1 Chabauty space of $\mathbf{PSL}_2(\mathbb{R})$

Let $G = \mathrm{PSL}_2(\mathbb{R})$. The group G acts on $\mathbb{H}^2$ by fractional linear transformation on upper half plane. An elemenatary subgroup of G is a subgroup of G that has a finite orbit in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Let the set A be a finite orbit of an elementary subgroup $H \leq G$. First we can think about where the set A is and what element H can have. If H contains an element that is parabolic or hyperbolic, then A cannot contain any element in $\mathbb{H}^2$. Otherwise the parabolic or hyperbolic isometry will create an infinite orbit. So in the case where A is contained in $\mathbb{H}^2$, we know that H can only have elliptic elements. The following fact is from [Katok], where it is proved algebraically.

Fact 1. *If H contains only elliptic elements, then it has a fixed point in $\mathbb{H}^2$.*

The fact above tells us that if elementary subgroup H only has elliptical elements, then every element fixes the same point. In other words, all the elements are some rotation around that fixed point.

We want to argue that if $h \in H$ is parabolic or hyperbolic, then h also creates an infinite orbit of any $\xi \in \partial\mathbb{H}^2$ that is not a fixed point of h. For example, in upper half plane model, if h is the map $z \mapsto \lambda z$, it is a hyperbolic isometry leaving the imaginary axis invariant. Any other hyperbolic isometry is a conjugation of h. The map h has two fixed points 0 and ∞, for any nonzero ξ on the real line, the orbit is $\{\lambda^n \xi : n \in \mathbb{Z}\}$. Similarly if h is the map $z \mapsto z + 1$, it is a parabolic isometry fixing ∞, and any $\xi \in \mathbb{R}$ has infinite orbit $\{\xi + n : n \in \mathbb{Z}\}$ under $\langle h \rangle$. Any other parabolic map is a conjugation to h.

Knowing how different types of isometry determines what elements A can have, we can begin to classify all the closed elementary subgroup $H \leq G$. The following result is from [BLL21].

Theorem 1.1. *The closed, elementary subgroups of G are as follows.*

1. *The trivial group.*
2. *The group $K(p)$ of all elliptic isometries fixing some $p \in \mathbb{H}^2$, and the finite cyclic subgroup $k(p, 2\pi/n)$ generated by a $2\pi/n$-rotation around $p \in \mathbb{H}^2$.*
3. *The group $N(\xi)$ of all parabolic isometries fixing some $\xi \in \partial\mathbb{H}^2$, and its infinite cyclic subgroup.*
4. *The group $A(\alpha)$ of all hyperbolic type isometries that translate along geodesic $\alpha \subseteq \mathbb{H}^2$, and its infinite cyclic subgroup $a(\alpha, t)$ consisting of translations by multiple of $t \in \mathbb{R}$.*
5. *The group $A'(\alpha) \cong A(\alpha) \rtimes \mathbb{Z}/2\mathbb{Z}$ and its infinite dihedral subgroup $a'(\alpha, t, p) \cong a(\alpha, t) \rtimes \mathbb{Z}/2\mathbb{Z}$. Here the subgroup $\mathbb{Z}/2\mathbb{Z}$ is generated by a π-rotation around $p \in \alpha$. In the latter case, p is well-defined up to translations along α by multiple of t.*

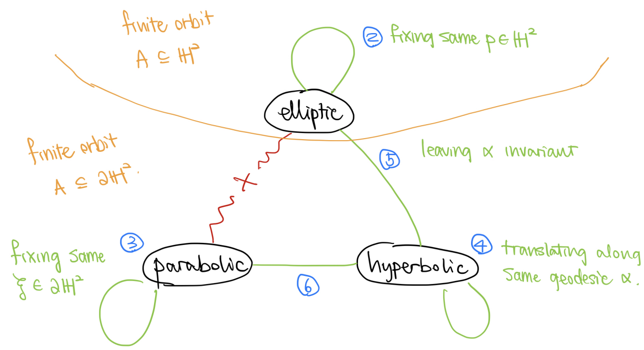

Figure 1: Compatibility chart between isometries

6. *The group $B(\xi) \cong N(\xi) \rtimes A(\alpha)$ of isometries fixing $\xi \in \partial\mathbb{H}^2$, and its subgroup $b(\xi, t) \cong N(\xi) \rtimes a(\alpha, t)$. Here α is a geodesic with ξ as one end.*

Proof. We will give a proof that looks different from the one in [BLL21]. We consider what kind of isometries plays well with each other and can coexist in the same subgroup H while still having finite orbit. Figure 1 summarizes the result.

If H only contains elliptic isometries, then by Fact 1 all the elements are rotations around the same point $p \in \mathbb{H}^2$. If H contains two parabolic isometries, H is elementary if and only if these two element fix the same point ξ on the boundary. Otherwise one will generate an infinite orbit of the fix point of the other. Similarly, if H contains two hyperbolic isometries, H is elementary if and only if these two elements are both translation along the same geodesic α. This paragraph covers cases $2 - 4$.

A parabolic isometry and an elliptic isometry cannot coexist in the same elementary subgroup. As we argued before, the finite orbit can only contain the fixed point ξ of the parabolic isometry, but the elliptic element will map ξ to some ξ' that is not fixed by the parabolic isometry, and therefore makes it impossible to find a finite orbit for H.

If H contains both an elliptic isometry and an hyperbolic isometry, H is elementary if and only if the elliptic permutes the two fixed points of the hyperbolic. In other word, the elliptic is a π-rotation around some $p \in \alpha$, where α is the geodesic the hyperbolic isometry translates along. This is case 5. If H contains both a hyperbolic isometry and a parabolic isometry, the hyperbolic isometry must fix the fixed point of the parabolic. This is case 6. $\square$

Commonly asked question (by me):

- the two semidirect products

- why no notation for cyclic subgroup of $N(\xi)$.
- why $B(\xi) \cong N(\xi) \rtimes A(\alpha)$ can be defined using just one α.

To do: maybe reorganize the structure of these two pages. but for now i need to move on to new stuff.

figure out what kind of hyperbolic isometries can converge to a parabolic.

Jorgenson first discovered the following interesting phenomenon.

Fact 2. *there exists a sequence of infinite cyclic groups generated by hyperbolic isometries that converges to a subgroup isomorphic to* $\mathbb{Z}^2$*, whose generators are both parabolic isometries.*

We discussed in the previous sections (if I actually write it down) about the convergence of a sequence of elliptic or hyperbolic cyclic subgroups to a parabolic infinite cyclic subgroup. The key is to control the ratio between how fast the fixed points converge to the desired fixed point and the how fast the rotation or translation slows down. The strategy for this case is similar.

Recall that any hyperbolic isometries of $\mathbb{H}^n$ is conjugate to a hyperbolic isometry fixing 0 and ∞ (my 2nd favorite one). This isometry boils down to two components, the translation along the axis and the rotation (an element in $SO(n-1)$) of plane $x_n = c$. For $\mathbb{H}^2$, $SO(1)$ is trivial. For our case in $\mathbb{H}^3$, $SO(2)$ is all the rotations around one point. This new rotation component is the reason why this phenomenon starts to appear in $\mathbb{H}^3$.

Let $\xi = ae^{i\theta/2}$, the matrix $\begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}$ is a hyperbolic isometry that translates by $2\ln(a)$ and rotates by θ. Conjugating it with the matrix $\begin{bmatrix} 1 & -n \\ 1 & n \end{bmatrix}$, we get a hyperbolic isometry $H_n = \frac{1}{2}\begin{bmatrix} \xi + \xi^{-1} & n(-\xi + \xi^{-1}) \\ \frac{1}{n}(-\xi + \xi^{-1}) & \xi + \xi^{-1} \end{bmatrix}$ that goes along arc n to $-n$. Taking $n \to \infty$, we want the matrix to converge to $z \mapsto z + bi$. Set $a = \sqrt[n]{\frac{n+1}{n}}$ and $\theta = 2\pi/n$

I hate writing matrices in latex so i will write it in goodnotes