Operador Tangente para Elasto-Visco-Plasticidade

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A determinação de uma expressão fechada e consistente para a matriz tangente do modelo de elasto-visco-plasticidade de Von-Mises incluindo os termos de encruamento isotrópico e cinemático dá-se através de uma perturbação de cada variável na expressão,

$$\dot{\boldsymbol{\sigma}}_{n+1} = \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}, \tag{1}$$

$$\dot{\boldsymbol{\sigma}}_{n+1} = \underbrace{2G\dot{\boldsymbol{\varepsilon}}^{tr}_{d_{n+1}}}_{(1)} - \underbrace{2G\dot{\Delta\gamma}\boldsymbol{N}^{tr}_{n+1}}_{(3)} + \underbrace{2G\Delta\gamma\dot{\boldsymbol{N}}^{tr}_{n+1}}_{(2)} + \underbrace{\dot{\boldsymbol{p}}^{tr}_{n+1}\mathbf{I}}_{(4)}. \tag{2}$$

Assumindo que a deformação no estado $n+1$ é dependente de um "tempo relativo", e que as taxas de deformação dependem linearmente de uma medida de tensão,

$$\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_{n+1}(t), \tag{3}$$

$$\dot{\Delta\gamma} = \frac{1}{\mu}\left[\left(\frac{q}{\sigma_y}\right)^{-\epsilon} - 1\right], \tag{4}$$

onde μ e ϵ são constantes do material. Observa-se que o termo em colchetes naa Eq. (4) consiste em uma medida de tensão.

A diferenciação do primeiro termo da Eq. (2) inicia-se por,

$$\dot{\boldsymbol{\varepsilon}}^{tr}_{d_{n+1}} = \dot{\boldsymbol{\varepsilon}}_{d_{n+1}}, \tag{5}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \overline{\boldsymbol{\varepsilon}_{n+1} - \frac{1}{3}tr(\boldsymbol{\varepsilon}_{n+1})\mathbf{I}}^{\,\cdot}, \tag{6}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \frac{\partial \boldsymbol{\varepsilon}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} : \dot{\boldsymbol{\varepsilon}}_{n+1} - \frac{1}{3}\left(\frac{\partial(\mathbf{I} : \boldsymbol{\varepsilon}_{n+1})}{\partial \boldsymbol{\varepsilon}_{n+1}} : \dot{\boldsymbol{\varepsilon}}_{n+1}\right)\mathbf{I}, \tag{7}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \mathbb{I} : \dot{\boldsymbol{\varepsilon}}_{n+1} - \frac{1}{3}(\mathbb{I} : \mathbf{I} : \dot{\boldsymbol{\varepsilon}}_{n+1})\mathbf{I}, \tag{8}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \mathbb{I} : \dot{\boldsymbol{\varepsilon}}_{n+1} - \frac{1}{3}(\mathbf{I} : \dot{\boldsymbol{\varepsilon}}_{n+1})\mathbf{I}, \tag{9}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \mathbb{I} : \dot{\boldsymbol{\varepsilon}}_{n+1} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I}) : \dot{\boldsymbol{\varepsilon}}_{n+1}, \tag{10}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \left[\mathbb{I} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I})\right] : \dot{\boldsymbol{\varepsilon}}_{n+1}, \tag{11}$$

$$\dot{\boldsymbol{\varepsilon}}_{d_{n+1}} = \mathbb{P} : \dot{\boldsymbol{\varepsilon}}_{n+1}. \tag{12}$$

A derivada do terceiro termo da Eq. (2) é tal que,

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \sqrt{\frac{3}{2}}\; \dot{\overline{\frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}}}, \tag{13}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \sqrt{\frac{3}{2}}\; \left[\frac{1}{||\boldsymbol{\eta}^{tr}_{n+1}||}\left(\mathbb{I} - \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||} \otimes \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}\right) : \dot{\boldsymbol{\eta}}^{tr}_{n+1}\right], \tag{14}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \sqrt{\frac{3}{2}}\frac{1}{||\boldsymbol{\eta}^{tr}_{n+1}||}\; \left[\left(\mathbb{I} - \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||} \otimes \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}\right) : \left(\dot{\boldsymbol{S}}^{tr}_{n+1} - \dot{\boldsymbol{\beta}}^{tr}_{n+1}\right)\right], \tag{15}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \sqrt{\frac{3}{2}}\frac{1}{||\boldsymbol{\eta}^{tr}_{n+1}||}\; \left[\left(\mathbb{I} - \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||} \otimes \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}\right) : 2G\; \mathbb{P} : \boldsymbol{\varepsilon}^{\dot{tr}}_{n+1}\right], \tag{16}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \sqrt{\frac{3}{2}}\frac{1}{||\boldsymbol{\eta}^{tr}_{n+1}||}\; \left[\left(\mathbb{I} - \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||} \otimes \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}\right) : 2G\; \mathbb{P} : \boldsymbol{\varepsilon}^{\dot{tr}}_{n+1}\right], \tag{17}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = 2G\sqrt{\frac{3}{2}}\frac{1}{||\boldsymbol{\eta}^{tr}_{n+1}||}\; \left[\left(\mathbb{I} - \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||} \otimes \frac{\boldsymbol{\eta}^{tr}_{n+1}}{||\boldsymbol{\eta}^{tr}_{n+1}||}\right) :\; \left(\mathbb{I} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I})\right) : \boldsymbol{\varepsilon}^{\dot{tr}}_{n+1}\right], \tag{18}$$

$$\dot{\boldsymbol{N}}^{tr}_{n+1} = \frac{3G}{q^{tr}_{n+1}}\; \left(\mathbb{I} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I}) - \frac{2}{3}\left(\boldsymbol{N}^{tr}_{n+1} \otimes \boldsymbol{N}^{tr}_{n+1}\right)\right) : \boldsymbol{\varepsilon}^{\dot{tr}}_{n+1}. \tag{19}$$

A derivada do segundo termo da Eq. (2) é deduzida a partir de,

$$\Delta\gamma = \frac{\Delta t}{\mu}\left[\left(\frac{q_{n+1}}{\sigma_y(\alpha_{n+1})}\right)^{\frac{1}{\epsilon}} - 1\right], \tag{20}$$

$$\Delta\gamma = \frac{\Delta t}{\mu}\left[\left(\frac{\sigma_y(\alpha_{n+1})}{q_{n+1}}\right)^{\epsilon} - 1\right], \tag{21}$$

$$\frac{\mu\Delta\gamma}{\Delta t} + 1 = \left(\frac{\sigma_y(\alpha_{n+1})}{q_{n+1}}\right)^{\epsilon}, \tag{22}$$

$$\left(\frac{\mu\Delta\gamma}{\Delta t}+1\right)^{-\epsilon}=\frac{\sigma_y(\alpha_{n+1})}{q_{n+1}}, \tag{23}$$

$$q_{n+1}\left(\frac{\mu\Delta\gamma}{\Delta t}+1\right)^{-\epsilon}-\sigma_y(\alpha_{n+1})=0, \tag{24}$$

$$q_{n+1}\left(\frac{\mu\Delta\gamma+\Delta t}{\Delta t}\right)^{-\epsilon}-\sigma_y(\alpha_{n+1})=0, \tag{25}$$

$$(q^{tr}_{n+1}-(3G+H)\Delta\gamma)\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}-\sigma_y(\alpha_{n+1})=0, \tag{26}$$

$$q^{tr}_{n+1}\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}-(3G+H)\Delta\gamma\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}-\sigma_y(\alpha_{n+1})=0. \tag{27}$$

Diferenciando a Eq. (27),

$$\dot{\overline{q^{tr}_{n+1}\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}}}-\dot{\overline{(3G+H)\Delta\gamma\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}}}-\dot{\overline{\sigma_y(\alpha_{n+1})}}=0, \tag{28}$$

$$\begin{split}\dot{\overline{q^{tr}_{n+1}}}\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}+q^{tr}_{n+1}\dot{\overline{\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}}}-(3G+H)\dot{\Delta\gamma}\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}\\-(3G+H)\Delta\gamma\dot{\overline{\left(\frac{\Delta t}{\mu\Delta\gamma+\Delta t}\right)^{\epsilon}}}-\frac{\partial\sigma_y}{\partial\alpha}\dot{\Delta\gamma}=0,\end{split} \tag{29}$$

Diferenciando os termos da Eq. (29) de forma separada, de modo a facilitar os algebrismos,

$$\dot{q}^{tr}_{n+1}=\boldsymbol{N}^{tr}_{n+1}:2G\,\mathbb{P}:\dot{\boldsymbol{\varepsilon}_{n+1}}, \tag{30}$$

$$\dot{\overline{\left(\frac{\Delta t}{\Delta\gamma\mu+\Delta t}\right)}}=-\left(\frac{\epsilon\mu}{\Delta t}\right)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon-1}\dot{\Delta\gamma}. \tag{31}$$

Substituindo na Eq. (29),

$$\begin{split}2G\boldsymbol{N}^{tr}_{n+1}:\dot{\boldsymbol{\varepsilon}}_{n+1}\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon}+q^{tr}_{n+1}\left(-\left(\frac{\epsilon\mu}{\Delta t}\right)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon-1}\dot{\Delta\gamma}\right)-\\(3G+H)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon}\dot{\Delta\gamma}-(3G+H)\,\Delta\gamma\left(\frac{-\epsilon\mu}{\Delta t}\right)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon-1}\dot{\Delta\gamma}-\\K(\alpha)\dot{\Delta\gamma}=0.\end{split} \tag{32}$$

Dividindo a equação por $\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{-\epsilon}$ e isolando $\dot{\Delta\gamma}$,

$$\dot{\Delta\gamma} = \frac{2G\boldsymbol{N}^{tr}_{n+1}}{\left(\frac{\epsilon\mu}{\Delta\gamma\mu+\Delta t}\right)\left(q^{tr}_{n+1}-(3G+H)\right)+(3G+H)+K(\alpha)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{\epsilon}} : \dot{\boldsymbol{\varepsilon}}_{n+1}, \quad (33)$$

$$\dot{\Delta\gamma} = \frac{2G\boldsymbol{N}^{tr}_{n+1}}{q_{n+1}\left(\frac{\epsilon\mu}{\Delta\gamma\mu+\Delta t}\right)+(3G+H)+K(\alpha)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{\epsilon}} : \dot{\boldsymbol{\varepsilon}}_{n+1}. \quad (34)$$

A derivada do último termo da Eq. (2) é dada por,

$$p_{n+1} = \kappa\ tr(\varepsilon_{n+1})\mathbf{I}, \quad (35)$$

$$\dot{p}_{n+1} = \kappa\ \dot{\overline{tr(\varepsilon_{n+1})\mathbf{I}}}, \quad (36)$$

$$\dot{p}_{n+1} = \kappa\ \left(\frac{\partial\left(\mathbf{I}:\boldsymbol{\varepsilon}_{n+1}\right)}{\partial\boldsymbol{\varepsilon}_{n+1}} : \dot{\boldsymbol{\varepsilon}}_{n+1}\right), \quad (37)$$

$$\dot{p}_{n+1} = \kappa\ \left(\mathbb{I}:\mathbf{I}:\dot{\boldsymbol{\varepsilon}}_{n+1}\right)\mathbf{I}, \quad (38)$$

$$\dot{p}_{n+1} = \kappa\ \left(\mathbf{I}\otimes\mathbf{I}\right) : \dot{\boldsymbol{\varepsilon}}_{n+1}. \quad (39)$$

Substituindo todas as derivadas na Eq. (2),

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_{n+1} = 2G\mathbb{P}:\dot{\boldsymbol{\varepsilon}}_{n+1} - 2G\frac{2G\boldsymbol{N}^{tr}_{n+1}}{q_{n+1}\left(\frac{\epsilon\mu}{\Delta\gamma\mu+\Delta t}\right)+(3G+H)+K(\alpha)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{\epsilon}} : \dot{\boldsymbol{\varepsilon}}_{n+1}\boldsymbol{N}^{tr}_{n+1} + \\ 2G\Delta\gamma\frac{3G}{q^{tr}_{n+1}}\ \left(\mathbb{I}-\frac{1}{3}(\mathbf{I}\otimes\mathbf{I})-\frac{2}{3}\left(\boldsymbol{N}^{tr}_{n+1}\otimes\boldsymbol{N}^{tr}_{n+1}\right)\right) : \dot{\boldsymbol{\varepsilon}^{tr}_{n+1}} + \kappa\ (\mathbf{I}\otimes\mathbf{I}):\dot{\boldsymbol{\varepsilon}}_{n+1}\mathbf{I},\end{aligned} \quad (40)$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_{n+1} = 2G\mathbb{P}:\dot{\boldsymbol{\varepsilon}}_{n+1} - 2G\frac{2G\boldsymbol{N}^{tr}_{n+1}}{q_{n+1}\left(\frac{\epsilon\mu}{\Delta\gamma\mu+\Delta t}\right)+(3G+H)+K(\alpha)\left(\frac{\Delta\gamma\mu+\Delta t}{\Delta t}\right)^{\epsilon}} : \dot{\boldsymbol{\varepsilon}}_{n+1}\boldsymbol{N}^{tr}_{n+1} + \\ 2G\Delta\gamma\frac{3G}{q^{tr}_{n+1}}\ \left(\mathbb{P}-\frac{2}{3}\left(\boldsymbol{N}^{tr}_{n+1}\otimes\boldsymbol{N}^{tr}_{n+1}\right)\right) : \dot{\boldsymbol{\varepsilon}^{tr}_{n+1}} + \kappa\ (\mathbf{I}\otimes\mathbf{I}):\dot{\boldsymbol{\varepsilon}}_{n+1}\mathbf{I}.\end{aligned} \quad (41)$$

Agrupando os termos em comum, e diferenciando em relação a $\boldsymbol{\varepsilon}_{n+1}$, obtêm-se uma equação para $\mathbb{D}^{EVP}$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \mathbb{D}^{EVP} = 2G\left(1 - \frac{3G\Delta\gamma}{q^{tr}_{n+1}}\right)\mathbb{P} + \\
6G^2 \left[\frac{\Delta\gamma}{q^{tr}_{n+1}} - \frac{1}{\left(\frac{q_{n+1}\epsilon\mu}{\mu\Delta\gamma + \Delta t}\right) + (3G + H) + K(\alpha)\left(\frac{\mu\Delta\gamma + \Delta t}{\Delta t}\right)^{\epsilon}}\right]\left(\boldsymbol{N}^{tr}_{n+1} \otimes \boldsymbol{N}^{tr}_{n+1}\right) + \\
\kappa \, \left(\mathbf{I} \otimes \mathbf{I}\right).
\end{aligned}
\tag{42}
$$