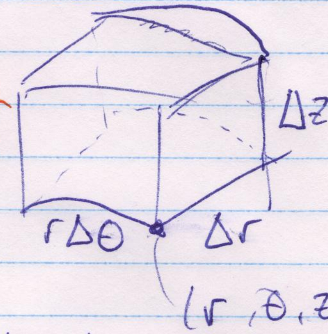
B) Consider a control volume (CV) with a fluid of density $\rho(t, r, \theta, z)$.

Take $U(t, r, \theta)$ and $V(t, r, \theta)$ not to be confused with volume! to be the fluid flows in the $\hat{r}$ and $\hat{\theta}$ directions.




Essentially what we need to do is determine which PDE is a consequence of conservation of mass:

$$\begin{pmatrix} \text{mass change} \\ \text{from } t \text{ to } t+\Delta t \end{pmatrix} + \begin{pmatrix} \text{mass leaving} \\ \text{CV} \end{pmatrix} = 0 .$$

The total mass in the CV can be approximated by using the density at $(r, \theta, z)$ as follows

$$\text{mass} = \int_V \rho(t, r, \theta, z)\, d\vec{x} \approx \rho(t, r, \theta, z) \underbrace{r\Delta\theta\, \Delta r\, \Delta z}_{\Delta V} .$$

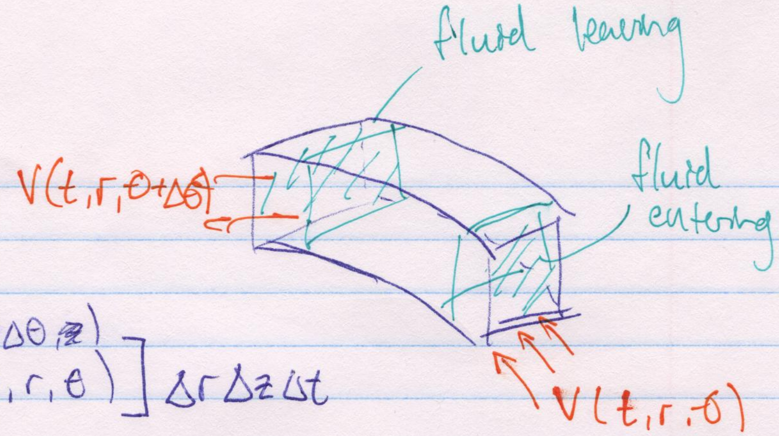
We can approximate the change in mass, then, with

$$\big(\rho(t+\Delta t, r, \theta, z) - \rho(t, r, \theta, z)\big)\Delta V$$

$$= \frac{\rho(t+\Delta t, r, \theta, z) - \rho(t, r, \theta, z)}{\Delta t} \Delta V \Delta t \simeq \frac{\partial \rho}{\partial t} \Delta V \Delta t$$

Now let's consider the mass leaving the CV during the $\Delta t$ time window. Let's consider the mass flow in the $\hat{\theta}$ direction.

As is shown in the figure on the following page, we can calculate the net mass flow in the $\hat{\theta}$ direction with

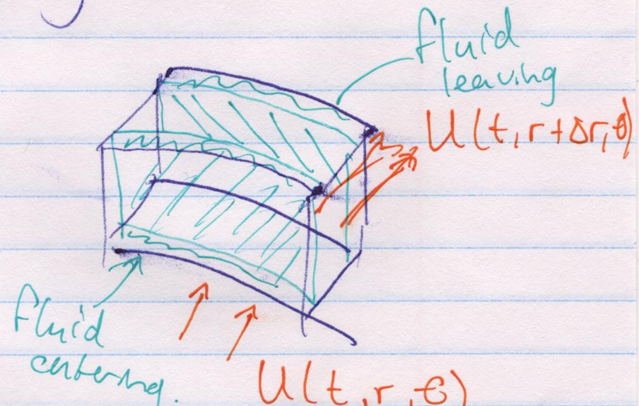



$$\Big[\rho(t,r,\theta+\Delta\theta,z)V(t,r,\theta+\Delta\theta,z) - \rho(t,r,\theta,z)V(t,r,\theta)\Big]\Delta r \Delta z \Delta t$$

$$= \frac{\rho(t,r,\theta+\Delta\theta,z)V(t,r,\theta+\Delta\theta) - \rho(t,r,\theta,z)V(t,r,\theta)}{\Delta\theta}\,\frac{\Delta V \Delta t}{r}$$

$$\simeq \frac{1}{r}\frac{\partial(\rho V)}{\partial\theta}\Delta V \Delta t.$$

Now let's consider the flow leaving in the $\hat{r}$ direction. The figure to the right isn't pretty, but hopefully it's clear enough. With the help of this figure we can caclulate the net mass flow in the $\hat{r}$ direction as




$$\rho(t,r+\Delta r,\theta,z)U(t,r+\Delta r,\theta)(r+\Delta r)\Delta\theta\Delta z\Delta t - \rho(t,r,\theta,z)U(t,r,\theta)\,r\Delta\theta\Delta z\Delta t$$

$$= \frac{\rho(t,r+\Delta r,\theta,z)U(t,r+\Delta r,\theta)(r+\Delta r) - \rho(t,r,\theta,z)U(t,r,\theta)r}{\Delta r}\,\frac{\Delta V \Delta t}{r}$$

$$\simeq \frac{1}{r}\frac{\partial(\rho U r)}{\partial r}\Delta V \Delta t.$$

We can now "fill in" the conservation of mass equation with our results:

$$\frac{\partial\rho}{\partial t}\Delta V\Delta t + \frac{1}{r}\frac{\partial(\rho V)}{\partial\theta}\Delta V\Delta t + \frac{1}{r}\frac{\partial(\rho U r)}{\partial r}\Delta V\Delta t = 0,$$

or, dividing through by $\Delta V \Delta t$,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho V)}{\partial \theta} + \frac{1}{r}\frac{\partial(\rho U r)}{\partial r} = 0 .$$

Using the identity (where $F$ is some vector field)

$$\frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z} F_z \equiv \nabla \cdot F$$

we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0,$$

where $\vec{U} = (\underset{r}{U}, \underset{\theta}{V}, \underset{z}{0})$. Hence our result is consistant with what we derived in Cartesian coordinates in lectures (divergence is coordinate free!).