Homology

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Contents

Part I

Homological Algebra, Lectures of Tim Van der Linden [1]

Chapter 1

Modules over a ring, exact sequences and homology

Based on lectures given by Tim Van der Linden in 2022-2023 [1] (might turn out to be a copy).

1.1 Modules over a ring

The rings are not assumed to be commutative, only to possess a unit 1.

Definition 1.1.1 (Ring)**.** A **ring** is an abelian group R (written additively) together with a binary operation

$$\cdot : R \times R \to R, (r, a) \to r \cdot a \stackrel{n}{=} ra$$

such that $(R, \cdot)$ is a monoid with unit 1, and such that $\cdot$ is distributive over $+$, meaning that for all $a, a', r, r' \in R$

(i) (Left-distributivity) $r(a + a') = ra + ra'$;

(ii) (Right-distributivity) $(r + r')a = ra + ra'$.

Example 1.1.1. By definition, a monoid is always with unit (or identity). A nonzero ring with commutative multiplication and multiplicative inverses for every non-zero element is called a **field**. Note that in any ring R we have $0a = (0 + 0)a = 0a + 0a$ which implies $0a = 0$ (and similarly we have $a0 = 0$), so unless $0 = 1$, 0 cannot admit a multiplicative inverse, and $1 = 0$ if and only if R is the **zero ring** (the ring with one element, denoted 0).

Modules are to rings what vector spaces are to fields.

Definition 1.1.2 (Module)**.** Let R be a ring. A **left R-module** is an abelian group A (written additively) together with an operation (scalar multiplication)

$$\cdot : R \times A \to A, (r, a) \to r \cdot a \overset{n}{=} ra$$

such that for all $r, r' \in R$, $a, a' \in A$

(i) (Distributivity) $(r + r')a = ra + r'a$ and $r(a + a') = ra + ra'$;

(ii) (Multiplicative compatibility) $(rr')a = r(r'a)$;

(iii) (Identity) $1a = a$.

Given two left R-modules, A and B, an **R-linear map** (or morphism of R-modules) $A \to B$ is a group morphism (for the abelian group structure of A and B) $\phi : A \to B$ such that $\phi(ra) = r\phi(a)$ for $r \in R$, $a \in A$. **Isomorphisms** are define in the usual way (as invertible R-linear maps with R-linear inverses), it turns out that the inverse of a bijective R-linear map is automatically R-linear, so that isomorphisms of R-modules are exactly the bijective R-linear maps.

Remark. We can define **right R-modules** in the obvious way. Given a ring R we can define the **opposite ring** R^{op} with the same underlying set, the same addition and reversed multplication, i.e. $r \cdot_{op} r' := r'r$ for $r, r' \in R$. We can then see that right R-modules are exaclty left R^{op}-modules. For a commutative ring $R = R^{op}$ and there is no distinction between left and right R-modules. In what follows we'll often use "R-modules" to refer to *left* R-modules.

Example 1.1.2. For a field $\mathbb{F}$, a (left or right) $\mathbb{F}$-module is a **vector space** over $\mathbb{F}$.

Example 1.1.3. For an R-module A, we have $0a = (0+0)a = 0a+0a$ implying $0a = 0$. Combining this with the identity ($1a = a$) we see that the only module over 0 (the zero ring) is the **zero module** (the module with one element, also denoted 0).

Example 1.1.4. By definition any $\mathbb{Z}$-module comes with an abelian group structure. Conversely, an abelian group A admits a unique scalar product making A into a $\mathbb{Z}$-module. So we see that the $\mathbb{Z}$-modules are the abelian group (there is an isomorphism of category Ab $\simeq$ $\mathbb{Z}$-Mod).

Definition 1.1.3 (Submodules)**.** Let A be an R-module. Given a subset S of A, we say that S is a **submodule** (or R-submodule) of A if S admits an R-module structure for which the injection $\iota : S \hookrightarrow A$ is R-linear. We can show that S is a submodule of A if and only if

(i) S is a subgroup of A (automatically abelian);

(ii) For all $r \in R$, $s \in S$, $rs \in S$.

In that case the R-module structure for which $\iota : S \hookrightarrow A$ is R-linear is unique, and obtained by restricting the operations of A to S.

Proposition 1.1.1. *Submodules are stable by intersection, meaning that for an R-module A and a family $(S_i)_{i\in I}$ of submodules of A, $\bigcap_{i\in I} S_i$ is a submodule of A.*

Definition 1.1.4 (Quotient by a submodule)**.** Let A be an R-module, and S be a submodule of A. For $a, a' \in A$ we write $a \sim_S a'$ if $(a - a') \in S$. $\sim_S$ is then an equivalence relation on A, and we can define the quotient $q : A \to A/S := A/\sim_S$. The **quotient** of A by S is then defined to be A/S equipped with the unique R-module structure for which $q : A \to A/S$ is R-linear. The image of $a \in A$ by $q : A \to A/S$ (so the equivalence class of a for $\sim_S$) is often denoted by $a + S$.

Definition 1.1.5 (Kernel, image and cokernel)**.** Given an R-linear map $\phi : A \to B$. The **image** $\operatorname{im}(\phi)$ of ϕ is a submodule of B, while the **kernel** of ϕ is defined to be submodule $\ker(\phi) := \{a \in A | \phi(a) = 0\}$. The **cokernel** of ϕ is the quotient $q : B \to B/\operatorname{im}(\phi)$ of B by the image of ϕ.

Proposition 1.1.2. *Given an R-linear map $\phi : A \to B$*

(i) ϕ is injective if and only if $\ker(\phi) = 0$*;*

(ii) ϕ is surjective if and only if $\operatorname{coker}(\phi) = 0$*.*

In particular ϕ is an isomorphism if and only if both $\ker(\phi)$ *and* $\operatorname{coker}(\phi)$ *are zero.*

Definition 1.1.6 (Span of a subset)**.** Let X be a subset of an R-module A. The span $\langle X\rangle$ is defined to be the smallest submodule of A containing X, so

$$\langle X\rangle := \bigcap\{S \text{ submodule of } A | X \subseteq S\}.$$

$\langle X\rangle$ consists of the (finite) linear combinations of elements of X (another possible definition of $\langle X\rangle$)

$$\langle X\rangle := \{\sum_{i=1}^{k} r_i x_i | 0 \leq k < \infty, r_j \in R, x_j \in X\}.$$

1.1.1 Free modules

Definition 1.1.7 (Free over a set)**.** Let A be an R-module and let $\delta : X \to R$ be a map from a set X to the underlying set of R. We say that R is **free over** X (or δ to be more precise) if for any map $\xi : X \to B$ there exists a unique R-linear map $\alpha : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[d, dotted, "\alpha"] \\ X \arrow[ur, "\delta"] \arrow[r, "\xi"'] & B \end{tikzcd}$$

Definition 1.1.8 (The free module over a set)**.** Given a set X there exists an R-module $R[X]$ together with an set map $\delta : X \to R[X]$ such that $R[X]$ is free over δ. The pair $(R[X], \delta)$ is unique up to isomorphism (as for any other object defined by means of a universal property) and δ is injective.

Proof. We'll construct a **standard version** of $(R[X], \delta)$. $R[X]$ consists of the **almost zero** functions $\phi : X \to R$, meaning that the support $\operatorname{supp}\phi := \{x \in X | \phi(x) \neq 0\}$ is finite, equipped with pointwise addition and scalar multiplication. The injection $\delta : X \to R[X]$ sends $x \in X$ to the indicator function of x

$$\delta(x) \overset{n}{=} \delta_x \overset{n}{=} x : X \to R, y \to \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

Any nonzero element ϕ of $R[X]$ is written uniquely $\phi = \sum_{i=1}^{k} r_i x_i$ for $x_1, ..., x_k \in X$ distinct and $r_i = \phi(x_i) \in R - \{0\}$. Given a map $\xi : X \to B$ from X to another R-module B, the unique factoring map $\alpha : R[X] \to B$ sends $\phi = \sum_{i=1}^{k} r_i x_i$ to $\sum_{i=1}^{k} r_i \xi(x_i)$. $\square$

Remark. From now on we'll use $(R[X], \delta)$ to refer to the standard free module over X.

Remark. The universal property of the free module over X can be stated as follows: for any R-module A, we have a canonical bijection

$$\operatorname{Set}(X, UA) \simeq \operatorname{Hom}(R[X], A)$$

where U denotes the forgetful functor $U : R\text{-Mod} \to \text{Set}$. So we see that the existence of free modules is equivalent to the existence of a left-adjoint to the forgetful functor.

Proposition 1.1.3. *Given an R-module A, a set X and a map $\xi : X \to A$. By the universal property the free module, there exists a unique R-linear map $\alpha : R[X] \to A$ such that*

$$\begin{tikzcd} & R[X] \arrow[d, dotted, "\alpha"] \\ X \arrow[ur, "\delta"] \arrow[r, "\xi"'] & A \end{tikzcd}$$

commutes. A is free over ξ if and only if α is an isomorphism. In particular ξ must be injective (for A to be free over ξ).

Definition 1.1.9 (Basis of a module)**.** Let A be an R-module. A subset $X \subseteq A$ is a **basis** of A if A is free over the injection $\iota : X \hookrightarrow A$. In other words, X is a basis of A if and only if for any R-module B, any map $\phi : X \to B$ extends uniquely to an R-linear map $\bar{\phi} : A \to B$.

Proposition 1.1.4. *Given an R-module A, a set X and a map $\xi : X \to A$, A is free over ξ if and only if ξ is injective and $\mathrm{im}(\xi) \subseteq A$ is a basis of A.*

Remark. Given a set X, X is a basis of $R[X]$ (when identified with its image by δ).

Proposition 1.1.5. *Let X be a subset of the R-module A. By the universal property of the free module, there exists a unique map $\alpha : R[X] \to A$ such that*

$$\begin{tikzcd} & R[X] \arrow[d, dotted, "\alpha"] \\ X \arrow[ur, "\delta"] \arrow[r, hook, "\iota"'] & A \end{tikzcd}$$

commutes. X is a basis of X if and only if α is an isomorphism.

Corollary 1.1.1. *A subset X of an R-module A is a basis of A if and only if*

(i) X is ***linearly independent****, meaning that for $1 \leq k$ and $x_j \in X$, $r_j \in R$*

$$\sum_{i=1}^{k} r_i x_i = 0$$

if and only if $r_1 = ... = r_k = 0$. This is equivalent to requiring that the unique factoring map $\alpha : R[X] \to A$ be injective.

(ii) X ***spans*** *A, meaning that $\langle X \rangle = A$ (i.e. that any element of A is a finite linear combination of elements of X). This is equivalent to requiring that the unique factoring map $\alpha : R[X] \to A$ be surjective.*

Definition 1.1.10 (Free module)**.** An R-module A is said to be free if and only if A

Bibliography

[1] Tim van der Linden. *Homological Algebra*. 2022-2023.