# MODULAR CURVES AS RIEMANN SURFACES

JINGXIANG MA

Contents



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# 1. Modular Curves As Riemann Surfaces

## 1.1. Introduction.

For a congruence subgroup $\Gamma$ of $SL(2,\mathbb{Z})$, we consider the quotient space $Y(\Gamma) = \Gamma \setminus \mathbb{H}$, this space is not a Riemann Surface in nature, neither it is compact, but there is a standard way to give a Riemann Surface structure to it and also to compactify it into a compact Riemann surface $X(\Gamma)$. This article is about the details about how to give $Y(\Gamma)$ a holomorphic chart, and how to add points and open sets. There are at least two reasons to study this problem:
When $\Gamma = SL(2,\mathbb{Z})$, the quotient space $Y(SL(2,\mathbb{Z}))$ is the moduli space $\mathcal{M}_1$ of genus 1 Riemann surfaces. Using the method in this article we compactify it into the Riemann sphere, but when it comes to higher genus, compactification of $\mathcal{M}_g$ becomes very difficult, so genus 1 case is the start point.
When $\Gamma$ is a general congruence subgroup, it turns to consider the moduli space of **enhanced elliptic curves**, there are more information we can get. In fact, the topology of the compactified spaces $X(\Gamma)$ will be used to determine the dimension of certain corresponding spaces of modular forms, which is also an interesting topic.
The reference is the first two chapters of [1].

## 1.2. Congruence Subgroups.

**Definition 1.1.** *Let $N$ a positive integer.*
*The **principal congruence subgroup of level N** is*

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, modN\}$$

*that is, the kernel of the canonical map $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}_N)$.*
*A subgroup $\Gamma$ of $SL(2,\mathbb{Z})$ is a **congruence subgroup** if $\Gamma(N) \subset \Gamma$ for some positive integer N, and in this case $\Gamma$ is a congruence subgroup of level N.*

**Proposition 1.2.**

$$[SL(2,\mathbb{Z}) : \Gamma(N)] < \infty$$

*Proof.* Consider the group homomorphism:

$$\pi : SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z}), A \mapsto [A]$$

then it's easy to see $\Gamma(N) = Ker\pi$, so

$$[SL(2,\mathbb{Z}) : \Gamma(N)] = |SL(2,\mathbb{Z}/N\mathbb{Z})| < \infty$$

□

*Remark.* In fact $|SL(2,\mathbb{Z}/N\mathbb{Z})| = N^3\Pi_{p|N}(1 - \frac{1}{p^2})$, but we don't need this number in this topic.

**Corollary 1.3.** *All congruence subgroup are of finite index.*

**Definition 1.4.** *For a congruence group $\Gamma$, we denote the quotient space $\Gamma \setminus SL(2,\mathbb{Z})$ by $Y(\Gamma)$.*

1.3. **Fundamental Domain.**

**Theorem 1.5.** *$SL(2,\mathbb{Z})$ is generated by*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*$S(\tau) = -1/\tau, T(\tau) = \tau + 1$,*
*and the fundamental domain of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is*

$$D = \{z \in \mathbb{H}, |Re(z)| \leqslant \frac{1}{2}, |z| \geqslant 1\}$$

*Proof.* We denote the subgroup of $SL(2,\mathbb{Z})$ by $\Gamma$First, to show:$\Gamma(D) = \mathbb{H}$: $z \in \mathbb{H}$Suppose $A \in SL(2,\mathbb{Z})$, then:

$$ImA(z) = \frac{1}{|cz+d|^2} Imz$$

Note that, when $max\{|c|,|d|\} \to \infty, ImA(z) \to 0$, so $\{ImA(z)|A \in SL(2,\mathbb{Z})\}$ has maximum, and for $\gamma \in \Gamma$ we also get a maximum, Suppose $Im\Gamma(w)$ is maximal, we may assume $|Rew| \leqslant 1/2$ by translation, so $|w| \geqslant 1$ i.e $w \in D$, if $|w| < 1$, set $w = re^{i\theta}$, then $S(w) = \frac{1}{r}e^{i\pi - \theta}$, note that:

$$Im(S(w)) = \frac{1}{r}sin(\pi - \theta) = \frac{1}{r}sin\theta > rsin\theta = Imw$$

which contradicts to choice of w, so $\Gamma(D) = \mathbb{H}$
Then we can show, if $z, w \in D, A \in SL(2,\mathbb{Z}), Az = w$, then $A \in \Gamma$ and what's more we claim that $z, w \in \partial D$ or $z = w \in D \setminus \partial D$
Firstly, we may assume $Imw \geqslant Imz, c > 0$, and since $ImA(z) = \frac{1}{|cz+d|^2}Imz$, we get $|cz+d| \leqslant 1$ note also that $z \in D$ impiles $Imz \geqslant \sqrt{3}/2$, hence:

$$1 \geqslant |cz+d| \geqslant Im(cz+d) = cImz \geqslant \sqrt{3}/2$$

Since $c \in \mathbb{Z}_{\geqslant 0}$, $c = 0$ or1;
If $c = 0$, then $detA = ad = 1$ and $a = d = \pm 1$, so we may assume $a = d = 1$ in this case A is a translation and $z, w$ are at two sides of D;
If $c = 1$, then:

$$1 \geqslant |cz+d| = |z+d| = \sqrt{(Rez+d)^2 + Imz^2} \geqslant \sqrt{(Rez+d)^2 + 3/4}$$

note that $|Rez| \leqslant 1/2$, $d = 0, -1, 1$
Suppose $d = 0$then $b = -1, c = 1$, $A = T^aS$, note that $|z| = |cz+d| \leqslant 1$, so $|z| = 1$, in this case z is in the blew boundary of D, check that $S(z) \in \partial D a = 0, \pm 1$
Suppose $d = \pm 1$, then $|cz+d| = |z \pm 1| \leqslant 1$, note that $z \in D$, only two possible cases left:
(1),$d = 1, z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \rho$, (2)$d = -1, z = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \rho + 1$,
(1):$b = a - 1$, and $A\rho = a - 1/(\rho + 1) = \rho + a$ so $a = 0, A(z) = z, A = ST$

or $a=1, A(z)=\rho+1, A=TST$;
(2):$b=-a-1$, and $A(z)=a-1/\rho=a+z$, in this case $a=0, A(z)=z=\rho+1, A=ST^{-1}$ or $a=-1, A(z)=\rho, A=T^{-1}ST^{-1}$
So we have shown that D is a fundamentaldomain, it leaves to show $\Gamma=SL(2,\mathbb{Z})$: take $z_0 \in inner(D)$, then $\forall B \in SL(2,\mathbb{Z}), \exists A \in \Gamma$ such that $A(B(z_0)) \in D$ and now that $z_0 \in inner(D)$, $AB$ must be $\pm 1$ note that $S^2=-1 \in \Gamma$, so $B \in \Gamma$. ☐

*Remark.*

$$\Gamma_\tau = \begin{cases} \{\pm I\} < S > & \tau = i \\ \{\pm I\} < ST > & \tau = \rho \\ \{\pm I\} & otherwise \end{cases}$$

$S^4=(ST)^6=I_2$.

### 1.4. $Y(\Gamma)$ is Hausdorff.

**Proposition 1.6.** *$\forall \tau_1, \tau_2$, $\exists U_1 \supset \tau_1, U_2 \supset \tau_2$ with compact closure, such that:*
*$\forall \gamma \in SL(2,\mathbb{Z})$, if $\gamma(U_1) \cap U_2 \neq \emptyset$, then $\gamma(\tau_1)=\tau_2$.*

*Proof.* We first choose any open nbhd $U_1', U_2'$ with bounded closure.
Step 1, We can show for only finite many $\gamma \in SL(2,\mathbb{Z}), \gamma(U_1') \cap U_2' \neq \emptyset$. choose $\mathbb{H} \supset \overline{U_i}' \supset U_i' \ni \tau_i$, then by

$$Im(\gamma(z)) = \frac{1}{|cz+d|^2} Imz$$

since $\{Imz, Rez, z \in U_1'\}$ is bounded, for all but finite many $(c,d)$ s.t $gcd(c,d)=1$

$$sup\{Im(\gamma(z)), z \in U_1'\} < inf\{Im(z), \tau \in U_2'\}$$

for any of these (c,d), those $\gamma \in SL(2,\mathbb{Z})$ of the form

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

is in an orbit

$$\gamma_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so $\gamma_k(U_1') = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + k$, so for fixed $(c,d)$, there are also finite $\gamma SL(2,\mathbb{Z})$ of the form $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ s.t $\gamma(U_1') \cap U_2 \neq \emptyset$.
Step 2, construction of $U_i$.
Set

$$F = \{\gamma \in SL(2,\mathbb{Z}), \gamma(U_1') \cap U_2' \neq \emptyset, \gamma(\tau_1) = \tau_2\}$$

for any $\gamma \in F$, choose $\gamma(\tau_1) \in U_{1,\gamma}, \tau_2 \in U_{2,\gamma}, U_{1,\gamma} \cap U_{2,\gamma} = \emptyset$, now, let:

$$U_1 = U_1' \cap \{\cap_{\gamma \in F} \gamma^{-1}(U_{1,\gamma})\}$$

$$U_2 = U_2' \cap \{\cap_{\gamma \in F} U_{2,\gamma}\}$$

we show $U_i$ satisfies the property.
if $\gamma(U_1) \cap U_2 \neq \emptyset$ then $\gamma(U_1') \cap U_2' \neq \emptyset$, if also $\gamma(\tau_1) \neq \tau_2$, then $\gamma \in F$, and $\gamma^{-1}(U_{1,\gamma}) \supset U_1, U_{2,\gamma} \supset U_2$.
Consequently, $U_{1,\gamma} \cap U_{2,\gamma} \supset \gamma(U_1) \cap U_2 \neq \emptyset$, which is a contradiction. □

**Corollary 1.7.** *$Y(\Gamma)$ is hausdorff*

*Proof.* note that $\pi : \mathbb{H} \to Y(\Gamma) = \Gamma \setminus \mathbb{H}$ is an open mapping. □

## 1.5. **Elliptic Points.**

**Definition 1.8.** *For $\Gamma$, a cong subgroup of $SL(2,\mathbb{Z})$, for $\tau \in \mathbb{H}$, we denote the isotropy group by $\Gamma_\tau$, then $\tau$ is called elliptic, iff $\pm I$ is a proper subgroup of $\{\pm I\}\Gamma_\tau$. The image of elliptic points are also called elliptic.*

**Proposition 1.9.** *$\tau$ elliptic implies $\Gamma_\tau$ finite cyclic, and there are finite many elliptic points.*

*Proof.*

$$SL(2,\mathbb{Z}) = \cup_{i=1}^{d} \Gamma\gamma_j$$

so

$$\mathbb{H} = SL(2,\mathbb{Z})D = \cup_{i=1}^{d} \Gamma\gamma_j D = \Gamma(\cup_{i=1}^{d} \gamma_j D)$$

since $\Gamma_\tau \leqslant SL(2,\mathbb{Z})_\tau$, subset of finite cyclic group is also finite cyclic, then all elliptic points are a subset of

$$E_\tau = \{\Gamma\gamma_j(i), \Gamma\gamma_j(\mu_3), 1 \leqslant j \leqslant d\}$$

which is finite. □

## 1.6. $Y(\Gamma)$ **as Riemann Surface.**

We have shown that $Y(\Gamma)$ is C2 and Hausdorff, now we give it holomorphic charts.

**Definition 1.10.** *(period)*
*For $\tau \in \mathbb{H}$*

$$h_\tau = |\{\pm I\}\Gamma_\tau : \{\pm I\}|$$

*Remark.* $h_\tau$ only depends on $\Gamma\tau$, so is well defined on pts on $Y(\Gamma)$.

Now, given $\tau \in \mathbb{H}$, we choose the chart near $\pi(\tau)$ as following steps:
Step1.
Let

$$\delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix}$$

check that $\delta_\tau(\tau) = 0, \delta_\tau(\overline{\tau}) = \infty$.
The group

$$\delta_\tau(\{\pm I\}\Gamma_\tau/\{\pm I\})\delta_\tau^{-1} = (\delta_\tau\{\pm I\}\Gamma\delta_\tau^{-1})_0/\{\pm I\}$$

is cyclic of order $h_\tau$, we choose a generator $\gamma_\tau$ of $(\delta_\tau\{\pm I\}\Gamma\delta_\tau^{-1})_0 \leqslant GL(2,\mathbb{C})$, since $\gamma_\tau$ is a fractional linear trans and also $\gamma_\tau$ fix 0 and $\infty$, $\gamma_\tau$ is must of

the form $\pm\begin{pmatrix}\mu_h & 0\\ 0 & 1\end{pmatrix}$ whose underlying action is $\gamma_\tau(z)=\mu_h z$, and $\mu_h$ is a primary $h_\tau$-th unit root.
Step2. Now, $\forall\tau$, take the open nbhd U as in 1.6, and define:

$$\psi: U\to\mathbb{C}, p\mapsto(\delta_\tau(p))^{h_\tau}$$

note that by choice of U, for any $\tau_1,\tau_2\in U$, if $\tau_1\in\Gamma\tau_2$, which means $U\cap\gamma U\neq\emptyset$, then $\gamma\in\Gamma_\tau$, so

$$\pi(\tau_1)=\pi(\tau_2)\Leftrightarrow$$
$$\tau_1\in\Gamma\tau_2\Leftrightarrow\tau_1\in\Gamma_\tau\tau_2$$
$$\Leftrightarrow\delta\tau_1\in(\delta\Gamma_\tau\delta^{-1})(\delta\tau_2)\Leftrightarrow\delta\tau_1=\mu_h^d(\delta(\tau_2)),\exists d$$
$$\Leftrightarrow\psi(\tau_1)=\psi(\tau_2)$$

so we have shown that:

**Proposition 1.11.** *$\psi$ induce a homeomorphism $\varphi:\pi(U)\to V=\psi(U)\subset V$ such that the following diagram commutes:*

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ \pi(U) \arrow[rr, "\varphi"] & & V \end{tikzcd}$$

*Proof.* Note that $\psi$ is open mapping and $\varphi$ is onto. □

*Remark.* From

$$\delta(\tau_1)\in(\delta\Gamma_\tau\delta^{-1})(\delta(\tau_2))\Leftrightarrow\delta(\tau_1)=\mu_h^d(\delta(\tau_2)),\exists d$$

we see that expect for $\pi(\tau)$, there are exactly $h_\tau$ points on U over a given point in $\pi(U)$, so the definition of period is more natural.

It leaves to show all charts are compatible:

**Theorem 1.12.** *The charts given by 1.11 makes $Y(\Gamma)$ a Riemann surface.*

*Proof.* Suppose $\pi(U_1)\cap\pi(U_2)\neq\emptyset$, $U_1,U_2$ are two charts, we have the following maps and sets:

$$\begin{tikzcd} & \pi(U_1)\cap\pi(U_2) \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ V_{1,2} \arrow[rr, "\varphi_{2,1}"] & & V_{2,1} \end{tikzcd}$$

we want show that $\varphi_{2,1}$ is holomorphic:
For $x\in\pi(U_1)\cap\pi(U_2)$, suppose $x=\pi(\tau_i),\tau_i\in U_i,\tau_2=\gamma\tau_1$, let $U_{1,2}=U_1\cap\gamma^{-1}(U_2)$, we consider the restriction of $\varphi_{2,1}$ to $\pi(U_{1,2})$.
We denote the resp. coordinates by $\delta_i^{h_i}$, firstly, note that holomorphic is local property, and $\varphi_{2,1}=\varphi_{2,3}\varphi_{3,1}$ in smaller nbhd when $x\in\pi(U_3)$.

So we may assume $\delta_1 = \delta_{\tau_1}$, i.e $\varphi_1(x) = 0$.
Now $\forall x' = \pi(\tau') \in \pi(U_{1,2})$, $\varphi_{2,1}(\varphi_1 x') = \varphi_2(x')$, we compute using two coordinate maps:
On the one side,

$$q = \varphi_1(x') = (\delta_{\tau_1}(\tau'))^{h_{\tau_1}}$$

On the other side, for $U_2$, suppose $\varphi_2(\widetilde{\tau_2}) = 0$, $h_2$ the period, then:

$$\varphi_2(x') = (\delta_2(\gamma(\tau')))^{h_2} = [(\delta_2 \cdot \gamma \cdot \delta_{\tau_1}^{-1})(\delta_{\tau_1}(\tau'))]^{h_2}$$

$$[(\delta_2 \cdot \gamma \cdot \delta_{\tau_1}^{-1})q^{\frac{1}{h_1}}]^{h_2}$$

So to show $\varphi_{2,1}$ is holomorphic, it suffices to consider the case $\tau_1$ is elliptic. then $\tau_2 = \gamma(\tau_1) \in U_2$ is also elliptic with the same period.
But $U_2$ contains at most 1 elliptic pt by choice, so $\tau_2 = \widetilde{\tau_2}$ and hence $h_2 = h_1$, now that:

$$0 \xrightarrow{\delta_1^{-1}} \tau_1 \xrightarrow{\gamma} \tau_2 \xrightarrow{\delta_2} 0$$

$$\infty \xrightarrow{\delta_1^{-1}} \overline{\tau_1} \xrightarrow{\gamma} \overline{\tau_2} \xrightarrow{\delta_2} \infty$$

so $\delta_2\gamma\delta_1^{-1}$ much be of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{C}^*$, so

$$\varphi_{2,1}(q) = [\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} q^{\frac{1}{h}}]^h = (\frac{\alpha}{\beta})^h q$$

is holomorphic. □

1.7. **Adding Cusps.** The idea comes from compactification of $SL(2,\mathbb{Z})\mathbb{H}$, using the fractional linear transformation

$$f(z) = \frac{z-i}{z+i} : \mathbb{H} \to D^2$$

the fundamental domain is mapped to a triangle with one vertex $f(\infty) = 1$ missing, by adding this point, $Y(1)$ is compactified to $X(1) = S^2$.

**Definition 1.13.** *(cusp point)*
*A cusp point is a* $\Gamma$*-equivalent class of* $\mathbb{Q} \cup \{\infty\} = SL(2,\mathbb{Z})\{\infty\}$

*Remark.* We set

$$SL(2,\mathbb{Z}) = \cup_{j=1}^d \Gamma\gamma_j$$

then $\mathbb{Q} \cup \{\infty\} = SL(2,\mathbb{Z})\{\infty\} = \cup_{j=1}^d \Gamma\{\gamma_j\infty\}$, so there are only finite many cusps.

**Definition 1.14.**

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$$

$$X(\Gamma) := \Gamma \setminus \mathbb{H}^*$$

1.8. **Adding Open Sets.**

**Definition 1.15.** *$\forall M > 0$, we set*

$$\mathcal{N}_M = \{\tau \in \mathbb{H}, Im(\tau) > M\}$$

*We add $\alpha(\mathcal{N}_M \cup \{\infty\}), M > 0, \alpha \in SL(2, \mathbb{Z})$ as open sets of $\mathbb{H}^*$ and take the quotient topology of $X(\Gamma)$.*

Under this topology, we show:

**Proposition 1.16.** *$X(\Gamma)$ is Hausdorff.*

*Proof.* By

$$Im(\gamma\tau) = \frac{1}{|cz + d|^2} Im\tau$$

we get

$$Im(\gamma\tau) \leqslant max\{Im\tau, \frac{1}{Im\tau}\} \tag{1.1}$$

For $x_1 = \pi(s_1) \neq x_2 = \pi(s_2)$, we show they have disjoint nbhs:
(1)$s_1, s_2 \in \mathbb{H}$, this have been proved;
(2)$s_1 \in Q \cup \{\infty\}, s_2 \in \mathbb{H}$, we first take a nbhd $U_2$ of $s_2$ with closure compact, then there exists M such that

$$sup\{Imz|\tau \in U\} < M, inf\{Imz|\tau \in U\} > 1/M$$

let $s_1 = \alpha\infty$, then by the inequality 1.1, $\pi(\alpha(\mathcal{N}_M \cup \infty))$ and $\pi(U)$ are disjoint nbhd.
(3)$s_1, s_2 \in Q \cup \{\infty\}$, set $s_1 = \alpha_1(\infty), s_2 = \alpha_2(\infty)$, consider

$$\pi(\alpha_1(\mathcal{N}_2 \cup \infty)), \pi(\alpha_2(\mathcal{N}_2 \cup \infty))$$

they are in fact disjoint in $X(\Gamma)$, since if $\gamma\alpha(\tau_1)\alpha_2(\tau_2), \tau_1, \tau_2 \in \mathcal{N}_2$, then $\alpha_2^{-1}\gamma\alpha_1(\tau_1) = \tau_2$, but $\tau_1, \tau_2$ are all in nbhd of $\infty$, $\alpha_2^{-1}\gamma\alpha_1$ must be of the form $\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, that is, a translation, and also we get $\alpha_2^{-1}\gamma\alpha_1(\infty) = \infty$, this impies $\gamma(s_1) = s_2$, which is impossible since $s_i$ are in different $\Gamma$-orbits. □

**Proposition 1.17.** *$X(\Gamma)$ is connectness and compact.*

*Proof.* Since $\mathbb{H}^*$ is connect, its image is also connect.
note that $D^* = \mathbb{D} \cup \infty$ is compact, then by finite union,

$$\mathbb{H}^* = SL(2, \mathbb{Z})D^* = \cup_{j=1}^{d} \Gamma_j(\gamma_j D^*)$$

is also compact, so its image is also compact. □

1.9. $X(\Gamma)$ **as Compact Riemann Surface.**

**Proposition 1.18.**

$$SL(2,\mathbb{Z})_\infty = \{\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathbb{Z}\}$$

**Definition 1.19.** *(width)*
*For* $s \in \mathbb{Q} \cup \{\infty\}$*, choose* $\delta_s \in SL(2,\mathbb{Z}), s.t \delta_s(s) = \infty$*, we define the width of* $s$ *as:*

$$h_s = h_{s,\Gamma} = |SL(2,\mathbb{Z})_\infty : (\delta\{\pm I\}\Gamma\delta^{-1})_\infty|$$

**Proposition 1.20.** $h_s \leqslant \infty$ *and is independent of* $\delta$*, in fact,*

$$h_s = |SL(2,\mathbb{Z})_s : \{\pm I\}\Gamma_s|$$

*Proof.* since

$$(\delta\{\pm I\}\Gamma\delta^{-1})_\infty = \delta\{\pm I\}\Gamma_s\delta^{-1}$$

$\delta_s$ is independent of $\delta$, and it leaves to show the finiteness.
Suppose $SL(2,\mathbb{Z}) = \cup_{j=1}^{d}\Gamma\gamma_j$, it's easy to check that $h_\infty$ is finite:

$$SL(2,\mathbb{Z})_\infty = \cup_{\gamma_j\infty=\gamma_j'\infty,\exists\gamma'\in\Gamma}\Gamma_\infty(\gamma_j'\gamma_j)$$

so $h_\infty \leqslant d$. □

Now we give charts around cusps.
$s, \delta, h_s$ as before, we set $U = \delta^{-1}(\mathcal{N}_2 \cup \{\infty\})$ and

$$\psi : U \to \mathbb{C}, \tau \mapsto e^{2\pi i \frac{\delta(\tau)}{h}}$$

Note that:

$$\pi(\tau_1) = \pi(\tau_2) \Leftrightarrow$$
$$\tau_1 = \gamma\tau_2, \exists\gamma \Leftrightarrow \delta\tau_1 = (\delta\gamma\delta^{-1})(\delta(\tau_2))$$

note that $\delta\tau_i \in \mathcal{N}_2 \cup \infty$, $\delta\gamma\delta^{-1}$ must be a translation, note also that

$$\delta\gamma\delta^{-1} \in \delta\Gamma\delta^{-1} \cap SL(2,\mathbb{Z})_\infty = (\delta\Gamma\delta^{-1})_\infty \leqslant \pm < \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} >$$

so

$$\pi(\tau_1) = \pi(\tau_2) \Leftrightarrow \delta(\tau_1) = \delta(\tau_2) + mh, h \in \mathbb{Z} \leqslant \psi(\tau_1) = \psi(\tau)$$

so we have shown that:

**Proposition 1.21.** $\psi$ *induce a homeomorphism* $\varphi : \pi(U) \to V = \psi(U) \subset V$ *such that the following diagram commutes:*

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ \pi(U) \arrow[rr, "\varphi"] & & V \end{tikzcd}$$

It leaves to show all charts are compatible:

**Theorem 1.22.** *The charts given by 1.11 and 1.21 makes $X(\Gamma)$ a compact Riemann surface.*

*Proof.* To check the transition maps are holomorphic, we only need to consider when at least one patch is a cusp nbhd.
take $x \in \pi(U_1) \cap \pi(U_2) \neq \emptyset$,
(1)If $U_1 \in \mathbb{H}, \infty \in U_2$, suppose $U_1$ corresponds to $\tau_1, \delta_1 = \delta_{\tau_1}$, and $U_2 = \delta_2^{-1}(\mathcal{N}_2 \cup \infty)$, suppose $x = \pi(q_1) = \pi(q_2), q_i \in U_i$, then:

$$q_2 = \gamma q_1, \exists \gamma$$

we consider the nbhd $U_{1,2} = U_1 \cap \gamma^{-1}(U_2)$ of $q_1$, now for any $x' = \pi(\tau'), \tau' \in U_{1,2}$, the transition is given by:

$$p = \delta_1(\tau')^{h_1} \mapsto e^{2\pi i \frac{\delta_2\gamma(\tau')}{h_2}} = e^{2\pi i \frac{\delta_2\gamma\delta_1^{-1}(\delta_1\tau')}{h_2}} = e^{2\pi i \frac{\delta_2\gamma\delta_1^{-1} p^{\frac{1}{h}}(\tau')}{h_2}}$$

if $h_1 = 1$, it is well defined, if $h_1 > 1$, since $\delta_2\gamma(\tau_1)$ is an elliptic point, and $\mathcal{N}_2$ contains no elliptic point, we deduce that $\tau_1 \notin U_{1,2}$, so

$$0 \notin \varphi_1(\pi(U_{1,2})) \ni p$$

so in this case $p \mapsto p^{\frac{1}{h}}$ is also holomorphic.
(2)If

$$U_i = \delta_i^{-1}(\mathcal{N}_2 \cup \{\infty\}), \delta_i(s_i) = \infty, i = 1, 2$$

then

$$\pi(U_1) \cap \pi(U_2) \neq \emptyset$$

so there exists $\gamma \in \Gamma$, $\delta_2\gamma\delta_1^{-1}(\mathcal{N} \cup \{\infty\}) \cap (\mathcal{N}_2 \cup \{\infty\}) \neq \emptyset$, so $\delta_2\gamma\delta_1^{-1}$ has to be of the form:

$$\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

note that:

$$\gamma(s_1) = \gamma\delta_1^{-1}(\infty) = \pm\delta_2^{-1}(\infty + m) = s_2$$

so $h_1 = h_2 = h$, now we check the transition: $\forall \tau \in U_{1,2}$, the transition is given by:

$$\begin{aligned} p = e^{2\pi i \frac{\delta_1(\tau)}{h}} \mapsto e^{2\pi i \frac{\delta_2\gamma(\tau)}{h}} &= e^{2\pi i \frac{\delta_2\gamma\delta_1^{-1}(\delta_1(\tau))}{h}} \\ &= e^{2\pi i \frac{\delta_1(\tau)+m}{h}} = e^{2\pi i \frac{m}{h}} p \end{aligned}$$

which is obviously holomorphic, so we finish the proof. □

## References

[1] Fred Diamond and Jerry Michael Shurman. *A first course in modular forms*, volume 228. Springer, 2005.