GEOMETRY OF QUOTIENTS

JINGXIANG MA

Contents

Keywords: Lie Group Action, Cohomology, Moment map.

1. Introduction

This is a summary of Kirwan's article [4] about cohomology of quotients, a brief introduction can be found in Mumford and Kirwan's book [5] 8.6 and 8.7.

Consider a compact symplectic manifold X with a Hamiltonian compact connected Lie group K action which admit a moment map μ, by fixing a K invariant metric, the K equivariant Morse theory of $f = ||\mu||^2$ is studied, and it turns out that although f is not a Morse Bott function, one can still get a Morse-like stratification of X. What's more, it turns out that this stratification is equivariantly perfect so we can compute the equivariant cohomology of X as a contribution of critical sets. One should mention that the torus action case was studied by Atiyah in [1], and Kirwan's work depends partly on this.

Since $\mu^{-1}(0)$ is always a critical set, we can compute $H_K(\mu^{-1}(0))$, when the stabiliser on $\mu^{-1}(0)$ is finite, $H_K(\mu^{-1}(0)) = H(\mu^{-1}(0)/K)$ in rational coeciffients. So we get a formula for $H(\mu^{-1}(0)/K;\mathbb{Q})$.

What's more, if X is a polirized Kahler manifold, or simply a smooth projective variaty, and the action extend to the complexification $G_{\mathbb{C}}$, which preserve the Kahler form, then one can consider the GIT quotient $X//G$.

Now using Kempf Ness theorem $X//G \cong \mu^{-1}(0)/K$, our formula can be used to compute $H(X//G)$.

One can also consider this picture in algebraic side, in this case the stratification is discussed totally algebraiclly, this stratification was originally studied by other mathematicians like Kempf, and is coincides with S_β as before. But since in this picture every thing is algebraic, one may use algebra tools. For example, the intersection cohomology can be used to doscuss the case the quotient is not a orbifold, and what's more, in the discussion of variation of GIT quotient, these discussions of stratification was used.

2. Preliminaries

2.1. **Settings For Two Moment Maps.** To avoid mistakes in coefficients, we fix our settings for two special moment map:

Consider the $U(n)$ action on $\mathbb{P}^{n-1}$, we take the symplectic form ω_{FS} as

$$\omega_{FS}|_{U_i} = \frac{i}{2}\partial\overline{\partial}log(1+\sum_{j\neq i}|z_j|^2)$$

where U_i is the chart $(x_i \neq 0)$.

Proposition 2.1. *Under our settings, the moment map is given by:*

$$< \mu[x_1, \cdots, x_n], \xi >= \frac{i}{2}\frac{z^*\xi z}{z^*z}$$

Consider the action of $SU(2)$ on $(\mathbb{P}^1, \omega_{FS})$, we choose the T as the diagonal maximal torus and $\mathbb{R} \cong Lie(T)$ by $\theta \mapsto \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}$

An element of $su(2)$ is of the form $\begin{pmatrix} i\omega & -\overline{z} \\ z & -i\omega \end{pmatrix}$ this gives a linear isomorphism $\varphi : su(2) \cong \mathbb{R}^3$ through

$$\begin{pmatrix} i\omega & -\overline{z} \\ z & -i\omega \end{pmatrix}, \mapsto (z, w)$$

, using the inner product on $su(2) :< a, b >:= \frac{1}{2}tr(a^*b)$, it's easy to check that:

Proposition 2.2. $< a, b >=< \varphi(a), \varphi(b) >_{\mathbb{E}^3}$.

Now we choose a homeomorphism $\psi : \mathbb{P}^1 \to S^2$:

$$[z_0, z_1] \mapsto (2i\frac{z_0\overline{z}_1}{|z|^2}, \frac{z_0\overline{z}_0 - z_1\overline{z}_1}{|z|^2})$$

Proposition 2.3. *Under the above choices, the moment map of $SU(2)$ on $\mathbb{P}^2 \cong S^2$ is given satisfies:*

$$\forall \beta \in \mathbb{R} \cong LieT, \mu_\beta(x) =< \mu(x), \beta >= -\frac{1}{2}x \cdot (0, 0, \theta)$$

2.2. Stratification of Manifolds.

Definition 2.4. *A finite collection $\{S_\beta, \beta \in \mathcal{B}\}$ of subsets from a stratification of X if X is the disjoint union of the strata S_β and there is a strict partial order $>$ on the indexing set $\mathcal{B}$ such that:*

$$\overline{S_\beta} \subset S_\beta \cup \bigcup_{\gamma > \beta} S_\gamma$$

for every $\beta \in B$.
The stratification is called smooth if every S_β is locally-closed submanifold of X.

Remark. This is not a general definition, for example, the stratification is not necessarily a Milnor stratification.

Example 2.5. *For a Morse (Bott) function on a closed manifold X, the underlying manifold X_λ where λ is a critical value gives a stratification of X.*

2.3. Equivariant Cohomology.

Definition 2.6. *If a topological group G acts on a space X, the* ***classifying space*** *EG is a principle G bundle $EG \to BG$ whose total space is contractible, we define*

$$X_G \to BG : X_G = EG \times_G X = \{EG \times X\}/\{(v, gx) \sim (vg, x)\}$$

as the associated bundle, and since BG is unique up to homotopy

$$H_G(X) := H(X_G)$$

only depends on G and is called equivariant cohomology of X.

Example 2.7. *The inductive limit of $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ is $S^1 \to ES^1 \to BS^1$.*

Proposition 2.8. *If K is a closed normal subgroup of G, then*

$$H_G(X) \cong H_{G/H}(X/K)$$

Proposition 2.9. *In the case of free action, $H_G(X) := H(X/G)$ in any coefficient, since $X_G \to X/G$ have contractible fiber.*
If any point only have finite stabilizer, then $H_G(X;\mathbb{Q}) := H(X/G;\mathbb{Q})$, since $H^*(BK;\mathbb{Q})$ is trivial when K is a finite group, here the map $X \times_G EG \to X/G$ is considered.

Proposition 2.10. *(Equivariant Thom-Gysin sequence)*
For any smooth oriented K invariant submanifold Y of X with codimension k, then there is a long exact sequence:

$$\to H_K^{m-k}(Y) \to H_K^m(X) \to H_K^m(X-Y) \to H_K^{m-k-1}(Y) \to$$

which is called the equivariant Thom-Gysin sequence. The first map is given by multiply the so called ***equivariant Euler class****.*

Proposition 2.11. *Suppose $\{S_\beta : \beta \in \mathcal{B}\}$ is a smooth K-invariant stratification of X, such that for each β, the equivariant Euler class of the normal bundle to S_β in X is not a zero-divisor in $H_K^*(S_\beta;\mathbb{Q})$, then the stratification is equivariantly perfect over $\mathbb{Q}$.*

Proposition 2.12. *If N is a complex vector bundle over a connected space Y and a compact group K acts as a group of bundle automorphisms. Suppose there is a subtorus T of K, which acts trivially on Y, and the representation of T on the fiber of N at any point of Y has no fixed point, then the equivariant Euler class of N in $H_K^*(Y;\mathbb{Q})$ is not a zero divisor.*

Proposition 2.13. *Suppose K is a compact connected Lie group acts on a compact symplectic manifold, and moment map exists, then for rational equivariant cohomology, we have:*

$$P_t^K(X;\mathbb{Q}) = P_t(X;\mathbb{Q})P_t(BK;\mathbb{Q})$$

2.4. **Convexity and Commuting Hamiltonians.**

Theorem 2.14. *(Atiyah & Guilenberg)*
For a Hamiltonian torus action on a compact symplectic manifold $T \curvearrowright (X,\omega)$, and suppose μ_T is a moment map, then the image of fixed point of T under μ_T is a finite set $\mathbb{A} \subset \mathfrak{t}^*$ and $\mu_T(X)$ is the convex hull $Conv(\mathbb{A})$ of $\mathbb{A}$.

Proof. See, for example, [1]. □

Theorem 2.15. *The moment map of a circle action is Morse Bott.*

2.5. Morse Theory of Minimally Degenerate Functions.

Definition 2.16. *A smooth function $f : X \to \mathbb{R}$ on a compact manifold X is called minimally degenerate if the following conditions hold:*
(1) The set of critical points for f on X is a finite union of disjoint closed subsets $\{C_\beta, \beta \in B\}$ on each of which f takes a constant value $f(C_\beta)$. The subsets are called critical subsets of f. If the critical set of f is reasonably well behaved we can take the subsets $\{C_\beta\}$to be its connected components.
(2) For every C_β there is a locally closed submanifold Σ_β containing C_β and with orientable normal bundle in X such that:
(a) C_β is the subset of ΣC_β on which f takes its minimum value.
(b) at every point $x \in C_\beta$ the tangent space $T_x\Sigma C_\beta$ is maximal among all subspaces T_xX on which the Hessian $H_x(f)$ is positive-definite.
A submanifold satisfying these properties is called a minimising manifold for f along C_β.

Proposition 2.17. *Let $f : X \to \mathbb{R}$ ba a minimally degenerate Morse function with critical subsets $\{C_\beta, \beta \in B\}$ on a compact manifold X, then we there exists a polynomial $R(t)$ with nonnegative integer coefficients, such that the Poincare polynomial satisfies:*

$$\sum_{\beta \in B} t^{\lambda(C_\beta)} P_t(C_\beta) - P_t(X) = (1+t)R(t)$$

Where $\lambda(C_\beta)$ is the index of f along C_β (we may assume C_β is connected so that λ is well defined).

Definition 2.18. *Suppose $f : X \to \mathbb{R}$ is a minimally degenerate Morse function with critial subsets $\{C_\beta, \beta \in B\}$ and suppose that X is given a fixed Riemannian metric. Then for each β let S_β be the subset of X consisting of all points $x \in X$ such that the limit set $\omega(x)$ of the trajectory of $-gradf$ from x is contained in C_β. X is the disjoint union of the subsets S_β.*

Theorem 2.19. *Let f be a minimally degenerate Morse function with critical subsets $\{C_\beta, \beta \in B\}$ on a compact Riemannian manifold. Suppose that the gradient flow of f is tangential to each of the minimising manifolds $\{\Sigma_\beta : \beta \in B\}$. Then the subsets $S_\beta, \beta \in B$ defined above form a smooth stratification of X called the Morse stratification of the function f on X.*
For each β the stratum S_β coincides with the minimising submanifold Σ_β in some neighbourhood of C_β.
Moreover each inclusion $C_\beta \to \Sigma_\beta$ is an equivalence of Cech cohomology. If there is a compact group K acting on X such that f, the minimising manifolds and the metric are all invariant under K then these inclusions are also equivalences of equivariant cohomology.

2.6. Lie Group theory needed.

2.6.1. *On compact connected Lie group.*

Definition 2.20. *A subgroup T of G is a* ***maximal torus*** *if T is a torus is maximal in the sense of subgroup.*

Proposition 2.21. *Any two maximal torus of G are conjugate to each other.*

Now fix a maximal torus T.

Definition 2.22. *Let $N = N_G(T) \leqslant G$, then the group $W = N/T$ is called a Weyl group.*

Theorem 2.23. *The Weyl group W satisfied:*
(1) W is finite;
(2) W acts effectively on T through conjugation. By effectively we mean the hom $W \to Aut(T)$ is injective.
(3) For $x, y \in T$, if $gxg^{-1} = y$, then there exists $n \in W$ such that $nxn^{-1} = y$.

Corollary 2.24. *There is a canonical homeomorphism:*

$$\kappa : T/W \cong CongG : Wt \mapsto c(G)t)$$

Proposition 2.25. *Any finite dimensional linear representation of T is determined by its weight. In another word, the representation space can be decomposed into 1 dimensional subspaces (weight spaces):*

$$V = \oplus_i V_{\alpha_i}, \alpha_i \in \mathbb{Z}^l$$

The restriction of T on V_{α_i} is of the form: $(t_1, \cdots, t_l)(v) = t_1^{\alpha_1} \cdots t_l^{\alpha_l} v = t^{\alpha} v$.

Note tha a representation satisfies $\varphi(ftg^{-1})$, we turn to study the conjugate representation of T on $\mathfrak{k}$, then

Definition 2.26. *since for $\vartheta_\alpha : T \to U(1) : exp(H) \to e^{2\pi i\alpha(H)}$ satisfies: $\vartheta_{-\alpha} = \overline{\vartheta_\alpha}$, we get two decomposition:*

$$\mathfrak{k}_{\mathbb{C}} = L_0 \bigoplus_{\alpha \in R^+} L_\alpha$$

$$\mathfrak{k} = M_0 \bigoplus_{\alpha \in R} M_\alpha, M_\alpha = (L_\alpha \oplus L_{-\alpha}) \cap \mathfrak{k}$$

Here R^+ is a choose in $\{-\alpha, \alpha\}$.

Proposition 2.27.

$$M_0 = \mathfrak{t}, L_0 = \mathfrak{t}_{\mathbb{C}}$$

Definition 2.28. *(1) $U_\alpha = Ker\vartheta_\alpha = Ker\alpha$, here we consider $\alpha \in \mathfrak{t}^*$;*
(2) $\mathcal{H}_\alpha = Lie(U_\alpha) = Ker\alpha$;
(3) $\mathcal{H}_\alpha$ separates $\mathfrak{t}$ into disjoint connected components, which are called ***Weyl chambers****, every chamber is of the form*

$$C_\epsilon = \bigcap_{\alpha \in R^+} \{v \in \mathfrak{t}, \epsilon_\alpha \alpha(v) > 0\}$$

here $\epsilon = (\epsilon_\alpha = \pm 1)$;
(4) For any pair of chamber and hypersurface $(C, \mathcal{H}_\alpha)$, $\overline{C} \cap \mathcal{H}_\alpha$ is called a ***wall****.*

Proposition 2.29. *Fix a K inv inner product on $\mathfrak{k}$, set S_α as reflection w.r.t $\mathcal{H}_\alpha$, consider the conjugate action of W on T, then:*
(1)$S_\alpha \in W$;
(2)Given any chamber C, the reflection in the walls of C generates W;
(3)The action of W on the chambers is simply transitive.

Corollary 2.30. *Fix a chamber C, then any conjugate class of $\mathfrak{k}$ intersects with a unique element in $\overline{C}$.*

2.6.2. *On reductive group.*

Definition 2.31. *If a complex Lie group G is the complexification of a maximal compact subgroup K, then G is called a reductive group.*

Proposition 2.32.

$$G = Kexp(i\mathfrak{k})$$

Definition 2.33. *(Borel Subgroup)*
Fix a maximal torus of K and a positive Weyl chamber $\mathfrak{t}_+$, and suppose the root space decomposition is given by

$$\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha} \mathfrak{g}^\alpha$$

*Then $B = exp\mathfrak{b}$ is called the **Borel subgroup** associated to $\mathfrak{t}_+$, where*

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha+} \mathfrak{g}^\alpha$$

Proposition 2.34.

$$G = BK$$

Proposition 2.35. *(Bruhat decomposition)*

$$G = BN_K(T)B$$

Definition 2.36. *(Parabolic Subgroup)*

Definition 2.37. *For $\beta \in \mathfrak{t}$, we define*

$$Stab\beta = k \in K : Adk(\beta) = \beta, stab\beta := Lie(Stab\beta) = \{a \in \mathfrak{k}, [a, \beta] = 0\}$$

Proposition 2.38. *For $\beta \in \mathfrak{t}_+$, Let*

$$P_\beta = \{g \in G, \exists \lim_{t \to +\infty} (exp(it\beta)g(expit\beta)^{-1})\}$$

Then P_β is a parabolic subgroup of G and $P_\beta = BStab\beta$.

2.7. **Some GIT theory.** In this case X is a projective variety and $R(X) = k[x_0, \cdots, x_n]/I(X)$, I(X) is the homogeneous ideal of X.

Definition 2.39. *For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we define the GIT quotient:*

$$X//G = Proj(R^G)$$

Definition 2.40. *A point $x \in X$ is:*
(1) ***semistable*** *if there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $f(x) \neq 0$.*
(2) ***polystable*** *if $x \in X^{ss}$ and $\overline{G \cdot x} = G \cdot x$.*
(2) ***stable*** *if $x \in X^{ss}$ and $dim G_x = 0$.*
(3) ***unstable*** *if it is not semistable.*
We denote the set of stable points by X^s and the set of semistable points by X^{ss} .

Proposition 2.41. *(Topological criterion for semistability)*
Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For $x \in \mathbb{P}^n$, choose a non-zero lift $\widetilde{x}$ in the affine cone $\widetilde{X} \in \mathbb{A}^{n+1}$. Then the following statements hold.
(1) x is ***semistable*** *if and only if $0 \notin \overline{G \cdot \widetilde{x}}$.*
(2) x is ***polystable*** *if and only if $G \cdot \widetilde{x}$ is closed.*
(3) x is ***stable*** *if and only if $G \cdot \widetilde{x}$ is closed and $dim G_x = 0$.*

Proposition 2.42. *$x \in X^{ss}_G$ iff for all 1-parameter subgroup (1-PS) $\lambda : \mathbb{C}^* \to G$, $x \in X^{ss}_{\mathbb{C}^*}$*

Proposition 2.43. *If $\lambda : \mathbb{C}^* \to GL(n+1)$ is given by $z \mapsto diag(z^{r_0}, \cdots, z^{r_n}$, then $x \in \mathbb{P}^n$ is semistable iff*

$$min\{r_j : x_j \neq 0\} \leqslant 0 \leqslant max\{r_j : x_j \neq 0\}$$

Definition 2.44. *(Hilbert-Mumford criterion)*

$$\mu(x, \lambda) := -min\{m_i : a_i \neq 0\}$$

and this number is independent of the choice of the lift and the basis e_i.

Theorem 2.45. *(Hilbert-Mumford)*
Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$, then

$$x \in X^{ss} \Leftrightarrow \mu(x, \lambda) \geqslant 0, \forall \lambda$$
$$x \in X^{s} \Leftrightarrow \mu(x, \lambda) > 0, \forall \lambda$$

Theorem 2.46. *(Kempf-Ness theorem I)*
Let $G = K^{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ such that its maximal compact subgroup K acts unitarily with moment map $\mu : X \to \mathfrak{k}$. Then the following statements hold for closed points $x \in X$:
i) $\overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset \Leftrightarrow x \in X^{ss}$;

ii) $G \cdot x \cap \mu^{-1}(0) \neq \emptyset \Leftrightarrow x \in X^{ps}$*;*
iii) If $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$*, then this intersection is a single* K *orbit.*

Theorem 2.47. *(Kempf-Ness theorem II)*
i) $G\mu^{-1}(0) = X^{ps}$ *.*
ii) If $x \in X$ *is polystable, then its orbit* $G \cdot x$ *meets* $\mu^{-1}(0)$ *in a single* K*-orbit.*
iii) $x \in X$ *is semistable if and only if its orbit closure* $\overline{G \cdot x}$ *meets* $\mu^{-1}(0)$*.*

Corollary 2.48. *The inclusion* $\mu^{-1}(0) \subset X^{ss}$ *induce a homeomorphism* $\mu^{-1}(0)/K \to X//G$*.*

3. Compact Connected Group Action

In this section, X is a compact symplectic manifold with a Hamiltonian compact connected Lie group K action. We fix the following choices:
(1)A moment map μ w.r.t the K action;
(2)A K invariant metric g on X;
(3)A K invariant metric $<,>$ on $\mathfrak{k}$;
(4)A maximal torus T and a Weyl chamber $\mathfrak{t}_+$.
And we have some definitions:

Definition 3.1. *Assume we have fix the above things,*
(1)For $\beta \in \mathfrak{t}$, $T_\beta := \overline{exp(\mathbb{R}\beta)}$;
(2)$\mu_\beta = \mu \cdot \beta$ w.r.t the metric on $\mathfrak{k}$.
(3)The Hamiltonian vector field generated by an element $\beta \in \mathfrak{k}$ is denoted by X_β;
(4)The induced moment map of T is denoted by μ_T.

3.1. **Critical Sets of Square of Moment Map.**

Definition 3.2.
$$f = ||\mu||^2 : X \to \mathbb{R}, f_T := ||\mu_T||^2$$

Proposition 3.3. *Suppose $\mu(x) = \beta$, then TFAE:*
(1)x is a critical point of f ;
(2)$X_\beta(x) = 0$;
(3)x is a critical point of μ_β;
(4)$x \in FixT_\beta$.

Definition 3.4. *For $\beta \in \mathfrak{t}$, we set*
$$Z_\beta = \{x \in cr.(\mu_\beta), \mu_\beta(x) = ||\beta||^2\}$$
here $cr.(\mu_\beta)$ means the critical set of μ_β.

Corollary 3.5. *Suppose $\mu(x) = \beta \in \mathfrak{t}$, then x is a critical point of f_T iff $x \in Z_\beta$, since f_T is T invariant, Z_β is T invariant.*

Remark. In fact, Z_β is a symplectic submanifold.

Note that For torus action ($T \curvearrowright Z_\beta$), we have the convexity theorem, so we define:

Definition 3.6. *For $\beta \in \mathfrak{t}$, we call β a minimal combination of weights of T, if for some subset $\mathbb{A}'$ of $\mathbb{A}$, β is the closest point to 0 of $Conv(\mathbb{A}')$.*
we denote by $\mathcal{B}$ the (finite) set of all minimal combinations of weights lying in $\mathfrak{t}_+$.

Remark. Since the vertices of A are all integral points, any $\beta \in \mathcal{B}$ is a rational point.

Corollary 3.7. *If x is a critical point of f_T, then $\mu(x)$ is a minimal combination of weights of T.*

Definition 3.8. *For $\beta \in \mathcal{B}$, we set:*

$$C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$$

Proposition 3.9. *The critical set of f is $\{C_\beta, \beta \in \mathcal{B}\}$*

Remark. In particular, for torus action case, the critical set of f is

$$\{Z_\beta \cap \mu^{-1}(\beta), \beta \in \mathcal{B}\}$$

3.2. **A Description of The Morse Stratification S_β.**

Definition 3.10. *We set the gradient flow of $-gradf$ by $\varphi(x,t)$ and define:*

$$S_\beta = \{x \in X, \overline{\{\varphi(x,\mathbb{R}^+)\}} \setminus \{\varphi(x,\mathbb{R}^+)\} \subset C_\beta\}$$

Proposition 3.11. *The inclusion $i_\beta : C_\beta \to S_\beta$ is an equiv of Cech cohomology and K-equivariant cohomology.*

Definition 3.12. *We set the gradient flow of $-grad\mu_\beta$ by $\varphi_\beta(x,t)$ and define:*

$$Y_\beta = \{x \in X, \lim_{t\to+\infty} \varphi_\beta(x,t) \in Z_\beta\}$$

remember we have defined

$$Stab\beta = \{k \in K : Adk(\beta) = \beta\}, stab\beta := Lie(Stab\beta) = \{a \in \mathfrak{k}, [a,\beta] = 0\}$$

Proposition 3.13. *The restriction of μ (hence $\mu - \beta$) onto Z_β maps to $stab\beta$ and can be considered as a moment map for the action of $Stab\beta$ on Z_β.*

Lemma 3.14. *For any $x \in Z_\beta \cap \mu^{-1}(\beta)$,*

$$\{k \in K, kx \in Y_\beta\} = Stab\beta, \{a \in \mathfrak{k}, X_a(x) \in T_x Y_\beta\} = stab\beta$$

Corollary 3.15. *Remeber that $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$, then C_β is homeomorphism to $K \times_{Stab\beta} (Z_\beta \cap \mu^{-1}(\beta))$.*

Proposition 3.16. *KY_β is a smooth submanifold when restrict to some K-invariant nbhd Σ_β of C_β, to be explicit , near any $x \in Z_\beta \cap \mu^{-1}(\beta)$, KY_β is locally homeomorphic to $K \times_{Stab\beta} Y_\beta$.*

By apply morse theory to $||\mu||^2$, Kirwan shows:

Theorem 3.17. *S_β coincides in a nbhd with Σ_β (hence KY_β) hence is smooth, and S_β gives a Morse stratification in our sense. What's more, the inclusion $i_\beta : C_\beta \to S_\beta$ is an equivalent of Cech and K-equivariant cohomology.*

Definition 3.18. *For $x \in Z_\beta$, the index of μ_β is defined as the dimension of maximal subspace of T_xX, on which $H_x(\mu_\beta)$ is negative definite.*

Definition 3.19. *Since Z_β maybe not connected, so to fix the codimension, we set for $m \in \mathbb{Z}^{\geqslant 0}$:*

$$Z_{\beta,m} = \{x \in Z_\beta, ind\mu_\beta(x) = m\}$$

and

$$C_{\beta,m} = K(Z_{\beta,m} \cap \mu^{-1}(\beta)) \cong K \times_{Stab\beta} (Z_{\beta,m} \cap \mu^{-1}(\beta))$$

Proposition 3.20. *The index of $H_x(f)$ at any $x \in C_{\beta,m}$ is*

$$d(\beta, m) = m - dimK + dimStab\beta$$

which is the codim of the component of Σ_β containg x.

3.3. A Equivariant Cohomological Formula.

Proposition 3.21. *By the description 3.15 of C_β we get:*

$$H_K^*(C_\beta) \cong H_{Stab\beta}(Z_\beta \cap \mu^{-1}(\beta))$$

$$H_K^*(C_{\beta,m}) \cong H_{Stab\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

Theorem 3.22. *The stratification $\{S_\beta\}$ defined is equivariantly perfect hence*

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^K(C_{\beta,m}) = \sum_{\beta,m} t^{d(\beta,m)} P_t^{Stab\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

Remark. In [2] Atiyah and Bott applied equivariant Morse theory to Yang Mills functional to study the moduli space of stable bundle, it turns out that stratification and cohomology formula still works.

By using 2.13, and note that $\mu^{-1}(0)$ is always a critical set, one get:

Corollary 3.23.

$$P_t^K(\mu^{-1}(0)) = P_t(X)P_t(BK) - \sum_{\beta \neq 0,m} t^{d(\beta,m)} P_t^{Stab\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

Remark. Consider the following two natural maps:

$$i : \mu^{-1}(0) \to X; \pi : \mu^{-1}(0) \times_K EK \to \mu^{-1}(0)/K$$

which induces morphism on cohomology level:

$$i^* : H_K^*(X) \to H_K^*(\mu^{-1}(0)); \pi^* : H^*(\mu^{-1}(0)/K) \to H_K(\mu^{-1}(0))$$

When stabilizers are finite and considering $\mathbb{Q}$, then π^* is an isom, hence we get the following surjective ring hom:

$$\kappa = (\pi^*)^{-1} \circ i^* : H_K^*(X;\mathbb{Q}) \to H^*(\mu^{-1}(0)/K;\mathbb{Q})$$

In the language of GIT, this is the case $X^{ss} = X^s$ and topologically $\mu^{-1}(0)/K \cong X//G$ for its complexification hence this can be rewritten as

$$\kappa : H^K(X;\mathbb{Q}) \to H(X//G;\mathbb{Q})$$

this map is often called **Kirwan map**.

3.4. **Torus Action Case.** In special, we consider the case $K = T$.
In this case, $\mathbb{A} = \mu(Fix(T))$, we don't need to choose a chamber, and $\mathcal{B}$ is directly those minimal combinations, then we use the definition (see 3.4 and 3.12)

$$Z_\beta = \{x \in cr.(\mu_\beta), \mu_\beta(x) = ||\beta||^2\}$$
$$Y_\beta = \{x \in X, \lim_{t\to+\infty} \varphi_\beta(x,t) \in Z_\beta\}$$

to compute Z_β and Y_β, also we compute C_β and S_β (see 3.8 and 3.10):

$$C_\beta = Z_\beta \cap \mu^{-1}(\beta)$$
$$S_\beta = \{x \in X, \overline{\{\varphi(x,\mathbb{R}^+)\}} \setminus \{\varphi(x,\mathbb{R}^+)\} \subset C_\beta\}$$

In this case they are all T invariant, so in fact, by 3.15,

$$C_\beta = Z_\beta \cap \mu^{-1}(\beta)$$

since in this case the moment map is morse bott, we can directly compute the indexes, and

$$Z_\beta = \coprod Z_{\beta,m}, S_\beta = \coprod S_{\beta,m}$$

indexing by indexes, then the cohomology formula becomes:

$$P_t^T(X) = \sum_{\beta,m} t^m P_t^T(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

In conclusion, we just need to compute the image of fixed points under moment map, and then compute

4. Reductive Group Acts on Kahler Manifold

In this section, G will be the complexification of a compact connected lie group K, and K preserves the Kahler form. We choose the complex structure on TX such that $J = i$.
For example, we may consider the $U(n+1)$ action on $(\mathbb{P}^n, \omega_{FS})$.
By definition, we have:

Proposition 4.1.

$$grad\mu_\beta(x) = iX_\beta(x), gradf(x) = 2iX_{\mu(x)}(x)$$

Hence the flow of $-grad\mu_\beta$ is given by:

$$\varphi_\beta(x,t) = exp(-it\beta)(x)$$

4.1. **Another Description of S_β.**

Definition 4.2. *$T, \mathbb{A}, \mathfrak{t}_+, \mathcal{B}, S_\beta, C_\beta, Z_\beta$, are the same as defined before.*
For $\beta \in \mathfrak{t}_+$, remember we have defined subgroup $Stab_\beta$ of K and its Lie algebra, we also have a geometric description of them (see 3.14),
also we have Borel subgroup B associated to $\mathfrak{t}^+$, with Lie algebra $\mathfrak{b}$ (see 2.33), and the parabolic subgroup P_β (see 2.38)of G.

Definition 4.3. *Remember that Y_β is defined as stable manifold of Z_β with torus T_β action (see 3.12), in this case μ_β is Morse Bott (see 2.15) , so we can define a retraction:*

$$p_\beta : Y_\beta \to Z_\beta$$

remember also that $\mu - \beta$ can be considered as the moment map of the action of $Stab\beta$ on Z_β (see 3.13) We define the flow of $-grad||\mu - \beta||^2$ by $\psi_\beta(x,t)$, and define:

$$Z_\beta^{min} = \{x \in Z_\beta, \overline{\{\psi_\beta(x, \mathbb{R}^+)\}} \setminus \{\psi_\beta(x, \mathbb{R}^+)\} \subset Z_\beta \cap \mu^{-1}(\beta)\}$$

and

$$Y_\beta^{min} = p_\beta^{-1} Z_\beta^{min}$$

Proposition 4.4. *Y_β, Y_β^{min} are all P_β invariant.*

Lemma 4.5. *If $x \in GY_\beta^{min}$, then β is the unique closet point to 0 of $\mu(\overline{Gx}) \cap \mathfrak{t}_+$.*

Proposition 4.6. *Let $x \in Y_\beta^{min}$, then*

$$\{g \in G, gx \in Y_\beta^{min}\} = P_\beta, \{a \in LieG, X_a(x) \in T_x Y_\beta^{min}\} = LieP_\beta$$

Theorem 4.7. *GY_β^{min} is smooth and*

$$GY_\beta^{min} \cong G \times_{P_\beta} Y_\beta^{min}$$

Theorem 4.8. *$S_\beta = GY_\beta^{min}$ so that*

$$H_G^*(S_\beta; \mathbb{Q}) \cong H_{P_\beta}(Y_\beta^{min}; \mathbb{Q})$$

4.2. **Relationship with GIT.**

Remark. In this part, we may assume the stabilizer of K on $\mu^{-1}(0)$ is finite, we will mention this assumption again if we need it.

Definition 4.9.

$$X^{min} := S_0$$

Lemma 4.10. *If $x \in \mu^{-1}(0)$, then $Gx \cap \mu^{-1}(0) = Kx$.*

Lemma 4.11. *If $x \neq y$ in $\mu^{-1}(0)/K$, then $x \neq y$ in X/G*

Theorem 4.12. *If the stabilizer of K on $\mu^{-1}(0)$ is finite, then the natural map*

$$\mu^{-1}(0)/K \to X^{min}/G$$

is a homeomorphism.

Remark. If we remove the assumption on stabilizers, change the right hand side by $X//G$ the consequence is still true.

The next step to identify X^{min} with X^{ss}.

Lemma 4.13. *When $G = \mathbb{C}^*$, $X^{ss} = X^{min}$*

Lemma 4.14. *for $x \in X$, then $0 \in \mu(\overline{Gx})$ iff $0 \in \mu_\lambda(\overline{\lambda(\mathbb{C})x})$ for any 1-PS λ compatible with K (that is, a complexification of some 1-PS: $S^1 \to K$).*

Theorem 4.15.

$$X^{ss} = X^{min}$$

4.3. **Finite Stablizer Case: A Cohomological Formula for GIT quotient.** In this part, we always assume the stabilizer of K on $\mu^{-1}(0)$ is finite. First observation is to combine 2.9 and 4.12.

Corollary 4.16.

$$H^*(X//G;\mathbb{Q}) = H^*(X^{ss}/G;\mathbb{Q}) \cong H^*_G(X^{ss};\mathbb{Q}) \cong H^*(\mu^{-1}(0)/K;\mathbb{Q}) \cong H^*_K(\mu^{-1}(0);\mathbb{Q})$$

Also use 2.13, formula 3.22 and note that $X^{ss} = S_0$, we get a formula to calculate the

Corollary 4.17.

$$P_t(X//G;\mathbb{Q}) = H^*_G(S_0;\mathbb{Q}) = P_t(X;\mathbb{Q})P_t(BG;\mathbb{Q}) - \sum_{\beta \in \mathcal{B}\setminus\{0\},m} t^{d(\beta,m)} P_t^{Stab\beta}(Z^{ss}_{\beta,m};\mathbb{Q})$$

Remark. Here we use the same argument in 3.11 to the inclusion

$$Z^{ss}_{\beta,m} = Z^{min}_{\beta,m} \to Z_{\beta,m} \cap \mu^{-1}(\beta)$$

Remark. When the stabilisers are not necessarily finite, one may consider the intersection cohomology, which was discussed in Kirwan's another paper [3]

5. More About Kirwan's Article

5.1. **A Refined Stratification of S_β.**

5.2. **An Algebraic Description of S_β.** In this part, $X \subset \mathbb{P}^n$ will be a variety over an algebraic closed field $\mathbb{C}$, G is a reductive group acts linearly on X.

Definition 5.1. *For any $\lambda : \mathbb{C}^* \to GL(n+1)$ of the form: $z \to diag(z^{r_0}, \cdots, z^{r_n}), r_i \in \mathbb{Z}$, for any $x = [x_i] \in X$, we define $m(x,\lambda) \in \mathbb{Z}^{\geqslant 0}$ by:*

$$m(x;\lambda) = max\{0, min\{r_j : x_j \neq 0\}\}$$

i.e, if $min\{r_j : x_j \neq 0\} \geqslant 0$, then $m(x;\lambda) = min\{r_j : x_j \neq 0\}$; otherwise define $m(x;\lambda) = 0$.

Corollary 5.2. *$x \in X^{ss}$ iff $m(x,\lambda) = 0$ for every 1-PS λ of G.*
$x \in X^{us}$ iff $m(x,\lambda) > 0$ for some 1-PSλ of G.

Definition 5.3.

$Y(G)=\{\lambda:\mathbb{C}^*\to G\}, M(G)=\{(\lambda,l),\lambda\in Y(G)\}/\{(\lambda,l)\sim(\mu,m)\Leftrightarrow\lambda(t^m)=\mu(t^l)\}$

We consider them as $\mathbb{Z}$ module and $\mathbb{Q}$ module.
We extend $m(x,\lambda)$ to $M(G)$ so that it satisfies $m(x,r\lambda)=rm(x,\lambda)$ for any $r\in\mathbb{Q}$.

Proposition 5.4. *(Hes)*
One can define a norm on $M(G)$ such that:
(1) $q:M(G)\to\mathbb{Q}$ is G-invariant;
(2) For any torus $T\subset G$, $q|_{M(T)}$ is a quadratic form.

Remark. We may fix a maximal torus and find a W inv norm on T, where W is the Weyl group.

Definition 5.5. *For any $x\in X$, let*

$$q^{-1}(x)_G=inf\{q(\lambda):\lambda\in M(G),m(x,\lambda)\geqslant 1\}$$
$$\Lambda_G(x)=\{\lambda\in M(G):m(x,\lambda)>0,q(\lambda)=q^{-1}(x)_G\}$$

Corollary 5.6. *$x\in X^{us}$ iff $q^{-1}(x)_G<\infty$, iff $\Lambda_G(x)\neq\emptyset$.*

Definition 5.7. *Let T be a maximal torus of G, then the representation of T on $\mathbb{C}^{n+1}$ splits as sum of scalar reps by characters $\alpha_0,\cdots,\alpha_n\in M(G)^*$, we use q to identify them as elements in $M(G)$.*

Definition 5.8. *We call the closest point to 0 of the convex hull in $M(T)$ of any nonempty subset of $\{\alpha_0,\cdots,\alpha_n\}$ a **minimal combination of weights**. Let $\mathcal{B}$ be the sets of all minimal combinations of weights lying in some positive Weyl chamber.*

Definition 5.9. *A subgroup H of G is called **optimal** for x if*

$$q_H^{-1}(x)=q_G^{-1}(x)$$

Proposition 5.10. *For any $x\in X$, there exists some $g\in G$ such that T is optimal for gx.*

Definition 5.11. *If $\lambda\in M(G)$, consider the conjugate action of G on G let*

$$P_\lambda=\{g\in G,m(g,\lambda)\geqslant 0\}$$

(note that now $m(g,\lambda)$ values in $\mathbb{Q}$).

Proposition 5.12. *(Hes)*
For each $\lambda\in M(G)$, P_λ is a parabolic subgroup of G

Proposition 5.13. *(Kempf)*
(1) For each $x\in X^{us}$, there is a unique parabolic subgroup $P(x)$, such that $P(x)=P_\lambda$ for all $\lambda\in\Lambda_G(x)$;
(2) $\Lambda_G(x)$ is a single $P(x)$ orbit under the adjoint action of G on $M(G)$;
(3) If $\lambda\in\Lambda_G(x)$ then we have: $g^{-1}\lambda g$ also lies in $\Lambda_G(x)$ iff $g\in P(x)$, in particular, $Stab_G(x)\subset P(x)$;

(4) $\Lambda_G(x) \subset M(P(x))$*;*
(5) If T *is optimal for* x *and* $\Lambda_T(x) = \{\beta/q(\beta)\}$*, then* $P(x) = P_\beta$*.*

Definition 5.14. *For each nonzero* $\beta \in M(T)$ *let*

$$S_\beta = G\{x \in X : \beta q(\beta) \in \Lambda_G(x)\}$$

$$S_0 = G\{x \in X : \Lambda_G(x) = \emptyset\}$$

Theorem 5.15. $\{S_\beta, \beta \in \mathcal{B}\}$ *gives a stratification of* X*.*

Definition 5.16.

$$Z_\beta = \{[x_0, \cdots, x_n] \in X : x_j = 0 \, if \, \alpha_j \cdot \beta \neq q(\beta)\}$$

$$Y_\beta = \{[x_0, \cdots, x_n] \in X : x_j = 0 \, if \, \alpha_j \cdot \beta < q(\beta), \exists j, \, s.t \, x_j \neq 0 \& \alpha_j \cdot \beta = q(\beta)\}$$

$$p_\beta : Y_\beta \to Z_\beta : [x_0, \cdots, x_n] \to [p_\beta x_0, \cdots, p_\beta x_n]$$

where $p_\beta x_j = x_j$ *if* $\alpha_j \cdot \beta = q(\beta)$*,* $p_\beta x_j = 0$ *otherwise.*

Definition 5.17.

$$Z_\beta^{ss} = \{x \in Z_\beta : \beta/q(\beta) \in \Lambda_G(x)\}$$

$$Y_\beta^{ss} = p_\beta^{-1} Z_\beta$$

Lemma 5.18. *Suppose* $\beta \neq 0$*,* $y \in Y_\beta$ *and* $x = p_\beta(y)$*, then TFAE:*
(1) T *is optimal for* y *and* $\Lambda_T(y) = \{\beta/q(\beta)\}$*;*
(2) $y \in S_\beta$*;*
(3) $y \in Y_\beta^{ss}$*;*
(4) $x \in X_\beta^{ss}$*;*
(5) $x \in S_\beta$

Corollary 5.19. *For any* $\beta \in \mathcal{B}$*,* $S_\beta = GY_\beta^{ss}$*.*

Proposition 5.20. *If* X *is nonsingular,* S_β *and* Y_β^{ss} *coincides with those defined in previous parts.*

5.3. **Hesselink Strata.** Remember that β is a rational point, we define λ_β by the 1-PS corresponds to $n\beta \in \chi_*(G)$ where n is choice such that $n\beta$ is primitive.

Theorem 5.21. *There is a bijective correspondence between the moment map strata* $\{S_\beta\}$ *and the Hesselink strata* $\{S^L_{d,<\tau>}\}$*, given by* $S_\beta \to S^L_{||\beta||,<\lambda_\beta>}$*, under this correspondence:*
(i) $P(\beta) = P(\lambda_\beta), Stab_\beta = L(\lambda)$*;*
(ii) $Z_\beta = X^{\lambda_\beta}_{||\beta||}, Z_\beta^{min} = Z^L_{||\beta||,\lambda}$*;*
(iii) $Y_\beta^{min} = S^L_{||\beta||,\lambda_\beta}$*.*

5.4. **Noncompact Manifold Case.**

6. Examples

6.1. $SU(2)$ acts on $(\mathbb{P}^1)^n$ Diagonally.

Proposition 6.1. *Consider $SU(2)$ acts on $(\mathbb{P}^1)^n$ Diagonally,*
We choose the diagonal elements as maximal torus, and the Weyl chamber $\mathfrak{t}_+$ as the positive part $\mathbb{R}^+$.
Under the isom $\mathbb{P}^1 \cong S^2, su(2) \cong \mathbb{R}^3$, the moment map is given by:

$$\mu : (S^2)^n \to su(2), (x_1, \cdots, x_n) \mapsto -\frac{1}{2}\sum_{i=1}^{n} x_i$$

Then

$$Fix(T) = \{(\epsilon_1, \cdots, \epsilon_n), \epsilon_k \in \{0, \infty\}\}$$

so

$$\mathbb{A} = \mu(Fix(T)) = \{2r - n; 0 \leqslant r \leqslant n\}$$

and

$$\mathcal{B} = \{r; n/2 \leqslant r \leqslant n\}$$

For each $\beta = r \in \mathcal{B}$,
$Z_\beta = \{(x_1, \cdots, x_n)$ precisely r many 0's and (n-r) many ∞'s$\}$
and
$Y_\beta = \{(x_1, \cdots, x_n)$ precisely r many 0's$\}$.
compute the index of μ_β we see

$$Z_{r,m} \neq 0 \Leftrightarrow m = 2r$$

in this case

$$d(r, m) = 2r - 3 + 1 = 2r - 2$$

and the cohomology formula is written as:

$$P_t((\mathbb{P}^1)^n)P_t(BSU(2)) = P_t(\mu^{-1}(0)/SU(2)) + \sum_{n/2 \leqslant r \leqslant n} \binom{n}{r} t^{2(r-1)} P_t(BS^1)$$

References

[1] Michael Francis Atiyah. Convexity and commuting hamiltonians. *Bulletin of the London Mathematical Society*, 14(1):1–15, 1982.

[2] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983.

[3] Frances Kirwan. Rational intersection cohomology of quotient varieties. *Inventiones mathematicae*, 86(3):471–505, 1986.

[4] Frances Clare Kirwan, John N Mather, and Phillip Griffiths. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31. Princeton university press, 1984.

[5] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.