

Rotational Equations of Motion

$$I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) = M_x$$

$$I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) = M_y$$

$$I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) = M_z$$

Three 1st-order EOMs in terms of angular velocities, not in terms of angles
 → We can't use them to solve for the orientation/attitude of a body!

Euler Angle Sequences

- Orientation/attitude/angles specified using three sequential rotations
- For example, we could use a "3-1-3" sequence:
 - Start with an inertial "a" frame
 - Rotate about the "3" axis by ϕ to get a primed frame
 - Rotate about the primed "1" axis by θ to get a double primed frame
 - Rotate about the double primed "3" axis by ψ to get the body-fixed "b" frame

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{3rd}}}{[R_3(\psi)]} \underset{\substack{\uparrow \\ \text{2nd}}}{[R_1(\theta)]} \underset{\substack{\uparrow \\ \text{1st}}}{[R_3(\phi)]} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [R] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Angular Velocity and Euler Angle Rates

- $\phi, \theta, \psi \rightarrow$ "Euler angles" (about individual axes)
- $\dot{\phi}, \dot{\theta}, \dot{\psi} \rightarrow$ "Euler Angle rates" (about individual axes)
- $\omega_1, \omega_2, \omega_3 \rightarrow$ "Angular velocities" (about body-fixed axes)
- We can write the angular velocity of the body in terms of $\dot{\phi}, \dot{\theta}, \dot{\psi}$:

$$\vec{\omega} = \dot{\phi} \hat{a}_3 + \dot{\theta} \hat{a}_1' + \dot{\psi} \hat{a}_3''$$

- Written in the body frame:

$$\vec{\omega} = (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) \hat{b}_1 + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) \hat{b}_2 + (\dot{\phi} \cos \theta + \dot{\psi}) \hat{b}_3$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin \theta \sin \psi & \cos \psi & 0 \\ \sin \theta \cos \psi & -\sin \psi & 0 \\ \cos \theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Three More Rotational EOMs

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = [B] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

- Solve for the Euler Angle rates:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\begin{aligned} \dot{\phi} &= \frac{1}{s\theta} (\omega_x s\psi + \omega_y c\psi) \\ \dot{\theta} &= \omega_x c\psi - \omega_y s\psi \\ \dot{\psi} &= \frac{1}{s\theta} (-\omega_x c\theta s\psi - \omega_y c\theta c\psi) + \omega_z \end{aligned}$$

Six 1st-Order Rotational EOMs

$$\begin{aligned} I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) &= M_x \\ I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) &= M_y \\ I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) &= M_z \end{aligned}$$

$$\begin{aligned} \dot{\phi} &= \frac{1}{s\theta}(\omega_x s\psi + \omega_y c\psi) \\ \dot{\theta} &= \omega_x c\psi - \omega_y s\psi \\ \dot{\psi} &= \frac{1}{s\theta}(-\omega_x c\theta s\psi - \omega_y c\theta c\psi) + \omega_z \end{aligned}$$

$$\text{Or } \{\dot{\omega}\} = [I]^{-1}\{F\} \text{ and } \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1}\{\omega\}$$

- Solve these six EOMs simultaneously (numerically) for body-fixed angular velocity $(\omega_x, \omega_y, \omega_z)$ and Euler angles (ϕ, θ, ψ)

Three 2nd-Order Rotational EOMs

$$\begin{aligned} I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) &= M_x \\ I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) &= M_y \\ I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) &= M_z \end{aligned}$$

$$\vec{\omega} = (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) \hat{b}_1 + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) \hat{b}_2 + (\dot{\phi} \cos \theta + \dot{\psi}) \hat{b}_3$$

$$\vec{\omega} = (\dots) \hat{b}_1 + (\dots) \hat{b}_2 + (\dots) \hat{b}_3$$

- Plug three components of $\vec{\omega}$ and $\vec{\dot{\omega}}$ into equations of motion above
→ Three 2nd-order ODEs in terms of $\ddot{\theta}, \ddot{\phi}, \ddot{\psi}, \dot{\theta}, \dot{\phi}, \dot{\psi}, \theta, \phi, \psi$
- Define six state variables $x_1 \dots x_6$
- Solve six state equations simultaneously (numerically) for $\theta(t), \phi(t), \psi(t)$

Singularities

- There's a problem....
- Look at the case where $\theta = 0$ (the middle rotation)
 - It's now like having a "3" rotation followed by another "3" rotation
 - We can't distinguish the 1st rotation from the 3rd rotation
- The problem shows up when we look at the $[B]$ matrix:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = [B] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Singularities

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- When doesn't this work?
 - When $[B]^{-1}$ doesn't exist \rightarrow when $[B]$ is "singular"

$$[B] = \begin{bmatrix} \sin \theta \sin \psi & \cos \psi & 0 \\ \sin \theta \cos \psi & -\sin \psi & 0 \\ \cos \theta & 0 & 1 \end{bmatrix} \rightarrow [B]^{-1} = \begin{bmatrix} \frac{\sin \psi}{\sin \theta} & \frac{\cos \psi}{\sin \theta} & 0 \\ \cos \psi & -\sin \psi & 0 \\ \frac{-\sin \psi}{\tan \theta} & \frac{-\cos \psi}{\tan \theta} & 1 \end{bmatrix}$$

- $[B]^{-1}$ breaks down when $\theta = 0, \pm\pi, \pm2\pi, \dots$

Singularities

- Could also see this from the determinant of $[B]$:

$$|B| = -\sin \theta \sin^2 \psi - \sin \theta \cos^2 \psi = -\sin \theta = 0$$

$$\rightarrow \theta = 0, \pm\pi, \pm2\pi, \dots$$

Euler Angle Sequences

- There are 12 possible Euler Angle sequences:

1-2-1	2-1-2	3-1-2
1-2-3	2-1-3	3-1-3
1-3-1	2-3-1	3-2-1
1-3-2	2-3-2	3-2-3

- Each sequence has different $[R]$, $[B]$, and singularities

Attitude Estimation (WAY oversimplified)

- Frequently systems (robots, aircraft, spacecraft) have an inertial measurement unit (IMU) $\rightarrow \omega_1, \omega_2, \omega_3$
- How can we estimate the attitude of a system from IMU (angular velocity) data?
 - We can't just integrate the angular velocities $\omega_1, \omega_2, \omega_3$ to get the angles ϕ, θ, ψ
 - Why?
- Use $[B]$ to estimate Euler angle rates from angular velocities:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- Then integrate Euler angle rates to get Euler angles:

$$\phi = \int \dot{\phi} dt \quad \text{or} \quad \phi_{k+1} = \phi_k + \dot{\phi}_k \Delta t$$

$$\theta = \int \dot{\theta} dt \quad \text{or} \quad \theta_{k+1} = \theta_k + \dot{\theta}_k \Delta t$$

$$\psi = \int \dot{\psi} dt \quad \text{or} \quad \psi_{k+1} = \psi_k + \dot{\psi}_k \Delta t$$

- In practice, MUCH more sophisticated methods are used (sensor fusion, Kalman filters, etc.)

Body-fixed frame B at G

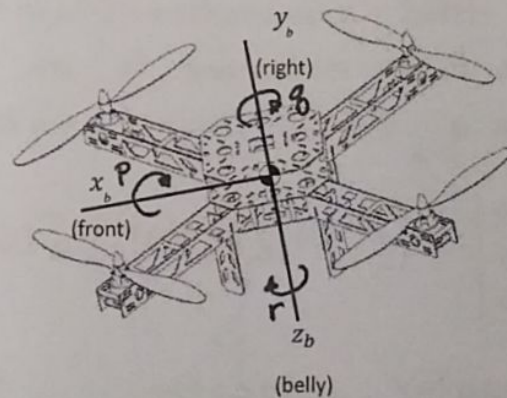
Angular velocity:

$$\vec{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \text{or} \quad \vec{\omega} = p \hat{i}_B + q \hat{j}_B + r \hat{k}_B$$

Roll
Pitch
Yaw

Angular acceleration:

$$\vec{\alpha} = \left(\frac{d\vec{\omega}}{dt} \right)_{\text{rel}} + \vec{\omega} \times \vec{\omega} = \dot{p} \hat{i}_B + \dot{q} \hat{j}_B + \dot{r} \hat{k}_B$$



Moment acting on body:

$$\vec{M} = (F_3 + F_4 - F_1 - F_2)L\hat{x}_B + (F_1 + F_3 - F_2 - F_4)L\hat{y}_B + (\tau_2 + \tau_3 - \tau_1 - \tau_4)\hat{z}_B$$

Assume planes of symmetry

Euler's equations:

$$I_x \dot{p} - (I_y - I_z)q\dot{r} = (F_3 + F_4 - F_1 - F_2)L$$

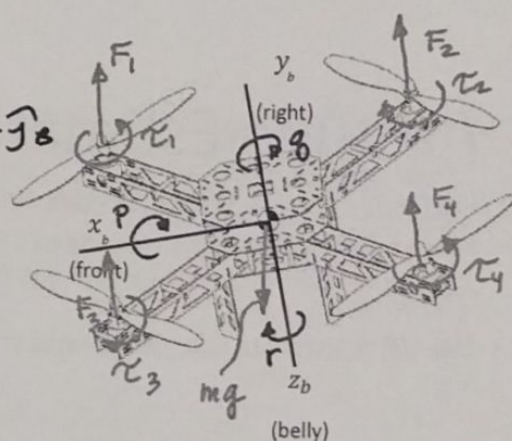
$$I_y \dot{q} - (I_z - I_x)p\dot{r} = (F_1 + F_3 - F_2 - F_4)L$$

$$I_z \dot{r} - (I_x - I_y)p\dot{q} = \tau_2 + \tau_3 - \tau_1 - \tau_4$$

$$\dot{p} = \frac{I_y - I_z}{I_x} q\dot{r} + \frac{L}{I_x} (F_3 + F_4 - F_1 - F_2)$$

$$\dot{q} = \frac{I_z - I_x}{I_y} p\dot{r} + \frac{L}{I_y} (F_1 + F_3 - F_2 - F_4) \quad (1)$$

$$\dot{r} = \frac{I_x - I_y}{I_z} p\dot{q} + \frac{1}{I_z} (\tau_2 + \tau_3 - \tau_1 - \tau_4)$$



3 rotational EOMs

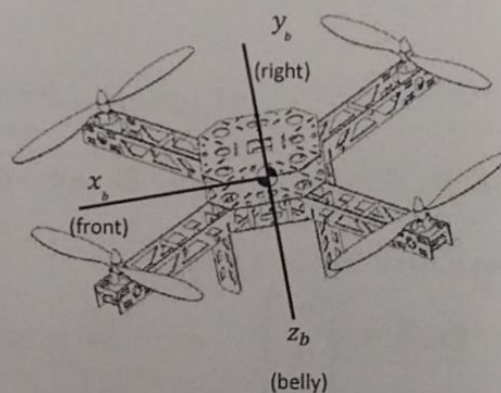
To find orientation ("attitude") of the body relative to an inertial frame, use a 3-2-1 Euler angle sequence

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = [R_x(\phi)][R_y(\theta)][R_z(\psi)] \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}$$

Angular velocity:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = [B] \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \Rightarrow \begin{cases} \dot{\psi} = \frac{1}{c\theta} (q s\phi + r c\phi) \\ \dot{\theta} = q c\phi - r s\phi \\ \dot{\phi} = \frac{1}{c\theta} (q s\theta s\phi + r s\theta c\phi) + p \end{cases} \quad (2)$$

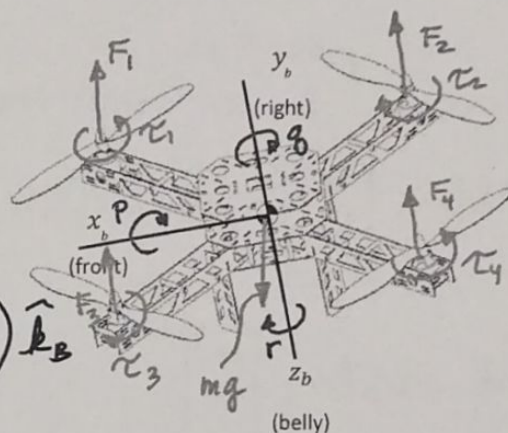


2nd rotational EOMs

Forces acting on body:

$$\vec{F} = -(F_1 + F_2 + F_3 + F_4)\hat{k}_B + mg\hat{k}_I$$

$$\vec{F} = -mg\sin\theta\hat{x}_B + mg\cos\theta\sin\phi\hat{y}_B + (mg\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4))\hat{k}_B$$



Acceleration:

$$\vec{a}_G = \frac{d\vec{v}_G}{dt} = \left(\frac{d\vec{v}_G}{dt}\right)_{rel} + \vec{\omega} \times \vec{v}_G$$

$$\vec{v}_G = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

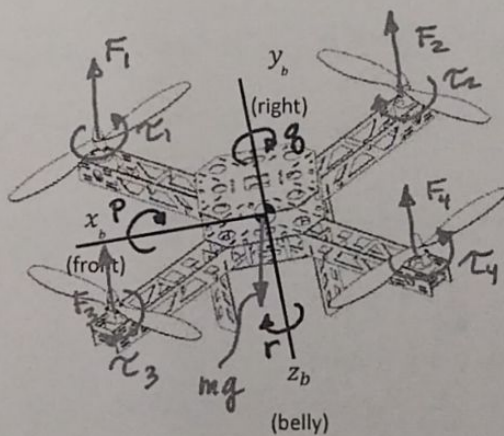
$$\vec{a}_G = (\dot{u} + qw - rv)\hat{x}_B + (\dot{v} + ru - pw)\hat{y}_B + (\dot{w} + pr - qu)\hat{k}_B$$

$$[\vec{F} = m\vec{a}_G]$$

$$\hat{x}_B: m(\dot{u} + qw - rv) = -mg\sin\theta$$

$$\hat{y}_B: m(\dot{v} + ru - pw) = mg\cos\theta\sin\phi$$

$$\hat{k}_B: m(\dot{w} + pr - qu) = mg\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)$$



$$\dot{u} = rv - qw - g\sin\theta$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi$$

$$\dot{w} = qu - pr + g\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)/M$$

(3)

1st translational
EOMs

To find position of the body relative to an inertial frame:

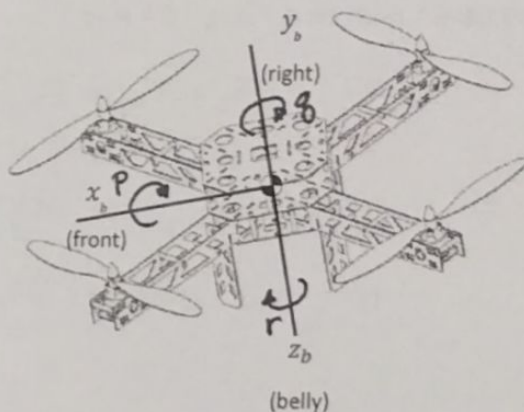
$$\xi r_G \xi = \begin{bmatrix} p_n \\ p_e \\ -h \end{bmatrix} \Rightarrow \xi \dot{r}_G \xi = \begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ -\dot{h} \end{bmatrix}$$

$$\text{and } \xi v_G \xi = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\xi v_G \xi = [R] \xi \dot{r}_G \xi \Rightarrow \xi \dot{r}_G \xi = [R]^{-1} \xi v_G \xi$$

$$\begin{aligned} \dot{p}_n &= u \cos \phi \cos \psi + v (s \phi s \theta \cos \psi - c \phi s \psi) + w (c \phi s \theta \cos \psi + s \phi s \psi) \\ \dot{p}_e &= u \cos \theta \sin \psi + v (s \phi s \theta \sin \psi + c \phi c \psi) + w (c \phi s \theta \sin \psi - s \phi c \psi) \\ -\dot{h} &= -u s \theta + v s \phi c \theta + w c \phi c \theta \end{aligned}$$

2nd
translation
EOMS



drone

$$\dot{p} = \frac{I_y - I_z}{I_x} q r + \frac{1}{I_x} (F_3 + F_4 - F_1 - F_2)$$

$$\dot{q} = \frac{I_z - I_x}{I_y} p r + \frac{1}{I_y} (F_1 + F_3 - F_2 - F_4)$$

$$\dot{r} = \frac{I_x - I_y}{I_z} p q + \frac{1}{I_z} (\tau_2 + \tau_3 - \tau_1 - \tau_4)$$

rotational
velocity
relative to body
accl.

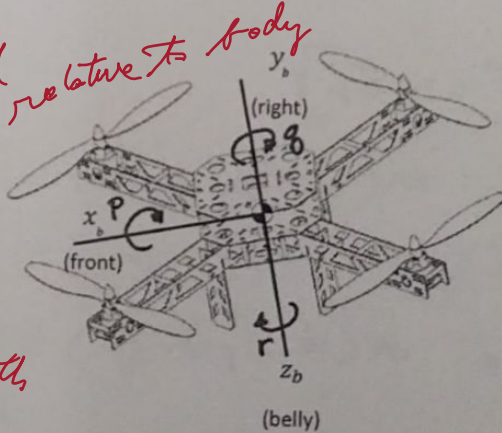
north
east
down

$$\dot{\psi} = \frac{1}{c \theta} (q s \phi + r c \phi)$$

$$\dot{\theta} = q c \phi - r s \phi$$

$$\dot{\phi} = \frac{1}{c \theta} (q s \theta s \phi + r s \theta c \phi) + p$$

rotational
velocity
rel. to earth



drone

$$\dot{u} = r v - q w - g \sin \theta$$

$$\dot{v} = p w - r u + g \cos \theta \sin \phi$$

$$\dot{w} = q u - p v + g \cos \theta \cos \phi - (F_1 + F_2 + F_3 + F_4)/m$$

translational
velocity
acceleration

$$\dot{p}_n = u \cos \phi \cos \psi + v (s \phi s \theta \cos \psi - c \phi s \psi) + w (c \phi s \theta \cos \psi + s \phi s \psi)$$

$$\dot{p}_e = u \cos \theta \sin \psi + v (s \phi s \theta \sin \psi + c \phi c \psi) + w (c \phi s \theta \sin \psi - s \phi c \psi)$$

$$-\dot{h} = -u s \theta + v s \phi c \theta + w c \phi c \theta$$

trans. velocity

rotation
matrix
applied
earth