Why can't I make the title look nice

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Abstract

Future me will put some abstract here

1 Introduction

2 Chab Chab

Let G be a Lie group, and A_n and A be its closed subgroups. We want to find a notion of convergence for closed subgroups. Chabauty topology gives such notion on the set $\mathrm{Sub}(G)$ of all closed subgroups of G.

Definition 2.1. A sequence of closed subgroups (A_n) converge to A in Chabauty topology if

1. Any $x \in A$ is the limit of some sequence (x_n) where $x_n \in A_n$
2. Any accumulation point of a sequence (x_n) where $x_n \in A_n$ lies in A.

For this section, we let $G = \mathrm{PSL}_2(\mathbb{R})$. The group G is the group of isometries acting on the upper half plane model of $\mathbb{H}^2$ by fractional linear transformation. Using Chabauty topology, we are able to describe the shape of the collection of all closed subgroups of G, denoted by $S(G)$. To do so, We want to break $S(G)$ into pieces that we know the shape of, and then try to glue it back together by looking for convergence.

A sequence of cyclic subgroup generated by a hyperbolic isometry can converge to a cyclic subgroup generated by a parabolic isometry, as shown in the following example.

Example 2.2. Let $\xi_n = 1 + 1/n$, Consider

$$H_n = \frac{1}{2}\begin{bmatrix} \xi_n + \xi_n^{-1} & n(-\xi_n + \xi_n^{-1}) \\ \frac{1}{n}(-\xi_n + \xi_n^{-1}) & \xi_n + \xi_n^{-1} \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

The sequence of subgroups $(\langle H_n \rangle)$ converges to the subgroup $\langle P \rangle$.

Here, H_n is a hyperbolic isometry that fixes $n, -n \in \partial\mathbb{H}^2$, leaves geodesic connecting $n, -n$ invariant, and travels along the geodesic by $2\ln(\xi_n)$; P is a parabolic isometry that fixes $\infty \in \partial\mathbb{H}^2$, and maps $z \mapsto z - 1$. Figure 1 gives an idea of the fundamental domains for each of these groups.

Proof. It is not hard to show $H_n \to P$ as n goes to ∞. However, as a noob in Chabauty topology, one might (for example I did) think we are done with this proof by showing that the sequence of generators converges to the generator. To complete the proof, it suffices to show the following fact:

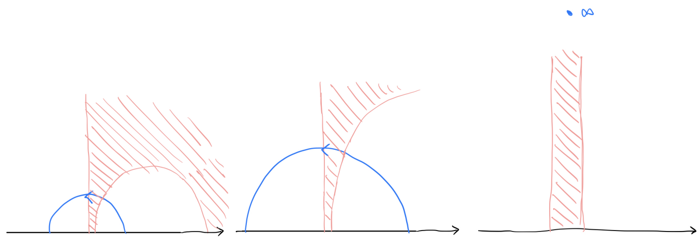

Figure 1: Fundamental domains of H_n's and P.

Fact 2.3. *$(H_n^{a_n})$ converges if and only if (a_n) converges. Moreover, if $a_n \to a$, then $H_n^{a_n} \to P^a$. Here a_n are integers*

It might be helpful to mention that

$$H_n^{a_n} = \frac{1}{2}\begin{bmatrix} \xi_n^{a_n} + \xi_n^{-a_n} & n(-\xi_n^{a_n} + \xi_n^{-a_n}) \\ \frac{1}{n}(-\xi_n^{a_n} + \xi_n^{-a_n}) & \xi_n^{a_n} + \xi_n^{-a_n} \end{bmatrix}$$

For top left and bottom right coefficients,

$$\xi_n^{a_n} + \xi_n^{-a_n} = \frac{(n+1)^{a_n}}{n^{a_n}} + \frac{n^{a_n}}{(n+1)^{a_n}} = \frac{(n+1)^{2a_n} + n^{2a_n}}{(n+1)^{a_n} n^{a_n}} \to 2$$

because the leading terms of numerator and denominators have the same power $2a_n$.

For the bottom left coefficient,

$$\begin{aligned} \frac{1}{n}(-\xi_n^{a_n} + \xi_n^{-a_n}) &= \frac{1}{n}\left(-\frac{(n+1)^{a_n}}{n^{a_n}} + \frac{n^{a_n}}{(n+1)^{a_n}}\right) \\ &= \frac{-(n+1)^{2a_n} + n^{2a_n}}{(n+1)^{a_n} n^{a_n+1}} \to 0 \end{aligned}$$

because the numerator is an at most $(2a_n - 1)$ degree polynomial while the denominator is degree $(2a_n + 1)$. These three coefficients converge for large n no matter what a_n is.

Now here is where "if and only if" comes from. For the top right coefficient,

$$\begin{aligned} n(-\xi_n^{a_n} + \xi_n^{-a_n}) &= n\left(-\frac{(n+1)^{a_n}}{n^{a_n}} + \frac{n^{a_n}}{(n+1)^{a_n}}\right) \\ &= \frac{n(-(n+1)^{2a_n} + n^{2a_n})}{(n+1)^{a_n} n^{a_n}} \end{aligned}$$

By binomial expansion, the leading term of the numerator is $2a_n n^{2a_n}$ and the leading term of the denominator is n^{2a_n}. So this term converges if and only if a_n converges. $\square$

Everyone pretend I wrote a really smooth transitional paragraph to the next example here `:-)`. BTW as I wrote this paragraph I finally figured out how to inverse search with my LaTeX setup. W.

Example 2.4. Consider

$$K_n = \begin{bmatrix} \cos\frac{\pi}{n} & -n\sin\frac{\pi}{n} \\ \frac{1}{n}\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -\pi \\ 0 & 1 \end{bmatrix}$$

The sequence of subgroups $(\langle K_n \rangle)$ converges to the subgroup $\langle P \rangle$.

Here, K_n is rotation $2\pi/n$ around ni; P is a parabolic isometry that fixes $\infty \in \partial\mathbb{H}^2$, and maps $z \mapsto z - \pi$.

3 The plot thickens

Jorgenson first discovered the following interesting phenomenon.

Fact 3.1. *There exists a sequence of infinite cyclic groups $\langle H_n \rangle$, where H_n is an isometry of hyperbolic type, such that the sequence converges to a subgroup isomorphic to $\mathbb{Z}^2$, whose generators are both parabolic isometries.*

In this section, we will construct one such sequence explicitly. We will use the upper half-plane model $\mathbb{C} \times \mathbb{R}^+$ for $\mathbb{H}^3$. The isometry group $\mathcal{I}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ acts on the boundary $\overline{\mathbb{C}}$ by fractional linear transformation.

4 Quotient things up

5 Convergence of the quotient

6 Closet of necessary backgrounds

References

[PH] Author names. Article Titles. *Journal Title*, ??? some mysterious numbers.