightarrow |e^{pt}| = |e^{j\omega t}| = 1 \quad \forall t \geq 0$$

i.e. the magnitude is constant

$\Rightarrow$ neither increasing nor decreasing with time

$\Rightarrow$ neither stable nor unstable

"Marginally stable"

Repeated MODES with $Re\{p\} = \emptyset$ will increase

in magnitude polynomially in t

$\Rightarrow$ Not as "bAd" as exponential Growth

An alternate decomposition of $y(t)$

$$y(t) = y_h(t) + y_f(t)$$

$$= \underbrace{y_{tr}(t) + y_{ss}(t)}_{} \text{ (regroup terms)}$$

$y_{tr}(t)$ is the "transient response", which satisfies:

$$\lim_{t \to \infty} |y_{tr}(t)| \to 0$$

$y_{ss}(t)$ is the "steady-state" response, which is all remaining terms in $y(t)$.

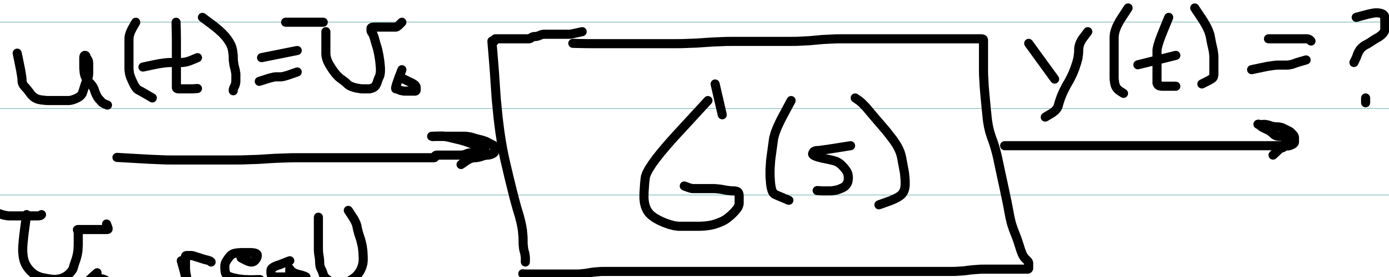
Notes:

1. If system is stable, $y_{tr}(t)$ contains $y_h(t)$ but $y_{tr}(t)$ would also contain decaying terms in $y_f(t)$ (if any).

2. Conversely, marginally stable terms in $y_h(t)$ (if any) would be part of $y_{ss}(t)$.

3. "steady-state" is not a useful concept if system is unstable.

Example: Stable system with constant input



(U_0 real)

constant!

$$y(t) = y_h(t) + y_f(t) = y_h(t) + \boxed{G(0)U_0}$$

Since system is stable, $y_h(t) \to 0$ as $t \to \infty$

So here: $y_{tr}(t) = y_h(t)$

$y_{ss}(t) = G(0)\,U_0$ (constant)

Very common and important case!

A Different Example

$$G(s) = \frac{s+2}{s(s+1)}, \quad u(t) = e^{-3t}$$

$$y_h(t) = C_1 + C_2 e^{-t}$$

$$y_f(t) = G(-3)e^{-3t} = -\tfrac{1}{6}e^{-3t}$$

$$y_{tr}(t) = C_2 e^{-t} - \tfrac{1}{6}e^{-3t}$$

$$y_{ss}(t) = C_1$$

Note: system is <u>not</u> stable here.

Convergence metrics

Useful to quantify how quickly stable modes decay to $\emptyset$.

"2% criterion": Defines the settling time t_s for a mode to be such that

$$|e^{pt}| \leq .02 \quad \forall t \geq t_s$$

For a 1st order mode ($p=\sigma$, real)

$$t_s = \frac{\ln(.02)}{\sigma} \approx \frac{4}{|\sigma|} = \frac{4}{|Re\{p\}|}$$

2^{nd} order settling time

For a 2^{nd} order mode $e^{pt}, e^{\bar{p}t}$ with $p = \sigma + j\omega$, $\omega \neq 0$, the calculation is more complicated due to the oscillations.

However:

$$t_s \approx \frac{4}{|\sigma|} = \frac{4}{|Re\{p\}|}$$

is still a useful approximation in these cases also.

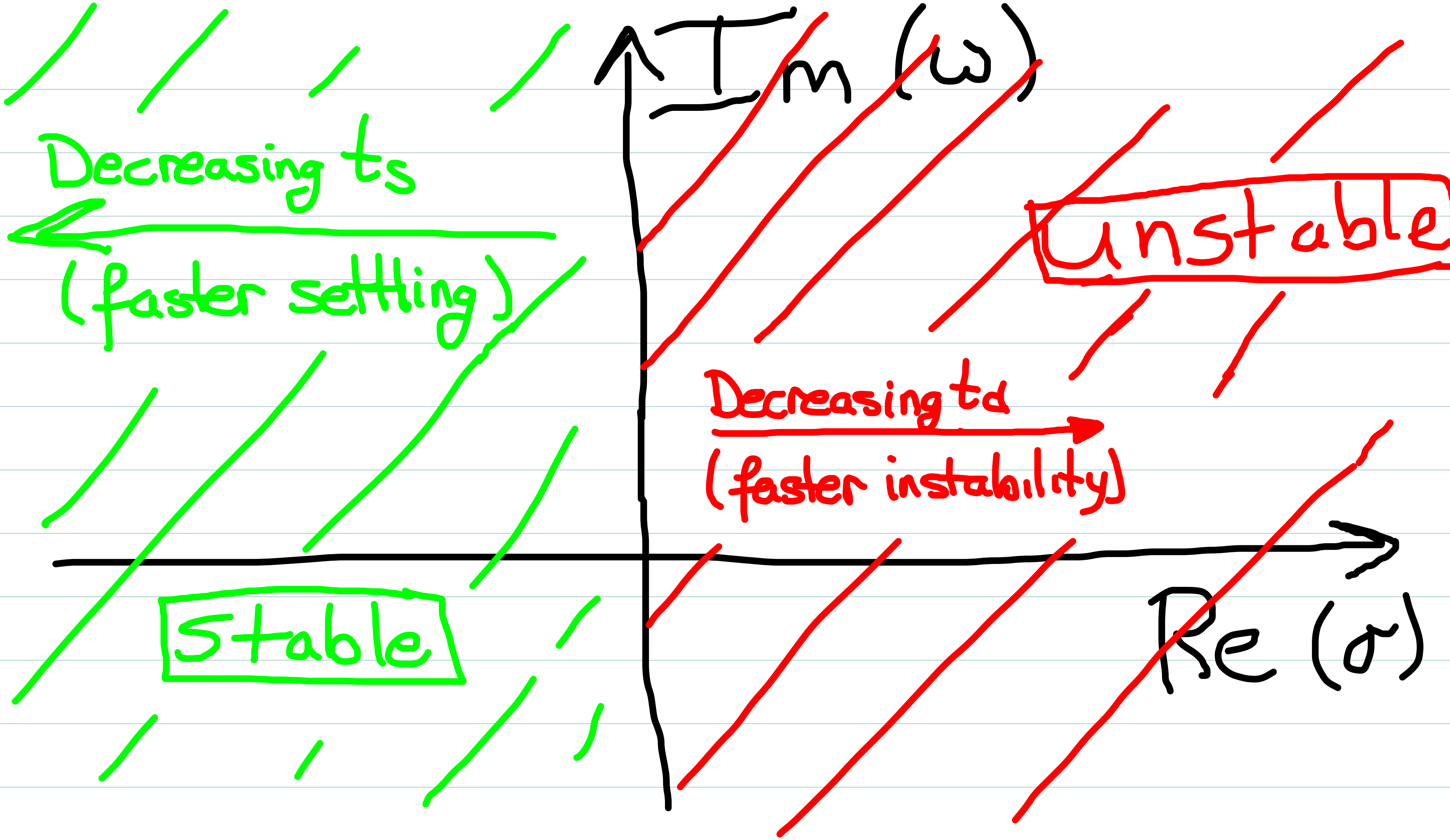
"Doubling time" of unstable modes

When $\sigma > 0$, the doubling time t_d is such that

$$|e^{\sigma t_d}| = 2 \quad \Rightarrow \quad \boxed{t_d \approx \frac{0.7}{\sigma}}$$

Smaller $t_d \Leftrightarrow$ "more unstable" system

$\Rightarrow$ Faster rate of increase for amplitude



Settling times decrease the further left of the imag axis the root p is.

To a first approximation, the settling time of a system is the settling time of its slowest mode

⇒ Mode closest to imag Axis determines settling time

⇒ Called the "dominant mode"

⇒ Only a "first cut." Will refine later

2nd Order "Damping ratio"

for 2nd order modes we also define the

damping ratio

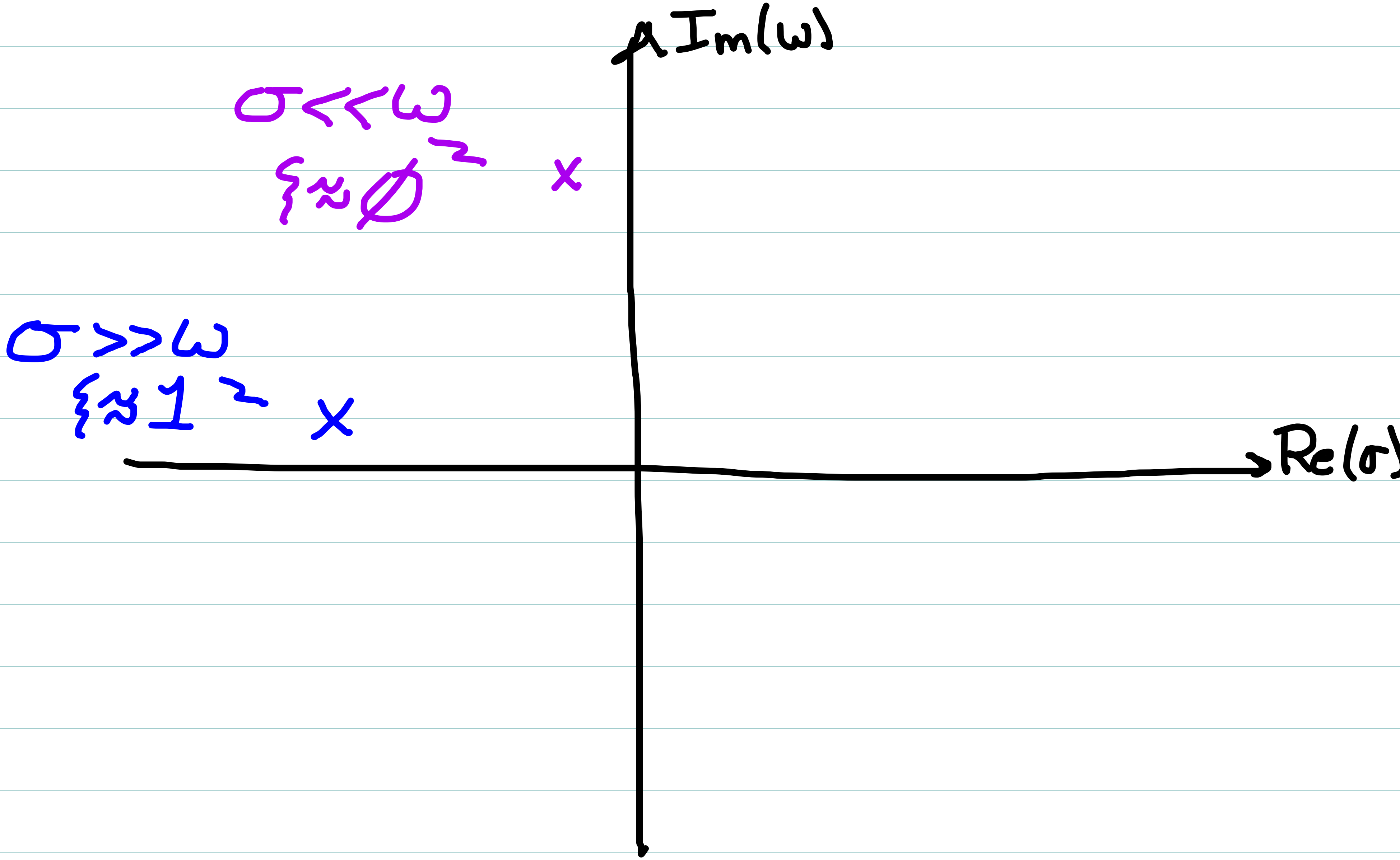
$$\boxed{\zeta = \left|\frac{\sigma}{p}\right| = \frac{|\sigma|}{\sqrt{\sigma^2+\omega^2}}}$$

A non-dimensional comparison of convergence rate to oscillation frequency

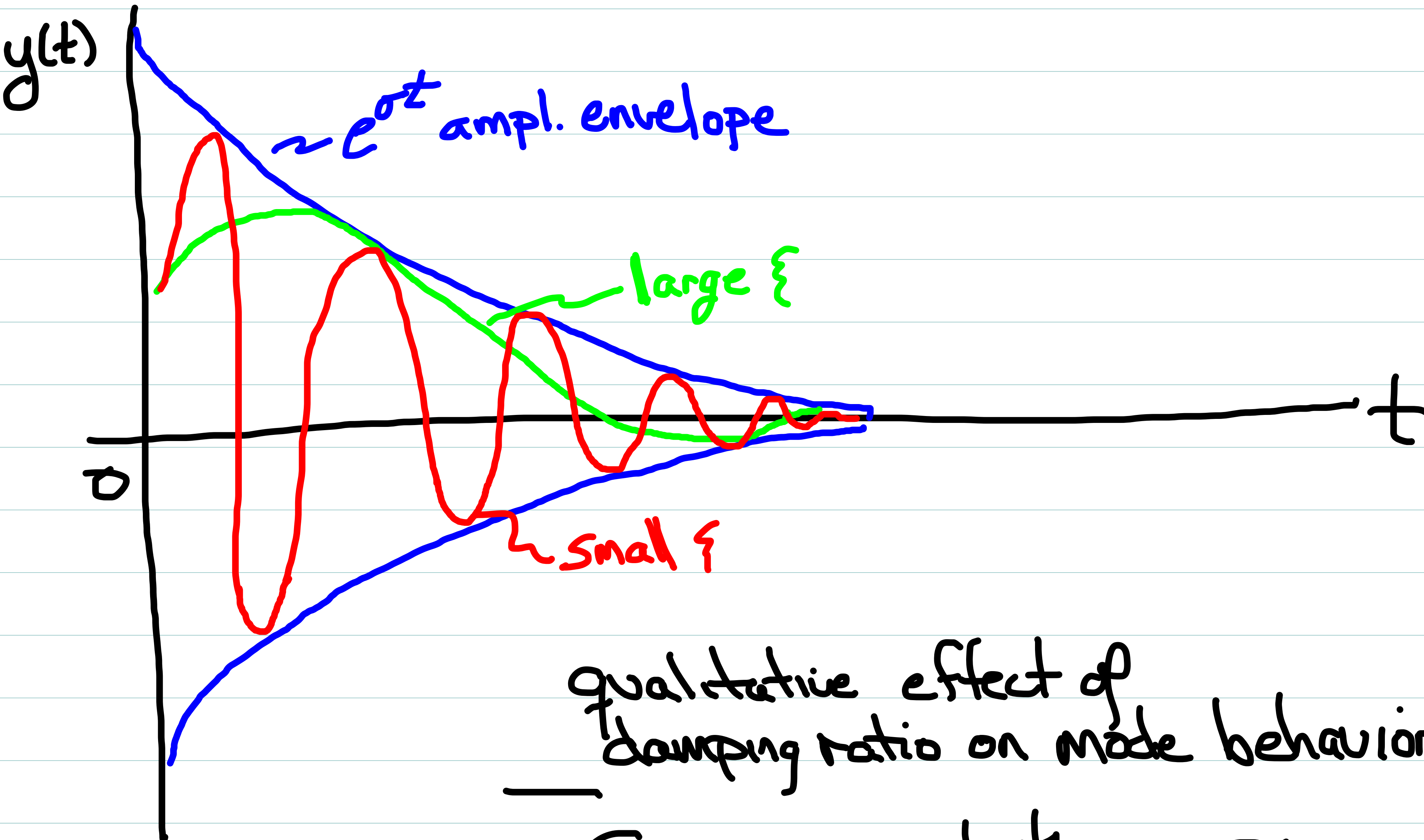
$0 \le \zeta \le 1$ for a stable mode

$\zeta \approx 0 \iff$ many oscillations before 2% criterion reached

$\zeta \approx 1 \iff$ less than one complete oscillation before 2% criterion reached.



Will explore in greater detail later



qualitative effect of damping ratio on mode behavior

Same σ in both CASES, different ω

Transfer functions

$$G(s) = \frac{q(s)}{r(s)}$$

Compactly gives us all information we need to predict major features of system response

- $y_h(t)$, modes, stability : all from $r(s)$ the denominator polynomial of $G(s)$

$$r(s) = \alpha_n \prod_{k=1}^{n} (s - p_k)$$

- forced response : Evaluate $G(s)$ at specific complex values of s.

Numerator Terms

Can also factor $q(s)$:

$$q(s) = \beta_m (s - z_1)(s - z_2)\cdots(s - z_m)$$

where $q(z_i) = 0$ for $i = 1, \ldots, m$

The values z_i are called the zeros of $G(s)$

since $G(z_i) = \dfrac{\cancel{q(z_i)} \rightarrow 0}{r(z_i)} = 0$

The values p_k are called the poles of $G(s)$

since $G(p_k) = \dfrac{q(p_k)}{0 \leftarrow \cancel{r(p_k)}} = \infty$

Zero/Pole/Gain (ZPK) form

$$G(s) = K \left[\frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{k=1}^{n} (s - p_k)} \right]$$

Poles p_k satisfy $r(p_k) = \emptyset$

Zeros z_i satisfy $q(z_i) = \emptyset$

Gain: $K = \frac{\beta_m}{\alpha_n}$ (always real)

Alternate ZPK form:

When $G(s)$ has complex poles and/or zeros, we commonly combine the conjugate roots of $r(s)$ or $q(s)$ into 2nd order polynomials.

For example, if $p = \sigma + j\omega$ and $\bar{p} = \sigma - j\omega$ are complex roots of $r(s)$:

$$\underbrace{(s-p)(s-\bar{p})}_{\text{replace with}} = \underline{s^2 - 2\sigma s + (\sigma^2 + \omega^2)}$$

replace with ↑ in $G(s)$