Lattices

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August 26, 2024

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1 Lattices

1.1 Posets

Definition 1 (Poset)**.** A *partially ordered set* or *poset* is a set P (required to be nonvoid i.e. non-empty in Grätzer) equipped with a binary relation $\leq$ with the following properties

1. Reflexivity: $\forall a \in P, a \leq a$
2. Transitivity: $\forall a, b, c \in P, (a \leq b \wedge b \leq c) \implies a \leq c$
3. Antisymmetry: $\forall a, b \in P, (a \leq b \wedge b \leq a) \implies a = b$

Given a poset $(P, \leq)$ we can define the *dual poset* $(P, \geq)$ equipped with the relation $\leq_d$ defined by

$$\forall a, b \in P, a \leq_d b \iff a \geq b$$

one easily verifies that $(P, \geq)$ is also a poset which brings us to the *duality principle.*

Proposition 1 (Duality principle)**.** *If a statement Φ is true for all posets, then its dual is also true for all posets.*

Definition 2 (Bounds and suprema)**.** Let P be a poset and consider $H \subseteq P$.

- An element $a \in P$ is an *upper bound* of H if $h \leq a$ for all $h \in H$.
- An upper bound a of H is the *least upper bound*, or *supremum* of H if $a \leq m$ for any upper bound m of H. H can have at most one supremum.

lower bounds and *greatest lower bounds* or *infima* are defined dually (i.e. they correspond to upper bounds and suprema in the dual poset).

Proof. Assume that a, a' are both suprema of H then $a \leq a'$ and $a' \leq a$, so $a = a'$ by antisymmetry. $\square$

When they exist, we use $\bigvee H = \sup H$ and $\bigwedge H = \inf H$ to denote the supremum and infimum of H respectively.

Proposition 2 (Associativity of suprema and infima)**.** *Let P a poset and let $(H_i)_{i\in I}$ be a family of subsets of P. Assume that for all $i \in I$, $a_i = \sup H_i$ and assume that $a = \sup_{i\in I} a_i$. Then $a = \sup \bigcup_{i\in I} H_i$.*

Naturally, we have the dual statement for infima.

Proof. We start by showing that a is an upper bound of $\bigcup_{i\in I} H_i$. For $i \in I$ and $h \in H_i$, $h \leq a_i \leq a$ so $h \leq a$ by transitivity.

We conclude by showing that a is indeed the least upper bound of $\bigcup_{i\in I} H_i$. Let m be an upper bound of $\bigcup_{i\in I} H_i$.Then for all $i \in I$, m is an upper bound of H_i so $a_i \leq m$. m is thus an upper bound of $\{a_i\}_{i\in I}$ and $a \leq m$. □

Proposition 3 (Complementarity of suprema and infima)**.** *Let P be a poset and let $H \subseteq P$. Define $S := \{b \in P | \forall h \in H, b \leq h\}$ the set of lower bounds of H, and assume that $a = \sup S$. Then a is the infimum of H.*

As always, we also have the dual statement.

Proof. For all $h \in H$, h is an upper bound of S by construction, so $a \leq h$. This shows that a is a lower bound of H.

For a lower bound m of H, $m \in S$ by definition of S so $m \leq a = \sup H$. This shows that a is indeed the greatest lower bound of H. □

This helps us answer the question "what are the supremum and infimum of the empty set ?". Any element of P is both an upper and a lower bound of $\emptyset$ (since the conditions become vacuous in that case), so the supremum and infimum of $\emptyset$ must be the smallest and greatest elements of P respectively. Formally

$$\sup \emptyset = \inf P$$
$$\inf \emptyset = \sup P$$

(i.e. if any member of these equations exist, the other exists and both members are equal).

1.2 Lattices as Posets

Definition 3 (Lattice)**.** A poset L is called a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for any pair of elements $a, b \in L$.

Definition 4 (Join and Meet)**.** Let L be a lattice, we define two binary operations $\vee$ and $\wedge$ called *join* and *meet* respectively.

$$\vee : L^2 \to L, (a, b) \to a \vee b := \sup\{a, b\}$$
$$\wedge : L^2 \to L, (a, b) \to a \wedge b := \inf\{a, b\}$$

Proposition 4 (Duality)**.** *Let $(L, \leq)$ be a lattice with join $\vee$ and meet $\wedge$. Then the dual poset $(L, \geq)$ is also a lattice. Its join is $\wedge$ and its meet $\vee$.*

Proposition 5 (Elementary properties of join and meet)**.** *Let L be a lattice and let $a, b, c \in L$.*

1. *Idempotency: $a \vee a = a$, $a \wedge a = a$.*
2. *Commutativity: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$.*
3. *Associativity: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$*
4. *Determines order: $a \leq b$ iff $a \vee b = b$, $a \leq b$ iff $a \wedge b = a$.*
5. *Absorption identities: $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$*

Proof. For each property the two statements are duals of each other, so it suffices to prove only one of the two.

1. and 2. are obvious and the associativity of suprema gives us $a \vee (b \vee c) = (a \vee b) \vee c = \sup\{a, b, c\}$.

Now for the way $\vee$ or $\wedge$ determine the order $\leq$. Clearly if $a \leq b$ then $b = a \vee b$. And if $b = a \vee b$ then $a \leq a \vee b = b$.

The absorption identities follow immediately from 4. once we notice $a \wedge b \leq a$. □

A slightly more general definition of lattices.

Proposition 6. *A poset L is a lattice iff all nonvoid finite subsets of L have a supremum and an infimum.*

Proof. Clearly L is a lattice if all finite nonvoid meets and joins exist. Now, if L is a lattice and $H = (a_1, ..., a_n)$ is a nonvoid finite subset of L (i.e. $n \geq 1$), then by associativity of the suprema and infima we get

$$\bigvee H = a_1 \vee a_2 \vee ... \vee a_n$$
$$\bigwedge H = a_1 \wedge a_2 \wedge ... \wedge a_n$$

□

1.3 Lattices as Algebras

An important consequence of the proposition 5 is that the order $\leq$ is completely determined by the knowledge of $\vee$ (or $\wedge$). This suggests that lattices may be defined algebraically.

Definition 5 (Algebraic definition of Lattice)**.** A nonvoid set L equipped with two binary operations $\vee$, $\wedge$ is called a lattice if $\vee$ and $\wedge$ are both idempotent, associative, commutative and satisfy both absorption identities.

Definition 6 (Algebraic Dual Lattice)**.** Let the algebra $(L, \vee, \wedge)$ be a lattice. Then the algebra $(L, \wedge, \vee)$ is also a lattice called the *dual lattice* ($\wedge$ and $\vee$ play completely symmetric roles in the algebraic definition).

Proposition 7. *If a statement Φ is true for all algebraic lattices, then its dual (obtained by exchanging $\vee$ and $\wedge$) is also true.*

Theorem 1. *Equivalence of the two definitions of lattice:*

1. *Let the poset $(L, \leq)$ be a lattice with $\vee$ and $\wedge$ defined through* sup *and* inf*. Then the algebra $L^a := (L, \vee, \wedge)$ is a lattice.*

2. *Let the algebra $(L, \vee, \wedge)$ be a lattice. Define the relation $a \leq b \iff b = a \vee b \iff a = a \wedge b$. Then the poset $L^p := (L, \leq)$ is a lattice.*

3. *Let the poset $(L, \leq)$ be a lattice. Then $(L^a)^p = L$.*

4. *Let the algebra $(L, \vee, \wedge)$ be a lattice. Then $(L^p)^a = L$*

Proof. We have shown 1. and 3. in proposition 5.

To show 2. and 4. we simply show that for $a, b \in L$, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ with sup and inf taken in L^p.
We start by showing that the two definitions of $\leq$ are in fact equal. For $a, b \in L$, $a \vee b = b$ implies $a \wedge b = a \wedge (a \vee b) = a$. So $a \vee b = b \implies a \wedge b = a$ and dually $(a \wedge b) = a \implies (a \vee b) = b$. In particular, we see that for the algebraic dual $L_d = (L, \wedge, \vee)$ we have $(L_d)^p = (L^p)_d$ in other words the poset induced by L_d is the dual of the poset induced by L.
Next we show that L^p is a poset.

Reflexivity: by idempotency, for all $a \in L$, $a \vee a = a$ so $a \leq a$.

(b) Transitivity: for $a, b, c \in L$, $a \vee b = b$ and $b \vee c = c$ implies by associativity $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = a \vee c = c$. This shows that $a \leq b$ and $b \leq c$ implies $a \leq c$.

(c) Antisymmetry: for $a, b \in L$ such that $a \leq b$ and $b \leq a$, commutativity gives us $a = a \vee b = b$.

Take $a, b \in L$. Commutativity and the absorption identities give us $a \wedge (a \vee b) = a$ and $b \wedge (a \vee b) = b$ so $a \vee b$ is an upper bound of $\{a, b\}$. If m is an upper bound of $\{a, b\}$, then $m = (a \vee m)$ and $m = (b \vee m)$ so $(a \vee b) \vee m = a \vee (b \vee m) = a \vee m = m$ and $a \vee b \leq m$. This shows that $a \vee b = \sup\{a, b\}$. Dually $(a \wedge b) = \inf\{a, b\}$[1].

□

1.4 Types of Lattices

1.4.1 Distributive Lattice

Definition 7 (Distributive Lattice)**.** A lattice L is distributive if for all $a, b, c \in L$

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A lattice is actually distributive as soon as it possesses one of those two properties.

Proposition 8 (Equivalence of two distributivities)**.** *In any lattice L the two following properties are equivalent*

1. $\forall a, b, c \in L, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

2. $\forall a, b, c \in L, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Proof. 2. $\Longrightarrow$ 1. is the dual of 1. $\Longrightarrow$ 2., so we only need to prove one of the implications.

Assume 1. then for $a, b, c \in L$, using associativity, commutativity and absorption identities we get

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \wedge c) \vee (b \wedge c)] \\ &= [a \vee (a \wedge c)] \vee (b \wedge c) \\ &= a \vee (b \wedge c)\end{aligned}$$

[1]In some detail, $a \wedge b = \inf\{a, b\}$ is simply $a \vee b = \sup\{a, b\}$ stated for the algebraic dual lattice L_d which induces the poset dual to L^p.

$\square$

1.4.2 Bounded Lattice

Definition 8. A lattice L is bounded if it possesses a top (i.e. greatest) element $1_L = \sup L$ and a bottom (i.e. smallest) element $0_L = \inf L$.

1.4.3 Complete Lattice

Definition 9. A lattice L is complete if it verifies one of the following two equivalent properties:

1. Any subset of L has a supremum.
2. Any subset of L has an infimum.

Proof of Equivalence. Assume that any subset of L has a supremum. Then, given a subset $H \subseteq L$, we define the set S of lower bounds of H. Then, by proposition 3, $\sup S = \inf H$. This shows that any subset of L has an infimum.

The reverse implication is obtained dually. $\square$

Proposition 9. *Let L be a complete lattice. Then L is bounded.*

Proof. The top and bottom elements are given by the empty meet and join respectively. $\square$

1.5 Morphisms

Definition 10. Given two lattices L, M a morphism of lattice from L to M is a map $f : L \to M$ preserving the join and the meet, that is for all $a, b \in L$

$$f(a \vee b) = f(a) \vee f(b)$$
$$f(a \wedge b) = f(a) \wedge f(b)$$

More specific types of morphisms.

Definition 11. Let L, M be lattices and $f : L \to M$ be a lattice morphism.

1. f is a bounded lattice morphism if L and M are bounded and f preserves the top and bottom elements, that is

$$f(0) = 0$$
$$f(1) = 1$$

2. f is a complete morphism if L and M are complete and f preserves arbitrary meet and joins. That is for all $H \subseteq L$

$$\bigvee f(H) = f(\bigvee H)$$
$$\bigwedge f(H) = f(\bigwedge H)$$

Remark 1. A complete morphism is always a morphism of bounded lattices since the top and bottom elements are the empty wedge and join.

Proposition 10 (Link with Order-Preserving maps)**.** *Relationship with morphisms of posets.*

1. *A morphism of lattice is always order preserving.*
2. *An isomorphism of posets $f : L \to M$ (bijective with f and f^{-1} order preserving) is a morphism of lattices. Further more if L and M are bounded (complete) f is a morphism of bounded (complete) lattices.*

Proof. We only show 1. For $a, b \in L$ such that $a \leq b$, $a = a \wedge b$ so $f(a) = f(a \wedge b) = f(a) \wedge f(b)$ and $f(a) \leq f(b)$. $\square$

2 Locales

2.1 Frames and Locales: Basic definitions

Definition 12 (Frame)**.** A frame is a poset L that has arbitrary joins (suprema) and finite meets (infima), and verifies the infinite distributive law, that is for all $b \in L$ and any family $(a_i)_{i \in I} \subseteq L$,

$$b \wedge \left(\bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} (b \wedge a_i)$$

Definition 13 (Frame Morphism)**.** Given two frames L, M a frame homomorphism $L \to M$ is a map $f : L \to M$ which preserves arbitrary joins and finite meets.

Definition 14 (Locales)**.** The frames together with their homomorphisms form a category **Frm** and the category of locales is the category opposite to **Frm**, i.e. $\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}}$.

Proposition 11. *Some basic properties of frames.*

1. *Frames are complete lattices (they also possess arbitrary meets/infima).*
2. *In particular, frames are bounded lattices and are therefore non-empty (they must contain a top and bottom element - which may be equal).*
3. *Frames are distributive lattices.*
4. *Frame homomorphisms are bounded lattice homomorphisms and order-preserving in particular.*

Proof. For 3. $\forall a, b, c \in L$, the infinite distributive law gives us

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

For 4. frame homomorphisms preserve the top and bottom elements because they are the empty meet (which is finite) and the empty join respectively. □

Example 1 (Frame of Opens of a Space). Let X be a topological space and let $(\mathcal{O}(X), \subseteq)$ be the poset of the opens of X ordered by inclusion.

$\mathcal{O}(X)$ is a frame. Indeed, arbitrary joins correspond to unions of sets, finite meets correspond to intersections of sets and these two operations (union and intersection of sets) satisfy the infinite distributive law.

$\mathcal{O}(X)$ is therefore a complete lattice and must have arbitrary meets as well. For an arbitrary family of opens $(U_i)_{i \in I}$ however, the meet is not simply given by the intersection, since this intersection is not open in general. It corresponds instead to the *interior* of the intersection[2]:

$$\bigwedge_{i \in I} U_i = \mathrm{int}\left(\bigcap_{i \in I} U_i\right)$$

[2] We can verify that the meet is constructed according to the procedure described in proposition 3: the interior of the intersection is the union of all opens contained in $\bigcap_{i \in I} U_i$, in other words it is the supremum of all lower bounds of $(U_i)_{i \in I}$.

For finite I, $\bigcap_{i\in I} U_i$ is open and the expression above simply corresponds to the intersection, as expected.

Observe, however, that in general frames do not verify the infinite distributive law for arbitrary meets:

$$V \vee \bigwedge_{i\in I} U_i = \bigwedge_{i\in I} (V \vee U_i)$$

As a counter-example, take the space $\mathbb{R}$ with the euclidean topology and consider its frame of opens $\mathcal{O}(\mathbb{R})$. If we take the open $V :=]-1,0[$ and the family of opens $(U_n :=]-1/n, 1[)_{n\geq 1}$, we get

$$\operatorname{int}\left(\bigcap_{n\geq 1} U_n\right) = \operatorname{int}[0,1[=]0,1[$$
$$\operatorname{int}\left(\bigcap_{n\geq 1} (V \cup U_n)\right) = \operatorname{int}]-1,1[=]-1,1[$$

and from this we deduce

$$V \vee \bigwedge_{n\geq 1} U_n =]-1,0[\cup]0,1[\subsetneq]-1,1[= \bigwedge_{n\geq 1} (V \vee U_n)$$

References

[1] Garrett Birkhoff. *Lattice Theory.* AMS, 1967.

[2] George Grätzer. *Lattice Theory: First Concepts and Distributive Lattices.* Manitoba, Dover, 2009.

[3] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Oxford, 2014.