Philosophical Question: What is $t=0$?

⇒ The instant we start acting on the system with external input.

⇒ In control Theory, we assume these inputs are completely "off" for $t<0$.

⇒ $u(t), \dot{u}(t), \ddot{u}(t)$, etc all <u>zero</u> for $t<0$

⇒ <u>Discontinuities</u> exist when $u(0) \neq 0$

$$u(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) = \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{u(t) = e^{pt}\,\mathbb{1}(t)}$$

where

$$\boxed{\mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

"Unit step function" (Very important!)

Now, Laplace is concerned about behavior of functions only for $t \geq 0$.

For all intents and purposes, functions in Laplace are considered 0 for $t < 0$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\,\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \varnothing & t \neq \varnothing \\ \end{cases}$$

Implication

Formally:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}\,\mathbb{1}(t)$$

$$= \begin{cases} e^{pt}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now generally, our diff'l eq'ns will involve derivatives of these discontinuous functions

$\Rightarrow$ creates singularities in analysis at $t=0$

$$\frac{d}{dt}\mathbb{1}(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \quad (???) \end{cases}$$

Theoretical problems in integrals when discontinuities or singularities at one of the end points.

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

$\Rightarrow$ possible problem here

Resolve these by taking lower limit at $t = \phi^{-}$ (the instant before $t = \phi$).

$\Rightarrow$ integral "sees" effect of singularities at $t = \phi$.

Starting the integral at 0^- instead of 0

- Avoids singularities at endpoints
- Causes transform to "see" singularities and discontinuities at $t=0$, so their effects will be reflected in the solutions for $y(t)$.

Hence:

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

Implications:

Assumed ICs for $y(t)$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - \boxed{y(0^-)}$$

$$\mathcal{L}\{\ddot{y}(t)\} = s^2 Y(s) - \boxed{\dot{y}(0^-) - s y(0^-)}$$

etc.

$$\mathcal{L}\{\dot{u}(t)\} = sU(s) - \boxed{u(0^-)}$$

$$\mathcal{L}\{\ddot{u}(t)\} = s^2 U(s) - \boxed{\dot{u}(0^-) - s u(0^-)}$$

etc.

Always $= 0$ in our analysis!

Thus:

$$Y(s) = \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s) - \cancel{b(s)}^{\,\rightarrow \emptyset}}{r(s)}\right]$$

$$= \left[\frac{q(s)}{r(s)}\right]U(s) + \left[\frac{c(s)}{r(s)}\right]$$

$\Rightarrow$ IC polynomial $b(s)$ for input vanishes

$\Rightarrow$ specific to controls convention for $u(t)$

$\Rightarrow$ Not a common assumption in regular math classes.

$\Rightarrow$ In controls, want to know effect of discontinuities

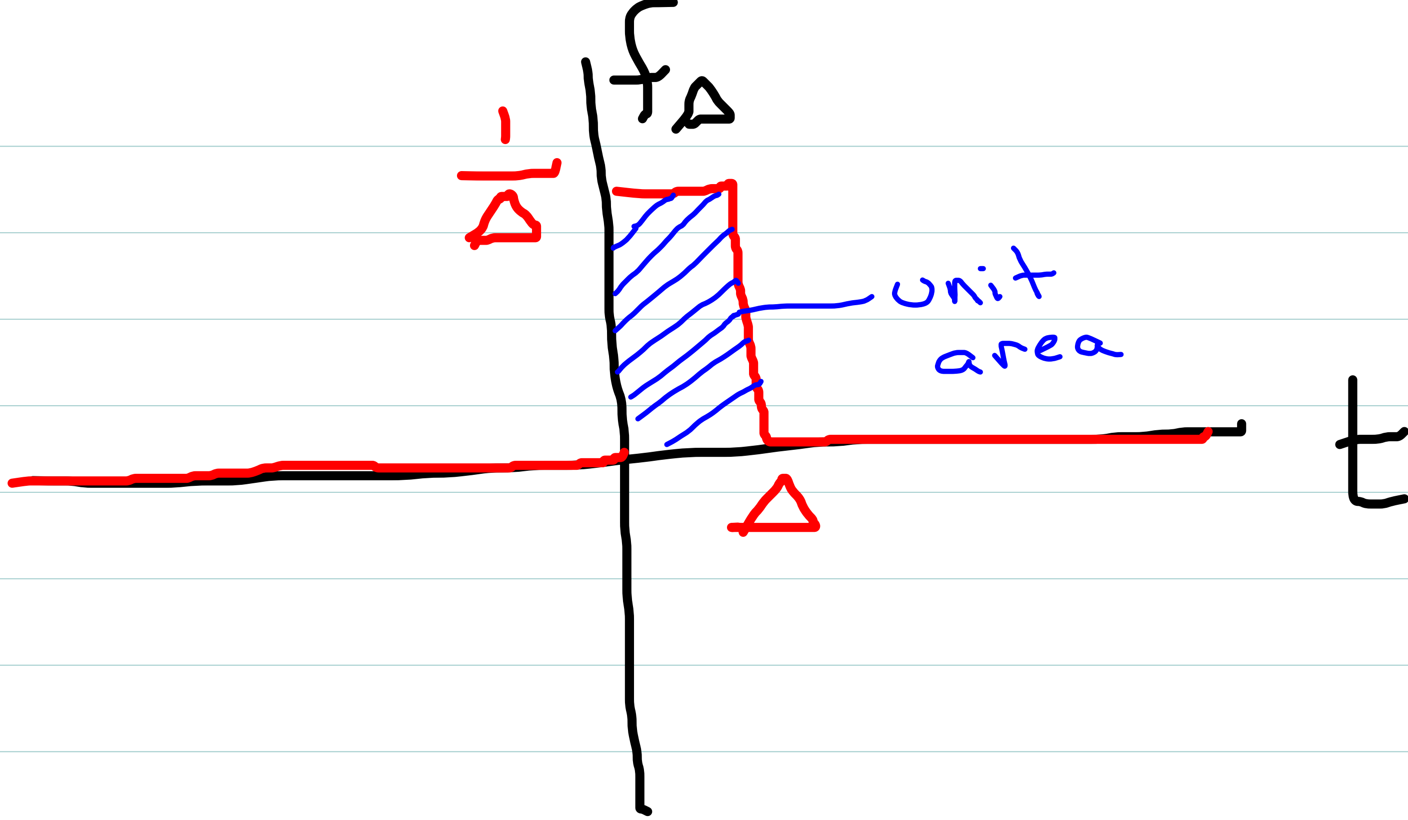
Common, discontinuous "test functions"

$u(t) = \mathbb{1}(t)$ (unit step)

$$u(t) = \cos(\omega t)\,\mathbb{1}(t)$$
$$= \begin{cases} \cos(\omega t) & t \geq \phi \\ \phi & t < \phi \end{cases}$$

$$u(t) = f_\Delta(t)$$
$$= \begin{cases} \frac{1}{\Delta} & \phi \leq t \leq \Delta \\ \phi & \text{otherwise} \end{cases}$$

"Unit pulse function"



Note: for <u>any</u> $\Delta > 0$

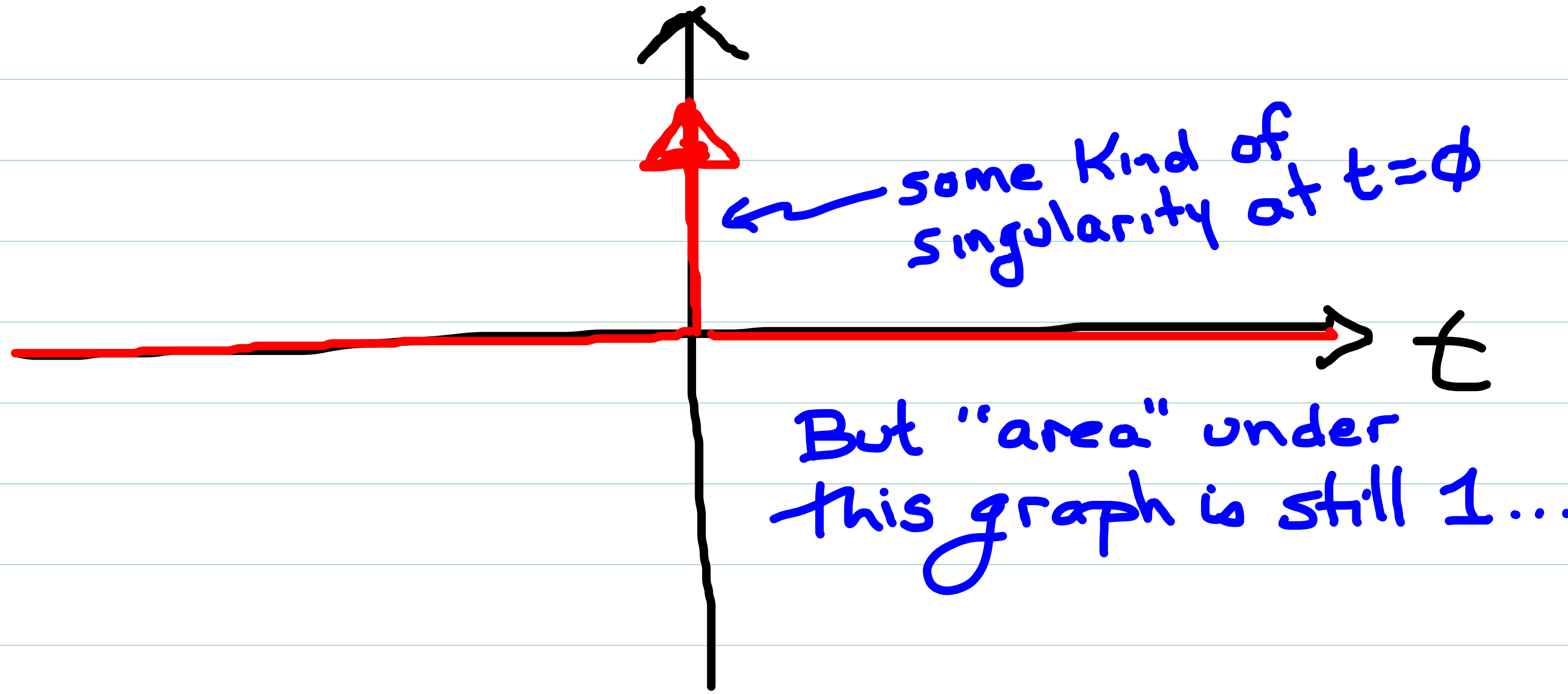
$$\int_{0^-}^{\infty} f_\Delta(t)\,dt = \int_{0^-}^{\Delta} \left(\frac{1}{\Delta}\right) dt$$

$$= 1$$

What is $\lim\limits_{\Delta \to 0} f_\Delta(t)$?

$$= \lim_{\Delta \to 0} \begin{cases} \frac{1}{\Delta} & \phi \le t \le \Delta \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \infty & t = \phi \\ \phi & \text{otherwise} \end{cases}$$



Define:

$$\boxed{\delta(t) = \lim_{\Delta \to 0} f_\Delta(t)}$$

"ideal impulse": models delivering a unit of input energy over negligibly small time.

(sharp "kick")

Alternate names:

"delta function"
"impulse function"
"Dirac delta"

Note: Not really a meaningful function at all!

More formally, belongs to a class of mathematical objects called "distributions" or "generalized functions"

Suppose $\delta(t)$ appears in an integral

$$\int_{-\infty}^{\infty} \delta(t)\, h(t)\, dt \quad , \quad h(t) \text{ arbitrary f'n}$$

$$= \int_{-\infty}^{\infty} \left[\lim_{\Delta \to 0} f_\Delta(t) \right] h(t)\, dt$$

$$= \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f_\Delta(t)\, h(t)\, dt$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \int_{0^-}^{\Delta} h(t)\, dt \right\}$$

$$= \left(\frac{1}{\Delta}\right) \left(\Delta\, h(\phi)\right)$$

$$= h(\phi)$$

Note: with $h(t) = 1$ for all t, we get

$$\int_{0^-}^{\infty} \delta(t)\, dt = 1$$

Defining Property of $\delta(t)$

$$\int_a^b \delta(t)h(t)\,dt = \begin{cases} h(0) & \text{if } 0 \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

"sifting property"

$\delta(t)$ is defined by what it does in an integral

Not as an ordinary function

Now we can compute:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt$$

$$= \left[e^{-st}\right]_{t=0} = 1$$

Thus:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\phi \\ \phi & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\left\{\frac{d}{dt}\mathbb{1}(t)\right\} = s\,\mathcal{L}\{\mathbb{1}(t)\} - \mathbb{1}(\phi^-)$$

Thus:

$$\mathcal{L}\{\delta(t)\} = 1$$

and by Linearity:

$$\mathcal{L}\{c\,\delta(t)\} = c \quad \text{for any constant } c.$$

Now recall $\frac{d}{dt}\mathbb{1}(t) = \begin{cases} \infty & t=\emptyset \\ \emptyset & \text{otherwise} \end{cases}$

which looks like $\frac{d}{dt}\mathbb{1}(t) = \delta(t)$.

Is this formally true?

$$\mathcal{L}\{\tfrac{d}{dt}\mathbb{1}(t)\} = s\,\underbrace{\mathcal{L}\{\mathbb{1}(t)\}}_{1/s} - \underbrace{\cancel{\mathbb{1}(\emptyset^-)}}_{\emptyset}$$

$$= 1 = \mathcal{L}\{\delta(t)\}$$

YES

Recap: Unit Impulse

$$\delta(t) = \lim_{\Delta \to 0} f_\Delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

"sifting property":

$$\int_a^b \delta(t) h(t)\, dt = \begin{cases} h(0) & \text{if } 0 \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Laplace Transform:

$$\mathcal{L}\{\delta(t)\} = 1$$

Useful property:

$$\frac{d}{dt} \mathbb{1}(t) = \delta(t)$$

Impulse Response

The <u>impulse response</u> of a system is the output $y(t)$ when $u(t) = \delta(t)$ and all ICs on $y(t)$ are zero.

$$Y(s) = G(s)U(s) + \frac{[c(s) - b(s)]}{r(s)}$$

$\Rightarrow u(t) = \delta(t) \Rightarrow b(s) = \emptyset$ and $U(s) = 1$

$\Rightarrow$ all ICs on $y(t)$ zero $\Rightarrow c(s) = \emptyset$

So:

$$Y(s) = G(s)$$

and thus

$$y(t) = \mathcal{L}^{-1}\{G(s)\} \triangleq g(t)$$

The <u>impulse response</u> $g(t)$ is the inverse transform of the transfer function $G(s)$

Conversely, knowledge (or <u>measurement</u>) of $g(t)$ tells us what the transfer function is, and hence the governing diff'l eq'ns.

$\Rightarrow$ Foundation of "system identification" theory.