Rotational Equations of Motion

$$I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) = M_x$$

$$I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) = M_y$$

$$I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) = M_z$$

Three 1st-order EOMs in terms of angular velocities, not in terms of angles
→ We can't use them to solve for the orientation/attitude of a body!

Euler Angle Sequences

- Orientation/attitude/angles specified using three sequential rotations
- For example, we could use a "3-1-3" sequence:
  - Start with an inertial "a" frame
  - Rotate about the "3" axis by ϕ to get a primed frame
  - Rotate about the primed "1" axis by θ to get a double primed frame
  - Rotate about the double primed "3" axis by ψ to get the body-fixed "b" frame

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underset{\text{3rd}}{[R_3(\psi)]}\underset{\text{2nd}}{[R_1(\theta)]}\underset{\text{1st}}{[R_3(\phi)]} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [R] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Angular Velocity and Euler Angle Rates

- $\phi, \theta, \psi \rightarrow$ "Euler angles" (about individual axes)
- $\dot{\phi}, \dot{\theta}, \dot{\psi} \rightarrow$ "Euler Angle rates" (about individual axes)
- $\omega_1, \omega_2, \omega_3 \rightarrow$ "Angular velocities" (about body-fixed axes)

- We can write the angular velocity of the body in terms of $\dot{\phi}, \dot{\theta}, \dot{\psi}$:

$$\vec{\omega} = \dot{\phi}\hat{a}_3 + \dot{\theta}\hat{a}_1' + \dot{\psi}\hat{a}_3''$$

- Written in the body frame:

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{b}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{b}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{b}_3$$

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}\sin\theta\sin\psi & \cos\psi & 0\\ \sin\theta\cos\psi & -\sin\psi & 0\\ \cos\theta & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = [B]\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}$$

Three More Rotational EOMs

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = [B]\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}$$

- Solve for the Euler Angle rates:

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = [B]^{-1}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

$$\dot{\phi} = \frac{1}{\mathrm{s}\,\theta}(\omega_x\,\mathrm{s}\,\psi + \omega_y\,\mathrm{c}\,\psi)$$

$$\dot{\theta} = \omega_x\,\mathrm{c}\,\psi - \omega_y\,\mathrm{s}\,\psi$$

$$\dot{\psi} = \frac{1}{\mathrm{s}\,\theta}(-\omega_x\,\mathrm{c}\,\theta\,\mathrm{s}\,\psi - \omega_y\,\mathrm{c}\,\theta\,\mathrm{c}\,\psi) + \omega_z$$

Six 1st-Order Rotational EOMs

$$I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) = M_x$$
$$I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) = M_y$$
$$I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) = M_z$$

$$\dot{\phi} = \frac{1}{s\,\theta}(\omega_x\, s\,\psi + \omega_y\, c\,\psi)$$
$$\dot{\theta} = \omega_x\, c\,\psi - \omega_y\, s\,\psi$$
$$\dot{\psi} = \frac{1}{s\,\theta}(-\omega_x\, c\,\theta\, s\,\psi - \omega_y\, c\,\theta\, c\,\psi) + \omega_z$$

Or $\{\dot{\omega}\} = [I]^{-1}\{F\}$ and $\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = [B]^{-1}\{\omega\}$

- Solve these six EOMs simultaneously (numerically) for body-fixed angular velocity ($\omega_x, \omega_y, \omega_z$) and Euler angles ($\phi, \theta, \psi$)

Three 2nd-Order Rotational EOMs

$$I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}(\omega_y^2 - \omega_z^2) = M_x$$
$$I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}(\omega_z^2 - \omega_x^2) = M_y$$
$$I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}(\omega_x^2 - \omega_y^2) = M_z$$

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{b}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{b}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{b}_3$$

$$\dot{\vec{\omega}} = (\ldots)\hat{b}_1 + (\ldots)\hat{b}_2 + (\ldots)\hat{b}_3$$

- Plug three components of $\vec{\omega}$ and $\dot{\vec{\omega}}$ into equations of motion above
  → Three 2nd-order ODEs in terms of $\ddot{\theta}, \ddot{\phi}, \ddot{\psi}, \dot{\theta}, \dot{\phi}, \dot{\psi}, \theta, \phi, \psi$
- Define six state variables $x_1 \ldots x_6$
- Solve six state equations simultaneously (numerically) for $\theta(t), \phi(t), \psi(t)$

Singularities

- There's a problem….
- Look at the case where $\theta = 0$ (the middle rotation)
  - It's now like having a "3" rotation followed by another "3" rotation
  - We can't distinguish the 1st rotation from the 3rd rotation
- The problem shows up when we look at the $[B]$ matrix:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = [B] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Singularities

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- When doesn't this work?
  - When $[B]^{-1}$ doesn't exist → when $[B]$ is "singular"

$$[B] = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \quad \rightarrow \quad [B]^{-1} = \begin{bmatrix} \dfrac{\sin\psi}{\sin\theta} & \dfrac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ \dfrac{-\sin\psi}{\tan\theta} & \dfrac{-\cos\psi}{\tan\theta} & 1 \end{bmatrix}$$

- $[B]^{-1}$ breaks down when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Singularities

- Could also see this from the determinant of $[B]$:

$$|B| = -\sin\theta\sin^2\psi - \sin\theta\cos^2\psi = -\sin\theta = 0$$

$$\rightarrow \theta = 0, \pm\pi, \pm 2\pi, \ldots$$

Euler Angle Sequences

- There are 12 possible Euler Angle sequences:

1-2-1	2-1-2	3-1-2
1-2-3	2-1-3	3-1-3
1-3-1	2-3-1	3-2-1
1-3-2	2-3-2	3-2-3

- Each sequence has different $[R]$, $[B]$, and singularities

Relates body frame to inertial frame

Relates angular velocity to Euler angle rates

Where it all breaks down

From the $[B]$ matrix

	3-1-3 (Historically Significant)	ard Aerospace)	3-2-1 (NASA Standard Airplane)
Sequence of Angles	ϕ (precession), θ (nutation), ψ (spin)	tation), ϕ (spin)	ψ (heading/yaw), θ (attitude/pitch), ϕ (bank/roll)
Application	celestial mechanics; spinning or rolling bodies	ling bodies	vehicle motion; attitude dynamics
Transformation Matrices	1st: $[R_3] = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2nd: $[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$ 3rd: $[R_3] = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1st: [illegible] $\begin{bmatrix} c\psi & s\psi & 0 \end{bmatrix}$ 2nd: [illegible] 3rd: [illegible] $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$	1st: $[R_3] = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2nd: $[R_2] = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$ 3rd: $[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$
Combined Transformation Matrix $[R]$	$\begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & s\phi c\psi + c\phi c\theta s\psi & s\theta s\psi \\ -c\phi s\psi - s\phi c\theta c\psi & -s\phi s\psi + c\phi c\psi c\theta & s\theta c\psi \\ s\phi s\theta & -c\phi s\theta & c\theta \end{bmatrix}$	$\begin{bmatrix} c\psi c\theta c\phi - s\psi s\phi & s\psi c\theta c\phi + c\psi s\phi & -s\theta c\phi \\ -c\psi c\theta s\phi - s\psi c\phi & -s\psi c\theta s\phi + c\psi c\phi & s\theta s\phi \\ c\psi s\theta & s\psi s\theta & c\theta \end{bmatrix}$	$\begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}$
$[B]$ Matrix	$[B] = \begin{bmatrix} s\theta s\psi & c\psi & 0 \\ s\theta c\psi & -s\psi & 0 \\ c\theta & 0 & 1 \end{bmatrix}$	$[B] = \begin{bmatrix} -s\theta c\phi & s\phi & 0 \\ s\theta s\phi & c\phi & 0 \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \end{bmatrix}$	$[B] = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix}$
Singularities	$\theta = 0, \pm\pi$	[illegible]	$\theta = \pm\pi/2$
Angular Velocities in Terms of Euler Angle Rates	$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot\phi s\theta s\psi + \dot\theta c\psi \\ \dot\phi s\theta c\psi - \dot\theta s\psi \\ \dot\phi c\theta + \dot\psi \end{bmatrix}$	[illegible]	$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -\dot\psi s\theta + \dot\phi \\ \dot\psi c\theta s\phi + \dot\theta c\phi \\ \dot\psi c\theta c\phi - \dot\theta s\phi \end{bmatrix}$
Euler Angle Rates in Terms of Angular Velocities	$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} \frac{1}{s\theta}(\omega_1 s\psi + \omega_2 c\psi) \\ \omega_1 c\psi - \omega_2 s\psi \\ \frac{1}{s\theta}(-\omega_1 c\theta s\psi - \omega_2 c\theta c\psi) + \omega_3 \end{bmatrix}$	$\begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{s\theta}(-\omega_1 c\phi + \omega_2 s\phi) \\ \omega_1 s\phi + \omega_2 c\phi \\ \frac{1}{s\theta}(\omega_1 c\theta c\phi - \omega_2 c\theta s\phi) + \omega_3 \end{bmatrix}$	$\begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{c\theta}(\omega_2 s\phi + \omega_3 c\phi) \\ \omega_2 c\phi - \omega_3 s\phi \\ \frac{1}{c\theta}(\omega_2 s\theta s\phi + \omega_3 s\theta c\phi) + \omega_1 \end{bmatrix}$

From $[B]^{-1}$

Each Euler angle sequence has specific singularity values of the middle angle

	3-1-3 (Historically Significant)	ard Aerospace)	3-2-1 (NASA Standard Airplane)
Sequence of Angles	ϕ (precession), θ (nutation), ψ (spin)	tation), ϕ (spin)	ψ (heading/yaw), θ (attitude/pitch), ϕ (bank/roll)
Application	celestial mechanics; spinning or rolling bodies	ling bodies	vehicle motion; attitude dynamics
Transformation Matrices	1st: $[R_3] = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2nd: $[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$ 3rd: $[R_3] = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1st: [illegible] $\begin{bmatrix} \text{[illegible]} & s\psi & 0 \\ \text{[illegible]} & c\psi & 0 \\ \text{[illegible]} & 0 & 1 \end{bmatrix}$ 2nd: [illegible] $\begin{bmatrix} \text{[illegible]} & 0 & -s\theta \\ \text{[illegible]} & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$ 3rd: $[R_3] = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1st: $[R_3] = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2nd: $[R_2] = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$ 3rd: $[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$
Combined Transformation Matrix $[R]$	$\begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & s\phi c\psi + c\phi c\theta s\psi & s\theta s\psi \\ -c\phi s\psi - s\phi c\theta c\psi & -s\phi s\psi + c\phi c\psi c\theta & s\theta c\psi \\ s\phi s\theta & -c\phi s\theta & c\theta \end{bmatrix}$	$\begin{bmatrix} c\psi c\theta c\phi - s\psi s\phi & s\psi c\theta c\phi + c\psi s\phi & -s\theta c\phi \\ -c\psi c\theta s\phi - s\psi c\phi & -s\psi c\theta s\phi + c\psi c\phi & s\theta s\phi \\ c\psi s\theta & s\psi s\theta & c\theta \end{bmatrix}$	$\begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}$
$[B]$ Matrix	$[B] = \begin{bmatrix} s\theta s\psi & c\psi & 0 \\ s\theta c\psi & -s\psi & 0 \\ c\theta & 0 & 1 \end{bmatrix}$	$[B] = \begin{bmatrix} -s\theta c\phi & s\phi & 0 \\ s\theta s\phi & c\phi & 0 \\ c\theta & 0 & 1 \end{bmatrix}$	$[B] = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix}$
Singularities	$\theta = 0, \pm\pi$	$\theta = 0, \pm\pi$	$\theta = \pm\pi/2$
Angular Velocities in Terms of Euler Angle Rates	$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot\phi s\theta s\psi + \dot\theta c\psi \\ \dot\phi s\theta c\psi - \dot\theta s\psi \\ \dot\phi c\theta + \dot\psi \end{bmatrix}$	$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -\dot\psi s\theta c\phi + \dot\theta s\phi \\ \dot\psi s\theta s\phi + \dot\theta c\phi \\ \dot\psi c\theta + \dot\phi \end{bmatrix}$	$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -\dot\psi s\theta + \dot\phi \\ \dot\psi c\theta s\phi + \dot\theta c\phi \\ \dot\psi c\theta c\phi - \dot\theta s\phi \end{bmatrix}$
Euler Angle Rates in Terms of Angular Velocities	$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} \frac{1}{s\theta}(\omega_1 s\psi + \omega_2 c\psi) \\ \omega_1 c\psi - \omega_2 s\psi \\ \frac{1}{s\theta}(-\omega_1 c\theta s\psi - \omega_2 c\theta c\psi) + \omega_3 \end{bmatrix}$	$\begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{s\theta}(-\omega_1 c\phi + \omega_2 s\phi) \\ \omega_1 s\phi + \omega_2 c\phi \\ \frac{1}{s\theta}(\omega_1 c\theta c\phi - \omega_2 c\theta s\phi) + \omega_3 \end{bmatrix}$	$\begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{c\theta}(\omega_2 s\phi + \omega_3 c\phi) \\ \omega_2 c\phi - \omega_3 s\phi \\ \frac{1}{c\theta}(\omega_2 s\theta s\phi + \omega_3 s\theta c\phi) + \omega_1 \end{bmatrix}$

Attitude Estimation (WAY oversimplified)

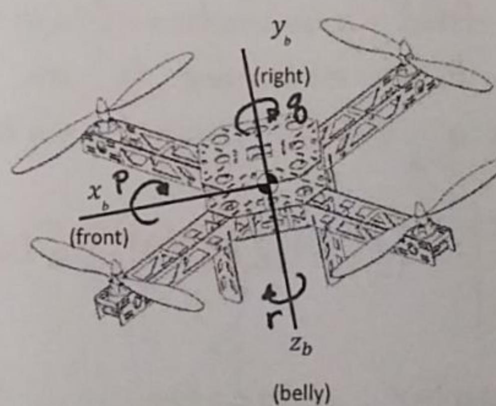
- Frequently systems (robots, aircraft, spacecraft) have an inertial measurement unit (IMU) → $\omega_1, \omega_2, \omega_3$
- How can we estimate the attitude of a system from IMU (angular velocity) data?
  - We can't just integrate the angular velocities $\omega_1, \omega_2, \omega_3$ to get the angles ϕ, θ, ψ
  - Why?
- Use $[B]$ to estimate Euler angle rates from angular velocities:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = [B]^{-1} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- Then integrate Euler angle rates to get Euler angles:

$$\phi = \int \dot{\phi} dt \quad \text{or} \quad \phi_{k+1} = \phi_k + \dot{\phi}_k \Delta t$$
$$\theta = \int \dot{\theta} dt \quad \text{or} \quad \theta_{k+1} = \theta_k + \dot{\theta}_k \Delta t$$
$$\psi = \int \dot{\psi} dt \quad \text{or} \quad \psi_{k+1} = \psi_k + \dot{\psi}_k \Delta t$$

- In practice, MUCH more sophisticated methods are used (sensor fusion, Kalman filters, etc.)



Body-fixed frame B at G

Angular velocity:

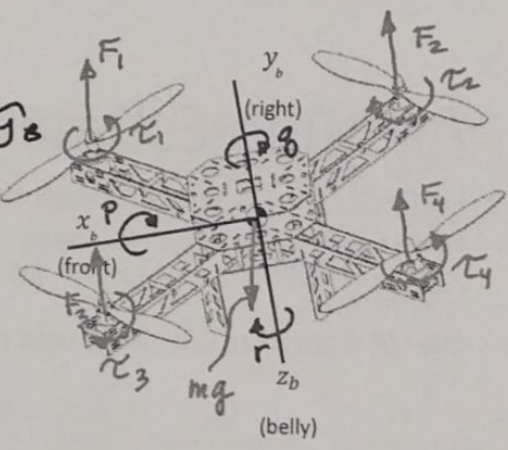
$$\{\omega\} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \text{or} \quad \vec{\omega} = \underbrace{p\hat{i}_B}_{\text{Roll}} + \underbrace{q\hat{j}_B}_{\text{Pitch}} + \underbrace{r\hat{k}_B}_{\text{Yaw}}$$

Angular acceleration:

$$\vec{\alpha} = \left(\frac{d\vec{\omega}}{dt}\right)_{rel} + \underbrace{\vec{\omega} \times \vec{\omega}}_{0} = \dot{p}\hat{i}_B + \dot{q}\hat{j}_B + \dot{r}\hat{k}_B$$

Moment acting on body:

$$\vec{M} = (F_3 + F_4 - F_1 - F_2)L\hat{i}_B + (F_1 + F_3 - F_2 - F_4)L\hat{j}_B + (\tau_2 + \tau_3 - \tau_1 - \tau_4)\hat{k}_B$$



Assume planes of symmetry

Euler's equations:

$$I_x\dot{p} - (I_y - I_z)qr = (F_3 + F_4 - F_1 - F_2)L$$

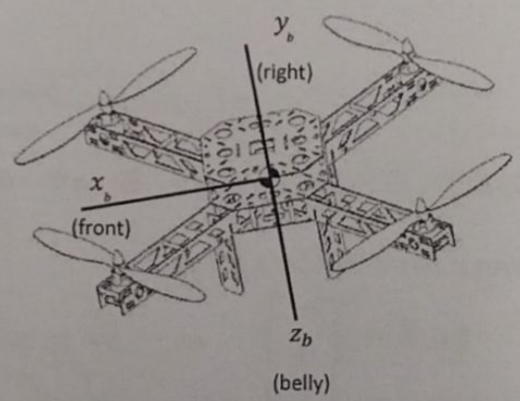
$$I_y\dot{q} - (I_z - I_x)pr = (F_1 + F_3 - F_2 - F_4)L$$

$$I_z\dot{r} - (I_x - I_y)pq = \tau_2 + \tau_3 - \tau_1 - \tau_4$$

$$\boxed{\begin{aligned} \dot{p} &= \frac{I_y - I_z}{I_x}qr + \frac{L}{I_x}(F_3 + F_4 - F_1 - F_2) \\ \dot{q} &= \frac{I_z - I_x}{I_y}pr + \frac{L}{I_y}(F_1 + F_3 - F_2 - F_4) \\ \dot{r} &= \frac{I_x - I_y}{I_z}pq + \frac{1}{I_z}(\tau_2 + \tau_3 - \tau_1 - \tau_4) \end{aligned}} \quad (1)$$

← 3 rotational EOMs

To find orientation ("attitude") of the body relative to an inertial frame, use a 3-2-1 Euler angle sequence



$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = [R_x(\phi)][R_y(\theta)][R_z(\psi)]\begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}$$

Angular velocity:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = [B]\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix}\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

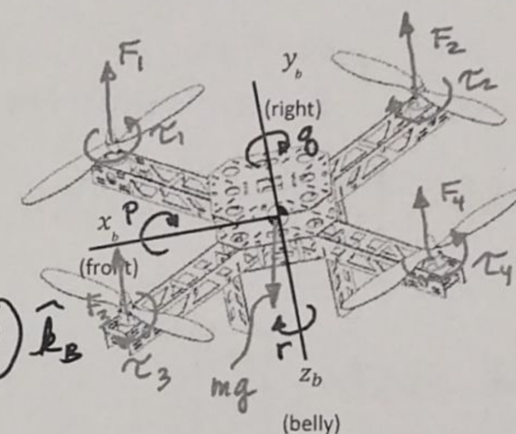
2nd rotational EOMs

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = [B]^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \Rightarrow \boxed{\begin{aligned} \dot{\psi} &= \frac{1}{c\theta}(qs\phi + rc\phi) \\ \dot{\theta} &= qc\phi - rs\phi \\ \dot{\phi} &= \frac{1}{c\theta}(qs\theta s\phi + rs\theta c\phi) + p \end{aligned}} \quad (2)$$

Forces acting on body:

$$\vec{F} = -(F_1 + F_2 + F_3 + F_4)\hat{k}_B + mg\,\hat{k}_I$$

$$\vec{F} = -mg\sin\theta\,\hat{\imath}_B + mg\cos\theta\sin\phi\,\hat{\jmath}_B + \left(mg\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)\right)\hat{k}_B$$



Acceleration:

$$\vec{a}_G = \frac{d\vec{v}_G}{dt} = \left(\frac{d\vec{v}_G}{dt}\right)_{rel} + \vec{\omega} \times \vec{v}_G \qquad \vec{V}_G = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

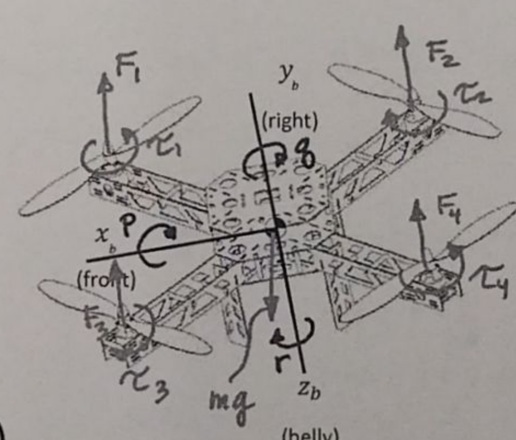
$$\vec{a}_G = (\dot{u} + qw - rv)\hat{\imath}_B + (\dot{v} + ru - pw)\hat{\jmath}_B + (\dot{w} + pv - qu)\hat{k}_B$$

$$\boxed{\vec{F} = m\vec{a}_G}$$

$\hat{\imath}_B$: $m(\dot{u} + qw - rv) = -mg\sin\theta$

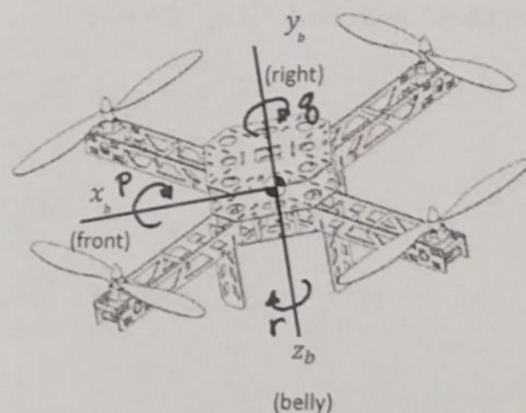
$\hat{\jmath}_B$: $m(\dot{v} + ru - pw) = mg\cos\theta\sin\phi$

$\hat{k}_B$: $m(\dot{w} + pv - qu) = mg\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)$



$$\boxed{\begin{aligned} \dot{u} &= rv - qw - g\sin\theta \\ \dot{v} &= pw - ru + g\cos\theta\sin\phi \\ \dot{w} &= qu - pv + g\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)/M \end{aligned}} \quad (3) \longleftarrow \text{1st translational EOMs}$$

To find position of the body relative to an inertial frame:



$$\{r_G\} = \begin{bmatrix} P_n \\ P_e \\ -h \end{bmatrix} \Rightarrow \{\dot{r}_G\} = \begin{bmatrix} \dot{P}_n \\ \dot{P}_e \\ -\dot{h} \end{bmatrix}$$

and $\{v_G\} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$$\{v_G\} = [R]\{\dot{r}_G\} \Rightarrow \{\dot{r}_G\} = [R]^{-1}\{v_G\}$$

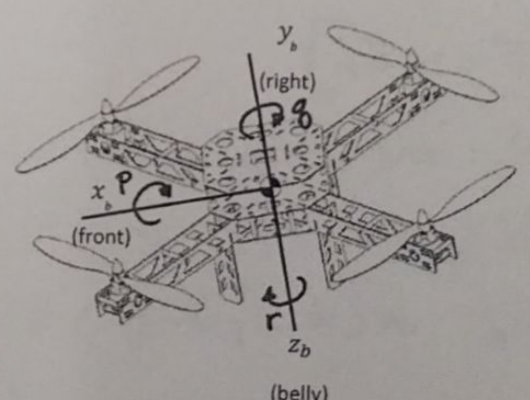
2nd translation EOMS

$$\boxed{\begin{aligned} \dot{P}_n &= u\,c\phi c\psi + v\,(s\phi s\theta c\psi - c\phi s\psi) + w\,(c\phi s\theta c\psi + s\phi s\psi) \\ \dot{P}_e &= u\,c\theta s\psi + v\,(s\phi s\theta s\psi + c\phi c\psi) + w\,(c\phi s\theta s\psi - s\phi c\psi) \\ -\dot{h} &= -u\,s\theta + v\,s\phi c\theta + w\,c\phi c\theta \end{aligned}} \quad (4)$$

$$\dot{p} = \frac{I_y - I_z}{I_x} qr + \frac{L}{I_x}(F_3 + F_4 - F_1 - F_2)$$

$$\dot{q} = \frac{I_z - I_x}{I_y} pr + \frac{L}{I_y}(F_1 + F_3 - F_2 - F_4)$$

$$\dot{r} = \frac{I_x - I_y}{I_z} pq + \frac{1}{I_z}(\tau_2 + \tau_3 - \tau_1 - \tau_4)$$



$$\dot{\psi} = \frac{1}{c\theta}(q s\phi + r c\phi)$$

$$\dot{\theta} = q c\phi - r s\phi$$

$$\dot{\phi} = \frac{1}{c\theta}(q s\theta s\phi + r s\theta c\phi) + p$$

$$\dot{u} = rv - qw - g\sin\theta$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi - (F_1 + F_2 + F_3 + F_4)/m$$

$$\dot{P}_n = u\,c\phi c\psi + v\,(s\phi s\theta c\psi - c\phi s\psi) + w\,(c\phi s\theta c\psi + s\phi s\psi)$$

$$\dot{P}_e = u\,c\theta s\psi + v\,(s\phi s\theta s\psi + c\phi c\psi) + w\,(c\phi s\theta s\psi - s\phi c\psi)$$

$$-\dot{h} = -u\,s\theta + v\,s\phi c\theta + w\,c\phi c\theta$$