

STABLE BUNDLES OVER RIEMANN SURFACES

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1. INTRODUCTION

We fix a compact Riemann surface Σ .

2. HOLOMORPHIC STRUCTURES AND $U(n)$ CONNECTIONS

2.1. Complex vector bundle over a Riemann surface.

Proposition 2.1. *The equivalent complex vector bundle over Σ is totally determined by its degree and rank, in another word, two vector bundle with the same degree and rank is isomorphic.*

This is the direct corollary of the following two lemmas:

Lemma 2.2. *Two line bundle over Σ are isomorphic iff they have the same degree, and there exists line bundle of any given degree.*

Proof. This is the canonical Abel-Jacobi theorem, note that we can even make it in the category of holomorphic line bundles. \square

Lemma 2.3. *Let $E \rightarrow \Sigma$ be a vector bundle with $n = \text{rank}(E) > \dim \Sigma = m$, then E have a non-vanishing section.*

Proof. We can perturb an arbitrary section s into a non-vanishing one: First, we may assume there are only finite many zeros by the compactness of X ;

Next, locally we may assume near every zero p , we can find local sections $s_1, \dots, s_m, s_{m+1}, \dots, s_n$ such that $\text{im}(s|_U) \subset \text{span}\{s_1, \dots, s_m\}$ by implicit function theorem: locally s is $\mathbb{R}^m \rightarrow \mathbb{R}^n, m < n$;

Secondly, we can perturb s to s_{m+1}, \dots, s_n directions near p , note that this operation doesn't add zeros. \square

Remark. Splitting of short exact sequence doesn't always exist in the category of holomorphic bundles.

As we have shown, because the structure of line bundle over a Riemann surface is quite easy and the dimension of a Riemann surface is 1, there is only one orbit fixing the rank and degree, so we may fix a complex bundle of type (r, d) and consider the classification of complex structures on it, which equivalently gives the classification of holomorphic bundles of type (r, d) .

2.2. Holomorphic structures and $U(n)$ connections. First we review some basic concepts in gauge theory:

In the category of G (principal) bundles, then we may consider the morphisms:

Definition 2.4. *A morphism between two G bundles P_1, P_2 is a bundle map which is G -equivariant, that is:*

$$\forall p \in P_1, g \in G, \varphi_2 f(p_1) = \varphi_2 f(p_1 g), f(pg) = f(p)g$$

Fix a G bundle $P \rightarrow \Sigma$, then we can consider the automorphism group, say $\mathcal{G}(P) = \text{Aut}(P)$.

Proposition 2.5. *Consider the adjoint action of G on \mathfrak{g} , and the associated bundle $\text{Ad}P = P \times_G G$, then*

$$\mathcal{G}(P) = \text{Aut}(P) \cong \Gamma \text{Ad}P$$

Proposition 2.6. *Consider the co-adjoint action of G on \mathfrak{g} , and the associated bundle $\text{ad}P = P \times_G \mathfrak{g}$, we denote the (affine) space of G connections on P by $\mathcal{A}(P)$, then:*

$$\mathcal{A}(P) \cong \Omega^1(\Sigma, \text{ad}P)$$

Definition 2.7. *The action of gauge group $\mathcal{G}(P)$ on the space of G connections is given by (locally, one can check it's global):*

$$u(A) = A - d_A u u^{-1}$$

Now we turn to our topic, as we have shown in the last section, we just need to fix a complex bundle E of type (n, d) and study holomorphic structures on it.

Natrually we may think about the space of holomorphic structures on E , denoted $\mathcal{C}(E)$, which has an affine structure associated with $\Omega^{0,1}(E)$, and the group of bundle automorphism, say $\text{Aut}(E)$, locally determined by a left action of $GL(n, \mathbb{C})$, so the moduli of equivalent holomorphic bundles is $\mathcal{C}/\text{Aut}(E)$.

It turns out it's better to consider the space of $U(n)$ connections since we have Yang Mills functional on this space, which is the gauge theoretical point of view, but the first observation is:

Proposition 2.8. *On E , there is an affine-linear isomorphism between the space of unitary connections, say \mathcal{A} , to the space of holomorphic structures on it, say \mathcal{C} .*

Proof. Given a unitary connection A , the operator $d_A : \Omega^0(E) \rightarrow \Omega^1(E)$ defines $\bar{\partial}_A : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$, which determines a holomorphic structure on E in the sense that a (smooth) section s of E is holomorphic if and only if $\bar{\partial}_A s = 0$.

Conversely, a holomorphic structure on E defines an operator $D'' : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$. We first fix a hermitian structure on E and then by checking locally, we find an unique unitary connection $d_A : \Omega^0(E) \rightarrow \Omega^1(E)$ such that $d_A^{0,1} = D''$. \square

Remark. In this sence, we can view $\text{Aut}(E)$ as the complexification $\mathcal{G}^{\mathbb{C}}$ of \mathcal{G} , and extend the gauge group action of \mathcal{G} to $\mathcal{G}^{\mathbb{C}}$ (See [2], P2):

$$g(A) = A - (\bar{\partial}_A g)g^{-1} + ((\bar{\partial}_A g)g^{-1})^*, g \in \mathcal{G}^{\mathbb{C}}, A \in \mathcal{A}$$

Corollary 2.9. *The space of equivalent class of holomorphic bundle of type (n, d) , is given by the orbit space:*

$$\mathcal{A}/\mathcal{G}^{\mathbb{C}}$$

3. STABLE BUNDLES AND HARDER-NARASIMHAN THEOREM

3.1. (Semi)stable bundle and Harder-Narasimhan filtration.

Theorem 3.1. *(Harder-Narasimhan)*

Any holomorphic bundle admits a canonical Harder-Narasimhan filtration:

$$0 = F_0 \subset F_1 \subset F_2 \cdots \subset F_r = E$$

with $D_i = F_i/F_{i-1}$ semi-stable and $\mu(D_1) > \mu(D_2) > \cdots \mu(D_r)$.

Remark. Fix the Rie surface and fix type (n, d) , then if n, d are coprime, semistable and stable are equivalent.

Proposition 3.2. *(Seshadri P102)*

stable \Rightarrow simple ($\text{End}(E) = \mathbb{C}^$) \Rightarrow indecomposable.*

semistable doesn't imply simple or indecomposable.

Remark. A polystable vector bundle over a smooth curve is direct sum of stable bundles with the same slope.

3.2. $\Gamma_{\mathbb{R}}$ and Yang Mills connection. Let Γ be the free group generated by the generators of $\pi_1(\Sigma)$, denoted $A_1, B_1, \dots, A_g, B_g$. We use an element J which commutes with A_i, B_i to get a center expansion of $\pi_1(M)$, i.e:

$$1 \rightarrow \mathbb{Z}J \rightarrow \Gamma \rightarrow \pi_1(M) \rightarrow 1$$

we may extending Γ to $\Gamma_{\mathbb{R}}$ by extending the centre to \mathbb{R} in the sense that:

$$1 \rightarrow \mathbb{R}J \rightarrow \Gamma_{\mathbb{R}} \rightarrow \pi_1(M) \rightarrow 1$$

To understand $\Gamma_{\mathbb{R}}$ we need to introduce the concept of path group. For the following discussions we refer to [3]

Definition 3.3. *(path group) We denote by $\Phi_0(M)$ the subset of $\Phi(M)$ contains of contractible loops and by $\Phi_{\omega}(M)$ the subset of $\Phi_0(M)$ with zero area in the interior.*

Proposition 3.4. *For $g \geq 1$, we get $\Gamma_{\mathbb{R}} \cong \Phi(M)/\Phi_{\omega}(M)$*

The relationship between path group and connections are related by holonomy representation:

Proposition 3.5. *Fixing a lie group K . There is a one-to-one correspondence between the equivalent classes of a pair $(K \rightarrow P \rightarrow M, \mathcal{A})$ of K -bundle and a connection on it, and the conjugate classes of group homomorphism: $\Phi(M) \rightarrow K$.*

Finally we relate the Yang Mills connections to group homomorphism through similar perspective:

Proposition 3.6. (*[1], P39*)

There is a one-to-one correspondence between the equivalent classes of a pair $(U(n) \rightarrow P \rightarrow M, \mathcal{A})$ of $U(n)$ -bundle and a Yang Mills connection on it, and the conjugate classes of group homomorphism: $\Phi(M)/\Phi_\omega(M) \rightarrow U(n)$.

3.3. Harder Narasimhan theorem.

Theorem 3.7. (*Narasimhan & Seshadri*)

A holomorphic bundles of rank n is stable if and only if it arises from an irreducible representation $\rho : \Gamma_{\mathbb{R}} \rightarrow U(n)$, moreover isomorphic bundles correspond to equivalent irreducible representations.

Remark. In Narasimhan & Seshadri's original paper (Ref1,2), the statement is a little different, here we use Atiyah-Bott's statement.

In the language of gauge theory, we can say

Corollary 3.8. (*[1], P49*)

Let $\mathcal{N} \subset \mathcal{A}$ denote the set of connections giving the minimum for the Yang Mills functional, they are \mathcal{G} equivalent to those representations $\rho : \Gamma_{\mathbb{R}}$ with $\rho(\mathbb{R})$ central. Let \mathcal{N}^s be those given by irreducible representations (irreducible implies central, See Ref1, P43), then under identification of \mathcal{A} with \mathcal{C} , we have $\mathcal{N}^s \subset \mathcal{C}^s$ and the induced map of quotient spaces:

$$\mathcal{N} // \mathcal{G} = \mathcal{N}^s // \mathcal{G} \rightarrow \mathcal{C}^s // \mathcal{G}^{\mathbb{C}}$$

is a homeomorphism, in another word, the equivalent class of stable bundle of type (n, d) is $\mathcal{N}^s // \mathcal{G}$.

4. AN ANALOGUE TO KEMPF NESS THEOREM

4.1. The Kempf Ness theorem.

Theorem 4.1. (*Kempf-Ness theorem*)

Let $G = K^{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ and suppose its maximal compact subgroup K is connected and acts symplectically on X (where the restriction of the Fubini Study form on \mathbb{P}^n is used to give X its symplectic structure). Let $\mu : X \rightarrow \mathfrak{K}^$ denote the associated moment map; then:*

i) $G\mu^{-1}(0) = X^{ps}$.

ii) If $x \in X$ is polystable, then its orbit $G \cdot x$ meets $\mu^{-1}(0)$ in a single K -orbit.

iii) $x \in X$ is semistable if and only if its orbit closure $\overline{G \cdot x}$ meets $\mu^{-1}(0)$.

iv) The inclusion $\mu^{-1}(0) \subset X^{ss}$ induce a homeomorphism $\mu^{-1}(0)/K \rightarrow X//G$.

4.2. Infinite dimensional Kempf Ness theorem. From the last subsection we see the moduli space of stable bundle is given by: $\mathcal{N} // \mathcal{G}$, we can explain the isomorphism

$$\mathcal{N} // \mathcal{G} = \mathcal{N}^s / \mathcal{G} \rightarrow \mathcal{C}^s / \mathcal{G}^{\mathbb{C}}$$

as a infinitely version of Kempf Ness theorem by understanding the curvature as a moment map:

Giving a principal bundle $P \rightarrow M$, consider the affine space of connections $\mathcal{A} = \Omega^1(M, ad(P))$:

Using an invariant inner product on \mathfrak{g} we can construct an skew product on \mathcal{A} , and the compatible complex structure is given by Hodge star $*$ (see Ref1, 94):

$$(\theta, \varphi) = \int_M (\theta \wedge * \varphi)$$

The gauge group $\mathcal{G} = \Gamma(AdP) = Aut(P)$ acts on it, with the Lie algebra $\Gamma(adP) = \Omega^0(M, adP)$.

Note that $\Omega^2(M, adP)$ is dual to $\Omega^0(M, adP)$, $\forall \varphi \in \Omega^2(M, adP)$ we may consider φ as an element of Lie algebra, and we define:

$$F_{\varphi}(A) = (F_A, \varphi), \forall A \in \mathcal{A}, \forall \varphi \in \Omega^2(M, adP)$$

We want to show F_{φ} is a Hamiltonian function, in fact we can show, for $\forall \psi \in \Omega^1(M, ad(P))$,

$$(df_{\varphi}, \psi) = \int (d_A \varphi) \wedge \psi = \ell_{X_{\varphi}} \omega$$

(Since \mathcal{A} is affine, the vector field determined by φ is φ itself),

So we have show that: the action of \mathcal{G} is hamiltonian and the hamiltonian function is F_{φ} ,

By definition, the moment map μ is determined by $(\mu(A), \varphi) = F_{\varphi}(A)$, but the definition of F_{φ} is exactly $F_{\varphi}(A) = (F_A, \varphi)$, so:

The moment map is given by $A \rightarrow F_A$!!!

Remember (3.8) the moduli space is given by $\mathcal{N} // \mathcal{G}$, and \mathcal{N} contains those connections that minimise the moment map (curvature), we take this minimum as c , then $\mathcal{N} = F_A^{-1}(c)$, and the homeomorphism:

$$\mathcal{N} // \mathcal{G} = \mathcal{N}^s / \mathcal{G} \rightarrow \mathcal{C}^s / \mathcal{G}^{\mathbb{C}}$$

can be translate into :

$$F_A^{-1}(c) // \mathcal{G} = \mathcal{N}^s / \mathcal{G} \cong \mathcal{C}^s / \mathcal{G}^{\mathbb{C}} = \mathcal{C} // \mathcal{G}^{\mathbb{C}}$$

4.3. Donaldson's proof of Harder- Narasimhan theorem.

Theorem 4.2. (Donaldson, [2])

*An indecomposable holomorphic bundle E over a Riemann surface X is stable if and only if there is a unitary connection on E having constant central curvature $*F = 2\pi i \mu(E)$. Such a connection is unique up to isomorphism.*

Proof. (Sketch) Given a Hermitian vector bundle E , we have a surjective map from the space of unitary connections, say \mathcal{A} , to the space of holomorphic map.

The gauge group G of unitary automorphism acts on \mathcal{A} , and the action extends to the complexification $\mathcal{A} \subset \mathcal{A}^{\mathbb{C}}$, two unitary bundles gives the same holomorphic structure iff they are in the same \mathcal{A} orbit.

Given a holomorphic bundle \mathcal{E} we write $\mathcal{O}(\mathcal{E})$ for the orbit of connections of the appropriate C^∞ bundle.

Consider the functional

$$J(A) = N\left(\frac{*F}{2\pi i} + \mu \cdot 1\right)$$

where $N : \Gamma(\Omega^0(\text{End}(E))) \rightarrow \mathbb{R}$ is defined as:

$$N(s) = \left(\int_X v(s)^2\right)^{\frac{1}{2}}$$

with the trace norm: $v(M) = \text{Tr}(M * M)^{1/2} = \sum |\lambda_i|$, F is the curvature and $\mu = \mu(E)$.

The existence of $\inf_J|_{\mathcal{O}(\mathcal{E})}$ in $\mathcal{O}(\mathcal{E})$ is relative to the stability condition. \square

Remark. Such corresponding theorem also appears in at least two other cases, one is the Hitchin-Kobayashi corresponding, the other is Yau-Tian-Donaldson theorem, in this philosophy, one can always find a stability condition corresponds to the existence for a good metric(connection).

REFERENCES

- [1] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983.
- [2] Simon K Donaldson. A new proof of a theorem of narasimhan and seshadri. *Journal of Differential Geometry*, 18(2):269–277, 1983.
- [3] Kent Morrison. Yang-mills connections on surfaces and representations of the path group. *Proceedings of the American Mathematical Society*, 112(4):1101–1106, 1991.