A First Look at Fractals

Michael Wieck-Sosa
Professor Yuliy Baryshnikov
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Abstract

In this exposé, we will introduce the theory of fractals. There are five sections. First, we will introduce space-filling curves. Second, we will introduce basic definitions and theorems about fractals. Third, we will give examples of the mathematical construction, dimensions, and visualizations of several fractals. Fourth, we will give two examples where fractals emerge in probability theory. Most of the results and definitions about fractals were taken from Chapter 9 of Hans Sagan's *Space-Filling Curves*, while other supporting definitions were taken from various sources found online. Minor details and clarifications were added to proofs when the authors thought them to be helpful. Fifth, we conclude with a list of suggested further readings for those interested in the geometry of fractals and their intersections with probability theory. The expected level of mathematical background for this material is an undergraduate course in elementary mathematical analysis and some exposure to basic measure-theoretic probability.

1 Introduction to Space-Filling Curves

What are space-filling curves, and why do mathematicians study them? In the late 19th century, George Cantor proved that two finite-dimensional smooth manifolds have the same cardinality, which implies that there exists a bijection between $[0, 1]$ and $[0, 1] \times [0, 1]$.

This discovery opened the door to many questions, such as:
1. Is this bijection continuous?
2. Is there a curve that passes through every point of $[0, 1] \times [0, 1]$?
3. Can $[0, 1]$ be mapped bijectively and continuously to a region with positive outer measure?

First, to understand what a space-filling curve is, we will start with some basic definitions.

Definition If $f : [0, 1] \longrightarrow (\mathbb{R}^n, d)$ is a continuous function where $[0, 1] \subset \mathbb{R}$ and d is the Euclidean norm, i.e. $\|x\|_2 = d((x_1, ..., x_n)) = \sqrt{x_1^2 + ... + x_n^2}$, then the image $f([0, 1])$ is called a curve, denoted by C, where $f(0)$ is the beginning point and $f(1)$ is the endpoint.

Definition $f(t) = x, t \in [0, 1]$ is a parametric representation of the curve $C = f[0, 1]$.

Definition J_n is the n-dimensional Jordan content of a Jordan measurable subset $(\mathbb{R}^n, d)$.

Example 1 $J_1([0, 1]) = 1$, $J_2([0, 1]^2) = 1$, and $J_3([0, 1]^3) = 1$ are the Jordan contents of dimension 1, 2, and 3, respectively. In English, these are typically called the length, area, and volume of the unit length line, unit square, and unit cube, respectively.

Definition $f([0,1])$ is called a space-filling curve if $f : [0,1] \longrightarrow (\mathbb{R}^n, d)$ is continuous and $J_n(f([0,1])) > 0$.

Now, what on earth do space-filling curves have to do with fractals?

2 Mathematical Background for Fractals

You have likely encountered images of fractals before, but you may not have made the connection between fractals and space-filling curves. Indeed, the explosion of interest in fractals in the last half-century has shed new light on space-filling curves. As we will show later, fractals can be understood as a general framework for generating space-filling curves through recursion.

In this section, we will introduce the definition of fractals and precise notions of distance and convergence for complete metric spaces so that we can prove one of the jewels of the theory of fractals. That is, for each function system that generates a fractal, there exists a unique attractor set for that function system with a non-integer fractal dimension (Hausdorff dimension) that exceeds its topological dimension (Lebesgue covering dimension).

First, we will start off with some preliminary definitions such that the theorems proven below and the definition of fractals (in terms of their topological and fractal dimensions) can be appreciated.

Definition[1] A topological space is a set X equipped with a collection of open subsets T such that:
1. $\emptyset \in T$
2. $X \in T$
3. $O_1, O_2, ..., O_n \in T \implies \bigcap_{i=1}^{n} O_i \in T$
4. $O_1, O_2, ... \in T \implies \bigcup_{i=1}^{\infty} O_i \in T$

Definition[2,3] An open cover $\mathcal{C} = \{O_i\}_{i \in I}$ of a topological space X equipped with the topology T is a collection of subsets $O_i \in T$ such that $X = \bigcup_{i \in I} O_i$. If $Y \subset X$, then an open cover $\mathcal{C} = \{O_i\}_{i \in I}$ of Y is a collection of subsets $O_i \in T$ such that $Y \subseteq \bigcup_{i \in I} O_i$.

Definition[4,5] A refinement of an open cover $\mathcal{C} = \{O_i\}_{i \in I}$ of a topological space X is an open cover $\mathcal{D} = \{U_j\}_{j \in J}$ such that $\forall j \in J, \exists i \in I$ such that $U_j \subseteq O_i$.

[1] Lipp, Johannes and Weisstein, Eric W. "Topological Space." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/TopologicalSpace.html

[2] Barile, Margherita. "Open Cover." From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. https://mathworld.wolfram.com/OpenCover.html

[3] Wikipedia contributors. "Cover (topology)." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 17 Feb. 2021. Web. 14 May. 2021.

[4] Ibid.

[5] Refinement. Encyclopedia of Mathematics. URL: http://encyclopediaofmath.org/index.php?title=Refinementoldid=51516

Definition[6,7,8,9] A subset S of a topological space X has Lebesgue covering dimension m, denoted $\dim_L(S)$, if all open covers $\mathcal{C} = \{O_i\}_{i\in I}$ have a refinement $\mathcal{D} = \{U_j\}_{j\in J}$ such that no $x \in S$ is in more than $m + 1$ sets U_j in the refinement $\mathcal{D}$ and if m is the minimum value for which this is true. If there does not exist a minimal m, then S is said to have infinite Lebesgue covering dimension.

Example[10] The circle with radius r centered at the origin

$$S^1(r) = \{x \in \mathbb{R}^2 : \|x\| = r\}$$

has Lebesgue covering dimension 1 because no point is contained in more than two sets in the refinement.

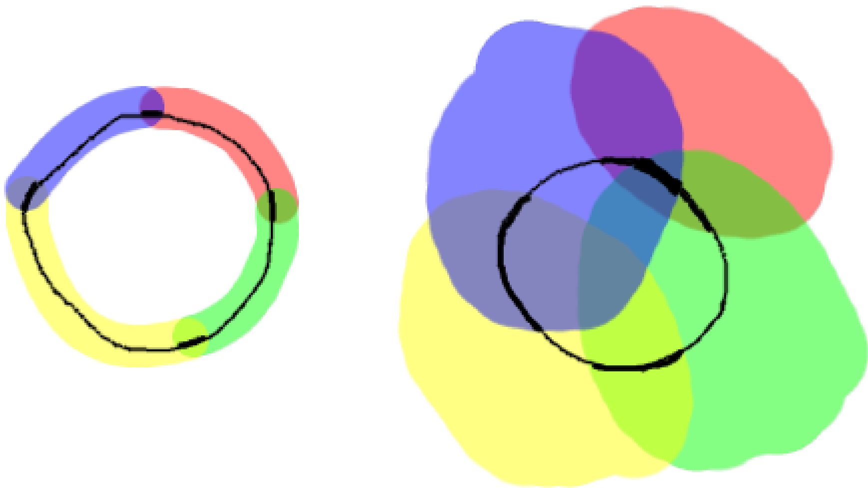

Figure 1: Left Image: Refinement, Right Image: Open cover of the circle

Example[11] The filled-in square with center (a, b) and side length r

$$R(r) = \{(x, y) \in \mathbb{R}^2 : |(x - a) + (y - b)| + |(x - a) - (y - b)| \le r\}$$

has Lebesgue covering dimension 1 because no point is contained in more than two sets in the refinement.

[6] Iyanaga, S. and Kawada, Y. (Eds.). Encyclopedic Dictionary of Mathematics. Cambridge, MA: MIT Press, p. 414, 1980.

[7] Munkres, J. R. Topology: A First Course, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 2000.

[8] Weisstein, Eric W. "Lebesgue Covering Dimension." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/LebesgueCoveringDimension.html

[9] Wikipedia contributors. "Cover (topology)." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 17 Feb. 2021. Web. 14 May. 2021.

[10] Wikipedia contributors. "Lebesgue covering dimension." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 13 May. 2021. Web. 14 May. 2021.

[11] Ibid.

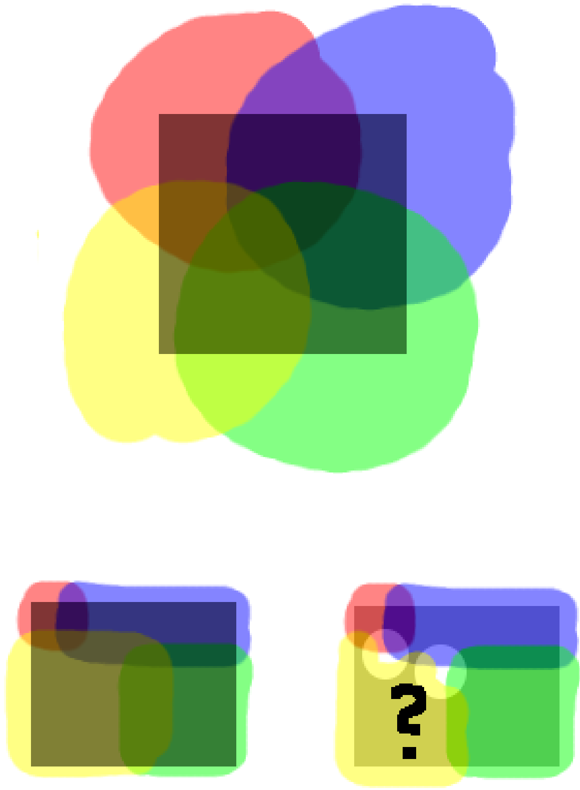

Figure 2: Bottom Image: Refinement, Top Image: Open cover of the square

Example $\mathbb{R}^n$ has Lebesgue covering dimension n because no point is contained in more than $n + 1$ sets in the refinement of any open cover.

Next, we will introduce the Hausdorff dimension. Afterwards, we will be ready to understand what a fractal is in terms of the topological and Hausdorff dimensions, which the reader will not forget is what motivates the introduction of these definitions in the first place.

Definition[12] A metric d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ such that $\forall x, y \in X$:
1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

These are often referred to as the distance axioms.

Definition[13] A metric space is a set X equipped with a metric d on X, denoted (X, d).

Definition[14,15] Let X be a metric space. If $S \subset X$ and $d \in [0, \infty)$, then the d-dimensional unlimited Hausdorff content of S is the infimum of the set of numbers $\delta \geq 0$ such that there

[12] Wikipedia contributors. "Metric (mathematics)." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 7 May. 2021. Web. 14 May. 2021.

[13] Wikipedia contributors. "Metric space." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 8 May. 2021. Web. 14 May. 2021.

[14] Wikipedia contributors. "Hausdorff dimension." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 11 Jan. 2021. Web. 14 May. 2021.

[15] Wikipedia contributors. "Ball (mathematics)." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 27 Jan. 2021. Web. 14 May. 2021.

is a collection of balls $B(p_i, r_i) = \{x_i \in S \mid d(x, p_i) < r_i\}$ covering S with radii $r_i > 0$ such that $\forall i \in I$ we have $\sum_{i \in I} r_i^d < \delta$. Formally, we have:

$$C_H^d(S) := \inf\{\sum_i r_i^d \mid \exists C = \{B(r_i, p_i)\}_{i \in I}, S \subseteq \bigcup_{i \in I} B(r_i, p_i), r_i > 0\}$$

Note: for the next definition, we will use the term outer measure. Without getting bogged down in introducing basic measure theory (sigma-algebras, measures, etc) we will point the reader unfamiliar with these definitions to Chapter 1 of Folland's *Real Analysis*. For now, readers unfamiliar with basic measure theory can think of a measure as a way of measuring objects of d dimensions. In one and two dimensions, you will almost surely have encountered the Lebesgue measure, namely $\ell([a, b]) = b - a$ and $\ell([a, b] \times [c, d]) = (b - a)(d - c)$.

Definition[16] The Hausdorff outer measure is similar to the Hausdorff content, but bounding the radii $r_i < r$ as $r \longrightarrow 0$. For $d \geq 0$, we define the d-dimensional Hausdorff outer measure of a subset S of a metric space X as

$$\mathcal{H}^d(S) := \lim\inf_{r \longrightarrow 0}\{\sum_{i \in I} r_i^d \mid \exists C = \{B(r_i, p_i)\}_{i \in I}, S \subseteq \bigcup_{i \in I} B(r_i, p_i), 0 < r_i < r\}$$

Definition[17,18] The Hausdorff dimension, or fractal dimension, of a subset S of a metric space X is defined as

$$\dim_H(S) := \inf\{d \geq 0 \mid \mathcal{H}^d(S) = 0\}$$

Definition[19] A subset S of $\mathbb{R}^n$ is self-similar, or affine, if a subset S' of S is mapped to S by a self-similarity transformation of the form

$$f(x) = Ax + b$$

where A is an $n \times n$ invertible matrix and b is an n dimensional vector.

Definition[20] A fractal is a subset S of $\mathbb{R}^n$ that is self-similar and has a fractal dimension $\dim_H(S)$ exceeds its topological dimension $\dim_L(S)$.

In the next section, we will give several examples of fractals with non-integer dimensions with images below. Specifically, we will give an exposition of the Cantor set, the von Koch Curve, the Sierpiński Triangle, the Mazurkiewicz Continuum (the Sierpiński Carpet), and the Heighway Dragon.

In general, a self-similar fractal is generated by iterated similarity transformations of the form

$$f(x) = x' = b_i A_i x + c_i$$

where $i \in \mathbb{N}, x, x', b_i, c_i \in (\mathbb{R}^m, d), b_i \in (0, 1)$, and the A_i are orthogonal matrices. In recursive fashion, at the i-th iteration, apply the similarity transformation to the union of the images from the $(i - 1)$-th iteration. This iterative process will be more rigorously defined next.

[16] Ibid.

[17] Wikipedia contributors. "Hausdorff dimension." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 11 Jan. 2021. Web. 14 May. 2021.

[18] Wikipedia contributors. "Ball (mathematics)." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 27 Jan. 2021. Web. 14 May. 2021.

[19] Carberry, Emma. Lecture 11: Fractals and Dimension. https://ocw.mit.edu/courses/mathematics/18-091-mathematical-exposition-spring-2005/lecture-notes/lecture11part1.pdf. Spring 2005.

[20] Ibid.

It is worth noting now that this general framework of self-similar fractals converges to a compact, non-empty subset of $(\mathbb{R}^m, d)$ that is unique, called the invariant attractor set which does not depend on the initial set $A_0 \subset (\mathbb{R}^m, d)$, which we will formally introduce and prove next.

Definition An iterated function system for a set of similarity transformation is given by

$$\mathcal{F}_i(x) := b_i A_i x + c_i$$

where $\mathcal{F}_i(x) = x'$.

Definition b_i is called the reduction ratio of $\mathcal{F}_i(x)$.

Observe that

$$||\mathcal{F}_i(x) - \mathcal{F}_i(y)|| = ||b_i A_i x + c_i - b_i A_i y + c_i|| = ||b_i A_i x - b_i A_i y|| = b_i ||A_i x - A_i y|| = b_i ||x - y||$$

where the equality $b_i||A_i x - A_i y|| = b_i||x - y||$ holds because the orthogonal transformations A_i preserve length the length of x, y.

The finite union of continuous images of compact sets from the iterated function system $\mathcal{F}_i$ is initially applied to a non-empty compact set $A_0 \subset (\mathbb{R}^m, d)$. In recursive fashion, this is applied the finite union of continuous images of compact sets from the iterated function system $\mathcal{F}_i$ applied to A_j resulting in A_{j+1}, which is given by the following iterative procedure

$$A_{j+1} = \bigcup_{i=1}^{n} \mathcal{F}_i(A_j)$$

where $j = 0, 1, 2, ...$

Below, we will develop results to prove that this sequence converges to a unique limit set, called the invariant attractor set, which does not depend on the choice of the initial set A_0.

Next, we will develop a rigorous definition of the distance between two non-empty compact subsets of $(\mathbb{R}^m, d)$ in the more general terms of complete metric spaces.

Definition[21] Given a metric space (X, d), the sequence $\{x_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall m, n > N, d(x_m, x_n) < \varepsilon$.

Definition[22] (X, d) is a complete metric space if every Cauchy sequence $\{x_i\}_{i \in \mathbb{N}}$ in (X, d) converges to an element $x \in (X, d)$.

Going forward, we will denote the complete metric space by S and the set of all non-empty compact subsets of S by $K(S)$.

Note that the results we will develop below are proven for complete metric spaces, which are more general than Euclidean spaces.

Definition Let $\rho(x, y)$ be the distance from x to y in the complete metric space set S. Then x is in the δ-neighborhood of S, denoted by $N_\delta(A)$, if there exists a $y \in S$ such that $\rho(x, y) < \delta$. Formally, $\exists y \in S$ s.t. $\rho(x, y) < \delta \implies x \in N_\delta(A)$.

[21] Wikipedia contributors. "Cauchy sequence." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 1 Apr. 2021. Web. 14 May. 2021.

[22] Wikipedia contributors. "Complete metric space." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 26 Mar. 2021. Web. 14 May. 2021.

Definition The Hausdorff distance between two non-empty compact sets $A, B \subset K(S)$ is

$$d(A, B) = \inf\{\delta \mid A \subset N_\delta(B), B \subset N_\delta(A)\}$$

Lemma 1 The Hausdorff distance satisfies the distance postulates.

Proof: This simple proof is left to the reader. See Lemma 9.2.1 on page 151 of Hans Sagan's *Space-Filling Curves.*

□

Lemma 2: $d(A, B) < \varepsilon \iff A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)$

Proof: This follows from the definition of the ε-neighborhood $N_\varepsilon(\cdot)$.

□

Equipped with a definition of the distance between two non-empty compact subsets of S, which is more general than $(\mathbb{R}^m, d)$, where d is the Euclidean distance, we are prepared to show that self-similar fractals converge to a compact, non-empty subset that is unique.

Now, we will prove the existence and uniqueness of the attractor set by showing that the mapping

$$\mathbb{F}(A) = \bigcup_{i=1}^{n} \mathcal{F}_i(A)$$

where the $\mathcal{F}_i : S \longrightarrow S, i = 1, 2, ..., n$ are similarity transformations with reduction ratios $b_i \in (0, 1)$. Then, we will show that the $\mathcal{F}_i$ are contraction mappings, and use the contraction mapping theorem to establish the existence and uniqueness of the attractor set. First, we will introduce several results that will be helpful to prove the contraction mapping theorem.

Lemma 3 If $A, B \subseteq K(S)$ and $d(A, B) < \varepsilon$ then for any fixed $a \in A$, $\exists b \in B$ such that $\rho(a, b) < \varepsilon$.

Proof: By hypothesis $\inf\{\delta \mid A \subseteq N_\delta(B), B \subseteq N_\delta(A)\} < \varepsilon$. Thus, $a \in A \subseteq N_\varepsilon(B)$, thus there is some $b \in B$ such that $\rho(a, b) < \varepsilon$.

To show that $K(S)$ is complete we must show that every Cauchy sequence $\{A_n\}$ converges to a non-empty compact subset of S. The limit set will consist of all Cauchy sequences of points from S that can be created by picking the nth element from A_n where

$$A = \{x \in S \mid x \text{ is the limit sequence of } \{X_n\}, x_n \in A_n, n \in \mathbb{N}\}$$

□

Lemma 4 If $\{A_n\}$ is a Cauchy sequence in the metric space $K(S)$ where d is the metric defined by the Hausdorff distance, then there exists a sequence $\{x_n\} \longrightarrow x \in S, x_n \in A_n$.

Proof: If $\{A_n\}$ is a Cauchy sequence in $K(S)$ then $\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $d(A_n, A_m) < \varepsilon, \forall m, n > N_\varepsilon$. Thus, we can pick an increasing sequence $N_1 < ... < N_j < ...$ s.t.

$$d(A_n, A_{N_j}) < \varepsilon/2^j \forall n > N_j, j \in \mathbb{N}$$

Pick $x_{N_1} \in A_{N_1}$. Then $d(A_{N_1}, A_{N_2}) < \varepsilon/2$, and by the Lemma 3, $\exists x_{N_2} \in A_{N_2}$ s.t. $\rho(x_{N_1}, x_{N_2}) < \varepsilon$, and so on for all $x_{N_{j-1}} \in A_{N_{j-1}}$ there exists a $x_{N_j} \in A_{N_j}$ s.t. $\rho(x_{N_{j-1}}, x_{N_j}) < \varepsilon/2^{j-1}$ such that

$$\rho(x_{N_m}, x_{N_n}) \leq \rho(x_{N_m}, x_{N_{m+1}}) + ... + \rho(x_{N_{n-1}}, x_{N_n}) \leq \varepsilon/2^m + ...\varepsilon/2^{n-1} < \varepsilon/2^{m-1}$$

Therefore we can pick $\bar{x}_n \in A_n$ in the way we did above to construct a Cauchy sequence such that $\rho(\bar{x}_{N_j}, \bar{x}_n) < \varepsilon/2^j$, and thus S is complete since $\lim_{n\to\infty} \bar{x}_n = x$ exists, where $x \in S$.

□

Corollary 5 If $\{A_n\}$ is a Cauchy sequence in $K(S)$ then

$$A = \{x \in S \mid x \text{ is the limit of a sequence } \{x_n\}, x_n \in A_n, n \in \mathbb{N}\}$$

is not empty.

Theorem 6 The space $K(S)$ of compact non-empty subsets of S where d is the Hausdorff metric is complete. In other words, every Cauchy sequence $\{A_n\}$ with $A_n \in K(S)$ converges to some $A \in K(S)$

Proof: The full proof of Theorem 9.2 from Sagan (2012) is on on pages 152-154. A sketch of the proof is given here. First, to show that $\{A_n\} \longrightarrow A$, we will prove that $\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $d(A_n, A) < \varepsilon$, which by Lemma 2 is equivalent to showing that $A \subseteq N_\varepsilon(A_n)$ and $A_n \subseteq N_\varepsilon(A)$ for all $n > N_\varepsilon$. Second, to show that A is compact, prove that A is bounded and closed separately, and use the Heine-Borel theorem.

□

Now, we will prove two theorems that will be directly used to prove the main result that fractals converge to a unique attractor set.

Theorem 7 The mapping

$$\mathbb{F}(X) = \bigcup_{i=1}^{n} \mathcal{F}_i(X)$$

where $\mathcal{F}_i : S \longrightarrow S, i = 1, 2, ..., n$ are similarity transformations with reduction ratios $b_i \in (0, 1)$, is a contraction mapping with $d(\mathbb{F}(A), \mathbb{F}(B)) \leq b \cdot d(A, B)$ and $b = \max(b_1, b_2, ..., b_n) \in (0, 1)$.

Proof: Let $A, B \subset K(S)$ and choose some $\delta > d(A, B)$. If $x \in \mathbb{F}(A)$ then $x = \mathcal{F}_i(x')$ for some $i \in \{1, 2, ..., n\}$ where $x' \in A$. Since $d(A, B) < \delta$, then by Lemma 3 there exists a point $y' \in B$ s.t. $\rho(x', y') < \delta$ and $y = \mathcal{F}_i(y') \in \mathbb{F}(B)$. Thus, $\rho(x, y) \leq b \cdot \rho(x', y') < b \cdot d(A, B) < b \cdot \delta$, $\forall x \in \mathbb{F}(A)$. Hence, $\mathbb{F}(A) \subseteq N_{b\delta}(\mathbb{F}(B))$. Analogously, $\mathbb{F}(B) \subseteq N_{b\delta}(A)$, so by Lemma 2 we have $d(\mathbb{F}(A), \mathbb{F}(B)) \leq b\delta, \forall \delta > d(A, B)$, and thus $d(\mathbb{F}(A), \mathbb{F}(B)) \leq b \cdot d(A, B)$.

□

Theorem 8 (Contraction Mapping Theorem) A contraction mapping

$$\mathbb{F} : K(S) \longrightarrow K(S)$$

where $d(\mathbb{F}(A), \mathbb{F}(B))$ and $b = \max(b_1, b_2, ..., b_n) \in (0, 1)$ in the complete metric space $K(S)$ has a unique fixed point A such that $\mathbb{F}(A) = A$.

Proof: Let $A_0 \in K(S)$ with $A_{k+1} = \mathbb{F}(A_k), k = 0, 1, 2, ...$ If $d(A_0, A_1) = \delta$ then the following inequalities hold

$$d(A_1, A_2) = d(\mathbb{F}(A_0), \mathbb{F}(A_1)) \leq b\delta$$

$$d(A_2, A_3) = d(\mathbb{F}(A_1), \mathbb{F}(A_2) \leq b^2\delta$$

$$...$$

$$d(A_n, A_{n+1}) = d(\mathbb{F}(A_{n-1}), \mathbb{F}(A_n)) \leq b^n\delta$$

$$\ldots$$

and so on.

Therefore, $\{A_n\}$ is a Cauchy sequence because

$$d(A_m, A_n) \leq d(A_m, A_n) \leq d(A_m, A_{m+1}) + d(A_{m+1}, A_{m+2}) + \ldots + d(A_{n-1}, A_n)$$

$$\leq b^m\delta + b^{m+1}\delta + \ldots + b^{n-1}\delta = b^m\delta(1 + b + b^2 + \ldots + b^{n-m-1}) < \delta b^m/(1 - r)$$

Since $K(S)$ is complete we must have that $\lim_{n\to\infty} A_n = A \in K(S)$ exists. By taking the limit of both sides of $A_{n+1} = \mathbb{F}(A_n)$ we obtain $A = \mathbb{F}(A)$, thus there exists a fixed point A.

Next, suppose for the purposes of contradiction that A is not unique. Then there are two distinct fixed points A, B such that $A = \mathbb{F}(A)$ and $B = \mathbb{F}(B)$ and thus

$$d(A, B) = d(\mathbb{F}(A), \mathbb{F}(B)) \leq b \cdot d(A, B)$$

so we must have $b \geq 1$, which contradicts our assumption that $0 < b < 1$, where the reader will not forget that $b = \max(b_1, b_2, \ldots, b_n) \in (0, 1)$ (the max of the reduction ratios).

□

Finally, we are ready to prove the main result of this exposé.

Theorem 9 If S is a complete metric space and the $\mathcal{F}_i : S \longrightarrow S$ are a similarity transformations with reduction ratios $b_i \in (0, 1)$, then the mapping $\mathbb{F} : K(S) \longrightarrow K(S)$ defined by

$$\mathbb{F}(A) = \bigcup_{i=1}^{n} \mathcal{F}_i(A_j)$$

where $A_j \in K(S), j \in \mathbb{N}_0$, has a unique compact attractor set A such that $\mathbb{F}(A) = A$.

Proof: This follows immediately from Theorems 7 and 8.

□

Note that the attractor set does not depend on the initial set A_0. Hence, if we choose a different initial set (as long as it is a non-empty compact subset of $\mathbb{R}$ with Euclidean distance metric), then we will generate the same fractal by using the iterated function system explained previously.

Moreover, if the attractor set A of an iterated function system is pathwise connected and there are distinct points $(\xi_k, v_k) \in A, k = 0, 1, 2, \ldots, n$ such that $\mathcal{F}_k = (\xi_0, v_0) = (\xi_{k-1}, v_{k-1})$ and $\mathcal{F}_k(\xi_n, v_n) = (\xi_k, v_k), k \in \mathbb{N}$, then there is a continuous function

$$f : [0, 1] \xrightarrow{onto} A$$

which interpolates $(\xi_0, v_0), (\xi_1, v_1), \ldots, (\xi_n, v_n)$ where $f(0) = (\xi_0, v_0)$ is the beginning point and $f(1) = (\xi_n, v_n)$ is the end point, and if $J_2(A) > 0$ i.e. if there is positive 2-dimensional Jordan content (area), then, remarkably, f is a space-filling curve.

3 Examples of Fractals

In this section, we will give several examples of fractals by introducing their mathematical construction, their similarity dimension (defined below), and computer-generated visualizations.

Definition A self-similar fractal generated by a catalogue of n similarity transformations $x' = \mathcal{F}_i(x)$, $i = 1, 2, ..., n$ with reduction ratio $b = \max(b_1, b_2, ..., b_n) \in (0, 1)$ is said to have a similarity dimension of

$$\dim_S = \frac{\log(n)}{\log(1/b)}$$

Moreover, if $\mathcal{F}_i$ has reduction ratio $b_i \in (0, 1)$ then the similarity dimensions $\dim_S$ of the self-similar fractal is the unique solution of $b_1^s + b_2^s + ... + b_n^s = 1$. Note that the definition of the similarity dimension is associated with the iterated function system that generates it and not the with the self-similar fractal, since we can have different iterated function systems that generate the same fractal.

If there is an open set O such that $\mathcal{F}_i(O) \subseteq O$ for all $i = 1, 2, ..., n$ and $\mathcal{F}_i(O) \cap \mathcal{F}_j(O) = \emptyset$ for $i \neq j$ (this condition is known as Moran's open set condition) then the similarity dimension $\dim_S$ is equal to the fractal or Hausdorff dimension $\dim_H$, which is associated with the self-similar fractal itself and not the iterated function system that generates the fractal.

Moreover, the following inequalities hold

$$\dim_L \leq \dim_H \leq \dim_S$$

where $\dim_L$ is the Lebesgue covering dimension or topological dimension defined in the previous section, $\dim_H$ is the Hausdorff dimension or fractal dimension defined in the previous section, and $\dim_S$ is the similarity dimension defined above.

Recall that an iterated function system for a set of similarity transformation is given by

$$\mathcal{F}_i(\vec{x}) = b_i A_i \vec{x} + c_i$$

where $\mathcal{F}_i(x) = x'$. For clarity, when we are dealing with dimensions greater than 1, we will denote the vector x by $\vec{x}$.

Example The Cantor Set

The Cantor set is obtained by the similarity transformations

$$\mathcal{F}_1(x) = \frac{1}{3}x$$

$$\mathcal{F}_2(x) = \frac{1}{3}(x + 2)$$

to the interval $[0, 1]$.

1. The topological dimension of the Cantor set is

$$\dim_L(\text{Cantor set}) = 0$$

since the Cantor set does not contain any open intervals.

2. The similarity dimension of the Cantor set is

$$\dim_S(\text{Cantor set}) = \frac{\log(2)}{\log(3)} \approx 0.631$$

since the reduction ratio is $b = 1/3$ and there are $n = 2$ similarity transformations.

3. The fractal dimension of the Cantor set is

$$\dim_H(\text{Cantor set}) = \frac{\log(2)}{\log(3)} \approx 0.631$$

since if we choose $O = (0, 1)$, then $\mathcal{F}_1(O) = (0, 1/3) \subseteq (0, 1)$ and $\mathcal{F}_2(O) = (2/3, 1) \subseteq (0, 1)$ and $\mathcal{F}_1(O) \cap \mathcal{F}_2(O) = (0, 1/3) \cap (2/3, 1) = \emptyset$ thus $\dim_H = \dim_S$.

Observe four iterations of the Cantor set following the iteration rule

Figure 3: Iteration Rule

Figure 4: 1 Iteration

Figure 5: 2 Iterations

Figure 6: 3 Iterations

Figure 7: 4 Iterations

Example The von Koch Curve

The von Koch set is obtained by the similarity transformations

$$\mathcal{F}_1(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x}$$

$$\mathcal{F}_2(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\vec{x} + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_3(x) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3/2 \\ \sqrt{3}/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\vec{x} + \frac{1}{3}\begin{pmatrix} 3/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\mathcal{F}_4(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \frac{1}{3}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

to the interval $[0,1]^2$.

1. The topological dimension of the von Koch curve is

$$\dim_L(\text{von Koch curve}) = 1$$

since each point is in no more than 2 sets in the refinement of any open covering.

2. The similarity dimension of the von Koch curve is

$$\dim_S(\text{von Koch curve}) = \frac{\log(4)}{\log(3)} \approx 1.262$$

since the reduction ratio is $b = 1/3$ and there are $n = 4$ similarity transformations.

3. The fractal dimension of the von Koch curve is

$$\dim_H(\text{von Koch curve}) = \frac{\log(4)}{\log(3)} \approx 1.262$$

since if we choose O to be the interior of the initial triangle in the figure below, then $\mathcal{F}_i(O) \subseteq O$ for $i = 1, 2, 3, 4$ and $\bigcap_{i=1}^4 \mathcal{F}_i(O) = \emptyset$ thus $\dim_H = \dim_S$.

0.4

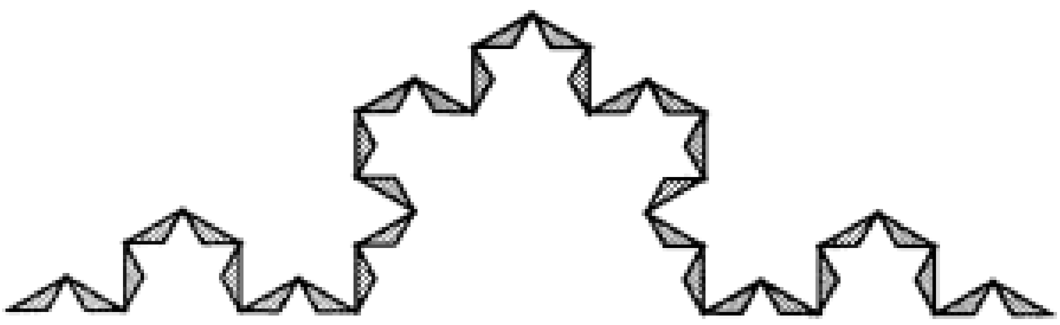

Figure 8: Fourth Step in Knopp's generation of theh von Koch curve

Observe four iterations of the von Koch curve following the iteration rule

Figure 9: Iteration Rule

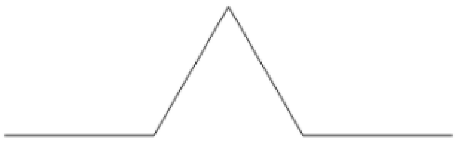

Figure 10: 1 Iteration

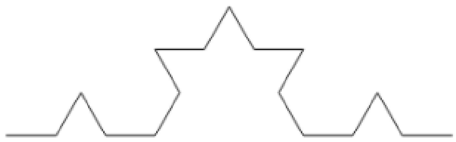

Figure 11: 2 Iterations

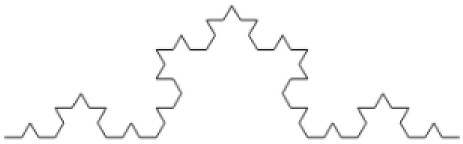

Figure 12: 3 Iterations

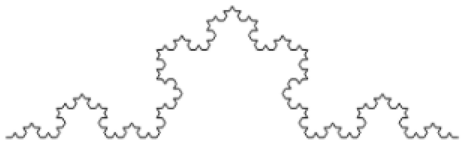

Figure 13: 4 Iterations

Example The Sierpiński Triangle

The Sierpiński Triangle is obtained by the similarity transformations

$$\mathcal{F}_1(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x}$$

$$\mathcal{F}_2(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_3(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \frac{1}{2}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

to the interval $[0,1]^2$.

1. The topological dimension of the Sierpiński Triangle is

$$\dim_L(\text{Sierpiński Triangle}) = 1$$

since each point is in no more than 2 sets in the refinement of any open covering.

2. The similarity dimension of the Sierpiński Triangle is

$$\dim_S(\text{Sierpiński Triangle}) = \frac{\log(3)}{\log(2)} \approx 1.585$$

since the reduction ratio is $b = 1/2$ and there are $n = 3$ similarity transformations.

3. The fractal dimension of the Sierpiński Triangle is

$$\dim_H(\text{Sierpiński Triangle}) = \frac{\log(3)}{\log(2)} \approx 1.585$$

since if we choose O to be the interior of the initial triangle, then $\mathcal{F}_i(O) \subseteq O$ for $i = 1, 2, 3$ and $\bigcap_{i=1}^{3} \mathcal{F}_i(O) = \emptyset$ thus $\dim_H = \dim_S$.

Observe four iterations of the Sierpiński Triangle following the iteration rule

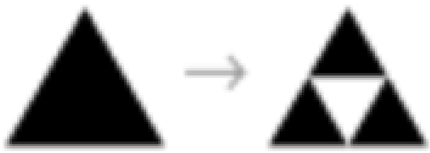

Figure 14: Iteration Rule

Figure 15: 1 Iteration

Figure 16: 2 Iterations

Figure 17: 3 Iterations

Figure 18: 4 Iterations

Example The Mazurkiewicz Continuum (Sierpiński Carpet)

The Mazurkiewicz Continuum (Sierpiński Carpet) is obtained by the similarity transformations

$$\mathcal{F}_1(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x}$$

$$\mathcal{F}_2(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

$$\mathcal{F}_3(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$$

$$\mathcal{F}_4(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_5(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$\mathcal{F}_6(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_7(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$\mathcal{F}_8(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$$

to the interval $[0, 1]^2$.

1. The topological dimension of the Mazurkiewicz Continuum is

$$\dim_L(\text{Mazurkiewicz Continuum}) = 1$$

since each point is in no more than 2 sets in the refinement of any open covering.

2. The similarity dimension of the Mazurkiewicz Continuum is

$$\dim_S(\text{Mazurkiewicz Continuum}) = \frac{\log(8)}{\log(3)} \approx 1.893$$

since the reduction ratio is $b = 1/3$ and there are $n = 8$ similarity transformations.

3. The fractal dimension of the Mazurkiewicz Continuum is

$$\dim_H(\text{Mazurkiewicz Continuum}) = \frac{\log(8)}{\log(3)} \approx 1.893$$

since if we choose O to be the interior of the square, then $\mathcal{F}_i(O) \subseteq O$ for $i = 1, 2, ..., 8$ and $\bigcap_{i=1}^{8} \mathcal{F}_i(O) = \emptyset$ thus $\dim_H = \dim_S$.

Observe four iterations of the Mazurkiewicz Continuum following the iteration rule

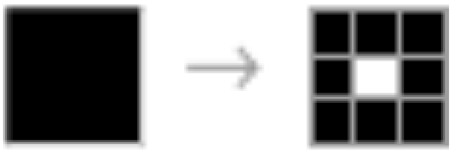

Figure 19: Iteration Rule

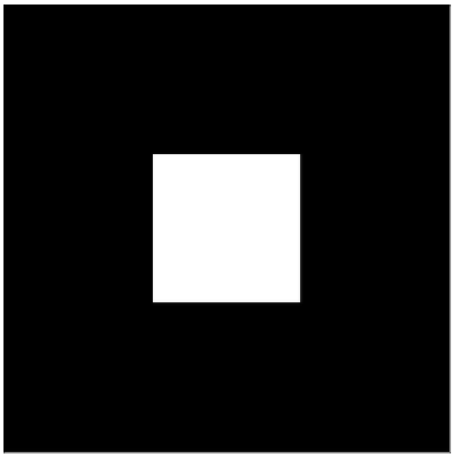

Figure 20: 1 Iteration

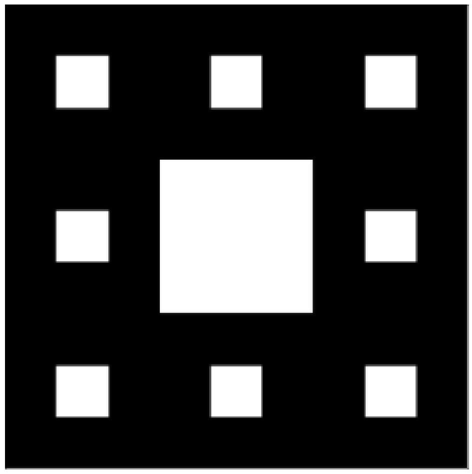

Figure 21: 2 Iterations

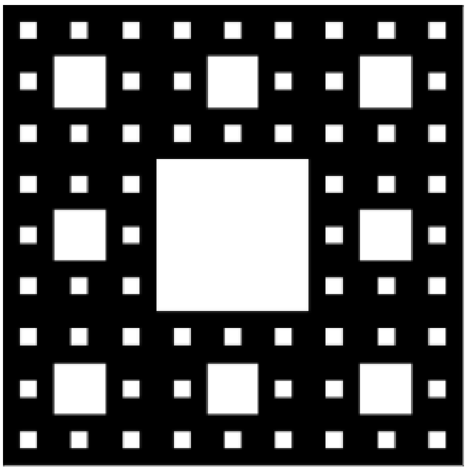

Figure 22: 3 Iterations

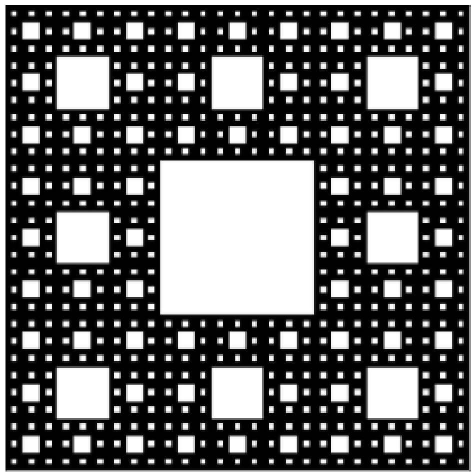

Figure 23: 4 Iterations

Example The Heighway Dragon

The Heighway Dragon is obtained by the similarity transformations

$$\mathcal{F}_1(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{x}$$

$$\mathcal{F}_2(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\vec{x} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathcal{F}_3(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathcal{F}_4(\vec{x}) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\vec{x} + \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

to the interval $[0, 1]^2$.

1. The topological dimension of the Heighway Dragon is

$$\dim_L(\text{Heighway Dragon}) = 2$$

since each point is in no more than 3 sets in the refinement of any open covering.

2. The similarity dimension of the Heighway Dragon is

$$\dim_S(\text{Heighway Dragon}) = \frac{\log(4)}{\log(2)} = 2$$

since the reduction ratio is $b = 1/2$ and there are $n = 4$ similarity transformations.

3. The fractal dimension of the Heighway Dragon is

$$\dim_H(\text{Heighway Dragon}) = \frac{\log(4)}{\log(2)} = 2$$

since if we choose O to be the interior of the initial line, then $\mathcal{F}_i(O) \subseteq O$ for $i = 1, 2, 3, 4$ and $\bigcap_{i=1}^{4} \mathcal{F}_i(O) = \emptyset$ thus $\dim_H = \dim_S$.

Observe four iterations of the Heighway Dragon

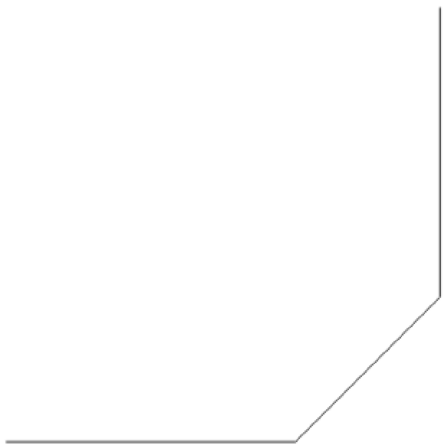

Figure 24: 1 Iteration

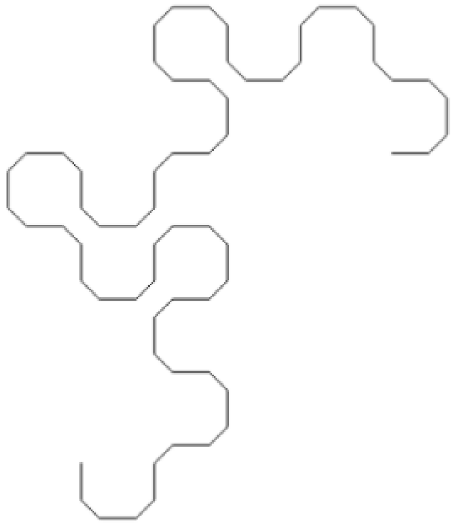

Figure 25: 5 Iterations

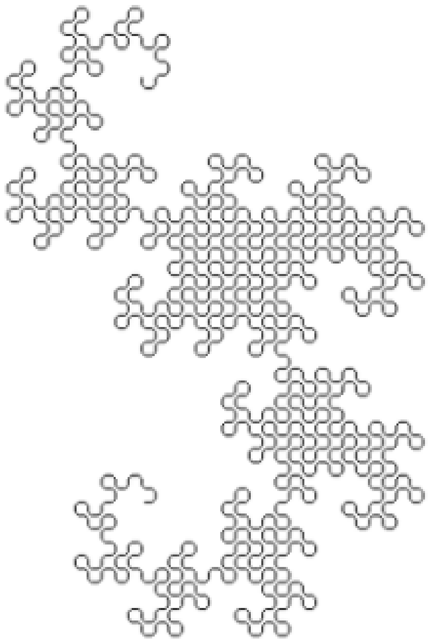

Figure 26: 10 Iterations

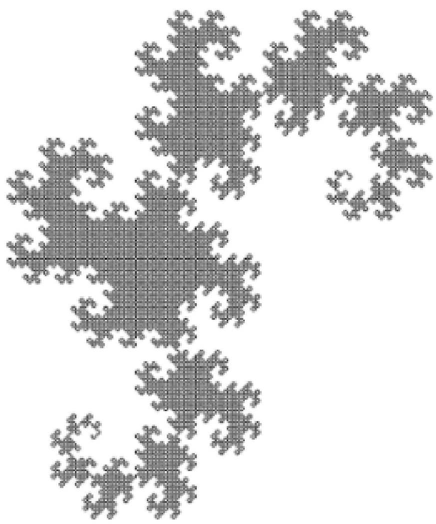

Figure 27: 15 Iterations

4 Applications of Fractals in Probability Theory

Thus far, we have introduced basic theorems about fractals, and we have given an exposition of the Cantor set, the von Koch Curve, the Sierpiński Triangle, the Mazurkiewicz Continuum (the Sierpiński Carpet), and the Heighway Dragon.

Now, we will conclude this exposé with some examples of the applications of fractals in probability theory. In fact, these two theories which often attract the attention of undergraduates studying mathematics are intimately related, and some mathematicians would argue that they are inseparably intertwined. While an entire introduction to the relationships of the theories of fractals and probability is out of the scope of this exposition, we will give some mostly self-contained[23] examples to give readers an idea of their relationships. Moreover, we will give suggestions for further reading in the next and final section.

Definition[24] A number $x \in (0,1]$ is called simply normal is base 2 if

$$f(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{\{0\}}(x_j) = \frac{1}{2}$$

where x_j denotes the jth digit in the binary expansion of x.

Example[25] Borel's normal-number theorem states that Lebesgue-almost all $x \in (0,1]$ are simply normal in base 2. Let $X := \sum_{j=1}^{\infty} X_j 2^{-j}$, where $\{X_j\}_{j=1}^{\infty}$ are independent random variables uniformly distributed on $\{0,1\}$. Then X is uniformly distributed on $[0,1]$, thus $\mathbb{P}(X \in A)$ is the Lebesgue measure of $A \subseteq [0,1]$. By the strong law of large numbers (SLLN), we have that $P(X \in N_{1/2}) = 1$ where $N_{1/2}$ denotes the collection of all $x \in (0,1]$ that are simply normal in base 2. Therefore, $N_{1/2}$ has full measure, so we have proven that Lebesgue-almost all $x \in (0,1]$ are simply normal in base 2.

Now, what does this have to do with fractals? Many non-normal numbers form collections of fractals. Fix $p \in (0,1)$ and let

$$N_p := \{x \in (0,1] \mid f(x) = p\}$$

Then $\forall x \in N_p$, the digits of x have the prescribed asymptotic frequencies p and $1-p$. Next, we will show that the set N_p has fractal dimension.

Theorem[26] The theorem of Eggleston states that

$$\dim_H N_p = p\log_2(\frac{1}{p}) + (1-p)\log_2(\frac{1}{1-p})$$

Therefore, the set N_p has non-integer Hausdorff dimension.

Second, we will conclude with an example of fractal Brownian motion.

Definition[27] Define $\mathbf{B} := \{B(t)\}_{t\geq 0}$ as the Brownian motion in $\mathbb{R}^d$. Then $\mathbf{B}$ is a collection of random variables such that $B(0) = 0, B(t+s) - B(s)$ is independent of $\{B(u)\}_{0\leq u\leq s}, \forall s,t \geq 0$, and

[23]For the reader already acquainted with basic measure-theoretic probability.

[24]From Fractals and Probability to Lévy Processes and Stochastic PDEs. Davar Khoshnevisan. https://www.math.utah.edu/ davar/PPT/GREIFSWALD/GreifswaldSurvey2.pdf. 2000.

[25]Ibid.

[26]Ibid.

[27]Ibid.

the coordinates of $B(t+s) - B(s)$ are independent of normally distributed random variables with mean zero and variance t, $\forall s, t \geq 0$.

One of the theorems of Wiener, which will not be proven here, states that **B** can be constructed such that the random function $t \longmapsto B(t)$ is almost surely Hölder continuous with an index less than $\frac{1}{2}$. Then the following limit

$$\lim_{\varepsilon \longrightarrow 0} \sup_{s \in [0,T], t \in (s, s+\varepsilon]} \frac{|B(t) - B(s)|}{\sqrt{2(t-s)\ln(t-s)}} = 1 \ \forall T > 0$$

exists almost surely.

Theorem[28] One of Lévy's Theorems states that $\dim_H B[0,b] = \min(d, 2)$ almost surely for $b > 0$.

In other words, for dimensions two and higher, the range of the Brownian motion has Hausdorff dimension two, and for any dimension $d < 2$ where d is not necessarily an integer, the range of the Brownian motion has Hausdorff dimension d. In fact, this can be generalized further into multifractal system in which the dynamics of the Brownian motion are best described by a continuous spectrum.

5 Suggested Further Reading

After reading this exposition, readers will be more or less prepared to read Falconer's *Fractal Geometry* and *Techniques in Fractal Geometry.* In the first text, a proper introductory graduate level text, readers will be exposed to how fractals can be applied to self-similar and affine sets, graphs of functions, with examples from number theory, dynamical systems, Julia sets, random fractals, and physics. In the second text, a sequel to the first, readers will be introduced to various advanced techniques used by mathematicians and scientists that study areas where fractal phenomena naturally emerge.

For an introductory text on the relationship of the theories of fractals and probability, consider Bishop's *Fractals in Probability and Analysis.* In that text, readers will be introduced to fractal sets that emerge from probability theory and the mathematical tools used to study them, a few of which we have introduced here. Second, readers may consider Khoshnevisan's *From Fractals and Probability to Lévy Processes and Stochastic PDEs* for a short (32 pages) survey on how random processes, such as Lévy processes, naturally produce fractals. For both texts, a basic introduction to measure theoretic probability, such as the first five chapters of Durrett's *Probability: Theory and Examples*, will be enough mathematical background to appreciate the material.

[28] Ibid.

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