1 Chabauty space of $\mathrm{PSL}_2(\mathbb{R})$

Let $G = \mathrm{PSL}_2(\mathbb{R})$. The group G acts on $\mathbb{H}^2$ by fractional linear transformation on upper half plane. An elemenatary subgroup of G is a subgroup of G that has a finite orbit in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Let the set A be a finite orbit of an elementary subgroup $H \leq G$. First we can think about where the set A is and what element H can have. If H contains an element that is parabolic or hyperbolic, then A cannot contain any element in $\mathbb{H}^2$. Otherwise the parabolic or hyperbolic isometry will create an infinite orbit. So in the case where A is contained in $\mathbb{H}^2$, we know that H can only have elliptic elements. The following fact is from [Katok], where it is proved algebraically.

Fact 1. *If H contains only elliptic elements, then it has a fixed point in $\mathbb{H}^2$.*

The fact above tells us that if elementary subgroup H only has elliptical elements, then every element fixes the same point. In other words, all the elements are some rotation around that fixed point.

We want to argue that if $h \in H$ is parabolic or hyperbolic, then h also creates an infinite orbit of any $\xi \in \partial\mathbb{H}^2$ that is not a fixed point of h. For example, in upper half plane model, if h is the map $z \mapsto \lambda z$, it is a hyperbolic isometry leaving the imaginary axis invariant. Any other hyperbolic isometry is a conjugation of h. The map h has two fixed points 0 and ∞, for any nonzero ξ on the real line, the orbit is $\{\lambda^n \xi : \ n \in \mathbb{Z}\}$. Similarly if h is the map $z \mapsto z+1$, it is a parabolic isometry fixing ∞, and any $\xi \in \mathbb{R}$ has infinite orbit $\{\xi + n : \ n \in \mathbb{Z}\}$ under $\langle h \rangle$. Any other parabolic map is a conjugation to h.

Knowing how different types of isometry determines what elements A can have, we can begin to classify all the closed elementary subgroup $H \leq G$. The following result is from [BLL21].

Theorem 1.1. *The closed, elementary subgroups of G are as follows.*

1. *The trivial group.*

2. *The group $K(p)$ of all elliptic isometries fixing some $p \in \mathbb{H}^2$, and the finite cyclic subgroup $k(p, 2\pi/n)$ generated by a $2\pi/n$-rotation around $p \in \mathbb{H}^2$.*

3. *The group $N(\xi)$ of all parabolic isometries fixing some $\xi \in \partial\mathbb{H}^2$, and its infinite cyclic subgroup.*

4. *The group $A(\alpha)$ of all hyperbolic type isometries that translate along geodesic $\alpha \subseteq \mathbb{H}^2$, and its infinite cyclic subgroup $a(\alpha, t)$ consisting of translations by multiple of $t \in \mathbb{R}$.*

5. *The group $A'(\alpha) \cong A(\alpha) \rtimes \mathbb{Z}/2\mathbb{Z}$ and its infinite dihedral subgroup $a'(\alpha, t, p) \cong a(\alpha, t) \rtimes \mathbb{Z}/2\mathbb{Z}$. Here the subgroup $\mathbb{Z}/2\mathbb{Z}$ is generated by a π-rotation around $p \in \alpha$. In the latter case, p is well-defined up to translations along α by multiple of t.*
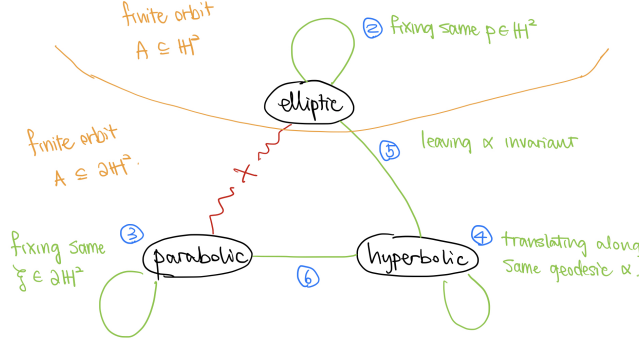
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Figure 1: Compatibility chart between isometries

6. *The group $B(\xi) \cong N(\xi) \rtimes A(\alpha)$ of isometries fixing $\xi \in \partial\mathbb{H}^2$, and its subgroup $b(\xi, t) \cong N(\xi) \rtimes a(\alpha, t)$. Here α is a geodesic with ξ as one end.*

Proof. We will give a proof that looks different from the one in [BLL21]. We consider what kind of isometries plays well with each other and can coexist in the same subgroup H while still having finite orbit. Figure 1 summarizes the result.

If H only contains elliptic isometries, then by Fact 1 all the elements are rotations around the same point $p \in \mathbb{H}^2$. If H contains two parabolic isometries, H is elementary if and only if these two element fix the same point ξ on the boundary. Otherwise one will generate an infinite orbit of the fix point of the other. Similarly, if H contains two hyperbolic isometries, H is elementary if and only if these two elements are both translation along the same geodesic α. This paragraph covers cases $2 - 4$.

A parabolic isometry and an elliptic isometry cannot coexist in the same elementary subgroup. As we argued before, the finite orbit can only contain the fixed point ξ of the parabolic isometry, but the elliptic element will map ξ to some ξ' that is not fixed by the parabolic isometry, and therefore makes it impossible to find a finite orbit for H.

If H contains both an elliptic isometry and an hyperbolic isometry, H is elementary if and only if the elliptic permutes the two fixed points of the hyperbolic. In other word, the elliptic is a π-rotation around some $p \in \alpha$, where α is the geodesic the hyperbolic isometry translates along. This is case 5. If H contains both a hyperbolic isometry and a parabolic isometry, the hyperbolic isometry must fix the fixed point of the parabolic. This is case 6. $\square$

Commonly asked question (by me):

- the two semidirect products

- why no notation for cyclic subgroup of $N(\xi)$.

- why $B(\xi) \cong N(\xi) \rtimes A(\alpha)$ can be defined using just one α.

To do: maybe reorganize the structure of these two pages. but for now i need to move on to new stuff.

I want to figure out what kind of hyperbolic isometries can converge to a parabolic.