



$$\ddot{\mathbf{z}} = \begin{pmatrix} mI & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} R_F(\theta) \begin{pmatrix} 0 \\ \sum u_i \end{pmatrix} \\ R_T(\theta) \begin{pmatrix} u_2 - u_4 \\ u_1 - u_3 \\ \lambda(u_1 + u_3 - u_2 - u_4) \end{pmatrix} \end{pmatrix}$$

R_F and R_T depend on \mathbf{z} . Not linear

Start with small angle approximations on rotation matrices

"Convert between inertial and non-inertial reference frames

$$J[u] = \int_0^{t_f} (\|u\|_2^2 + \alpha \|s\|_2^2) dt$$

$$s(0) = s_0$$

$$s(t_f) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq 0.$$

$$\sigma = \begin{pmatrix} m \cdot 10 \\ 10 \cdot 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$H = p \cdot \dot{\sigma} - \left[\|u\|_2^2 + \alpha \|s\|_2^2 \right]$$

For optimal control u ,

$$0 = \frac{dH}{du} = \frac{d}{du} \left(\sum_{i=1}^{12} p_i \dot{\sigma}_i \right) - 2u^T - 2\alpha \|s\|_2 \frac{d}{du} \|s\|_2$$

$$= \sum_{i=1}^{12} p_i \frac{d}{du} \dot{\sigma}_i - 2u^T$$

$$\dot{\sigma} = \begin{pmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{pmatrix} = \begin{pmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{pmatrix} = \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} F(s, u) \\ \tau(s, u) \end{pmatrix}$$

$$\dot{s} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$\Rightarrow \dot{\sigma} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$s = \begin{pmatrix} x \\ y \\ z \\ \theta \end{pmatrix}$$

$$= \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} u_1 + u_2 - u_3 - u_4 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} R_F(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R_T(s) \begin{pmatrix} u_1 - u_2 \\ u_1 - u_3 \\ \lambda(u_1 + u_2 - u_3 - u_4) \end{pmatrix} \end{pmatrix}$$

$$\frac{d}{du_1} \sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\frac{d}{du_2} \sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\frac{d}{du_3} \sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$\frac{d}{du_4} \sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$\text{Let } \delta_j = \begin{cases} (0, 1, 1) & \text{if } j=1 \\ (1, 0, -1) & \text{if } j=2 \\ (0, -1, 1) & \text{if } j=3 \\ (-1, 0, 1) & \text{if } j=4 \end{cases}$$

$$\frac{d}{du_j} \dot{\sigma}_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, 6\} \\ \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \begin{bmatrix} 0 & u \end{bmatrix} R(s) \delta_j \end{cases} \text{ if } i \in \{7, \dots, 12\}$$

$$2u^T = \sum_{i=1}^{12} p_i \frac{d}{du} \dot{\sigma}_i$$

$$\Rightarrow \begin{cases} u_1 = \frac{1}{2} \sum_{i=7}^{12} p_i \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \Big|_{i=6} \\ u_2 = \frac{1}{2} \sum_{i=7}^{12} p_i \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix} \Big|_{i=6} \\ u_3 = \frac{1}{2} \sum_{i=7}^{12} p_i \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \Big|_{i=6} \\ u_4 = \frac{1}{2} \sum_{i=7}^{12} p_i \begin{bmatrix} mI & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{pmatrix} R(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ R(s) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \Big|_{i=6} \end{cases}$$

Now I have $u(\sigma, p)$.
Need evolution of costate.

$$H = p \cdot \dot{s} - \left[\|u\|_2^2 + \alpha \|s\|_2^2 \right]$$

$$H = p \cdot \dot{s} - \left[\|u\|_2^2 + \alpha \|s\|_2^2 \right]$$

$$p' = - \frac{DH}{D\sigma}$$

$$- \frac{DH}{D\sigma_i} = \frac{D}{D\sigma_i} \left(\|u\|_2^2 + \alpha \|s\|_2^2 \right)$$

$$= \begin{cases} 0 & \text{if } i \geq 7 \\ \text{something otherwise} \end{cases}$$

So let's say $k < 7$, k instead of i

$$\sigma = \begin{pmatrix} s \\ \dot{s} \end{pmatrix} =$$

$$\begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \text{der.} \end{pmatrix}$$

$$\textcircled{1} \frac{\partial}{\partial \sigma_k} (\|u\|_2^2)$$

$$= \frac{\partial}{\partial \sigma_k} (\|u\|_2^2)$$

$$\sigma_k = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\underline{15} =$$

$$\begin{pmatrix} x \\ y \\ y \\ 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \frac{\partial}{\partial \sigma_k} \left[\left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(2) \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2 \right. \\ \left. + \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(2) \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2 \right. \\ \left. + \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(2) \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2 \right. \\ \left. + \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(2) \begin{pmatrix} -1 \\ \lambda \end{pmatrix} \end{pmatrix} \right]_{i-6} \right)^2 \right]$$

$$= \frac{\partial}{\partial s_k} \sum_{j=1}^4 \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= \sum_{j=1}^4 \frac{\partial}{\partial s_k} \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right)^2$$

$$= \sum_{j=1}^4 \left(\left[\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right] \frac{\partial}{\partial s_k} \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^4 \left(\left[\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right] \left(\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial s_k} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial s_k} R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^4 \left(\left[\frac{1}{2} \sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{L} \end{pmatrix}^{-1} \begin{pmatrix} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right] \left(\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{L} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial J_k} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial J_k} R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right) \right)$$

$$= \sum_{j=1}^4 \left(u_j \left(\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{L} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial J_k} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial J_k} R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right) \right) \quad S = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{cases} 0 & \text{if } k \in \{1, 2, 3\} \\ \sum_{j=1}^4 \left(u_j \left(\sum_{i=7}^{12} p_i \left[\begin{pmatrix} mI & 0 \\ 0 & \mathcal{L} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial J_k} R(\underline{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\partial}{\partial J_k} R(\underline{2}) \delta_j \end{pmatrix} \right]_{i-6} \right) \right) & \text{if } k \in \{4, 5, 6\} \end{cases}$$

where

$$\frac{d}{ds_k} R(s) = \frac{d}{ds_k} \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

0 if $k \in \{1, 2, 3\}$

```
In[10]:= D[rotation[phi, theta, psi], phi] // MatrixForm
```

```
Out[10]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\theta] + \cos[\phi] \sin[\theta] \sin[\psi] & \cos[\theta] \cos[\phi] \\ -\cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & -\cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\theta] \sin[\phi] \end{pmatrix}$$

if $k=4$

```
In[11]:= D[rotation[phi, theta, psi], theta] // MatrixForm
```

```
Out[11]//MatrixForm=
```

$$\begin{pmatrix} -\cos[\psi] \sin[\theta] & -\sin[\theta] \sin[\psi] & -\cos[\theta] \\ \cos[\theta] \cos[\psi] \sin[\phi] & \cos[\theta] \sin[\phi] \sin[\psi] & -\sin[\theta] \sin[\phi] \\ \cos[\theta] \cos[\phi] \cos[\psi] & \cos[\theta] \cos[\phi] \sin[\psi] & -\cos[\phi] \sin[\theta] \end{pmatrix}$$

if $k=5$

```
In[12]:= D[rotation[phi, theta, psi], psi] // MatrixForm
```

```
Out[12]//MatrixForm=
```

$$\begin{pmatrix} -\cos[\theta] \sin[\psi] & \cos[\theta] \cos[\psi] & 0 \\ -\cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & 0 \\ \cos[\psi] \sin[\theta] - \cos[\phi] \sin[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] & 0 \end{pmatrix}$$

if $k=6$

$$\frac{\partial}{\partial \sigma_k} \left(\alpha \|s\|_2^2 \right) \quad (2)$$

$$= \frac{\partial}{\partial \sigma_k} \alpha \left(\sum_{i=1}^6 s_i^2 \right)$$

$$= \alpha \sum_{i=1}^6 \frac{\partial}{\partial \sigma_k} s_i^2$$

$$(9) = \begin{pmatrix} 15 \\ 15 \end{pmatrix}$$

$$= \alpha \sum_{i=1}^6 \frac{\partial}{\partial \sigma_k} \sigma_i^2$$

$$= \begin{cases} 2\alpha \sigma_k & \text{if } k < 7 \\ 0 & \text{if } k \geq 7 \end{cases}$$

so

$$p' = - \frac{DH}{D\sigma} = - \frac{D}{D\sigma} \left(- \overset{(1)}{\|u\|_2^2} - \alpha \overset{(2)}{\|s\|_2^2} \right)$$

depends only on σ, p .

solve for