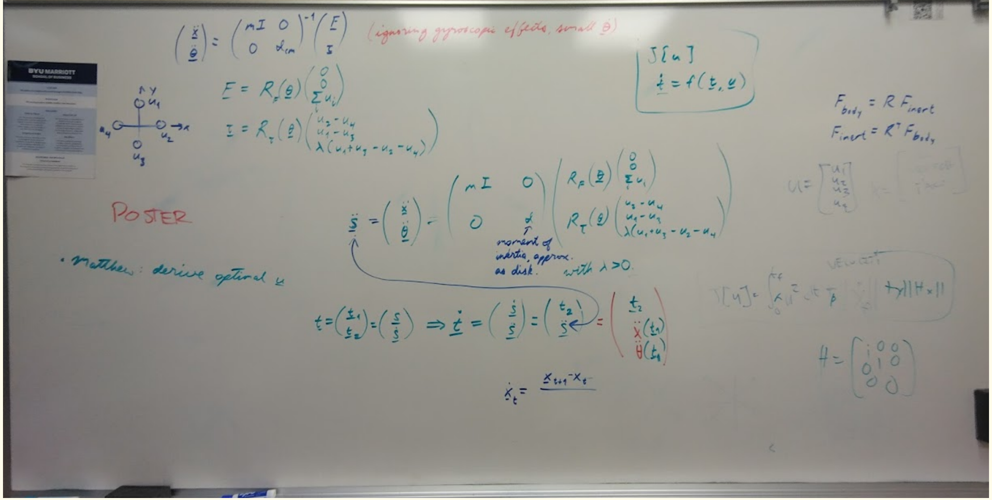
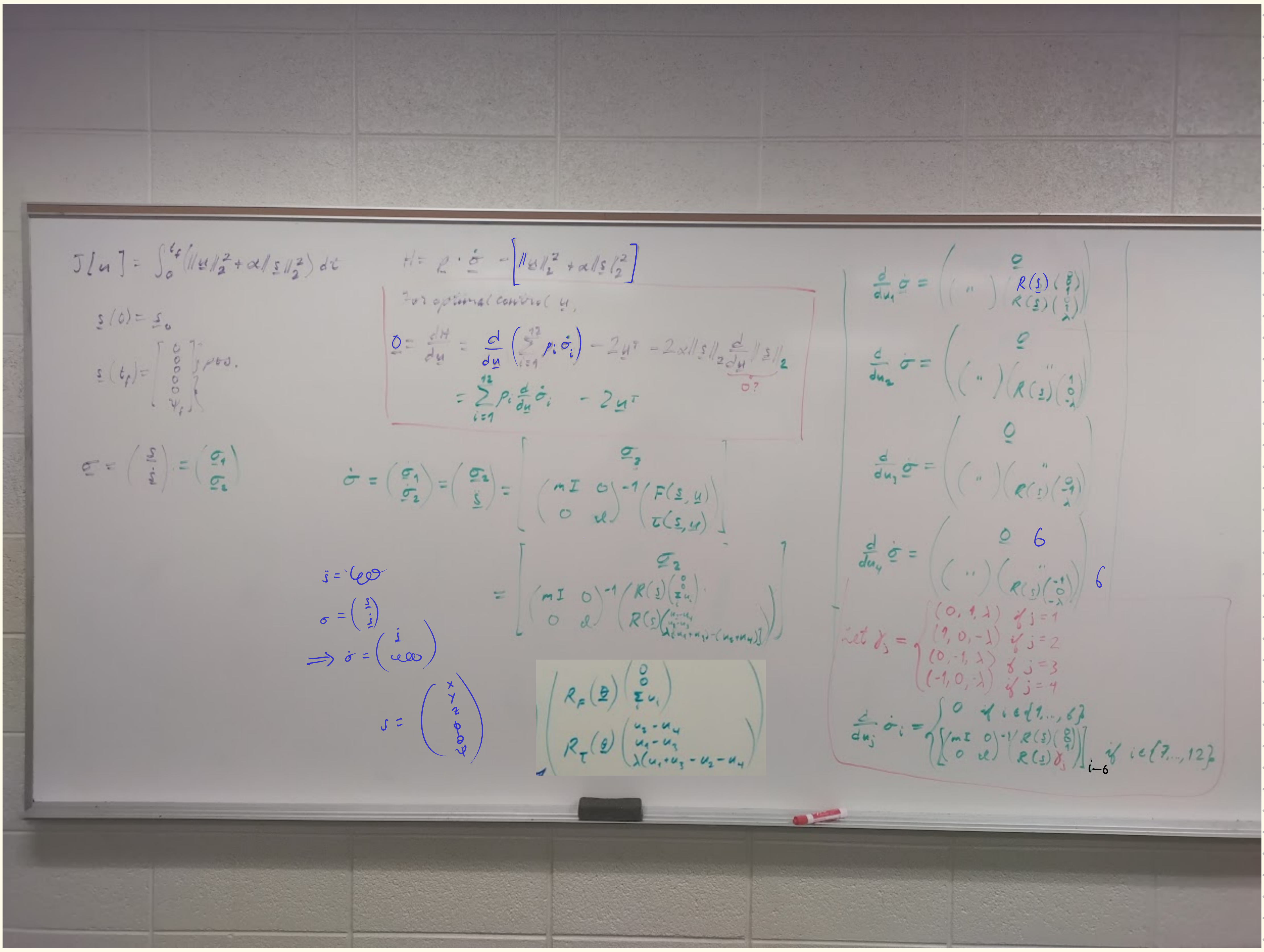



$$\underline{\ddot{s}} = \begin{pmatrix} mI & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} R_F(\underline{\theta}) \begin{pmatrix} 0 \\ 0 \\ \sum_i u_i \end{pmatrix} \\ R_\tau(\underline{\theta}) \begin{pmatrix} u_2 - u_4 \\ u_1 - u_3 \\ \lambda(u_1 + u_3 - u_2 - u_4) \end{pmatrix} \end{pmatrix}$$

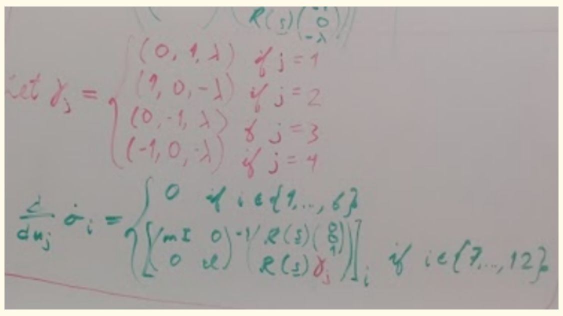
$R_F$ and $R_\tau$ depend on $\underline{s}$. Not linear

Start with small angle approximations on rotation matrices

"Convert between inertial and non-inertial reference frames




$$2\underline{u}^T = \sum_{i=1}^{12} p_i \frac{d}{du}\dot{\sigma}_i$$




$$\Rightarrow \begin{cases} u_1 = \frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{J} \end{pmatrix}^{-1}\begin{pmatrix} R(\underline{s})\begin{pmatrix}0\\0\\1\end{pmatrix} \\ R(\underline{s})\begin{pmatrix}0\\1\\\lambda\end{pmatrix}\end{pmatrix}\right]_{i-6} \\ u_2 = \frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{J} \end{pmatrix}^{-1}\begin{pmatrix} R(\underline{s})\begin{pmatrix}0\\0\\1\end{pmatrix} \\ R(\underline{s})\begin{pmatrix}1\\0\\-\lambda\end{pmatrix}\end{pmatrix}\right]_{i-6} \\ u_3 = \frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{J} \end{pmatrix}^{-1}\begin{pmatrix} R(\underline{s})\begin{pmatrix}0\\0\\1\end{pmatrix} \\ R(\underline{s})\begin{pmatrix}0\\-1\\\lambda\end{pmatrix}\end{pmatrix}\right]_{i-6} \\ u_4 = \frac{1}{2}\sum_{i=7}^{12} \underbrace{p_i}_{\text{costate}} \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{J} \end{pmatrix}^{-1}\begin{pmatrix} R(\underline{s})\begin{pmatrix}0\\0\\1\end{pmatrix} \\ R(\underline{s})\begin{pmatrix}-1\\0\\-\lambda\end{pmatrix}\end{pmatrix}\right]_{i-6} \end{cases}$$

$$\underbrace{\underline{\sigma}}_{\text{state}} = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix}$$

Now I have $u(\underline{\sigma}, \underline{p})$.
Need evolution of costate.

$$H = \underline{p} \cdot \dot{\underline{\sigma}} - \left[ \|\underline{u}\|_2^2 + \alpha \|\underline{s}\|_2^2 \right]$$

$$H = \underline{p} \cdot \dot{\underline{\sigma}} - \left[ \|\underline{u}\|_2^2 + \alpha \|\underline{s}\|_2^2 \right]$$

$$\underline{\sigma} = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \\ \text{der.} \\ \\ \end{pmatrix}$$

$$\underline{p}' = -\frac{DH}{D\underline{\sigma}}$$

$$-\frac{DH}{D\sigma_i} = \frac{D}{D\sigma_i}\left( \|\underline{u}\|_2^2 + \alpha \|\underline{s}\|_2^2 \right)$$

$$= \begin{cases} 0 & \text{if } i \geq 7 \\ \text{something} & \text{otherwise} \end{cases}$$

So let's say $k < 7$. $k$ instead of $i$

(1) $\frac{\partial}{\partial \sigma_k}\left( \|\underline{u}\|_2^2 \right)$ $\qquad \sigma_k = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix}$ $\qquad \underline{s} = \begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{pmatrix}$

$$= \frac{\partial}{\partial \sigma_k}\left( \|\underline{u}\|_2^2 \right)$$

$$= \frac{\partial}{\partial s_k}\Bigg[ \left( \frac{1}{2}\sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2$$

$$+ \left( \frac{1}{2}\sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2$$

$$+ \left( \frac{1}{2}\sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ -1 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2$$

$$+ \left( \frac{1}{2}\sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} -1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2 \Bigg]$$

$$= \frac{\partial}{\partial s_k} \sum_{j=1}^{4} \left( \frac{1}{2} \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= \sum_{j=1}^{4} \frac{\partial}{\partial s_k} \left( \frac{1}{2} \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= \sum_{j=1}^{4} \left( \left[ \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right] \frac{\partial}{\partial s_k} \left( \frac{1}{2} \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^{4} \left( \left[ \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right] \left( \frac{1}{2} \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^{4} \left( \left[ \frac{1}{2} \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} m I & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right] \left( \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} m I & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^{4} \left( u_j \left( \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} m I & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right) \qquad s = \begin{pmatrix} x \\ y \\ z \\ | \end{pmatrix}$$

$$= \begin{cases} 0 & \text{if } k \in \{1, 2, 3\} \\ \displaystyle\sum_{j=1}^{4} \left( u_j \left( \sum_{i=7}^{12} p_i \left[ \begin{pmatrix} m I & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right) & \text{if } k \in \{4, 5, 6\} \end{cases}$$

where

$$\frac{d}{ds_k} R(\underline{s}) = \frac{d}{ds_k}\begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$= \Big\{$

$\underline{0}$ if $k \in \{1, 2, 3\}$

```
In[10]:= D[rotation[ϕ, θ, ψ], ϕ] // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] \end{pmatrix}$$

if $k = 4$

```
In[11]:= D[rotation[ϕ, θ, ψ], θ] // MatrixForm
```

Out[11]//MatrixForm=

$$\begin{pmatrix} -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] & -\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\theta] \\ \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] & \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] \\ \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] \end{pmatrix}$$

if $k = 5$

```
In[12]:= D[rotation[ϕ, θ, ψ], ψ] // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi] & 0 \\ -\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & 0 \\ \mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & 0 \end{pmatrix}$$

if $k = 6$

$$\frac{\partial}{\partial \sigma_k} \left( \alpha \|\underline{s}\|_2^2 \right)$$ 2

$$= \frac{\partial}{\partial \sigma_k} \alpha \left( \sum_{i=1}^{6} s_i^2 \right)$$

$$= \alpha \sum_{i=1}^{6} \frac{\partial}{\partial \sigma_k} s_i^2$$

$$\underline{\sigma} = \begin{pmatrix} \underline{s} & \vdots & \underline{s} \end{pmatrix}$$

$$= \alpha \sum_{i=1}^{6} \frac{\partial}{\partial \sigma_k} \sigma_i^2$$

$$= \begin{cases} 2\alpha\sigma_k & \text{if } k < 7 \\ 0 & \text{if } k \geq 7 \end{cases}$$

So

$$\underline{p}' = -\frac{DH}{D\underline{\sigma}} = -\frac{D}{D\sigma_i}\left(-\|\underline{u}\|_2^2 - \alpha\|\underline{s}\|_2^2\right)$$

depends only on $\underline{\sigma}$, $\underline{p}$.

solve bvp