Basics of Differential Geometry

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Contents

Part I

Topology

Chapter 1

Intro

Main references are [6, 5].

1.1 Things left to learn

The different sets of axioms one can use to define a topological space, as in [1]. A topological space is most commonly defined by specifying its open sets. But one can also define a topology in the following ways:

(i) By specifying its neighborhoods or closed sets.

(ii) By specifying the interior, closure, exterior, boundary or 'derived set' operators.

(iii) Through nets or filters (which are equivalent).

The notions of **nets** and **filters**, their equivalence and their relationship with the notion of a topology, should be explored (see [3]). The different notions of convergence that can be defined in a topology, and the degree to which they determine this topology is also an interesting question [4, 2].

General topology (more set-theoretic than algebraic, and not focused on finite-dimensional topological manifolds) as in [6].

Chapter 2

Topological spaces

Definition 2.0.1 (Topology)**.** A **topology** on a set X is a collection τ of subsets of X such that

(i) τ contains $\emptyset$ and X;

(ii) The union of the elements of any subset of τ is again in τ;

(iii) The intersection of the elements of any finite subset of τ is again in τ.

A **topological space** is a pair (X, τ) consisting of a set X together with a topology τ on X. The elements of τ are called the **open sets** of (X, τ).

Given two topological spaces (X, τ), (X', τ'), a map $f : (X, \tau) \to (X', \tau')$ is said to be **continuous** if for any $U \in \tau'$, $f^{-1}(U) \in \tau$. In most instances we can omit τ when referring to the topological space (X, τ) with no risk of confusion.

Definition 2.0.2 (Neighborhoods)**.** Let X be a topological space.

(i) Let K be a subset of X, another subset N of X is said to be a **neighborhood** of K if there exists an open subset U of X such that $K \subseteq U \subseteq N$. An **open neighborhood** of K is an open subset of X that contains K.

(ii) Let x be a point of X. An (open) neighborhood of x is an (open) neighborhood of the singleton $\{x\}$.

Definition 2.0.3 (Closed subsets)**.** A subset F of X is said to be **closed** if its complement $X - F$ is open.

Proposition 2.0.1. *Let X be a topological space.*

(i) $\emptyset$ and X are closed.

(ii) Any intersection of closed subsets of X is closed.

(iii) A finite unions of closed subsets of X is closed.

Proposition 2.0.2. *A map between topological spaces is continuous if and only the preimage of any closed subset is closed.*

Proof. This is because for any map $f : X \to Y$ and any subset $A \subseteq Y$, $f^{-1}(Y - A) = X - f^{-1}(A)$. □

Definition 2.0.4 (Closure and interior)**.** Let A be a subset of a topological space X.

(i) The **closure** of A, denoted $\bar{A}$ is the smallest closed subset containing A.

$$\bar{A} := \bigcap \{F \subseteq X | S \text{is closed and } A \subseteq F\}.$$

(ii) The **interior** of A, denoted $\mathrm{Int}(A)$ is the largest open subset contained in A.

$$\mathrm{Int}(A) := \bigcup \{U \subseteq X | U \text{is open and } U \subseteq A\}.$$

(iii) The **exterior** of A, denoted by $\mathrm{Ext}(A)$, is defined to by $\mathrm{Ext}(A) := X - \bar{A}$, it is the complement of the closure, that is the largest open that does not overlap with A.

(iv) The **boundary** of A, denoted by ∂A is defined by $\partial A := \bar{A} - \mathrm{Int}(A)$.

Proposition 2.0.3. *Let A be a subset of an topological space X.*

(i) A point is in Int(A) if and only if it has a neighborhood contained in A.

(ii) A point is in Ext(A) if and only if it has a neighborhood contained in $X - A$.

(iii) A point is in ∂A if and only if any neighborhood of it contains both a point of A and a point of $X - A$.

(iv) A point is in $\bar{A}$ if and only if any neighborhood of it contains a point of A.

(v) The following are equivalent:

- *A is open.*
- *$A =$ Int(A).*
- *A contains none of its boundary points (hence the 'open terminology').*
- *Any point of A has a neighborhood contained in A.*

(vi) The following are equivalent:

- *A is closed.*
- *$A = \bar{A}$.*
- *A contains all of its boundary points (hence the 'closed' terminology).*
- *Any point of $X - A$ has a neighborhood contained in $X - A$.*

Definition 2.0.5 (Limit and isolated points)**.** Let A be a subset of a topological space X.

(i) A point $p \in X$ (not necessarily in A) is a **limit point** of A if any neighborhood of p contains a point of A other than p. Limit points are also called **cluster points** and **accumulation** points.

(ii) A point $p \in A$ is **isolated in** A if p has a neighborhood N such that $N \cap A = \{p\}$.

Observe that any point of A is either isolated in A or a limit point of A.

Proposition 2.0.4. *A set is closed if and only if it contains all of its limit points.*

Definition 2.0.6 ((Nowhere) dense sets)**.** A subset A of a topological space X is said to be **dense** in X if $\bar{A} = X$. Given a subset S of X, A is said to be dense in S if $A \cap S$ is dense in S (with the subset topology). It is said to be **nowhere dense** or **rare** in X if $\bar{A}$ has empty interior. Equivalently A is rare if it is not dense in any nonempty open subset of X. Equivalently, A is rare if its exterior is dense in X.

Part II

Manifolds

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