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PHYS 230 Module 1c

1) Calculations to support my plots: (K_r and K_N as functions of $\frac{v}{c}$)

a) $\frac{K_N}{E_0} = \frac{\frac{1}{2}mv^2}{mc^2} = \frac{1}{2}\frac{v^2}{c^2} \Rightarrow$ For plot: $\boxed{\frac{1}{2}x^2}$

b) $\frac{K_r}{E_0} = \frac{E - E_0}{E_0} = \frac{E}{E_0} - 1 = \frac{\gamma\left(\frac{v}{c}\right)mc^2}{mc^2} - 1 = \frac{1}{\sqrt{1-\beta^2}} - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1$

$\Rightarrow$ For plot: $\boxed{\frac{1}{\sqrt{1-x^2}} - 1}$

2) $K_N = \frac{1}{2}mv^2$; $K_R = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}mc^2 - mc^2 = \gamma(\beta)mc^2 - mc^2$

We know: $(1+x)^p \simeq 1 + px + \frac{p(p-1)}{2}x^2 + \ldots$ for $|x| << 1$ (A)

$\gamma(\beta) \equiv (1-\beta^2)^{-1/2}$, and for $|\beta| << 1$, we have: $\boxed{(x = -\beta^2),\ p = -1/2}$

$\gamma(\beta) \simeq 1 + \left(-\frac{1}{2}\right)(-\beta^2) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}(-\beta^2)^2$ → In this line, I use the Taylor series expansion given in (A), with $x = -\beta^2$ and $p = -1/2$

$\Rightarrow \gamma(\beta) \simeq 1 + \frac{1}{2}\beta^2 - \frac{3}{8}\beta^4$

So $K_R \simeq \left(1 + \frac{1}{2}\beta^2 - \frac{3}{8}\beta^4\right)mc^2 - mc^2$

$\Rightarrow K_R \simeq mc^2\left(1 + \frac{1}{2}\beta^2 - \frac{3}{8}\beta^4 - 1\right)$

$\Rightarrow K_R \simeq mc^2\frac{1}{2}\frac{v^2}{c^2} - mc^2\frac{3}{8}\frac{v^4}{c^4}$

$\Rightarrow K_R \simeq \frac{1}{2}mv^2 - \frac{3}{8}m\frac{v^4}{c^2}$

Therefore $\boxed{\delta K = -\frac{3}{8}m\frac{v^4}{c^2}}$

and $\boxed{K_r \simeq K_N + \delta K = \frac{1}{2}mv^2 - \frac{3}{8}m\frac{v^4}{c^2}}$

So $\frac{K_r}{E_0} \simeq \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} \Rightarrow$ $\boxed{\text{For plot: } \frac{1}{2}x^2 - \frac{3}{8}x^4}$ $(E_0 = mc^2)$

3) $\vec{a}_N = \frac{\vec{F}}{m}$ $\qquad \vec{a}_r = \frac{\vec{F} - (\vec{F}\cdot\vec{\beta})\vec{\beta}}{\gamma(\beta)\, m}$

For $\vec{F} \perp \vec{v}$, we have $\vec{a}_r = \frac{\vec{F}}{\gamma(\beta)\, m}$ $\qquad (\vec{F}\cdot\vec{\beta} = 0$ for $\vec{F} \perp \vec{\beta})$

$\Rightarrow |\vec{a}_N| = \frac{F}{m}$ and $|\vec{a}_r| = \frac{F}{\gamma(\beta)\, m}$

$$\Rightarrow \frac{|\vec{a}_r|}{|\vec{a}_n|} = \frac{\frac{F}{\gamma(\beta)m}}{\frac{F}{m}} = \frac{1}{\gamma(\beta)} = \frac{1}{\frac{1}{\sqrt{1-\beta^2}}} = \boxed{\sqrt{1-\beta^2}}$$

For $\vec{F} \parallel \vec{v}$, we have $\vec{a}_r = \frac{\vec{F} - F\beta\vec{\beta}}{\gamma(\beta)\, m}$ $\qquad (\vec{F}\cdot\vec{\beta} = F\beta$ for $\vec{F} \parallel \vec{\beta})$

$\Rightarrow |\vec{a}_r| = \frac{F - F\beta^2}{\gamma(\beta)\, m}$ $\qquad$ (Note we cannot have $|\vec{a}_r| = \frac{-F + F\beta^2}{\gamma(\beta)m}$ since $|\beta| < 1$, and that would give $|\vec{a}_r| < 0$, which is impossible)

$$\Rightarrow \frac{|\vec{a}_r|}{|\vec{a}_n|} = \frac{\frac{F - F\beta^2}{\gamma(\beta)m}}{\frac{F}{m}} = \frac{mF - mF\beta^2}{F\gamma(\beta)\, m} = \frac{1-\beta^2}{\gamma(\beta)} = \frac{1-\beta^2}{\frac{1}{\sqrt{1-\beta^2}}} = \boxed{(1-\beta^2)^{3/2}}$$

4) Use relativistic expression for momentum to find speed of relativistic particle accelerated from rest as function of time.

$$\vec{p}_r = \gamma\left(\frac{v}{c}\right) m\vec{v}$$

$$\Rightarrow \vec{p}_r = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} m\vec{v}$$

$F = \frac{dp}{dt}$, if a particle is initially at rest and a constant force $\vec{F}$ is applied for time t. (We remove vector signs since all motion is along 1 direction)

So $$\int_0^t F\,dt = \int_0^p dp'$$

$$\Rightarrow Ft = p = \gamma\left(\frac{v}{c}\right) mv$$

$$\Rightarrow Ft = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} mv$$

$$\Rightarrow \frac{Ft}{m} = \frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\Rightarrow \left(\frac{Ft}{m}\right)^2 = \frac{v^2}{1-\left(\frac{v}{c}\right)^2}$$

$$\Rightarrow \left(\frac{Ft}{m}\right)^2 = \frac{v^2}{\frac{c^2-v^2}{c^2}}$$

$$\Rightarrow \left(\frac{Ft}{m}\right)^2 = \frac{v^2c^2}{c^2-v^2}$$

$$\Rightarrow \left(\frac{Ft}{mc}\right)^2 = \frac{v^2}{c^2-v^2}$$

$$\Rightarrow \tau^2c^2 - \tau^2v^2 = v^2, \text{ where } \tau = \frac{Ft}{mc}$$

$$\Rightarrow \tau^2c^2 = v^2 + \tau^2v^2$$

$$\Rightarrow \tau^2c^2 = v^2(1+\tau^2)$$

$$\Rightarrow \frac{\tau^2}{(1+\tau^2)} = \frac{v^2}{c^2}$$

$$\Rightarrow \frac{\tau}{\sqrt{1+\tau^2}} = \frac{v}{c}$$