Scattering of a Plane Wave from a Rigid Sphere

We seek the resultant pressure wave field in Spherical coordinates. Following the analysis of Beranek and Mellow

$$\tilde{p}(r,\theta) = \tilde{p}_I(r,\theta) + \tilde{p}_S(r,\theta) \tag{1}$$

where I is the incident wave and s is the scattered wave.

Notebook Imports and Configuration

```
%pylab inline
#%pylab notebook
#%matplotlib qt
import sk_dsp_comm.sigsys as ss
import scipy.signal as signal
import scipy.special as special
from IPython.display import Audio, display
from IPython.display import Image, SVG
```

```
Populating the interactive namespace from numpy and matplotlib
```

Figure Rendering Options

```
pylab.rcParams['savefig.dpi'] = 100 # default 72
#pylab.rcParams['figure.figsize'] = (6.0, 4.0) # default (6,4)
#%config InlineBackend.figure_formats=['png'] # default for inline viewing
%config InlineBackend.figure_formats=['svg'] # SVG inline viewing
#%config InlineBackend.figure_formats=['pdf'] # render pdf figs for LaTeX
```

Define the Spherical Hankel Functions

```
def spherical_hn1(n,z,derivative=False):
    """ Spherical Hankel Function of the First Kind """
    return special.spherical_jn(n,z,derivative=False)+1j \
           *special.spherical_yn(n,z,derivative=False)

def spherical_hn2(n,z,derivative=False):
    """ Spherical Hankel Function of the Second Kind """
    return special.spherical_jn(n,z,derivative=False)-1j \
           *special.spherical_yn(n,z,derivative=False)
```

The Incident Plane Wave Expansion

We start with a spherical harmonics expansion of a plane wave source:

$$\tilde{p}_I(r,\theta) = \tilde{p}_0 \sum_{n=0}^{\infty} (-j)^n (2n+1) j_n(kr) P_n\big(\cos(\theta)\big) \tag{2}$$

where the wavenumber k is given by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c/f} = \frac{2\pi f}{c} \text{ radians/m} \tag{3}$$

where c is the speed of sound in air, given by 344.4 m/s at an ambient pressure of 10^5 Pa and at 22° C. The variable f is frequency in Hz. The special functions are spherical Bessel functions of the first king, $j_n(kr)$, and the Legendre functions $P_n^m(\cos(\theta))|_{m=0} = P_n(\cos(\theta))$.

```
def pI(w, z, f, R = 1, N = 50):
    """

    Mark Wickert November 2018
    """
    p_0 = 1
    k = 2*pi/(344.4/f)
    p_polar = zeros((len(z),len(w)),dtype=complex128)
    for n,wn in enumerate(w):
        for m,zm in enumerate(z):
            r = sqrt(zm**2 + wn**2)
            cos_theta = zm/sqrt(zm**2 + wn**2)
            for kk in range(N+1):
                if r <= R:
                    p_polar[m,n] = 0.0
                else:
                    p_polar[m,n] += p_0*(-1j)**kk * (2*kk+1) \
                    * special.spherical_jn(kk,k*r) \
                    * special.lpmv(0,kk,cos_theta)
    return p_polar
```

The Scattered Field from the Sphere

Next we move on to finding the scattered field:

$$\tilde{p}_S(r,\theta) = -\tilde{p}_0 \sum_{n=0}^{\infty} (-j)^n (2n+1) \frac{j_n'(kR)}{h_n'^{(2)}(kR)} h_n^{(2)}(kr) P_n\big(\cos(\theta)\big) \tag{4}$$

where R is the sphere radius. The new special functions are spherical Hankel functions of the second kind, $h_n^{(2)}(kr)$, and their derivatives, $h_n'^{(2)}(kr)$.

```
def pS(w, z, f, R = 1, N = 50):
    """

    Mark Wickert November 2018
    """
    p_0 = 1
    k = 2*pi/(344.4/f)
```

```
    p_polar = zeros((len(z),len(w)),dtype=complex128)
    for n,wn in enumerate(w):
        for m,zm in enumerate(z):
            r = sqrt(zm**2 + wn**2)
            cos_theta = zm/sqrt(zm**2 + wn**2)
            for kk in range(N+1):
                if r <= R:
                    p_polar[m,n] = 0.0
                else:
                    p_polar[m,n] += p_0*(-1j)**kk * (2*kk+1) \
                    * special.spherical_jn(kk,k*R,True)/\
                    spherical_hn2(kk,k*R,True) \
                    * spherical_hn2(kk,k*r) \
                    * special.lpmv(0,kk,cos_theta)
    return -p_polar
```

The Resultant Sound Pressure Over a Rectangular Region

Get a view of the sound pressure magnitude in a rectangular regions that includes the rigid sphere.

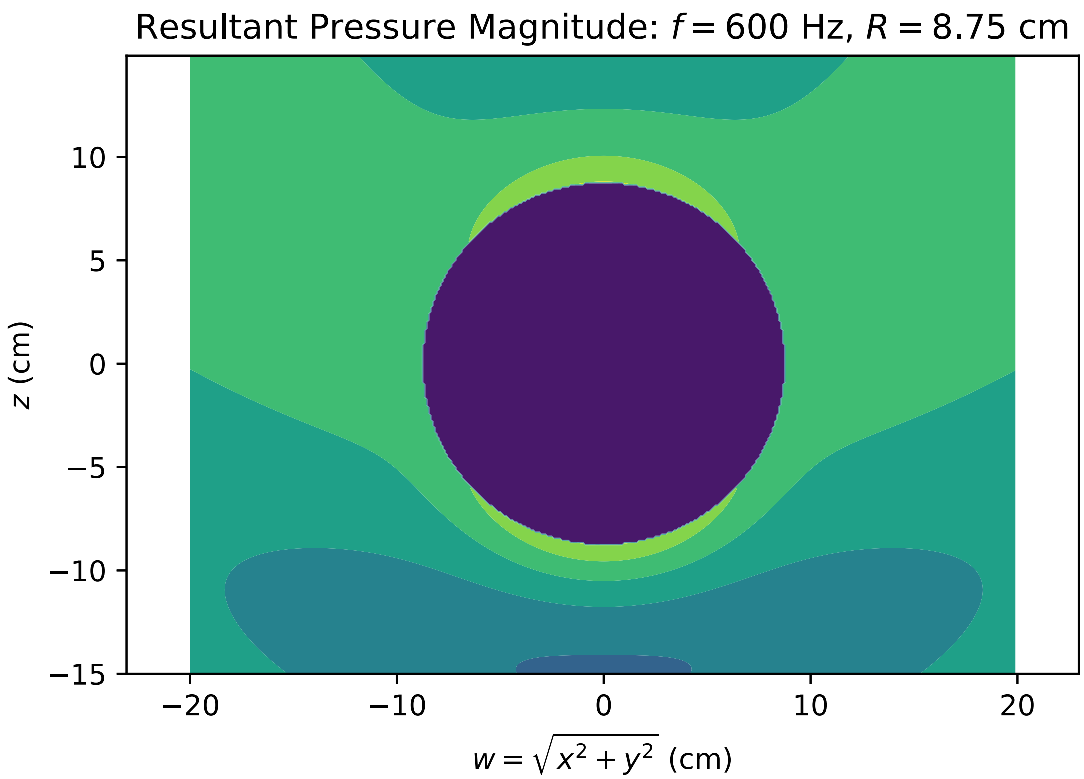
Pressure Near Spherical Head at 600 Hz

```
# Calculation is time consuming so an archive is available too
w = arange(-0.20,0.20,.001) # 005
z = arange(-0.15,0.15,0.001) #004
#pI_polar = pI(w,z,600,0.0875,50)
#pS_polar = pS(w,z,600,0.0875,50)
```

```
# Save the results of the above if desired.
#savez_compressed('p_polar_600_R875',a=w,b=z,c=pI_polar,d=pS_polar)
```

```
# Restore from the archive to make a plot
p_600_R875 = load('p_polar_600_R875.npz')
w_600_R875 = p_600_R875['a']
z_600_R875 = p_600_R875['b']
pI_600_R875_polar = p_600_R875['c']
pS_600_R875_polar = p_600_R875['d']
```

```
wM,zM = meshgrid(w,z)
contourf(wM*100,zM*100, abs(pI_600_R875_polar+pS_600_R875_polar))
title(r'Resultant Pressure Magnitude: $f=600$ Hz, $R=8.75$ cm')
ylabel(r'$z$ (cm)')
xlabel(r'$w = \sqrt{x^2 + y^2}$ (cm)')
axis('equal');
#savefig('mag_600_R375.pdf')
```



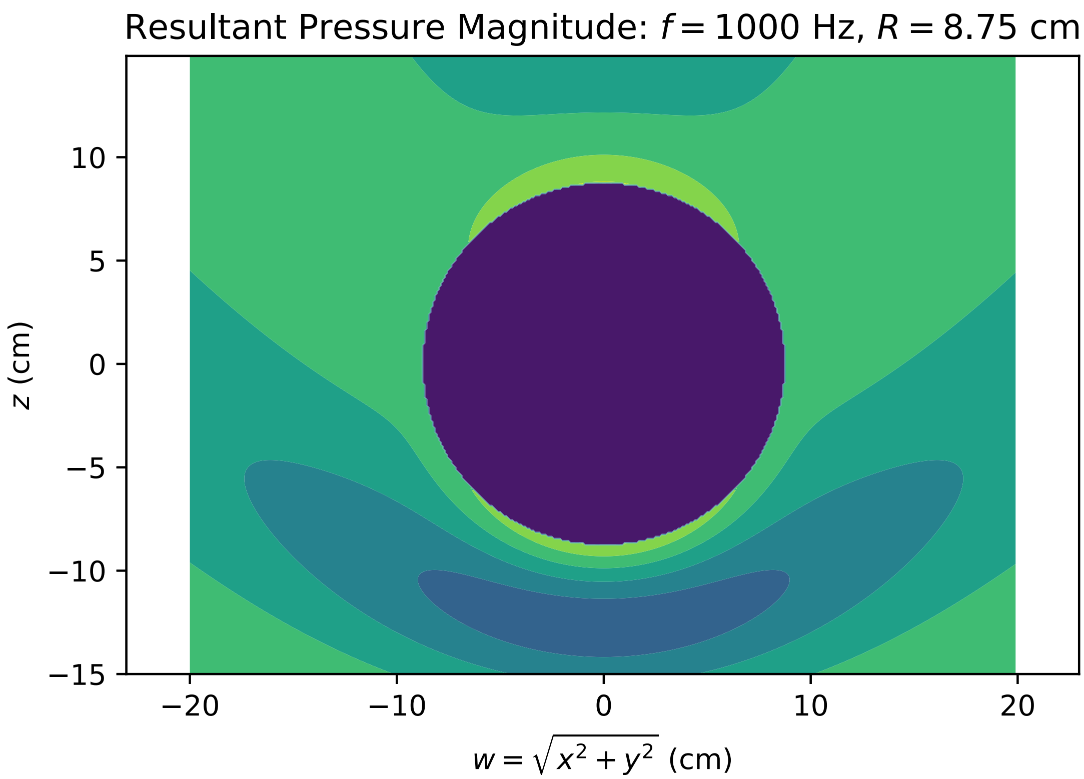
Pressure Near Spherical Head at 1000 Hz

```
w = arange(-0.20,0.20,.001) # 005
z = arange(-0.15,0.15,0.001) #004
# pI_polar = pI(w,z,1000,0.0875,50)
# pS_polar = pS(w,z,1000,0.0875,50)
```

```
savez_compressed('p_polar_1000_R875',a=w,b=z,c=pI_polar,d=pS_polar)
```

```
p_1000_R875 = load('p_polar_1000_R875.npz')
w_1000_R875 = p_600_R875['a']
z_1000_R875 = p_600_R875['b']
pI_1000_R875_polar = p_1000_R875['c']
pS_1000_R875_polar = p_1000_R875['d']
```

```
wM,zM = meshgrid(w,z)
contourf(wM*100,zM*100, abs(pI_1000_R875_polar+pS_1000_R875_polar))
title(r'Resultant Pressure Magnitude: $f=1000$ Hz, $R=8.75$ cm')
ylabel(r'$z$ (cm)')
xlabel(r'$w = \sqrt{x^2 + y^2}$ (cm)')
axis('equal');
#savefig('mag_1000_R375.pdf')
```



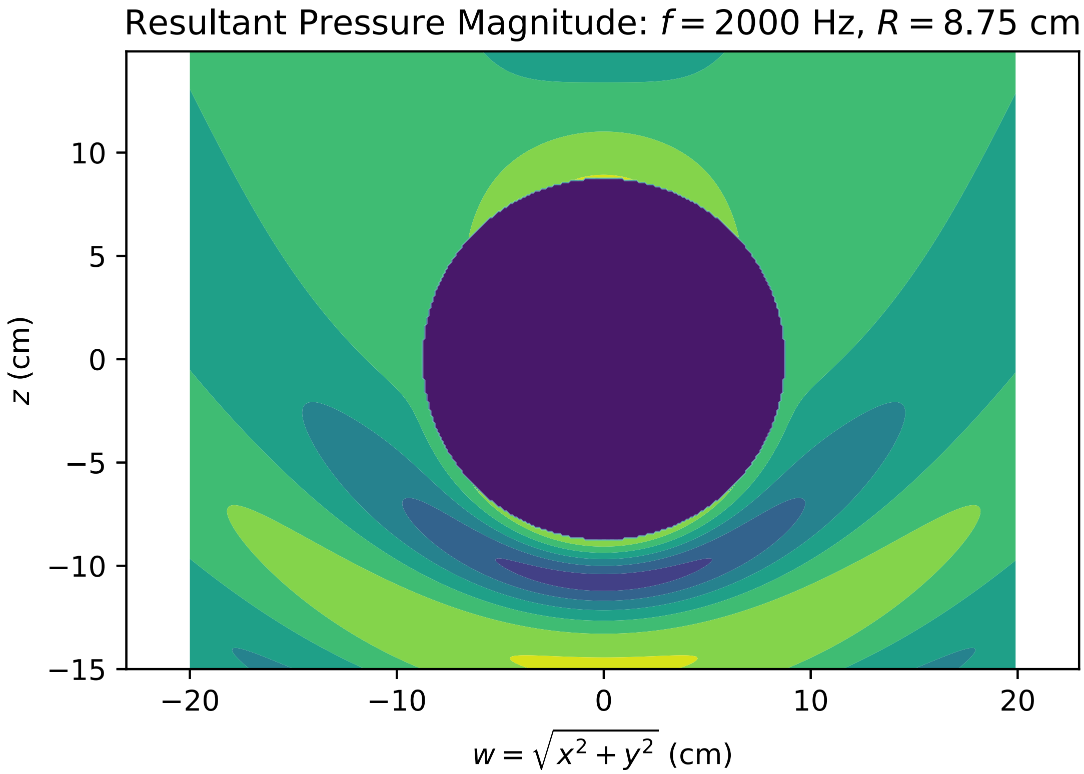
Pressure Near Spherical Head at 2000 Hz

```
w = arange(-0.20,0.20,.001) # 005
z = arange(-0.15,0.15,0.001) #004
# pI_polar = pI(w,z,2000,0.0875,50)
# pS_polar = pS(w,z,2000,0.0875,50)
```

```
savez_compressed('p_polar_2000_R875',a=w,b=z,c=pI_polar,d=pS_polar)
```

```
p_2000 = load('p_polar_2000_R875.npz')
w_2000 = p_2000['a']
z_2000 = p_2000['b']
pI_2000_R875_polar = p_2000['c']
pS_2000_R875_polar = p_2000['d']
```

```
wM,zM = meshgrid(w,z)
contourf(wM*100,zM*100, abs(pI_2000_R875_polar+pS_2000_R875_polar))
title(r'Resultant Pressure Magnitude: $f=2000$ Hz, $R=8.75$ cm')
ylabel(r'$z$ (cm)')
xlabel(r'$w = \sqrt{x^2 + y^2}$ (cm)')
axis('equal');
#savefig('mag_2000_R375.pdf',transparent=True)
```


Resultant Pressure Magnitude: $f = 2000$ Hz, $R = 8.75$ cm
10
5
0
−5
−10
−15
z (cm)
−20
−10
0
10
20
$w = \sqrt{x^2 + y^2}$ (cm)