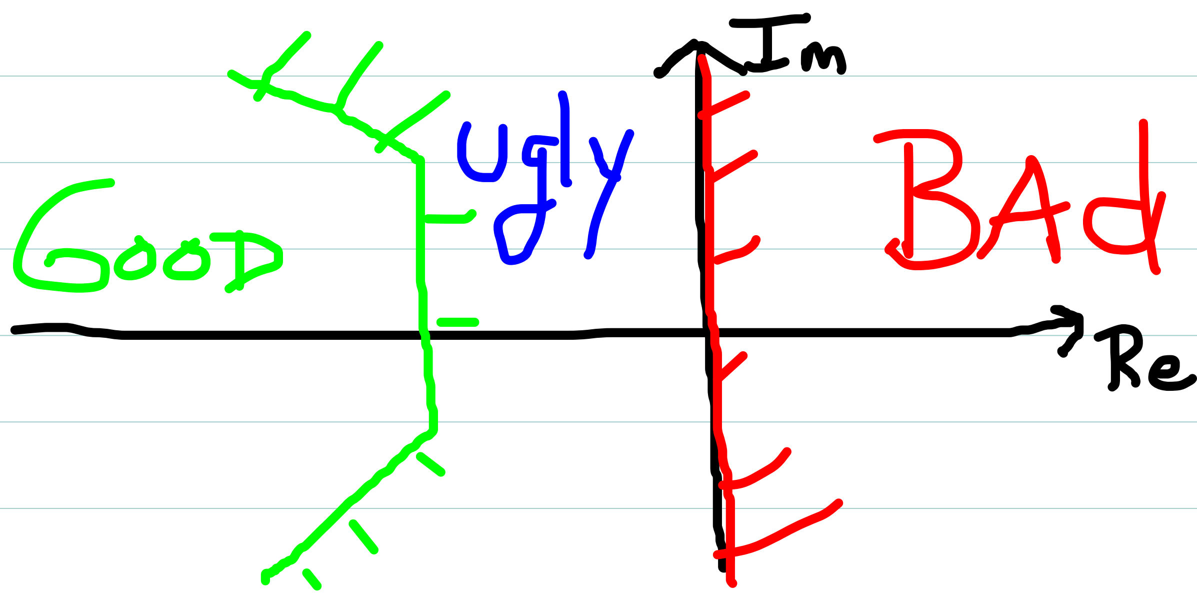
Fundamental Consideration: Closed-loop Stability

$\boxed{\text{Most basic design consideration:}}$ ⇐ !!

Closed-loop poles should be "good", and certainly <u>must be stable</u>.

Thus, sol'ns of $\boxed{\text{CE: } 1+L(s)=\emptyset}$ must be in left half of complex plane, preferably in "good region" (far from imag Axis, relatively close to or on the real Axis).



A crucial Observation:

If $L(j\omega) = -1$ for some ω, then

$1 + L(s) = \emptyset$ has a sol'n $s = j\omega$ for some ω

$\Rightarrow$ closed-loop dynamics has poles at $\pm j\omega$, on imag Axis

$\Rightarrow$ Such poles are on the boundary between bad and ugly

$\Rightarrow$ This situation must be avoided!!!

Now if $\boxed{\angle L(j\omega) = -1}$ for some $\omega > \phi$, then:

$\Rightarrow$ Polar plot of $L(j\omega)$ passes through -1

$\Rightarrow$ $\omega_a = \omega_\gamma$ (both crossover freqs same)

$\Rightarrow$ $\boxed{a = \phi\ dB,\ \gamma = \phi^\circ}$ (both margins ϕ)

$\boxed{\text{Any such feedback loop is bad!}}$

Now, suppose $\exists \omega \geq \phi \ni$: $L(j\omega) \approx -1$ (i.e. close to, but not exactly -1)

By continuity of $L(s)$, $1 + L(s) = 0$ would have a sol'n very near (but not exactly on) the imag axis.

Some poles of $T(s)$ would be in bad or ugly region
$\Rightarrow$ Also undesireable!

Now, if $L(j\omega) \approx -1$ for some $\omega \geq 0$

$\Rightarrow$ Polar plot of $L(j\omega)$ comes very close to -1 but doesn't pass exactly through it

$\Rightarrow$ (typically) $|a_{dB}|$ and $|\gamma|$ very small (small margins)

$\Rightarrow$ $\boxed{\text{This should Also be avoided.}}$

Thus, for $T(s)$ to have only good poles, we need conditions:

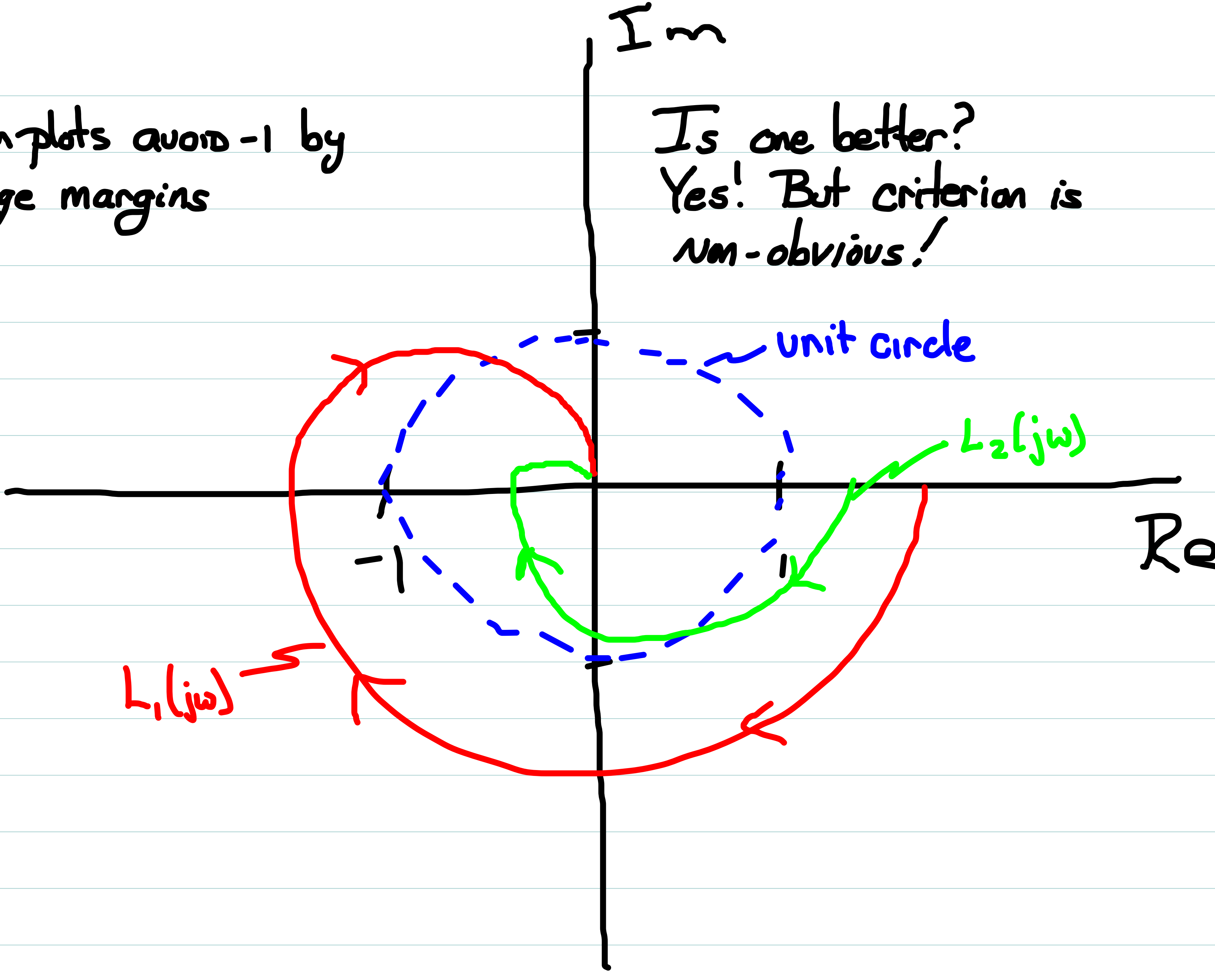
$\Rightarrow$ Gain and phase margins $\boxed{\text{of } L(s)}$ $\Leftarrow$!!! to be Large

$\Rightarrow$ polar plot of $L(j\omega)$ avoids -1 by wide margins

Necessary, but <u>Not sufficient</u>!

Both plots avoid -1 by large margins

Is one better?
Yes! But criterion is non-obvious!



Nyquist Stability Criterion

All roots of $1+L(s)=\phi$ are in LHP if.

the Nyquist diagram (a modified polar plot) $\boxed{\text{of } L(j\omega)}$ circles the -1 point the correct number of times).

$\Rightarrow$ Major Theoretical result! Used extensively in control theory

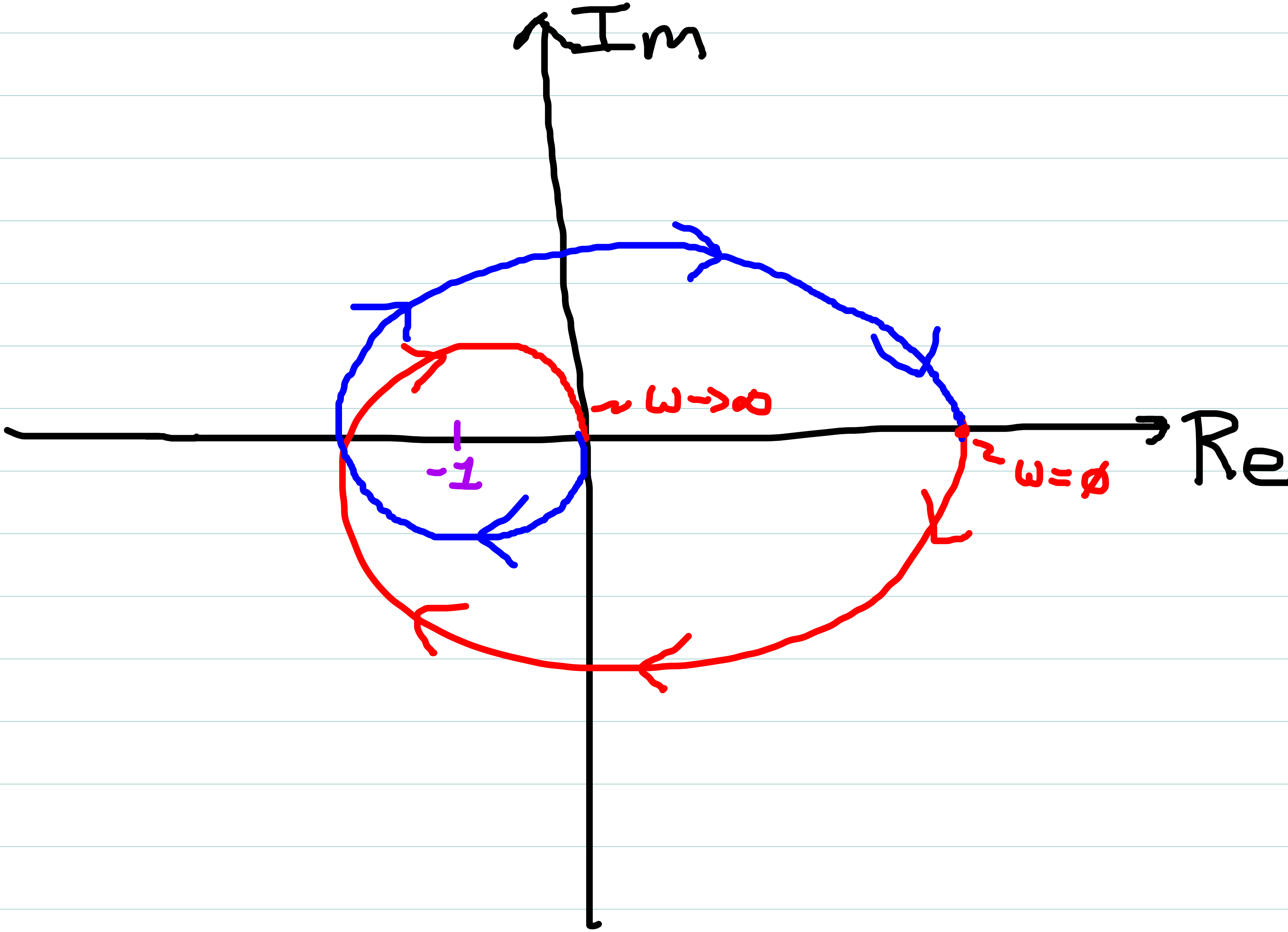
$\Rightarrow$ Questions to answer

$\Rightarrow$ How to create diagram from polar?

$\Rightarrow$ How to count encirclements of -1?

$\Rightarrow$ How many encirclements needed?

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $\quad K_B, \tau > \emptyset$



Nyquist Diagram

When $L(s)$ is type $N \leq \emptyset$ (no poles at origin)

$\Rightarrow$ Draw polar of $L(j\omega)$

$\Rightarrow$ "Flip" polar of L about real axis
(this is the polar of $L(-j\omega)$, i.e. for negative frequencies)

$\Rightarrow$ Put arrows on flipped plot whose direction is consistent with direction of arrows on original polar plot
(i.e. arrows show direction of increasing frequency, from $\omega = -\infty$, through $\omega = \emptyset$, to $\omega = \infty$).

$\boxed{\text{(We will modify for } N > \emptyset \text{ after we examine complete stability condition.}}$

Counting Encirclements

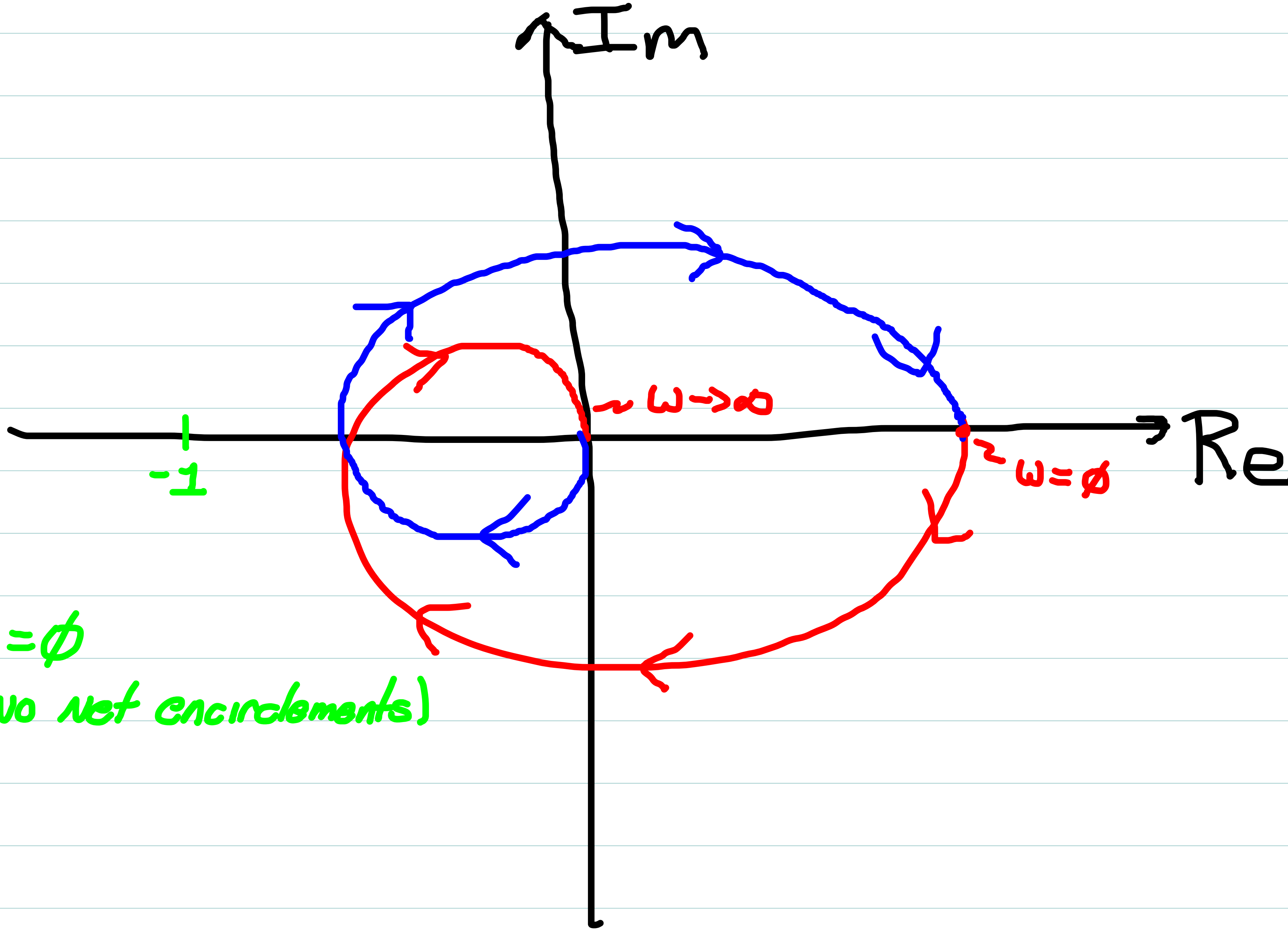
⇒ Count the number of complete loops the diagram makes around −1.

⇒ A clockwise loop counts as +1 encirclement
A counter-clockwise loop counts as −1 encirclement

⇒ Diagrams may have both CW or CCW loops around −1

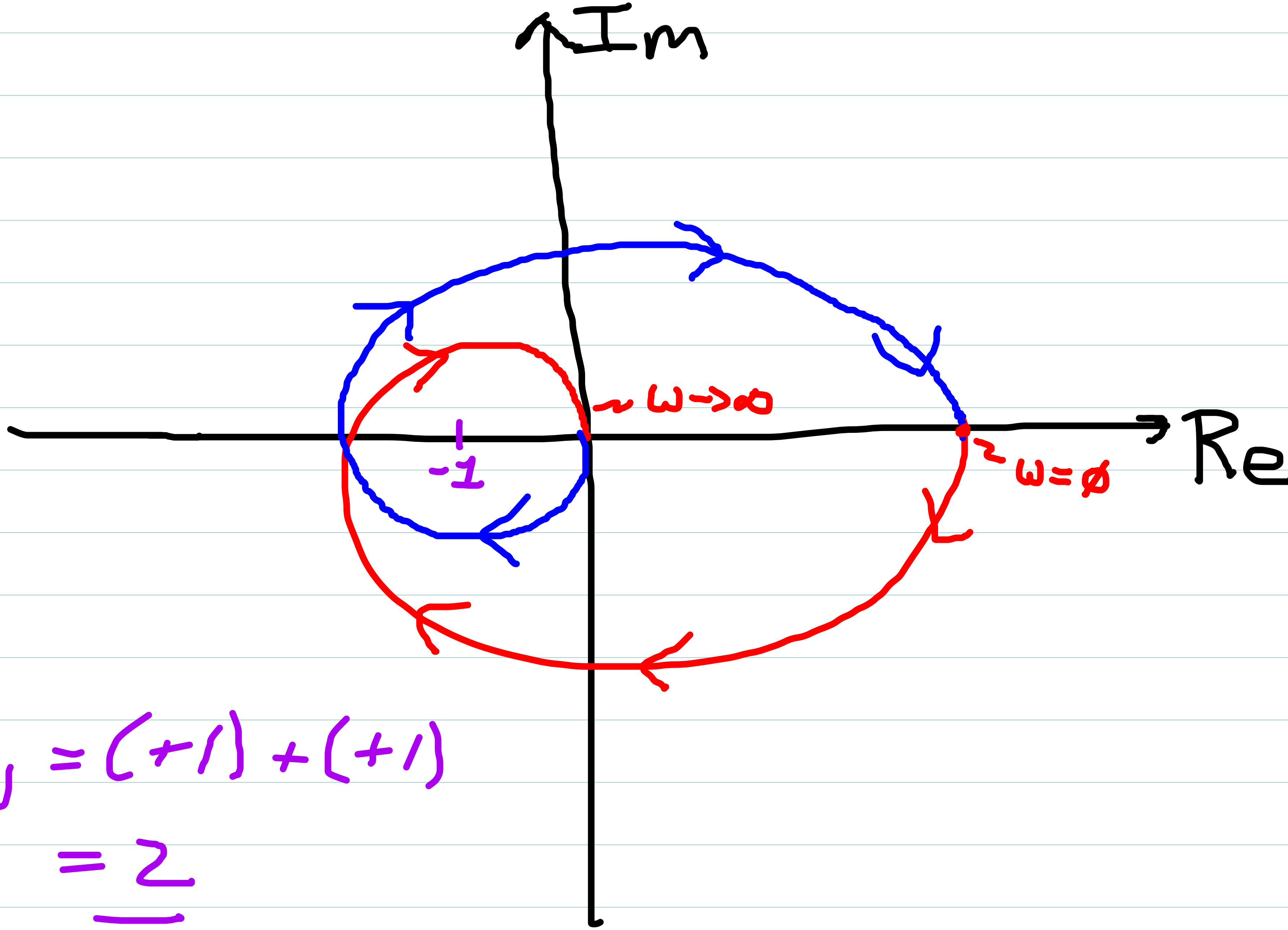
⇒ Let $N_{cw}(L)$ be the net number of CW encirclements for Nyquist diagram of L
(i.e. result of adding contribution of each loop using the ±1 convention above).

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $\quad K_B, \tau > 0$



$N_{CW} = 0$
(no net encirclements)

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$ $K_B, \tau > 0$



$N_{CW} = (+1) + (+1)$
$= \underline{2}$

Easy Way to Count Encirclements

"Ray trick"

=> Draw a ray radially outward from -1 in any direction

=> Looking along the ray, away from -1

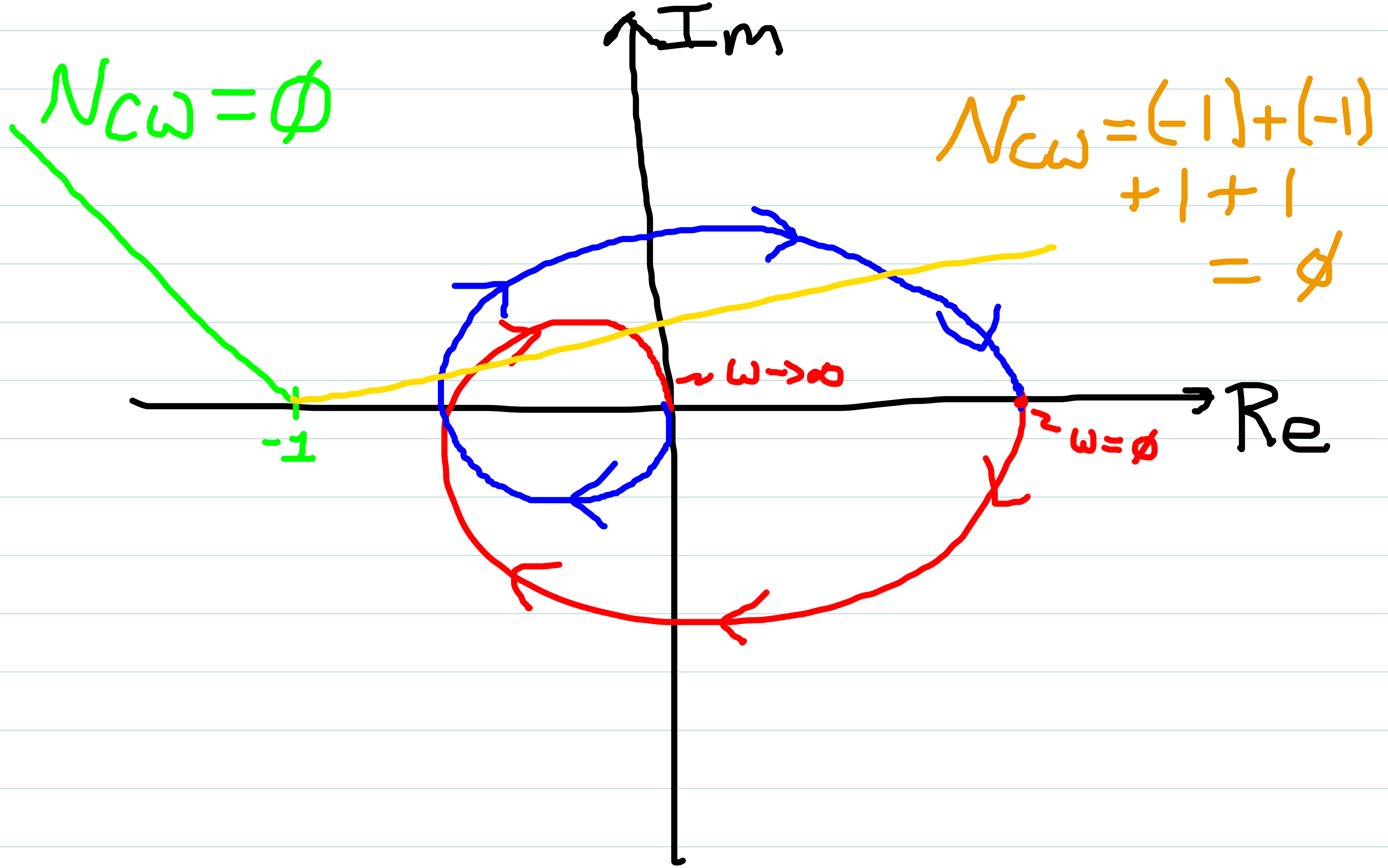
=> Count $+1$ each time diagram crosses ray from left to right.

=> Count -1 each time ray is crossed right to left.

=> Same result regardless of ray direction

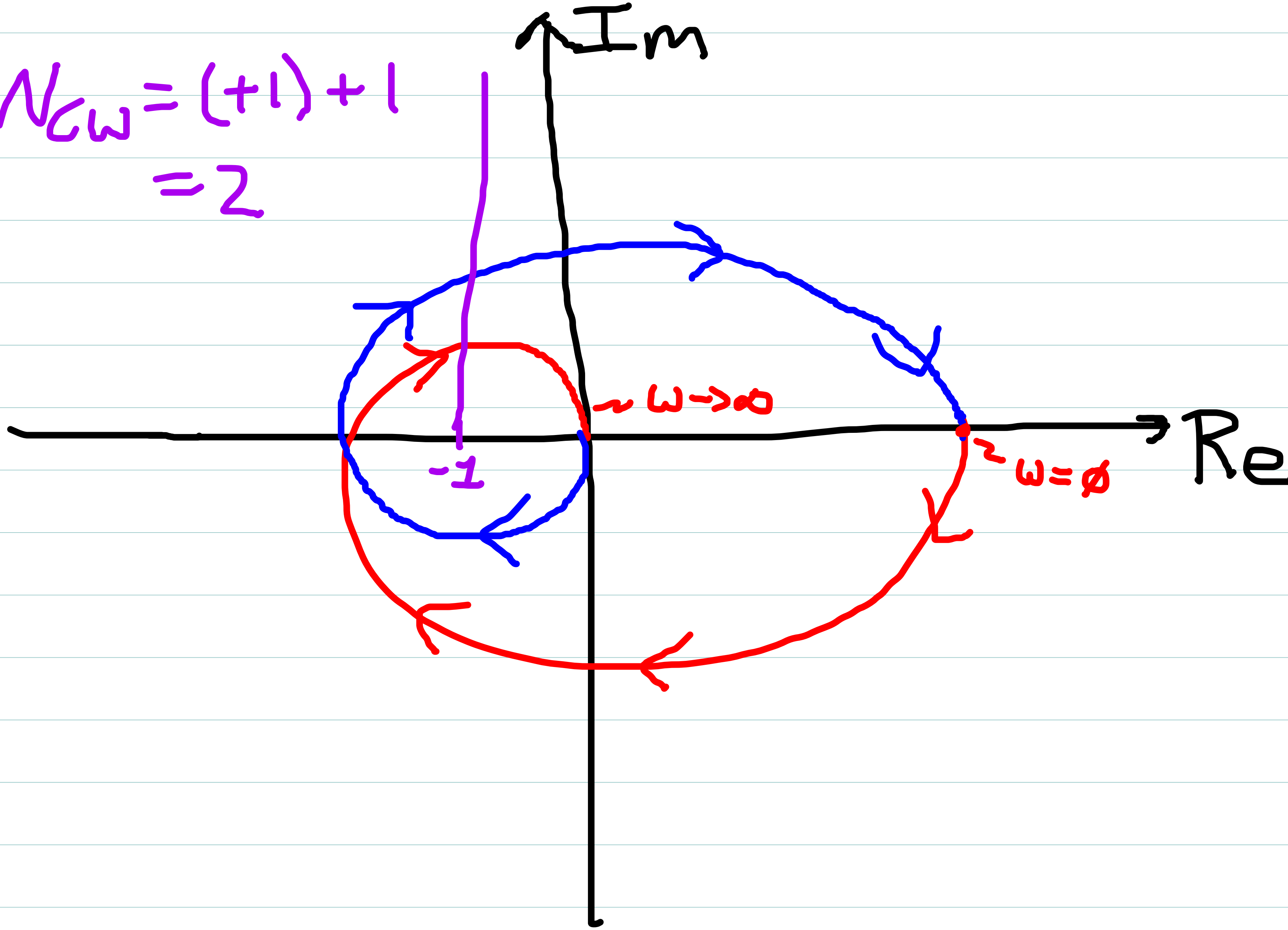
=> Choose direction with least number of intersections for easiest counting.

Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$



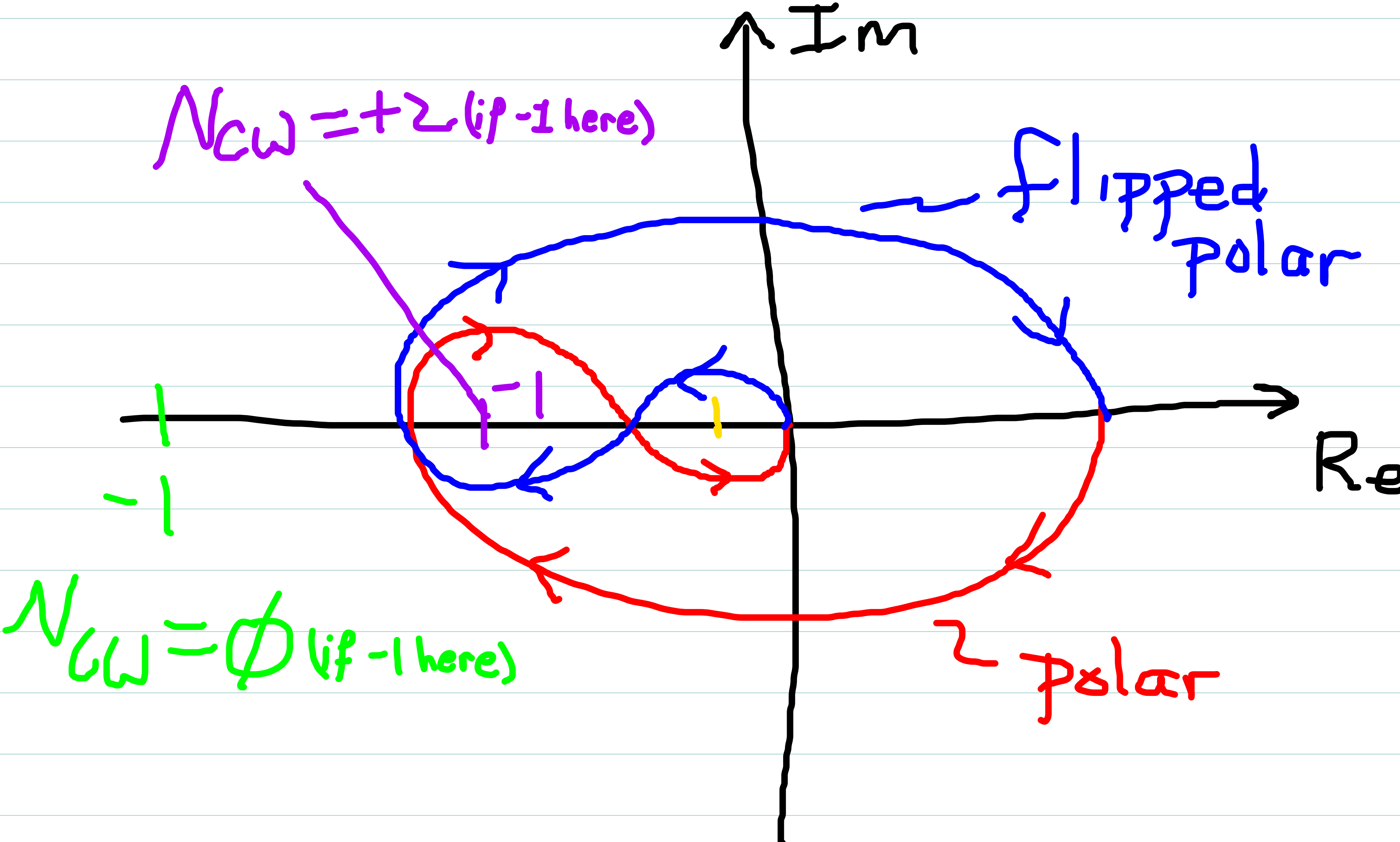
Example: $L(s) = \dfrac{K_B}{(\tau s+1)^3}$

$N_{cw} = (+1) + 1$
$= 2$



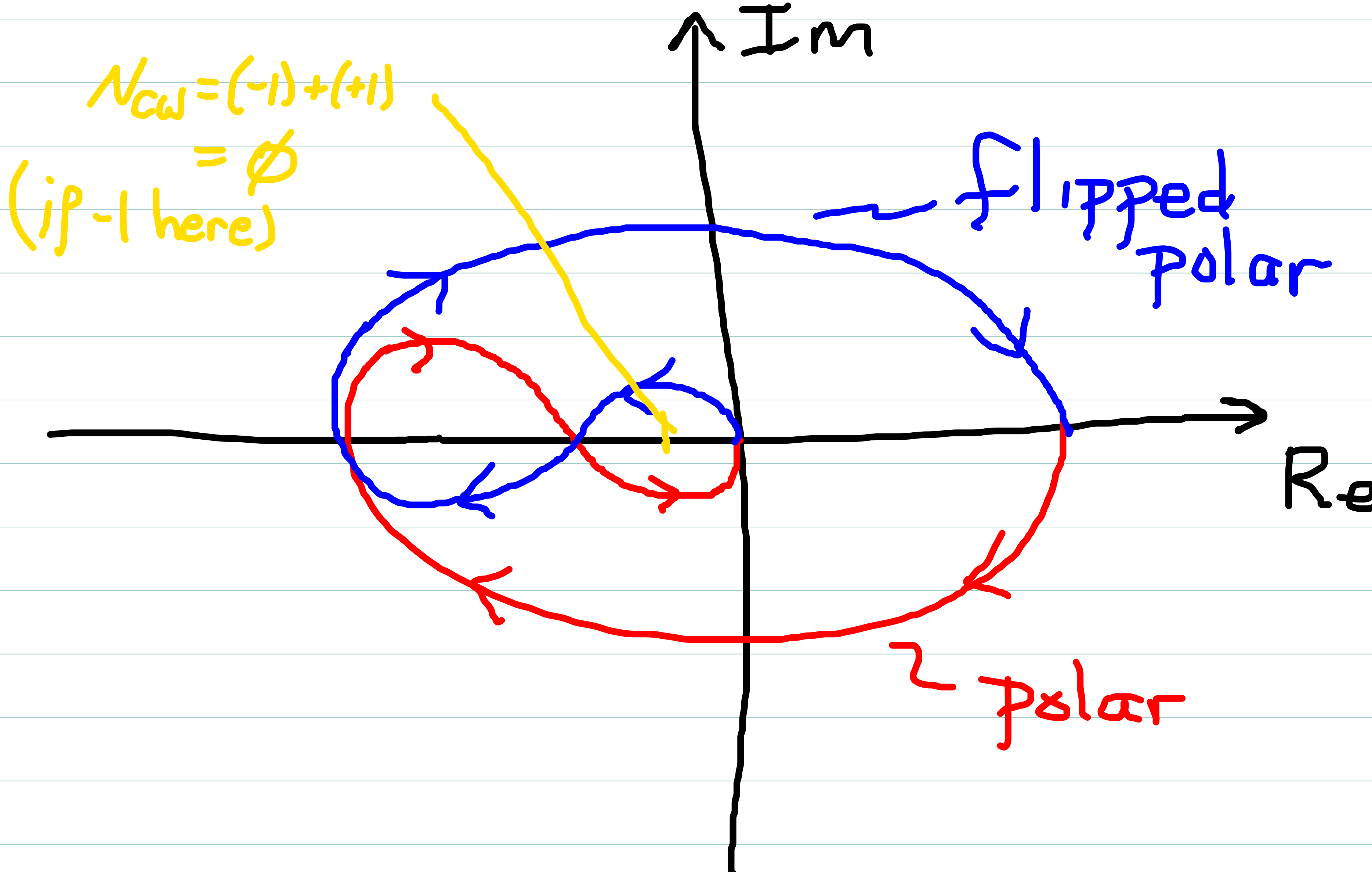
A more complicated Example:

$$L(s) = \frac{K_B(\tau_1 s+1)^2}{(\tau_2 s+1)^3} \qquad \tau_2 \gg \tau_1 > \emptyset$$



A more complicated Example:

$$L(s) = \frac{K_B(T_1 s+1)^2}{(T_2 s+1)^3} \qquad T_2 \gg T_1 > \emptyset$$



Nyquist Stability Theorem

For an arbitrary transfer function $G(s)$, define

$$P_R(G) = \#\text{RHP (unstable) poles of } G(s)$$

Nyquist showed:

$$\boxed{N_{cw}(L) = P_R(T) - P_R(L)}$$

Re-arranging:

$$\underbrace{P_R(T)}_{\text{want to predict}} = \underbrace{P_R(L) + N_{cw}(L)}_{\text{Known}}$$

Note: $P_R(T) \geq \emptyset$ <u>always</u>. If you compute $P_R(T) < \emptyset$

$\Rightarrow$ you have drawn the diagram incorrectly, or

$\Rightarrow$ you have counted encirclements incorrectly.

Implication

$\Rightarrow$ We <u>must</u> have $P_R(T) = \emptyset$ (stable closed-loop system)

$$\Rightarrow \boxed{N_{cw}(L) = -P_R(L)}$$ (Stability condition)

i.e. Nyquist diagram must show a net <u>negative</u> number of encirclements, equal to number of unstable poles of $L(s)$.

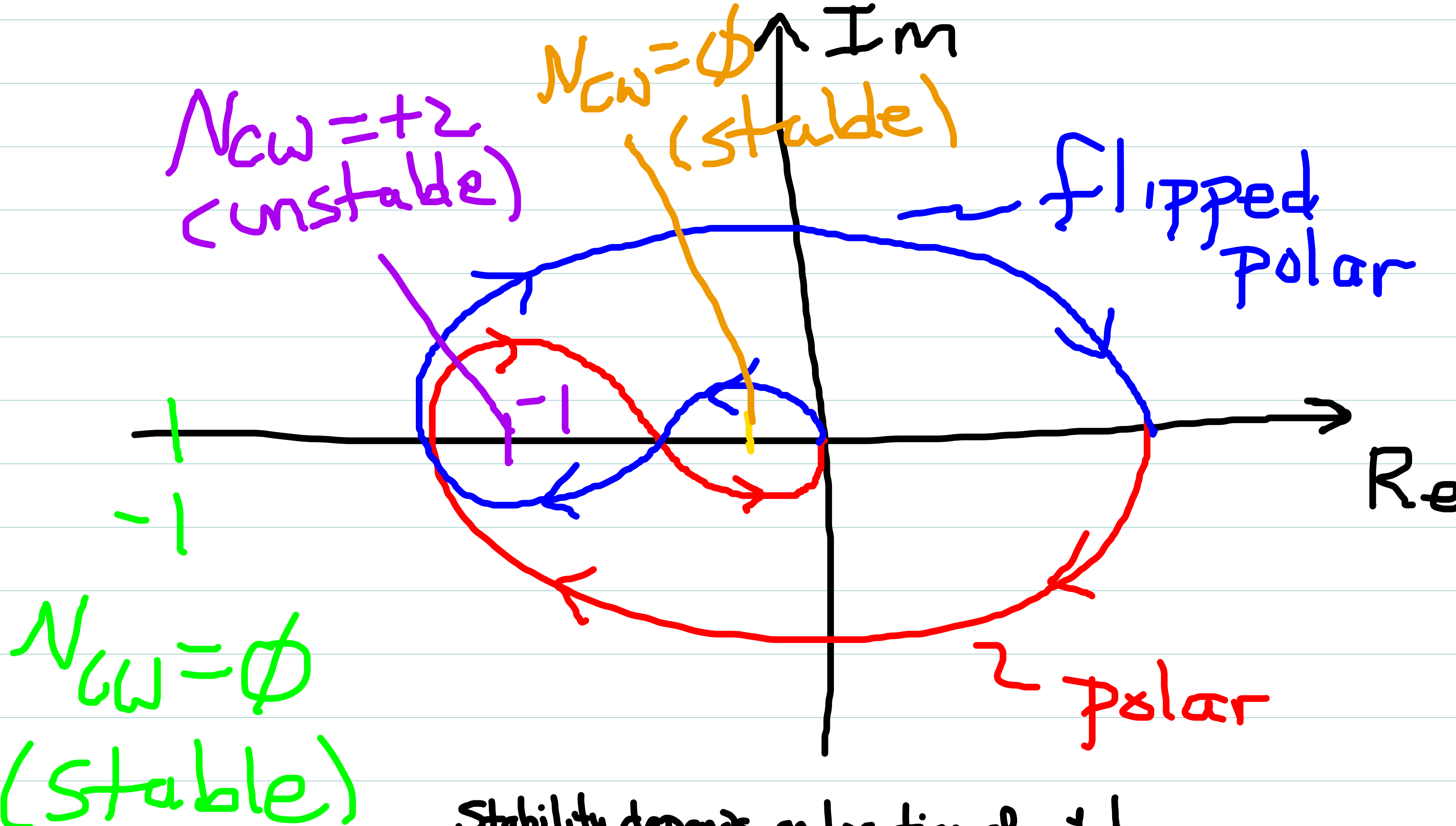
Recall negative CW encirclements are CCW encirclements

$\Rightarrow$ Nyquist diagram must show a net number of <u>CCW</u> encirclements equal to # unstable poles of $L(s)$

<u>Note</u>: if $P_R(L) = \emptyset$ ($L(s)$ is stable) then the diagram must show <u>NO</u> ($\emptyset$) net encirclements

A more complicated Example:

$$L(s) = \frac{K_B(T_1 s+1)^2}{(T_2 s+1)^3} \qquad T_2 >> T_1 > \emptyset \quad \Big] \; P_R(L) = \emptyset$$



Stability depends on location of -1!