Basics of Differential Geometry

Maxime Willaert

August 26, 2024

Contents

Part I

Topology

Chapter 1

Intro

Main references are [6, 5].

1.1 Things left to learn

The different sets of axioms one can use to define a topological space, as in [1]. A topological space is most commonly defined by specifying its open sets. But one can also define a topology in the following ways:

(i) By specifying its neighborhoods or closed sets.

(ii) By specifying the interior, closure, exterior, boundary or 'derived set' operators.

(iii) Through nets or filters (which are equivalent).

The notions of **nets** and **filters**, their equivalence and their relationship with the notion of a topology, should be explored (see [3]). The different notions of convergence that can be defined in a topology, and the degree to which they determine this topology is also an interesting question [4, 2].

General topology (more set-theoretic than algebraic, and not focused on finite-dimensional topological manifolds) as in [6].

Chapter 2

Topological spaces

2.1 Basic definitions

Definition 2.1.1 (Topology)**.** A **topology** on a set X is a collection τ of subsets of X such that

(i) τ contains $\emptyset$ and X;

(ii) The union of the elements of any subset of τ is again in τ;

(iii) The intersection of the elements of any finite subset of τ is again in τ.

A **topological space** is a pair (X, τ) consisting of a set X together with a topology τ on X. The elements of τ are called the **open sets** of (X, τ).

Given two topological spaces (X, τ), (X', τ'), a map $f : (X, \tau) \to (X', \tau')$ is said to be **continuous** if for any $U \in \tau'$, $f^{-1}(U) \in \tau$. In most instances we can omit τ when referring to the topological space (X, τ) with no risk of confusion.

Definition 2.1.2 (Neighborhoods)**.** Let X be a topological space.

(i) Let K be a subset of X, another subset N of X is said to be a **neighborhood** of K if there exists an open subset U of X such that $K \subseteq U \subseteq N$. An **open neighborhood** of K is an open subset of X that contains K.

(ii) Let x be a point of X. An (open) neighborhood of x is an (open) neighborhood of the singleton $\{x\}$.

Definition 2.1.3 (Closed subsets)**.** A subset F of X is said to be **closed** if its complement $X - F$ is open.

Proposition 2.1.1. *Let X be a topological space.*

(i) $\emptyset$ and X are closed.

(ii) Any intersection of closed subsets of X is closed.

(iii) A finite unions of closed subsets of X is closed.

Proposition 2.1.2. *A map between topological spaces is continuous if and only the preimage of any closed subset is closed.*

Proof. This is because for any map $f : X \to Y$ and any subset $A \subseteq Y$, $f^{-1}(Y - A) = X - f^{-1}(A)$. $\square$

Definition 2.1.4 (Closure and interior)**.** Let A be a subset of a topological space X.

(i) The **closure** of A, denoted $\bar{A}$ is the smallest closed subset containing A.

$$\bar{A} := \bigcap\{F \subseteq X | S \text{is closed and } A \subseteq F\}.$$

(ii) The **interior** of A, denoted $\mathrm{Int}(A)$ is the largest open subset contained in A.

$$\mathrm{Int}(A) := \bigcup\{U \subseteq X | U \text{is open and } U \subseteq A\}.$$

(iii) The **exterior** of A, denoted by $\mathrm{Ext}(A)$, is defined to by $\mathrm{Ext}(A) := X - \bar{A}$, it is the complement of the closure, that is the largest open that does not overlap with A.

(iv) The **boundary** of A, denoted by ∂A is defined by $\partial A := \bar{A} - \mathrm{Int}(A)$.

Proposition 2.1.3. *Let A be a subset of an topological space X.*

(i) A point is in $Int(A)$ if and only if it has a neighborhood contained in A.

(ii) A point is in $Ext(A)$ if and only if it has a neighborhood contained in $X - A$.

(iii) A point is in ∂A if and only if any neighborhood of it contains both a point of A and a point of $X - A$.

(iv) A point is in $\bar{A}$ if and only if any neighborhood of it contains a point of A.

(v) The following are equivalent:

- *A is open.*
- *$A = Int(A)$.*
- *A contains none of its boundary points (hence the 'open terminology').*
- *Any point of A has a neighborhood contained in A.*

(vi) The following are equivalent:

- *A is closed.*
- *$A = \bar{A}$.*

- *A contains all of its boundary points (hence the 'closed' terminology).*
- *Any point of $X - A$ has a neighborhood contained in $X - A$.*

Definition 2.1.5 (Limit and isolated points)**.** Let A be a subset of a topological space X.

(i) A point $p \in X$ (not necessarily in A) is a **limit point** of A if any neighborhood of p contains a point of A other than p. Limit points are also called **cluster points** and **accumulation** points.

(ii) A point $p \in A$ is **isolated in** A if p has a neighborhood N such that $N \cap A = \{p\}$.

Observe that any point of A is either isolated in A or a limit point of A.

Proposition 2.1.4. *A set is closed if and only if it contains all of its limit points.*

Definition 2.1.6 ((Nowhere) dense sets)**.** A subset A of a topological space X is said to be **dense** in X if $\bar{A} = X$. Given a subset S of X, A is said to be dense in S if $A \cap S$ is dense in S (with the subset topology). It is said to be **nowhere dense** or **rare** in X if $\bar{A}$ has empty interior. Equivalently A is rare if it is not dense in any nonempty open subset of X. Equivalently, A is rare if its exterior is dense in X.

2.2 Maps between topological spaces

Proposition 2.2.1. *Some basic properties of continuous maps.*

(i) A constant map is continuous.

(ii) Compositions of continuous maps are continuous.

(iii) Let X and Y be two topological spaces. A map $f : X \to Y$ is continuous if and only if for any open subset U of X, $f|_U$ is continuous. In particular the restriction of a continuous map to any open subset is continuous.

Definition 2.2.1. Let $f : X \to Y$ be a map between two topological spaces.

(i) f is **closed** if it takes closed subsets of X to closed subsets of Y.

(ii) f is **open** if it takes open subsets of X to open subsets of Y.

(iii) f is a **homeomorphism** if f is a continuous bijection whose inverse is continuous.

(iv) f is a **local homeomorphism** if any point of X admits an open neighborhood U such that $f|_U : U \to f(U)$ is a homeomorphism. Clearly a homeomorphism is a local homeomorphism.

Proposition 2.2.2 (Properties of local homeomorphisms)**.** *Let $f : X \to Y$ be a local homeomorphism.*

(i) f is open and closed.

(ii) If f is bijective, then f is a homeomorphism.

Proposition 2.2.3 (Characterization of homeomorphism)**.** *Let $f : X \to Y$ be a bijective continous map. The following are equivalent:*

(i) f is a homeomorphism.

(ii) f is open.

(iii) f is cosed.

Proposition 2.2.4 (Characterizing maps)**.** *Let $f : X \to Y$ be a map between two topological spaces.*

(i) f is continous if and only if $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

(ii) f closed if and only if $f(\bar{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.

(iii) f is continous if and only if $f^{-1}(IntB) \subseteq Int(f^{-1}(B))$ for all $B \subseteq Y$.

(iv) f is open if and only if $f^{-1}(IntB) \supseteq Int(f^{-1}(B))$ for all $B \subseteq Y$.

Proof. (i) Assume that f is continuous. Then $f^{-1}(\overline{f(A)})$ is closed and contains A, so it must contain $\bar{A}$ and $f(\bar{A}) \subseteq \overline{f(A)}$. Conversely, assume that f satisfies the condition given in (i). For $F \subseteq Y$ closed, set $A := f^{-1}(F)$. Then

$$f(\bar{A}) \subseteq \overline{f(A)} \subseteq F$$

which implies $\bar{A} \subseteq f^{-1}(F) = A$, so A is closed. □

2.3 Generating topologies

Definition 2.3.1 (Comparing topologies)**.** Let τ, τ' be topologies on a set X. When $\tau \subseteq \tau'$ we say that τ is **coarser** (or **smaller**) than τ', or that τ' is **finer** (or **larger**) than τ. We say that τ and τ' are **comparable** if $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$.

Example 2.3.1. The finest topology on a set X is the **discrete topology** $\tau = \mathcal{P}(X)$ (all subsets of X are open for the discrete topology). The coarsest topology on a set X is the **trivial** or **indiscrete topology** $\tau := \{\emptyset, X\}$.

Proposition 2.3.1. *Let $(\tau_i)_{i\in I}$ be a family of topologies on a set X. Then the intersection $\tau := \bigcap_{i\in I} \tau_i$ is a topology on X.*

Remark. The empty intersection would yield the discrete topology.

Definition 2.3.2 (Preorder)**.** A binary relation $\leq$ on a set A is a **preorder** if it is **reflexive** and **transitive**, meaning that for all $a, b, c \in A$

(i) (Reflexivity) $a \leq a$;

(ii) (Transivity) $a \leq b$ and $b \leq c$ implies $a \leq c$.

We'll often use the word 'preorder' to refer to a pair $(A, \leq)$ consisting of a set A and a preorder $\leq$ on A.

Definition 2.3.3 (Partial order)**.** A partial order on a set A is a preoder $\leq$ on A which is **antisymmetric** meaning that for all $a, b \in A$, $a \leq b$ and $b \leq a$ implies $a = b$. A **partially ordered set** or **poset** is a set together with a partial order (sometimes required to be nonempty).

See personal notes on **lattices** [7]. Given a subset S of a poset $(P, \leq)$, the **supremum** of S is sometimes called its **join**, while the **infimum** of S is sometimes called its **meet**.

Now let $\mathrm{Top}(X)$ denote the set of topologies on X, which becomes a poset when equipped with the inclusion relation $\subseteq$. $(\mathrm{Top}(X), \subseteq)$ admits top and bottom elements in the form of the discrete and indiscrete topologies. Furthermore, by proposition 2.3.1, $(\mathrm{Top}(X), \subseteq)$ admits arbitrary meets (infima), which automatically implies that $(\mathrm{Top}(X), \subseteq)$ admits arbitrary joins (the supremum of a subset S of $\mathrm{Top}(X)$ will be obtained as the infimum of the upper bounds of S, see [7]). Thus we see that $(\mathrm{Top}(X), \subseteq)$ is a complete lattice.

Proposition 2.3.2. *$(Top(X), \subseteq)$ is a complete lattice.*

(i) Its upper bound is $\mathcal{P}(X)$, its lower bound is $\{\emptyset, X\}$.

(ii) The infimum of a family $(\tau_i)_{i\in I}$ of topologies on X is given by their intersection $\bigcap_{i\in I} \tau_i$.

(iii) The supremum $\bigvee_{i\in I} \tau_i$ of a family $(\tau_i)_{i\in I}$ of topologies on X is the infimum of its upperbounds, i.e.

$$\bigvee_{i\in I} \tau_i = \bigcap\{\tau \in Top(X) | \tau_i \subseteq \tau \text{ for all } i \in I\}.$$

Here the $\bigvee$ notation is warranted because $\bigvee_{i\in I} \tau_i$ will not *coincide with $\bigcup_{i\in I} \tau_i$ in general.*

Definition 2.3.4 (Generated topology)**.** Let ξ be a collection of subsets of X. The **topology generated by** ξ, denoted by $\langle\xi\rangle$, is defined to be the coarsest topology containing ξ, i.e.

$$\langle\xi\rangle = \bigcap\{\tau \in \mathrm{Top}(X) | \xi \subseteq \tau\}.$$

Given a topology τ on X, a collection ξ of subsets of X that generates τ in the above sense is called a **subbasis** of τ.

Proposition 2.3.3. *The topology generated by ξ is the collection of all unions of finite (possibly empty) intersections of elements of ξ.*

Remark. Allowing for empty intersections of elements of ξ is of crucial importance, as X (the empty intersection) belongs to the generated topology.

Definition 2.3.5 (Basis for a given topology)**.** Let τ be a topology on a set X. A **basis** for τ is a subcollection $\beta \subseteq \tau$ such that any open of τ is a union of elements of β (note that the empty union is the empty set). In that case it is easy to see that τ is the topology generated by β.

Proposition 2.3.4. *$\beta \subseteq \tau$ is a basis of τ if and only if for any open U and any point $p \in U$ there exists $B \in \beta$ such that $p \in B \subseteq U$.*

A criterion that determines whether a collection of subsets is the basis of some topology.

Proposition 2.3.5. *Let β be a collection of subsets of X. Then β is the basis of the topology $\langle\beta\rangle$ it generates if and only if*

(i) β covers X, meaning that $X = \bigcup\beta$;

(ii) For any pair $A, B \in \beta$ and any point $p \in A \cap B$ there exists $C \in \beta$ such that $p \in C \subseteq A \cap B$.

2.4 Metrizable spaces and convergence

Definition 2.4.1 (Convergence)**.** Let X be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X **converges** to a point $x \in X$ if $(x_n)_n$ is **eventually** in any neighborhood of x, i.e. if for any open neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_m \in U$ for all $m \geq N$. We then write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 2.4.2 (Metric)**.** A **metric** on a set X is a map $d : X \times X \to \mathbb{R}$ with the following property:

(i) (Positivity) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) (Symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

A **metric space** is a set equipped with a metric.

Definition 2.4.3 (Induced topology)**.** Let (X,d) be a metric space.

(i) For $r > 0$ and $x \in X$, the **open ball of radius r around x** is the set

$$B_r(x) := \{y \in X | d(x,y) < r\}.$$

(ii) For $r \geq 0$ and $x \in X$, the **closed ball of radius r around x** is the set

$$\bar{B}_r(x) := \{y \in X | d(x,y) \leq r\}.$$

(iii) The topology on X induced by d is the topology generated by the open balls, which form a basis for that topology. In other words, a subset of X is open for the induced topology if it contains an open ball around each of its points.

Definition 2.4.4 (Metrizable space)**.** A topology τ on a set X is said to be **metrizable** if it there exists a metric on X that induces τ.

Proposition 2.4.1. *A sequence $(x_n)_n$ in a metric space (X,d) converges to x if and only if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x) < \epsilon$ for all $m \geq N$.*

Part II

Manifolds

Bibliography

[1] Wikipedia contributors. *Axiomatic foundations of topological spaces*. 2024. URL: https://en.wikipedia.org/w/index.php?title=Axiomatic_foundations_of_topological_spaces&oldid=1237268389 (visited on 08/25/2024).

[2] Wikipedia contributors. *Convergence spaces*. 2024. URL: https://en.wikipedia.org/w/index.php?title=Convergence_space&oldid=1240724450 (visited on 08/25/2024).

[3] Wikipedia contributors. *Filters in topology*. 2024. URL: https://en.wikipedia.org/w/index.php?title=Filters_in_topology&oldid=1238941858 (visited on 08/25/2024).

[4] Wikipedia contributors. *Sequential space*. 2024. URL: https://en.wikipedia.org/w/index.php?title=Sequential_space&oldid=1237307585 (visited on 08/25/2024).

[5] John M. Lee. *Introduction to Topological Manifolds*. 2nd ed. Springer, 2010.

[6] James Munkres. *Topology*. 2nd ed. Pearson, 2014.

[7] Maxime Willaert. *Lattices*. 2022-2023.