Thus finally, the Case 1 step response is:

$$\boxed{y(t) = G(0)\left[1 - \left(\frac{\omega_n}{\omega_d}\right) e^{\sigma t} \sin(\omega_d t + \cos^{-1}\zeta)\right]}$$

We can now solve for important transient parameters

$\Rightarrow$ t_c: Solve for first $t > 0$ such that

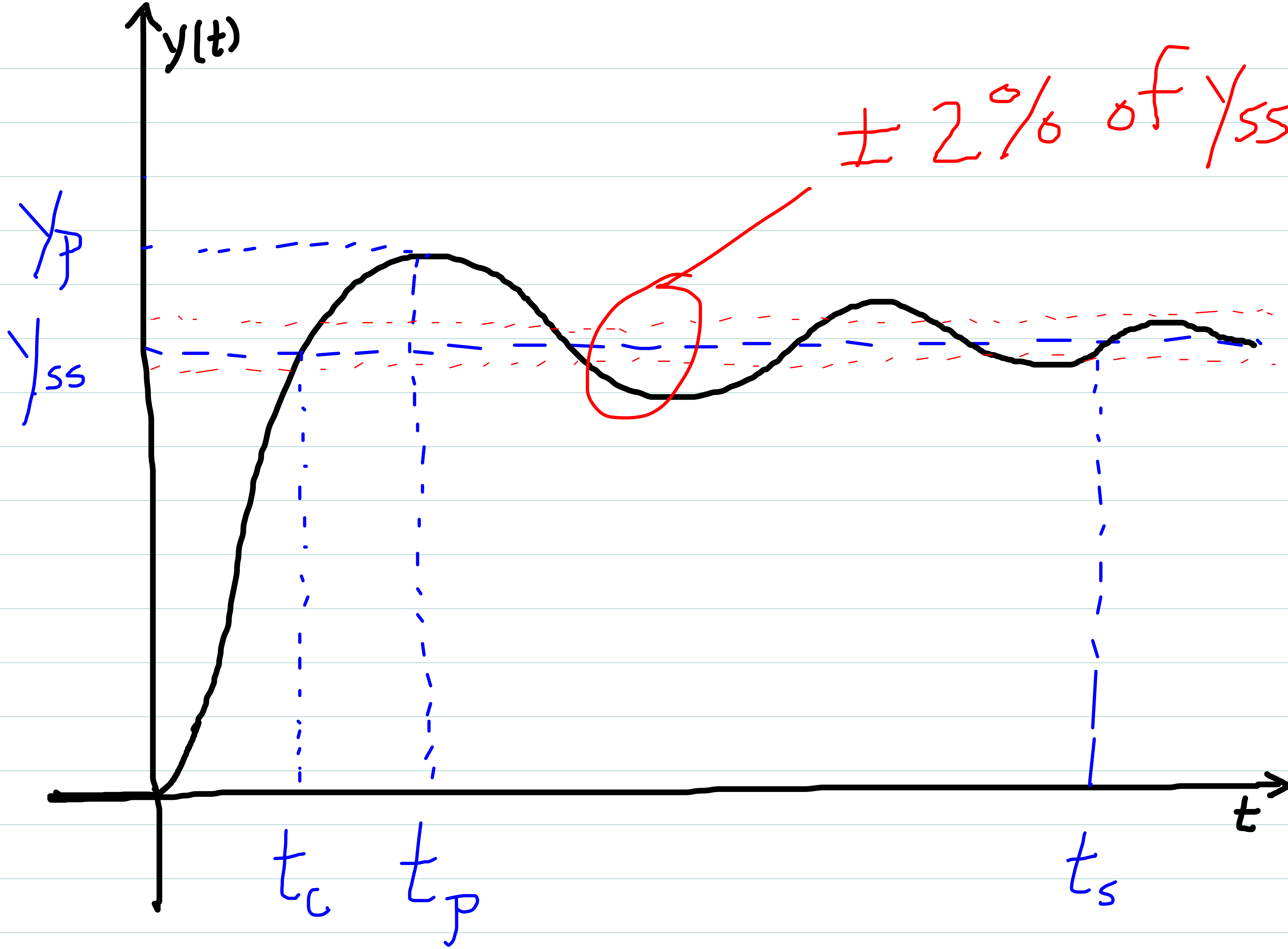
$$y(t) = y_{ss}(t) = G(0)$$

$$\Rightarrow \sin(\omega_d t + \cos^{-1}\zeta) = 0$$

$$\Rightarrow t_c = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

or:

$$\boxed{t_c = \frac{\pi - \rho}{\omega_d}}$$



$y(t)$
$\pm 2\%$ of y_{ss}
y_p
y_{ss}
t
t_c
t_p
t_s

$\Rightarrow$ For t_p, y_p

Solve for first $t>0$ such that

$$\dot{y}(t) = 0$$

$$\Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Substituting:

$$y_p = y(t_p) = G(0)\left[1 + e^{(\sigma\pi/\omega_d)}\right]$$

<u>Define</u>:

$$\boxed{\begin{aligned} M_p &= e^{(\sigma\pi/\omega_d)} \\ \text{then: } \quad y_p &= G(0)[1 + M_p] \end{aligned}}$$

Peak Overshoot

⇒ M_p is the Normalized peak overshoot

$$y_p = G(0)[1+M_p] \Longrightarrow M_p = \frac{y_p - G(0)}{G(0)} = \frac{y_p - y_{ss}}{y_{ss}}$$

⇒ M_p is entirely determined by damping ratio ξ

$$M_p = \exp\left[\frac{\sigma\pi}{\omega_d}\right]$$

$$= \exp\left[\frac{(-\xi\omega_n)\pi}{\omega_n\sqrt{1-\xi^2}}\right]$$

OR

$$\boxed{M_p = \exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]}$$

$\%OS = 100 \times M_p$

Settling Time

As usual, we can use the approximation

$$\boxed{t_s \approx \frac{4}{|Re\{p_i\}|} = \frac{4}{|\sigma|}}$$

But t_s is actually a function of ξ also here:

$$t_s = \frac{C(\xi)}{|\sigma|}$$

with $3 \leq C(\xi) \leq 5$ for most $0 \leq \xi < 0.9$

so 4 is an "average" value for $C(\xi)$

However for $\boxed{0.95 \leq \xi \leq 1}$ a better approximation is: $\boxed{t_s \approx \frac{6}{|\sigma|}}$

Summary: Case I step response; $p_1 = \sigma + j\omega_d$

"Natural" frequency: $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = |p_1|$

Damping ratio: $\xi = \dfrac{|\sigma|}{\omega_n}$

1st crossing: $t_c = \dfrac{\pi - \cos^{-1}\xi}{\omega_d} = \dfrac{\pi - \nu}{\omega_d}$, $\xi = \cos\nu$

1st peak: $t_p = \dfrac{\pi}{\omega_d}$

Normalized overshoot: $M_p = \exp\left[\dfrac{\sigma\pi}{\omega_d}\right] = \exp\left[\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$$M_p = \left[\frac{y_p - y_{ss}}{y_{ss}}\right]$$

Peak response: $y_p = y_{ss}[1 + M_p]$

$y_{ss} = G(0)$ for unit step

Limiting case: $\xi \to 0$

$\xi \to 0 \Rightarrow \sigma = -\xi\omega_n \to 0 \Rightarrow P_1 = j\omega_d$ (pure imaginary)

Overshoot $M_p = e^{(\sigma\pi/\omega_d)} \to 1$ (100% OS)

Peak: $y_p = G(0)[1 + M_p]$

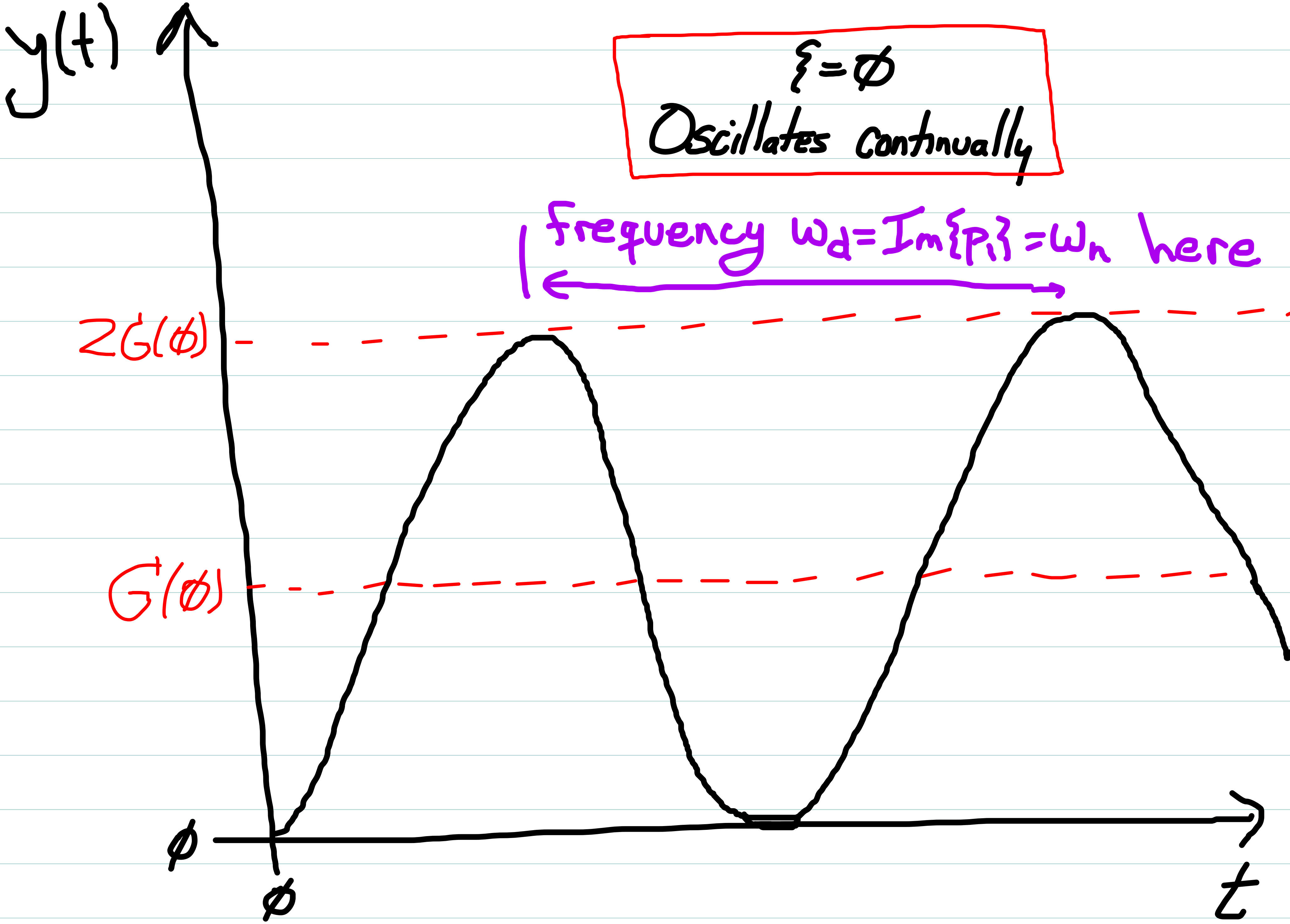
or $y_p = 2y_{ss}$

Settling time: $t_s \approx \dfrac{4}{|\sigma|} = \infty$

Never settles!

Response oscillates infinitely between 0 and $2G(0)$ with frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = \omega_n$

"Undamped"



y(t)
$\zeta = \emptyset$
Oscillates continually
frequency $\omega_d = Im\{p_i\} = \omega_n$ here
$2G(\emptyset)$
$G(\emptyset)$
$\emptyset$
$\emptyset$
t

Limiting Case, $\zeta \to 1$

$\zeta \to 1 \Rightarrow \quad \sigma = -\zeta\omega_n \to -\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \to \phi$$

Response does not oscillate!

Overshoot:

$$M_p = e^{(\sigma\pi/\omega_d)} = e^{-\omega_n\pi/\phi} = \phi$$

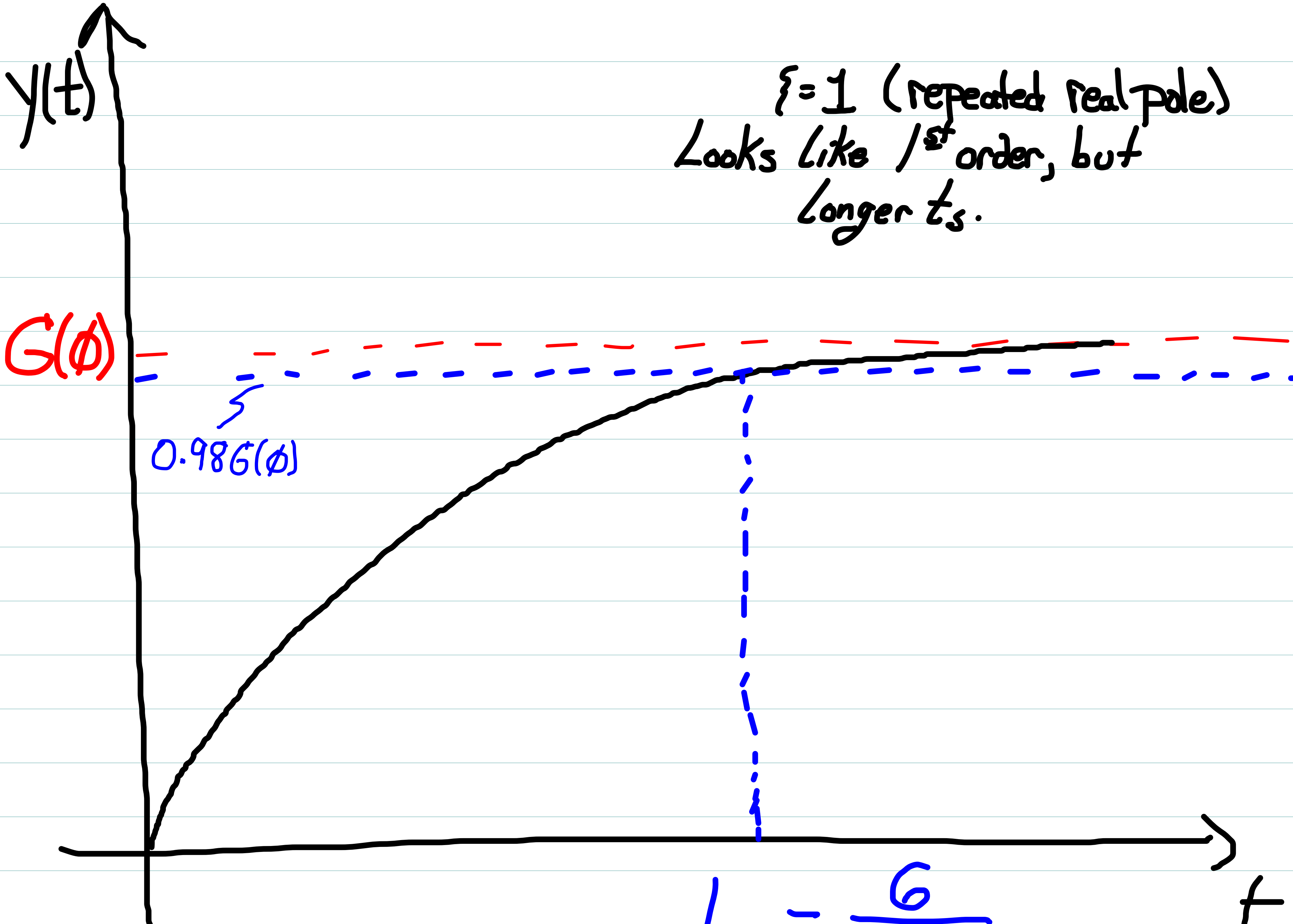
No overshoot

1st crossing: $\quad t_c = \dfrac{\pi - \cos^{-1}\zeta}{\omega_d} = \pi/2 / \phi = \infty$

$\Rightarrow$ response asymptotes to y_{ss} from below

Settling: $\quad t_s \approx \dfrac{6}{|\sigma|}$ use 6 here

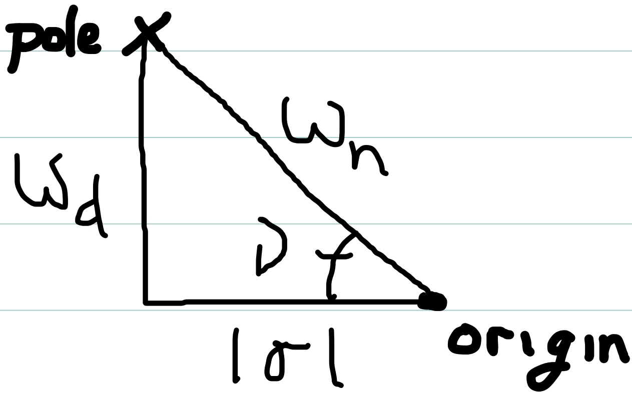
$\zeta = 1$ (repeated real pole)
Looks like 1^{st} order, but longer t_s.



$$t_s = \frac{6}{|\sigma|}$$

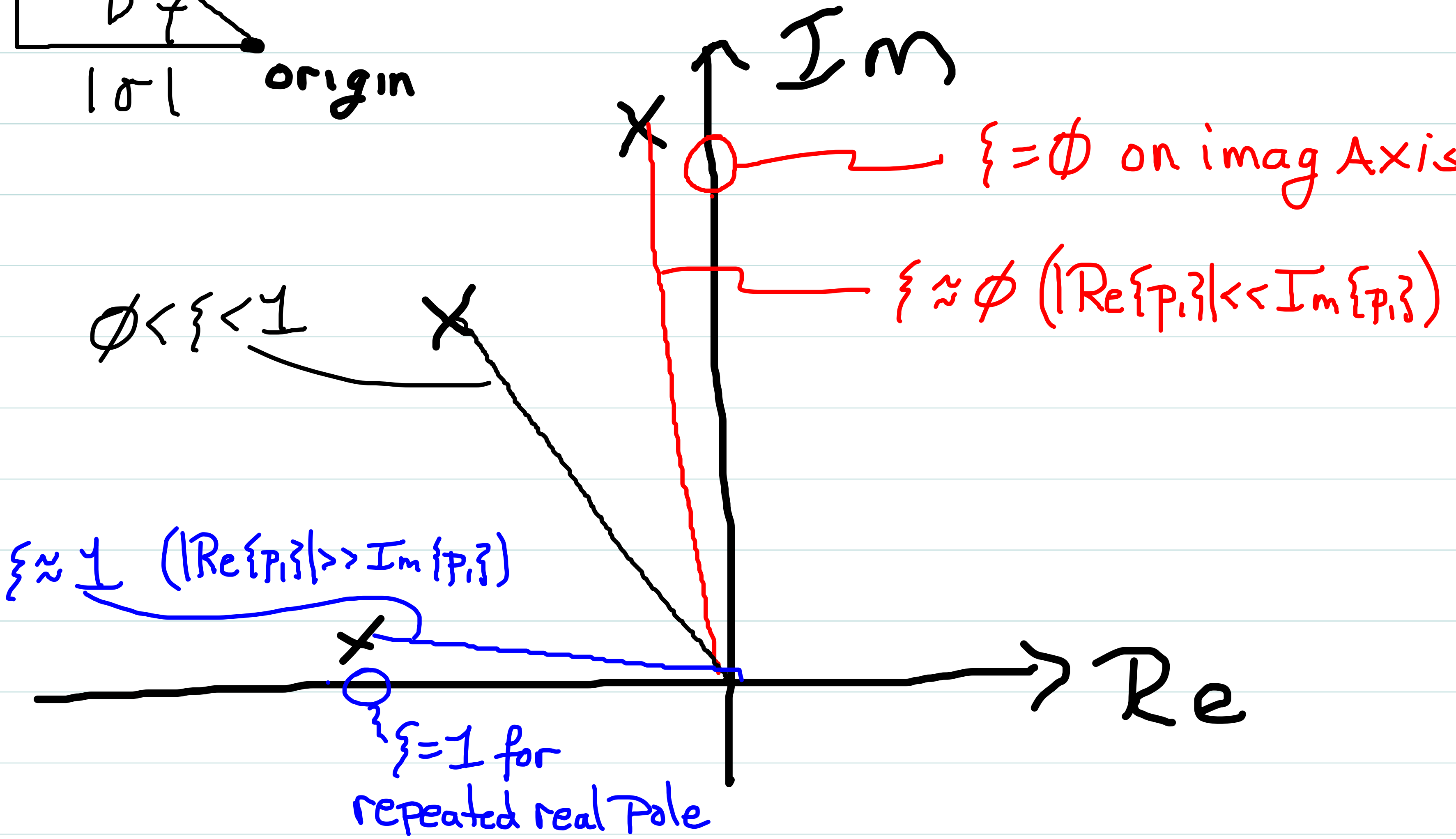
Graphical Interpretation of ζ:

$\boxed{\zeta = \cos\nu :}$

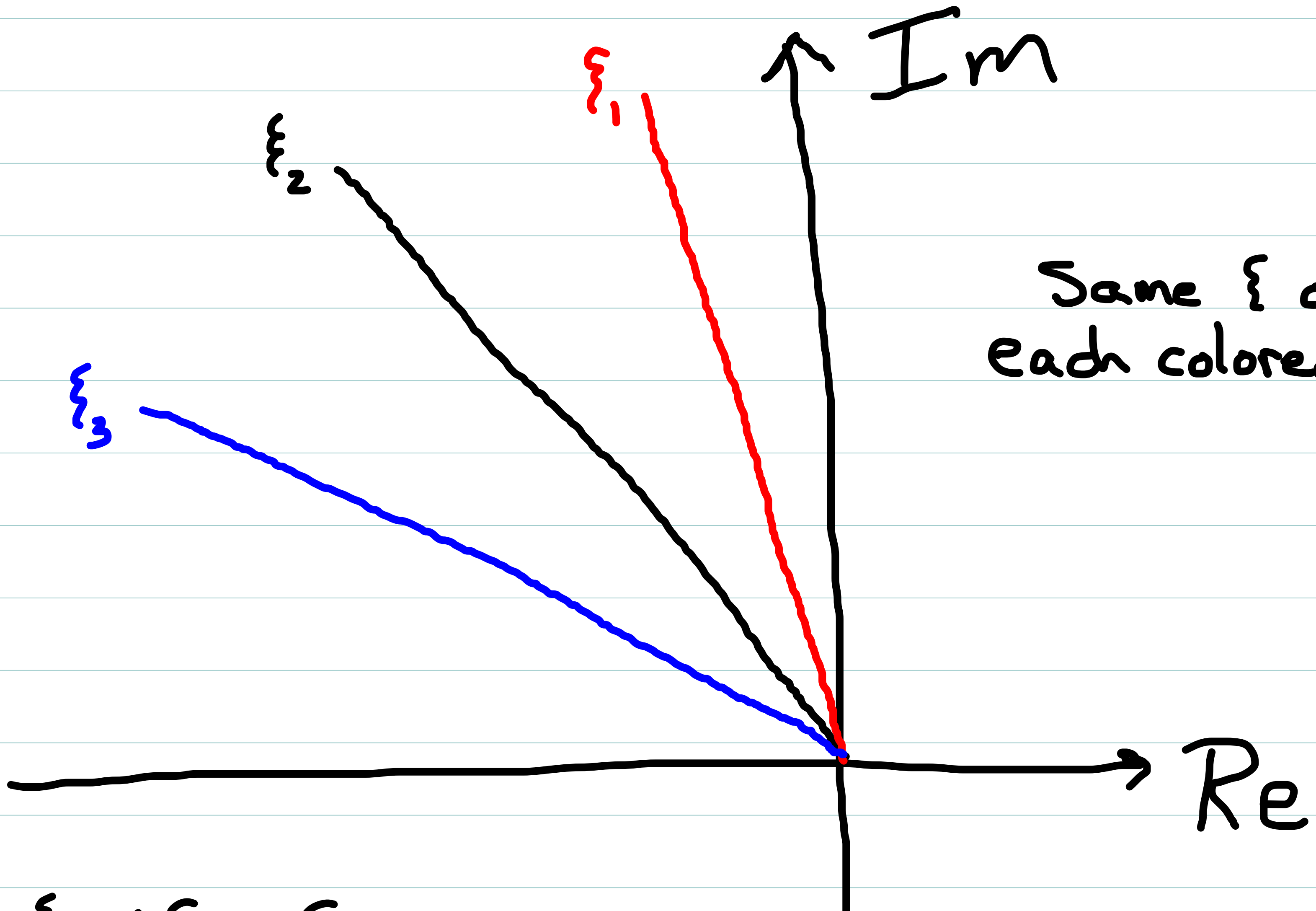


$\zeta \to 0 \Rightarrow \nu \to \pi/2$

$\zeta \to 1 \Rightarrow \nu \to 0$



Lines of constant ξ lie on rays in upper left quadrant of complex plane:



Same ξ on each colored line

$\xi_1 < \xi_2 < \xi_3$

⇒ 1st and 2nd order step responses are "building blocks" by which we can understand response of more complex systems

⇒ each real pole introduces a new decaying exponential into transient response.

⇒ each complex pole pair introduces a decaying oscillation into the transient

⇒ An arbitrary number of poles of different types will typically require numerical simulation to quantify y_p, t_r, t_p, t_s

⇒ __However__ in some cases we can still accurately predict these features.

Suppose:

$$G(s) = \frac{K}{(s-p_1)(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \text{with } \zeta<1$$

$$= \frac{K}{(s-p_1)(s-p_2)(s-\bar{p}_2)}$$

for a unit step input $u(t) = \mathbb{1}(t)$ we know

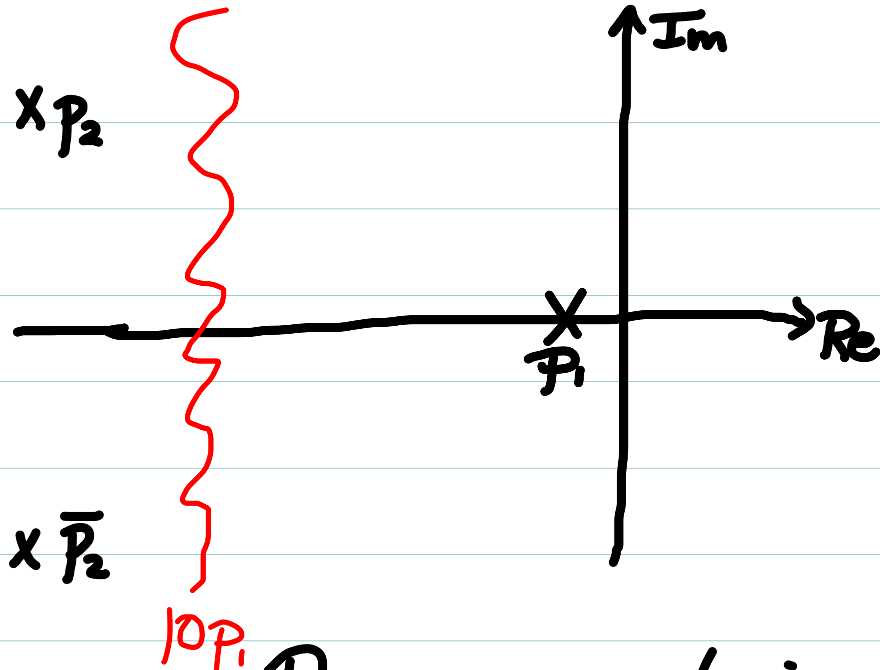
$$y_{ss} = G(0) = \frac{K}{-\omega_n^2 p_1}$$

But what can we say about y_p, t_p, t_c, t_s?

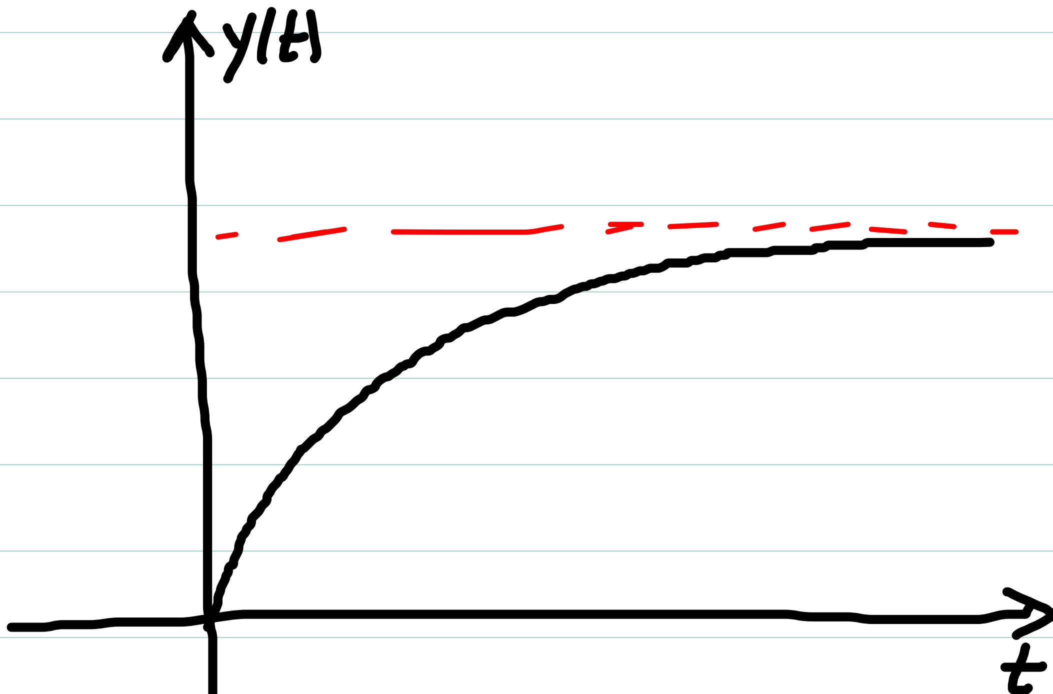
In general, not much unless either

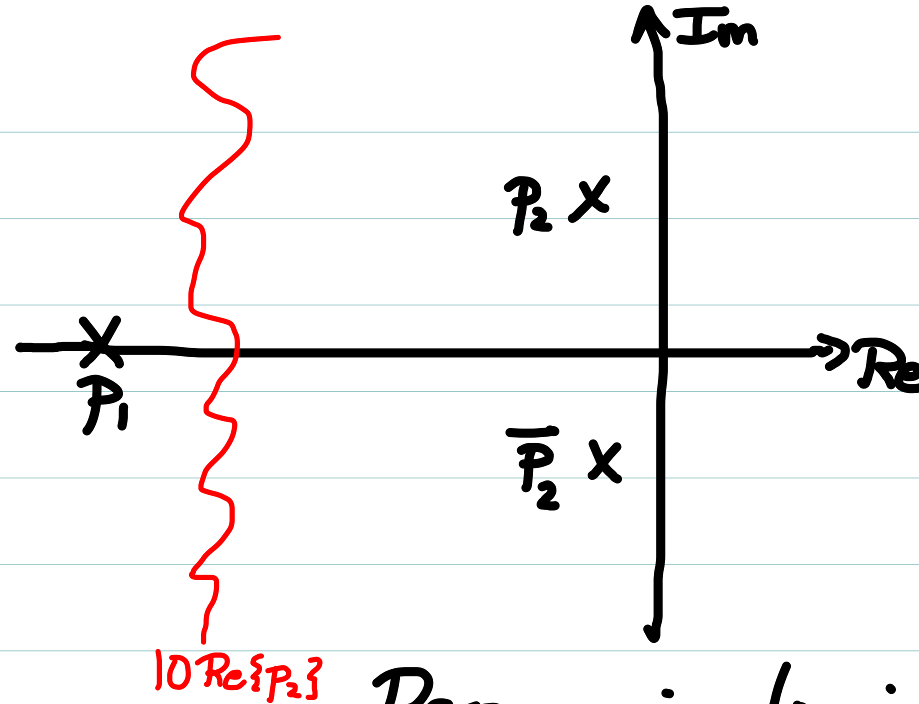
$$|p_1| > 10|Re\{p_2\}| \quad \text{or} \quad |Re\{p_2\}| > 10|p_1|$$

i.e. if one of the modes is dominant.

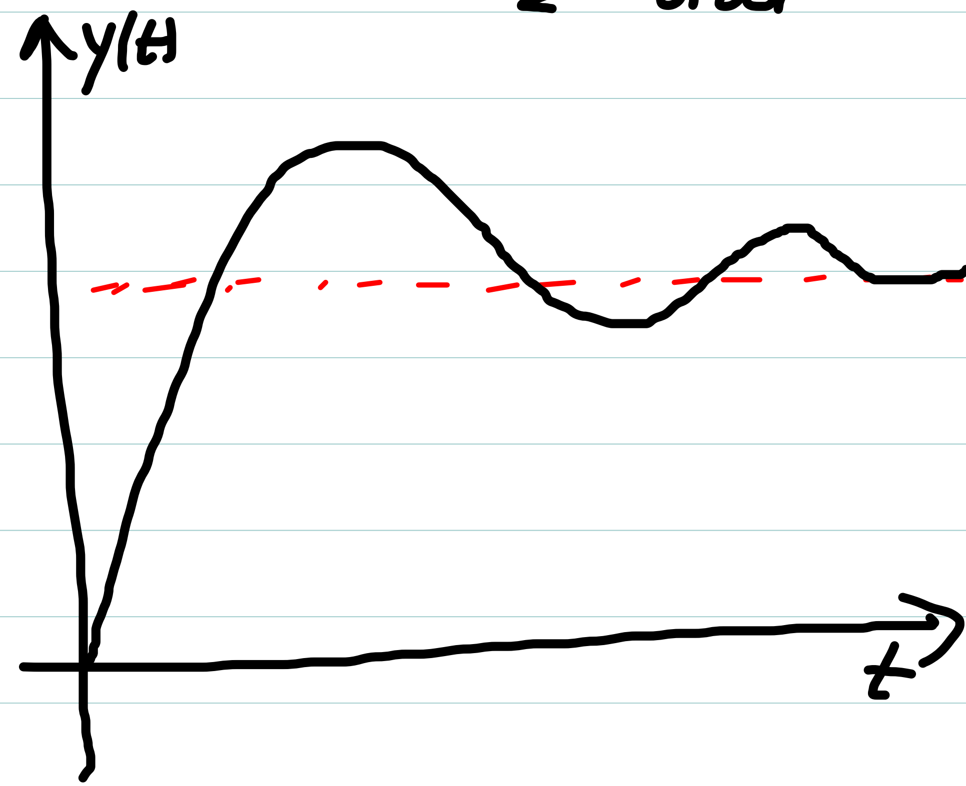


Response is dominantly 1^{st} order





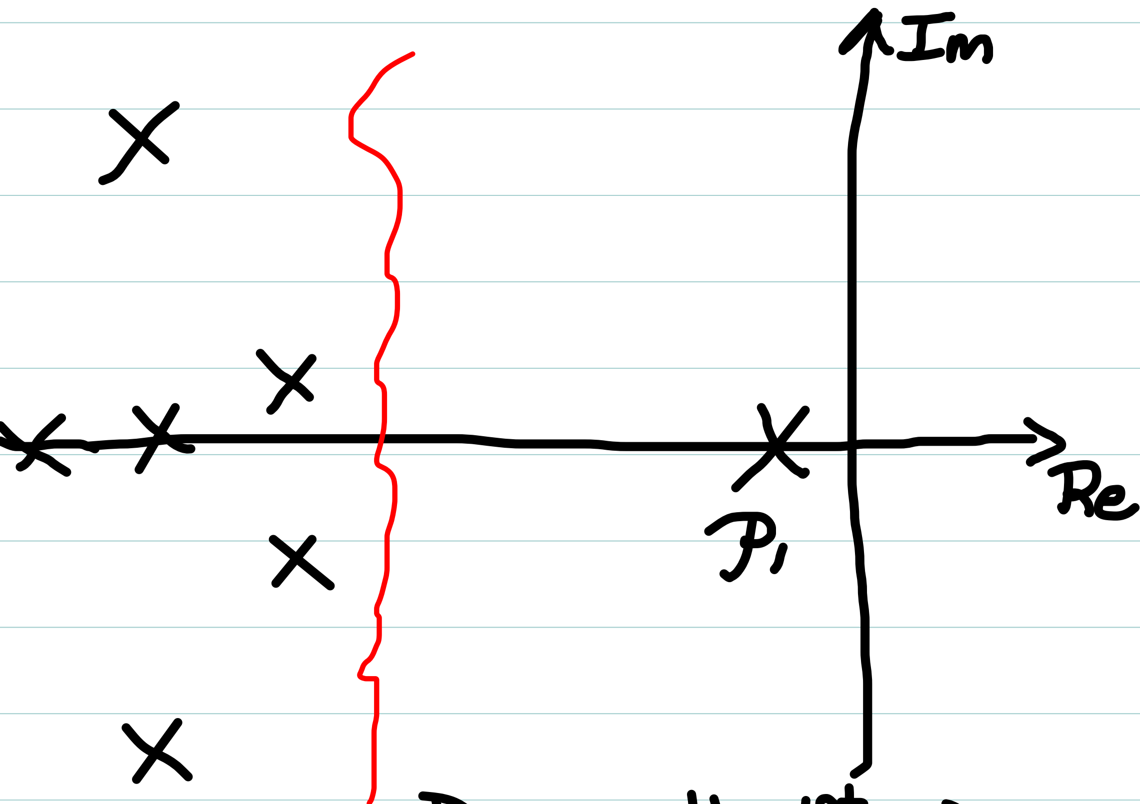
Response is dominant[ly] 2^{nd} order



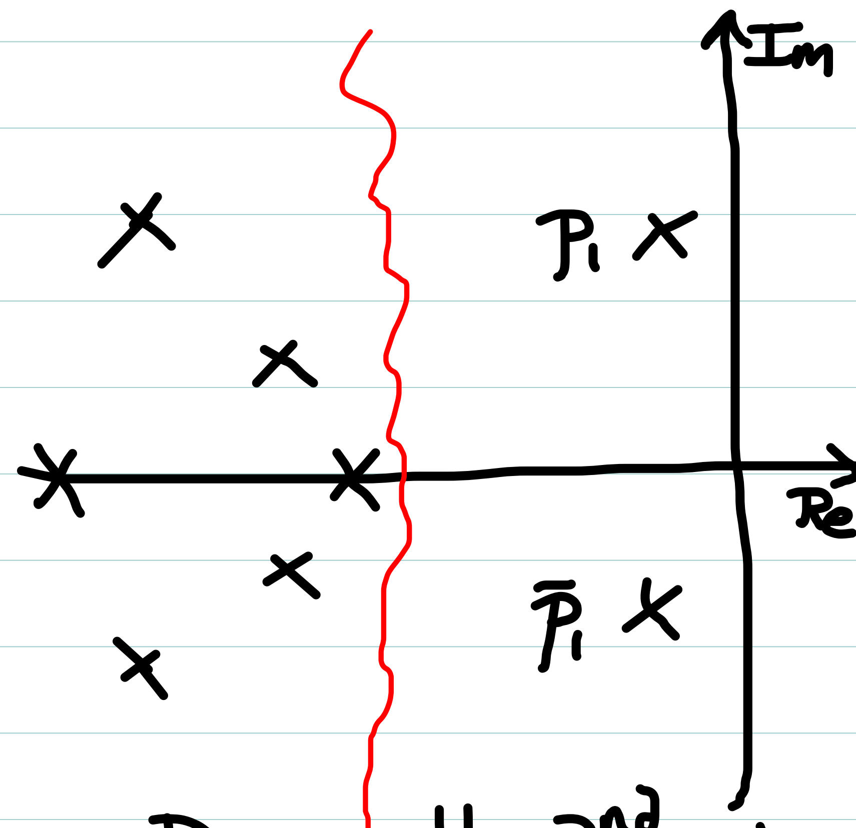
Dominant modes revisited

When a single mode is dominant, we can approximate the features of the response using just that mode

An arbitrarily complex system can be well approximated in this fashion.



Dominantly 1st order



Dominantly 2nd order

Effect of Zeros

Step response of

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Zero at $z_1 = -\beta_0/\beta_1$

3 important effects:

1. "Input absorbing" property
2. Transient suppression
3. Transient amplification

Both? Yes!

Depending on system

Note:

=> Zeros do not contribute new modes to the response.

=> Zeros do not affect stability!

=> zeros do affect the coefficient of each mode in general sol'n (change residues).

=> Hence zeros will affect the transient (and possibly steady-state) parts of a step response.

(I.) Input absorbtion

For unit step response of stable system

$$y_{ss}(t) = G(\emptyset)$$

Suppose $z_1 = -\beta_0/\beta_1 = \emptyset \Rightarrow \beta_0 = \emptyset$

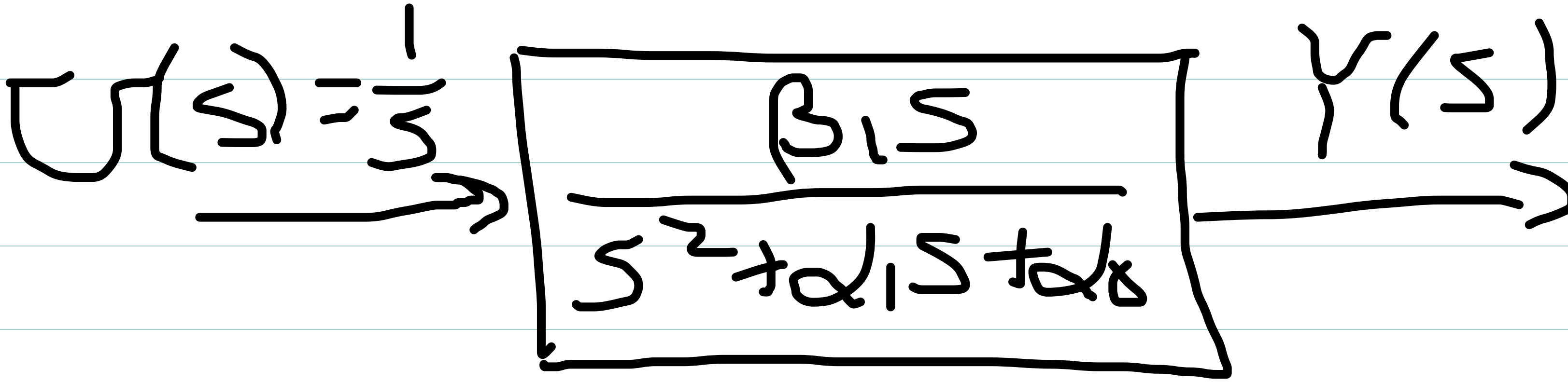
zero at origin

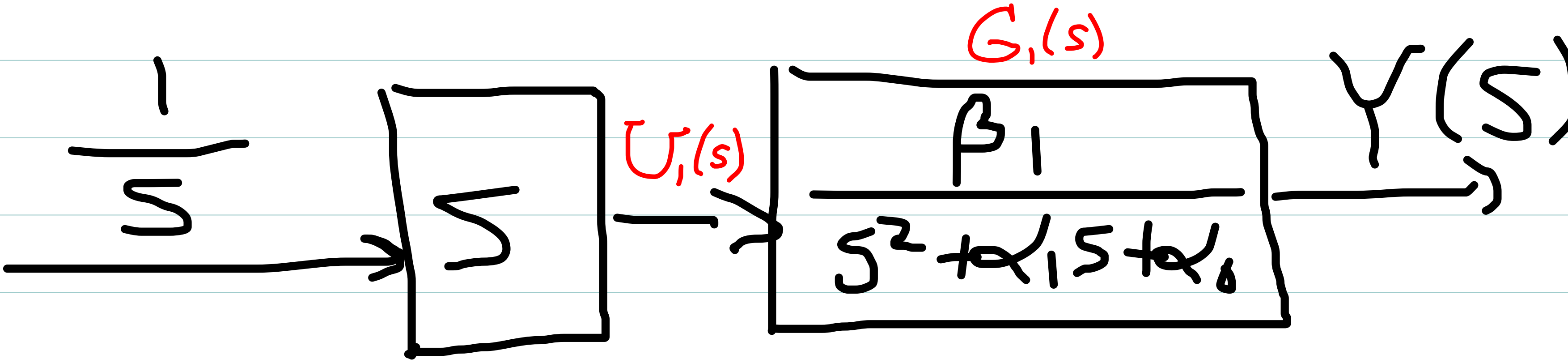
$$G(s) = \frac{\beta_1 s}{s^2 + \alpha_1 s + \alpha_0}$$

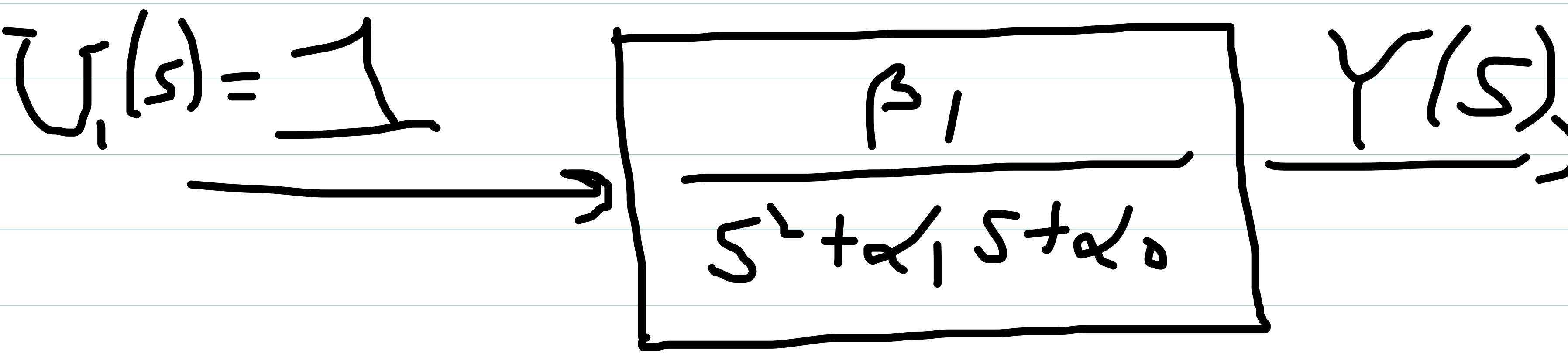
Then $y_{ss}(t) = G(\emptyset) = \emptyset \Leftarrow$ steady-state is zero response contains only transient terms

In fact, $y(t)$ is the impulse response of

$$G_1(s) = \frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$$


$U(s) = \frac{1}{s}$
$\frac{\beta_1 s}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$


$\frac{1}{s}$
s
$U_1(s)$
$G_1(s)$
$\frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$


$U_1(s) = 1$
$\frac{\beta_1}{s^2 + \alpha_1 s + \alpha_0}$
$Y(s)$