Online Tensor Robust Principal Component Analysis

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A thesis submitted in partial fulfillment for the degree of Honours in Statistics

at the

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The Australian National University

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October 30, 2018

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, contains no material published or written by another person, except where due reference is made in the thesis.

Acknowledgements

I would like to express my deepest gratitude to my advisor Dr. Dale Roberts for the valuable guidance he provided me in writing this thesis. I would also like to thank my amazing girlfriend Juliana Zubovic for being great. And I would like to thank my close friend Nishanth Pathy for the excellent job he did editing this thesis.

Abstract

Tensor Robust Principal Component Analysis (TRPCA) is a procedure for recovering a data structure that has been corrupted by noise. In this thesis, a proof (inspired by Lu et al. (2018)) is given that TRPCA successfully performs this operation. An online optimisation algorithm to perform this procedure for p-dimensional tensors is proposed (based on a similar algorithm for the 3-dimensional case from Z. Zhang, Liu, Aeron, & Vetro (2016)). The required tensor identities to apply a proof of convergence (similar to the approach of Feng, Xu, & Yan (2013) for the matrix case) are derived and then applied. Examples using satellite image data and synthetic data are provided throughout to demonstrate the utility of the theoretical work and examine hypotheses.

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Chapter 1

Introduction

In many fields of modern data analysis, the observed data are not necessarily the prime objects of interest. The computer steering a driverless car should pay more attention to objects moving across the foreground (for example, a kangaroo about to jump onto the road) rather than less relevant aspects of the background (for example, a tree far from the road). In the opposite scenario, we might be interested primarily in the background: when we view satellite images, we are often much more interested in the land that lies beneath clouds than we are in the clouds themselves. In addition to this, sensor failures or malicious tampering may also lead us to doubt the quality of observed data. One way of framing this problem is that if the object of our interest, L_0, becomes corrupted by some type of noise, C_0, what we observe is then $M_0 = L_0 + C_0$. A surprising result is that it is possible to recover L_0 and C_0, provided that L_0 is of low rank and C_0 is sparse. To do this, we solve the optimisation problem

$$\min_{L,C} \|L\|_* + \lambda \|C\|_1, \quad \text{s.t.} \;\; M_0 = L + C.$$

This was proven by Candès, Li, Ma, & Wright (2011) for matrices and Lu et al. (2018) generalised this to tensors. This generalisation required extending many of the things we know about matrices to tensors. An important example is the nuclear norm, which for matrices is given by the sum of the singular values. Until recently (Martin, Shafer, & Larue, 2013), however, we had no suitable way of multiplying tensors, let alone a singular value decomposition (SVD) or nuclear norm.

1.1 Literature Review

There is a rich body of literature from robust statistics concerning the problem of a lack of faith in the model that is assumed to have generated the observed data (Huber & Ronchetti, 2009; Maronna, Martin, & Yohai, 2006). However, the approach from this literature is not always optimal for two reasons. First, these approaches do not return the best answer. Typically under the regime described above, the goal is to derive estimators to make inferences about the object of interest (L_0 in our above formulation), based on observed data (M_0) that are robust to perturbations from noise (C_0). In most cases, while these estimators preserve some desirable properties in the presence of perturbations, they rarely return the 'correct' answer that would be returned if there were no perturbations (T. Zhang & Lerman, 2014). Nor should they: based on the assumptions behind these estimators (which typically do not include our assumption that the object of interest is low-rank and the noise is sparse), expecting a 'correct' answer would be unreasonable. Second, there are many cases where the object of interest is the perturbation, so in this case, most traditional approaches from robust statistics are not helpful.

As alluded to above, the goal we have in mind is exact recovery of the low rank matrix L_0, which in turn implies exact recovery of the sparse matrix C_0. How might one go about doing this? In the case shown in Figure 1.1, performing this decomposition appears very straight forward.

Figure 1.1: Decomposition into low rank and sparse parts from A. C. Sobral (2017).

The low-rank matrix and the sparse matrix can be split just by looking at them. One way to formalise the perhaps subconscious process of splitting these components is the following optimisation problem

$$\begin{aligned} \text{Minimise :} \quad & \text{rank}(L) + \lambda \|C\|_0 \\ \text{Subject to :} \quad & M = L + C \end{aligned}$$

where $\|\cdot\|_0$ is the l_0 'norm' (this function returns the number of non-zero entries in its argument, but despite its notation it is actually not a norm). This approach seems intuitive: we put as much of M as possible into L while keeping the rank of L low. However, this optimisation is non-convex, and determining the smallest number of entries that need to be changed to reduce a matrix's rank (known as a its rigidity) is in general an NP-hard problem (Codenotti, 2000).

We can relax this intractable problem to a tractable convex optimisation by replacing the rank with the nuclear norm (given by the sum of the matrix's singular values) and by replacing the l_0 norm with the l_1 norm (given by the sum of the absolute values of the elements of a matrix). One definition of the rank of a matrix is the number of its non-zero singular values, so the rank of a matrix is, in a sense, a version of the nuclear norm that ignores the magnitudes of the singular values. We can be more explicit about the relation between the rank and the nuclear norm: the nuclear norm is the lower convex envelope of the rank (Fazel, 2002). Similarly, the difference between the l_0 norm and the l_1 norm is that the l_1 norm includes the magnitudes of the elements. For many large underdetermined systems of linear equations, the l_1 minimum is also often the sparsest solution (Donoho, 2006). So this convex relaxation should perform similarly to the original optimisation problem given here. We then get the following optimisation problem (Wright, Ganesh, Rao, Peng, & Ma, 2009; Candès et al., 2011; Chandrasekaran, Sanghavi, Parrilo, & Willsky, 2011)

$$\begin{aligned} \text{Minimise}: &\quad \|L\|_* + \lambda\|C\|_1 \\ \text{Subject to}: &\quad M = L + C \end{aligned}$$

where $\|\cdot\|_*$ is the nuclear norm and $\|\cdot\|_1$ is the l_1 norm. This method has become known in the literature as Robust Principal Component Analysis, though its only relation to Robust Statistics is in the problem's set-up and its only relation to Principal Component Analysis is its exploitation of low-dimensional representations.

Under this convex relaxation, many results (Wright et al., 2009; Candès et al., 2011; Chandrasekaran et al., 2011; Xu, Caramanis, & Sanghavi, 2012) have been derived that, under certain assumptions on the structure of the data, the matrix L can be exactly recovered with a high probability so long as its rank is sufficiently small. Another useful contribution to this theory was an online stochastic optimisation algorithm developed by Feng et al. (2013) that removes the requirement to load the entire dataset into memory at once, which is particularly useful for large or high-dimensional datasets.

All of these results were derived for matrices (which are 2-dimensional tensors), but many datasets are not 2-dimensional. A standard image, for instance, is a matrix of pixels, and each pixel is typically 3-dimensional (red, green and blue). Pixels from satellite images frequently have many more dimensions (common dimensions range from 5, 7 and 10 up to hundreds). The storage of temporal data (which is common for video streams and images taken over time) can add another dimension. Thus, to apply these 2-dimensional approaches to higher-dimensional data, some type of flattening is applied (often to black and white, or working with a single of band of colour). This flattening removes structure from the data, resulting in information loss, so an approach that does not rely on flattening would be advantageous.

Since these 2-dimensional methods were developed, our understanding of tensors and multidimensional data has grown. In particular, the development of a tensor multiplication known as the 't-product' that generalises matrix multiplication has spurred the development the 't-SVD', which generalises the Singular Value Decomposition to the tensor case (Kilmer & Martin, 2011; Martin et al., 2013). These new methods have facilitated the derivation of a new tensor nuclear norm (Lu et al., 2018) that can then be used to extend the previously derived Robust Principal Component Analysis methods to the tensor case. Inspired by Feng et al. (2013), Z. Zhang et al. (2016) also extended the online optimisation algorithm from the 2-dimensional case to the 3-dimensional case, but no proof of convergence was given.

There have also been other approaches employed to apply the Robust Principal Component Analysis technique to multidimensional data without using the t-product. These are primarily based on the higher-order singular value decomposition (Tucker, 1966), including online approaches (A. Sobral, Javed, Jung, Bouwmans, & Zahzah, 2015; Li et al., 2018). Though relevant, these techniques are unfortunately beyond the scope of this thesis.

1.2 Aims

This thesis aims to collate and then build upon the recent advancements in the understanding of multidimensional data based on the t-product. After developing the required tools for working with tensors in the Chapter 2, a proof that the tensor Robust Principal Component Analysis procedure works will be given in Chapter 3. Then in Chapter 4, an online optimisation algorithm for tensors of arbitrary dimension will be proposed and its convergence analysed.

Chapter 2

Tensors

A real vector lives in $\mathbb{R}^{n_1}$ and a real matrix in $\mathbb{R}^{n_1 \times n_2}$. We can generalise this further:

Definition 2.1. A p-dimensional real tensor is an element of the vector space $\mathbb{R}^{n_1 \times \ldots \times n_p}$.

In the same way that matrices are much more than just elements of a vector space (for one, they are also linear functions), tensors are also more than just data structures. Our aim in this chapter is to extend to tensors many of the properties that matrices can have, such as invertibility, orthogonality and various factorisations, as well as to derive some basic results for use throughout the later chapters. Much of this work is based on the work of Martin et al. (2013).

First, some notation. In the matrix case, if we have a matrix A in $n_1 \times n_2$, we identify an element of A by writing $A(i,j)$ for the element in the ith row and jth column. Similarly we can select the ith row vector of A by writing $A(i,:)$. We extend this idea to tensors in the same way. If $\mathcal{A}$ is a tensor in $n_1 \times \ldots \times n_p$, and $J = j_1, \ldots, j_p$ we can select the element in the Jth position $\mathcal{A}_J = \mathcal{A}(j_1, \ldots, j_p)$. We can also select a lower-dimensional data structure by fixing some indices and varying over the others. If we only fix the qth index to i we will denote the $p-1$ dimensional data structure as $\mathcal{A}_i^q$. Often we will fix only the last dimension, so for simplicity we will write $\mathcal{A}_i^{n_p} = \mathcal{A}_i$. We will also call $\mathcal{A}(:,:,J)$ (that is, extracting a matrix from $\mathcal{A}$ by varying over the first two coordinates and fixing all others) a frontal slice. All indices for tensor elements will start at 0, and indices will be taken modulo the size of that dimension. Finally, it will often be convenient to use the following notation: $\Pi_{i=k}^{p} n_i \equiv \tilde{n}_k$.

2.1 Basics

To talk about things like invertibility, orthogonality and factorisations in a tensor context, we will need a way of multiplying tensors. The following definitions establish the basics needed to do this.

Definition 2.2. The function $\texttt{unfold} : n_1 \times ... \times n_p \to n_1 n_p \times n_2 \times ... \times n_{p-1}$ is defined by

$$\texttt{unfold}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_{n_p} \end{bmatrix}$$

and has inverse $\texttt{fold} : n_1 n_p \times n_2 \times ... \times n_{p-1} \times \mathbb{N} \to n_1 \times ... \times n_p$ defined by

$$\texttt{fold}(\texttt{unfold}(\mathcal{A}), n_p) = \mathcal{A}.$$

In most cases, the second argument of `fold` will be implied to be n_p by the fact that it is folding up `unfold`'s output. Unless it is ambiguous, this argument will not be included.

Next, we have the circulant operator.

Definition 2.3. The function $\texttt{bcirc} : n_1 n_p \times n_2 \times n_3 \times ... \times n_{p-1} \to n_1 n_p \times n_2 n_p \times n_3 \times ... \times n_{p-1}$ is defined for a block vector of tensors (or just a normal vector)

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_{n_p} \end{bmatrix}$$

where each $\mathcal{A}_i$ is in $n_1 \times ... \times n_{p-1}$ (and in all cases we will be considering, $\mathcal{A}$ will be the output of `unfold`) as

$$\texttt{bcirc}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_{n_p} & \mathcal{A}_{n_{p-1}} & \dots & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_{n_p} & \dots & \mathcal{A}_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{A}_{n_p} & \mathcal{A}_{n_{p-1}} & \dots & \mathcal{A}_2 & \mathcal{A}_1 \end{bmatrix}.$$

We also note that we can define an inverse on the range of `bcirc` by $\texttt{bcirc}^{-1}(\texttt{bcirc}(\mathcal{A})) = \mathcal{A}$, which is well defined because $\texttt{bcirc}(\mathcal{A}) = \texttt{bcirc}(\mathcal{B})$ if and only if $\mathcal{A} = \mathcal{B}$. Now we can define the t-product.

Definition 2.4. The operation $*$ takes a tensor in $n_1 \times n_2 \times ... \times n_p$ and another in $n_2 \times l \times n_3 \times ... \times n_p$ and then returns a tensor in $n_1 \times l \times n_3 \times ... \times n_p$. It is defined for tensors of dimension-3 as

$$\mathcal{A} * \mathcal{B} = \texttt{fold}\Big(\texttt{bcirc}\left(\texttt{unfold}(\mathcal{A})\right) \cdot \texttt{unfold}(\mathcal{B})\Big)$$

where $\cdot$ is normal matrix multiplication. For tensors in dimension greater than 3, the t-product requires the $*$ operation and is defined as

$$\mathcal{A} * \mathcal{B} = \texttt{fold}\Big(\texttt{bcirc}\left(\texttt{unfold}(\mathcal{A})\right) * \texttt{unfold}(\mathcal{B})\Big).$$

Here, it worth noting which tensors we can multiply with the t-product. They must both be of the same dimension (just like matrices, which are always dimension-2). The size of the second dimension of the first tensor must be same size as the first dimension of the second tensor (just like matrices). The second dimension of the second tensor can be anything it wants (just like matrices). The remaining dimensions must all be the same, which is vacuously true of matrices since the additional dimensions do not exist.

Note that Definition 2.4 is recursive, with a base case of matrix multiplication. In almost all cases to which we apply the $*$ operator in the following analysis, we will apply `unfold` and `bcirc` many times to get to the base case. So unless stated otherwise, for $\mathcal{A}$ in $n_1 \times ... \times n_p$, we will write $\texttt{bcirc}(\mathcal{A}) = (\texttt{bcirc} \circ \texttt{unfold})^{p-2}(\mathcal{A})$ and $\texttt{unfold}(\mathcal{A}) = \texttt{unfold}^{p-2}(\mathcal{A})$.

Lemma 2.5. *For $\mathcal{A}$ in $n_1 \times n_2 \times ... \times n_p$, when considering $\texttt{bcirc}(\mathcal{A})$ as an $\tilde{n}_3 \times \tilde{n}_3$ block matrix with blocks of size $n_1 \times n_2$, the ijth block of $\texttt{bcirc}(\mathcal{A})$ is given by*

$$\texttt{bcirc}(\mathcal{A})_{ij} = \mathcal{A}\left(:, :, J\right)$$

where the kth dimension of J is given by

$$\left(\left\lfloor \frac{i \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{j \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor\right) \bmod_{n_k}$$

and $\tilde{n}_{p+1} = 1$.

Proof. We proceed by induction. For the 3-dimensional case this is saying that

$$\texttt{bcirc}(\mathcal{A})_{ij} = \mathcal{A}\left(:, :, \left(\left\lfloor \frac{i}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{j}{\tilde{n}_{k+1}} \right\rfloor\right) \bmod_{n_k}\right)$$

which follows directly from the definition of the circulant. Now we assume this is true for any $p-1$ dimensional tensor. Considering blocks of size $n_1\tilde{n}_4 \times n_2\tilde{n}_4$, we have that $\texttt{bcirc}(\mathcal{A})_{ij}$ belongs to the block in row $\left\lfloor \frac{i}{\tilde{n}_4} \right\rfloor$ and column $\left\lfloor \frac{j}{\tilde{n}_4} \right\rfloor$, which by the circulant structure is given by the $\left(\left\lfloor \frac{i}{\tilde{n}_4} \right\rfloor - \left\lfloor \frac{j}{\tilde{n}_4} \right\rfloor\right) \bmod_{n_3}$th frontal slice of $\texttt{bcirc}^{p-3}(\mathcal{A})$. We then need the element in row $i \bmod_{\tilde{n}_4}$ and column $j \bmod_{\tilde{n}_4}$ of this matrix. But this is the block circulant of the $p-1$ dimensional tensor

$$\mathcal{A}\left(:, :, \left(\left\lfloor \frac{i}{\tilde{n}_4} \right\rfloor - \left\lfloor \frac{j}{\tilde{n}_4} \right\rfloor\right) \bmod_{n_3}, :, \ldots, :\right)$$

so we can apply the inductive hypothesis and we are done. $\square$

We also have the following characterisation of the t-product.

Lemma 2.6. *The* Jth frontal face of $\mathcal{C} = \mathcal{A} * \mathcal{B}$ is given by

$$\mathcal{C}(:,:,J) = \sum_I \mathcal{A}(:,:,I)\mathcal{B}(:,:,J-I).$$

Proof. This follows because the indices of the Mth row of `bcirc`$(\mathcal{A})$ are given by

$$\left(\left\lfloor \frac{M \ \mathrm{mod}_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{j \ \mathrm{mod}_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor \right) \ \mathrm{mod}_{n_k}$$

and the indices for the column of `unfold`$(\mathcal{B})$ are given by

$$\left(\left\lfloor \frac{i \ \mathrm{mod}_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{0 \ \mathrm{mod}_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor \right) \ \mathrm{mod}_{n_k} = \left(\left\lfloor \frac{i \ \mathrm{mod}_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor \right) \ \mathrm{mod}_{n_k}.$$

So adding these indices together when $i = j$ (as would be the case in this multiplication) we get the sum of the indices of the frontal slices that are multiplied is constant. □

Proposition 2.7. *The t-product is associative:* $\mathcal{A} * (\mathcal{B} * \mathcal{C}) = (\mathcal{A} * \mathcal{B}) * \mathcal{C}$.

Proof. Matrix multiplication, which is the base case of the recursion of the t-product, is associative. □

We can now start talking about tensor extensions of familiar matrix properties. First, we define the equivalent of an identity matrix.

Definition 2.8. The order-p identity tensor in $n_1 \times n_1 \times n_3 \times ... \times n_p$ is the tensor $\mathcal{I}$ such that $\mathcal{I}(:,:,\mathbf{0})$ is the $n_1 \times n_1$ identity and any other frontal slice $\mathcal{I}(:,:,J)$ is 0.

To see that this does indeed behave like an identity, we note that `bcirc`$(\mathcal{I}) = I$.

Second, we define the tensor equivalent of a matrix inverse. Like matrices, tensors must be square in the first two dimensions to be invertible.

Definition 2.9. If $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ are in $n_1 \times n_1 \times n_3 \times ... \times n_p$ and

$$\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A} = \mathcal{I},$$

then $\mathcal{B}$ is the inverse of $\mathcal{A}$ (and vice versa).

Lemma 2.10. *Just like matrices,* $(\mathcal{A} * \mathcal{B})^{-1} = \mathcal{B}^{-1} * \mathcal{A}^{-1}$.

Proof. $\mathcal{I} = \mathcal{A} * \mathcal{B} * \mathcal{B}^{-1} * \mathcal{A}^{-1} = \mathcal{B}^{-1} * \mathcal{A}^{-1} * \mathcal{A} * \mathcal{B}$. □

Third, we define the equivalent of a matrix transpose. At first glance, this appears to be a weird definition. We introduce it in line with Martin et al. (2013), and then show it is equivalent to another (perhaps nicer) definition.

Definition 2.11. If $\mathcal{A}$ is an order-3 tensor in $n_1 \times n_2 \times n_3$ then $\mathcal{A}^T$ is the order-3 tensor in $n_2 \times n_1 \times n_3$ found by taking the transpose of each frontal slice and then reversing the ordering of all but the first frontal slice (see diagram below).

If $\mathcal{A}$ is an order-p tensor with $p > 3$ in $n_1 \times ... \times n_p$ then $\mathcal{A}^T$ is the tensor in $n_2 \times n_1 \times n_3 \times ... \times n_p$ found by reversing the ordering of all but the first frontal block and then by taking the transpose of each frontal block, which gives

$$\mathcal{A}^T = \texttt{fold}\left(\begin{bmatrix} (\mathcal{A}_0)^T \\ (\mathcal{A}_{n_p-1})^T \\ (\mathcal{A}_{n_p-2})^T \\ \vdots \\ (\mathcal{A}_1)^T \end{bmatrix}\right). \tag{2.1.1}$$

Another characterisation is as follows.

Proposition 2.12. *If $\mathcal{A}$ is an order-p tensor with in $n_1 \times ... \times n_p$ then $\mathcal{A}^T$ is the tensor in $n_2 \times n_1 \times n_3 \times ... \times n_p$ given by*

$$\mathcal{A}^T(:,:,J) = \mathcal{A}(:,:,-J)^T. \tag{2.1.2}$$

Proof. We denote the transpose given by (2.1.2) as $(\cdot)^{T'}$ and we will show that it is equivalent to (2.1.1). For dimension-3 tensors, inspecting the definition of the transpose shows that this is immediately true. Now we assume it is true for $p-1$ dimensional tensors and let $\mathcal{A}$ be a p-dimensional tensor. By the applying the first definition, the inductive hypothesis and recalling that we take all indices modulo that dimension (so in the ith dimension, $-j = n_i - j$), we have that

$$\mathcal{A}^T = \texttt{fold}\left(\begin{bmatrix} (\mathcal{A}_0)^T \\ (\mathcal{A}_{n_p-1})^T \\ (\mathcal{A}_{n_p-2})^T \\ \vdots \\ (\mathcal{A}_1^{n_p})^T \end{bmatrix}\right) = \texttt{fold}\left(\begin{bmatrix} (\mathcal{A}_{n_p-0})^T \\ (\mathcal{A}_{n_p-1})^T \\ (\mathcal{A}_{n_p-2})^T \\ \vdots \\ (\mathcal{A}_{n_p-(n_p-1)})^T \end{bmatrix}\right) = \texttt{fold}\left(\begin{bmatrix} (\mathcal{A}_0)^{T'} \\ (\mathcal{A}_{-1})^{T'} \\ (\mathcal{A}_{-2})^{T'} \\ \vdots \\ (\mathcal{A}_{-(n_p-1)})^{T'} \end{bmatrix}\right) = \mathcal{A}^{T'}.$$

□

Remark 2.13. Note that $\texttt{bcirc}(\mathcal{A}^T) = \texttt{bcirc}(\mathcal{A})^T$, since

$$\texttt{bcirc}(\mathcal{A}^T)_{i,j} = \mathcal{A}^T\,(:,:,J) = \mathcal{A}\,(:,:,\mathbf{n}-J)^T = \texttt{bcirc}(\mathcal{A})^T_{j,i}$$

where the index switch follows because, from Lemma 2.5, the kth dimension of J is given by

$$\left(\left\lfloor \frac{i \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{j \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor\right) \bmod_{n_k}$$

so

$$n_k - \left(\left\lfloor \frac{i \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{j \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor\right) \bmod_{n_k} = \left(\left\lfloor \frac{j \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor - \left\lfloor \frac{i \bmod_{\tilde{n}_k}}{\tilde{n}_{k+1}} \right\rfloor\right) \bmod_{n_k}.$$

Now we have all the machinery we need to define orthogonality, which is now simple.

Definition 2.14. A tensor $\mathcal{Q}$ in $n_1 \times n_1 \times n_3 \times \ldots \times n_p$ is orthogonal if $\mathcal{Q}^T * \mathcal{Q} = \mathcal{Q} * \mathcal{Q}^T = \mathcal{I}$.

Proposition 2.15. *A tensor $\mathcal{A}$ is invertible (orthogonal) if and only if* $\texttt{bcirc}(\mathcal{A})$ *is invertible (orthogonal). Also,* $\texttt{bcirc}(\mathcal{A}^{-1}) = \texttt{bcirc}(\mathcal{A})^{-1}$.

Proof. From Corollary 2.22 (proved below), we have that $\texttt{bcirc}(\mathcal{A}*\mathcal{B}) = \texttt{bcirc}(\mathcal{A})\,\texttt{bcirc}(\mathcal{B})$. Then the following calculation demonstrates the point of invertible tensors

$$I = \texttt{bcirc}(\mathcal{I}) = \texttt{bcirc}(\mathcal{A} * \mathcal{A}^{-1}) = \texttt{bcirc}(\mathcal{A})\,\texttt{bcirc}(\mathcal{A}^{-1})$$

The same calculation yields the desired result for orthogonal tensors. $\square$

2.2 Diagonalisation

Now we define an important matrix operation that will be used to diagonalise circulant matrices. This will save is large amounts of computation time and will also help to derive an SVD for tensors. First, we need the following definitions.

Definition 2.16. The Kronecker product $\otimes : (n_1 \times n_2) \times (n_3 \times n_4) \to n_1 n_3 \times n_2 n_4$ is given by

$$A \otimes B = \begin{bmatrix} A_{00}B & \dots & A_{0(n_2-1)}B \\ \vdots & \ddots & \vdots \\ A_{(n_1-1)0}B & \dots & A_{(n_1-1)(n_2-1)}B \end{bmatrix}$$

Definition 2.17. The discrete Fourier transform of dimension n, F_n is the matrix with elements $(F_n)_{jk} = \omega_n^{jk}$ for j, k in $0, 1, ...n-1$ where $\omega_n = e^{\frac{i2\pi}{n}}$, so

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

We will slightly abuse notation and write

$$\tilde{F} = \frac{1}{\sqrt{\tilde{n}_3}} \left(F_{n_p} \otimes ... \otimes F_{n_3} \otimes I_{n_1} \right), \qquad \tilde{F} = \frac{1}{\sqrt{\tilde{n}_3}} \left(F_{n_p} \otimes ... \otimes F_{n_3} \otimes I_{n_2} \right),$$

where the dimension of the matrix on the end will be implied by what it is taking the product with. These matrices are orthogonal (Martin et al., 2013). We now have the following important theorem.

Theorem 2.18. [Martin et al. (2013)] *If $\mathcal{A}$ in $n_1 \times ... \times n_p$ then*

$$\texttt{tdiag}(\mathcal{A}) := \tilde{F}\, \texttt{bcirc}(\mathcal{A}) \tilde{F}^* \tag{2.2.1}$$

is a block diagonal matrix with $\tilde{n}_3$ blocks, each of size $n_1 \times n_2$.

Corollary 2.19. *The ith block (for i in $0...\tilde{n}_3$) on the diagonal of $\texttt{tdiag}(\mathcal{A})$ is given by $D_i = \tilde{F}_i \, \texttt{unfold}(\mathcal{A})$ where $\tilde{F}_i$ is the ith row of $\tilde{F}$.*

Proof. First we note that from Theorem 2.18 we can write

$$\texttt{tdiag}(\mathcal{A}) = \tilde{F} \, \texttt{bcirc}(\mathcal{A}) \tilde{F}^* = \texttt{bdiag}(\{D_i\}_{i=0}^{\tilde{n}_3})$$

so, as $\tilde{F}^* \tilde{F} = I$, we have

$$\tilde{F} \, \texttt{bcirc}(\mathcal{A}) = \texttt{bdiag}(\{D_i\}_{i=0}^{\tilde{n}_3}) \tilde{F}$$

The 0th block column of $\tilde{F}$ is a column of identities, so the 0th block column of $\texttt{bdiag}(\{D_i\}_{i=0}^{\tilde{n}_3}) \tilde{F}$ is $\{D_i\}_{i=0}^{\tilde{n}_3}$ arranged as a block column. And the ith entry of the 0th column of $\tilde{F} \, \texttt{bcirc}(\mathcal{A})$ is the ith row of $\tilde{F}$ times the 0th column of $\texttt{bcirc}(\mathcal{A})$ which is given by $\texttt{unfold}(\mathcal{A})$ (Lemma 2.5). □

Lemma 2.20. [Martin et al. (2013)] *If $D = \texttt{bdiag}(\{D_i\}_{i=0}^{\tilde{n}_3})$ then $\tilde{F} D \tilde{F}^*$ has the required block circulant structure so that $\texttt{bcirc}^{-1} \tilde{F} D \tilde{F}^* = \mathcal{D}'$ for some tensor $\mathcal{D}'$.*

Lemma 2.21. [Martin et al. (2013)] *If $\mathcal{C} = \mathcal{A} * \mathcal{B}$ then $\texttt{tdiag}(\mathcal{C}) = \texttt{tdiag}(\mathcal{A}) \, \texttt{tdiag}(\mathcal{B})$.*

Corollary 2.22. *The $\texttt{bcirc}$ operation has the following property: $\texttt{bcirc}(\mathcal{A} * \mathcal{B}) = \texttt{bcirc}(\mathcal{A}) \, \texttt{bcirc}(\mathcal{B})$.*

Proof. We have that $\texttt{tdiag}(\mathcal{C}) = \texttt{tdiag}(\mathcal{A}) \, \texttt{tdiag}(\mathcal{B})$, so cancelling the Fourier transforms gives $\texttt{bcirc}(\mathcal{A}) \, \texttt{bcirc}(\mathcal{B}) = \texttt{bcirc}(\mathcal{C}) = \texttt{bcirc}(\mathcal{A} * \mathcal{B})$. □

2.3 Factorisation

Now we can derive the t-SVD. As we will see, it shares many of the same properties of the regular SVD.

Theorem 2.23. *A p-dimensional tensor $\mathcal{A}$ in $n_1 \times n_2 \times ... \times n_p$ can be factored as*

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$$

where $\mathcal{U}$ is in $n_1 \times n_1 \times n_3 \times ... \times n_p$ and $\mathcal{V}$ is in $n_2 \times n_2 \times n_3 \times ... \times n_p$ are orthogonal and $\mathcal{S}$ is in $n_1 \times n_2 \times ... \times n_p$ with the properties that $\mathcal{S}(i, j, .., k) = 0$ unless $i = j$ and if $\mathcal{S}(a, a, :, ..., :) = \mathbf{0}$, then $\mathcal{S}(b, b, : ..., :) = \mathbf{0}$ for all $b \geq a$.

Proof. To see this we use Theorem 2.18 to block-diagonalise and then take the SVD of each block on the diagonal:

$$\begin{aligned}
\tilde{F}\,\texttt{bcirc}(\mathcal{A})\tilde{F}^* &= \texttt{bdiag}(\{D_i\}_{i=1}^{\tilde{n}_3}) \\
&= \texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{V_i^T\}_{i=1}^{\tilde{n}_3}) \\
&= \texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\tilde{F}\tilde{F}^*\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\tilde{F}\tilde{F}^*\,\texttt{bdiag}(\{V_i^T\}_{i=1}^{\tilde{n}_3})
\end{aligned}$$

where $D_i = U_i S_i V_i^T$. Then we define $\mathcal{U} = \texttt{bcirc}^{-1}(\tilde{F}^*\,\texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\tilde{F})$, $\mathcal{S} = \texttt{bcirc}^{-1}(\tilde{F}^*\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\tilde{F})$, and $\mathcal{V}^T = \texttt{bcirc}^{-1}(\tilde{F}^*\,\texttt{bdiag}(\{V_i^T\}_{i=1}^{\tilde{n}_3})\tilde{F})$ (which are all well defined by Lemma 2.20). Then we have that:

$$\begin{aligned}
\mathcal{U} * \mathcal{S} * \mathcal{V}^T &= \mathcal{U} * \mathcal{S} * \mathcal{V}^T * \mathcal{I} \\
&= \texttt{fold}\left(\texttt{bcirc}(\mathcal{U})\,\texttt{bcirc}(\mathcal{S})\,\texttt{bcirc}(\mathcal{V}^T)\,\texttt{unfold}(\mathcal{I})\right) \\
&= \texttt{fold}\left(\tilde{F}^*\,\texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\tilde{F}\tilde{F}^*\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\tilde{F}\tilde{F}^*\,\texttt{bdiag}(\{V_i^*\}_{i=1}^{\tilde{n}_3})\tilde{F}\,\texttt{unfold}(\mathcal{I})\right) \\
&= \texttt{fold}\left(\tilde{F}^*\,\texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{V_i^*\}_{i=1}^{\tilde{n}_3})\tilde{F}\,\texttt{unfold}(\mathcal{I})\right) \\
&= \texttt{fold}\left(\tilde{F}^*\,\texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{S_i\}_{i=1}^{\tilde{n}_3})\,\texttt{bdiag}(\{V_i^*\}_{i=1}^{\tilde{n}_3})\tilde{F}\,\texttt{unfold}(\mathcal{I})\right) \\
&= \texttt{fold}\left(\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{I})\right) \\
&= \mathcal{A} * \mathcal{I} \\
&= \mathcal{A}.
\end{aligned}$$

To see that $\mathcal{U}$ is orthogonal, we have that

$$\begin{aligned}\mathcal{U} * \mathcal{U}^T &= \mathcal{U} * \mathcal{U}^T * \mathcal{I} = \texttt{fold}(\texttt{bcirc}(\mathcal{U})\,\texttt{bcirc}(\mathcal{U}^T)\,\texttt{unfold}(\mathcal{I})) \\ &= \texttt{fold}(\tilde{F}^*\,\texttt{bdiag}(\{U_i\}_{i=1}^{\tilde{n}_3})\tilde{F}\tilde{F}^*\,\texttt{bdiag}(\{U_i^T\}_{i=1}^{\tilde{n}_3})\tilde{F}\,\texttt{unfold}(\mathcal{I})) = \mathcal{I}\end{aligned}$$

and a nearly identical calculation shows that $\mathcal{V} * \mathcal{V}^T = \mathcal{I}$. The fact that $\mathcal{S}(i, j, .., k) = \mathbf{0}$ unless $i = j$ follows because the Fourier transform acts on the blocks of size $n_1 \times n_2$, and so the diagonal structure within the blocks is preserved. The tensor $\mathcal{S}$ also has the property that if $\mathcal{S}(a, a, :, ..., :) = \mathbf{0}$, then $\mathcal{S}(b, b, : ..., :) = \mathbf{0}$ for all $b \geq a$. This also follows because the Fourier transform preserves the diagonal structure within the blocks, and these blocks are diagonal matrices from the middle of SVDs (which possess the property that $S_{aa} = 0 \implies S_{bb} = 0$ for all $b \geq a$). $\square$

Remark 2.24. We can define square roots and inverse square roots (when the tensor is invertible) by taking the square roots and inverse square roots of the diagonal matrix with positive values $\texttt{bdiag}(\{D_i\}_{i=1}^{\tilde{n}_3})$, and then apply the Fourier transform and inverse block circulant in the same way that the t-SVD was defined.

Remark 2.25. If $\mathcal{A}, \mathcal{B}$ are in the set $\{\mathcal{S} | \mathcal{S}(i, j, .., k) = 0 \text{ unless } i = j\}$ then $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$.

Proof. Using Lemma 2.6, we can write the t-product as follows, and then use the fact that diagonal matrices commute to get

$$\mathcal{A} * \mathcal{B}(:, :, J) = \sum_I \mathcal{A}(:, :, I)\mathcal{B}(:, :, J - I) = \sum_I \mathcal{B}(:, :, J - I)\mathcal{A}(:, :, I) = \mathcal{B} * \mathcal{A}(:, :, J)$$

$\square$

For this thesis, there are two ways of defining the rank of a tensor that will be useful for us.

Definition 2.26. The tensor average rank of $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$ in $n_1 \times ... \times n_p$ is given by

$$\text{rank}_\text{a}(\mathcal{A}) = \frac{1}{\tilde{n}_3}\texttt{rank}(\texttt{bcirc}(\mathcal{A})) = \frac{1}{\tilde{n}_3}\texttt{rank}(\tilde{F}\,\texttt{bcirc}(\mathcal{S})\tilde{F}^*).$$

Definition 2.27. The tensor tubal rank of $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$ in $n_1 \times ... \times n_p$ is given by the number of non-zero tubes of $\mathcal{S}$:

$$\mathrm{rank_t}(\mathcal{A}) = \#\{i, \mathcal{S}(i, i, :, ..., :) \neq 0\} = \min\{\mathtt{rank}(S_i)\}.$$

The tensor tubal rank will be used a lot more in this thesis, so we will use the word rank to refer to the tensor tubal rank. The two are connected though, with the tensor average rank providing a lower bound for the tensor tubal rank (Lu et al., 2018). So low-rank tensors (referring to low tubal rank), necessarily have low average rank.

Remark 2.28. We can also define a skinny t-SVD. We derive this in the same way as the regular t-SVD except that we use skinny SVDs instead of regular. To ensure that the dimensions line up, we keep $\mathrm{rank_t}(\mathcal{A})$ singular values and their associated rows and columns, so these skinny SVDs may be of dimension slightly larger than normal skinny SVDs and may have some zero singular values. In particular, for $\mathrm{rank_t}(\mathcal{A}) = r$ we have $\mathcal{A} = \mathcal{U}' * \mathcal{S}' * \mathcal{V}'^T$; however, $\mathcal{U}'$ is in $n_1 \times r \times n_3 ... \times n_p$, $\mathcal{V}'^T$ is in $r \times n_2 \times n_3 ... \times n_p$ and $\mathcal{S}'$ is in $r \times r \times n_3 ... \times n_p$, so $\mathcal{U}'$ and $\mathcal{V}'$ are not orthogonal, though $\mathcal{U}'^T * \mathcal{U}' = \mathcal{V}'^T * \mathcal{V}' = \mathcal{I}$ is in $r \times r \times n_3 ... \times n_p$.

2.4 Norms

Since the tensor optimisation problem at the heart of this thesis involves the minimisation of the sum of two tensor norms, the following definitions and basic results will be used frequently throughout. We start with two simple norms that are very similar to their matrix and vector counterparts.

Definition 2.29. The l_1 norm of a tensor $\mathcal{A}$ in $n_1 \times ... \times n_p$ with indices in J is defined as

$$\|\mathcal{A}\|_1 = \sum_J |\mathcal{A}_J|.$$

Definition 2.30. The infinity norm of a tensor $\mathcal{A}$ is given by

$$\|\mathcal{A}\|_\infty = \max_J |\mathcal{A}_J|.$$

Proposition 2.31. $\langle \mathcal{A}, \mathcal{B} \rangle = \sum_J \mathcal{A}_J^* \mathcal{B}_J \leq \max_J |\mathcal{B}_J| \sum_J |\mathcal{A}_J| = \|\mathcal{B}\|_\infty \|\mathcal{A}\|_1$.

Next, we define an inner product for tensors.

Definition 2.32. The inner product of A and B in $n_1 \times n_2$ is given by

$$\langle A, B \rangle = \mathtt{trace}(A^* B) = \sum_{i,j} A_{ij}^* B_{ij}.$$

and for $\mathcal{A}$ and $\mathcal{B}$ in $n_1 \times ... \times n_p$ it is given by

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i=0}^{n_p - 1} \langle \mathcal{A}_i^{n_p}, \mathcal{B}_i^{n_p} \rangle.$$

We equivalently can write that

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_J \mathcal{A}_J^* \mathcal{B}_J.$$

To see that these are equivalent, we note first that it is true for the 2-dimensional case by the definition above. Then, proceeding by induction, we assume it is true for the $p-1$ dimensional case. Then by applying the definition and the inductive hypothesis we get

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i=0}^{n_p - 1} \langle \mathcal{A}_i^{n_p}, \mathcal{B}_i^{n_p} \rangle = \sum_{i=0}^{n_p - 1} \sum_J (\mathcal{A}_i^{n_p})_J^* (\mathcal{B}_i^{n_p})_J = \sum_J \mathcal{A}_J^* \mathcal{B}_J.$$

Remark 2.33. Using the following definition of the trace, $\mathtt{trace}(\mathcal{A}) = \mathtt{trace}(\mathcal{A}(:, :, \mathbf{0}))$, we have that $\langle \mathcal{A}, \mathcal{B} \rangle = \mathtt{trace}(\mathcal{A}^* * \mathcal{B})$. We derive this result by using the characterisation of the tensor product from Lemma 2.6 and of the transpose from Proposition 2.12 and then noting that

$$\begin{aligned}
\texttt{trace}(\mathcal{A}^* * \mathcal{B}) &= \texttt{trace}(\mathcal{A}^* * \mathcal{B}(:,:,\mathbf{0})) \\
&= \texttt{trace}\left(\sum_J \mathcal{A}^*(:,:,J)\mathcal{B}(:,:,\mathbf{0}-J)\right) \\
&= \texttt{trace}\left(\sum_J \mathcal{A}(:,:,-J)^*\mathcal{B}(:,:,-J)\right) \\
&= \sum_J \texttt{trace}\left(\mathcal{A}(:,:,J)^*\mathcal{B}(:,:,J)\right) \\
&= \sum_J \mathcal{A}_J^*\mathcal{B}_J.
\end{aligned}$$

Remark 2.34. The adjoint of tensor multiplication is tensor multiplication by the Hermitian tensor: $\langle \mathcal{A} * \mathcal{B}, \mathcal{C}\rangle = \langle \mathcal{B}, \mathcal{A}^* * \mathcal{C}\rangle$. Using the characterisation of the inner product from above, we have that $\langle \mathcal{A} * \mathcal{B}, \mathcal{C}\rangle = \texttt{trace}(\mathcal{B}^* * \mathcal{A}^* * \mathcal{C}) = \langle \mathcal{B}, \mathcal{A}^* * \mathcal{C}\rangle$.

Remark 2.35. Because the circulant structure repeats the same elements multiple times, we also have that

$$\begin{aligned}
\langle \texttt{bcirc}(\mathcal{A}), \texttt{bcirc}(\mathcal{B})\rangle &= \langle \texttt{bcirc}^{p-2}(\mathcal{A}), \texttt{bcirc}^{p-2}(\mathcal{B})\rangle \\
&= n_3\langle \texttt{bcirc}^{p-3}(\mathcal{A}), \texttt{bcirc}^{p-3}(\mathcal{B})\rangle = \tilde{n}_3\langle \mathcal{A}, \mathcal{B}\rangle.
\end{aligned}$$

Definition 2.36. We define $\mathbf{e}_J$ as the tensor with a 1 in its Jth element and 0s elsewhere.

Remark 2.37. [Z. Zhang & Aeron (2017)] If $\mathbf{e}_J$ in $n_1 \times ... \times n_p$ with $J = j_1...j_p$ then for $\overset{c}{\mathbf{e}}_{j_1}$ in $n_1 \times 1 \times n_3 \times ... \times n_p$ with $\overset{c}{\mathbf{e}}_{j_1} = 0$ everywhere other than a 1 in $(j_1, 0, 0, ..., 0)$, $\overset{\bullet}{\mathbf{e}}_{J'}$ in $1 \times 1 \times n_3 \times ... \times n_p$ with $\overset{r}{\mathbf{e}}_{j_1} = 0$ everywhere other than a 1 in $(0, 0, j_3, ..., j_p)$, and $\overset{r}{\mathbf{e}}_{j_2}$ in $1 \times n_2 \times n_3 \times ... \times n_p$ with $\overset{r}{\mathbf{e}}_{j_1} = 0$ everywhere other than a 1 in $(0, j_2, 0, ..., 0)$, then $\mathbf{e}_J = \overset{c}{\mathbf{e}}_{j_1} * \overset{\bullet}{\mathbf{e}}_{J'} * \overset{r}{\mathbf{e}}_{j_2}$.

Remark 2.38. We have that $\langle \mathbf{e}_J, \mathcal{A}\rangle = \mathcal{A}(J)$ and that $\mathcal{A} = \sum_J \langle \mathbf{e}_J, \mathcal{A}\rangle \mathbf{e}_J$.

Next we have the norm induced by this inner product.

Definition 2.39. The Frobenius norm of a tensor $\mathcal{A}$ in $n_1 \times ... \times n_p$ is given by

$$\|\mathcal{A}\|_F = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle} = \sqrt{\sum_J |\mathcal{A}_J|^2}.$$

From this and the inner product above we get the Cauchy–Schwarz inequality

$$\langle \mathcal{A}, \mathcal{B} \rangle \leq \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle} \sqrt{\langle \mathcal{B}, \mathcal{B} \rangle} = \|\mathcal{A}\|_F \|\mathcal{B}\|_F. \tag{2.4.1}$$

Using Remark 2.35 for the second equality, we get

$$\| \mathtt{bcirc}(\mathcal{A}) \|_F = \sqrt{\langle \mathtt{bcirc}(\mathcal{A}), \mathtt{bcirc}(\mathcal{A}) \rangle} = \sqrt{\tilde{n}_3 \langle \mathcal{A}, \mathcal{A} \rangle} = \sqrt{\tilde{n}_3} \|\mathcal{A}\|_F. \tag{2.4.2}$$

Using the matrix bound $\|AB\|_F \leq \|A\|_F \|B\|_F$ and the calculation above we get that

$$\|\mathcal{A} * \mathcal{B}\|_F = \| \mathtt{bcirc}(\mathcal{A}) \, \mathtt{unfold}(\mathcal{B}) \|_F \leq \| \mathtt{bcirc}(\mathcal{A}) \|_F \| \mathtt{unfold}(\mathcal{B}) \|_F = \sqrt{\tilde{n}_3} \|\mathcal{A}\|_F \|\mathcal{B}\|_F. \tag{2.4.3}$$

Using the other characterisation of the dot product from Remark 2.33, we have that multiplication by orthogonal tensors preserves the Frobenius norm

$$\|\mathcal{Q} * \mathcal{A}\|_F^2 = \mathtt{trace}(\mathcal{A}^* * \mathcal{Q}^* * \mathcal{Q} * \mathcal{A}) = \mathtt{trace}(\mathcal{A}^* * \mathcal{A}) = \|\mathcal{A}\|_F^2. \tag{2.4.4}$$

And finally, using some of the above facts also gives

$$\| \mathtt{tdiag}(\mathcal{A}) \|_F^2 = \| \tilde{F} \, \mathtt{bcirc}(\mathcal{A}) \tilde{F}^* \|_F^2 = \tilde{n}_3 \|\mathcal{A}\|_F^2. \tag{2.4.5}$$

Now we use the Frobenius norm to define the spectral norm of a tensor.

Definition 2.40. The spectral norm of a tensor $\mathcal{A}$ in $n_1 \times ... \times n_p$ is given by

$$\|\mathcal{A}\| = \sup_{\|\mathcal{B}\|_F = 1} \|\mathcal{A} * \mathcal{B}\|_F,$$

where $\mathcal{B} \in n_2 \times 1 \times n_3 \times ... \times n_p$.

Note that this is the operator norm of tensor multiplication, measured by the Frobenius norm (which was induced by the inner product). From this we also note that

$$\|\mathcal{A}\| = \sup_{\|\mathcal{B}\|_F=1} \|\mathcal{A} * \mathcal{B}\|_F = \sup_{\|\,\mathtt{unfold}(\mathcal{B})\|_F=1} \|\,\mathtt{bcirc}(\mathcal{A})\,\mathtt{unfold}(\mathcal{B})\|_F = \|\,\mathtt{bcirc}(\mathcal{A})\|. \tag{2.4.6}$$

And since the spectral norm is induced the by Frobenius norm, which is invariant under multiplication by orthogonal tensors, the spectral norm is invariant under multiplication by orthogonal tensors. In particular,

$$\|\mathcal{A}\| = \|\,\mathtt{bcirc}(\mathcal{A})\| = \|\tilde{F}\,\mathtt{bcirc}(\mathcal{A})\tilde{F}^*\| = \|\,\mathtt{tdiag}\,(\mathcal{A})\|. \tag{2.4.7}$$

Also, we use the fact that $\|\mathcal{A} * \mathcal{B}\|_F \leq \sqrt{\tilde{n}_3}\|\mathcal{A}\|_F\|\mathcal{B}\|_F$ (Eq. (2.4.3)) to get

$$\|\mathcal{A}\| = \sup_{\|\mathcal{B}\|_F=1} \|\mathcal{A} * \mathcal{B}\|_F \leq \sup_{\|\mathcal{B}\|_F=1} \sqrt{\tilde{n}_3}\|\mathcal{A}\|_F\|\mathcal{B}\|_F = \sqrt{\tilde{n}_3}\|\mathcal{A}\|_F. \tag{2.4.8}$$

Finally, we are ready to introduce the tensor nuclear norm, which we define as the dual norm of the spectral norm.

Definition 2.41. The tensor nuclear norm of a tensor $\mathcal{A}$ in $n_1 \times ... \times n_p$ is given by

$$\|\mathcal{A}\|_* = \sup_{\|\mathcal{B}\| \leq 1} \langle \mathcal{A}, \mathcal{B} \rangle.$$

An important point to note here is that that the lower convex envelope (closest approximation from below by a convex function) of the tensor average rank is the tensor nuclear norm (Lu et al., 2018). And, as signposted throughout the introduction of these definitions, the tensor nuclear norm is not arbitrarily defined: rather, it is the dual of the operator norm that arises from tensor multiplication (when measured under the Frobenius norm), which is induced by the inner product.

Remark 2.42. We briefly note a few different ways of writing the tensor nuclear norm

$$\|\mathcal{A}\|_* = \sup_{\|\mathcal{B}\|\le 1} \langle \mathcal{A}, \mathcal{B} \rangle = \frac{1}{\tilde{n}_3} \sup_{\|\,\texttt{bcirc}(\mathcal{B})\|\le 1} \langle \texttt{bcirc}(\mathcal{A}), \texttt{bcirc}(\mathcal{B}) \rangle = \frac{1}{\tilde{n}_3} \|\,\texttt{bcirc}(\mathcal{A})\|_*,$$

where the third equality holds by the definition of matrix nuclear norm (as the dual of the matrix spectral norm). Writing the t-SVD of $\mathcal{A}$ as $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$, we also have by using the definition of the matrix nuclear norm as the sum of its singular values that

$$\|\mathcal{A}\|_* = \frac{1}{\tilde{n}_3} \langle \tilde{F}\, \texttt{bcirc}(\mathcal{S}) \tilde{F}^*, I \rangle = \frac{1}{\tilde{n}_3} \langle \texttt{bcirc}(\mathcal{S}), I \rangle = \langle \mathcal{S}, \mathcal{I} \rangle = \sum_{i=0}^{\mathrm{rank_t}(\mathcal{A})-1} \mathcal{S}(i, i, \mathbf{0}),$$

which interestingly only depends on the values of $\mathcal{S}$ in the first frontal slice. These values, however are related to all the values in $\mathcal{S}$ via its derivation using the Fourier transform. Finally, we also have that

$$\|\mathcal{A}\|_* = \frac{1}{\tilde{n}_3} \langle \tilde{F}\, \texttt{bcirc}(\mathcal{S}) \tilde{F}^*, I \rangle = \frac{1}{\tilde{n}_3} \langle \tilde{F}\, \texttt{bcirc}(\mathcal{S})^{\frac{1}{2}} \tilde{F}^*, \tilde{F}\, \texttt{bcirc}(\mathcal{S})^{\frac{1}{2}} \tilde{F}^* \rangle = \frac{1}{\tilde{n}_3} \|\tilde{F}\, \texttt{bcirc}(\mathcal{S})^{\frac{1}{2}} \tilde{F}^*\|_F^2.$$

Proposition 2.43. $\langle \mathcal{A}, \mathcal{B} \rangle \le \|\mathcal{B}\| \|\mathcal{A}\|_*$

Proof. Note that

$$\frac{\langle \mathcal{A}, \mathcal{B} \rangle}{\|\mathcal{B}\|} \le \sup_{\|\mathcal{C}\|\le 1} \langle \mathcal{A}, \mathcal{C} \rangle = \|\mathcal{A}\|_*.$$

□

Now that we have the tensor norms and a few basic results, we can start talking about how we will perform tensor optimisation.

2.5 Optimisation

For every differentiable convex function f, it holds that

$$f(y) \ge f(x) + \nabla f(x)^T (y - x).$$

We can generalise this idea to non-differentiable functions as follows.

Definition 2.44. g is a subgradient of f at x if

$$f(y) \geq f(x) + g(x)^T (y - x),$$

in which case we write $g \in \partial f$.

This definition allows us to use the below optimisation method, which does not require the objective function to be differentiable.

Theorem 2.45. [Boyd & Vandenberghe (2008)] *If f is convex then*

$$f(x^*) = \inf_x f(x) \iff \mathbf{0} \in \partial f(x^*).$$

In this thesis, two subgradients will be of particular use.

Lemma 2.46. [Lu et al. (2018)] *The subgradient of the tensor nuclear norm of a tensor $\mathcal{A}$ with skinny t-SVD given by $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^*$ is*

$$\{\mathcal{U} * \mathcal{V}^* + \mathcal{W} \,|\, \mathcal{U}^* * \mathcal{W} = 0,\, \mathcal{W} * \mathcal{V} = 0,\, \|\mathcal{W}\| \leq 1\}.$$

Lemma 2.47. [Boyd & Vandenberghe (2008)] *The subgradient of the l_1 norm at $\mathcal{A}$ is given by*

$$\partial \|\mathcal{A}\|_1 = \{\mathcal{G} \,|\, \|\mathcal{G}\|_\infty \leq 1, \langle \mathcal{G}, \mathcal{A} \rangle = \|\mathcal{A}\|_1\}.$$

2.6 Some Useful Derivatives

The following derivatives will be useful in the later chapters.

Lemma 2.48. $\dfrac{d}{d\mathcal{B}} \dfrac{1}{2} \|\mathcal{A} * \mathcal{B}\|_F^2 = \mathcal{A}^T * \mathcal{A} * \mathcal{B}.$

Proof. We have

$$\begin{aligned}
\frac{d}{d\mathcal{B}}\frac{1}{2}\|\mathcal{A}*\mathcal{B}\|_F^2 &= \frac{d}{d\,\texttt{unfold}(\mathcal{B})}\frac{1}{2}\|\,\texttt{fold}(\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B}))\|_F^2 \\
&= \frac{d}{d\,\texttt{unfold}(\mathcal{B})}\frac{1}{2}\|\,\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B})\|_F^2 \\
&= \frac{d}{d\,\texttt{unfold}(\mathcal{B})}\frac{1}{2}\texttt{trace}((\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B}))^T\,\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B})) \\
&= \frac{d}{d\,\texttt{unfold}(\mathcal{B})}\frac{1}{2}\texttt{trace}(\texttt{unfold}(\mathcal{B})^T\,\texttt{bcirc}(\mathcal{A})^T\,\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B})) \\
&= \texttt{bcirc}(\mathcal{A})^T\,\texttt{bcirc}(\mathcal{A})\,\texttt{unfold}(\mathcal{B}) \\
&= \mathcal{A}^T*\mathcal{A}*\mathcal{B},
\end{aligned}$$

where the second-last line follows from the standard matrix identity (e.g. Roweis (1999)).

□

Lemma 2.49. $\dfrac{d}{d\mathcal{A}}\dfrac{1}{2}\|\mathcal{A}*\mathcal{B}\|_F^2 = \mathcal{A}*\mathcal{B}*\mathcal{B}^T.$

Proof. To show that this is true, we show that every frontal slice is the same on each side of the equality. Using the other characterisation of the t-product from Lemma 2.6 and the standard matrix identity (e.g. Roweis (1999)), we get

$$\begin{aligned}
\frac{d}{d\mathcal{A}(:,:K)}\frac{1}{2}\|\mathcal{A}*\mathcal{B}\|_F^2 &= \frac{d}{d\mathcal{A}(:,:K)}\frac{1}{2}\sum_J\|\mathcal{A}*\mathcal{B}(:,;J)\|_F^2 \\
&= \frac{d}{d\mathcal{A}(:,:K)}\frac{1}{2}\sum_J\|\sum_I\mathcal{A}(:,:,I)\mathcal{B}(:,;J-I)\|_F^2 \\
&= \sum_J\sum_I\mathcal{A}(:,:,I)\mathcal{B}(:,;J-I)\mathcal{B}(:,;J-K)^T \\
&= \sum_J\sum_I\mathcal{A}(:,:,I)\mathcal{B}(:,;J-I)\mathcal{B}^T(:,;K-J) \\
&= \sum_J(\mathcal{A}*\mathcal{B})(:,;J)\mathcal{B}^T(:,;K-J) \\
&= (\mathcal{A}*\mathcal{B}*\mathcal{B}^T)(:,:K).
\end{aligned}$$

□

In this thesis, we will need to perform the tensor equivalent of a 'Taylor expansion'. As such, we make the following definition, with the function taking the intuitive definition.

Definition 2.50. The function $\texttt{tvec} : n_1 \times ... \times n_p \to \tilde{n}_1 \times 1$ flattens a tensor into a vector.

Remark 2.51. Suppose $f : n_1 \times .. \times n_p \to \mathbb{R}$ is infinitely differentiable with third derivative equal to 0. Then Taylor's theorem gives

$$\begin{aligned} f(\mathcal{B}) =& f(\mathcal{A}) + \texttt{tvec}(\mathcal{B} - \mathcal{A})^T (\nabla_{\mathcal{A}} f(\mathcal{A})) \\ &+ \frac{1}{2} \texttt{tvec}(\mathcal{B} - \mathcal{A})^T (\nabla_{\mathcal{A}} \nabla_{\mathcal{A}} f(\mathcal{A})) \, \texttt{tvec}(\mathcal{B} - \mathcal{A}). \end{aligned}$$

Formally, calculating the above expansion requires explicitly writing out the Hessian matrix and deciding on an ordering onto which `unfold` will map. In nearly every case in this thesis, this will not be necessary. Instead it is sufficient for our purposes to note that

$$\frac{d}{d\mathcal{A}} \texttt{unfold}(\mathcal{A} * \mathcal{B}) = \texttt{bcirc}(\mathcal{B}) \otimes I_{n_1}$$

since

$$\begin{aligned} \frac{d}{d\mathcal{A}(:,:,L)} (\mathcal{A} * \mathcal{C})(:,:,K) &= \frac{d}{d\mathcal{A}(:,:,L)} \sum_J \mathcal{A}(:,:,J) \mathcal{C}(:,:,K-J) \\ &= \frac{d}{d\mathcal{A}(:,:,L)} \mathcal{A}(:,:,L) \mathcal{C}(:,:,K-L) \\ &= \frac{d}{d\mathcal{A}(:,:,L)} \mathcal{A}(:,:,L) \mathcal{C}(:,:,K-L) \\ &= \mathcal{C}(:,:,K-L)^T \otimes I_{n_1}. \end{aligned}$$

Arranging these appropriately gives $\frac{d}{d\mathcal{A}} \texttt{unfold}(\mathcal{A} * \mathcal{C}) = \texttt{bcirc}(\mathcal{C}^T) \otimes I_{n_1}$. This means that when $\nabla_{\mathcal{A}} f(\mathcal{A}) = \mathcal{A} * \mathcal{C}$ for some tensor $\mathcal{C}$, by using the fact that $\| \texttt{bcirc}(\mathcal{A}) \|_F = \sqrt{\tilde{n}_3} \|\mathcal{A}\|_F$ (Eq. (2.4.2)), we get

$$\|\nabla_{\mathcal{A}} \nabla_{\mathcal{A}} f(\mathcal{A})\|_F = \| \texttt{bcirc}(\mathcal{C}^T) \otimes I_{n_1} \|_F = n_1 \| \texttt{bcirc}(\mathcal{C}^T) \|_F = n_1 \sqrt{\tilde{n}_3} \|\mathcal{C}\|_F.$$

We will need the explicit expression in one instance. Fortunately, it is the simplest case: $\nabla_{\mathcal{A}} f(\mathcal{A}) = \mathcal{A} * \mathcal{I} = \mathcal{A}$. In this case we have that $\frac{d}{d\mathcal{A}(J)} f(\mathcal{A}) = \mathcal{A}(J)$. Hence,

$\frac{d}{d\mathcal{A}(K)} \frac{d}{d\mathcal{A}(J)} f(\mathcal{A}) = 1$ when $K = J$ and 0 otherwise, so the Hessian is given by the identity, which in particular is positive definite.

2.7 SVD vs t-SVD Dimensionality Reduction

The following example is presented to demonstrate the utility of the t-SVD and its similarity with the SVD. The pixels of the satellite image in Figure 2.1 are 10-dimensional, but we can only view 3 dimensions at a time. As a result, projecting the $10 \times 565 \times 410$ tensor to $3 \times 565 \times 410$ in a way that minimises information loss is of great use.

Figure 2.1: The red, green and blue bands of a $10 \times 565 \times 410$ Sentinel-2 (recently launched satellite) observation of Denman Prospect, Wright, and Coombs (Canberra) at a single observation date. All of the other satellite images used in this thesis will be of the same area taken by the same satellite.

Selecting only 3 bands is not optimal, since it would ignore all the information captured in the other bands (such as infra-red and ultraviolet light). One method of reducing the dimensionality of the data is flattening it and then using Principal Component Analysis, where the data is projected onto the singular vectors corresponding to the largest singular values. We can use the t-SVD in an similar way that requires no flattening. If the t-SVD of $\mathcal{A}$ is given by $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^*$, then we can project $\mathcal{A}$ to a lower dimension by setting $\mathcal{A}' = \mathcal{U}^*(0:3, :, \ldots, :) * \mathcal{A}$ (see Hao, Kilmer, Braman, & Hoover (2013) for more details). There are many differences of course. For example, the objects corresponding to singular values in the tensor case are $p-1$ dimensional. So to measure their size we have to take a norm.

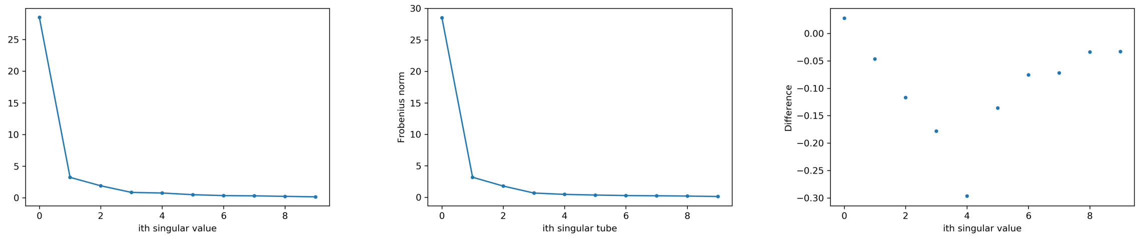

Figure 2.2: Left: the singular values of the matrix found by flattening the image in Fig. 2.1. Middle: the Frobenius norm of the singular tubes of the t-SVD of the image in Fig. 2.1. Right: the difference between these two sequences of values. While they are similar, they are not identical.

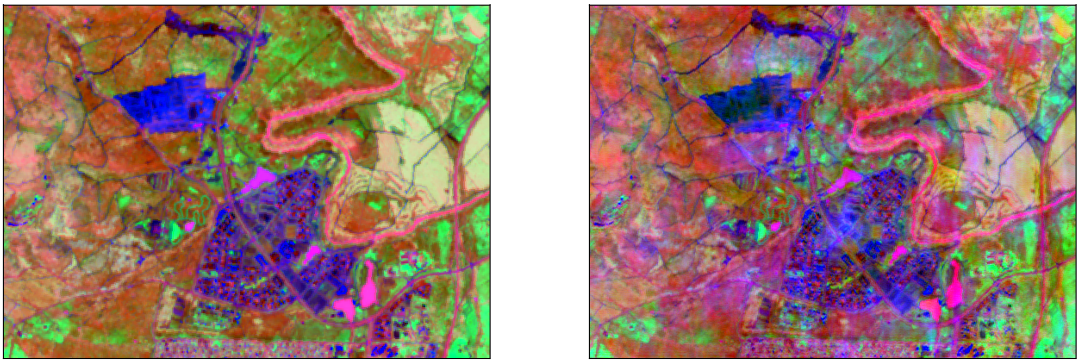

Figure 2.3: Left: matrix-based dimensionality reduction. Right: tensor-based dimensionality reduction. In both, we show the first principal component (PC) in the red channel, the second PC in the green channel, and the third PC in the blue channel. The reductions are similar and both do a good job of displaying information that was not clear before. For example, the contrast between urban (blue) and non-urban areas is made more stark, and the winding row of trees in the upper right-hand side of the image is turned bright pink — similar to the colour of the lake in the lower right-hand side — indicating that water is likely flowing there. To quantify the difference between the two approaches, the respective reconstruction errors were measured and it was found that the tensor-based approach performed better, with a reconstruction error of 0.9866 compared with the 1.3528 reconstruction error of the matrix-based approach.

Chapter 3

Tensor Robust Principal Component Analysis

The goal of this chapter is to prove that the Tensor Robust Principal Component Analysis (TRPCA) procedure does what it says it does: when we observe $\mathcal{M}_0$ that is the sum of a low-rank tensor $\mathcal{L}_0$ and a sparse noise tensor $\mathcal{C}_0$, the unique minimiser to

$$\min_{\mathcal{L},\mathcal{C}} \|\mathcal{L}\|_* + \lambda\|\mathcal{C}\|_1, \quad \text{s.t. } \mathcal{M}_0 = \mathcal{L} + \mathcal{C}$$

is $(\mathcal{L}_0, \mathcal{C}_0)$. The proof that we give is similar to the proof in Lu et al. (2018), but it differs in three ways. First, our proof is for tensors of arbitrary dimension rather than 3-dimensional tensors with square frontal faces. Second, our proof is more explicit as it gives specific constants in our bounds rather than leaving them unspecified. Third, our proof takes the probability that an element of the tensor contains noise (ρ) as exogenous (though bounded above by a relatively high bound), rather than relying on it being small to bound some terms. To achieve these results we need to rework many parts of the proofs in Lu et al. (2018). In particular, Lemma D.1 from Lu et al. (2018) was removed from the argument given here and an alternative approach was employed, resulting in a new bound at the end of Section 3.2.5.

To prove that the TRPCA procedure works, we need make some assumptions about the minimisation problem we are addressing. One of the main challenges is that the separation of the low-rank and sparse components of the data becomes intractable if the low-rank part is sparse or if the sparse noise is low rank. To avoid this situation, we assume that the support of the sparse noise is Bernoulli distributed (so then a low-rank sparse noise tensor is highly unlikely). We further assume that the orthogonal tensors in the skinny t-SVD of the low-rank part are not too 'incoherent' as measured by μ, which we define as the minimum m for which all of the following conditions (known as tensor incoherence conditions) hold simultaneously:

$$\max_{j\in 0\ldots n_1-1} \|\mathcal{U}^* * \overset{c}{\mathbf{e}}_j\|_F \leq \sqrt{\frac{mr}{n_1\tilde{n}_3}}, \tag{3.0.1}$$

$$\max_{j\in 0\ldots n_2-1} \|\mathcal{V}^* * \overset{r}{\mathbf{e}}_j\|_F \leq \sqrt{\frac{mr}{n_2\tilde{n}_3}}, \tag{3.0.2}$$

$$\|\mathcal{U} * \mathcal{V}^*\|_\infty \leq \sqrt{\frac{mr}{\tilde{n}_1\tilde{n}_3}}. \tag{3.0.3}$$

The minimum possible value of μ is 1; this would occur when the values of $\mathcal{U}(i, :, \ldots :)$, which are similar to rows of a matrix, are constant at $\frac{1}{\sqrt{n_1\tilde{n}_3}}$ and the values $\mathcal{V}(:, i, :, \ldots, :)$, which are similar to columns of a matrix, are constant at $\frac{1}{\sqrt{n_2\tilde{n}_3}}$. The maximum possible value of μ is $\frac{n_{(2)}\tilde{n}_3}{r}$; this would occur when one of the values of $\mathcal{U}(i, :, \ldots :)$ and $\mathcal{V}(:, i, :, \ldots, :)$ are 1. As μ gets smaller, we are seeing the information stored in each 'row' or 'column' 'spread out' across the matrix (and a 'spread out' matrix is the opposite of sparse), and so the problem gets easier to solve as μ gets smaller (Z. Zhang & Aeron, 2017). The third tensor incoherence condition above is less intuitive than the other two and is regarded by some as restrictive (Chen, 2015); it is, however, necessary for us to be able to solve this problem (Lu et al., 2018).

We are now ready to state and then prove the main theorem of this chapter.

Theorem 3.1. *Suppose that we observe $\mathcal{M}_0 = \mathcal{L}_0 + \mathcal{C}_0$ such that the signs of $\mathcal{C}_0$ are ± 1 with probability $\rho/2$ and 0 with probability $1-\rho$, the tensor $\mathcal{L}_0$ has the skinny t-SVD $\mathcal{U} * \mathcal{S} * \mathcal{V}^*$ and $\operatorname{rank}_t(\mathcal{L}_0) = r$, the tensor incoherence conditions Eq. (3.0.1) to Eq. (3.0.3) are satisifed with constant μ, and the following conditions are satisfied (where we write $n_{(1)} = \max\{n_1, n_2\}$ and $n_{(2)} = \min\{n_1, n_2\}$)*

$$\rho < \frac{436 - 6\sqrt{836}}{900} \approx 0.2917, \tag{3.0.4}$$

$$r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)}\tilde{n}_3}{(\log(n_{(1)}\tilde{n}_3))^2}, \tag{3.0.5}$$

then when

$$\lambda = \frac{1}{\sqrt{n_{(1)}\tilde{n}_3}} \tag{3.0.6}$$

we have that $(\mathcal{L}_0, \mathcal{C}_0)$ is the unique minimiser to

$$\min_{\mathcal{L},\mathcal{C}} \|\mathcal{L}\|_* + \lambda\|\mathcal{C}\|_1, \quad s.t.\ \mathcal{M}_0 = \mathcal{L} + \mathcal{C}$$

with 'high probability' (which for this thesis will mean greater than $1 - O((n_{(1)}\tilde{n}_3)^{-1})$).

Remark 3.2. We will fix $\varepsilon = \left(\frac{\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}\right)^{\frac{1}{4}}$ to yield the following inequalities that will be of use later. First we note that substituting in the bound for μr from Eq. (3.0.5) gives $\varepsilon < \frac{\sqrt{2\rho}}{5}$ since

$$\varepsilon = \left(\frac{\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}\right)^{\frac{1}{4}} \leq \left(\rho^2 \frac{4}{5^4}\right)^{\frac{1}{4}} = \sqrt{\rho \frac{2}{25}} < \frac{1}{5} < \exp(-1) \tag{3.0.7}$$

where the second-last inequality holds by the fact that we are assuming $\rho < \frac{1}{2}$. Rearranging the bound for r from Eq. (3.0.5) and substituting in the value we have fixed for ε gives

$$C_0 \frac{\epsilon^{-2}\mu r \log(n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3} \leq \rho \leq 1 - C_0 \frac{\epsilon^{-2}\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3} \tag{3.0.8}$$

where in this case, $C_0 = \frac{4}{5^4}$. In order to simplify some later analysis, on the left-hand side we have removed a power on the log (which makes the bound looser) and on the right-hand side we have used the fact that $\rho < 1 - \rho$ since $\rho < \frac{1}{2}$. For this analysis, we will fix

$$C_0 = \frac{32}{3} \tag{3.0.9}$$

which also loosens these bounds. There are many other choices for the constants that can be made to work. These ones were chosen above to obtain a success probability of $1 - O((n_{(1)}\tilde{n}_3)^{-1})$.

Inequalities Eq. (3.0.5) and Eq. (3.0.8) tell us that we need to have a lower bound on the amount of noise for our TRPCA procedure to work, which is counter-intuitive on face value. However, it is logical that for this procedure to succeed we will need a certain amount of noise: if there was no noise, we would observe $\mathcal{M}_0 = \mathcal{L}_0 + \mathcal{C}_0 = \mathcal{L}_0 + 0$, and when we performed the TRPCA procedure $\min\limits_{\mathcal{L},\mathcal{C}} \|\mathcal{L}\|_* + \lambda\|\mathcal{C}\|_1, \quad \text{s.t. } \mathcal{M}_0 = \mathcal{L} + \mathcal{C}$, we would need its minimum to be at $\mathcal{L} = \mathcal{M}_0$, $\mathcal{C} = 0$. It seems unlikely that this would be the case, since some of $\mathcal{C}_0$ (corresponding to finer detail/high rank information) will probably be put into $\mathcal{C}$ unless the rank of $\mathcal{L}_0$ is very small. We note further that if we really need a minimum amount of noise, we can just add more noise. The inequality from Eq. (3.0.5) also strangely says that as the amount of noise rises, the required rank can rise. This is very unintuitive, and there appears to be no particular reason for this to be the case. Whether this relationship actually holds will be among the questions explored at the end of this chapter.

3.1 Projections

The proof of Theorem 3.1 relies heavily on two projections, and how they interact. This section introduces these projections and provides some basic results to be used later.

Definition 3.3. We define the projection onto the set of indices $\mathbf{\Omega} := \text{support}(\mathcal{C}_0)$ as

$$P_{\mathbf{\Omega}}(\mathcal{A}) = \sum_J \delta_J \langle \mathbf{e}_J, \mathcal{A} \rangle \mathbf{e}_J,$$

where $\delta_J = 1$ when $J \in \mathbf{\Omega}$ and 0 otherwise.

Definition 3.4. We define the space $\mathbf{T}$, which is somewhat analogous to the column/row space of the low-dimensional object we are interested in:

$$\mathbf{T} := \{\mathcal{U} * \mathcal{Y}^* + \mathcal{W} * \mathcal{V}^* \mid \mathcal{Y}^* \in \mathbb{R}^{r \times n_2 \times \ldots \times n_p} \;\; \mathcal{W} \in \mathbb{R}^{n_1 \times r \times \ldots \times n_p}\}.$$

Definition 3.5. The projection onto $\mathbf{T}$ is given explicitly by

$$P_{\mathbf{T}}(\mathcal{A}) = \mathcal{U} * \mathcal{U}^* * \mathcal{A} + \mathcal{A} * \mathcal{V} * \mathcal{V}^* - \mathcal{U} * \mathcal{U}^* * \mathcal{A} * \mathcal{V} * \mathcal{V}^*,$$

so that

$$\begin{aligned} P_{\mathbf{T}^\perp}(\mathcal{A}) &= \mathcal{A} - P_{\mathbf{T}}(\mathcal{A}) \\ &= \mathcal{A} - (\mathcal{U} * \mathcal{U}^* * \mathcal{A} + \mathcal{A} * \mathcal{V} * \mathcal{V}^* - \mathcal{U} * \mathcal{U}^* * \mathcal{A} * \mathcal{V} * \mathcal{V}^*) \\ &= (\mathcal{I} - \mathcal{U} * \mathcal{U}^*) * \mathcal{A} * (\mathcal{I} - \mathcal{V} * \mathcal{V}^*). \end{aligned}$$

Lemma 3.6. *The operators $P_{\mathbf{\Omega}}$, $P_{\mathbf{\Omega}^\perp}$, $P_{\mathbf{T}}$ and $P_{\mathbf{T}^\perp}$ are linear, self-adjoint and have operator norm less than or equal to 1.*

Proof. These projections are all linear because the t-product and inner product are linear. As the sum and composition of self-adjoint operators are self-adjoint, and since $P_{\mathbf{T}}$ and $P_{\mathbf{T}^\perp}$ are defined by the sum of multiple t-products by a Hermitian tensor (which is self-adjoint), we have that $P_{\mathbf{T}}$ and $P_{\mathbf{T}^\perp}$ are self-adjoint. A very simple direct calculation also shows that $P_{\mathbf{\Omega}}$ and $P_{\mathbf{\Omega}^\perp}$ are self-adjoint. Finally we note that all self-adjoint projections have norm less than or equal to 1. □

Proposition 3.7. $\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2 \leq \frac{\mu r(n_1+n_2)}{\tilde{n}_1}$.

Proof. Since $P_{\mathbf{T}}$ is self-adjoint (Lemma 3.6) we have that

$$\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2 = \langle P_{\mathbf{T}}\mathbf{e}_I, P_{\mathbf{T}}\mathbf{e}_I\rangle = \langle P_{\mathbf{T}}P_{\mathbf{T}}\mathbf{e}_I, \mathbf{e}_I\rangle = \langle P_{\mathbf{T}}\mathbf{e}_I, \mathbf{e}_I\rangle,$$

which then becomes

$$
\begin{aligned}
&= \langle \mathcal{U} * \mathcal{U}^* * \mathbf{e}_I + \mathbf{e}_I * \mathcal{V} * \mathcal{V}^* - \mathcal{U} * \mathcal{U}^* * \mathbf{e}_I * \mathcal{V} * \mathcal{V}^*, \mathbf{e}_I \rangle \\
&= \langle \mathcal{U} * \mathcal{U}^* * \mathbf{e}_I, \mathbf{e}_I \rangle + \langle \mathbf{e}_I * \mathcal{V} * \mathcal{V}^*, \mathbf{e}_I \rangle - \langle \mathcal{U} * \mathcal{U}^* * \mathbf{e}_I * \mathcal{V} * \mathcal{V}^*, \mathbf{e}_I \rangle \\
&= \langle \mathcal{U}^* * \mathbf{e}_I, \mathcal{U}^* * \mathbf{e}_I \rangle + \langle \mathcal{V}^* * \mathbf{e}_I^*, \mathcal{V}^* * \mathbf{e}_I^* \rangle - \langle \mathcal{U}^* * \mathbf{e}_I * \mathcal{V}, \mathcal{U}^* * \mathbf{e}_I * \mathcal{V} \rangle \\
&= \|\mathcal{U}^* * \mathbf{e}_I\|_F^2 + \|\mathcal{V}^* * \mathbf{e}_I^*\|_F^2 - \|\mathcal{U}^* * \mathbf{e}_I * \mathcal{V}\|_F^2 \\
&\le \|\mathcal{U}^* * \mathbf{e}_I\|_F^2 + \|\mathcal{V}^* * \mathbf{e}_I^*\|_F^2 \\
&= \|\mathcal{U}^* * \overset{c}{\mathbf{e}}_{i_1}\|_F^2 + \|\mathcal{V}^* * \overset{r}{\mathbf{e}}_{i_2}\|_F^2
\end{aligned}
$$

where the last equality holds because $\mathbf{e}_J = \overset{c}{\mathbf{e}}_{j_1} * \overset{\bullet}{\mathbf{e}}_{J'} * \overset{r}{\mathbf{e}}_{j_2}$ (Remark 2.37). Now, since we have $\max_{j \in 0 \ldots n_1 - 1} \|\mathcal{U} * \overset{c}{\mathbf{e}}_j\|_F \le \sqrt{\frac{\mu r}{n_1 \tilde{n}_3}}$ and $\max_{j \in 0 \ldots n_2 - 1} \|\mathcal{V}^* * \overset{r}{\mathbf{e}}_j\|_F \le \sqrt{\frac{\mu r}{n_2 \tilde{n}_3}}$ (Eq. (3.0.1) and Eq. (3.0.2)) this is less than or equal to

$$
\frac{\mu r}{n_1 \tilde{n}_3} + \frac{\mu r}{n_2 \tilde{n}_3} = \frac{\mu r (n_1 + n_2)}{\tilde{n}_1}. \quad \square
$$

We will use the following theorem from Tropp (2011) to bound the norm of tensors. It is a generalisation of the classic Bernstein inequality to the non-commutative and random matrix setting.

Theorem 3.8. [Tropp (2011)] *Let $\{Z_k\}_{k=1}^K$ be a sequence of independent $n_1 \times n_2$ random matrices such that $\mathbb{E} Z_k = 0$ and $\|Z_k\| \le R$ almost surely. Then if we let $\sigma^2 = \max\left\{ \| \sum_{k=1}^K \mathbb{E}[Z_k^* Z_k] \|, \| \sum_{k=1}^K \mathbb{E}[Z_k Z_k^*] \| \right\}$ we have that for any $t \ge 0$,*

$$
\begin{aligned}
\mathbb{P}\left(\left\| \sum_{k=1}^K Z_k \right\| \ge t \right) &\le (n_1 + n_2) \exp\left(-\frac{t^2}{2\sigma^2 + \frac{2}{3} R t} \right) \\
&\le (n_1 + n_2) \exp\left(-\frac{3t^2}{8\sigma^2} \right), \textit{ for } t \le \frac{\sigma^2}{R}.
\end{aligned}
$$

Lemma 3.9. *If $\mathbf{\Omega}$ follows a Bernoulli distribution with parameter $\rho \ge C_0 \epsilon^{-2} \frac{\mu r \log(n_{(1)} \tilde{n}_3)}{n_{(2)} \tilde{n}_3}$, then*

$$
\|P_{\mathbf{T}} - \rho^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}\| \le \epsilon.
$$

with high probability.

Proof. Our approach will be to bound $P_{\mathbf{T}} - \rho^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}$ by writing its norm as the norm of a sum of random matrices and then bounding that expression using the non-commutative Bernstein inequality (Theorem 3.8).

First, by unravelling the definition of $P_{\mathbf{\Omega}}$ (Remark 2.38) and writing $\mathcal{Z}$ as a sum of its elements and basis tensors (Remark 2.38) and then by applying the linearity of $P_{\mathbf{T}}$ (Lemma 3.6) and rearranging, we can write

$$\begin{aligned}
(P_{\mathbf{T}} - \rho^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}})\mathcal{Z} &= P_{\mathbf{T}} \left(\sum_J \langle \mathbf{e}_J \mathcal{Z}_J \rangle \mathbf{e}_J - \rho^{-1} \delta_J \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle \mathbf{e}_J \right) \\
&= \sum_J \langle \mathbf{e}_J \mathcal{Z}_J \rangle P_{\mathbf{T}}(\mathbf{e}_J) - \rho^{-1} \delta_J \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle P_{\mathbf{T}} \mathbf{e}_J \\
&= \sum_J (1 - \rho^{-1} \delta_J) \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle P_{\mathbf{T}} \mathbf{e}_J \\
&\equiv \sum_J \mathcal{H}_J(\mathcal{Z}),
\end{aligned}$$

where $\mathcal{H}_J(\mathcal{Z}) = (1 - \rho^{-1} \delta_J) \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle P_{\mathbf{T}} \mathbf{e}_J$, which is self-adjoint as

$$\begin{aligned}
\langle \mathcal{A}, \mathcal{H}_J(\mathcal{Z}) \rangle &= \langle \mathcal{A}, (1 - \rho^{-1} \delta_J) \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle P_{\mathbf{T}} \mathbf{e}_J \rangle \\
&= (1 - \rho^{-1} \delta_J) \langle \mathbf{e}_J, P_{\mathbf{T}} \mathcal{Z} \rangle \langle \mathcal{A}, P_{\mathbf{T}} \mathbf{e}_J \rangle \\
&= \langle (1 - \rho^{-1} \delta_J) \langle P_{\mathbf{T}} \mathcal{A}, \mathbf{e}_J \rangle P_{\mathbf{T}} \mathbf{e}_J, \mathcal{Z} \rangle \\
&= \langle \mathcal{H}_J(\mathcal{A}), \mathcal{Z} \rangle.
\end{aligned}$$

Now we note that

$$\mathcal{H}_J(\mathcal{Z}) = H_J \, \mathtt{unfold}(\mathcal{Z})$$

for some $n_{(2)}\tilde{n}_3 \times n_{(2)}\tilde{n}_3$ matrix (here, we can use $n_{(2)}$, as transposing preserves the matrix spectral norm) H_J since $\mathcal{H}_J(\mathcal{Z})$ is a linear transformation. Since $\mathcal{H}_J$ is self-adjoint, we also have

$$H_J^* H_J \, \texttt{unfold}(\mathcal{Z}) = H_J H_J^* \, \texttt{unfold}(\mathcal{Z}) = H_J H_J \, \texttt{unfold}(\mathcal{Z})$$
$$= H_J \, \texttt{unfold}(\mathcal{H}_J(\mathcal{Z}))) = \mathcal{H}_J \mathcal{H}_J(\mathcal{Z}).$$

To apply the non-commutative Bernstein inequality (Theorem 3.8) we need to show that $\mathbb{E}H_J = 0$ and to bound $\|H_J\|$ and $\|\sum_J \mathbb{E}H_J^2\|$. On the first assumption, we note that $\mathbb{E}\delta_J = \rho$ so $\mathbb{E}(1 - \rho^{-1}\delta_J) = 1 - 1 = 0$; hence, $\mathbb{E}H_J = 0$. On the second assumption, since $P_{\mathbf{T}}$ is self-adjoint (Lemma 3.6), we have that

$$\begin{aligned}
\|H_J\| = \|\mathcal{H}_J\| &= \sup_{\|\mathcal{Z}\|_F = 1} \|(1 - \rho^{-1}\delta_J)\langle \mathbf{e}_J, P_{\mathbf{T}} Z\rangle (P_{\mathbf{T}}(\mathbf{e}_J))\|_F \\
&\leq \sup_{\|\mathcal{Z}\|_F = 1} \rho^{-1} |\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle| \;\; \|(P_{\mathbf{T}}(\mathbf{e}_J))\|_F \\
&= \sup_{\|\mathcal{Z}\|_F = 1} \rho^{-1} |\langle P_{\mathbf{T}}\mathbf{e}_J, \mathcal{Z}\rangle| \;\; \|P_{\mathbf{T}}(\mathbf{e}_J)\|_F.
\end{aligned}$$

Now we bound this using the Cauchy–Schwarz inequality (Eq. (2.4.1)) and the fact that $\|P_{\mathbf{T}}\mathbf{e}_J\|_F^2 \leq \frac{(n_1+n_2)\mu r}{\tilde{n}_1}$ (Proposition 3.7) to get

$$\begin{aligned}
&\leq \sup_{\|\mathcal{Z}\|_F = 1} \rho^{-1} \|P_{\mathbf{T}}\mathbf{e}_J\|_F \|\mathcal{Z}\|_F \;\; \|P_{\mathbf{T}}\mathbf{e}_J\|_F \\
&= \rho^{-1} \|P_{\mathbf{T}}\mathbf{e}_J\|_F^2 \\
&\leq \frac{(n_1+n_2)\mu r}{\tilde{n}_1 \rho}.
\end{aligned}$$

Next, we have that

$$\begin{aligned}
\mathcal{H}_J^2 = \mathcal{H}_J(\mathcal{H}_J(\mathcal{Z})) &= \mathcal{H}_J\left((1 - \rho^{-1}\delta_J)\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J\right) \\
&= (1 - \rho^{-1}\delta_J) \left\langle \mathbf{e}_J, P_{\mathbf{T}}\left((1 - \rho^{-1}\delta_J)\langle \mathbf{e}_J, P_{\mathbf{T}}(\mathcal{Z})\rangle P_{\mathbf{T}}\mathbf{e}_J\right)\right\rangle P_{\mathbf{T}}\mathbf{e}_J \\
&= (1 - \rho^{-1}\delta_J)^2 \langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle \; \langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_J\rangle \; P_{\mathbf{T}}\mathbf{e}_J
\end{aligned}$$

and also that

$$
\begin{aligned}
\mathbb{E}(1-\rho^{-1}\delta_J)^2 &= \mathbb{E}(1-2\rho^{-1}\delta_J+\rho^{-2}\delta_J) \\
&= \rho^{-1}-1.
\end{aligned}
$$

So we can combine these to get that

$$
\mathbb{E}\mathcal{H}_J^2(\mathcal{Z}) = (\rho^{-1}-1)\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle \left\langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_J\right\rangle P_{\mathbf{T}}\mathbf{e}_J,
$$

and so we can rearrange this using the linearity of $P_{\mathbf{T}}$ (Lemma 3.6) as follows

$$
\begin{aligned}
\left\|\sum_J \mathbb{E}H_J^2(\mathcal{Z})\right\|_F = \left\|\sum_J \mathbb{E}\mathcal{H}_J^2(\mathcal{Z})\right\|_F &= \left\|\sum_J (\rho^{-1}-1)\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle \left\langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_J\right\rangle P_{\mathbf{T}}\mathbf{e}_J\right\|_F \\
&= \left\|\sum_J (\rho^{-1}-1)\left\langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_J\right\rangle P_{\mathbf{T}}(\mathbf{e}_J\langle \mathbf{e}_J, P_{\mathbf{T}}(\mathcal{Z})\rangle)\right\|_F,
\end{aligned}
$$

and we denote this final expression as (I). Then, to bound (I) we apply the triangle inequality. We write $\max_J |\left\langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_J\right\rangle| = |\left\langle \mathbf{e}_M, P_{\mathbf{T}}\mathbf{e}_M\right\rangle|$ and recall that $P_{\mathbf{T}}(\mathcal{Z}) = \sum_J \mathbf{e}_J\langle \mathbf{e}_J, P_{\mathbf{T}}(\mathcal{Z})\rangle$ (Remark 2.38). From this, we use the fact that $P_{\mathbf{T}}$ is a projection to get that

$$
\begin{aligned}
\text{(I)} \leq& (\rho^{-1}-1)|\left\langle \mathbf{e}_M, P_{\mathbf{T}}\mathbf{e}_M\right\rangle| \left\|\sum_J P_{\mathbf{T}}(\mathbf{e}_J\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle)\right\|_F \\
=& (\rho^{-1}-1)|\left\langle \mathbf{e}_M, P_{\mathbf{T}}\mathbf{e}_M\right\rangle| \left\|P_{\mathbf{T}}\left(\sum_J \mathbf{e}_J\langle \mathbf{e}_J, P_{\mathbf{T}}\mathcal{Z}\rangle\right)\right\|_F \\
=& (\rho^{-1}-1)|\left\langle \mathbf{e}_M, P_{\mathbf{T}}P_{\mathbf{T}}\mathbf{e}_M\right\rangle| \left\|P_{\mathbf{T}}P_{\mathbf{T}}\mathcal{Z}\right\|_F \\
=& (\rho^{-1}-1)|\left\langle \mathbf{e}_M, P_{\mathbf{T}}P_{\mathbf{T}}\mathbf{e}_M\right\rangle| \left\|P_{\mathbf{T}}\mathcal{Z}\right\|_F,
\end{aligned}
$$

and we denote this final expression as (II). Then, using the fact that $P_{\mathbf{T}}$ is self-adjoint with norm less than or equal to 1 (Lemma 3.6), the Cauchy–Schwarz inequality

(Eq. (2.4.1)) and the fact that $\|P_{\mathbf{T}}\mathbf{e}_M\|_F^2 \leq \frac{(n_1+n_2)\mu r}{\tilde{n}_1}$ (Proposition 3.7), we derive

$$\begin{aligned}
\text{(II)} \leq & (\rho^{-1}-1)|\left\langle P_{\mathbf{T}}\mathbf{e}_M, P_{\mathbf{T}}\mathbf{e}_M\right\rangle| \, \|P_{\mathbf{T}}\mathcal{Z}\|_F \\
= & (\rho^{-1}-1)\|P_{\mathbf{T}}\mathbf{e}_M\|_F^2 \, \|P_{\mathbf{T}}\mathcal{Z}\|_F \\
\leq & (\rho^{-1}-1)\|P_{\mathbf{T}}\mathbf{e}_M\|_F^2 \, \|\mathcal{Z}\|_F \\
\leq & \rho^{-1}\|P_{\mathbf{T}}\mathbf{e}_M\|_F^2 \, \|\mathcal{Z}\|_F \\
\leq & \frac{(n_1+n_2)\mu r}{\tilde{n}_1\rho}\|\mathcal{Z}\|_F
\end{aligned}$$

which implies that $\|\sum_J \mathbb{E}H_J^2(\mathcal{Z})\| = \|\sum_J \mathbb{E}\mathcal{H}_J^2(\mathcal{Z})\| \leq \frac{(n_1+n_2)\mu r}{\tilde{n}_1\rho}$. Putting the above results together, we get

$$\begin{aligned}
\|P_{\mathbf{T}} - \rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}P_{\mathbf{T}}\| &= \sup_{\|\mathcal{Z}\|_F=1} \|(P_{\mathbf{T}} - \rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}P_{\mathbf{T}})\mathcal{Z}\|_F \\
&= \sup_{\|\mathcal{Z}\|_F=1} \|\sum_J \mathcal{H}_J(\mathcal{Z})\|_F \\
&= \sup_{\|\mathcal{Z}\|_F=1} \|\sum_J H_J \, \texttt{unfold}(\mathcal{Z})\|_F \\
&= \|\sum_J H_J\|.
\end{aligned}$$

Using the bounds from above with $R = \frac{(n_1+n_2)\mu r}{n_1 n_2 \rho} = \sigma^2$ (so $\frac{\sigma^2}{R} = 1 > \epsilon$), we can apply the non-commutative Bernstein inequality (Theorem 3.8) to get

$$\mathbb{P}\left(\|P_{\mathbf{T}} - \rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}P_{\mathbf{T}}\| > \epsilon\right) = \mathbb{P}\left(\|\sum_J H_J\| > \epsilon\right) \leq 2\tilde{n}_{(2)}\tilde{n}_3 \exp\left(-\frac{3}{8}\frac{\epsilon^2}{\frac{(n_1+n_2)\mu r}{\tilde{n}_1\rho}}\right).$$

We rearrange this final expression and substitute in $n_1+n_2 \leq 2n_{(1)}$ to derive that the expression is equal to

$$2n_{(2)}\tilde{n}_3 \exp\left(-\frac{3}{8}\frac{\epsilon^2\tilde{n}_1\rho}{(n_1+n_2)\mu r}\right) \leq 2n_{(2)}\tilde{n}_3 \exp\left(-\frac{3}{16}\frac{\epsilon^2\tilde{n}_1\rho}{n_{(1)}\mu r}\right) = 2n_{(2)}\tilde{n}_3 \exp\left(-\frac{3}{16}\frac{\epsilon^2 n_{(2)}\tilde{n}_3\rho}{\mu r}\right).$$

Provided that $\rho \geq C_0 \epsilon^{-2} \frac{\mu r \log(n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3}$, this expression is then

$$\leq 2n_{(2)}\tilde{n}_3 \exp\left(-C_0 \frac{3}{16} \log(n_{(1)}\tilde{n}_3)\right) = 2(n_{(2)}\tilde{n}_3)(n_{(1)}\tilde{n}_3)^{-C_0 \frac{3}{16}}.$$

Since we chose $C_0 = \frac{32}{3}$ (Eq. (3.0.9)) we have that this is less than or equal to $2n_{(2)}n_{(1)}^{-2}\tilde{n}_3^{-1}$, which is itself less than or equal to $O((n_{(1)}\tilde{n}_3)^{-1})$. $\square$

We now prove a proposition that will give us a useful corollary from the lemma above.

Proposition 3.10. $\|P_{\mathbf{\Omega}} P_{\mathbf{T}}\|^2 = \|P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}\|$.

Proof. Using the definitions of the relevant norms we get

$$\|P_{\mathbf{\Omega}} P_{\mathbf{T}}\|^2 = \left(\sup_{\|\mathcal{A}\|_F = 1} \|P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}\|_F\right)^2 = \sup_{\|\mathcal{A}\|_F = 1} \|P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}\|_F^2 = \sup_{\|\mathcal{A}\|_F = 1} \langle P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}, P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}\rangle$$

and then, using the fact that these projections are self-adjoint (Lemma 3.6), we get our desired result:

$$= \sup_{\|\mathcal{A}\|_F = 1} \langle \mathcal{A}, P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}\rangle = \sup_{\|\mathcal{A}\|_F = 1} \langle \mathcal{A}, P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}} \mathcal{A}\rangle = \|P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}\|. \quad \square$$

Corollary 3.11. $\|P_{\mathbf{\Omega}} P_{\mathbf{T}}\|^2 \leq \rho + \epsilon := \sigma < \frac{4}{9}$.

Proof. To see this, we first note that by taking complements in Lemma 3.9, when $1 - \rho \geq C_0 \epsilon^{-2} \frac{\mu r \log(n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3}$, we get

$$\|P_{\mathbf{T}} - (1-\rho)^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}^\perp} P_{\mathbf{T}}\| \leq \epsilon.$$

Next, by rearranging and using the fact that $P_{\mathbf{\Omega}^\perp} = I - P_{\mathbf{\Omega}}$ we note that

$$\begin{aligned}
P_{\mathbf{T}} - (1-\rho)^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}^\perp} P_{\mathbf{T}} &= (1-\rho)^{-1}((1-\rho) P_{\mathbf{T}} - P_{\mathbf{T}} P_{\mathbf{\Omega}^\perp} P_{\mathbf{T}}) \\
&= (1-\rho)^{-1}(P_{\mathbf{T}} - \rho P_{\mathbf{T}} - P_{\mathbf{T}}(I - P_{\mathbf{\Omega}}) P_{\mathbf{T}}) \\
&= (1-\rho)^{-1}(P_{\mathbf{T}} - \rho P_{\mathbf{T}} - P_{\mathbf{T}} + P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}) \\
&= (1-\rho)^{-1}(P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}} - \rho P_{\mathbf{T}})
\end{aligned}$$

so

$$P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}} = (1-\rho)(P_{\mathbf{T}} - (1-\rho)^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}^\perp} P_{\mathbf{T}}) + \rho P_{\mathbf{T}}.$$

We can then use the triangle inequality and the fact that $P_{\mathbf{T}}$ has norm less than or equal to 1 (Lemma 3.6) to get that

$$\begin{aligned}
\|P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}\| &\leq (1-\rho)\|(P_{\mathbf{T}} - (1-\rho)^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}^\perp} P_{\mathbf{T}})\| + \rho\|P_{\mathbf{T}}\| \\
&\leq (1-\rho)\epsilon + \rho \\
&\leq \epsilon + \rho.
\end{aligned}$$

From Proposition 3.10, we have that

$$\|P_{\mathbf{\Omega}} P_{\mathbf{T}}\|^2 = \|P_{\mathbf{T}} P_{\mathbf{\Omega}} P_{\mathbf{T}}\| \leq \rho + \epsilon := \sigma.$$

Then, since from Eq. (3.0.7) we have that $\epsilon \leq \frac{\sqrt{2\rho}}{5}$ and from Theorem 3.1 $\rho < \frac{436-6\sqrt{836}}{900}$, we have that

$$\sigma = \rho + \epsilon \leq \rho + \frac{\sqrt{2\rho}}{5} < \frac{436-6\sqrt{836}}{900} + \frac{\sqrt{2\frac{436-6\sqrt{836}}{900}}}{5} = \frac{4}{9}. \square$$

With these results in hand, we are now ready to begin the main section of this proof.

3.2 Proof that TRPCA Works

To prove that TRPCA works (Theorem 3.1), we will prove the below lemma, which provides a sufficient condition for TRPCA to succeed. Later in this chapter, we will show that this sufficient condition is satisfied when the conditions from Theorem 3.1 are satisfied.

Lemma 3.12. *The tensor pair $(\mathcal{L}_0, \mathcal{C}_0)$ is the unique minimiser to the TRPCA minimi-*

sation problem with $\lambda = \frac{1}{\sqrt{n_{(1)}\tilde{n}_3}}$ *so long as there exists a tensor* $\mathcal{W}$ *satisfying*

$$\mathcal{W} \in \mathbf{T}^{\perp}, \tag{3.2.1}$$

$$\|\mathcal{W}\| < \frac{1}{2}, \tag{3.2.2}$$

$$\|P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W} - \lambda\,\mathrm{sign}(\mathcal{C}_0))\|_F \leq \frac{\lambda}{4}, \tag{3.2.3}$$

$$\|P_{\mathbf{\Omega}^{\perp}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W})\|_{\infty} < \frac{\lambda}{2}. \tag{3.2.4}$$

Proof. We will show for any $\mathcal{H} \neq 0$, the objective function that we minimise in TRPCA given by $\|\cdot\|_* + \lambda\|\cdot\|_1$ is larger when evaluated at $(\mathcal{L}_0 + \mathcal{H}, \mathcal{C}_0 - \mathcal{H})$ than $(\mathcal{L}_0, \mathcal{C}_0)$ for any $\mathcal{H}$. It is sufficient to only consider perturbations of $\pm\mathcal{H}$ as we still have to satisfy the constraint that $\mathcal{L} + \mathcal{C} = \mathcal{M}$.

First, we fix a $\mathcal{W}$ satisfying Eq. (3.2.1) to Eq. (3.2.4). Then we set

$$\mathcal{E} = \frac{1}{\lambda}\mathcal{U} * \mathcal{V}^* + \frac{1}{\lambda}\mathcal{W} - \mathrm{sign}(\mathcal{C}_0). \tag{3.2.5}$$

Then we write $\mathcal{F} = P_{\mathbf{\Omega}^{\perp}}\mathcal{E}$ and $\mathcal{D} = P_{\mathbf{\Omega}}\mathcal{E}$, so by using Eq. (3.2.4), we have that

$$\|\mathcal{F}\|_{\infty} \leq \frac{1}{2} \tag{3.2.6}$$

and by using Eq. (3.2.3), we have that

$$\|\mathcal{D}\|_F \leq \frac{1}{4}. \tag{3.2.7}$$

Substituting this into the above and rearranging, we have

$$\mathcal{U} * \mathcal{V}^* + \mathcal{W} = \lambda(\mathrm{sign}(\mathcal{C}_0) + \mathcal{F} + \mathcal{D}). \tag{3.2.8}$$

Note that the left-hand side of the equation above is a subgradient of the tensor nuclear norm while the right-hand side is a subgradient of the l_1 norm, both of which will be set equal to 0 at the minimum.

Now let $\mathcal{U} * \mathcal{V}^* + \mathcal{W}_0$ be another subgradient of the tensor nuclear norm at $\mathcal{L}_0$ satisfying $\langle \mathcal{W}_0, \mathcal{H} \rangle = \|P_{\mathbf{T}^\perp} \mathcal{H}\|_*$. The existence of such a tensor is guaranteed, as the definition of the tensor nuclear norm is

$$\|P_{\mathbf{T}^\perp} \mathcal{H}\|_* = \sup_{\|\mathcal{A}\|=1} \langle \mathcal{A}, P_{\mathbf{T}^\perp} \mathcal{H} \rangle.$$

Since we are taking this supremum over a closed set, there exists a tensor $\mathcal{A}$ that attains the supremum above. Letting $\mathcal{W}_0 = P_{\mathbf{T}^\perp} \mathcal{A}$, since $P_{\mathbf{T}^\perp}$ is self-adjoint (Lemma 3.6), we get

$$\langle \mathcal{W}_0, \mathcal{H} \rangle = \langle P_{\mathbf{T}^\perp} \mathcal{A}, \mathcal{H} \rangle = \langle \mathcal{A}, P_{\mathbf{T}^\perp} \mathcal{H} \rangle = \|P_{\mathbf{T}^\perp} \mathcal{H}\|_*.$$

We also note that $\|\mathcal{W}_0\| = \|P_{\mathbf{T}^\perp} \mathcal{A}\| \leq \|\mathcal{A}\| = 1$, so $\mathcal{U} * \mathcal{V}^* + \mathcal{W}_0$ is an element of the subgradient of the tensor nuclear norm (Lemma 2.46).

Next, we let $\operatorname{sign}(\mathcal{C}_0) + \mathcal{F}_0$ be a subgradient of the the l_1 norm at $\mathcal{C}_0$ satisfying $\langle \mathcal{F}_0, \mathcal{H} \rangle = -\|P_{\mathbf{\Omega}^\perp} \mathcal{H}\|_1$. The tensor $\mathcal{F}_0 = -\operatorname{sign}(P_{\mathbf{\Omega}^\perp} \mathcal{H})$ is an example of such a tensor as

$$\langle -\operatorname{sign}(P_{\mathbf{\Omega}^\perp} \mathcal{H}), \mathcal{H} \rangle = -\langle \operatorname{sign}(P_{\mathbf{\Omega}^\perp} \mathcal{H}), P_{\mathbf{\Omega}^\perp} \mathcal{H} \rangle = -\|P_{\mathbf{\Omega}^\perp} \mathcal{H}\|_1.$$

We also note that the elements of $\mathcal{F}_0$ are 0 anywhere $\operatorname{sign}(\mathcal{C}_0)$ is non-zero (as $\mathcal{F}_0$ is only supported on $\mathbf{\Omega}^\perp$, the complement of the support of $\operatorname{sign}(\mathcal{C}_0)$); otherwise, the elements of $\mathcal{F}_0$ only equal -1, 0 or 1, so the tensor $\operatorname{sign}(\mathcal{C}_0) + \mathcal{F}_0$ is in the subgradient of the l_1 norm at $\mathcal{C}_0$ (Lemma 2.47). Then, by applying the definition of subgradients and the equalities above, we have that

$$\begin{aligned}
&\|\mathcal{L}_0 + \mathcal{H}\|_* + \lambda \|\mathcal{C}_0 - \mathcal{H}\|_1 \\
\geq &\|\mathcal{L}_0\|_* + \lambda \|\mathcal{C}_0\|_1 + \langle \mathcal{U} * \mathcal{V}^* + \mathcal{W}_0, \mathcal{H} \rangle - \lambda \langle \operatorname{sign}(\mathcal{C}_0) + \mathcal{F}_0, \mathcal{H} \rangle \\
= &\|\mathcal{L}_0\|_* + \lambda \|\mathcal{C}_0\|_1 + \langle \mathcal{U} * \mathcal{V}^* - \lambda \operatorname{sign}(\mathcal{C}_0), \mathcal{H} \rangle + \|P_{\mathbf{T}^\perp} \mathcal{H}\|_* + \lambda \|P_{\mathbf{\Omega}^\perp} \mathcal{H}\|_1.
\end{aligned}$$

So to show that the objective function $\|\cdot\|_* + \lambda\|\cdot\|_1$ is larger when evaluated at $(\mathcal{L}_0 + \mathcal{H}, \mathcal{C}_0 - \mathcal{H})$ than $(\mathcal{L}_0, \mathcal{C}_0)$, it is sufficient to show that

$$\langle \mathcal{U} * \mathcal{V}^* - \lambda \operatorname{sign}(\mathcal{C}_0), \mathcal{H}\rangle + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_* + \lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 \geq 0.$$

To do this, we first note that since all the projections in the below expression have norm less than or equal to 1 (Lemma 3.6), and since $\|P_{\mathbf{\Omega}}P_{\mathbf{T}}\| < \sqrt{\sigma}$ ($< \frac{2}{3}$) (Corollary 3.11), we have that

$$\begin{aligned}
\|P_{\mathbf{\Omega}}\mathcal{H}\|_F &= \|P_{\mathbf{\Omega}}P_{\mathbf{T}}\mathcal{H} + P_{\mathbf{\Omega}}P_{\mathbf{T}^\perp}\mathcal{H}\|_F \\
&\leq \|P_{\mathbf{\Omega}}P_{\mathbf{T}}\mathcal{H}\|_F + \|P_{\mathbf{\Omega}}P_{\mathbf{T}^\perp}\mathcal{H}\|_F \\
&\leq \sqrt{\sigma}\|\mathcal{H}\|_F + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_F \\
&\leq \sqrt{\sigma}\|P_{\mathbf{\Omega}}\mathcal{H}\|_F + \sqrt{\sigma}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_F + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_F.
\end{aligned}$$

We recall that $\langle \mathcal{A}, \mathcal{A}\rangle \leq \|\mathcal{A}\|\|\mathcal{A}\|_*$ (Proposition 2.43) and $\|\mathcal{A}\| \leq \sqrt{\tilde{n}_3}\|\mathcal{A}\|_F$ (Eq. (2.4.8)), which we can combine to get

$$\|\mathcal{A}\|_F^2 = \langle \mathcal{A}, \mathcal{A}\rangle \leq \|\mathcal{A}\|\|\mathcal{A}\|_* \leq \sqrt{\tilde{n}_3}\|\mathcal{A}\|_F\|\mathcal{A}\|_*,$$

which gives us $\|\mathcal{A}\|_F \leq \sqrt{\tilde{n}_3}\|\mathcal{A}\|_*$. Then, using this and the fact that $\|\mathcal{A}\|_F \leq \|\mathcal{A}\|_1$, we have that

$$\begin{aligned}
(1 - \sqrt{\sigma})\|P_{\mathbf{\Omega}}\mathcal{H}\|_F &\leq \sqrt{\sigma}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_F + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_F \\
&\leq \sqrt{\sigma}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 + 2\sqrt{\tilde{n}_3}\|P_{\mathbf{T}^\perp}\mathcal{H}\|_*.
\end{aligned}$$

Rearranging Eq. (3.2.8), $\mathcal{U} * \mathcal{V}^* + \mathcal{W} = \lambda(\operatorname{sign}(\mathcal{C}_0) + \mathcal{F} + \mathcal{D})$ becomes $\mathcal{U} * \mathcal{V}^* - \lambda \operatorname{sign}(\mathcal{C}_0) = \lambda\mathcal{F} + \lambda\mathcal{D} - \mathcal{W}$. Applying the triangle and Cauchy–Schwarz inequalities and the statements that $P_{\mathbf{T}}\mathcal{W} = 0$ (Eq. (3.2.1)), $P_{\mathbf{\Omega}}\mathcal{F} = P_{\mathbf{\Omega}}P_{\mathbf{\Omega}^\perp}\mathcal{E} = 0$, $P_{\mathbf{\Omega}^\perp}\mathcal{D} = P_{\mathbf{\Omega}^\perp}P_{\mathbf{\Omega}}\mathcal{E} = 0$, $\langle \mathcal{X}, \mathcal{Y}\rangle \leq$

$\|X\|\|\mathcal{Y}\|_*$ (Proposition 2.43), and $\langle \mathcal{X}, \mathcal{Y}\rangle \leq \|\mathcal{X}\|_1\|\mathcal{Y}\|_\infty$ (Proposition 2.31), we get

$$\begin{aligned}
|\langle \mathcal{U} * \mathcal{V}^* - \lambda\,\mathrm{sign}(\mathcal{C}_0), \mathcal{H}\rangle| &= |\langle \lambda\mathcal{F} + \lambda\mathcal{D} - \mathcal{W}, \mathcal{H}\rangle| \\
&\leq |\langle \lambda\mathcal{F}, \mathcal{H}\rangle| + |\langle \lambda\mathcal{D}, \mathcal{H}\rangle| + |\langle \mathcal{W}, \mathcal{H}\rangle| \\
&= \lambda|\langle P_{\mathbf{\Omega}^\perp}\mathcal{F}, \mathcal{H}\rangle| + \lambda|\langle P_{\mathbf{\Omega}}\mathcal{D}, \mathcal{H}\rangle| + |\langle P_{\mathbf{T}^\perp}\mathcal{W}, \mathcal{H}\rangle| \\
&\leq \lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\|\mathcal{F}\|_\infty + \lambda\|P_{\mathbf{\Omega}}\mathcal{H}\|_F\|P_{\mathbf{\Omega}}\mathcal{D}\|_F + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\|\mathcal{W}\|,
\end{aligned}$$

where the last line follows from self-adjointness of $P_{\mathbf{\Omega}^\perp}$, $P_{\mathbf{\Omega}}$, and $P_{\mathbf{T}^\perp}$. Now we use $\|\mathcal{F}\|_\infty \leq \frac{1}{2}$ (Eq. (3.2.6)), $\|P_{\mathbf{\Omega}}\mathcal{D}\|_F \leq \frac{1}{4}$ (Eq. (3.2.7)), $\|\mathcal{W}\| \leq \frac{1}{2}$ (Eq. (3.2.2)) and the result from above that $(1-\sqrt{\sigma})\|P_{\mathbf{\Omega}}\mathcal{H}\|_F \leq \sqrt{\sigma}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 + 2\sqrt{\tilde{n}_3}\|P_{\mathbf{T}^\perp}\mathcal{H}\|_*$ to get that

$$\begin{aligned}
&\lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\|\mathcal{F}\|_\infty + \lambda\|P_{\mathbf{\Omega}}\mathcal{H}\|_F\|P_{\mathbf{\Omega}}\mathcal{D}\|_F + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\|\mathcal{W}\| \\
&\leq \lambda\frac{1}{2}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 + \lambda\frac{1}{4}\|P_{\mathbf{\Omega}}\mathcal{H}\|_F + \frac{1}{2}\|P_{\mathbf{T}^\perp}\mathcal{H}\|_* \\
&\leq \lambda\frac{1}{2}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 + \lambda\frac{1}{4}\frac{\sqrt{\sigma}\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 + 2\sqrt{\tilde{n}_3}\|P_{\mathbf{T}^\perp}\mathcal{H}\|_*}{1-\sqrt{\sigma}} + \frac{1}{2}\|P_{\mathbf{T}^\perp}\mathcal{H}\|_* \\
&= \|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\lambda\left(\frac{1}{2} + \frac{1}{4}\frac{\sqrt{\sigma}}{1-\sqrt{\sigma}}\right) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\frac{1}{2}\left(1 + \lambda\frac{\sqrt{\tilde{n}_3}}{1-\sqrt{\sigma}}\right).
\end{aligned}$$

Now we calculate

$$\begin{aligned}
&\langle \mathcal{U} * \mathcal{V}^* - \lambda\,\mathrm{sign}(\mathcal{C}_0), \mathcal{H}\rangle + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_* + \lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 \\
&\geq -|\langle \mathcal{U} * \mathcal{V}^* - \lambda\,\mathrm{sign}(\mathcal{C}_0), \mathcal{H}\rangle| + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_* + \lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 \\
&\geq -\left(\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\lambda\left(\frac{1}{2} + \frac{1}{4}\frac{\sqrt{\sigma}}{1-\sqrt{\sigma}}\right) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\frac{1}{2}\left(1 + \lambda\frac{\sqrt{\tilde{n}_3}}{1-\sqrt{\sigma}}\right)\right) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_* + \lambda\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1 \\
&=\|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\lambda\left(1 - \left(\frac{1}{2} + \frac{1}{4}\frac{\sqrt{\sigma}}{1-\sqrt{\sigma}}\right)\right) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\left(1 - \frac{1}{2}\left(1 + \lambda\frac{\sqrt{\tilde{n}_3}}{1-\sqrt{\sigma}}\right)\right).
\end{aligned}$$

For $\lambda = \frac{1}{\sqrt{n_{(1)}\tilde{n}_3}}$ and $\sqrt{\sigma} < \frac{2}{3}$, (Corollary 3.11) this becomes

$$> \|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\lambda\left(1 - \left(\frac{1}{2} + \frac{1}{4}\frac{\frac{2}{3}}{1-\frac{2}{3}}\right)\right) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\left(1 - \frac{1}{2}\left(1 + \frac{1}{\sqrt{n_{(1)}}}\frac{1}{(1-\frac{2}{3})}\right)\right) \tag{3.2.9}$$

$$= \|P_{\mathbf{\Omega}^\perp}\mathcal{H}\|_1\lambda\,(1-1) + \|P_{\mathbf{T}^\perp}\mathcal{H}\|_*\left(\frac{1}{2} - \frac{1}{2}\frac{3}{\sqrt{n_{(1)}}}\right), \tag{3.2.10}$$

which is greater than or equal to 0 for every $\mathcal{H}$ (provided $n_{(1)}$ is greater than 9), delivering our desired result. $\square$

3.2.1 Construction and Verification of the Dual Certificate

We will now construct a $\mathcal{W}$ that satisfies Eq. (3.2.1) to Eq. (3.2.4) (in the literature, this is known as a dual certificate). To do this, we first note that because the support of $\mathcal{C}_0$ ($\mathbf{\Omega}$) is assumed to be Bernoulli-distributed with parameter ρ, the distribution of $\mathbf{\Omega}^c$ is Bernoulli-distributed with parameter $1-\rho$, or equivalently

$$\mathbf{\Omega}^c = \bigcup_{i=1}^{j_0} \Omega_i,$$

for Ω_i Bernoulli-distributed with parameter q such that $\rho = (1-q)^{j_0}$. To see this we note

$$\rho = \mathbb{P}(J \in \mathbf{\Omega}) = \mathbb{P}(J \notin \Omega_i \, \forall i) = \mathbb{P}(\mathrm{Bin}(j_0, q) = 0) = (1-q)^{j_0}.$$

Rearranging this gives that $q = 1 - \rho^{\frac{1}{j_0}} \geq \frac{1-\rho}{j_0}$. Then we choose $j_0 = \lceil \log(n_{(1)}\tilde{n}_3) \rceil$ so that (from Eq. (3.0.8)) we have $\rho \leq 1 - C_0 \frac{\epsilon^{-2}\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}$, which once rearranged gives us $1-\rho \geq C_0 \frac{\epsilon^{-2}\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}$. Hence,

$$q \log(n_{(1)}\tilde{n}_3) \geq 1 - \rho \geq C_0 \frac{\epsilon^{-2}\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3},$$

and in particular

$$q \geq C_0 \frac{\epsilon^{-2}\mu r(\log n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3}. \tag{3.2.11}$$

The last thing we need to do before we construct $\mathcal{W}$ is state the following remark.

Remark 3.13. Since $\|P_{\mathbf{T}}P_{\mathbf{\Omega}}\| < \frac{2}{3} < 1$ (Corollary 3.11), on $\mathbf{\Omega}$, $P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}} = I - P_{\mathbf{\Omega}}P_{\mathbf{T}}$ is invertible and $(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^{-1}$ is given by the convergent Neumann series $\sum_{k=0}^{\infty}(P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^k$.

Now, we are ready to construct $\mathcal{W}$. To do this, we let

$$\mathcal{W} = \mathcal{W}^L + \mathcal{W}^S \tag{3.2.12}$$

where

$$\mathcal{W}^S = \lambda P_{\mathbf{T}^\perp} \left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^{-1} \operatorname{sign}(\mathcal{C}_0) = \lambda P_{\mathbf{T}^\perp} \sum_{k=0}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k \operatorname{sign}(\mathcal{C}_0) \tag{3.2.13}$$

(with its inverse being calculated using Remark 3.13) and

$$\mathcal{W}^L = P_{\mathbf{T}^\perp} \mathcal{Y}_{J_0} \tag{3.2.14}$$

for

$$\mathcal{Y}_j = \mathcal{Y}_{j-1} + q^{-1} P_{\mathbf{\Omega}_j} P_{\mathbf{T}} (\mathcal{U} * \mathcal{V}^* - \mathcal{Y}_{j-1}) \tag{3.2.15}$$

with $\mathcal{Y}_0 = 0$. Substituting $\mathcal{W} = \mathcal{W}^L + \mathcal{W}^S$ into the conditions Eq. (3.2.1) to Eq. (3.2.4), the conditions we need to prove become

$$\mathcal{W}^L + \mathcal{W}^S \in \mathbf{T}^\perp \tag{3.2.16}$$

$$\|\mathcal{W}^L + \mathcal{W}^S\| < \frac{1}{2} \tag{3.2.17}$$

$$\|P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L + \mathcal{W}^S - \lambda \operatorname{sign}(\mathcal{C}_0))\|_F \leq \frac{\lambda}{4} \tag{3.2.18}$$

$$\|P_{\mathbf{\Omega}^\perp}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L + \mathcal{W}^S)\|_\infty < \frac{\lambda}{2}, \tag{3.2.19}$$

which are satisfied when we satisfy the following sets of conditions for $\mathcal{W}^L$

$$\|\mathcal{W}^L\| < \frac{1}{4} \tag{3.2.20}$$

$$\|P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_F \leq \frac{\lambda}{4} \tag{3.2.21}$$

$$\|P_{\mathbf{\Omega}^\perp}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_\infty < \frac{\lambda}{4}, \tag{3.2.22}$$

and for $\mathcal{W}^S$

$$\|\mathcal{W}^S\| < \frac{1}{4} \tag{3.2.23}$$

$$\|P_{\mathbf{\Omega}^\perp}\mathcal{W}^S\|_\infty < \frac{\lambda}{4}. \tag{3.2.24}$$

To see this, $\mathcal{W}^L$ and $\mathcal{W}^S$ are projected onto $\mathbf{T}^\perp$, so the constraint in Eq. (3.2.16) will be satisfied by these new conditions by construction. By the triangle inequality, Eq. (3.2.20) and Eq. (3.2.23) jointly imply Eq. (3.2.17), just as Eq. (3.2.22) and Eq. (3.2.24) jointly imply Eq. (3.2.19). That Eq. (3.2.21) implies Eq. (3.2.18) is a little more complicated, as it relies on the fact that $P_{\mathbf{\Omega}}(\mathcal{W}^S) = \lambda\,\mathrm{sign}(\mathcal{C}_0)$:

$$\begin{aligned}
P_{\mathbf{\Omega}}\mathcal{W}^S &= \lambda P_{\mathbf{\Omega}} P_{\mathbf{T}^\perp} \left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^{-1} \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda P_{\mathbf{\Omega}}(\mathcal{I} - P_{\mathbf{T}}) \left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^{-1} \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}}) \left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^{-1} \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}}) \left(\sum_{k=0}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k\right) \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}}) \left(\sum_{k=0}^{\infty} P_{\mathbf{\Omega}} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k\right) \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}) \left(\sum_{k=0}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k\right) \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}})(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}})^{-1} \mathrm{sign}(\mathcal{C}_0) \\
&= \lambda\,\mathrm{sign}(\mathcal{C}_0).
\end{aligned}$$

We will now show that the constructed $\mathcal{W}^L$ and $\mathcal{W}^S$ satisfy these required constraints, completing the proof of Theorem 3.1.

3.2.2 $\mathcal{W}^L$ Part 1

To do this, first we need to prove the following two long lemmas.

Lemma 3.14. *If $\mathcal{Z} \in \mathbf{T}$ then $\|\mathcal{Z} - \rho^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}} \mathcal{Z}\|_\infty \le \epsilon \|\mathcal{Z}\|_\infty$ with high probability.*

Proof. To prove this, we will write the expression we want to bound as a sum of random matrices and then bound that sum using the non-commutative Bernstein inequality (Theorem 3.8). We note that by unravelling the definition of $P_{\mathbf{\Omega}}$ and applying the linearity of $P_{\mathbf{T}}$ we have that

$$\begin{aligned}\rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}(\mathcal{Z}) &= \rho^{-1}P_{\mathbf{T}}\left(\sum_J \delta_J \mathcal{Z}_J \mathbf{e}_J\right)\\ &= \sum_J \rho^{-1}\delta_J\mathcal{Z}_J P_{\mathbf{T}}(\mathbf{e}_J).\end{aligned}$$

$\mathcal{Z} \in \mathbf{T}$, so $P_{\mathbf{T}}\mathcal{Z} = \mathcal{Z}$. Noting this, we unravel the definition of $P_{\mathbf{\Omega}}$ (Definition 3.3), write $\mathcal{Z}$ as a sum of its elements and basis tensors (Remark 2.38) and then rearrange to get

$$\begin{aligned}\langle \mathbf{e}_K, \mathcal{Z} - \rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}\mathcal{Z}\rangle &= \left\langle \mathbf{e}_K, P_{\mathbf{T}}\mathcal{Z} - \sum_J \rho^{-1}\delta_J\langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J\right\rangle\\ &= \left\langle \mathbf{e}_K, P_{\mathbf{T}}\left(\sum_J \langle \mathbf{e}_J, \mathcal{Z}\rangle \mathbf{e}_J\right) - \sum_J \rho^{-1}\delta_J\langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J\right\rangle\\ &= \left\langle \mathbf{e}_K, \sum_J \langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J - \sum_J \rho^{-1}\delta_J\langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J\right\rangle\\ &= \sum_J \left\langle \mathbf{e}_K, \langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J - \rho^{-1}\delta_J\langle \mathbf{e}_J, \mathcal{Z}\rangle P_{\mathbf{T}}\mathbf{e}_J\right\rangle\\ &= \sum_J (1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \left\langle \mathbf{e}_K, P_{\mathbf{T}}\mathbf{e}_J\right\rangle\\ &\equiv \sum_J \mathbf{t}_J.\end{aligned}$$

We observe that the $\mathbf{t}_J$s are independently distributed as they are only stochastic in δ_J. We want to apply the non-commutative Bernstein inequality (Theorem 3.8) (viewing these $\mathbf{t}_J$s as 1×1 dimensional matrices), so we need to show that $|\mathbf{t}_J|$ and $|\sum_J \mathbb{E}[\mathbf{t}_J^2]|$ are bounded and that $\mathbb{E}\mathbf{t}_J = 0$. First, we note that $\mathbb{E}\mathbf{t}_J = 0$ since $\mathbb{E}(1-\rho^{-1}\delta_J) = 0$. Second, to show that $|\mathbf{t}_J|$ is bounded we use that $P_{\mathbf{T}}$ is self-adjoint (Lemma 3.6), the Cauchy–Schwarz inequality (Eq. (2.4.1)) and that $\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2 \leq \frac{\mu r(n_1+n_2)}{\tilde{n}_1}$ (Proposition 3.7) to get that

$$
\begin{aligned}
|\mathbf{t}_J| &= \left|(1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \, \langle P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle\right| \\
&\leq \left|(1-\rho^{-1}\delta_J)\right| \, |\langle \mathbf{e}_J, \mathcal{Z}\rangle| \, |\, \langle P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle \,| \\
&\leq \rho^{-1}\|\mathcal{Z}\|_\infty | \, \langle P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle \,| \\
&= \rho^{-1}\|\mathcal{Z}\|_\infty | \, \langle P_{\mathbf{T}}P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle \,| \\
&= \rho^{-1}\|\mathcal{Z}\|_\infty | \, \langle P_{\mathbf{T}}\mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_K\rangle \,| \\
&\leq \rho^{-1}\|\mathcal{Z}\|_\infty \|P_{\mathbf{T}}\mathbf{e}_J\|_F \|P_{\mathbf{T}}\mathbf{e}_K\|_F \\
&\leq \frac{\mu r(n_1+n_2)}{\tilde{n}_1\rho}\|\mathcal{Z}\|_\infty.
\end{aligned}
$$

Finally, we bound $|\sum_J \mathbb{E}[\mathbf{t}_J^2]|$ by using the fact that $\mathbb{E}[(1-\rho^{-1}\delta_J)^2] = \rho^{-1} - 1 \leq \rho^{-1}$, the definition of the Frobenius norm and that $\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2 \leq \frac{\mu r(n_1+n_2)}{\tilde{n}_1}$ (Proposition 3.7) to get that

$$
\begin{aligned}
\left|\sum_J \mathbb{E}[\mathbf{t}_J^2]\right| &= \left|\sum_J \mathbb{E}\left[((1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \, \langle P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle)^2\right]\right| \\
&\leq \sum_J \rho^{-1}\langle \mathbf{e}_J, \mathcal{Z}\rangle^2 \, \langle P_{\mathbf{T}}\mathbf{e}_J, \mathbf{e}_K\rangle^2 \\
&\leq \rho^{-1}\|\mathcal{Z}\|_\infty^2 \sum_J \langle \mathbf{e}_J, P_{\mathbf{T}}\mathbf{e}_K\rangle^2 \\
&= \rho^{-1}\|\mathcal{Z}\|_\infty^2 \|P_{\mathbf{T}}\mathbf{e}_K\|_F^2 \\
&\leq \frac{\mu r(n_1+n_2)}{\rho\tilde{n}_1}\|\mathcal{Z}\|_\infty^2.
\end{aligned}
$$

Having proved that its conditions hold, we apply the non-commutative Bernstein inequality (Theorem 3.8) using the bounds found above with $R = \frac{\mu r(n_1+n_2)}{\tilde{n}_1\rho}\|\mathcal{Z}\|_\infty$ and $\sigma^2 = \frac{\mu r(n_1+n_2)}{\rho\tilde{n}_1}\|\mathcal{Z}\|_\infty^2$ (so $\frac{\sigma^2}{R} = \|\mathcal{Z}\|_\infty > \epsilon\|\mathcal{Z}\|_\infty$) and get that

$$
\mathbb{P}\left(\left|\langle \mathbf{e}_K, \mathcal{Z} - \rho^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}}\mathcal{Z}\rangle\right| > \epsilon\|\mathcal{Z}\|_\infty\right) = \mathbb{P}\left(\left|\sum_J \mathbf{t}_J\right| > \epsilon\|\mathcal{Z}\|_\infty\right) \leq 2\exp\left(-\frac{3}{8}\frac{\epsilon^2\|\mathcal{Z}\|_\infty^2}{\frac{\mu r(n_1+n_2)}{\rho\tilde{n}_1}\|\mathcal{Z}_J\|_\infty^2}\right).
$$

We rearrange this final expression and use the fact that $n_1 + n_2 \leq 2n_{(1)}$ to get

$$= 2\exp\left(-\frac{3}{8}\frac{\epsilon^2\rho\tilde{n}_1}{\mu r(n_1+n_2)}\right) \leq 2\exp\left(-\frac{3}{16}\frac{\epsilon^2\rho\tilde{n}_1}{\mu r n_{(1)}}\right) = 2\exp\left(-\frac{3}{16}\frac{\epsilon^2\rho n_{(2)}\tilde{n}_3}{\mu r}\right).$$

Now because $\rho \geq C_0\frac{\epsilon^{-2}\mu r\log(n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3}$ and we chose $C_0 = \frac{32}{3}$ (Eq. (3.0.9)) this becomes

$$\leq 2\exp\left(-C_0\frac{3}{16}\log(n_{(1)}\tilde{n}_3)\right) = 2(n_{(1)}\tilde{n}_3)^{-C_0\frac{3}{16}} = 2(n_{(1)}\tilde{n}_3)^{-2}$$

which is less than $O((n_{(1)}\tilde{n}_3)^{-1})$, giving us our desired result. $\square$

Lemma 3.15. $\|\mathcal{Z} - \rho^{-1}P_{\mathbf{\Omega}}\mathcal{Z}\| \leq \sqrt{\frac{C_0 n_{(1)}\tilde{n}_3\log(n_{(1)}\tilde{n}_3)}{\rho}}\|\mathcal{Z}\|_\infty$ *with high probability.*

Proof. Similar to our proof of Lemma 3.14, we will write this expression as a sum of random matrices and then apply the non-commutative Bernstein inequality (Theorem 3.8). We note that by writing $\mathcal{Z}$ as a sum of its elements and basis tensors (Remark 2.38), unravelling the definition of $P_{\mathbf{\Omega}}$ and noting that `tdiag` preserves the spectral norm (Eq. (2.4.7)), we have

$$\|\mathcal{Z} - \rho^{-1}P_{\mathbf{\Omega}}\mathcal{Z}\| = \left\|\sum_J(1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\mathbf{e}_J\right\| = \left\|\sum_J(1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)\right\|$$

which is a sum of $n_1\tilde{n}_3 \times n_2\tilde{n}_3$ random matrices. We want to apply the non-commutative Bernstein inequality (Theorem 3.8), so we need to bound $\|(1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)\|$ and show it has an expected value of 0, and to bound $\|\sum_J\left((1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)\right)^*(1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)\|$. First, we note that $\mathbb{E}((1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)) = 0$ since $\mathbb{E}((1-\rho^{-1}\delta_J) = 0$. Second, since `tdiag` preserves the spectral norm Eq. (2.4.7), we also have that

$$\|(1-\rho^{-1}\delta_J)\langle\mathbf{e}_J,\mathcal{Z}\rangle\,\texttt{tdiag}(\mathbf{e}_J)\| \leq \rho^{-1}\|\mathcal{Z}\|_\infty\|\,\texttt{tdiag}(\mathbf{e}_J)\| = \rho^{-1}\|\mathcal{Z}\|_\infty\|\mathbf{e}_J\| = \rho^{-1}\|\mathcal{Z}\|_\infty.$$

Finally,

$$
\begin{aligned}
&\left\|\sum_J \mathbb{E}\left[\left((1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right)^* (1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right]\right\| \\
=&\left\|\sum_J \mathbb{E}\left[(1-\rho^{-1}\delta_J)(1-\rho^{-1}\delta_J)\right] \left(\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right)^* \langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right\| \\
=&\frac{1-\rho}{\rho}\left\|\sum_J \left(\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right)^* \langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right\| \\
\leq&\frac{1-\rho}{\rho}\|\mathcal{Z}\|_\infty^2\left\|\sum_J \left(\texttt{tdiag}(\mathbf{e}_J)\right)^* \texttt{tdiag}(\mathbf{e}_J)\right\| \\
\leq&\frac{1-\rho}{\rho}\|\mathcal{Z}\|_\infty^2\left\|\sum_J \mathbf{e}_J^* * \mathbf{e}_J\right\| \\
&\leq \frac{1}{\rho}\|\mathcal{Z}\|_\infty^2 n_1\tilde{n}_3.
\end{aligned}
$$

The last inequality holds because a direct calculation yields that $\mathbf{e}_J^* * \mathbf{e}_J(j_2, j_2, \mathbf{0}) = 1$ and $\mathbf{e}_J^* * \mathbf{e}_J$ is 0 elsewhere, so $\sum_J \mathbf{e}_J^* * \mathbf{e}_J = n_1\tilde{n}_3\mathcal{I}$. By a virtually identical calculation,

$$
\left\|\sum_J \mathbb{E}\left[\left((1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right)\left((1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right)^*\right]\right\| \;\leq \frac{1}{\rho}\|\mathcal{Z}\|_\infty^2 n_2\tilde{n}_3.
$$

Having verified that its conditions hold, we can apply the non-commutative Bernstein inequality (Theorem 3.8) with $R = \rho^{-1}\|\mathcal{Z}\|_\infty$ and $\sigma^2 = \rho^{-1}\|\mathcal{Z}\|_\infty^2 n_{(1)}\tilde{n}_3$ (so $\frac{\sigma^2}{M} = \|\mathcal{Z}\|_\infty n_{(1)}\tilde{n}_3 > \sqrt{\frac{C_0 n_{(1)}\tilde{n}_3\log(n_{(1)}\tilde{n}_3)}{\rho}}\|\mathcal{Z}\|_\infty$ by substituting) and get

$$
\begin{aligned}
&\mathbb{P}\left(\|(\mathcal{Z} - \rho^{-1}P_{\mathbf{\Omega}}\mathcal{Z}\| > \sqrt{\frac{C_0 n_{(1)}\tilde{n}_3\log(n_{(1)}\tilde{n}_3)}{\rho}}\|\mathcal{Z}\|_\infty\right) \\
=&\mathbb{P}\left(\left\|\sum_J (1-\rho^{-1}\delta_J)\langle \mathbf{e}_J, \mathcal{Z}\rangle \,\texttt{tdiag}(\mathbf{e}_J)\right\| > \sqrt{\frac{C_0 n_{(1)}\tilde{n}_3\log(n_{(1)}\tilde{n}_3)}{\rho}}\|\mathcal{Z}\|_\infty\right) \\
&\leq (n_1\tilde{n}_3 + n_2\tilde{n}_3)\exp\left(-\frac{3}{8}\frac{\left(\sqrt{\frac{C_0 n_{(1)}\tilde{n}_3\log(n_{(1)}\tilde{n}_3)}{\rho}}\|\mathcal{Z}\|_\infty\right)^2}{\frac{1}{\rho}\|\mathcal{Z}\|_\infty^2 n_{(1)}\tilde{n}_3}\right) \\
&\leq 2n_{(1)}\tilde{n}_3\exp\left(-\frac{3}{8}C_0\log(n_{(1)}\tilde{n}_3)\right) \leq (2n_{(1)}\tilde{n}_3)^{1-\frac{3}{8}C_0},
\end{aligned}
$$

where the third-last inequality uses $n_1+n_2 \leq 2n_{(1)}$. Since we chose $C_0 = \frac{32}{3}$ (Eq. (3.0.9)), the above expression is less than or equal to $(2n_{(1)}\tilde{n}_3)^{-2}$, which is less than $O((n_{(1)}\tilde{n}_3)^{-1})$, giving us our desired result. $\square$

We are finally ready to show that the $\mathcal{W}^L$ constructed in Eq. (3.2.14) satisfies the condition from Eq. (3.2.20) that $\|\mathcal{W}^L\| \leq \frac{1}{4}$. We let

$$\mathcal{Z}_j = \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_j, \tag{3.2.25}$$

where $\mathcal{Y}_j$ is defined as in Eq. (3.2.15) as

$$\mathcal{Y}_j = \mathcal{Y}_{j-1} + q^{-1}P_{\mathbf{\Omega}_j}P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - \mathcal{Y}_{j-1}).$$

We note that $\mathcal{Z}_i \in \mathbf{T}$ as $\mathcal{U} * \mathcal{V}^* \in \mathbf{T}$ and subspaces are closed under scalar multiplication and addition. We also have

$$\mathcal{Z}_j = (P_{\mathbf{T}} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}})\mathcal{Z}_{j-1}$$

which follows by substituting the definition of $\mathcal{Y}_i$ as follows:

$$\begin{aligned}
(P_{\mathbf{T}} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}})\mathcal{Z}_{j-1} &= (P_{\mathbf{T}} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}})(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1}) \\
&= P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1}) - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1}) \\
&= \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1}) \\
&= \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}(\mathcal{Y}_{j-1} + q^{-1}P_{\mathbf{\Omega_j}}P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j-1})) \\
&= \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_j \\
&= \mathcal{Z}_{j+1}.
\end{aligned}$$

Since $P_{\mathbf{T}}\mathcal{Z}_i = \mathcal{Z}_i$ and $\|\mathcal{Z} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}_i}\mathcal{Z}\|_\infty \leq \epsilon\|\mathcal{Z}\|_\infty$ (Lemma 3.14), we have

$$\|\mathcal{Z}_j\|_\infty = \|(P_{\mathbf{T}} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega_j}}P_{\mathbf{T}})\mathcal{Z}_{j-1}\|_\infty = \|\mathcal{Z}_{j-1} - q^{-1}P_{\mathbf{T}}P_{\mathbf{\Omega}_j}\mathcal{Z}_{j-1}\|_\infty \leq \epsilon\|\mathcal{Z}_{j-1}\|_\infty$$

and so

$$\|\mathcal{Z}_j\|_\infty \le \epsilon \|\mathcal{Z}_{j-1}\|_\infty \le \epsilon^j \|\mathcal{Z}_0\|_\infty = \epsilon^j \|\mathcal{U} * \mathcal{V}^*\|_\infty. \tag{3.2.26}$$

Now we note that as $\mathcal{Z}_j \in \mathbf{T}$ for all j, we have that

$$\mathcal{Y}_j = \mathcal{Y}_{j-1} + q^{-1} P_{\mathbf{\Omega}_j} P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - \mathcal{Y}_{j-1}) = \mathcal{Y}_{j-1} + q^{-1} P_{\mathbf{\Omega}_j} P_{\mathbf{T}} \mathcal{Z}_{j-1} = \mathcal{Y}_{j-1} + q^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1}$$

so inductively,

$$\mathcal{Y}_{j_0} = \sum_{j=1}^{j_0} q^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1}. \tag{3.2.27}$$

Then, to show that $\|\mathcal{W}^L\| < \frac{1}{4}$, we note that as $P_{\mathbf{T}^\perp} \mathcal{Z}_{j-1} = 0$,

$$\begin{aligned}
\|\mathcal{W}^L\| &= \|P_{\mathbf{T}^\perp} \mathcal{Y}_{j_0}\| \\
&= \left\| P_{\mathbf{T}^\perp} \sum_{j=1}^{j_0} q^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} \right\| \\
&= \left\| \sum_{j=1}^{j_0} q^{-1} P_{\mathbf{T}^\perp} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} \right\| \\
&= \left\| \sum_{j=1}^{j_0} (q^{-1} P_{\mathbf{T}^\perp} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} - P_{\mathbf{T}^\perp} \mathcal{Z}_{j-1}) \right\| \\
&\le \sum_{j=1}^{j_0} \|(q^{-1} P_{\mathbf{T}^\perp} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} - P_{\mathbf{T}^\perp} \mathcal{Z}_{j-1})\|
\end{aligned}$$

and then, since $P_{\mathbf{T}^\perp}$ is a norm-1 operator, we can rearrange this as follows

$$\begin{aligned}
&= \sum_{j=1}^{j_0} \|P_{\mathbf{T}^\perp}(q^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} - \mathcal{Z}_{j-1})\| \\
&\le \sum_{j=1}^{j_0} \|q^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}_{j-1} - \mathcal{Z}_{j-1}\| \\
&= \sum_{j=1}^{j_0} \|(I - q^{-1} P_{\mathbf{\Omega}_j}) \mathcal{Z}_{j-1}\|.
\end{aligned}$$

We use the fact that $\|\mathcal{Z} - \rho^{-1} P_{\mathbf{\Omega}_j} \mathcal{Z}\| \le \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{q}} \|\mathcal{Z}\|_\infty$ (Lemma 3.15) and that

$\|\mathcal{Z}_j\|_\infty \leq \epsilon^j \|\mathcal{U} * \mathcal{V}^*\|_\infty$ (Eq. (3.2.26)) to get

$$\begin{aligned}
&\leq \sum_{j=1}^{j_0} \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{q}} \|\mathcal{Z}_{j-1}\|_\infty \\
&\leq \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{q}} \sum_{j=1}^{j_0} \|\mathcal{Z}_{j-1}\|_\infty \\
&\leq \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{q}} \sum_{j=1}^{j_0} \epsilon^{j-1} \|\mathcal{U} * \mathcal{V}^*\|_\infty \\
&\leq \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{q}} \|\mathcal{U} * \mathcal{V}^*\|_\infty (1-\epsilon)^{-1}.
\end{aligned}$$

Now we use the fact that $\|\mathcal{U} * \mathcal{V}^*\|_\infty \leq \sqrt{\frac{\mu r}{\tilde{n}_1 \tilde{n}_3}}$ (Eq. (3.0.3)) and that $q \geq C_0 \frac{\epsilon^{-2} \mu r (\log n_{(1)} \tilde{n}_3)}{n_{(2)} \tilde{n}_3}$ (Eq. (3.2.11)) to get

$$\leq \sqrt{\frac{C_0 n_{(1)} \tilde{n}_3 \log(n_{(1)} \tilde{n}_3)}{C_0 \frac{\epsilon^{-2} \mu r (\log n_{(1)} \tilde{n}_3)}{n_{(2)} \tilde{n}_3}}} \sqrt{\frac{\mu r}{\tilde{n}_1 \tilde{n}_3}} (1-\epsilon)^{-1} = \epsilon (1-\epsilon)^{-1} \leq \frac{1}{4}$$

where the last inequality holds because $\epsilon \leq \frac{1}{5}$ (Eq. (3.0.7)) and $\frac{1}{5}(1-\frac{1}{5})^{-1} = \frac{1}{4}$.

3.2.3 $\mathcal{W}^L$ Part 2

We will now prove that $\mathcal{W}^L$ satisfies the constraint from Eq. (3.2.21) that

$$\|P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_F \leq \frac{\lambda}{4}.$$

Since $\|P_{\mathbf{T}} - q^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega}_j} P_{\mathbf{T}}\| \leq \epsilon$ (Lemma 3.9), we have that

$$\left\|(P_{\mathbf{T}} - q^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega_j}} P_{\mathbf{T}})\mathcal{Z}\right\|_F \leq \epsilon \|\mathcal{Z}\|_F.$$

Hence, for $\mathcal{Z}_j$,

$$\|\mathcal{Z}_j\|_F = \|(P_{\mathbf{T}} - q^{-1} P_{\mathbf{T}} P_{\mathbf{\Omega_j}} P_{\mathbf{T}})\mathcal{Z}_{j-1}\|_F \leq \epsilon \|\mathcal{Z}_{j-1}\|_F$$

so

$$\|\mathcal{Z}_j\|_F \leq \epsilon\|\mathcal{Z}_{j-1}\|_F \leq \epsilon^j\|\mathcal{Z}_0\|_F = \epsilon^j\|\mathcal{U} * \mathcal{V}^*\|_F \leq \epsilon^j\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}}, \tag{3.2.28}$$

where for the last inequality we use the fact that $\|\mathcal{U} * \mathcal{V}^*\|_\infty \leq \sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}}$ by Eq. (3.0.3). As $\mathcal{Y}_0 = 0$, by induction we have

$$P_{\mathbf{\Omega}}\mathcal{Y}_j = P_{\mathbf{\Omega}}\mathcal{Y}_{j-1} + P_{\mathbf{\Omega}}q^{-1}P_{\Omega_j}P_{\mathbf{T}}(\mathcal{U} * \mathcal{V}^* - \mathcal{Y}_{j-1}) = 0,$$

as $\mathbf{\Omega} \cap \Omega_j = 0$. So then we have $P_{\mathbf{\Omega}}\mathcal{W}^L = P_{\mathbf{\Omega}}P_{\mathbf{T}^\perp}\mathcal{Y}_j = P_{\mathbf{\Omega}}(\mathcal{Y}_j - P_{\mathbf{T}^\perp}\mathcal{Y}_j) = P_{\mathbf{\Omega}}P_{\mathbf{T}^\perp}\mathcal{Y}_j$, which means that

$$P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L) = P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_j) = P_{\mathbf{\Omega}}\mathcal{Z}_j.$$

Now we can use the fact that $P_{\mathbf{\Omega}}$ has norm less than or equal to 1 by Lemma 3.6, that $\|\mathcal{Z}_j\|_F \leq \epsilon^j\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}}$ (Eq. (3.2.28)) and that we chose $j_0 = \lceil\log(n_{(1)}\tilde{n}_3)\rceil$ to get

$$\|P_{\mathbf{\Omega}}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_F = \|P_{\mathbf{\Omega}}\mathcal{Z}_{j_0}\|_F \leq \|\mathcal{Z}_{j_0}\|_F \leq \epsilon^{j_0}\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}} \leq \epsilon^{\log(n_{(1)}\tilde{n}_3)}\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}}.$$

Now, we note that $\epsilon \leq \exp(-1)$ by Eq. (3.0.7), that $r \leq \frac{n_{(2)}\tilde{n}_3}{\mu(\log(n_{(1)}\tilde{n}_3))^2}$ by Eq. (3.0.5) and that $\lambda = \frac{1}{\sqrt{n_{(1)}\tilde{n}_3}}$ to get that the above expression is less than or equal to

$$\begin{aligned}
\exp\left(-\log(n_{(1)}\tilde{n}_3)\right)\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}} &\leq (n_{(1)}\tilde{n}_3)^{-1}\sqrt{\frac{\mu\left(\frac{n_{(2)}\tilde{n}_3}{\mu(\log(n_{(1)}\tilde{n}_3))^2}\right)}{\tilde{n}_1\tilde{n}_3}} \\
&= \frac{1}{(n_{(1)}\tilde{n}_3)^{\frac{3}{2}}\log(n_{(1)}\tilde{n}_3)} \\
&= \frac{\lambda}{(n_{(1)}\tilde{n}_3)\log(n_{(1)}\tilde{n}_3)} \\
&\leq \frac{\lambda}{4},
\end{aligned}$$

giving us our desired result.

Remark 3.16. We quickly note that the above calculation also yields $\|\mathcal{Z}_{j_0}\|_F \leq \frac{\lambda}{8}$

3.2.4 $\mathcal{W}^L$ Part 3

Now we will show that $\mathcal{W}^L$ satisfies the third constraint Eq. (3.2.22) that

$$\|P_{\mathbf{\Omega}^\perp}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_\infty < \frac{\lambda}{4}.$$

Since $\mathcal{Z}_j = \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_j$ (Eq. (3.2.25)) and $\mathcal{W}^L = P_{\mathbf{T}^\perp}\mathcal{Y}_{J_0}$ (Eq. (3.2.14)) we can write

$$\mathcal{Z}_{j_0} + \mathcal{Y}_{j_0} = \mathcal{U} * \mathcal{V}^* - P_{\mathbf{T}}\mathcal{Y}_{j_0} + \mathcal{Y}_{j_0} = \mathcal{U} * \mathcal{V}^* + P_{\mathbf{T}^\perp}\mathcal{Y}_{j_0} = \mathcal{U} * \mathcal{V}^* + \mathcal{W}^L.$$

Making this substitution, we get

$$\|P_{\mathbf{\Omega}^\perp}(\mathcal{U} * \mathcal{V}^* + \mathcal{W}^L)\|_\infty = \|P_{\mathbf{\Omega}^\perp}(\mathcal{Z}_{j_0} + \mathcal{Y}_{j_0})\|_\infty \le \|P_{\mathbf{\Omega}^\perp}\mathcal{Z}_{j_0}\|_\infty + \|P_{\mathbf{\Omega}^\perp}\mathcal{Y}_{j_0}\|_\infty.$$

Now we note that we can use the result from above (Remark 3.16) to get that

$$\|P_{\mathbf{\Omega}^\perp}\mathcal{Z}_{j_0}\|_\infty \le \|\mathcal{Z}_{j_0}\|_\infty \le \|\mathcal{Z}_{j_0}\|_F \le \frac{\lambda}{8}.$$

All we have left to show is that $\|P_{\mathbf{\Omega}^\perp}\mathcal{Y}_{j_0}\|_\infty \le \frac{\lambda}{8}$. From Eq. (3.2.27) we have that

$$\mathcal{Y}_{j_0} = \sum_{j=1}^{j_0} q^{-1} P_{\mathbf{\Omega}^\perp}\mathcal{Z}_{j-1}.$$

Then we have that

$$\|P_{\mathbf{\Omega}^\perp}\mathcal{Y}_{j_0}\|_\infty = \left\|\sum_{j=1}^{j_0} q^{-1} P_{\mathbf{\Omega}^\perp}\mathcal{Z}_{j-1}\right\|_\infty \le q^{-1}\sum_{j=1}^{j_0}\|P_{\mathbf{\Omega}^\perp}\mathcal{Z}_{j-1}\|_\infty \le q^{-1}\sum_{j=1}^{j_0}\|\mathcal{Z}_{j-1}\|_\infty,$$

which means we can apply the fact that $\|\mathcal{Z}_{j-1}\|_\infty \le \epsilon^{j-1}\|\mathcal{U} * \mathcal{V}^*\|_\infty$ (by Eq. (3.2.26)) to get that

$$q^{-1}\sum_{j=1}^{j_0}\|\mathcal{Z}_{j-1}\|_\infty \le q^{-1}\sum_{j=1}^{j_0}\epsilon^{j-1}\|\mathcal{U} * \mathcal{V}^*\|_\infty \le q^{-1}(1-\epsilon)^{-1}\|\mathcal{U} * \mathcal{V}^*\|_\infty.$$

From Eq. (3.0.3) and Eq. (3.2.11), we have that $\|\mathcal{U} * \mathcal{V}^*\|_\infty \leq \sqrt{\frac{\mu r}{\tilde{n}_1 \tilde{n}_3}}$ and that $q \geq C_0 \frac{\epsilon^{-2}\mu r(\log n_{(1)}\tilde{n}_3)}{n_{(2)}\tilde{n}_3}$, which we apply to get

$$\begin{aligned} q^{-1}(1-\epsilon)^{-1}\|\mathcal{U} * \mathcal{V}^*\|_\infty &\leq \frac{n_{(2)}\tilde{n}_3}{C_0\epsilon^{-2}\mu r(\log n_{(1)}\tilde{n}_3)}(1-\epsilon)^{-1}\sqrt{\frac{\mu r}{\tilde{n}_1\tilde{n}_3}} \\ &\leq \frac{\sqrt{n_{(2)}}\epsilon^2}{C_0\sqrt{n_{(1)}}\sqrt{\mu r}(\log n_{(1)}\tilde{n}_3)}(1-\epsilon)^{-1}. \end{aligned}$$

We use the fact that $\varepsilon \leq \left(\frac{\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}\right)^{\frac{1}{4}}$ by Eq. (3.0.7) and that $\lambda = \frac{1}{\sqrt{n_{(1)}\tilde{n}_3}}$ by Eq. (3.0.6) to get

$$\begin{aligned} \frac{\sqrt{n_{(2)}}\epsilon^2}{C_0\sqrt{n_{(1)}}\sqrt{\mu r}(\log n_{(1)}\tilde{n}_3)}(1-\epsilon)^{-1} &\leq \frac{\sqrt{n_{(2)}}\sqrt{\frac{\mu r(\log(n_{(1)}\tilde{n}_3))^2}{n_{(2)}\tilde{n}_3}}}{C_0\sqrt{n_{(1)}}\sqrt{\mu r}(\log n_{(1)}\tilde{n}_3)}(1-\epsilon)^{-1} \\ &= \frac{1}{C_0\sqrt{n_{(1)}\tilde{n}_3}}(1-\epsilon)^{-1} \\ &= \frac{\lambda}{C_0}(1-\epsilon)^{-1}. \end{aligned}$$

Since we chose $C_0 = \frac{32}{3}$ by Eq. (3.0.9) and $\varepsilon < \frac{1}{5}$ by Eq. (3.0.7) we have the above expression is less than

$$\lambda\frac{3}{32}\left(1-\frac{1}{5}\right)^{-1} \leq \lambda\frac{3}{32}\frac{5}{4} < \frac{\lambda}{8}.$$

3.2.5 $\mathcal{W}^S$ Part 1

We will now show that the $\mathcal{W}^S$ constructed in Eq. (3.2.13) satisfies the condition from Eq. (3.2.23) that $\|\mathcal{W}^S\| \leq \frac{1}{4}$. To do this, we will write $\mathcal{W}^S$ as an operator times a random tensor, which we will then bound. We let $\mathcal{B} = \text{sign}(\mathcal{C}_0)$ and note that the distribution of the element of $\mathcal{B}$ with index J is given by

$$\mathcal{B}_J = \begin{cases} 1, & \text{w.p. } \frac{\rho}{2} \\ 0, & \text{w.p. } 1-\rho \\ -1, & \text{w.p. } \frac{\rho}{2}. \end{cases} \tag{3.2.29}$$

Then we note that

$$\mathcal{W}^S = \lambda P_{\mathbf{T}^\perp} \left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^{-1} \mathcal{B} = \lambda P_{\mathbf{T}^\perp} \sum_{k=0}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k \mathcal{B}.$$

So to bound $\mathcal{W}^S$ we let $R(\mathcal{B}) = \sum_{k=0}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k \mathcal{B}$, which, as the sum of compositions of self-adjoint operators, is self-adjoint. Then we use the fact that $P_{\mathbf{T}}$ has norm less than or equal to 1 (see Lemma 3.6) and that `tdiag` preserves the operator norm (see Eq. (2.4.7)) to get

$$\begin{aligned}\|\lambda P_{\mathbf{T}^\perp} \sum_{k=1}^{\infty} \left(P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\right)^k \mathcal{B}\| = \|\lambda P_{\mathbf{T}^\perp} R(\mathcal{B})\| \leq \lambda \|R(\mathcal{B})\| &= \lambda \| \texttt{tdiag}(R(\mathcal{B}))\| \\ &= \max_{i \in 0,\ldots,\tilde{n}_3 - 1} \lambda \| \texttt{tdiag}(R(\mathcal{B}))_i\|,\end{aligned}$$

where the last inequality holds by the fact that the spectral norm of a block diagonal matrix is simply given by the largest spectral norm of the matrices on the block diagonal. To bound this expression, we use the following lemma.

Lemma 3.17 (Net bound on the spectral norm)**.** [Vershynin (2007, 2011)] *The spectral norm of a matrix $A \in \mathbb{C}^{m_1 \times m_2}$ is bounded by*

$$\|A\| \leq 2 \max_{x \in N} \|Ax\|_2 \leq 4 \max_{x,y \in N} |\langle y, Ax\rangle|$$

where N (is a half net for the unit sphere that) has less than or equal to $\min\{5^{2n_1}, 5^{2n_2}\}$ *elements, all with Frobenius norm 1.*

Applying the above lemma, rearranging using the the formula from Corollary 2.19 for the

ith matrix on the block diagonal and using the fact that R is self-adjoint, we have that

$$\begin{aligned}
\max_{i\in 0,\ldots,\tilde{n}_3-1} \lambda\| \texttt{tdiag}(R(\mathcal{B}))_i\| &\leq 4\lambda \max_{i\in 0,\ldots,\tilde{n}_3-1} \max_{x,y\in N} |\langle x, \texttt{tdiag}(R(\mathcal{B}))_i y\rangle| \\
&= 4\lambda \max_{i\in 0,\ldots,\tilde{n}_3-1} \max_{x,y\in N} |\langle xy^*, \tilde{F}_i\, \texttt{unfold}(R(\mathcal{B}))\rangle| \\
&= 4\lambda \max_{i\in 0,\ldots,\tilde{n}_3-1} \max_{x,y\in N} |\langle \tilde{F}_i^* xy^*, \texttt{unfold}(R(\mathcal{B}))\rangle| \\
&= 4\lambda \max_{i\in 0,\ldots,\tilde{n}_3-1} \max_{x,y\in N} |\langle \texttt{fold}(\tilde{F}_i^* xy^*), R(\mathcal{B})\rangle| \\
&= \max_{i\in 0,\ldots,\tilde{n}_3-1} \max_{x,y\in N} |\langle 4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*)), \mathcal{B}\rangle|.
\end{aligned}$$

To bound this final expression, we will use a corollary of the non-commutative Bernstein inequality. The argument Lu et al. (2018) gave for this section used Hoeffding's inequality, but it appears to only be valid for real valued tensors. However, it seems possible that complex entries could appear. So this section was changed, inspired by a discussion on Math Overflow (Suvrit, 2013).

Lemma 3.18. *If the signs of $\mathcal{B}$ are distributed as Eq.* (3.2.29) *(*± 1 *with probability* $\frac{\rho}{2}$ *and 0 with probability* $1-\rho$*) then*

$$\mathbb{P}\left(|\langle \mathcal{A}, \mathcal{B}\rangle| > t\right) \leq 4 \exp\left(-\frac{3t^2}{8\|\mathcal{A}\|_F^2}\right).$$

Proof. To prove this, we will use the non-commutative Bernstein inequality. First we note that by writing complex numbers in their equivalent matrix form, we have

$$|\langle \mathcal{A}, \mathcal{B}\rangle| = |\sum_J \mathcal{A}_J \mathcal{B}_J| = \left\|\sum_J \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J)\end{pmatrix} \mathcal{B}_J\right\|.$$

The elements of this sequence of matrices is only stochastic in $\mathcal{B}_J$, which has an expected value of 0. The norm of each element is also clearly bounded by the infinity norm of $\mathcal{A}$. Next we note that

$$\begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J)\end{pmatrix}^* \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J)\end{pmatrix} = \begin{pmatrix} r(\mathcal{A}_J) & -i(\mathcal{A}_J) \\ i(\mathcal{A}_J) & r(\mathcal{A}_J)\end{pmatrix} \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J)\end{pmatrix} = |\mathcal{A}_J|^2 I$$

and that

$$\begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix} \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix}^* = \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix} \begin{pmatrix} r(\mathcal{A}_J) & -i(\mathcal{A}_J) \\ i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix} = |\mathcal{A}_J|^2 I.$$

So then we have that

$$\left\| \sum_J \mathbb{E} \left(\left(\begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix} \mathcal{B}_J \right)^* \begin{pmatrix} r(\mathcal{A}_J) & i(\mathcal{A}_J) \\ -i(\mathcal{A}_J) & r(\mathcal{A}_J) \end{pmatrix} \mathcal{B}_J \right\| = \rho \|\mathcal{A}\|_F^2.$$

So we can apply the inequality from Theorem 3.8 (setting $M = \|\mathcal{A}\|_\infty$, $\sigma^2 = \|\mathcal{A}\|_F^2$) to get

$$\mathbb{P}\left(|\langle \mathcal{A}, \mathcal{B} \rangle| > t\right) \leq 4 \exp\left(-\frac{3t^2}{8\|\mathcal{A}\|_F^2}\right). \square$$

So, to use this lemma to bound $|\langle 4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*)), \mathcal{B} \rangle|$, we need to bound the squared Frobenius norm of $4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*))$, which is straightforward, since $\tilde{F}_i^*$, x and y all have Frobenius norm 1:

$$\|4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*))\|_F^2 \leq 16\lambda^2 \|R\|^2 \|(\texttt{fold}(\tilde{F}_i^* xy^*))\|_F^2 \leq 16\lambda^2 \|R\|^2.$$

Remark 3.19. Since $\|P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\| = \|P_{\mathbf{T}} P_{\mathbf{\Omega}}\|^2 = \sigma < 1$ (Corollary 3.11) we have that

$$\|R\| = \| \sum_{k=0}^{\infty} (P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}})^k \| \leq \sum_{k=0}^{\infty} \|P_{\mathbf{\Omega}} P_{\mathbf{T}} P_{\mathbf{\Omega}}\|^k = \sum_{k=0}^{\infty} \|P_{\mathbf{T}} P_{\mathbf{\Omega}}\|^{2k} \leq \sum_{k=0}^{\infty} \sigma^{2k} = \frac{1}{1-\sigma^2} \equiv \sigma',$$

where we know the sum converges since $\sigma < \frac{4}{9} < 1$ (Corollary 3.11). Combining these and substituting $\lambda = \frac{1}{\sqrt{n_{(1)} \tilde{n}_3}}$ (Eq. (3.0.6)) gives

$$\|4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*))\|_F^2 \leq 16\lambda^2 \sigma'^2 = \frac{16\sigma'^2}{n_{(1)} \tilde{n}_3}.$$

So now we can apply Lemma 3.18 for a fixed x, y and i to get

$$\begin{aligned}\mathbb{P}\left(|\langle 4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*)), \mathcal{B}\rangle| > \frac{1}{4}\right) &\leq 4\exp\left(-\frac{3}{8}\frac{\left(\frac{1}{4}\right)^2}{\|4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*))\|_F^2}\right)\\ &\leq 4\exp\left(-\frac{3}{8}\frac{1}{16}\frac{n_{(1)}\tilde{n}_3}{16\sigma'^2}\right) = 4\exp\left(-\frac{3n_{(1)}\tilde{n}_3}{2^{11}\sigma'^2}\right).\end{aligned}$$

Putting this together and taking the union bound over x and y (of which there are no more than $5^{2n_{(2)}}5^{2n_{(2)}} = 5^{4n_{(2)}}$ elements) gives

$$\begin{aligned}\mathbb{P}\left(\|\lambda P_{\mathbf{T}^\perp}\sum_{k=1}^{\infty}\left(P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^k M\| > \frac{1}{4}\right) &\leq \mathbb{P}\left(\max_{i\in 0,\ldots,\tilde{n}_3-1}\max_{x,y\in N}|\langle 4\lambda R(\texttt{fold}(\tilde{F}_i^* xy^*)), \mathcal{B}\rangle| > \frac{1}{4}\right)\\ &\leq 4\tilde{n}_3 5^{4n_{(2)}}\exp\left(-\frac{3n_{(1)}\tilde{n}_3}{2^{11}\sigma'^2}\right)\end{aligned}$$

which, provided that

$$\frac{3\tilde{n}_3}{2^{11}\sigma'^2} - 4\log(5) > 0, \tag{3.2.30}$$

is less than $O((n_{(1)}\tilde{n}_3)^{-1})$, giving us our desired result.

3.2.6 $\mathcal{W}^S$ Part 2

We will now show that the $\mathcal{W}^S$ constructed in Eq. (3.2.13) satisfies the conditions of Eq. (3.2.24) that $\|P_{\mathbf{\Omega}^\perp}\mathcal{W}^S\|_\infty < \frac{\lambda}{4}$. First we note that

$$\begin{aligned}\mathcal{W}^S &= \lambda P_{\mathbf{T}^\perp}\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\\ &= \lambda(\mathcal{I} - P_{\mathbf{T}})\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\\ &= \lambda\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B} - \lambda P_{\mathbf{T}}\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}.\end{aligned}$$

Using the other characterisation of the inverse of $P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}$ (Remark 3.13) we have

$$P_{\mathbf{\Omega}^\perp}\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1} = P_{\mathbf{\Omega}^\perp}\left(\sum_{k=0}^{\infty}(P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^k\right) = 0 \tag{3.2.31}$$

and also that

$$P_{\mathbf{\Omega}}\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}=P_{\mathbf{\Omega}}\left(\sum_{k=0}^{\infty}(P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^{k}\right)=\left(\sum_{k=0}^{\infty}(P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^{k}\right)=\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}. \tag{3.2.32}$$

We can use the first of these to get

$$P_{\mathbf{\Omega}^{\perp}}\mathcal{W}^{S}=-\lambda P_{\mathbf{\Omega}^{\perp}}P_{\mathbf{T}}\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}.$$

For any index I in $\mathbf{\Omega}^{\perp}$, we can substitute in the above equality to get that

$$\mathcal{W}_I^S=\langle\mathbf{e}_I,\mathcal{W}^S\rangle=\langle P_{\mathbf{\Omega}^{\perp}}\mathbf{e}_I,\mathcal{W}^S\rangle=\langle\mathbf{e}_I,P_{\mathbf{\Omega}^{\perp}}\mathcal{W}^S\rangle=\langle\mathbf{e}_I,-\lambda P_{\mathbf{\Omega}^{\perp}}P_{\mathbf{T}}\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\rangle$$

and then we can use the fact these are all self-adjoint operators (Lemma 3.6) to get

$$\begin{aligned}
&=\lambda\langle P_{\mathbf{\Omega}^{\perp}}\mathbf{e}_I,-P_{\mathbf{T}}\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\rangle\\
&=\lambda\langle P_{\mathbf{T}}\mathbf{e}_I,-\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\rangle\\
&=\lambda\langle P_{\mathbf{T}}\mathbf{e}_I,-P_{\mathbf{\Omega}}\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\mathcal{B}\rangle\\
&=\langle-\lambda\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}P_{\mathbf{\Omega}}P_{\mathbf{T}}\mathbf{e}_I,\mathcal{B}\rangle.
\end{aligned}$$

Since $\|(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}})^{-1}\|\leq\frac{1}{1-\sigma^2}$ (Remark 3.19), $\|P_{\mathbf{\Omega}}P_{\mathbf{T}}\|^2\leq\sigma$ (Corollary 3.11) and $\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2\leq\frac{\mu r(n_1+n_2)}{\tilde{n}_1}$ (Proposition 3.7), we have

$$\begin{aligned}
\|\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}P_{\mathbf{\Omega}}P_{\mathbf{T}}\mathbf{e}_I\|_F^2 &\leq\|\left(P_{\mathbf{\Omega}}-P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}\|^2\|P_{\mathbf{\Omega}}P_{\mathbf{T}}\|^2\|P_{\mathbf{T}}\mathbf{e}_I\|_F^2\\
&\leq\left(\frac{1}{1-\sigma^2}\right)^2\sigma^2\frac{\mu r(n_1+n_2)}{\tilde{n}_1}\\
&=\frac{\sigma^2}{(1-\sigma^2)^2}\frac{\mu r(n_1+n_2)}{\tilde{n}_1}.
\end{aligned}$$

Now we use that $r \leq \frac{n_{(2)}\tilde{n}_3}{\mu(\log(n_{(1)}\tilde{n}_3))^2}$ (Eq. (3.0.5)) to get that the above expression is

$$\begin{aligned}
&\leq \frac{\sigma^2}{(1-\sigma^2)^2} \frac{\frac{n_{(2)}\tilde{n}_3}{(\log(n_{(1)}\tilde{n}_3))^2}(n_1+n_2)}{\tilde{n}_1} \\
&\leq \frac{\sigma^2}{(1-\sigma^2)^2} \frac{n_{(1)}n_{(2)}\tilde{n}_3}{\tilde{n}_1(\log(n_{(1)}\tilde{n}_3))^2} \\
&= \frac{\sigma^2}{(1-\sigma^2)^2} \frac{1}{(\log(n_{(1)}\tilde{n}_3))^2} \\
&= \frac{\sigma''}{(\log(n_{(1)}\tilde{n}_3))^2}
\end{aligned}$$

where $\sigma'' = \frac{\sigma^2}{(1-\sigma^2)^2}$. We therefore apply Lemma 3.18 and get that

$$\begin{aligned}
\mathbb{P}\left(|\mathcal{W}_I^S| > \frac{\lambda}{4}\right) &\leq 4\exp\left(-\frac{3}{8}\frac{(\frac{\lambda}{4})^2}{\|\lambda\left(P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}}P_{\mathbf{T}}P_{\mathbf{\Omega}}\right)^{-1}P_{\mathbf{\Omega}}P_{\mathbf{T}}\mathbf{e}_I\|_F^2}\right) \\
&\leq 4\exp\left(-\frac{3}{2^7}\frac{1}{\frac{\sigma''}{(\log(n_{(1)}\tilde{n}_3))^2}}\right) \\
&\leq 4\exp\left(-\frac{3(\log(n_{(1)}\tilde{n}_3))^2}{2^7\sigma''}\right) = 4(n_{(1)}\tilde{n}_3)^{-\frac{3\log(n_{(1)}\tilde{n}_3)}{2^7\sigma''}}
\end{aligned}$$

so taking the union bound gives that

$$\begin{aligned}
\mathbb{P}\left(\|P_{\mathbf{\Omega}^\perp}\mathcal{W}^S\|_\infty \geq \frac{\lambda}{4}\right) = \mathbb{P}\left(\max_I |\mathcal{W}_I^S| > \frac{\lambda}{4}\right) &\leq \tilde{n}_1 4(n_{(1)}\tilde{n}_3)^{-\frac{3\log(n_{(1)}\tilde{n}_3)}{2^7\sigma''}} \\
&= 4n_{(2)}(n_{(1)}\tilde{n}_3)^{1-\frac{3\log(n_{(1)}\tilde{n}_3)}{2^7\sigma''}}.
\end{aligned}$$

which, provided that

$$\frac{3\log(n_{(1)}\tilde{n}_3)}{2^7\sigma''} > 2, \tag{3.2.33}$$

is less than $O((n_{(1)}\tilde{n}_3)^{-1})$, giving us our desired result.

3.2.7 Q.E.D.

Under the assumptions made by Theorem 3.1, the dual certificate constructed at Eq. (3.2.12) satisfies all the required constraints, meaning that we can apply Lemma 3.12. This proves Theorem 3.1.

3.3 Discussion and Numerical Experiments

Several potential limitations of the TRPCA procedure were suggested in the proof of Theorem 3.1; the aim of the following empirical study is to investigate these limitations. The Alternating Direction Method of Multipliers as outlined by (Lu et al., 2018) was used to implement the TRPCA procedure for this section. The low-rank recovery error used was $\frac{\|L-L_0\|_F}{\|L_0\|_F}$, where L_0 is the low-rank data structure we are trying to recover and L is the output of the algorithm. We also measured the recovery error for the noise, but in all observed cases, this error was found to behave similarly, so for neatness of presentation these results have been omitted.

Figure 3.1: Left: the input (M_0) to the TRPCA algorithm. Gaussian noise (mean 0, varience $\frac{1}{25}$) was added to 10 % of the elements of the original image. Middle: the low rank output (L) of the TRPCA algorithm. Right: the sparse noise output (C) of the TRPCA algorithm.

One of the weaknesses in the proof given above is the hard upper bound on ρ (the probability that there is noise in a given index). From Theorem 3.1, we require that $\rho < \frac{436-6\sqrt{836}}{900} \approx 0.2917$. This is specifically to ensure that $\sqrt{\sigma} < \frac{2}{3}$ (Corollary 3.11) so that Eq. (3.2.9) is positive. It makes sense that as the amount of noise gets higher, the problem gets harder to solve, but it would be nicer (and more intuitive) if rather than having a hard cut-off, we had a trade-off between the amount of noise and the size of the tensor (for a fixed rank), as we do in other parts of this proof, such as in Eq. (3.2.30) and Eq. (3.2.33). We test below whether this trade-off is plausible along with testing whether or not the problem gets easier to solve as the amount of noise increases — as suggested by Eq. (3.0.5), which states $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)}\tilde{n}_3}{(\log(n_{(1)}\tilde{n}_3))^2}$. Fig. 3.2 shows that the expected trade-off (rather than hard cut-off) between ρ and the size of the tensor (for a fixed rank) is empirically demonstrable and continues to work for values of ρ that are greater than the hard bound required above. It also shows that the problem does not get easier to solve as the amount of noise increases.

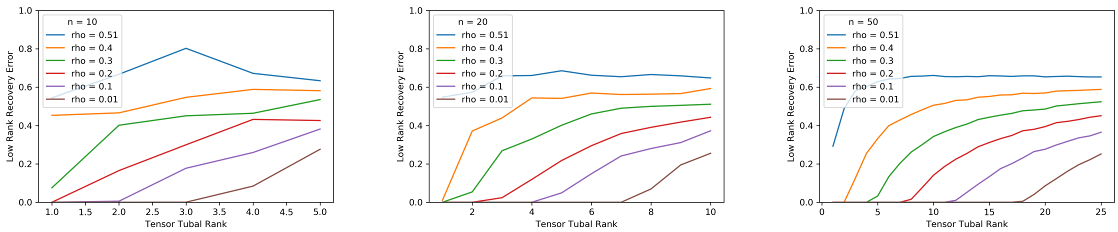

Figure 3.2: For each n and ρ, tensors of varying rank r were created by multiplying a $n \times r \times n$ and a $r \times n \times n$ tensor, each with random entries (Gaussian, mean 0, standard deviation $\frac{1}{n}$). Then noise (Gaussian, mean 0, standard deviation $\frac{1}{n}$) was added to ρ % of the entries. These were then put into the TRPCA algorithm, and the recovery error was measured. The recovery error improves as n increases and deteriorates as ρ increases.

Another weakness in the proof above is the assumption that the tensor of the noise's signs has entries that are ± 1 with probability $\frac{\rho}{2}$ and 0 with probably $1-\rho$. This assumption is much stronger than what is intuitively needed to solve this problem (the assumption that the noise tensor is sparse). In applications, the intuition alone serves us well, and violating the formality of this assumption (rather than its implication of sparsity) does not prevent the procedure from working, as demonstrated in Fig. 3.3. However, the performance of the algorithm is somewhat affected.

Something peculiar about the the t-product and the associated analysis is that the first two dimensions of any tensor are treated differently to the remainder. Since for $\mathcal{C} = \mathcal{A} * \mathcal{B}$ $\mathcal{C}(:,:,J) = \sum_I \mathcal{A}(:,:,I)\mathcal{B}(:,:,J-I)$, (Lemma 2.6) we can see that if were to switch any of the dimensions from 3 or above, the t-product would be unchanged (up to relabelling). Since the t-product would be unchanged, the tensor spectral norm (which is the operator norm of the t-product) and the tensor nuclear norm (the dual norm of the spectral

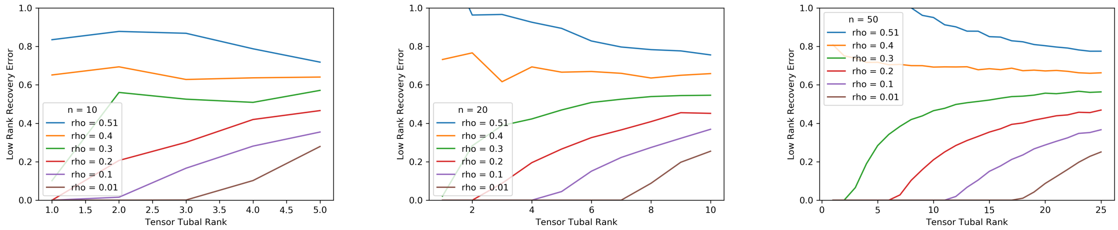

Figure 3.3: For each n and ρ, tensors of varying rank r were created by multiplying a $n \times r \times n$ and a $r \times n \times n$ tensor, each with random entries (Gaussian, mean 0, standard deviation $\frac{1}{n}$). Then noise (absolute value of a Gaussian distribution with mean 0 and standard deviation $\frac{1}{n}$) was added to ρ % of the entries. These were then put into the TRPCA algorithm, and the recovery error measured. As with Fig. 3.2, the recovery error improves as n increases and deteriorates as ρ increases, though the performance looks to be somewhat worse overall.

norm) would also be unchanged, leaving the entire procedure unchanged. This is reflected in the analysis above in the way that the size of any of the dimensions from 3 or above (n_i for $i \geq 3$) only appear as a product of all the dimensions greater than or equal to 3 ($\tilde{n}_3$). Given this, what if we wanted to fix different dimensions in the first 2 positions? This could provide some theoretical benefits since Eq. (3.0.5) requires that $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)}\tilde{n}_3}{(\log(n_{(1)}\tilde{n}_3))^2}$. While we certainly can permute the dimensions to maximise the number on the right, there is no guarantee that r and μ will not also change. For instance, we could consider a $n_1 \times n_2 \times n_3$ dimensional tensor as a $n_1 \times n_2 \times n_3 \times 1 \times 1$ dimensional tensor and then permute it to a $1 \times 1 \times n_1 \times n_2 \times n_3$ dimensional tensor. In this case, (crucially) ignoring μ, the number on the right of the above inequality is maximised, and the number on the left is minimised (since the rank is upper-bounded by the $n_{(2)} = 1$ and lower-bounded by 1). With the exception of Eq. (3.2.10), this also improves the likelihood of all the other bounds holding (since they are $O((n_{(1)}\tilde{n}_3)^{-1})$). A similar re-framing can be done without violating Eq. (3.2.10) by only switching 1 of the first 2 dimensions. This easy accomodation of our bounds feels too good to be true, and the results from Fig. 3.4 indicate that it is: under the new orientation, μ may increase.

$n_1 \times n_2 \times n_3 \times 1 \times 1$	$n_3 \times n_2 \times n_1 \times 1 \times 1$	$n_1 \times 1 \times n_3 \times n_2 \times 1$	$1 \times 1 \times n_3 \times n_2 \times n_1$
3.139e-08 (3, 380)	0.6394 (20, 350)	0.8511 (1, 32579)	0.6871 (1, 569299)

Figure 3.4: In the table above, the row corresponds to a single 5-dimensional tensor that has 2 dimensions of size 1 and 3 dimensions of size 20. Each column then corresponds to a different orientation of that tensor. The recovery error of the tensor when the TRPCA algorithm is applied to it is displayed alongside that tensor's tubal rank (left in parentheses) and the μ that satisfies the incoherence conditions for that tensor (right in parentheses). While we can reorient the tensor, the rank may increase (as in the second column) or μ might increase (as in columns 3 and 4). Recall that the inequalty we need to hold is $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)} \tilde{n}_3}{(\log(n_{(1)} \tilde{n}_3))^2}$.

Another important question to ask is which low-rank tensor is returned when the inequality from Eq. (3.0.5) that $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)} \tilde{n}_3}{(\log(n_{(1)} \tilde{n}_3))^2}$ is violated. A logical hypothesis to test is that it could be the low-rank tensor created by setting the smallest (in norm) $n - \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)} \tilde{n}_3}{(\log(n_{(1)} \tilde{n}_3))^2}$ singular tubes to 0. This would be a nice result since Kilmer & Martin (2011) showed that this low-rank approximation minimises the reconstruction error under the Frobenius norm. A preliminary analysis indicated that this was unfortunately not the case, though further investigation is required.

Chapter 4

An Online Algorithm

The goal of this chapter is to solve the following optimisation problem

$$\min \|\mathcal{L}\|_* + \lambda\|\mathcal{C}\|_1 \text{ subject to } \mathcal{M}_0 = \mathcal{L} + \mathcal{C}.$$

For sufficiently large λ_3, this is equivalent to solving the unconstrained problem

$$\min_{\mathcal{L},\mathcal{C}} \frac{1}{2}\lambda_3\|\mathcal{M} - \mathcal{L} - \mathcal{C}\|_F^2 + \|\mathcal{L}\|_* + \lambda\|\mathcal{C}\|_1,$$

which we can divide through by λ_3 so that $\lambda_1 = \frac{1}{\lambda_3}$ and $\lambda_2 = \frac{\lambda}{\lambda_3}$ and we solve

$$\min_{\mathcal{L},\mathcal{C}} \frac{1}{2}\|\mathcal{M} - \mathcal{L} - \mathcal{C}\|_F^2 + \lambda_1\|\mathcal{L}\|_* + \lambda_2\|\mathcal{C}\|_1. \tag{4.0.1}$$

An online approach would improve computational efficiency, as it would allow the procedure to be applied to larger datasets in a shorter amount of time. The biggest challenge in developing an online algorithm is that the tensor nuclear norm needs all the values of the tensor to already be computed, so the algorithm cannot be computed incrementally. Thus, our aim is to replace the tensor nuclear norm by something that can be computed online and gives the same result as the batch case. Once this is done, we will propose an algorithm to minimise the reformulated problem (4.0.1) and then analyse the convergence of this algorithm. This chapter primarily builds on the work of Feng et al. (2013) (for matrices) and Z. Zhang et al. (2016) (for 3-dimensional tensors); however, we expand upon these works in a number of ways. First, we propose a more general result that holds for p- rather than only 3-dimensional tensors. Second, unlike Z. Zhang et al. (2016), we analyse convergence. Our approach mainly follows Feng et al. (2013) and consequently inherits some the flaws of that paper. We have attempted to fix many of these and conclude the chapter with a discussion of these flaws.

Throughout this chapter we will make the following assumptions:

Assumption 4.1. $\|\mathcal{A}_t\|_F$ (which is defined in the next section) and $\|\mathcal{M}\|_F$ are uniformly bounded. And $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ (also defined in the next section) is bounded away from 0.

4.1 Reframing the Nuclear Norm

We can reformulate the nuclear norm as follows.

Lemma 4.2. *For $\mathcal{L}$ in $n_1 \times T \times n_3 \times \ldots \times n_p$ with $\mathrm{rank_t}(\mathcal{L}) \leq r$ we have*

$$\|\mathcal{L}\|_* = \inf_{\substack{\mathcal{A} \in \mathbb{R}^{n_1 \times r \times n_3 \times \ldots \times n_p} \\ \mathcal{B} \in \mathbb{R}^{T \times r \times n_3 \times \ldots \times n_p}}} \left\{ \frac{1}{2}(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2) : \mathcal{L} = \mathcal{A} * \mathcal{B}^T \right\}.$$

Notice that we have replaced n_2 with T to highlight that we expect the size of this dimension to vary (it will typically represent time). A version of this lemma for 3-dimensional tensors was stated without proof (to save space) by Z. Zhang et al. (2016), but theirs seems to be (incorrectly) scaled by n_3. Additionally, the empirical work by Z. Zhang et al. (2016) found $\lambda_1 = \lambda_2$ to be the tuning that performed the best, which is different to what the theory suggests should be optimal. It is however, much closer to what the theory suggests when the incorrect scaling in Z. Zhang et al. (2016) is accounted for. The proof below was adapted from the matrix version (e.g. Fritz (2018)).

Proof. For all $\mathcal{A} * \mathcal{B}^T = \mathcal{L}$, we can write $\|\mathcal{L}\|_*$ using its definition as the dual of the spectral norm and then apply the fact that $\langle \mathcal{A} * \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{B}, \mathcal{A}^* * \mathcal{C} \rangle$ (Remark 2.34) to get that

$$\|\mathcal{L}\|_* = \|\mathcal{A} * \mathcal{B}^T\|_* = \sup_{\|\mathcal{V}\|\leq 1} \langle \mathcal{A} * \mathcal{B}^T, \mathcal{V}\rangle = \sup_{\|\mathcal{V}\|\leq 1} \langle \mathcal{B}^T, \mathcal{A}^* * \mathcal{V}\rangle.$$

Then we use Cauchy–Schwarz (Eq. (2.4.1)) and apply the fact that $\|\mathcal{V}\| \leq 1$ to get

$$\sup_{\|\mathcal{V}\|\leq 1} \langle \mathcal{B}^T, \mathcal{A}^* * \mathcal{V}\rangle \leq \sup_{\|\mathcal{V}\|\leq 1} \|\mathcal{B}^T\|_F \|\mathcal{A}^* * \mathcal{V}\|_F \leq \|\mathcal{B}\|_F \|\mathcal{A}\|_F \leq \frac{1}{2}(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2).$$

So

$$\|\mathcal{L}\|_* \leq \inf_{\substack{\mathcal{A} \in \mathbb{R}^{n_1\times r\times n_3\times\ldots\times n_p} \\ \mathcal{B} \in \mathbb{R}^{T\times r\times n_3\times\ldots\times n_p}}} \left\{\frac{1}{2}(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2) : \mathcal{L} = \mathcal{A} * \mathcal{B}^T\right\}.$$

Now we note that, in line with Remark 2.24, we can write the t-SVD $\mathcal{L} = \mathcal{U} * \mathcal{S} * \mathcal{V}^* = \mathcal{U} * \mathcal{S}^{\frac{1}{2}} * \mathcal{S}^{\frac{1}{2}} * \mathcal{V}^*$. Using the fact that orthogonal operators do not change the Frobenius norm (Eq. (2.4.4)), $\|\mathcal{A}\|_F^2 = \frac{1}{\tilde{n}_3}\|\,\mathtt{bcirc}(\mathcal{A})\|_F^2$ (Eq. (2.4.2)) and that $\frac{1}{\tilde{n}_3}\|\tilde{F}^*\,\mathtt{bcirc}(\mathcal{S}^{\frac{1}{2}})\tilde{F}\|_F^2 = \|\mathcal{L}\|_*$ (Remark 2.42), we have that

$$\frac{1}{2}\left(\|\mathcal{U} * \mathcal{S}^{\frac{1}{2}}\|_F^2 + \|\mathcal{V} * \mathcal{S}^{\frac{1}{2}*}\|_F^2\right) = \|\mathcal{S}^{\frac{1}{2}}\|_F^2 = \frac{1}{\tilde{n}_3}\|\,\mathtt{bcirc}(\mathcal{S}^{\frac{1}{2}})\|_F^2 = \frac{1}{\tilde{n}_3}\|\tilde{F}^*\,\mathtt{bcirc}(\mathcal{S}^{\frac{1}{2}})\tilde{F}\|_F^2 = \|\mathcal{L}\|_*$$

and $\mathcal{L} = (\mathcal{U} * \mathcal{S}^{\frac{1}{2}}) * (\mathcal{S}^{\frac{1}{2}} * \mathcal{V}^*)$. So

$$\|\mathcal{L}\|_* \geq \inf_{\substack{\mathcal{A} \in \mathbb{R}^{n_1\times r\times n_3\times\ldots\times n_p} \\ \mathcal{B} \in \mathbb{R}^{T\times r\times n_3\times\ldots\times n_p}}} \left\{\frac{1}{2}(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2) : \mathcal{L} = \mathcal{A} * \mathcal{B}^T\right\}.$$

Putting these together implies that

$$\|\mathcal{L}\|_* = \inf_{\substack{\mathcal{A} \in \mathbb{R}^{n_1\times r\times n_3\times\ldots\times n_p} \\ \mathcal{B} \in \mathbb{R}^{T\times r\times n_3\times\ldots\times n_p}}} \left\{\frac{1}{2}(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2) : \mathcal{L} = \mathcal{A} * \mathcal{B}^T\right\}. \quad \square$$

Using this lemma, we can substitute $\mathcal{L} = \mathcal{A} * \mathcal{B}^T$ into Eq. (4.0.1) and solve a new minimisation problem:

$$\min_{\substack{\mathcal{A} \in \mathbb{R}^{n_1\times r\times n_3\times\ldots\times n_p} \\ \mathcal{B} \in \mathbb{R}^{T\times r\times n_3\times\ldots\times n_p} \\ \mathcal{C} \in \mathbb{R}^{n_1\times T\times n_3\times\ldots\times n_p}}} \frac{1}{2}\|\mathcal{M} - \mathcal{A} * \mathcal{B}^T - \mathcal{C}\|_F^2 + \frac{\lambda_1}{2}\left(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2\right) + \lambda_2\|\mathcal{C}\|_1. \qquad (4.1.1)$$

The lemma below shows that this delivers the same result, provided that $\mathcal{A}$ is full rank and a technical condition holds.

Lemma 4.3. *If $\mathcal{A}$ is full rank and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ minimises Eq.* (4.1.1)*, then ($\mathcal{L} = \mathcal{A} * \mathcal{B}^T, \mathcal{C}$) minimises Eq.* (4.0.1)*, provided that $\|\frac{1}{\lambda_1}\mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T\| \leq 1$ (where these terms are defined below).*

Proof. Our goal is to show that if $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ minimises

$$\frac{1}{2}\|\mathcal{M} - \mathcal{A} * \mathcal{B}^T - \mathcal{C}\|_F^2 + \frac{\lambda_1}{2}\left(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2\right) + \lambda_2\|\mathcal{C}\|_1 \tag{4.1.2}$$

then $(\mathcal{L} = \mathcal{A} * \mathcal{B}^T, \mathcal{C})$ minimises

$$\frac{1}{2}\|\mathcal{M} - \mathcal{L} - \mathcal{C}\|_F^2 + \lambda_1\|\mathcal{L}\|_* + \lambda_2\|\mathcal{C}\|_1. \tag{4.1.3}$$

After taking the derivative with respect to $\mathcal{C}$, these equations are the same, so we only have the examine with respect to $\mathcal{A}$ and $\mathcal{B}$. For the rest of this lemma, we will write $\mathcal{M} - \mathcal{E} = \tilde{\mathcal{M}}$ to tidy up our equations. Now, differentiating Eq. (4.1.3) and setting it equal to 0 implies that $\mathcal{L} = \mathcal{A} * \mathcal{B}^T$ is a minimiser if

$$\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T \in \partial\lambda_1\|\mathcal{A} * \mathcal{B}^T\|_* \tag{4.1.4}$$

so our goal will be to show that the $\mathcal{A}$ and $\mathcal{B}$ that minimise Eq. (4.1.2) satisfy Eq. (4.1.4).

Using Lemma 2.48 and Lemma 2.49 to differentiate Eq. (4.1.2) with respect to $\mathcal{A}$ and $\mathcal{B}$ and then setting it equal to 0 (which is sufficient for minimisation as this is a convex problem) gives

$$(\mathcal{A} * \mathcal{B}^T - \tilde{\mathcal{M}}) * \mathcal{B} + \lambda_1\mathcal{A} = 0 \tag{4.1.5}$$

$$(\mathcal{B} * \mathcal{A}^T - \tilde{\mathcal{M}}^T) * \mathcal{A} + \lambda_1\mathcal{B} = 0 \tag{4.1.6}$$

Multiplying Eq. (4.1.5) by $\mathcal{A}^T$ and Eq. (4.1.6) by $\mathcal{B}^T$ gives

$$\mathcal{A}^T * \mathcal{A} * \mathcal{B}^T * \mathcal{B} - \mathcal{A}^T * \tilde{\mathcal{M}} * \mathcal{B} + \lambda_1 \mathcal{A}^T * \mathcal{A} = 0$$
$$\mathcal{B}^T * \mathcal{B} * \mathcal{A}^T * \mathcal{A} - \mathcal{B}^T * \tilde{\mathcal{M}}^T * \mathcal{A} + \lambda_1 \mathcal{B}^T * \mathcal{B} = 0$$

and then transposing the second line gives

$$\mathcal{A}^T * \mathcal{A} * \mathcal{B}^T * \mathcal{B} - \mathcal{A}^T * \tilde{\mathcal{M}} * \mathcal{B} + \lambda_1 \mathcal{A}^T * \mathcal{A} = 0$$
$$\mathcal{A}^T * \mathcal{A} * \mathcal{B}^T * \mathcal{B} - \mathcal{A}^T * \tilde{\mathcal{M}} * \mathcal{B} + \lambda_1 \mathcal{B}^T * \mathcal{B} = 0.$$

So at the minimum we have that

$$\mathcal{A}^T * \mathcal{A} = \mathcal{B}^T * \mathcal{B}.$$

Let $\mathcal{A}$ have t-SVD $\mathcal{X} * \mathcal{S} * \mathcal{Z}^T$, where we note that $\mathcal{X}$ is in $n_1 \times n_1 \times n_3 \times ... \times n_p$, $\mathcal{S}$ is in $n_1 \times r \times n_3 \times ... \times n_p$ and $\mathcal{Z}$ is in $r \times r \times n_3 \times ... \times n_p$. Then we have that

$$\mathcal{Z} * \mathcal{S}^T * \mathcal{X}^T * \mathcal{X} * \mathcal{S} * \mathcal{Z}^T = \mathcal{Z} * \mathcal{S}^T * \mathcal{S} * \mathcal{Z}^T = \mathcal{A}^T * \mathcal{A} = \mathcal{B}^T * \mathcal{B}$$

and in particular, $\mathcal{S}^T * \mathcal{S}$ is in $r \times r \times n_3 \times ... \times n_p$ and is invertible since $\mathcal{A}$ is full rank. We note that $\|\mathcal{Q} * \mathcal{A}\|_F = \|\mathcal{A}\|_F$ for any orthogonal $\mathcal{Q}$ (Eq. (2.4.4)) and that the only other place $\mathcal{A}$ and $\mathcal{B}$ appear in Eq. (4.1.2) is as $\mathcal{A} * \mathcal{B}^T$. As such, for any solution $(\mathcal{A}, \mathcal{B})$ and orthogonal tensor $\mathcal{Q}$, we also have an equivalent factorisation $(\mathcal{A} * \mathcal{Q}, \mathcal{B} * \mathcal{Q})$ of the solution. So we substitute $\mathcal{A}$ with $\mathcal{A} * \mathcal{Z}$ and $\mathcal{B}$ with $\mathcal{B} * \mathcal{Z}$ and get

$$\mathcal{S}^T * \mathcal{S} = \mathcal{A}^T * \mathcal{A} = \mathcal{B}^T * \mathcal{B}.$$

Noting this substitution, we continue with the condition from Eq. (4.1.5) to get

$$
\begin{aligned}
(\mathcal{A} * \mathcal{B}^T - \tilde{\mathcal{M}}) * \mathcal{B} + \lambda_1 \mathcal{A} &= 0 \\
\mathcal{A} * \mathcal{B}^T * \mathcal{B} - \tilde{\mathcal{M}} * \mathcal{B} + \lambda_1 \mathcal{A} &= 0 \\
\mathcal{A} * (\mathcal{B}^T * \mathcal{B} + \lambda_1 \mathcal{I}) &= \tilde{\mathcal{M}} * \mathcal{B} \\
\mathcal{A} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) &= \tilde{\mathcal{M}} * \mathcal{B} \\
\mathcal{A} &= \tilde{\mathcal{M}} * \mathcal{B} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})^{-1}
\end{aligned}
$$

where $(\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})$ is invertible because $\texttt{bcirc}(\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})$ is invertible (Proposition 2.15) since the eigenvalues of $\texttt{bcirc}(\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})$ are all greater than 0. Similarly for $\mathcal{B}$, taking the condition from Eq. (4.1.6) and substituting from above, we get

$$
\begin{aligned}
(\mathcal{B} * \mathcal{A}^T - \tilde{\mathcal{M}}^T) * \mathcal{A} + \lambda_1 \mathcal{B} &= 0 \\
\mathcal{B} * \mathcal{A}^T * \mathcal{A} - \tilde{\mathcal{M}}^T * \mathcal{A} + \lambda_1 \mathcal{B} &= 0 \\
\mathcal{B} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) &= \tilde{\mathcal{M}}^T * \mathcal{A} \\
\mathcal{B} &= \tilde{\mathcal{M}}^T * \mathcal{A} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})^{-1} \\
\mathcal{B} &= \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 I)^{-1} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})^{-1} \\
\mathcal{B} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) &= \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{A} &= \tilde{\mathcal{M}} * \mathcal{B} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})^{-1} \\
\mathcal{A} &= \tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T * \mathcal{A} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 I)^{-1} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I})^{-1} \\
\mathcal{A} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) &= \tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T * \mathcal{A}.
\end{aligned}
$$

As noted above, $\mathcal{S}^T * \mathcal{S}$ is full rank so we can take the square root of its inverse (Remark 2.24). Then we note that $\mathcal{B} * (\mathcal{S}^T * \mathcal{S})^{-\frac{1}{2}}$ is 'nearly orthogonal', as $\mathcal{B}^T * \mathcal{B} = \mathcal{I}$ in

$r \times r \times n_3 \times ... \times n_p$. Then substituting $\mathcal{B}' = \mathcal{B} * (\mathcal{S}^T * \mathcal{S})^{-\frac{1}{2}}$ we get that

$$\begin{aligned}
&\tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B} * (\mathcal{S}^T * \mathcal{S})^{-\frac{1}{2}} * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} \\
&= \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B}' * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} \\
&= \mathcal{B} * (\mathcal{S}^T * \mathcal{S})^{-\frac{1}{2}} * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) \\
&= \mathcal{B}' * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}),
\end{aligned}$$

and now applying the fact that diagonal tensors commute (Remark 2.25), cancelling $(\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}}$ on each side and then multiplying each side by $\mathcal{B}'^T$ we get

$$\begin{aligned}
\tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B}' * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} &= \mathcal{B}' * (\mathcal{S}^T * \mathcal{S})^{\frac{1}{2}} * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) \\
\tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B}' &= \mathcal{B}' * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) \\
\mathcal{B}'^T * \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{B}' &= (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}) * (\mathcal{S}^T * \mathcal{S} + \lambda_1 \mathcal{I}).
\end{aligned}$$

Then by replacing $\mathcal{A}$ by $\mathcal{A}' = \mathcal{A} * (\mathcal{S}^T * \mathcal{S})^{-\frac{1}{2}}$ and executing a virtually identical calculation we get

$$\begin{aligned}
\tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T * \mathcal{A}' &= \mathcal{A}' * (\mathcal{S} * \mathcal{S}^T + \lambda_1 I) * (\mathcal{S} * \mathcal{S}^T + \lambda_1 \mathcal{I}) \\
\mathcal{A}'^T * \tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T * A' &= (\mathcal{S} * \mathcal{S}^T + \lambda_1 I) * (\mathcal{S} * \mathcal{S}^T + \lambda_1 \mathcal{I}).
\end{aligned}$$

Now, we denote the t-SVD of $\tilde{\mathcal{M}}$ by $\mathcal{U} * \Sigma * \mathcal{V}^T$ so that

$$\tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T = \mathcal{U} * \Sigma * \Sigma^T * \mathcal{U}^T \qquad \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} = \mathcal{V} * \Sigma^T * \Sigma * \mathcal{V}^T$$

and note that these factorisations are unique (up to permutation) by Theorem 2.23. So it follows that the factorisations

$$\mathcal{U}^T * \tilde{\mathcal{M}} * \tilde{\mathcal{M}}^T * \mathcal{U} = \Sigma * \Sigma^T \qquad \mathcal{V}^T * \tilde{\mathcal{M}}^T * \tilde{\mathcal{M}} * \mathcal{V} = \Sigma^T * \Sigma$$

are unique (up to permutation) as well. Now we also note that we can write the t-SVD of $\tilde{\mathcal{M}}$ uniquely (up to permutation) as

$$\tilde{\mathcal{M}} = \mathcal{U}_1 * \Sigma_1 * \mathcal{V}_1^T + \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T \tag{4.1.7}$$

where $\mathcal{U}_1$ is in $n_1 \times r \times n_3 \times ... \times n_p$, $\mathcal{U}_2$ is in $n_1 \times n_1 - r \times n_3 \times ... \times n_p$, $\mathcal{V}_1$ is in $n_2 \times r \times n_3 \times ... \times n_p$, $\mathcal{V}_2$ is in $n_2 \times n_2 - r \times n_3 \times ... \times n_p$, Σ_1 is in $r \times r \times n_3 \times ... \times n_p$ and Σ_2 is in $n_1 - r \times n_2 - r \times n_3 \times ... \times n_p$ with

$$\mathcal{U}_1^T * \mathcal{U}_2 = 0 \qquad\qquad \mathcal{V}_1^T * \mathcal{V}_2 = 0. \tag{4.1.8}$$

So, for a suitable permutation, by uniqueness we must have

$$\Sigma_1 = \mathcal{S} * \mathcal{S}^T + \lambda_1 \mathcal{I} \tag{4.1.9}$$

and

$$\begin{aligned} \mathcal{A}' &= \mathcal{U}_1 & \mathcal{B}' &= \mathcal{V}_1 \\ \mathcal{A} * (\mathcal{S} * \mathcal{S}^T)^{-\frac{1}{2}} &= \mathcal{U}_1 & \mathcal{B} * (\mathcal{S} * \mathcal{S}^T)^{-\frac{1}{2}} &= \mathcal{V}_1 \\ \mathcal{A} &= \mathcal{U}_1 * (\mathcal{S} * \mathcal{S}^T)^{\frac{1}{2}} & \mathcal{B} &= \mathcal{V}_1 * (\mathcal{S} * \mathcal{S}^T)^{\frac{1}{2}} \end{aligned}$$

Hence, we get

$$\mathcal{A} * \mathcal{B}^T = \mathcal{U}_1 * (\mathcal{S} * \mathcal{S}^T)^{\frac{1}{2}} * (\mathcal{S} * \mathcal{S}^T)^{\frac{1}{2}} * \mathcal{V}_1^T = \mathcal{U}_1 * \mathcal{S} * \mathcal{S}^T * \mathcal{V}_1^T. \tag{4.1.10}$$

Our goal (from Eq. (4.1.4)) was to show that

$$\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T \in \partial \lambda_1 \|\mathcal{A} * \mathcal{B}^T\|_*.$$

$\mathcal{A} * \mathcal{B}^T$ has t-SVD $\mathcal{U}_1 * \mathcal{S} * \mathcal{S}^T * \mathcal{V}_1^T$. So using Lemma 2.46, any element of $\partial \lambda_1 \|\mathcal{A} * \mathcal{B}^T\|_*$ is given by $\lambda_1(\mathcal{U}_1 * \mathcal{V}_1^T + \mathcal{W})$ where $\mathcal{U}_1^* * \mathcal{W} = 0$, $\mathcal{W} * \mathcal{V}_1 = 0$ and $\|\mathcal{W}\| \leq 1$. Using the results from above that $\mathcal{A} * \mathcal{B}^T = \mathcal{U}_1 * \mathcal{S} * \mathcal{S}^T * \mathcal{V}_1^T$ (Eq. (4.1.10)), $\tilde{\mathcal{M}} = \mathcal{U}_1 * \Sigma_1 * \mathcal{V}_1^T + \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T$

(Eq. (4.1.7)) and $\Sigma_1 = \mathcal{S} * \mathcal{S}^T + \lambda_1 \mathcal{I}$ (Eq. (4.1.9)), we have that

$$\begin{aligned}
\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T &= \mathcal{U}_1 * \Sigma_1 * \mathcal{V}_1^T + \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T - \mathcal{U}_1 * \mathcal{S} * \mathcal{S}^T * \mathcal{V}_1^T \\
&= \mathcal{U}_1 * (\mathcal{S} * \mathcal{S}^T + \lambda_1 I) * \mathcal{V}_1^T + \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T - \mathcal{U}_1 * \mathcal{S} * \mathcal{S}^T * \mathcal{V}_1^T \\
&= \lambda_1 \left(\mathcal{U}_1 * \mathcal{V}_1^T + \frac{1}{\lambda_1} \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T \right).
\end{aligned}$$

Then we have that $\mathcal{U}_1^* * \mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T = 0$ and $\mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T * \mathcal{V}_1 = 0$ by Eq. (4.1.8). And $\|\frac{1}{\lambda_1}\mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T\| \leq 1$ by assumption. So, by Lemma 2.46. $\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T \in \partial \lambda_1 \|\mathcal{A} * \mathcal{B}^T\|_*$ and we are done. $\square$

The assumption that $\|\frac{1}{\lambda_1}\mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T\| \leq 1$ was not made in Feng et al. (2013). They stated (for the matrix case) that it is easy to verify. However, to this student, it was unverifiable. One way to avoid this assumption is to make λ_2 smaller. Then, since we assumed as a first-order condition to the optimisation that $\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T \in \partial \lambda_1 \|\mathcal{C}\|_1$ and that the subgradient of the l_1 norm has entries between ± 1, we can bound $\tilde{\mathcal{M}} - \mathcal{A} * \mathcal{B}^T$. But a sufficiently small λ_2 was incompatible with the analysis given in Chapter 3. Given the other oversights found in the paper (discussed later), it is possible that this is actually not easy to verify.

Lemma 4.3 above tells us that minimising Eq. (4.1.1) delivers the same result as ministering Eq. (4.0.1). So now we will construct an algorithm to minimise Eq. (4.1.1).

4.2 Algorithm

Our aim is to construct an online algorithm that, upon observing $\mathcal{M}_0 \in \mathbb{R}^{n_1 \times T \times n_3 \times \ldots \times n_p}$, solves the following optimisation problem:

$$\min_{\substack{\mathcal{A} \in \mathbb{R}^{n_1 \times r \times n_3 \times \ldots \times n_p} \\ \mathcal{B} \in \mathbb{R}^{T \times r \times n_3 \times \ldots \times n_p} \\ \mathcal{C} \in \mathbb{R}^{n_1 \times T \times n_3 \times \ldots \times n_p}}} \frac{1}{2} \|\mathcal{M}_0 - \mathcal{A} * \mathcal{B}^T - \mathcal{C}\|_F^2 + \frac{\lambda_1}{2} \left(\|\mathcal{A}\|_F^2 + \|\mathcal{B}\|_F^2 \right) + \lambda_2 \|\mathcal{C}\|_1. \qquad (4.2.1)$$

To do this, first we note that we can write the optimisation problem above as

$$\min_{\substack{\mathcal{A} \in \mathbb{R}^{n_1 \times r \times n_3 \times \ldots \times n_p} \\ \mathbf{b}_i \in \mathbb{R}^{1 \times r \times n_3 \times \ldots \times n_p} \\ \mathbf{c}_i \in \mathbb{R}^{n_1 \times 1 \times n_3 \times \ldots \times n_p}}} \sum_{i=1}^{T} \frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\left(\|\mathcal{A}\|_F^2 + \|\mathbf{b}_i\|_F^2\right) + \lambda_2\|\mathbf{c}_i\|_1 \tag{4.2.2}$$

so that we only have to deal with once slice at a time. An interesting point here is that the only dimension in which we can do this is the second: if we were to use any other dimension, the t-product $\mathcal{A} * \mathbf{b}_i^T$ would have a dimension mismatch.

Next, we define a loss function for each observation $\mathbf{m}_i \in \mathbb{R}^{n_1 \times 1 \times n_3 \times \ldots \times n_p}$ and $\mathcal{A} \in \mathbb{R}^{n_1 \times r \times n_3 \ldots \times n_p}$ as

$$l(\mathbf{m}_i, \mathcal{A}) = \min_{\substack{\mathbf{b} \in \mathbb{R}^{1 \times r \times n_3 \times \ldots \times n_p} \\ \mathbf{c} \in \mathbb{R}^{n_1 \times 1 \times n_3 \times \ldots \times n_p}}} \frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1 \tag{4.2.3}$$

then let the total loss across the first $t \leq T$ observations be given by

$$f_t(\mathcal{A}) = \frac{1}{t}\sum_{i=1}^{t} l(\mathbf{m}_i, \mathcal{A}) + \frac{\lambda_1}{2t}\|\mathcal{A}\|_F^2 \tag{4.2.4}$$

and note that the total loss across all the observations (so setting $t = T$) is equal to $\frac{1}{T}$ times Eq. (4.1.1), which gives the following remark.

Remark 4.4. Minimising $f_t(\mathcal{A})$ is equivalent to minimising Eq. (4.1.1).

We also quickly note that

$$f(\mathcal{A}) = \mathbb{E}_{\mathbf{m}}\Big(l(\mathbf{m}, \mathcal{A})\Big) = \lim_{t \to \infty} f_t(\mathcal{A}) \tag{4.2.5}$$

where the last equality holds by the strong law of large numbers and the fact that $\mathcal{A}$ is bounded (Assumption 4.1). Next we let

$$g_t(\mathcal{A}) = \frac{1}{t}\sum_{i=1}^{t}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) + \frac{\lambda_1}{2t}\|\mathcal{A}\|_F^2 \tag{4.2.6}$$

so that g_t is a 'non-optimised' version of f_t that can be minimised as a way of minimising

f_t (here, $\mathbf{b}_i$ and $\mathbf{c}_i$ are given by the algorithm below). In particular, we have that for all $\mathcal{A}$

$$g_t(\mathcal{A}) \geq f_t(\mathcal{A}). \tag{4.2.7}$$

Now we are ready to define our algorithm. The idea is to minimise g_t (which we will show later is asymptotically equivalent to minimising f_t) by alternatively minimising it with respect to $\mathcal{A}_t$, and $\mathbf{b}_t$ and $\mathbf{c}_t$, as each new sample $\mathbf{m}_i$ is revealed. The tensors $\mathcal{Y}_t = \sum_{i=1}^{t} \mathbf{b}_i^T * \mathbf{b}_i$ and $\mathcal{X}_t = \sum_{i=1}^{t} (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i$ used in the algorithm below are used to store values for computational reasons that will soon become clear.

Algorithm 1 Online Robust Tensor Principal Component Analysis

Input $\mathcal{M}$ (observed data), r (rank of $\mathcal{M}$, usually 1)
$\lambda_1 = \frac{1}{100\sqrt{\max\{n_1,n_2\}\tilde{n}_3}}$, $\lambda_2 = \frac{1}{100}$, $\mathcal{Y}_0 = 0$, $\mathcal{X}_0 = 0$
$U, S, V = \text{t-SVD}(\mathcal{M})$, $\mathcal{A}_0 = U(:, 0 : r, :, ... :)$
for t = 0 to T **do**
Reveal sample $\mathbf{m}_t = \mathcal{M}(:, t, :, ..., :)$
Minimise with respect to $\mathbf{b}_t$ and $\mathbf{c}_t$ using Algorithm 2:

$$\{\mathbf{b}_t, \mathbf{c}_t\} = \arg\min_{\mathbf{b},\mathbf{c}} \frac{1}{2}\|\mathbf{m}_t - \mathcal{A}_{t-1} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1$$

Update $\mathcal{Y}$ and $\mathcal{X}$:

$$\mathcal{Y}_t = \mathcal{Y}_{t-1} + \mathbf{b}_t^T * \mathbf{b}_t \quad \mathcal{X}_t = \mathcal{X}_{t-1} + (\mathbf{m}_t - \mathbf{c}_t) * \mathbf{b}_t^T$$

Update $\mathcal{A}$ using Algorithm 3:

$$\mathcal{A}_t = \arg\min_{\mathcal{A}} \frac{1}{t}\sum_{i=1}^{t}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) + \frac{\lambda_1}{2t}\|\mathcal{A}\|_F^2$$

end for
return $\mathcal{A}$, $\mathbf{b}$ and $\mathbf{c}$

To find the $\mathbf{b}$ and $\mathbf{c}$ that minimise $\frac{1}{2}\|\mathbf{m}_t - \mathcal{A}_{t-1} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1$, we will use two lemmas:

Lemma 4.5. [Hale, Yin, & Zhang (2008)] *The unique solution to*

$$\min_{x \in \mathbb{R}^n} v\|x\|_1 + \frac{1}{2}\|x - y\|_2^2$$

is given by $s_v(y)$ *where*

$$S_v(y) = \begin{cases} y - v & y > v \\ y + v & y < -v \\ 0 & \textit{otherwise} \end{cases}$$

So the optimal $\mathbf{c}$ is given by $S_{\lambda_2}(\mathbf{m}_t + \mathcal{A}_{t-1} * \mathbf{b}^T)$.

Next we have from Lemma 4.7 (proven below) that the optimal $\mathbf{b}^T$ is given by

$$\mathbf{b}^T = (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}).$$

So we can use the following algorithm:

Algorithm 2 Minimise with respect to $\mathbf{b}_t$ and $\mathbf{c}_t$

```
Input A_{t-1}, m_t and b_{t-1} (starting value)
Ã = (A_{t-1}^T * A_{t-1} + λ_1 I)^{-1} * A_{t-1}^T
while not converged do
    c = S_{λ_2}(m_t + A_{t-1} * b^T)
    b^T = Ã * (m_t − c)
end while
return c, b^T
```

To find the $\mathcal{A}$ that minimises

$$\frac{1}{t}\sum_{i=1}^{t}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) + \frac{\lambda_1}{2t}\|\mathcal{A}\|_F^2,$$

we apply Lemma 2.49 to get that

$$
\begin{aligned}
0 = &\frac{d}{d\mathcal{A}} \sum_{i=1}^{t} \left(\frac{1}{2} \|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2} \|\mathbf{b}_i\|_F^2 + \lambda_2 \|\mathbf{c}_i\|_1 \right) + \frac{\lambda_1}{2} \|\mathcal{A}\|_F^2 \\
= &\sum_{i=1}^{t} -(\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i) * \mathbf{b}_i + \lambda_1 \mathcal{A} \\
= &\mathcal{A} * \sum_{i=1}^{t} \mathbf{b}_i^T * \mathbf{b}_i - \sum_{i=1}^{t} (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i + \lambda_1 \mathcal{A} \\
= &\mathcal{A} * \sum_{i=1}^{t} \mathbf{b}_i^T * \mathbf{b}_i - \sum_{i=1}^{t} (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i + \lambda_1 \mathcal{A} \\
\implies &\mathcal{A} * \left(\sum_{i=1}^{t} \mathbf{b}_i^T * \mathbf{b}_i + \lambda_1 \mathcal{I} \right) = \sum_{i=1}^{t} (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i
\end{aligned}
$$

So we want to solve $\mathcal{A} * (\mathcal{Y}_t + \lambda_1 \mathcal{I}) = \mathcal{X}_t$ where we have substituted $\mathcal{Y}_t = \sum_{i=1}^{t} \mathbf{b}_i^T * \mathbf{b}_i$ and $\mathcal{X}_t = \sum_{i=1}^{t} (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i$ (as per Algorithm 1). Directly solving this (in the Fourier domain and then transforming back) made Algorithm 2 unstable. So we implemented block-coordinate descent with warm restarts (Bertsekas, 2016; Feng et al., 2013) to minimise

$$
\texttt{tdiag}(\mathcal{A})\,\texttt{tdiag}((\mathcal{Y}_t + \lambda_1 \mathcal{I}))\,\texttt{tdiag}(\mathcal{A})^T - \texttt{tdiag}(\mathcal{X}_t)\,\texttt{tdiag}(\mathcal{A})^T
$$

with respect to $\mathcal{A}$, which has a minimum at the value for which we are trying to solve (in the Fourier domain, which we can then transform back).

Algorithm 3 Minimise with respect to $\mathcal{A}$

Input $\mathcal{A}_t$ $\mathcal{X}_t$, $\mathcal{Y}_t$ λ_1
$\tilde{\mathcal{X}} = \texttt{fft}(\mathcal{X}_t)$, $\tilde{\mathcal{Y}} = \texttt{fft}(\mathcal{Y}_t + \lambda_1 \mathcal{I})$
for each frontal slice I and column j **do**
  $\mathcal{A}_{t+1}(:, j, I) = \frac{\mathcal{X}(:,j,I) - \mathcal{A}(:,:,I)\mathcal{Y}(:,j,I)}{\mathcal{Y}(j,j,I)} + \mathcal{A}_t(:, j, I)$
end for
return $\mathcal{A}_I$

To save on computations, all t-products are done in the Fourier domain. We will now show that, under certain conditions, Algorithm 1 converges.

4.3 Analysis of Convergence

Theorem 4.6. *As $t \to \infty$, Algorithm 1 converges to the correct solution.*

To show this we will prove the following lemmas, which will show that $g_t(\mathcal{A}_t)$ and $f_t(\mathcal{A}_t)$ converge to the same limit and that $f_t(\mathcal{A}_t)$ is at its minimum at that limit. By Remark 4.4, this is sufficient to show that our our initial objective function (Eq. (4.0.1)) is minimised.

Lemma 4.7. *The tensors*

$$\begin{aligned}\mathbf{c}_{i*} &= \mathbf{m}_i - \left(\mathcal{I} + \frac{1}{\lambda_1}\mathcal{A} * \mathcal{A}^T\right) * \lambda_2 \partial \|\mathbf{c}_{i*}\|_1 \\ \mathbf{b}_{i*}^T &= (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}_{i*}).\end{aligned}$$

are a solution to Eq. (4.2.3)

$$l(\mathbf{m}_i, \mathcal{A}) = \min_{\substack{\mathbf{b} \in \mathbb{R}^{1 \times r \times n_3 \times \ldots \times n_p} \\ \mathbf{c} \in \mathbb{R}^{n_1 \times 1 \times n_3 \times \ldots \times n_p}}} \frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2 \|\mathbf{c}\|_1.$$

We also have that $\mathbf{b}_{j*}$ and $\mathbf{c}_{j*}$ are uniformly bounded.

Proof. Since Eq. (4.2.3) is convex with respect to $\mathbf{b}$, we can use Lemma 2.48 to differentiate Eq. (4.2.3) with respect to $\mathbf{b}$ and set this equal to 0, yielding

$$\begin{aligned}0 &= \frac{\partial}{\partial \mathbf{b}^T}\frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1 \\ &= -\mathcal{A}^T * (\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}) + \lambda_1 \mathbf{b}^T\end{aligned}$$

so we have that

$$\begin{aligned}\mathcal{A}^T * \mathcal{A} * \mathbf{b}^T + \lambda_1 \mathbf{b}^T &= \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}) \\ (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I}) * \mathbf{b}^T &= \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}) \\ \mathbf{b}^T &= (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c})\end{aligned}$$

where $(\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})$ is invertible because we can write

$$\texttt{bcirc}((\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})) = \texttt{bcirc}(\mathcal{A})^T \, \texttt{bcirc}(\mathcal{A}) + \lambda_1 I.$$

and then since the eigenvalues of the block circulant matrix above are strictly positive (Corollary 2.22), we can apply the fact that a tensor is invertible if and only if the `bcirc` of that tensor is invertible (Proposition 2.15). Now, similarly for $\mathbf{c}$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \mathbf{c}} \frac{1}{2} \|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2} \|\mathbf{b}\|_F^2 + \lambda_2 \|\mathbf{c}\|_1 \\ &= -(\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}) + \lambda_2 \partial \|\mathbf{c}\|_1. \end{aligned}$$

Then substituting $\mathbf{b}_{i*}^T$ into the above, we get that

$$0 = -\mathbf{m}_i + \mathcal{A} * (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}) + \mathbf{c} + \lambda_2 \partial \|\mathbf{c}\|_1$$

so then

$$\begin{aligned} \lambda_2 \partial \|\mathbf{c}\|_1 &= (\mathbf{m}_i - \mathbf{c}) - \mathcal{A} * (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}) \\ &= \left(\mathcal{I} - \mathcal{A} * (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T\right) * (\mathbf{m}_i - \mathbf{c}). \end{aligned}$$

A somewhat involved direct calculation (very similar to the calculation used to derive the Sherman-Morrison-Woodbury matrix identity, see Golub & Van Loan (2012)) shows that $\mathcal{I} - \mathcal{A} * (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T$ is invertible with inverse $\mathcal{I} + \frac{1}{\lambda_1} \mathcal{A} * \mathcal{A}^T$. So then we have

$$\mathbf{c}_{i*} = \mathbf{m}_i - \left(\mathcal{I} + \frac{1}{\lambda_1} \mathcal{A} * \mathcal{A}^T\right) * \lambda_2 \partial \|\mathbf{c}_{i*}\|_1.$$

To get that $\mathbf{b}_{i*}$ and $\mathbf{c}_{i*}$ are bounded for every i, we note that first for $\mathbf{c}_{i*}$,

$$\mathbf{c}_{i*} = \mathbf{m}_i - \left(\mathcal{I} + \frac{1}{\lambda_1} \mathcal{A} * \mathcal{A}^T\right) * \lambda_2 \partial \|\mathbf{c}_{i*}\|_1$$

and all of these terms are bounded, so $\mathbf{c}_{i*}$ must be as well. Similarly,

$$\mathbf{b}_{i*}^T = (\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1} * \mathcal{A}^T * (\mathbf{m}_i - \mathbf{c}_{i*})$$

is also the product of bounded tensors. To see that $(\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1}$ is bounded, we quickly note that since $\texttt{bcirc}(\mathcal{X}^{-1}) = \texttt{bcirc}(\mathcal{X})^{-1}$ (Proposition 2.15) and $\texttt{bcirc}(\mathcal{X} * \mathcal{Y}) = \texttt{bcirc}(\mathcal{X})\,\texttt{bcirc}(\mathcal{Y})$ (Corollary 2.22), we have that

$$\begin{aligned}\|(\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I})^{-1}\|_F &= \frac{1}{\tilde{n}_3}\|\,\texttt{bcirc}((\mathcal{A}^T * \mathcal{A} + \lambda_1 \mathcal{I}))^{-1}\|_F \\ &= \frac{1}{\tilde{n}_3}\|(\texttt{bcirc}(\mathcal{A})^T\,\texttt{bcirc}(\mathcal{A}) + \lambda_1 I)^{-1}\|_F.\end{aligned}$$

It follows that $\texttt{bcirc}(\mathcal{A})^T\,\texttt{bcirc}(\mathcal{A}) + \lambda_1 I$ is a symmetric matrix, so examining its eigenvalue decomposition reveals that it is bounded (since $\lambda_1 > 0$). $\square$

Corollary 4.8. $\frac{1}{t}\|\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}\|_F$ *and* $\frac{1}{t}\|\mathcal{X}_t - t(\mathbf{m}_{t+1} - \mathbf{c}_{t+1}) * \mathbf{b}_{t+1}\|_F$ *are uniformly bounded.*

Corollary 4.9. *The functions l, g and f are all Lipschitz continuous in $\mathcal{A}$.*

Proof. We start by showing that $l(\mathbf{m}_i, \cdot)$ is Lipschitz continuous with a constant that is independent of $\mathbf{m}_i$. Fist we recall that (Eq. (4.2.3))

$$l(\mathbf{m}_i, \mathcal{A}) = \min_{\substack{\mathbf{b} \in \mathbb{R}^{1 \times r \times n_3 \times \ldots \times n_p} \\ \mathbf{c} \in \mathbb{R}^{n_1 \times 1 \times n_3 \times \ldots \times n_p}}} \frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1.$$

So then

$$\begin{aligned}l(\mathbf{m}_i, \mathcal{A}_1) - l(\mathbf{m}_i, \mathcal{A}_2) &= \frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_1 * \mathbf{b}_{1*}^T - \mathbf{c}_{1*}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_{1*}\|_F^2 + \lambda_2\|\mathbf{c}_{1*}\|_1 \\ &\quad - \left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_2 * \mathbf{b}_{2*}^T - \mathbf{c}_{2*}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_{2*}\|_F^2 + \lambda_2\|\mathbf{c}_{2*}\|_1\right),\end{aligned}$$

where $\mathbf{b}_{j*}$ and $\mathbf{c}_{j*}$ minimise

$$\frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_j * \mathbf{b}_{j*}^T - \mathbf{c}_{j*}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_{j*}\|_F^2 + \lambda_2\|\mathbf{c}_{j*}\|_1.$$

Now, without loss of generality, we assume that $l(\mathbf{m}_i, \mathcal{A}_2) \leq l(\mathbf{m}_i, \mathcal{A}_1)$. So then if we evaluate $l(\mathbf{m}_i, \mathcal{A}_1)$ using $\mathbf{b}_{2*}$ and $\mathbf{c}_{2*}$, this can only increase this term, which is larger, so the gap $l(\mathbf{m}_i, \mathcal{A}_1) - l(\mathbf{m}_i, \mathcal{A}_2)$ can only grow. Then, applying the reverse triangle inequality and the fact that $\|\mathcal{X} * \mathcal{Y}\|_F \leq \sqrt{\tilde{n}_3}\|\mathcal{X}\|_F\|\mathcal{Y}\|_F$ (Eq. (2.4.3)), we get that

$$\begin{aligned}
l(\mathbf{m}_i, \mathcal{A}_1) - l(\mathbf{m}_i, \mathcal{A}_2) &= \frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_1 * \mathbf{b}_{1*}^T - \mathbf{c}_{1*}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_{1*}\|_F^2 + \lambda_2\|\mathbf{c}_{1*}\|_1 \\
&\quad - \left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_2 * \mathbf{b}_{2*}^T - \mathbf{c}_{2*}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_{2*}\|_F^2 + \lambda_2\|\mathbf{c}_{2*}\|_1\right) \\
&\leq \frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_1 * \mathbf{b}_{2*}^T - \mathbf{c}_{1*}\|_F^2 - \frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_2 * \mathbf{b}_{2*}^T - \mathbf{c}_{2*}\|_F^2 \\
&\leq \frac{1}{2}\|(\mathcal{A}_1 - \mathcal{A}_2) * \mathbf{b}_{2*}^T\|_F^2 \\
&\leq \frac{1}{2}\tilde{n}_3\|\mathbf{b}_{2*}\|_F^2\|\mathcal{A}_1 - \mathcal{A}_2\|_F\|\mathcal{A}_1 - \mathcal{A}_2\|_F \\
&\leq \frac{1}{2}\tilde{n}_3\|\mathbf{b}_{2*}\|_F^2(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F)\|\mathcal{A}_1 - \mathcal{A}_2\|_F \\
&\leq K\|\mathcal{A}_1 - \mathcal{A}_2\|_F
\end{aligned}$$

as $\mathbf{b}$ and $\mathcal{A}_i$ are bounded. We note that this inequality, combined with the reverse triangle inequality, implies that f_n is Lipschitz continuous:

$$\begin{aligned}
f_n(\mathcal{A}_1) - f_n(\mathcal{A}_2) &= \frac{1}{n}\sum_{i=1}^n l(\mathbf{m}_i, \mathcal{A}_1) + \frac{\lambda_1}{2n}\|\mathcal{A}_1\|_F^2 - \frac{1}{n}\sum_{i=1}^n l(\mathbf{m}_i, \mathcal{A}_2) - \frac{\lambda_1}{2n}\|\mathcal{A}_2\|_F^2 \\
&= \frac{1}{n}\sum_{i=1}^n (l(\mathbf{m}_i, \mathcal{A}_1) - l(\mathbf{m}_i, \mathcal{A}_2)) + \frac{\lambda_1}{2n}\|\mathcal{A}_1\|_F^2 - \frac{\lambda_1}{2n}\|\mathcal{A}_2\|_F^2 \\
&\leq \frac{1}{n}\sum_{i=1}^n K\|\mathcal{A}_1 - \mathcal{A}_2\|_F + \frac{\lambda_1}{2n}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 \\
&\leq K\|\mathcal{A}_1 - \mathcal{A}_2\|_F + \frac{\lambda_1}{2n}(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F)\|\mathcal{A}_1 - \mathcal{A}_2\|_F \\
&\leq K'\|\mathcal{A}_1 - \mathcal{A}_2\|_F.
\end{aligned}$$

Similar to the above, we also have that g_n is Lipschitz continuous as

$$
\begin{aligned}
g_n(\mathcal{A}_1) - g_n(\mathcal{A}_2) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_1 * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) + \frac{\lambda_1}{2n}\|\mathcal{A}_1\|_F^2 \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A}_2 * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) - \frac{\lambda_1}{2n}\|\mathcal{A}_2\|_F^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\|(\mathcal{A}_1 - \mathcal{A}_2) * \mathbf{b}_i^T\|_F^2\right) + \frac{\lambda_1}{2n}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2\|\mathbf{b}_i^T\|_F^2\tilde{n}_3\right) + \frac{\lambda_1}{2n}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 K''\right) + \frac{\lambda_1}{2n}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 \\
&\leq (K + \lambda_1)(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F)\|\mathcal{A}_1 - \mathcal{A}_2\|_F \\
&\leq K'''\|\mathcal{A}_1 - \mathcal{A}_2\|_F.
\end{aligned}
$$

□

Lemma 4.10. $\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F \leq O(1/t)$.

Proof. To show this we will use a Taylor expansion to upper bound $\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F$. By applying Lemma 2.49 and substituting $\mathcal{Y}_t = \sum_{i=1}^{t}\mathbf{b}_i^T * \mathbf{b}_i$ and $\mathcal{X}_i = \sum_{i=1}^{t}(\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i$, we have

$$
\begin{aligned}
\nabla_{\mathcal{A}} g_t(\mathcal{A}) &= \nabla_{\mathcal{A}}\frac{1}{t}\sum_{i=1}^{t}\left(\frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}_i\|_F^2 + \lambda_2\|\mathbf{c}_i\|_1\right) + \frac{\lambda_1}{2t}\|\mathcal{A}\|_F^2 \\
&= \frac{1}{t}\sum_{i=1}^{t} -\left(\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\right) * \mathbf{b}_i + \frac{\lambda_1}{t}\mathcal{A} \\
&= \frac{1}{t}\mathcal{A} * \sum_{i=1}^{t}\mathbf{b}_i^T * \mathbf{b}_i - \frac{1}{t}\sum_{i=1}^{t}(\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i + \frac{\lambda_1}{t}\mathcal{A} \\
&= \mathcal{A} * \frac{1}{t}\left(\lambda_1\mathcal{I} + \mathcal{Y}_t\right) - \frac{1}{t}\mathcal{X}_t.
\end{aligned} \tag{4.3.1}
$$

Now we need to differentiate this again using Remark 2.51 to get that the Hessian matrix is given by a matrix with the same elements as the following matrix:

$$\begin{aligned}\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} g_t(\mathcal{A}) &= \nabla_{\mathcal{A}}\mathcal{A} * \frac{1}{t}\left(\lambda_1 \mathcal{I} + \mathcal{Y}_t\right) - \frac{1}{t}\sum_{i=1}^{t}(\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i \\ &= \frac{1}{t}\left(\lambda_1 \, \texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t)\right) \otimes I_{n_1}. \end{aligned} \tag{4.3.2}$$

Now we take the Taylor expansion of g_t around $\mathcal{A}_t$ (which is minimised here so the gradient is 0 (Algorithm 1)) to get that

$$\begin{aligned} g_t(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) =& \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T (\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) \\ &+ \frac{1}{2}\texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T (\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} g_t(\mathcal{A}_t))\, \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t) \\ =& \frac{1}{2}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T (\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} g_t(\mathcal{A}_t))\, \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t). \end{aligned}$$

Substituting the near-Hessian matrix into the above and noting from Remark 2.51 that $\lambda_1 \, \texttt{bcirc}(\mathcal{I}) \otimes I_{n_1}$ is positive definite, we get that

$$\begin{aligned} & g_t(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) \\ &= \frac{1}{2}\texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T \,\texttt{tvec}\left(\frac{1}{t}\left(\lambda_1 \, \texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t)\right) \otimes I_{n_1}\right) \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t) \\ &\geq \frac{1}{2}\texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T \,\texttt{tvec}\left(\frac{1}{t}\texttt{bcirc}(\mathcal{Y}_t) \otimes I_{n_1}\right) \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t), \end{aligned}$$

where we have abused notation and used `tvec` of the near-Hessian to be the required Hessian matrix. Taking the Frobenius norm of each side and applying the fact that $\|\,\texttt{bcirc}(\mathcal{X})\|_F = \sqrt{\tilde{n}_3}\|\mathcal{X}\|_F$ (Eq. (2.4.2)) gives

$$\sqrt{\tilde{n}_3 n_1}\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F \|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F^2 \leq \|g_t(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)\|_F. \tag{4.3.3}$$

Here we rely on the assumption that $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ is bounded away from 0 to ensure that the term on the left does not go to 0, which would ruin our result. Because g_t is minimised at $\mathcal{A}_t$ (Algorithm 1), we have that $g_{t+1}(\mathcal{A}_t) - g_{t+1}(\mathcal{A}_{t+1}) \geq 0$. So we have that

$$\begin{aligned}
g_t(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) &\leq g_t(\mathcal{A}_{t+1}) + g_{t+1}(\mathcal{A}_t) - g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) \\
&= (g_t(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_{t+1})) - (g_t(A_t) - g_{t+1}(\mathcal{A}_t)) \\
&\equiv v_t(\mathcal{A}_{t+1}) - v_t(\mathcal{A}_t),
\end{aligned}$$

where have let $v_t(\cdot) = g_t(\cdot) - g_{t+1}(\cdot)$. In particular, we have that

$$\sqrt{\tilde{n}_3 n_1} \left\| \frac{1}{t}\mathcal{Y}_t \right\|_F \|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F^2 \leq \|g_t(\mathcal{A}_{t+1}) - g_t(A_t)\|_F \leq \|v_t(\mathcal{A}_{t+1}) - v_t(\mathcal{A}_t)\|_F. \quad (4.3.4)$$

Now we will bound this expression by taking the Taylor approximation of v_t around $\mathcal{A}_t$. First, using the result from Eq. (4.3.1) above that $\nabla_{\mathcal{A}} g_t(\mathcal{A}) = \mathcal{A} * \frac{1}{t}(\lambda_1 \mathcal{I} + \mathcal{Y}_t) - \frac{1}{t}\mathcal{X}_t$ and then manipulating based on the definitions of $\mathcal{Y}_t$ and $\mathcal{X}_t$ ($\mathcal{Y}_t = \sum_{i=1}^t \mathbf{b}_i^T * \mathbf{b}_i$ and $\mathcal{X}_t = \sum_{i=1}^t (\mathbf{m}_i - \mathbf{c}_i) * \mathbf{b}_i$ (Algorithm 1)), we have that

$$\begin{aligned}
\nabla_{\mathcal{A}_t} v_t(\mathcal{A}_t) &= \nabla_{\mathcal{A}_t} g_t(\mathcal{A}_t) - \nabla_{\mathcal{A}_t} g_{t+1}(\mathcal{A}_t) \\
&= \mathcal{A}_t * \frac{1}{t}(\lambda_1 \mathcal{I} + \mathcal{Y}_t) - \frac{1}{t}\mathcal{X}_t - \mathcal{A}_t * \frac{1}{t+1}(\lambda_1 \mathcal{I} + \mathcal{Y}_{t+1}) + \frac{1}{t+1}\mathcal{X}_{t+1} \\
&= \frac{1}{t(t+1)}\Big(\mathcal{A}_t * ((t+1)\mathcal{Y}_t - t\mathcal{Y}_{t+1}) + \mathcal{A}_t * ((t+1)\lambda_1\mathcal{I} - t\lambda_1\mathcal{I}) - ((t+1)\mathcal{X}_t - t\mathcal{X}_{t+1})\Big) \\
&= \frac{1}{t(t+1)}\Big(\mathcal{A}_t * ((t+1)\mathcal{Y}_t - t\mathcal{Y}_{t+1}) + \lambda_1\mathcal{A}_t - ((t+1)\mathcal{X}_t - t\mathcal{X}_{t+1})\Big) \\
&= \frac{1}{t(t+1)}\Big(\mathcal{A}_t * ((t+1)\mathcal{Y}_t - t\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}) + \lambda_1\mathcal{A}_t \\
&\quad - ((t+1)\mathcal{X}_t - t\mathcal{X}_t - t(\mathbf{m}_{t+1} - \mathbf{c}_{t+1}) * \mathbf{b}_{t+1})\Big) \\
&= \frac{1}{t(t+1)}\Big(\mathcal{A}_t * (\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}) + \lambda_1\mathcal{A}_t - (\mathcal{X}_t - t(\mathbf{m}_{t+1} - \mathbf{c}_{t+1}) * \mathbf{b}_{t+1})\Big).
\end{aligned}$$

We take the Frobenius norm of this expression, apply the triangle inequality, use the fact that $\|\mathcal{X} * \mathcal{Y}\|_F \leq \sqrt{\tilde{n}_3}\|\mathcal{X}\|_F\|\mathcal{Y}\|_F$ (Eq. (2.4.3)) and substitute the bounds from

Corollary 4.8 to get that

$$
\begin{aligned}
\|\nabla_{\mathcal{A}_t} v_t(\mathcal{A}_t)\|_F &= \left\| \frac{1}{t(t+1)} \left(\mathcal{A}_t * (\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}) + \lambda_1 \mathcal{A}_t - (\mathcal{X}_t - t\,(\mathbf{m}_{t+1} - \mathbf{c}_{t+1}) * \mathbf{b}_{t+1}) \right) \right\|_F \\
&\leq \frac{1}{t(t+1)} \Big(\sqrt{\tilde{n}_3} \|\mathcal{A}_t\|_F \|\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}\|_F \\
&\qquad\qquad + \lambda_1 \|\mathcal{A}_t\|_F + \|\mathcal{X}_t - t(\mathbf{m}_{t+1} - \mathbf{c}_{t+1}) * \mathbf{b}_{t+1}\|_F \Big) \\
&\leq \frac{1}{t(t+1)} \Big(tM_1 + M_2 + tM_3 \Big) \\
&\leq O\left(\frac{1}{t}\right).
\end{aligned}
$$

Next we have from Eq. (4.3.2) that $\nabla_{\mathcal{A}} \nabla_{\mathcal{A}} g_t(\mathcal{A}) = \frac{1}{t}\left(\lambda_1 \,\texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t)\right) \otimes I_{n_1}$. Manipulating this inequality in a similar manner to the above gives

$$
\begin{aligned}
&\nabla_{\mathcal{A}_t} \nabla_{\mathcal{A}_t} v_t(\mathcal{A}_t) \\
&= \nabla_{\mathcal{A}_t} \nabla_{\mathcal{A}_t} g_t(\mathcal{A}_t) - \nabla_{\mathcal{A}_t} \nabla_{\mathcal{A}_t} g_{t+1}(\mathcal{A}_t) \\
&= \frac{1}{t}\left(\lambda_1 \,\texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t)\right) \otimes I_{n_1} - \frac{1}{t+1}\left(\lambda_1 \,\texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_{t+1})\right) \otimes I_{n_1} \\
&= \frac{1}{t(t+1)} \big(\lambda_1 \,\texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1})\big) \otimes I_{n_1}.
\end{aligned}
$$

Taking the Frobenius norm of this, applying the triangle inequality, using the fact that $\|\,\texttt{bcirc}(\mathcal{X})\|_F = \sqrt{\tilde{n}_3}\|\mathcal{X}\|_F$ (Eq. (2.4.2)) and using the bounds from Corollary 4.8, we get that

$$
\begin{aligned}
\|\nabla_{\mathcal{A}_t} \nabla_{\mathcal{A}_t} v_t(\mathcal{A}_t)\|_F &= \frac{1}{t(t+1)} \| \big(\lambda_1 \,\texttt{bcirc}(\mathcal{I}) + \texttt{bcirc}(\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1})\big) \otimes I_{n_1}\|_F \\
&\leq \frac{1}{t(t+1)} \sqrt{\tilde{n}_3 n_1} (\|\mathcal{Y}_t - t\mathbf{b}_{t+1}^T * \mathbf{b}_{t+1}\|_F + \|\lambda_1 \mathcal{I}\|_F) \\
&= \frac{1}{t(t+1)} (tM_1 + \lambda_1 n_{(2)}) \\
&\leq O\left(\frac{1}{t}\right).
\end{aligned}
$$

So using the Taylor expansion (Remark 2.51) of v_t around $\mathcal{A}_t$, we have that

$$\begin{aligned}v_t(\mathcal{A}_{t+1}) - v_t(\mathcal{A}_t) =& \texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T(\nabla_{\mathcal{A}} v_t(\mathcal{A}_t)) \\ &+ \frac{1}{2}\texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)^T(\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} v_t(\mathcal{A}_t))\,\texttt{tvec}(\mathcal{A}_{t+1} - \mathcal{A}_t)\end{aligned}$$

so

$$\|v_t(\mathcal{A}_{t+1}) - v_t(\mathcal{A}_t)\|_F \leq \|\mathcal{A}_{t+1} - \mathcal{A}_t)\|_F\|\nabla_{\mathcal{A}} v_t(\mathcal{A}_t))\|_F + \frac{1}{2}\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F^2\|\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} v_t(\mathcal{A}_t)\|_F. \tag{4.3.5}$$

Then combing Eq. (4.3.4), Eq. (4.3.5) and the fact that $\|\nabla_{\mathcal{A}} v_t(\mathcal{A}_t))\|_F$ and $\|\nabla_{\mathcal{A}}\nabla_{\mathcal{A}} v_t(\mathcal{A}_t))\|_F$ are both $O(1/t)$ gives

$$\begin{aligned}\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F^2 &\leq O\left(\frac{1}{t}\right)\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F + O\left(\frac{1}{t}\right)\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F^2 \\ \|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F\left(1 - O\left(\frac{1}{t}\right)\right) &\leq O\left(\frac{1}{t}\right)\end{aligned}$$

so for large t, $0 \leq \left(1 - O\left(\frac{1}{t}\right)\right) \leq 1$ and then we have that

$$\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F \leq O\left(\frac{1}{t}\right).$$

□

Lemma 4.11. *As $t \to \infty$, $g_t(\mathcal{A}_t)$ converges almost surely when $\mathcal{A}_t$ is given by Algorithm 1.*

We will use the following lemma to show that $g_t(\mathcal{A}_t)$ converges.

Lemma 4.12 (Sufficient condition of convergence for a stochastic process Fisk (1965))**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, u_t, $t > 0$ a realisation of a stochastic process and $\mathcal{F}_t$ be the filtration determined by the past information at time t. Let*

$$\delta_t = \begin{cases} 1 & \textit{if } \mathbb{E}[u_{t+1} - u_t|\mathcal{F}_t] > 0, \\ 0 & \textit{otherwise.}\end{cases}$$

If for all t $u_t \geq 0$ and $\sum_{t=1}^{\infty}\mathbb{E}[\delta_t(u_{t+1} - u_t)] < \infty$, then u_t is a quasi-martingale and *converges almost surely and*

$$\sum_{t=1}^{\infty} \mathbb{E}[|u_{t+1} - u_t| \big| \mathcal{F}_t] < \infty \text{ almost surely.}$$

We will use this lemma by showing that the sum over all t of the expected value (given the past information up to t) of the positive variation of $g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)$ is finite, which will then imply that $g_t(\mathcal{A}_t)$ converges.

The next lemma will also be important in our proof.

Lemma 4.13. [Van der Vaart (2007)] *Let $F = f_\theta : X \to \mathbb{R}$ be a set of measurable functions that are Lipschitz continuous in and indexed by $\theta \in \Theta$, a bounded subset in $\mathbb{R}^n$, such that*

$$|f_{\theta_1}(x) - f_{\theta_2}(x)| \leq K \|\theta_1 - \theta_2\|_2.$$

Then for any $f \in F$ with

$$\mathbf{P}_n f = \frac{1}{n} \sum_{i=1}^{n} f(X_i) \qquad\qquad \mathbf{P}_f = \mathbb{E}_X[f(X)],$$

and $\mathbf{P}f^2 < \delta^2$ and $\|f\| < M$ and the X_i being Borel-measurable. Then we have that

$$\mathbb{E}_P \|\sqrt{n}(\mathbf{P}_n f - \mathbf{P}f)\|_F = O(1)$$

where $\|\mathbb{G}_n\|_F = \sup_{f \in F} \|\mathbb{G}_n f\|$.

Proof of Lemma 4.11. First we note that from Algorithm 1 we have that $\mathcal{A}_t = \arg\min_{\mathcal{A}} g_t(\mathcal{A})$, from Eq. (4.2.6) we have

$$g_t(\mathcal{A}) = \frac{1}{t} \sum_{i=1}^{t} \left(\frac{1}{2} \|\mathbf{m}_i - \mathcal{A} * \mathbf{b}_i^T - \mathbf{c}_i\|_F^2 + \frac{\lambda_1}{2} \|\mathbf{b}_i\|_F^2 + \lambda_2 \|\mathbf{c}_i\|_1 \right) + \frac{\lambda_1}{2t} \|\mathcal{A}\|_F^2$$

and from Eq. (4.2.3) we have

$$l(\mathbf{m}_i, \mathcal{A}) = \min_{\substack{\mathbf{b} \in \mathbb{R}^{1\times r\times n_3\times\ldots\times n_p} \\ \mathbf{c} \in \mathbb{R}^{n_1\times 1\times n_3\times\ldots\times n_p}}} \frac{1}{2}\|\mathbf{m}_i - \mathcal{A} * \mathbf{b}^T - \mathbf{c}\|_F^2 + \frac{\lambda_1}{2}\|\mathbf{b}\|_F^2 + \lambda_2\|\mathbf{c}\|_1.$$

So we can write $g_{t+1}(\mathcal{A}_t) = \frac{1}{t+1}(tg_t(\mathcal{A}_t) + l(\mathbf{m}_{t+1}, \mathcal{A}_t))$. Since $\mathcal{A}_t$ minimises g_t, we also have that $g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) \leq 0$, and, as mentioned above (Eq. (4.2.7)), $g_t \geq f_t$. Using these facts, we have

$$\begin{aligned}
g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) &= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + g_{t+1}(\mathcal{A}_t) - g_t(\mathcal{A}_t) \\
&= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{1}{t+1}(tg_t(\mathcal{A}_t) + l(\mathbf{m}_{t+1}, \mathcal{A}_t)) - g_t(\mathcal{A}_t) \\
&= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{tg_t(\mathcal{A}_t) + l(\mathbf{m}_{t+1}, \mathcal{A}_t) - (t+1)g_t(\mathcal{A}_t)}{t+1} \\
&= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - g_t(\mathcal{A}_t)}{t+1} \\
&= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t) + f_t(\mathcal{A}_t) - g_t(\mathcal{A}_t)}{t+1} \\
&= g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} + \frac{f_t(\mathcal{A}_t) - g_t(\mathcal{A}_t)}{t+1} \qquad (4.3.6) \\
&\leq \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1}.
\end{aligned}$$

We write the past information as

$$\mathcal{F}_t = \sigma(\{\mathbf{m}_1, ..., \mathbf{m}_t, \mathcal{A}_1, ..., \mathcal{A}_t, \mathbf{b}_1, ..., \mathbf{b}_t, \mathbf{c}_t, ..., \mathbf{c}_t\})$$

and then, using the inequality from Eq. (4.3.6) above note that

$$\begin{aligned}
\left[\mathbb{E}\left(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)|\mathcal{F}_t\right)\right]^+ &\leq \frac{1}{t+1}\left[\mathbb{E}\left(l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)|\mathcal{F}_t\right)\right]^+ \\
&= \frac{1}{t+1}\left[f(\mathcal{A}_t) - f_t(\mathcal{A}_t)\right]^+ \\
&= \frac{1}{t+1}\left[\mathbf{P}l(\cdot, \mathcal{A}_t) - \mathbf{P}_n l(\cdot, \mathcal{A}_t) - \frac{\lambda_1}{2t}\|\mathcal{A}_t\|_F^2\right]^+ \\
&\leq \frac{1}{\sqrt{t}(t+1)}\sqrt{t}\left|\mathbf{P}l(\cdot, \mathcal{A}_t) - \mathbf{P}_n l(\cdot, \mathcal{A}_t)\right| + \frac{\lambda_1}{2t(t+1)}\|\mathcal{A}_t\|_F^2.
\end{aligned}$$

Since $l(\cdot, \mathcal{A}_t)$ is Lipschitz in $\mathcal{A}_t$ (Corollary 4.9), we can apply Lemma 4.13 and get

$$\mathbb{E}\sqrt{t}\,|\mathbf{P}l(\cdot, \mathcal{A}_t) - \mathbf{P}_n l(\cdot, \mathcal{A}_t)| = M_l < \infty.$$

In particular, we have

$$\begin{aligned}
\mathbb{E}([\mathbb{E}\,(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)|\mathcal{F}_t)]^+) &\leq \mathbb{E}\left(\frac{1}{\sqrt{t}(t+1)}\sqrt{t}\,|\mathbf{P}l(\cdot, \mathcal{A}_t) - \mathbf{P}_n l(\cdot, \mathcal{A}_t)| + \frac{\lambda_1}{2t(t+1)}\|\mathcal{A}_t\|_F^2\right) \\
&\leq \frac{1}{\sqrt{t}(t+1)} M_l + \frac{\lambda_1}{2t(t+1)} M_{\mathcal{A}} \\
&\leq \frac{M'}{t^{\frac{3}{2}}}.
\end{aligned}$$

We quickly note (for use later) that the above two calculations also yield

$$\mathbb{E}\left[\frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1}|\mathcal{F}\right] \leq \frac{M'}{t^{\frac{3}{2}}}. \tag{4.3.7}$$

Now we define δ_t as

$$\delta_t = \begin{cases} 1 & \text{if } \mathbb{E}\,(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)|\mathcal{F}_t) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and then use the fact that geometric sums converge to get

$$\sum_{t=1}^{\infty} \mathbb{E}[\delta_t(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t))] = \sum_{t=1}^{\infty} \mathbb{E}([\mathbb{E}\,(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)|\mathcal{F}_t)]^+) \leq \sum_{t=1}^{\infty} \frac{M'}{t^{\frac{3}{2}}} < \infty.$$

So Lemma 4.12 says that $g_t(\mathcal{A}_t)$ converges almost surely. □

Remark 4.14. We also quickly note that the above proof also yields

$$\sum_{t=1}^{\infty} |\mathbb{E}\,(g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)|\mathcal{F}_t)\,| < \infty \text{ a.s.},$$

so in particular, the sum of the negative variation $\sum_{t=1}^{\infty} \mathbb{E}\,([g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)]^-|\mathcal{F}_t)$ also converges.

Lemma 4.15. *$g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t) \to 0$ and $f_t(\mathcal{A}_t)$ converges almost surely.*

To do this, we will use the following lemma to show that the sequence $g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t) \to 0$.

Lemma 4.16. [Bertsekas (2016)] *If a_n and b_n are two non-negative real sequences such that $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} a_n b_n < \infty$ and there exists a K so that $|b_{n+1} - b_n| < K a_n$ for all n, then $\lim_{n\to\infty} b_n = 0$.*

Proof of Lemma 4.15. We need to find a positive sequence that diverges when summed of diverges and that converges when multiplied term-wise by $g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)$ and then summed. For this we choose $\left\{\frac{1}{t+1}\right\}_{t\in\mathbb{Z}}$. To see that this choice is appropriate, we continue with the working from Eq. (4.3.6)

$$g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) = g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) + \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} + \frac{f_t(\mathcal{A}_t) - g_t(\mathcal{A}_t)}{t+1}$$

and since $g_{t+1}(\mathcal{A}_{t+1}) - g_{t+1}(\mathcal{A}_t) \leq 0$, we have

$$g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t) \leq \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} + \frac{f_t(\mathcal{A}_t) - g_t(\mathcal{A}_t)}{t+1}$$

so

$$\begin{aligned}\frac{g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} &\leq \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} - (g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)) \\ &\leq \frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} + [g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)]^-.\end{aligned}$$

Now we take the same filtration $\mathcal{F}_t$ as above and take the conditional expectation as follows

$$\begin{aligned}\frac{g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} &= \mathbb{E}\left(\frac{g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1}|\mathcal{F}_t\right) \\ &\leq \mathbb{E}\left(\frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1}|\mathcal{F}_t\right) + \mathbb{E}\left([g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)]^-|\mathcal{F}_t\right)\end{aligned}$$

and then summing this over t gives

$$\sum_{t=1}^{\infty} \frac{g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} \leq \sum_{t=1}^{\infty} \mathbb{E}\left(\frac{l(\mathbf{m}_{t+1}, \mathcal{A}_t) - f_t(\mathcal{A}_t)}{t+1} |\mathcal{F}_t\right) + \sum_{t=1}^{\infty} \mathbb{E}\left([g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)]^- |\mathcal{F}_t\right).$$

From Eq. (4.3.7) we have that $\mathbb{E}\left[\frac{l(\mathbf{m}_{t+1},\mathcal{A}_t)-f_t(\mathcal{A}_t)}{t+1}|\mathcal{F}_t\right] \leq \frac{M'}{t^{\frac{3}{2}}}$ almost surely, and from Remark 4.14 we have that $\sum_{t=1}^{\infty} \mathbb{E}\left([g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)]^- |\mathcal{F}_t\right) < \infty$ almost surely. As g_t and f_t are Lipschitz continuous by Corollary 4.9 and $\|\mathcal{A}_{t+1} - \mathcal{A}_t\| = O\left(\frac{1}{t}\right)$ by Lemma 4.10, we have

$$\begin{aligned}
&|g_{t+1}(\mathcal{A}_{t+1}) - f_{t+1}(\mathcal{A}_{t+1}) - (g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t))| \\
\leq &|g_{t+1}(\mathcal{A}_{t+1}) - g_t(\mathcal{A}_t)| + |f_{t+1}(\mathcal{A}_{t+1}) - f_t(\mathcal{A}_t))| \\
\leq &K\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F + K'\|\mathcal{A}_{t+1} - \mathcal{A}_t\|_F \\
\leq &K''\frac{1}{t}.
\end{aligned}$$

Hence, we can apply Lemma 4.16 with the divergent sequence as $\left\{\frac{1}{t+1}\right\}_{t\in\mathbb{Z}}$, and the sequence that we want to show converges as $g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t)$, which yields that $g_t(\mathcal{A}_t) - f_t(\mathcal{A}_t) \to 0$. From Lemma 4.11 we have that $g_t(\mathcal{A}_t)$ converges almost surely, so combining these we have that $g_t(\mathcal{A}_t)$ and $f_t(\mathcal{A}_t)$ converge almost surely to the same limit. □

Lemma 4.17. *For large t, $\mathcal{A}_t$ minimises f_t Eq.* (4.2.5).

Proof. From Lemma 4.11 we have that $g_t(\mathcal{A}_t)$ converges almost surely. We also have from Eq. (4.2.7) that $g_t(\mathcal{X}) \geq f_t(\mathcal{X})$ for all t and any tensor $\mathcal{X}$. Taking the first-order Taylor expansions around $\mathcal{A}_t + \mathcal{X}$ (Remark 2.51), we get that for all $\mathcal{X}$

$$\begin{aligned}
g_t(\mathcal{A}_t + \mathcal{X}) = &g_t(\mathcal{A}_t) + \mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F) \\
f_t(\mathcal{A}_t + \mathcal{X}) = &f_t(\mathcal{A}_t) + \mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} f(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F)
\end{aligned}$$

where h_t is a sequence converging to 0. $g_t(\mathcal{A}_t + \mathcal{X}) \geq f_t(\mathcal{A}_t + \mathcal{X})$ for all t (Eq. (4.2.7)), so for all $\mathcal{X}$ we get

$$\begin{aligned} & g_t(\mathcal{A}_t) + \mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F) \\ \geq\ & f_t(\mathcal{A}_t) + \mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} f_t(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F) \end{aligned}$$

For large t, we have that $g_t(\mathcal{A}_t) \to f_t(\mathcal{A}_t)$ (Lemma 4.15). Provided t is sufficiently large, we can therefore cancel these terms to get that

$$\mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F) \geq \mathtt{tvec}(\mathcal{X})^T(\nabla_{\mathcal{A}} f_t(\mathcal{A}_t)) + o(h_t\|\mathcal{A}_t\|_F)$$

and so

$$\mathtt{tvec}(\mathcal{X})^T\,\mathtt{tvec}(\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) \geq \mathtt{tvec}(\mathcal{X})^T\,\mathtt{tvec}(\nabla_{\mathcal{A}} f_t(\mathcal{A}_t)).$$

Then choosing $\mathcal{X}$ to be $\nabla_{\mathcal{A}_t} f_t(\mathcal{A}_t)$ and using the fact that $\nabla_{\mathcal{A}_t} g_t(\mathcal{A}_t) = 0$ (as $\mathcal{A}_t$ minimises $g_t(\cdot)$), we get

$$0 = \mathtt{tvec}(\nabla_{\mathcal{A}_t} f_t(\mathcal{A}_t))^T\,\mathtt{tvec}(\nabla_{\mathcal{A}} g_t(\mathcal{A}_t)) \geq \mathtt{tvec}(\nabla_{\mathcal{A}_t} f_t(\mathcal{A}_t))^T\,\mathtt{tvec}(\nabla_{\mathcal{A}} f_t(\mathcal{A}_t)) \geq 0,$$

which implies that $\nabla_{\mathcal{A}_t} f_t(\mathcal{A}_t) = 0$, so $\mathcal{A}_t$ minimises f_t. □

We have shown that $g_t(\mathcal{A}_t)$ and $f_t(\mathcal{A}_t)$ converge to the same limit and that $f_t(\mathcal{A}_t)$ is at its minimum at that limit, which by Remark 4.4 is sufficient to show that our our initial objective function (Eq. (4.0.1)) is minimised. This completes the proof of Theorem 4.6.

4.4 Discussion of Result

There are a number of unsatisfying aspects of the proof of convergence presented above, primarily that we have had to assume $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ is bounded away from 0, that $\|\mathcal{A}\|_F$ is bounded and that $\|\frac{1}{\lambda_1}\mathcal{U}_2 * \Sigma_2 * \mathcal{V}_2^T\| \leq 1$ (the problems with this final assumption were already discussed after Lemma 4.3).

The assumption that $\|\mathcal{A}\|_F$ is bounded is analogous to Assumption 1 in Feng et al. (2013) (and identical in the matrix case). In their presentation, this is justified with only a sentence saying that it *"usually is bounded"*. This assumption and its justification are both hidden in the middle of a proof in the supplementary material. The empirical experiments performed at the end of this chapter indeed do indicate that $\|\mathcal{A}\|_F$ is usually bounded, but this is far from a satisfying justification. Unfortunately, proving that $\|\mathcal{A}\|_F$ is bounded was found to be too difficult and thus was left as future work.

The assumption that $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ is bounded away from 0 is made to ensure that the lower bound in Lemma 4.10 does not go to 0. This assumption was not required in Feng et al. (2013) because they erroneously stated that the Hessian matrix was $I \otimes (\mathcal{Y}_t + \lambda_1)$, when in reality it was $\frac{1}{t} I \otimes (\mathcal{Y}_t + \lambda_1)$. This means that instead of showing that $\|\mathcal{A}_t - \mathcal{A}_{t+1}\|_F \leq O(\frac{1}{t})$, they only showed that it was $O(1)$, which was already implied by their Assumption 1. To fix this problem, $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ is assumed to be bounded away from 0, which is a more reasonable assumption since the only time we would expect $\left\|\frac{1}{t}\mathcal{Y}_t\right\|_F$ not to be bounded away from 0 is when our low-rank object of interest is the zero tensor.

Given the flaws discussed above, one should be left wondering if anything of value was achieved within this thesis in terms of the analysis of convergence. Although this thesis has inherited many of the same problems as those in Feng et al. (2013), what has been shown here is that, by replacing the matrix product with the t-product and matrices with tensors, most of Feng et al. (2013) can be done in a routine manner through heavy use of the identities derived in Chapter 2.

Aside from these issues, there is one problem that we were unable to circumvent in regards to the online TRPCA algorithm: the second dimension $n_2 = T$ is taken to be arbitrarily large to prove that Algorithm 1 converges, but Theorem 3.1 requires that $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)} \tilde{n}_3}{(\log(n_{(1)} \tilde{n}_3))^2}$ (recall that $n_{(1)} = \max(n_1, n_2)$ and $n_{(2)} = \min(n_1, n_2)$) and r

and μ are bounded below by 1. So as T goes to infinity, the right-hand side would go to 0. Hence, for arbitrarily large T, it is impossible not to violate this inequality. A preliminary numerical experiment did not find evidence that these implied problems with the algorithm emerged at high T (Fig. 4.1), but more analysis is required before we can be sure.

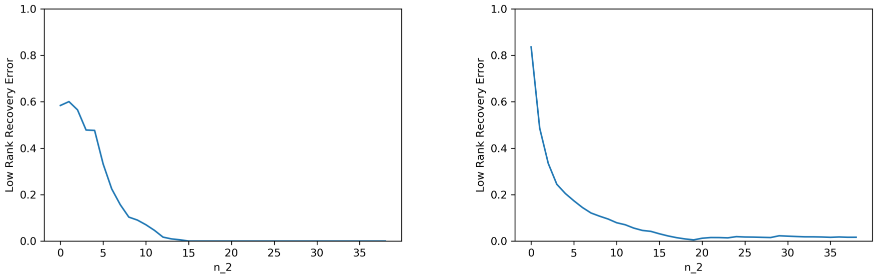

Figure 4.1: Left: a low-rank $20 \times 20 \times 3$ tensor (the shape of a small image) was created by generating a $20 \times 3 \times 3$ random tensor and multiplying it with a $3 \times 20 \times 3$ random tensor (both with Gaussian entries, mean 0, standard deviation $\frac{1}{20}$). This tensor was then copied and stacked along its second axis 40 times, producing a $20 \times 40 \times 20 \times 3$ tensor. Gaussian noise was then added (mean 0, standard deviation $\frac{1}{n_2}$) to 10 % of the entries. This tensor was then sliced as a $20 \times n_2 \times 20 \times 3$ tensor for n_2 ranging from 1 to 40 and the recovery error of the low-rank component was measured. Right: a low-rank $20 \times 40 \times 3$ tensor (the shape of a small image) was generated by generating a $20 \times 3 \times 3$ random tensor and multiplying it with a $3 \times 40 \times 3$ random tensor (both with Gaussian entries, mean 0, standard deviation $\frac{1}{20}$). Gaussian noise was then added then added (mean 0, standard deviation $\frac{1}{n_2}$) to 10 % of the entries. This tensor was then sliced as a $20 \times n_2 \times 3$ tensor for n_2 ranging from 1 to 40 and the recovery error of the low rank component was measured. In both cases, the batch TRPCA algorithm was used. As n_2 rose, the performance improved until n_2 hit 20, and then remained constant. Since we want $r \leq \rho^2 \frac{4}{5^4} \frac{1}{\mu} \frac{n_{(2)}\tilde{n}_3}{(\log(n_{(1)}\tilde{n}_3))^2}$ to hold, while n_2 is less than or equal to n_1, increasing n_2 makes the requirement on the size of the rank of the tensor easier to fulfill. However, once n_2 is larger than n_1, the opposite is true, but no degradation appeared in this case.

4.5 Implementation

Algorithm 1 was implemented and its usefulness demonstrated by applying it to a sequence of 18 satellite images, all of which have been corrupted by clouds. Fig. 4.2 shows the separation between the low rank component (the land) and the sparse component (the clouds) for 3 of these images. Its performance was also compared with the batch version in Fig. 4.3 and found to perform well. Recall that the low-rank recovery error we are using is $\frac{\|L-L_0\|_F}{\|L_0\|_F}$, where L_0 is the low-rank data structure we are trying to recover and L is the output of the algorithm.

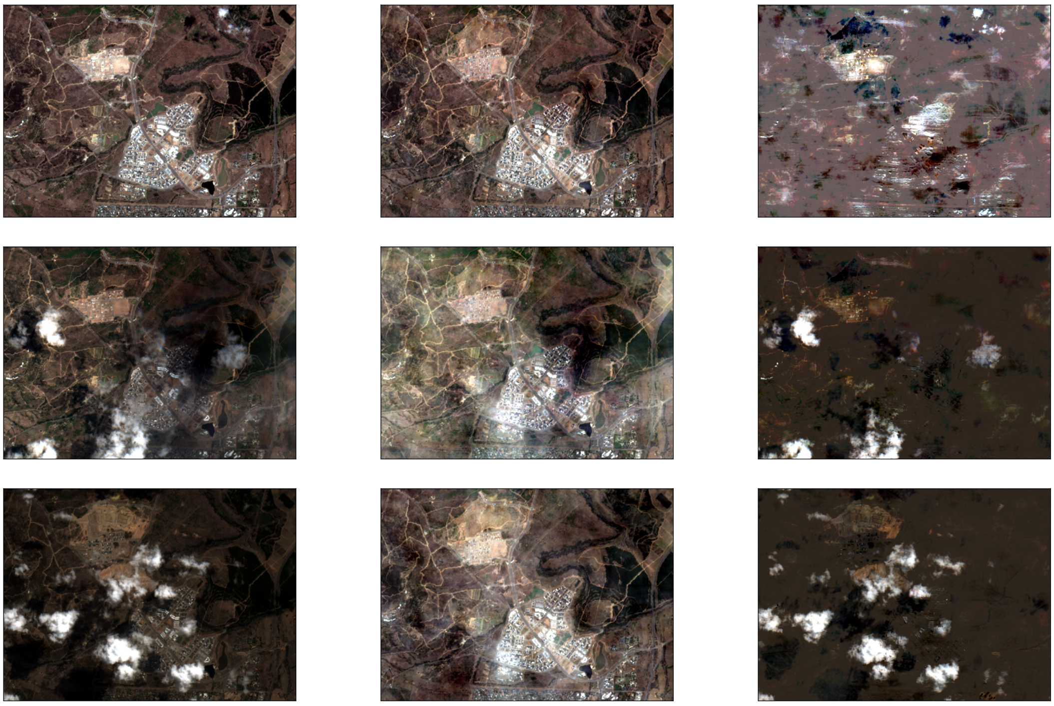

Figure 4.2: The red, green and blue bands from a sequence of images over time of Denman Prospect, Wright, and Coombs (Canberra) taken by Sentinel-2 were stacked in a tensor with time as the second axis. This was then put into the OTRCPA algorithm with the upper bound on the rank chosen to be 3, since the land is not changing much over time. Three of the images (left column) and its decomposition into a low-rank part (middle column) and sparse-noise part (right column) are displayed above. Since these images were taken over a long period of time, which included variations in seasons (affecting the greenery of the land) and significant construction, some of these changes were also removed from the image as sparse noise since they varied over time.

Large Image	$\rho = 0.01$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.51$
Online Total Error	0.2680	0.2873	0.3302	0.3409	0.3700	0.3826
Online Final Frame Error	0.0293	0.0726	0.0969	0.1185	0.1407	0.1719
Batch Final Frame Error	0.0959	0.1042	0.1154	0.1291	0.1531	0.2314

Small Image	$\rho = 0.01$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.51$
Online Total Error	0.0441	0.1205	0.2191	0.2839	0.3971	0.5939
Batch Total Error	0.0251	0.0289	0.0348	0.0414	0.0499	0.0650
Online Final Frame Error	0.0299	0.0847	0.1289	0.1725	0.2542	0.3973
Batch Final Frame Error	0.0251	0.0289	0.0348	0.0414	0.0499	0.0649

Figure 4.3: Top: for multiple values of ρ, Gaussian noise (mean 0, standard deviation 1) was added to ρ % of the entries of a large satellite image (the image from Fig. 2.1) that had been stacked 10 times along its second axis. Then this sequence of image was recovered using the OTPRCA algorithm. The final frame of this sequence was also recovered using the TRPCA algorithm (since it was impossible to run on the entire image). The recovery errors from these two procedures was then recorded. Bottom: for multiple values of ρ, Gaussian noise (mean 0, standard deviation 1) was added to ρ % of the entries of a small satellite image (a part of the image from Fig. 2.1) that had been stacked 10 times along its second axis. Then this sequence of image was recovered using the OTPRCA and TRPCA algorithm. The recovery errors from these two procedures was then recorded and found to be comparable for the lower values of ρ, especially when the time to converge was taken into account by analysing the final frames.

Chapter 5

Conclusion

There remain many directions for further research on this topic. First, there many aspects of our understanding of tensors and the t-product can be deepened: to oversimplify the future directions in this regard, a broad approach is to see just how much of our knowledge about matrices generalises to the tensor case. This student feels after finishing this thesis that the answer is: a lot. There also many aspects of the TRPCA procedure that could be explored, such as generalising the sign tensor of the noise (to a random variable that is not ± 1 with equal probability or to non-iid noise) or analysing which tensor is returned when the procedure fails. Much work remains to be done surrounding the online algorithm. One method that could be applied to demonstrate the boundedness of $\mathcal{A}$ is to note that the implicitly defined function that takes $\mathbf{m}_j$ and all the past information ($\mathbf{b}_i$ and $\mathbf{c}_i$ for $i \leq j$) and returns $\mathcal{A}_i$ is continuously differentiable and only stochastic in $\mathbf{m}_j$. Similarly, the implicitly defined function that takes $\mathbf{m}_{j+1}$ and $\mathcal{A}_i$ and returns $\mathbf{b}_{i+1}$ and $\mathbf{c}_{i+1}$ is also continuously differentiable and only stochastic in $\mathbf{m}_{j+1}$. We also know that $\mathbf{m}_j$ is bounded. So with suitable assumptions, a Markov process approach could be employed. Finally, benchmarking of the online algorithm proposed in this thesis with other procedures for solving the same problem is needed.

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