ON THE COMPLEX STRUCTURE OF S^n

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1. Introduction

It is well known that S^2 admits a complex (manifold) structure, and a natural question is: when does a smooth manifolds admit a complex structure?

To answer this question, we should know whether there are complex structures on common manifolds. If M is just a linear space, it's nothing to talk about, because it just depends on the real dimension: V^n admits a complex structure if and only if n is even. So we naturally turn to S^n. We can immediately say that n must be even, but what' s more we can say?

It seems that searching a more algebraic structure-almost complex structure is a suitable direction. An almost complex structure is a continuous complex structure on tangent space, so a complex structure is also an almost one and on linear space these two structures are the same. By Newlander and Nirenberg's theorem [7], if an almost structure has some integrability, then it can be induced by a complex structure, so it seems easier: to find almost complex structures and check their integrability.

In 1953 [2] , Borel and Serre finished the proof that S^{2n} admits an almost complex structure if and only if n = 1 or 3, which is maybe the most complete and necessary step. According to Atiyah's conclusion [1], many mathematicians also contributed to this part, such as Ehresmann, Hopf, Kirchhoff, Eckmann-Frohlicher and Ehresmann-Liberman. The tools for the non-existence part is related to topological tools like characteristic classes and K-theory, but the construction of the almost complex structure is more technical, which is related to a division algebra structure: octonions, a generalization of complex numbers with real dimension 8. Using this special almost complex structure, S^6 is seen as the unit sphere of the image part of octoions, and the almost complex structure is given by operation in octoions. In some sense the appearance of octoions is not strange, because an almost complex structure is an algebraic one, and there are only four kinds of finite dimension division algebra over real number field: real number, complex number, quaternions and octonions (Hurwitz's theorem, see [5] or [4]P143). Bad news is, this almost complex structure is non-integrable, so the story doesn't end!

Up to now, it is still an open problem whether S^6 admits a complex structure and most mathematicians think it doesn't exist. In 2016, Atiyah posted a proof [1], but is not generally admitted to be right. Another direction to solve this problem is to consider the compatibility of complex structure on S^6 with other structures. In 1987, Lebrun [6] showed that there is no complex structure compatible with the standard Euclidean metric of S^6 and the latest result, by S.S. Chern, tell us that there aren't any $\omega-$ compatible complex structures on S^6, using some further study on octonions, tools in Lie groups $Spin(7)$ and G_2 as subgroups of $End(\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O})$, where ω is also just a special 2-form, which will be introduced in the last part. This part is mainly from Bryant's review [3].

2. Almost complex structure of S^{2n}

2.1. Nonexistence of almost complex structure on $S^{2n}(n \neq 1, 3)$.

The idea is to use the results from characteristic classes.

We consider S^n as the unit sphere in the Euclidean space $\mathbb{E}^{n+1}$, and we denote the normal bundle NS^n is trivial line bundle on S^n, and also $TS^n \oplus NS^n \oplus \epsilon^{n+1}$ is trivial bundle. By Whitney product formula it's easy to get:

Proposition 2.1. $p(TS^n) = 1$.

The idea is to assume S^n admits an almost complex structure and then deduce a contradiction. The first step is to rule out $S^{4k}(k \geqslant 1)$:

Proposition 2.2. *$S^{4k}(k \geqslant 1)$ doesn't admit an almost complex structure.*

Proof. Suppose S^{4k} admit an almost complex structure J, then J induced a decomposition of eigenspace $TS^{4k} \otimes \mathbb{C} = E \oplus \overline{E}$, which we considered as direct sum of complex vector bundle. Now use the Whitney product formula we get:

$c(TS^{4k} \otimes \mathbb{C}) = c(E \oplus \overline{E}) = (1 + c_{2k}(E))(1 + c_{2k}(E)) = 1 + 2c_{2k}(E) = 1 + 2e(S^{4k})$

On the other hand by definition of Pontryagin classes,

$$c(TS^{4k} \otimes \mathbb{C}) = 1 + (-1)^k p_k(c(TS^{4k}))$$

so we get $(-1)^k p_k(TS^{4k}) = 2e(S^{4k})$. This is a contradiction since we already know $p_k(TS^{4k})$ while $e(S^{4k})$ is not: note that $< e(S^{4k}), [S^{4k}] >= \chi(S^{4k}) = 2$. □

Now assume there is an almost complex structure, then it induces a complex vector bundle structure on TS^{2n}, so we can define the Chern character of S^{2n}. The other cases are ruled out using Chern characters via the following fact:

Fact. *$ch(S^{2n})$ is an integral number.*

Based on this fact, we can easily get:

Proposition 2.3. *$S^{2n}(n > 3)$ doesn't admit an almost complex structure.*

Proof. From

$$ch(S^{2n}) = 1 + -1^n \frac{c_n(TS^{2n})}{(n-1)!} \in \mathbb{Z}$$

we get $(n-1)!| < c_n(TS^{2n}), [S^{2n}] >= \chi(S^{2n}) = 2$, this is possible only when $n \leqslant 3$. □

2.2. **Octonians.**

Definition 2.4. *(Quaternions)*

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k | x_i \in \mathbb{R}\}$$

where i,j,k satiesfies:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k; jk = -kj = i, ki = -ik = j$$

and conjugation in $\mathbb{H}$ *is defined as*

$$\overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k$$

Proposition 2.5.

$$(a + bi + cj + dk)(a' + b'i + c'j + d'k)$$
$$= (aa'-bb'-cc'-dd')+(ab'+ba'+cd'-dc')i+(ac'+ca'+db'-bd')j+(ad'+da'+bc'-cb')k$$
$$\forall q = a + bi + cj + dk, q \cdot \overline{q} = |q|^2 = a^2 + b^2 + c^2 + d^2$$

By directly computation or just check on basis, we get:

Proposition 2.6.

$$\overline{h_1 h_2} = \overline{h_2} \cdot \overline{h_1}$$

Remark. Quaternions is a associative, non-commutative (because $ij = -ji = k$) divisible algebra over $\mathbb{R}$.

Definition 2.7. *(Octonians)*

$$\mathcal{O} = \mathbb{H} \oplus \mathbb{H}$$
$$(q_1, q_2) \circ (a_1, a_2) = (q_1 a_1 - \overline{a_2} q_2, a_2 q_1 + q_2 \overline{a_1})$$

$h \in \mathcal{O}$ *has the form* $x_0 + x_1 e_1 + \cdots + x_7 e_7, x_i \in \mathbb{R}$*, where*

$$e_0 = (1, 0), e_1 = (i, 0), e_2 = (j, 0), e_3 = (k, 0), e_4 = (0, 1), e_5 = (0, i), e_6 = (0, j), e_7 = (0, k)$$

and conjugation in $\mathcal{O}$ *is defined as*

$$\overline{x_0 + x_1 e_1 + \ + x_7 e_7} = x_0 - x_1 e_1 - \ - x_7 e_7$$

check that $e_i e_4 = e_{i+4}, i = 0, 1, 2, 3$*, let* $e_4 = \epsilon$*, and* $\mathbb{H} = (\mathbb{H}, 0) \subset \mathcal{O}$*, an element*

$$x_0 + x_1 e_1 + \cdots + x_7 e_7$$

in $\mathcal{O}$ *can also be written as*

$$h_1 + h_2 \epsilon$$

where

$$h_1 = x_0 + x_1 i + x_2 j + x_3 k, h_2 = x_4 + x_5 i + x_6 j + x_7 k$$

and in this sense $\mathcal{O} = \mathbb{H} \oplus \mathbb{H}\epsilon$*, and*

$$(q_1 + q_2\epsilon) \circ (a_1 + a_2\epsilon) := (q_1 a_1 - \overline{a_2} q_2) + (a_2 q_1 + q_2 \overline{a_1})\epsilon$$

Proposition 2.8.

$$\overline{x \circ y} = \overline{y} \circ \overline{x}$$

Proof.

$$\overline{(q_1+q_2\epsilon)\circ(a_1+a_2\epsilon)} = \overline{(q_1a_1-\overline{a_2}q_2)+(a_2q_1+q_2\overline{a_1})\epsilon}$$

$$\overline{(q_1a_1-\overline{a_2}q_2)}-(a_2q_1+q_2\overline{a_1})\epsilon = (\overline{a_1}\cdot\overline{q_1}-\overline{q_2}\cdot a_2)-(a_2q_1+q_2\overline{a_1})\epsilon$$

$$\overline{y}\cdot\overline{x} = (\overline{a_1}-a_2\epsilon)\circ(\overline{q_1}-q_2\epsilon) = (\overline{a_1}\cdot\overline{q_1}-\overline{q_2}\cdot a_2)-(a_2q_1+q_2\overline{a_1})\epsilon$$

□

Remark. We can check that

$$h\circ\overline{h} = |h|^2$$

where $|\cdot|$ is the standard metric in $\mathbb{R}^8$, so a non-zero octonian has left and right inverse(and are the same).

Proof. Note that $\overline{(q_1,q_2)} = (\overline{q_1},-q_2)$,we have:

$$(q_1,q_2)\circ\overline{(q_1,q_2)} = (q_1,q_2)\circ(\overline{q_1},-q_2) = (q_1\overline{q_1}+\overline{q_2}q_2,-q_2q_1+q_2q_1)$$
$$= (|q_1|^2+|q_2|^2,0)$$

□

Remark. $\mathcal{O}$ is a non-associative (because $(ij)\epsilon = k\epsilon$ but $i(j\epsilon) = (ji)\epsilon = -k\epsilon$), non-commutative (because $\mathbb{H}\subset\mathcal{O}$) divisible algebra over $\mathbb{R}$, hence $(\mathcal{O},\circ)$ is not a group.

Proposition 2.9. *Define inner product on $\mathcal{O}$ by $< x,y >:= Re(x\circ\overline{y})$, then it's the standard Euclidean inner product under basis $\{e_i\}$, so if we see $\mathcal{O}$ as $\mathbb{R}^8$, the orthogonality are the same.*

Proof. By directly compute:

$$< (q_1,q_2),(a_1,a_2) >= Re((q_1,q_2)\circ(\overline{a_1},-a_2)) = Re((q_1\overline{a_1}+\overline{a_2}q_2,*)) = Re_{\mathbb{H}}(q_1\overline{a_1}+\overline{a_2}q_2)$$

Use prop1.1,

$$Re_{\mathbb{H}}((a+bi+cj+dk)\overline{(a'+b'i+c'j+d'k)}) = Re_{\mathbb{H}}((a+bi+cj+dk)(a'-b'i-c'j-d'k))$$
$$= aa'+bb'+cc'+dd'$$

so $<,>$ is exactly the Euclidean inner product. □

Corollary 2.10.

$$< xy >< xy >=< x,x >< y,y >$$

$$< xy,zy >=< x,z >< y,y >=< y,y >< x,z >=< yx,yz >$$

Proof. It suffices to show $< xy >< xy >=< x,x >< y,y >$ because if we consider $< (x+z)y,(x+z)y >=< x+z,x+z >< y,y >$ we get $< xy,zy >=< x,z >< y,y >$.
$< xy >< xy >=< x,x >< y,y >$ is right, by directly computation. □

Corollary 2.11.

$$< xw, y >=< x, y\overline{w} >, < wx, y =< x, \overline{w}y >$$

$$(xw)\overline{w} = x(w\overline{w}), (xw)w = xw^2$$

$$\overline{w}(wx) = (\overline{w}w)x, w(wx) = w^2x$$

$$(xu)\overline{v} + (xv)\overline{u} = 2x < u, v >$$

$$u(\overline{v}x) + v(\overline{u}x) = 2x < u, v >$$

Proof. Note that

$$< x(1 + w), y(1 + w) >=< x, y >< 1 + w, 1 + w >$$

and compare terms, we can get

$$< x, yw > + < xw, y >=< x, y > (2 < w, 1 >)$$

hence

$$< xw, y >=< x, y(2 < w, 1 > -w) >=< x, y\overline{w} >$$

similarly, we can show $< wx, y =< x, \overline{w}y >$ by considering

$$< (1 + w)x, (1 + w)y) >=< 1 + w, 1 + w >< x, y >$$

Now, using these two properties, we can show the left:

$$< (xw)\overline{w}, y >=< xw, yw >=< x, y >< w, w >=< x < w, w >, y >=< x(w\overline{w}), y >$$

Since $<, >$ is non-degenerated, $(xw)\overline{w} = x(w\overline{w})$, since $\overline{w} = 2 < w, 1 > -w$, $(xw)(2 < w, 1 > -w) = x(w(2 < w, 1 > -w))$ and we get

$$(xw)w = xw^2$$

Similarly, we can show the other equations. □

Corollary 2.12.

$$y(xy) = (yx)y$$

$$(xy)(zx) = x(yz)x, \forall x, y \in ImO$$

Remark. The second identity is one of the so called Moufang identities.

Proof. Use conclusions in Cor 1.2, we can get:

$$y(xy) + \overline{x}(\overline{y}y) = 2 < y, \overline{x} > y$$

$$(yx)y + (y\overline{y})\overline{x} = 2 < y, \overline{x} > y$$

$$(xy)(zx) + \overline{z}((\overline{y} \cdot \overline{x})x) = (xy)(zx) + \overline{z} \cdot \overline{y}|x|^2 = 2 < xy, \overline{z} > x$$

$$x((yz)x) + \overline{yz}|x|^2 = 2x < x, \overline{zy} > x = 2 < xy, \overline{z} > x$$

compare these two equations we get the identities. □

Definition 2.13. *(Standad basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$)*
Let

$$N = \frac{1}{2}(1 - i\epsilon), \overline{N} = \frac{1}{2}(1 + i\epsilon)$$

$$E_1 = jN, \overline{E}_1 = j\overline{N}$$

$$E_2 = kN, \overline{E}_2 = k\overline{N}$$

$$E_3 = (kj)N, E_3 = (kj)\overline{N}$$

then $\{N, E_1, E_2, E_3, \overline{N}, \overline{E}_1, \overline{E}_2, \overline{E}_3\} := \{N, E, \overline{N}, \overline{E}\}$ *is* $\mathbb{C}-$ *basis of* $\mathbb{C} \otimes \mathcal{O}$*, which is called the standard basis of* $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$*.*

Remark. In this definition, the i in $N = \frac{1}{2}(1 - i\epsilon), \overline{N} = \frac{1}{2}(1 + i\epsilon)$ is not the $\mathbf{i}$ in $\mathbb{H} \subset \mathcal{O}$ and to avoid mistake, we replace the $\mathbf{i}$ in $\mathbb{H} \subset \mathcal{O}$ by $jk = -kj$. For example,

$$E_3 = (kj)\overline{N} = \frac{1}{2}((kj) + i(kj)\epsilon)$$

The conjugation in $\mathbb{C} \otimes \mathcal{O}$ is just in coefficients $\mathbb{C}$, i.e

$$\overline{x + iy} = x - iy, \forall x, y \in \mathcal{O}$$

In this sence, we define:

$$Re : \mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}, x + iy \mapsto x$$

Definition 2.14. *(Cross product of $\mathcal{O}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$)*

$$\forall x, y \in \mathcal{O}, x \times y := \frac{1}{2}(\overline{y}x - \overline{x}y)$$

and in $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$ *the cross product is the natural linear extension.*

Proposition 2.15.

$$y \times x = -x \times y$$

$$x \times x = 0$$

If $< x, y >= 0$ *and* $< x, x >=< y, y >= 1$*, then:*

$$x(y \times x) = y$$

$$y(y \times x) = -x$$

2.3. **Almost complex structure on S^2 and S^6.**

Theorem 2.16. S^n *admits an almost complex structure iff n=2 or 6.*

Remark. An almost complex mfd must be of even dim.

Theorem 2.17. S^2 *admits a complex structure,* S^6 *admit an almost complex structure* $\mathcal{J}$*.*

Proof. S^2 has a standard complex structure: using stereographic projection $S^2-N \to \mathbb{C}, x \mapsto \varphi(x) = z$, and around N, let $\phi(x) = \frac{1}{\varphi(x)}, x \neq N, \phi(N) = 0$, we have a complex structure on S^2.
Consider $\mathbb{R}^7$ as the pure imaginary part of octonians, use product of octonians we can construct an almost-complex structure $\mathcal{J}$ on S^n as following:

$$\mathcal{J}_u(v) = v \circ u, \forall u \in S^6, v \in T_uS^6 = \{v \in Im\mathcal{O}, < v, u >= 0\}$$

it is well-defined because in $\mathcal{O}$, by Cor 1,1 in section 1,

$$Re(v \circ u) =< v \circ u, 1 >=< v \circ u, -u \circ u >= - < v, u >= 0$$
$$< v \circ u, u >=< v, 1 >< u, u >= Re(u) = 0$$

hence $\mathcal{J}_u$ maps T_uS^6 to itself, note that in $S^6 \subset Im\mathcal{O}$, $\overline{u} = -u$, so $(J_u)^2 \circ v = (v \circ u) \circ u = v \circ (u^2) = -v$ (By Cor1.2 in section1), so $\mathcal{J}$ is really an almost complex structure. □

Fact. *We $\mathcal{J}$ is not integrable hence does not induce a complex structure on S^6.*

3. Nonexistence of compatible complex structures on S^6

Remark. In this section, $\mathcal{J}$ is the standard almost complex structure on S^6, and g is the standard metric induced on S^6 by its inclusion into $\mathbb{R}^7$, ω is the nondegenerate 2-form compatible with $(\mathcal{J}, g)$, i.e ω satiesfies $\omega(v, \mathcal{J}w) = g(v, w)$.

3.1. $Spin(7)$ and G_2.

Definition 3.1. ($Spin(7)$)
For any $u \in S^6 \in \mathbb{R}^7 = Im\mathcal{O}$, $J_u \in SO(8)$, then $Spin(7)$ is defined as the subgroup of $SO(8)$ generated by all these $\{J_u | u \in S^6\}$.

Remark. Since $< xu, yu >=< x, y >< u, u >=< x, y >$ for $u \in S^6$, $J_u \in O(8)$, note also that S^6 is connected, $detJ_u \equiv 1$ or $detJ_u \equiv -1$, for all $u \in S^6$, but $\epsilon \in S^6$ and

$$(1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)\epsilon = (\epsilon, i\epsilon, j\epsilon, k\epsilon, -1, -i, -j, -k)$$

so under this basis,

$$J_\epsilon = \begin{pmatrix} 0 & -I \\ I & ,0 \end{pmatrix}$$

and $detJ_\epsilon = 1$, hence $Spin(7) \subset SO(8)$.

Remark. Note that $T_\epsilon S^6 = span\{i, j, k, i\epsilon, j\epsilon, k\epsilon\}$ and

$$J_\epsilon(i, j, k, i\epsilon, j\epsilon, k\epsilon) = (i\epsilon, j\epsilon, k\epsilon, -i, -j, -k)$$

so if we consider $T_{\mathbb{C},\epsilon}S^6 = \mathbb{C} \otimes_{\mathbb{C}} T_\epsilon S^6$, $\frac{1}{2}(j - ij\epsilon)$, $\frac{1}{2}(k - ik\epsilon)$ and $\frac{1}{2}((kj) - (kj)i\epsilon)$ becomes the basis of $T^{1,0}(\epsilon)$, which is eactly E_1, E_2, E_3, as we defined in Section1, and $\overline{E}_1, \overline{E}_2, \overline{E}_3$ is the basis of $T^{0,1}(\epsilon)$.
Recall that we have the standard basis $\{N, E, \overline{N}, \overline{E}\}$ of $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$, given any

$g \in Spin(7)$, we can get a new basis $(n, f, \overline{n}, \overline{f}) = (N, E, \overline{N}, \overline{E})g$, which is called an **admissible basis**.

Fact. *Spin*(7) *is a connected ,simply connected, compact Lie group of real dimension 21. Its center is* $\{\pm I_8\} \cong \mathbb{Z}_2$ *and* $Spin(7)/\{\pm I_8\}$ *is isomorphic to* $SO(7)$.

Proposition 3.2. *There is a group homomorphism* $\chi : Spin(7) \to SO(7) \subset GL(Im\mathcal{O})$ *such that* $\chi(J_u) = r_u, \forall u \in S^6$, *where* $r_u(x) := (\overline{u}x)u = \overline{u}(xu)$ *(may use Cor1.2) , further more, we have the following equivalents*

$$g(x \times y) = \chi(g)(x \times y), \forall g \in Spin(7), x, y \in \mathcal{O}$$

Proof. Since $Spin(7)$ is generated by J_u, the uniqueness is clear, it suffices to show $\chi(J_u) = r_u$ is well defined:

$$(J_u x) \times (J_u y) = (xu) \times (yu) = \frac{1}{2}[(\overline{yu})(xu) - (\overline{xu})(yu)]$$

$$= \frac{1}{2}[(\overline{u} \cdot \overline{y})(xu) - (\overline{u} \cdot \overline{x})(yu)] = \frac{1}{2}[\overline{u}(\overline{y}x - \overline{x}y)u]$$

$$= \overline{u}[\frac{1}{2}(\overline{y}x - \overline{x}y)]u = r_u(x \times y)$$

so we finish the proof. □

Corollary 3.3.

$$f \times \overline{n} = n \times \overline{f} = 0$$

Definition 3.4. (G_2)

$$G_2 = Stab_1(Spin(7)) = \{g \in Spin(7), g(1) = 1\}$$

Proposition 3.5. *The following are basic properties of* G_2*:*
G_2 *preserves the metric and orientation on* $\mathbb{R}$*;*
G_2 *acts transitively on* S^6 *, and the* G_2*-stabilizer of any* $u \in S^6$ *is isomorphic to* $SU(3) \subset SO(6)$ *thus,* $S^6 = G_2/SU_3$*;*
G_2 *acts transitively on* S^6*;*

Remark. Note that $Spin(7)$ and G_2 are all subgroup of Lie group $SO(7)$, $Spin(7)$ and G_2 are all Lie groups, we denote $spin(7)$ and g_2 as their Lie algebra respectly.

Proposition 3.6. *Under the standard basis of* $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$*(since* $Spin(7) \subset End(\mathcal{O}) \subset End(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O})$*),* $spin(7)$ *has the following representation:*

$$spin(7) = \left\{ \begin{pmatrix} ic & -{}^t\overline{b} & 0 & -{}^t a \\ b & d & a & [\overline{a}] \\ 0 & -{}^t\overline{a} & -ic & -{}^t b \\ \overline{a} & [a] & \overline{b} & \overline{d} \end{pmatrix} \middle| \begin{array}{l} a, b \in M_{3\times 1}(\mathbb{C}) \\ c \in \mathbb{R}, d \in M_{3\times 3}(\mathbb{C}) \\ d +^t \overline{d} = 0 \\ trd + ic = 0 \end{array} \right\} \tag{3.1}$$

Proof. By Prop 3.1, $spin(7)$ has $\mathbb{R}-$dim 21, so it suffices to find a $\mathbb{R}-$dim 21 subspace of $spin(7)$. The first claim is:

$$L = \{J_\epsilon \circ J_w | w \in Im\mathcal{O}, <\epsilon, w>= 0\} \subsetneq spin(7), dim_{\mathbb{R}} L = 6$$

To show this claim is true, first note that $\forall u \in S^6 \cap \epsilon^\perp, \lambda \in \mathbb{R}$, $cos\lambda\epsilon + sin\lambda u \in Im\mathcal{O}$ and

$$||cos\lambda\epsilon + sin\lambda u||^2 = cos^2\lambda + sin^2\lambda = 1,$$

So $cos\lambda\epsilon + sin\lambda u \in S^6$ and $J_{cos\lambda\epsilon+sin\lambda u} \in Spin(7)$, so is $J_\epsilon \circ J_{cos\lambda\epsilon+sin\lambda u} = -cos\lambda I + sin\lambda J_\epsilon \circ J_u$.
Note also that since $u \perp \epsilon$, by Cor1.2 in section 1, $\forall x \in \mathcal{O}, (xu)\epsilon + (x\epsilon)u = 2x < u, \epsilon >= 0$, so $J_u \circ J_\epsilon + J_\epsilon \circ J_u = 0$ and

$$(J_\epsilon \circ J_u)^2 = J_\epsilon \circ J_u \circ J_\epsilon \circ J_u = J_\epsilon \circ (J_u \circ J_\epsilon) \circ J_u = -(J_\epsilon)^2 \circ (J_u)^2 = -I$$

It follows that

$$exp(\lambda J_\epsilon \circ J_u) = \sum \frac{\lambda^n}{n!}(J_\epsilon \circ J_u)^n$$
$$= I + \lambda(J_\epsilon \circ J_u) + \frac{\lambda^2}{2!}(-I) - \frac{\lambda^3}{3!}(J_\epsilon \circ J_u) + \frac{\lambda^4}{4!}I + \frac{\lambda^5}{5!}(J_\epsilon \circ J_u) + \cdots$$
$$= (1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots)I + (\lambda - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots)(J_\epsilon \circ J_u)$$
$$= cos\lambda I + sin\lambda J_\epsilon \circ J_u = J_\epsilon \circ J_{-cos\lambda\epsilon+sin\lambda u} \in Spin(7)$$

hence $J_\epsilon \circ J_u \in spin(7)$ where $u \in S^6 \cap \epsilon$, note that L is a linear space, $L \subsetneq Spin(7)$.
$dim_R L = 6$ because $dim Im\mathcal{O} = 7$ and the orthognality $<\epsilon, w>= 0$ reduce dim of L to $7 - 1 = 6$. Since $spin(7)$ is a lie algebra, $[L, L] \subsetneq spin(7)$ too, and the second claim is:

$$L \oplus [L, L] = spin(7)$$

To show this claim, we need to consider the matrix representation of L using the standard basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$, $\{N, E, \overline{N}, \overline{E}\}$, note that $<\epsilon, Re(E_l)>=<\epsilon, Re(iE_l)>= 0, l = 1, 2, 3$(for example, $<\epsilon, Re(iE_j)>=<\epsilon, 1/2j\epsilon>= 0$) and $dim L = 6$, any $w \in \epsilon^\perp$ can be right as form

$$w = 2Re(a^1E_1 + a^2E_2 + a^3E_3), a^l \in \mathbb{C}, l = 1, 2, 3$$

now use the basis, by directly computation, we can show:

$$J_\epsilon \circ J_w(N, E_1, E_2, E_3) = (\overline{N}, \overline{E}_1, \overline{E}_2, \overline{E}_3) \begin{pmatrix} 0 & i\overline{a}^1 & i\overline{a}^2 & i\overline{a}^3 \\ -i\overline{a}^1 & 0 & ia^3 & -ia^2 \\ -i\overline{a}^2 & -ia^3 & 0 & ia^1 \\ -i\overline{a}^3 & ia^2 & -ia^1 & 0 \end{pmatrix} \tag{3.2}$$

For example, we compute $(Nw)\epsilon$:
At first, $N = \frac{1}{2}(1 - i\epsilon), \overline{N} = \frac{1}{2}(1 + \epsilon)$, so

$$E_1 = \frac{1}{2}(j - ij\epsilon), \overline{E_1} = \frac{1}{2}(j + ij\epsilon)$$

$$E_2 = \frac{1}{2}(k - ik\epsilon), \overline{E_2} = \frac{1}{2}(k + ik\epsilon)$$
$$E_3 = \frac{1}{2}((kj) - i(kj)\epsilon), \overline{E}_3 = \frac{1}{2}((kj) - i(kj)\epsilon)$$

and then

$$w = 2Re(a^1E_1 + a^2E_2 + a^3E_3) =$$
$$= 2Re[(a_1^1+ia_2^1))(\frac{1}{2}(j-ij\epsilon))+(a_1^2+ia_2^2)(\frac{1}{2}(k-ik\epsilon))+(a_1^3+ia_2^3)(\frac{1}{2}((kj)-i(kj)\epsilon))]$$
$$= a_1^1 j + a_2^1 j\epsilon + a_1^2 k + a_2^2 k\epsilon + a_1^3(kj) + a_2^3(kj)\epsilon$$

so

$$(Nw)\epsilon = [(\frac{1}{2}(1 - i\epsilon)) \circ (a_1^1 j + a_2^1 j\epsilon + a_1^2 k + a_2^2 k\epsilon + a_1^3(kj) + a_2^3(kj)\epsilon)] \circ \epsilon$$
$$= \{\frac{1}{2}[a_1^1 j + a_2^1 j\epsilon + a_1^2 k + a_2^2 k\epsilon + a_1^3(kj) + a_2^3(kj)\epsilon]$$
$$-\frac{1}{2}i[a_2^1 j - a_1^1\epsilon + a_2^2 k - a_1^2 k\epsilon + a_2^3(kj) - a_1^3(kj)\epsilon]\} \circ \epsilon$$
$$= \frac{1}{2}[a_1^1 j\epsilon - a_2^1 j + a_1^2 k\epsilon - a_2^2 k + a_1^3(kj)\epsilon - a_2^3(kj)]$$
$$-\frac{1}{2}i[a_1^1 j + a_2^1 j\epsilon + a_1^2 k + a_2^2 k\epsilon + a_1^3(kj) + a_2^3(kj)\epsilon]$$
$$= \frac{1}{2}[(j\epsilon-ij)a_1^1-(j+i(j\epsilon))a_2^2]+\frac{1}{2}[(k\epsilon-ik)a_1^2-(k+i(k\epsilon))]+\frac{1}{2}[((kj)\epsilon-i(kj))a_1^3-((kj)+i(kj)\epsilon)a_2^3]$$
$$= \frac{1}{2}(j+ij\epsilon)(-ia_1^1-a_2^1)+\frac{1}{2}(k+ik\epsilon)(-ia_1^2-a_2^2)+\frac{1}{2}((kj)+i(kj)\epsilon)(-ia_1^3-a_2^3)$$
$$= -i(\overline{E}_1\overline{a}^1 + \overline{E_2}\overline{a}^2 + \overline{E}_3\overline{a}^3)$$

Now we return to the equation (4), to simplifies the symbol, we define $a = (a^1, a^2, a^3)$ be any 3×1 colum vector, and

$$[a] = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix} \tag{3.3}$$

then (4) ca be write as:

$$J_\epsilon \circ J_w(N, E) = (\overline{N}, \overline{E}) \begin{pmatrix} 0 & -i^t\overline{a} \\ -i\overline{a} & [ia] \end{pmatrix}$$

take conjugation we get

$$J_\epsilon \circ J_w(\overline{N}, \overline{E}) = (N, E) \begin{pmatrix} 0 & -i^t a \\ -ia & [i\overline{a}] \end{pmatrix}$$

so expressed in $\{N, E, \overline{N}, \overline{E}\}$, we get:

$$J_\epsilon \circ J_w(N, E, \overline{N}, \overline{E}) = (N, E, \overline{N}, \overline{E}) \begin{pmatrix} 0 & 0 & 0 & -i^t a \\ 0 & 0 & ia & [-i\overline{a}] \\ 0 & i^t\overline{a} & 0 & 0 \\ -i\overline{a} & [ia] & 0 & 0 \end{pmatrix} \tag{3.4}$$

note that L is linear space and cancel i, we get:

$$L = \left\{ \left(\begin{array}{cccc} 0 & 0 & 0 & -{}^t a \\ 0 & 0 & a & [\overline{a}] \\ 0 & -{}^t\overline{a} & 0 & 0 \\ \overline{a} & [a] & 0 & 0 \end{array} \right) \middle| \; a \in \mathbb{C}^3 \right\} \tag{3.5}$$

now we can compute $[L, L]$, for any

$$A = \left(\begin{array}{cccc} 0 & 0 & 0 & -{}^t a \\ 0 & 0 & a & [\overline{a}] \\ 0 & -{}^t\overline{a} & 0 & 0 \\ \overline{a} & [a] & 0 & 0 \end{array} \right), B = \left(\begin{array}{cccc} 0 & 0 & 0 & -{}^t b \\ 0 & 0 & b & [\overline{b}] \\ 0 & -{}^t\overline{b} & 0 & 0 \\ \overline{b} & [b] & 0 & 0 \end{array} \right)$$

$$[A, B] = AB - BA = \left(\begin{array}{cc} \kappa & 0 \\ 0 & \overline{\kappa} \end{array} \right)$$

where

$$\kappa = \left(\begin{array}{cc} -{}^t a\overline{b} +{}^t b\overline{a} & 2{}^t b[a] \\ 2[\overline{a}]\overline{b} & -2a{}^t\overline{b} + 2b{}^t\overline{a} - 2({}^t\overline{a}b -{}^t \overline{b}a)I_3 \end{array} \right) \in M_{4\times 4}(\mathbb{C})$$

In fact, we can show

$$[L, L] = \left\{ \left(\begin{array}{cc} \kappa & 0 \\ 0 & \overline{\kappa} \end{array} \right) \middle| \; \begin{array}{c} \kappa +{}^t \overline{\kappa} = 0, tr\kappa \\ \kappa \in M_4(\mathbb{C}) \end{array} \right\} \tag{3.6}$$

so $dim_{\mathbb{R}}[L, L] = 15$.
Now, abviously $L \cap [L, L] = \{0\}$, since $dim_{\mathbb{R}} L + dim_{\mathbb{R}}[L, L] = 6 + 15 = 21 = dim_{\mathbb{R}} spin(7)$, we conclude that:

$$spin(7) = L \oplus [L, L]$$

and the equation (3) is the sum of (7) and (8). □

Definition 3.7. *(Maurer-Cartan form)*
For a Lie group G with unit e, the ***right fundamental form****, or the* ***Maurer-Cartan form*** *of G is a G_e valued 1-form on G, defined as the following formula:*

$$\Phi(X)(g) = (R_{g^{-1}})_*(X(g)), \forall X \in \mathscr{X}(G)$$

i.e the drawback of vector field to X_e.

Remark. Φ is right invariant, since

$$(R_x)^*(\Phi)(X)(a) = \Phi((R_x)_*X)(ax) = (R_{(ax)^{-1}})_*((R_x)_*(X))(e) = (R_{a^{-1}})_*(X)(e)$$

For any $g \in Spin(7)$, $\{n, f, \overline{n}, \overline{f}\} = \{N, E, \overline{N}, \overline{E}\}g$ is called an admissible basis, one can show the space of admissible bases (i.e the orbit of $\{N, E, \overline{N}, \overline{E}\}$ over $Spin(7)$ action) is diffeomorphism to Spin(7), so every $g \in Spin(7)$ induces a diffeomorphism of $Spin(7)$, and $(N, E, \overline{N}, \overline{E})$ and

$(n, f, \overline{n}, \overline{f})$ can be seen as frame at different point, now the Maurer-Cartan form is $g^{-1}dg$ since

$$d(n, f, \overline{n}, \overline{f}) = (N, E, \overline{N}, \overline{E})dg = (n, f, \overline{n}, \overline{f})g^{-1}dg$$

note that $g^{-1}dg$ is $spin(7)$ valued, by the last thm, $\Phi = g^{-1}dg$ is of the form

$$\begin{pmatrix} i\rho & -{}^t\overline{\mathfrak{b}} & 0 & -{}^t\theta \\ \mathfrak{b} & \kappa & \theta & [\overline{\theta}] \\ 0 & -{}^t\overline{\theta} & -i\rho & -{}^t\mathfrak{b} \\ \overline{\theta} & [\theta] & \overline{\mathfrak{b}} & \overline{\kappa} \end{pmatrix}$$

Proposition 3.8. Φ *satiesfies the first structure equations:*

$$d\Phi = -\Phi \wedge \Phi$$

Proof. Since $\Phi = g^{-1}dg$, and we use the identity $g^{-1}g = id$ to get:

$$d(g^{-1}) = d(g^{-1})gg^{-1} = -g^{-1}dg \wedge g^{-1}$$

So

$$d\Phi = d(g^{-1}dg) = d(g^{-1}) \wedge dg + 0 = -g^{-1}dg \wedge g^{-1}dg = -\Phi \wedge \Phi$$

$\square$

Now, note that $1 = N + \overline{N}$ and $(N + \overline{N})g = n + \overline{n}$, we define

$$p : Spin(7) \to \mathcal{O} : g \mapsto n + \overline{n}$$

then abviously, $G_2 = p^{-1}(1)$, then we can compute the lie algebra of G_2, using the matrix representation of $spin(7)$:

Proposition 3.9.

$$g_2 = \left\{ \begin{pmatrix} 0 & {}^t\overline{\theta} & 0 & -{}^t\theta \\ -\theta & \kappa & \theta & [\overline{\theta}] \\ 0 & -{}^t\overline{\theta} & 0 & {}^t\theta \\ \overline{\theta} & [\theta] & -\overline{\theta} & \overline{\kappa} \end{pmatrix} \middle| \begin{array}{l} \theta \in M_{3\times 1}(\mathbb{C}) \\ c \in \mathbb{R}, \kappa \in M_{3\times 3}(\mathbb{C}) \end{array} \right\} \tag{3.7}$$

Proof. Note that

$$p = n + \overline{n} = (n, f, \overline{n}, \overline{f}) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (N, E, \overline{N}, \overline{E})g \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$dp = (N, E, \overline{N}, \overline{E})dg \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (n, f, \overline{n}, \overline{f})g^{-1}dg \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (n, f, \overline{n}, \overline{f})\Phi \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

By prop3.3, we eventually have:

$$dp = i(n - \overline{n})\rho + f(\mathfrak{b} + \theta) + \overline{f}(\overline{\mathfrak{b}} + \overline{\theta})$$

so the left invariant 1 form of G_2 are those such that $\rho = \mathfrak{b} + \theta = 0$. $\square$

Proposition 3.10. *Recall that we have defined the cross product on $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{O}$ in Section1, for the admissible basis, we have:*

$$n(2i(n\times\overline{n}))=in$$

$$f(2i(n\times\overline{n}))=if$$

In special, if the basis is induced by an element in G_2, we have:

$$2i(n\times\overline{n})=i(n-\overline{n})$$

Proof. Suppose $n=n_1+in_2$,

$$n(2i(n\times\overline{n}))=2i(n_1+in_2)[(n_1+in_2)\times(n_1-in_2)]$$

$$=2i(n_1+in_2)[(n_1\times n_1+n_2\times n_2)+i(n_2\times n_1-n_1\times n_2)]$$

$$=-4(n_1+in_2)(n_2\times n_1)$$

since any $g\in Spin(7)$ keeps the metrics and $N=\frac{1}{2}(1-i\epsilon)$, $2n_1\&2n_2$ are unit orthogonal basis, so we can use conclusions in Prop 1.5:

$$=-\frac{1}{2}(2n_1+i2n_2)(2n_2\times 2n_1)=-\frac{1}{2}(2n_2-i2n_1)$$

$$=-n_2+in_1=i(n_1+in_2)=in$$

now, differentiating this equation, we get

$$dn(2i(n\times\overline{n}))+nd(2i(n\times\overline{n}))=idn$$

by the first structure equation of $Spin(7)$,

$$dn=n(i\rho)+f\cdot\mathfrak{b}+\overline{f}\cdot\overline{\theta}$$

and by Cor3.1

$$f\times\overline{n}=n\times\overline{f}=0$$

so

$$d(n\times\overline{n})=\overline{f}\times\overline{n}\cdot\overline{\theta}+n\times f\cdot\theta$$

and

$$[n(i\rho)+f\mathfrak{b}+\overline{f\theta}](2i(n\times\overline{n}))+n(\overline{f}\times\overline{n}\cdot\overline{\theta}+n\times f\cdot\theta)=i[n(i\rho)+f\mathfrak{b}+\overline{f\theta}]$$

compare coefficients of $\mathfrak{b}$, we get

$$f(2i(n\times\overline{n}))=if$$

Now we turn to the third equation:

$$2i(n\times\overline{n})=2i(i(n_2\times n_1-n_1\times n_2))=-4n_2\times n_1$$

note that $n+\overline{n}=1$ implies $n_1=\frac{1}{2}$, and $n_2=g(\frac{-1}{2}\epsilon)\in Im\mathcal{O}$

$$-4n_2\times n_1=-2n_2\times 1=-2n_2=i(n-\overline{n})$$

□

Proposition 3.11. *For any $g\in G_2$, $u(g)=i(n-\overline{n})\in S^6$, and*

$$T_{\mathbb{C}}S^6=span_{\mathbb{C}}\{f_1,f_2,f_3\}\oplus span_{\mathbb{C}}\{\overline{f}_1,\overline{f}_2,\overline{f}_3\}=T^{1,0}\oplus T^{0,1}$$

Proof. We have show in the last remark that over ϵ it's right, for any $u(g) \in S^6, g \in G_2$, since $G_2 \subset SO(Im\mathcal{O})$,

$$T_{\mathbb{C},u}S^6 = (T_{\mathbb{C},\epsilon}S^6)g = span_{\mathbb{C}}\{f, \overline{f}\}$$

It suffices to show $T^{1,0} = span_{\mathbb{C}}\{f_1, f_2, f_3\}$ □

Now, using the first structure equations of $Spin(7)$(Prop 3.4), and consider the new frame $\{u = i(n - \overline{n}), f, \overline{f}\}$, we can get the structure equations of G_2:

Proposition 3.12.

$$du = f(-2i\theta) + \overline{f}(2i\overline{\theta})$$

$$df = u(-i^t\overline{\theta}) + f\kappa + \overline{f}[\theta]$$

$$d\theta = -\kappa \wedge \theta + [\overline{\theta}] \wedge \overline{\theta}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge^t \overline{\theta} -^t \theta \wedge \overline{\theta} I_{3\times 3}$$

Proof. For the first equation, differentiate both sides of

$$u = i(n - \overline{n}) = i(n, f, \overline{n}, \overline{f}) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = i(N, E, \overline{N}, \overline{E})g \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

For the other, just use the first structure equations of Spin(7).
Note that we always use the relationship $\rho = \mathfrak{b} + \theta = 0$. □

Corollary 3.13. *Note that $f(\overline{f})$ are basis of $T^{1,0}(g(\epsilon))(T^{0,1}(g(\epsilon)))$ respectively, using the first equation, we can show, $\alpha \in \Omega^1(S^6)$ of type $(1,0)$ iff $u^*(\alpha)$ is linearly combination of $\theta^1, \theta^2, \theta^3$.*

Note that $N = \frac{1}{2}(1 - i\epsilon)$, then $i(N - \overline{N}) = i(-i\epsilon) = \epsilon$ has length 1, since $Spin(7) \subset SO(8)$, and J_u maps $Im\mathcal{O}$ to $Im\mathcal{O}$, $u(g) = i(n - \overline{n})$ maps to S^6, we consider u^*,use the structure equations, we can get the following properties:

Proposition 3.14.

$$u^*g = 4^t\theta \circ \overline{\theta}$$

$$u^*w = 2i^t\theta \wedge \overline{\theta}$$

$$dw = 3Im(\Upsilon)$$

where

$$u^*\Upsilon = 8\theta_1 \wedge \theta_2 \wedge \theta_3$$

Proof. Let
$$X = {}^T(j,k,kj), Y = {}^T(j,k,kj)\epsilon, Z = X + iY$$
then $E = \frac{1}{2}(X - iY)$, since
$$\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix}\begin{pmatrix} \frac{1}{2}I & -\frac{1}{2}iI \\ \frac{1}{2}I & \frac{1}{2}iI \end{pmatrix} = I$$
$$({}^TE^*, {}^T\overline{E}^*) = ({}^TX^*, {}^TY^*)\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix} = ({}^TX^* + i{}^TY^*, {}^TX^* - i{}^TY^*)$$
so
$${}^TE^* \circ \overline{E}^* = ({}^TX^* + i{}^TY^*) \circ (X^* - iY^*) = {}^TX^* \circ X^* + {}^TY^* \circ Y^*$$
is the standard Euclidean metric g reduced on $T_\epsilon S^6 = span\{j, k, kj, j\epsilon, k\epsilon, (kj)\epsilon\}$, i.e
$$g|_\epsilon = {}^TE^* \circ \overline{E}^*$$
Suppose $(n, f, \overline{n}, \overline{f}) = (N, E, \overline{N}, \overline{E})x$ for some $x \in Spin(7)$, then for any $V = (f, \overline{f})v \in T_uS^6$, $(x^{-1})_*(V) = (E, \overline{E})v \in T_\epsilon S^6$, note that x keeps the metrix g,
$$g_u((f,\overline{f})v_1, (f,\overline{f})v_2) = g_\epsilon((E,\overline{E})v_1, (E,\overline{E})v_2)$$
hence
$$g|_u = {}^Tf^* \circ \overline{f}^*$$
Now we can compute the draw back of g:
$$u^*(g) = {}^Tu^*(f^*) \circ u^*(\overline{f}) = {}^T< f, du > \circ < \overline{f}, du > = {}^T(-2i\theta) \circ (2i\overline{\theta}) = 4{}^T\theta \circ \overline{\theta}$$
Note that we have got the matrix of J_ϵ, so by the same method, we can prove the second equation, differentiate it and we get the third equation. □

Corollary 3.15. *The standard almost structure $\mathcal{J}$ on S^6 is not integrable.*

Proof. By the equations
$$d^{2,-1}\omega = \pi^{3,0}(3Im\Upsilon) = \pi^{3,0}[3\frac{-i}{2}(\Upsilon - \overline{\Upsilon})]$$
since $u^*\Upsilon = 8\theta_1 \wedge \theta_2 \wedge \theta_3$, Υ is of type $(3,0)$ and $\overline{\Upsilon}$ is of type $(0,3)$, so
$$d^{2,-1}\omega = \frac{-3i}{2}\Upsilon \neq 0$$
so $\mathcal{J}$ is non-integrable. □

Definition 3.16. *For an arbitrary smooth almost-complex structure J on S^6, $\pi : F_J \to S^6$ denotes the (right) principle $GL(3,\mathbb{C})$-bundle over S^6 whose fibers are J-linear isomorphism $u : T_{\pi(u)}S^6 \to \mathbb{C}^3$, the action of $GL(3,\mathbb{C})$ on F_J is given by $u \cdot A = A^{-1} \circ u$;*
The canonical $\mathbb{C}^3$ valued 1-form on F_J is defined by the formula $\eta(v) = u(\pi_*(v))$ where $v \in T_uF_J$;
$B_J \to S^6$ is the whitney sum of $u : G_2 \to S^6$ and $\pi : F_J \to S^6$, which is a principle $SU(3) \times GL(3,\mathbb{C})$-bundle.

Proposition 3.17. *The projection of B_J onto G_2 or F_J is a surjective submersion.*

Remark. From now on, all forms defined on either F_J, G_2 or S^6 will be regarded as pulled back to B_J without notating the pullback.

Proposition 3.18. *There exist unique mappings $r, s : B_J \to M_{3,3}(\mathbb{C})$ such that*

$$\theta = r \cdot \eta + s \cdot \overline{\eta}$$
$$\overline{\theta} = \overline{s} \cdot \eta + \overline{r} \cdot \overline{\eta}$$

and for the 3-form Υ, we have:

$$\Upsilon_J^{3,0} = 8det(r)\eta_1 \wedge \eta_2 \wedge \eta_3$$
$$\Upsilon_J^{0,3} = 8det(s)\overline{\eta_1} \wedge \overline{\eta_2} \wedge \overline{\eta_3}$$

Proposition 3.19. *For an arbitrary smooth almost-complex structure J on S^6, the J-type decomposition of $\omega = \omega_J^{(2,0)} + \omega_J^{(1,1)} + \omega_J^{(0,2)}$ are exactly:*

$$\omega_J^{(2,0)} = i^t\eta \wedge ({}^t r\overline{s} - {}^t\overline{s}r)\eta$$
$$\omega_J^{(1,1)} = 2i^t\eta \wedge ({}^t r\overline{r} - {}^t\overline{s}s)\overline{\eta}$$
$$\omega_J^{(0,2)} = i^t\overline{\eta} \wedge ({}^t\overline{r}s - {}^t s\overline{r})\overline{\eta}$$

3.2. **Chern's result.**

Proposition 3.20. $\mathbb{J}(S^6, \omega)$ *is disjoint union of the subbundles $\mathbb{J}_q(S^6, \omega), 0 \leqslant q \leqslant 3$, where the fiber of $\mathbb{J}_q(S^6, \omega)$ over x consists of the ω_x-compatible complex structures on T_xM have ω_x- index $(3-q, q)$, i.e $\omega_x(\cdot, J_x\cdot)$ has inertial index $(6-2q, 2q)$.*

Proof. Note that inertial index is a local invariant (at one point x, we can find $v_i \in T_xM$ such that $g_x(,)$ is diagonal, By prop 8.11 in Mac Lee's book , we can find a local frame in $TM|_U$ such that $s_i|_x = v_i$, and we then can use Schmidt procedure and reduce to smaller neighbor $x \in U' \subset U$ such that $g|_{U'}(,)$ is diagonal under this basis) and S^n is connected, it suffice to show if ω_x is compatible with some J_x, then $g(\cdot,\cdot) = \omega_x(\cdot, J_x\cdot)$ must have inertial index $(6-2q, 2q)$.
Note that if $g(e_1, e_1) = \pm 1$, then $g(Je_1, Je_1) = \pm 1$ $g(e_1, Je_1) = g(Je_1, e_1) = e(Je_1, J^2e_1) = -g(Je_1, e_1) = 0$, and since e_1 and Je_1 are $\mathbb{R}-$linearly independent, $g|_{span\{e_1, Je_1\}}$ have the form $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, by induction on $span\{e_1, Je_1\}^{\perp}$, we can finally show:
there exists basis $\{e_1, Je_1, \cdots, e_n, Je_n\}$ such that $g = diag\{\pm I_{2\times 2}, \cdots, \pm I_{2\times 2}\}$ over this basis, this is what we want to show. $\square$

Proposition 3.21. *$\mathcal{J}$ and $-\mathcal{J}$ are sections on $\mathbb{J}_0(S^6, \omega)$ and $\mathbb{J}_3(S^6, \omega)$ respectively.*

Proof. Since $\mathbb{J}_0(S^6,\omega) = -\mathbb{J}_3(S^6,\omega)$, it suffices to show $\mathcal{J}$ is a section on $\mathbb{J}_0(S^6,\omega)$, equivalent to show the g that $(\mathcal{J},\omega)$ induced is positive definite, i.e a metric. but this is the definition, note that ω is induced by g and $\mathcal{J}$, where g is restriction of standard matric in $\mathbb{R}^7$. ☐

Proposition 3.22. *There are exactly two homotopy equivalent classes of almost-complex structures on S^6, $[\mathcal{J}]$ and $[-\mathcal{J}]$.*

Lemma 3.23. *The subbundle $\mathbb{J}_1(S^6,\omega)$ and $\mathbb{J}_2(S^6,\omega)$ have no continuous sections over S^6.*

Proof. Since $\mathbb{J}_1(S^6,\omega) = -\mathbb{J}_2(S^6,\omega)$, it suffices to show $\mathbb{J}_1(S^6,\omega)$ has no continuous section. Such a section J has inertial index $(4,2)$, and have a orthogonal decomposition $TS^6 = E_4 \oplus E_2$, where $g|_{E_4}$ is positive definite and $g|_{E_2}$ is negative definite (use the same method as prop 3.1, begin with some local section $\{e_1, Je_1\}$ s.t $g(e_1,e_1)\neq 0$, if $g(s_i,s_i)=0$, then consider $s_1+\lambda s_i$ for some $\lambda\in\mathbb{R}$, where $g(s_1,s_i)\neq 0$). Now that E_4 and E_2 are all orientable, we compute the Euler class and we have:

$$0 \neq e(TS^6) = e(E_4)e(E_2) = 0$$

which is a contradiction, $e(E_2) = e(E_4) = 0$ since $e(E_2)\in H^2(S^6;\mathbb{Z}) = 0$ and $e(E_4)\in H^4(S^6;\mathbb{Z}) = 0$. ☐

Theorem 3.24. (*Chern*) *$\omega-$compatible complex structure on a connected open set $U\subset S^6$ is a section of either $\mathbb{J}_1(U,\omega)$ or $\mathbb{J}_2(U,\omega)$. In particular, there is no $\omega-$compatible complex structure on S^n.*

Proof. Suppose we have a integrable almost complex structure $J : U \to \mathbb{J}(U,\omega)$ over a connected open set $U\subset S^6$, then $T_{\mathbb{C}} = TS^6_{\mathbb{C}}$ and $\Omega^k_{\mathbb{C}} = \Omega^k_{S^6,\mathbb{C}}$ has J-type decomposition:

$$T_{\mathbb{C}} = T^{1,0}\oplus T^{0,1}, \Omega^k_{\mathbb{C}} = \underset{p+q=k}{\oplus}\Omega^{p,q}$$

since $\omega\subset\Omega^2_{\mathbb{R}}\subset\Omega^2_{\mathbb{C}}$, we have the decomposition:

$$\omega = \omega^{2,0}+\omega^{1,1}+\omega^{0,2}$$

Since J is integrable, $\omega^{0,2} = \overline{\omega^{2,0}} = 0$, and $d = d_J^{1,0}+d_J^{0,1}$, so

$$\begin{aligned} 0 = d_J^{2,-1}\omega = \pi_J^{(3,0)}(dw) = \pi_J^{(3,0)}(3Im(\Upsilon)) = \pi_J^{3,0}(-\frac{3}{2}i(\Upsilon-\overline{\Upsilon})) \\ = \frac{3}{2}i(\overline{\pi_J^{0,3}(\Upsilon)}-\pi_J^{3,0}(\Upsilon)) = 12i(det(\overline{s})-det(r))\eta_1\wedge\eta_2\wedge\eta_3 \end{aligned}$$

which means

$$det(\overline{s})) = det(r)$$

We claims that $\omega = \omega_J^{(1,1)}$ can't be positive or negative definite and consequently $\omega\in\mathbb{J}_1$ or $\mathbb{J}_2$, it suffice to show ω can't be positive definite, if so, since

$$\omega = \omega_J^{(1,1)} = 2i^t\eta\wedge({}^t r\overline{r} -^t \overline{s}s)\overline{\eta}$$

${}^t r\overline{r} > {}^t r\overline{r} -^t \overline{s}s$ is also positive definite, hence $det(\overline{s}) = det(r) \neq 0$, note that ${}^t r\overline{r}$ and ${}^t\overline{s}s$ are positive semi-definite (${}^t v^t r\overline{r}\overline{v} = ||rv||^2$), ${}^t\overline{s}s$ would also be positive definite, but since they have the same determinant, ${}^t r\overline{r} - {}^t\overline{s}s$ couldn't be positive definite:
Let $A = {}^t r\overline{r}, B = {}^t\overline{s}s$, since A is Hermitian, there exists $P_1 \in U(n)$ such that $P_1^t A\overline{P}_1 = diag\{\lambda_1, \cdots, \lambda_n\}, \lambda_i \in \mathbb{R}$, since A is positive definite, $\lambda_i > 0$, let $\Lambda = diag\{1/\sqrt{\lambda_1}, \cdots, 1/\sqrt{\lambda_n}\}, P_2 = P_1\Lambda$, then $P_2^t A\overline{P}_2 = I_n$ and note that $B' = P_2^t A\overline{P}_2$ is also hermition and positive definite, we use unitary diagonal decomposition again and get $P_3 \in U(n)$ such that $P_3^t B'\overline{P}_3 = diag\{\lambda'_1, \cdots, \lambda'_n\}, \lambda'_i \in \mathbb{R}$ and at the same time, $P_3^t I_n \overline{P}_3 = I_n$.
Now, let $P = P_2P_3$, then $P^t(A-B)\overline{P} = I_n - diag\{\lambda'_1, \cdots, \lambda'_n\}$, since $A - B$ is positive definite, so is $P^t(A-B)\overline{P}$, hence $\lambda'_i < 1$, but $detA = detB$ implies $\lambda'_1 \cdots \lambda'_n = 1$, contradiction. $\square$

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