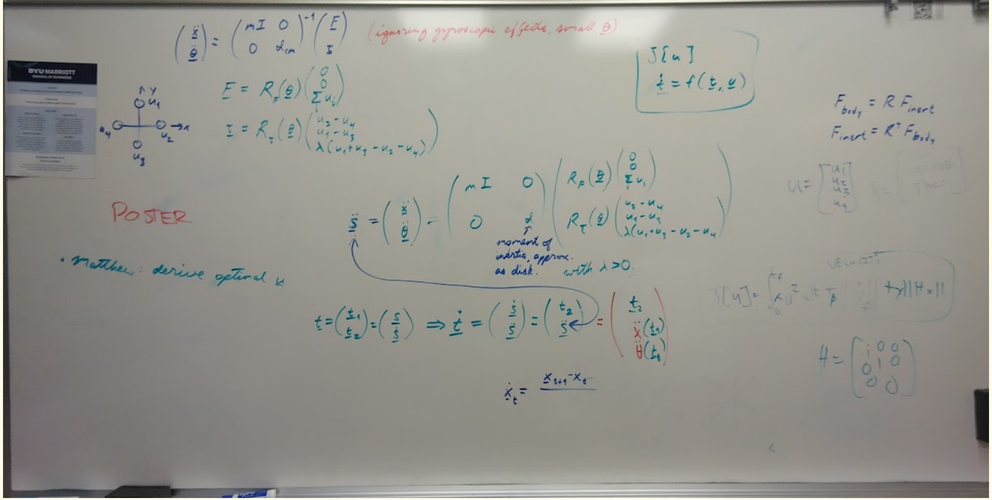
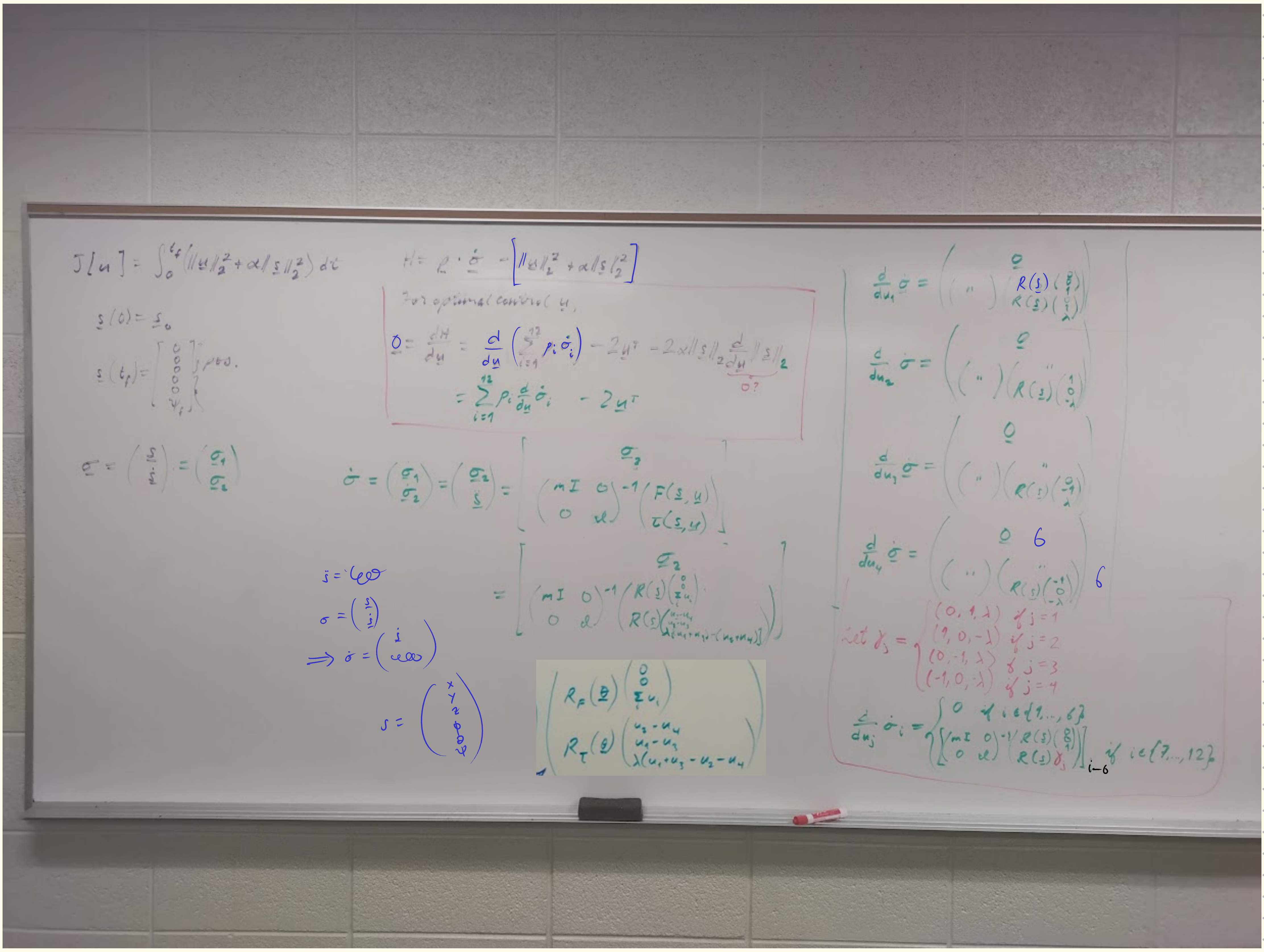


$$\ddot{\underline{s}} = \begin{pmatrix} mI & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} R_F(\underline{\theta}) \begin{pmatrix} 0 \\ 0 \\ \sum_i u_i \end{pmatrix} \\ R_\tau(\underline{\theta}) \begin{pmatrix} u_2 - u_4 \\ u_1 - u_3 \\ \lambda(u_1 + u_3 - u_2 - u_4) \end{pmatrix} \end{pmatrix}$$

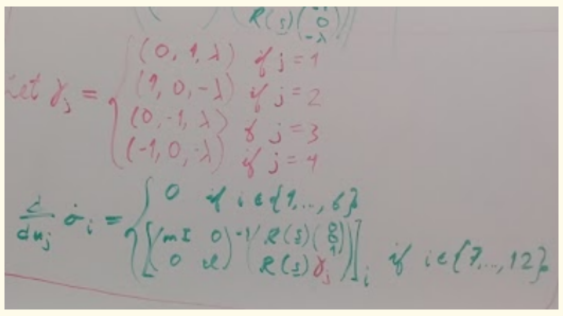
R_F and R_τ depend on $\underline{s}$. Not linear

Start with small angle approximations on rotation matrices

"Convert between inertial and non-inertial reference frames



$$2\underline{u}^T = \sum_{i=1}^{12} p_i \frac{d}{d\underline{u}} \dot{\sigma}_i$$



$$\Rightarrow \begin{cases} u_1 = \frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \\ u_2 = \frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \\ u_3 = \frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ -1 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \\ u_4 = \frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} -1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \end{cases}$$

costate

$\underline{\sigma} = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix}$

state

Now I have $u(\underline{\sigma}, \underline{p})$.

Need evolution of costate.

$$H = \underline{p} \cdot \dot{\underline{\sigma}} - \left[\|\underline{u}\|_2^2 + \alpha \|\underline{s}\|_2^2 \right]$$

$$H = \underline{p} \cdot \dot{\underline{\sigma}} - \left[\|\underline{u}\|_2^2 + \alpha \|\underline{s}\|_2^2 \right]$$

$$\underline{p}' = -\frac{DH}{D\underline{\sigma}}$$

$$\frac{DH}{D\sigma_i} = \frac{D}{D\sigma_i}\left(-\|\underline{u}\|_2^2 - \alpha \|\underline{s}\|_2^2 \right)$$

$$= \begin{cases} 0 & \text{if } i \geq 7 \\ \text{something} & \text{otherwise} \end{cases}$$

So let's say $k < 7$. k instead of i

$$\sigma = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \\ \text{der.} \\ \\ \end{pmatrix}$$

1 $$\frac{\partial}{\partial \sigma_k}\left(-\|\underline{u}\|_2^2\right) \qquad \sigma_k = \begin{pmatrix} \underline{s} \\ \dot{\underline{s}} \end{pmatrix} \qquad \underline{s} = \begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{pmatrix}$$

$$= -\frac{\partial}{\partial \sigma_k}\left(\|\underline{u}\|_2^2\right)$$

$$= -\frac{\partial}{\partial s_k}\Bigg[\left(\frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix} \end{pmatrix}\right]_{i-6}\right)^2$$

$$+ \left(\frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} 1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix}\right]_{i-6}\right)^2$$

$$+ \left(\frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \; \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \\ R(\underline{s})\begin{pmatrix} 0 \\ -1 \\ \lambda \end{pmatrix} \end{pmatrix}\right]_{i-6}\right)^2$$

$$+ \left(\frac{1}{2}\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{I} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\begin{pmatrix} -1 \\ 0 \\ -\lambda \end{pmatrix} \end{pmatrix}\right]_{i-6}\right)^2 \Bigg]$$

$$= -\frac{\partial}{\partial s_k} \sum_{j=1}^{4} \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= -\sum_{j=1}^{4} \frac{\partial}{\partial s_k} \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= -\sum_{j=1}^{4} \left(\left[\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right] \frac{\partial}{\partial s_k} \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= -\sum_{j=1}^{4} \left(\left[\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right] \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \ell \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{s}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{s})\, \gamma_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$\frac{d}{ds_k} R(\underline{s}) = \frac{d}{ds_k} \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$= \begin{cases} \underline{0} & \text{if } k \in \{1,2,3\} \\ \text{Out[10]} & \text{if } k = 4 \\ \text{Out[11]} & \text{if } k = 5 \\ \text{Out[12]} & \text{if } k = 6 \end{cases}$$

In[10]:= `D[rotation[ϕ, θ, ψ], ϕ] // MatrixForm`

Out[10]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] \end{pmatrix} \quad \text{if } k = 4$$

In[11]:= `D[rotation[ϕ, θ, ψ], θ] // MatrixForm`

Out[11]//MatrixForm=

$$\begin{pmatrix} -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] & -\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\theta] \\ \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] & \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] \\ \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] \end{pmatrix} \quad \text{if } k = 5$$

In[12]:= `D[rotation[ϕ, θ, ψ], ψ] // MatrixForm`

Out[12]//MatrixForm=

$$\begin{pmatrix} -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi] & 0 \\ -\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & 0 \\ \mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & 0 \end{pmatrix} \quad \text{if } k = 6$$

$$\frac{\partial}{\partial \sigma_k}\left(-\alpha \|\underline{s}\|_2^2\right) \quad 2$$

$$= \frac{\partial}{\partial \sigma_k} - \alpha\left(\sum_{i=1}^{6} s_i^2\right)$$

$$= -\alpha \sum_{i=1}^{6} \frac{\partial}{\partial \sigma_k} s_i^2 \qquad \underline{\sigma} = \begin{pmatrix} s_1 \\ \vdots \\ s_1 \end{pmatrix}$$

$$= -\alpha \sum_{i=1}^{6} \frac{\partial}{\partial \sigma_k} \sigma_i^2$$

$$= \begin{cases} -2\alpha\sigma_k & \text{if } k < 7 \\ 0 & \text{if } k \geq 7 \end{cases}$$

So

$$\underline{p}' = -\frac{DH}{D\underline{\sigma}} = -\frac{D}{D\sigma_i}\left(\overset{1}{-\|\underline{u}\|_2^2} - \overset{2}{\alpha\|\underline{s}\|_2^2}\right)$$

depends only on $\underline{\sigma}$, $\underline{p}$.

solve bvp