SOME EXAMPLES IN VARIATION OF GEOMETRIC INVARIANT THEORY

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Abstract. We focus on variation of geometric invariant theory in torus action case, and try to explain the ideas when we want to describe all possible quotients and their relatoionships.

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1. A brief history of GIT and VGIT

The origin of Geometric Invariant Theory(GIT) draws back to the nineteenth century, when Hilbert studied the action of $SL(V)$ on the polynomial ring $Sym(V^*)$. Such studies is now called (classical) invariant theory, which aims to find invariants for group action on a finitely generated algebra. It turns out that not all invariant rings are finitely generated, but for reductive groups it is always true (see [17]).

In 1960s, Mumford [16] started to study the quotient of a group action in algebraic category, he constructed the quotients and studied the topology of the quotients. The related studies are now called the Geometric Invariant Theory.

In the construction of quotient for an arbitrary projective variety, one have to fix an embedding into the projective space(or equivalently an ample bundle on the variety), use the embedding to throw out some points (called unstable points) and get a good quotient on the left (called semistable points). It is natural to ask how does the quotient depend on different embeddings, but it was until 1990s that the ignored problem was seriously treated independently by Thaddeus [18] and Dolgachev-Hu [6](also by Brion-Procesi [3] in the toric case). The related studies are now called Variation of Geometric Invariant Theory(VGIT).

There are also many interesting studies during the development of GIT and VGIT, for examples:

(1)The Kempf Ness theorem [12] tells us that GIT quotients are related to symplectic quotients.

(2)The cohomology of quotients are studied. See [15] for rational cohomology of orbifold quotients (i.e. all stabilizers are finite) and [14] [13]for intersection cohomology of general quotients.

(3)The idea of stability inspires the study of vector bundles. The so called Kobayashi-Hitchin correspondence, is thought to be an infinite dimensional version of the Kempf Ness theorem mentioned above. See Atiyah [1], Donaldson [7] and further developments.

(4)The VGIT theory has great relationship with Birational Geometry [10]. For example, it has been shown that certain birational models can be decomposd into VGIT wall crossing maps [11].

(5)VGIT also appears in the study of mirror symmetry in mathematical physics. In Gauged Linear Sigma Model(GLSM) introduced by Witten [19], different phases are related by a VGIT wall crossing map. For a mathematical theory of GLSM, see [8].

Conventions. We work on the field of complex numbers $\mathbb{C}$. Throughout the article G is a reductive group acting algebraicly on a variety X. When X is a (quasi)projective variety in $\mathbb{P}(V)$, we ask further that G acts linearly(that is, through a representation $G \to GL(V)$. By reductive group we mean complexification of a compact lie group.

2. Basic GIT

2.1. **Construction of quotients.** The idea is to find a natural way to construct a quotient variety.

First consider a reductive group G acts algebraically on an affine variety $X = SpecR$. By Nagata [17] R^G is finitely generated, so it's natural to define:

Definition 2.1. *Let G be a reductive group acting on an affine variety $SpecR$. Then the affine quotient is defined as $X//G = SpecR^G$.*

In the projective case, the idea is to treat it locally and return to the affine case. Again by Nagata [17] R^G is finitely generated, so we can use the construction of projective varieties: take the generators f_i of R^G and glue together $X_f//G$. It turns out that X_{f_i} may fail to cover the whole space X, so semistable points are introduced.

Definition 2.2. *For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ and R its homogenous coordinate ring,*
(1) A point $x \in X$ is semistable if there exists $r > 0$ and an invariant section $f \in R_+^G$ such that $f(x) \neq 0$. We denote the set of semistable points by X^{ss}, points that are not semistable are called unstable points;
(2) We define the GIT quotient $X//G = Proj(R^G)$.

Remark. Since $Proj(R^G)$ is the gluing of affine quotients $\{(R_f)_0//G, f \in R^G\}$, so the natural quotient map is only defined on semistable points.

Now consider a variety X and an G linearized ample bundle L on X, by G linearized we mean:

Definition 2.3. *(Linearization)*
Let $\pi : L \to X$ be a line bundle on X. A linearization of the action of G with respect to L is an action of G on L such that:
(1) For all $g \in G$ and $l \in L$, we have $\pi(g \cdot l) = g\pi(l)$, (2) For all $x \in X$ and$g \in G$ the map of fibres $L_x \to L_{g\cdot x}$ is a linear map.

Note that use the ample bundle we can define an G equivariant embedding $i : X \subset \mathbb{P}^N$ for some large enough N, such that $L = i^*(\mathcal{O}(1))$, and under this embedding $R(X) = H^0(i^*(\oplus_{n\geqslant 0}\mathcal{O}(n))) = H^0(\oplus_{n\geqslant 0}L^{\otimes n})$. The projective quotient generalize to polarized case:

Definition 2.4. *Let G be a reductive group acting on a projective variety X with respect to an ample linearization.*
(1) A point $x \in X$ is semistable with respect to L if there exists $r > 0$ and an invariant section $\sigma \in R(X, L^{\otimes r})^G$ such that $\sigma(x) \neq 0$. The set of semistable

points is denoted by $X^{ss}(L)$.
(2) The GIT quotient of G acting on X with respect to L is the morphism

$$\varphi : X^{ss}(L) \to X//_L G := ProjR(X, L)^G$$

obtained by resting the above rational map to its domain of definition.

There is a special kind of semistable points called stable points:

Definition 2.5. *We call a point is stable, if the orbit* Gx *is closed and the stabilizer* G_x *is of finite dimension.*

2.2. **Criterion for (semi)stablility.** By definition, to separate the semistable and unstable points one have to compute the invariant ring at first, which returns to classical invariant theory and is not easy in general case. Topological criterion and numberical criterion are two kinds of useful criterions in practice.

Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$, and the natural projection is denoted by $\pi : X^{ss} \to X//G$.

Proposition 2.6. *(Topological criterion for semistability). For* $x \in \mathbb{P}^n$*, choose a non-zero lift* $\widetilde{x}$ *in the affine cone* $\widetilde{X} \in \mathbb{A}^{n+1}$*. Then the following statements hold.*
(1) x is semistable if and only if $0 \notin \overline{G \cdot \widetilde{x}}$.
(2) x is stable if and only if $G \cdot \widetilde{x}$ *is closed and has dimension equal to the dimension of G.*

Proposition 2.7. *(Fibers of* π*)*
(1) $\pi(x) = \pi(y)$ *if and only if* $\overline{G \cdot x} = \overline{G \cdot y}$.
(2) $X^s \to Y^s = \pi(X^s)$ *is a geometric quotient, i.e. every fiber is a G orbit.*

The following criterion is more useful in practice, it tells us to check (semi)stability of a G-variety it suffices to check on each one parameter subgroup (1-=PS) of G.

Definition 2.8. *Suppose* λ *is a 1-PS of G (since every action of* $\mathbb{C}^*$ *on* $\mathbb{C}^{n+1}$ *is diagonized), under suitable basis one can write:*

$$\lambda(t) \cdot v = (t^{m_0} x_0, \cdots, t^{m_n} x_n)$$

The Hilbert-Mumford criterion is defined as:

$$\mu(x, \lambda) := min\{m_i : a_i \neq 0\}$$

and this number is independent of the choice of the lift and the basis e_i.

Remark. (1)In some places μ is defined as $-min\{m_i : a_i \neq 0\}$;(2) Using topological criterion for stability, it's easy to shown that $x \in X$ is (semi)stable iff $\mu(x, \lambda) < 0 (\leqslant 0)$ for all 1-PSs λ of G. The Hilbert-Mumford criterion gives the converse to this statement; (3)If we start from a linearized line bundle L, then we consider the weight of λ on L_{x_0} where

$x_0 = \lim_{t=0} \lambda(t)x$. It's easy to show it is exactly $\mu(x,\lambda)$ if we fix the embedding. To distinguish the line bundle we denote the criterion by $\mu^L(x,\lambda)$ in this case.

Theorem 2.9. *([16, Theorem 2.1]Hilbert-Mumford Criterion)*
Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$, then

$$x \in X^{ss} \Leftrightarrow \mu(x,\lambda) \leqslant 0, \forall \lambda$$
$$x \in X^{s} \Leftrightarrow \mu(x,\lambda) < 0, \forall \lambda \neq 1$$

Using the numberical criterion, we can get a combinational description on (semi)stable points in torus action case:

Definition 2.10. *For $\widetilde{x} = (\sum_j x_j e_j) \in V$ we define the **weight set** of $\widetilde{x}$ by*

$$\Pi(\widetilde{x}) = \{\vec{m}_j, x_j \neq 0\}$$

*and the **weight polytope** of $\widetilde{x}$, say $\overline{wt(\widetilde{x})}$ is defined as the convex hull of $wt(x)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$.*
we define

$$\mu^L(x,\lambda_{\vec{a}}) = min\{\vec{a}\vec{m}_j, x_j \neq 0\} = min_{\chi \in wt(x)} < \vec{a}, \chi >$$

The following result is just an application on numberical criterion:

Proposition 2.11. *Let G be a torus $\mathbb{C}^{*r}$ and L an ample G-linearized line bundle on a projective G-variety X, then:*

$$x \in X^{ss}(L) \Leftrightarrow 0 \in Conv(\Pi(x))$$
$$x \in X^{s}(L) \Leftrightarrow 0 \in IntConv(\Pi(x))$$

By considering all possible 1-PSs, we we can introduce a function $M(x)$:

Definition 2.12. *Let T a maximal torus of G and $W = N_G(T)/T$ its Weyl group. We fix a W invariant Euclidean norm $||\cdot||$ in $\mathfrak{X}_*(T) \otimes \mathbb{R}$. any $\lambda \in \mathfrak{X}_*(G)$, choose $g \in G$ such that $g\lambda g^{-1} \in \mathfrak{X}_*(T)$, we define $||\lambda|| = ||g\lambda g^{-1}||$, and define:*

$$\overline{\mu}^L(x,\lambda) := \frac{\mu^L(x,\lambda)}{||\lambda||}$$
$$M^L(x) = sup_{\lambda \in \mathfrak{X}_*(G)\overline{\mu}^L(x,\lambda)}$$

It was shown in [6] that $M^L(x)$ is always finite and hence:

Proposition 2.13.

$$X^{ss}(L) = \{x, M^L(x) \leqslant 0, \forall \lambda\}$$
$$X^{s}(L) = \{x, M^L(x) < 0, \forall \lambda \neq 1\}$$

3. Variation of affine quotients through a character

3.1. **Basic ideas.** Although the linearization problem is largely related to projective varieties, for affine case we can still consider the trivial line bundle and variate the G action on the line bundles.

Let $X = Spec(R)$ be a affine variety with a reductive group G acting on it. Given any character $\chi : G \to \mathbb{C}^*$, we can construct a GIT quotient with respect to χ: consider the trivial line bundle $L_\chi = X \times \mathbb{C}$, we extend G action to this trivial bundle by:

$$g(v,t) = (gv, \chi(g)t)$$

So

$$H^0(X, L_\chi^{\otimes m}) = R^G_{\chi^m} = \{f \in R | \forall g, f(gv) = \chi(g)^m f(x)\}$$

and

$$X//_\chi G = Proj(\oplus_{m \geqslant 0} R^G_{\chi^m})$$

The first observation is as the following:

Proposition 3.1. *If we consider the graded ring $R[u] = \oplus_{m\geqslant 0} Ru^m$ by adding a formal variable u, and define the G action on $R[u]$ by:*

$$g_\chi(f \otimes u^m) = g(f) \otimes \chi(g)^{-m} u^m$$

Then it's easy to check that $R[u]^G_\chi \cong \oplus_{m\geqslant 0} R^G_{\chi^m}$ as a graded ring, so

$$X//_\chi G = Proj R[u]^G_\chi$$

In special, when $\chi = 1$ is the trivial character, one get $R[u]^G_\chi = R^G[u]$ and hence

$$X//_1 G = Proj(R^G[u]) = Spec(R^G) = X//G$$

this is the original affine quotient.

There is another way to think about $X//_\chi G$, consider:

$$G_\chi = Ker(\chi : G \to \mathbb{C}^*)$$

If we give a grading of R through the weight decomposition of χ, then we find

$$R^G_{\chi^m} = (R^{G_\chi})_{(m)}$$

Hence we get:

Proposition 3.2.

$$X//_\chi G = Proj(\oplus_{m\geqslant 0} R^{G_\chi}_{(m)})$$

Here is an example to use the above proposition, consider a $GL(r)$ action on an affine variety and we consider the linerization L_{det} , then:

Lemma 3.3. *Let the subgroup $C = \mathbb{C}^* I \subset G = GL(r)$, and $F = FixC$, then:*
$x \in X^{ss}(det)$ iff $\overline{SL(r)x}$ does't meet F.

Proof. $\Rightarrow$) is obvious, since we could find a invariant f of positive weight, such that $f(x) \neq 0$, note that in this case f is constant on a $SL(r)$ orbit, hence $F(\overline{SL(r)x}) = c \neq 0$, but $F = FixC$ implies $f(x) = f(\lambda Ix) = \det(\lambda I)^m F(x)$, so F is a closed orbit with $f|F = 0$.
$\Leftarrow$) in this side, first note that they are all $SL(r)$ invariant and closed, so we could find $g \in R^{SL(r)}$ that separate them, i.e $f|_F = 0, f|_{\overline{SL(r)x}} \neq 0$, but $f|_F = 0$ implies that $f \in R_+^{SL(r)}$, so take homogeneous part we get it $\square$

Now consider the case $G = GL(r), \chi = det$, then have a better description of semistable points:

Example 3.4.

$$Gr(r,n) = M(r,n)//_{det}GL(r)$$

Proof. We use model in 3.3, So

$$M(r,n)//_{det}GL(r) = Proj(\oplus_{m \geqslant 0}\mathbb{C}[M(r,n)]_{(m)}^{SL(r)})$$

Now consider the (semi)stable, we would show:

$$X^s = X^{ss} = \{V, rkV = r\}$$

If $rkV < r-1$, then use an element of $GL(n)$, we get a matrix of the form:

$$\begin{pmatrix} 0 & \cdots & 0 \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix}$$

by multiplying $diag\{t^{-r+1}, t, \cdots, t\}$ we get

$$\begin{pmatrix} 0 & \cdots & 0 \\ t* & \cdots & t* \\ t* & \cdots & t* \end{pmatrix}$$

which have limit 0, but $0 \in F$ hence V is unstable point, this implies $V \in X^{ss} \Rightarrow rkV = r$;
Now for V with $rkV = r$, we can find a submatrix A_I, s.t $detA_I \neq 0$, note that $detAB_I = detA_I detB_I = detA_I$ for $B \in SL(r)$, we find $f = detA_I \in R^G$ and has positive weight r. hence $V \in X^{ss}$ Note also that in this case V have no fiexed point, and $SL(r)$ is closed, so $SL(r)V$ is closed and this implies $X^s = X^{ss}$ hence the GIT quotient is the geometric quotient, which is obviously $Gr(r,n)$. $\square$

3.2. **A first attempt.**

$\mathbb{C}^*$ *action on a vector space.* First consider the group action.

Under suitable choices we decomp $\mathbb{C}^n$ into weight spaces, the affine quotient totally depends on the weights $W = (w_1, \cdots, w_n)$, we may assume $w_p \leqslant w_{p+1}$.

Consider the choice of linearization, we assume the linearization comes from a character $\chi_d(t) = t^d$, then

$$\mathbb{C}^n//_{\chi_d}\mathbb{C}^* = Proj(\oplus_{m\geqslant 0}R_m^{\chi_d})$$

Where

$$R^\chi = \oplus_{m\in\mathbb{Z}}R_m^{\chi_d}$$

$R_m^{\chi_d} = \{f \in \mathbb{C}[x_1, \cdots, x_n], f(t(x_1, \cdots, x_n)) = \chi_d(t)^m f(x_1, \cdots, x_n) = t^{md}f(x_1, \cdots, x_n)\}$
It's easy to find $R_m^{\chi_d} = R_{md}^{\chi_1}$, so essentially there are only three kinds $d = 0, -1, +1$, which corresponds to $Y//0, Y//-, Y//+$. Note that in any case $R_0^\chi = R^G$, there are natrual maps

$$f_\pm : Y//\pm \to Y//0$$

Now for specific cases:

Example 3.5. $W = (-1, \cdots, -1, 1, \cdots, 1)$.

Suppose $t(x_1, \cdots, x_p, y_1, \cdots, y_q) = (t^{-1}x_1, \cdots, t^{-1}x_p, ty_1, \cdots, ty_q)$*,then*

$$R_0 = \mathbb{C}[x_i y_j]$$

so $Y//0$ *is the affine cone over the Segre embedding* $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \subset \mathbb{P}^{p+q-1}$*,*

we can use matrices to represent it:

(1) $Y//0$ *is the subvariety of affine space* $M_{p\times q}$ *of rank at most 1, the ideal of* $Y//0$ *is generated by determinants of submatrices.*

$$R_+ = R_0[y_1, \cdots, y_q], R_- = R_0[x_1, \cdots, x_q]$$

Then $Y// + (Y//-)$ *is gluing of* $Spec(R_+[1/y_i])_{(0)}(Spec(R_-[1/x_j])_{(0)})$*, which is a subvariety of*

$$Y//0 \times \mathbb{P}^{q-1}\,(\mathbb{P}^{p-1} \times Y//0)$$

and $f_\pm$ *is given by projection.*

Take $y \in Y//0$ *,* $f_+^{-1}(p)(f_-^{-1}(p))$ *is one single point representing the* $\mathbb{P}^{q-1}(\mathbb{P}^{p-1})$ *part of y, except that y=0 where* $f_+^{-1}(0)(f_-^{-1}(0))$ *is the total space* $\mathbb{P}^{q-1}(\mathbb{P}^{p-1})$*.*

We also use matrices to represent them:

(2) $Y//+$ *is the subvariety of* $Y//0 \times \mathbb{P}^{q-1}$ *defined by*

$$Y//+ = \{(Z, \vec{y} = (y_1, \cdots, y_q)), rank \begin{pmatrix} y \\ Z \end{pmatrix} = 1\}$$

(3) $Y//-$ *is the subvariety of* $\mathbb{P}^{p-1} \times (Y//0)$ *defined by*

$$Y//+ = \{(\vec{x} = (x_1, \cdots, x_p), Z), rank(\vec{x}^t, Z) = 1\}$$

Example 3.6. $W = (-1, \cdots, -1, 0, \cdots, 0, 1, \cdots, 1)$*;* $Y//0$ *is* $\mathbb{C}^r$ *product the affine cone given by Eg3.5*

Example 3.7. *Assume* $\omega_i \neq 0, \forall i, \omega_p < 0, \omega_{p+1} > 0$*.*

$Y//0$ *is the weighted cone over*

$$\mathbb{P}(\omega_1, \cdots, \omega_p) \times \mathbb{P}(\omega_{p+1}, \cdots, \omega_{p+q}) \subset \mathbb{P}(\omega_1\omega_{p+1}, \cdots, \omega_p\omega_{p+q})$$

For variations, we first consider the case $W = (-1, \cdots, -1, n)$*, where the* -1 *weight space is of* n *dimensional.*

$R_0 = \oplus_{m \geqslant 0}\mathbb{C}^{mn}[x_1, \cdots, x_n][y^m] \cong \oplus_{m \geqslant 0}\mathbb{C}^{mn}[x_1, \cdots, x_n]$*, then* $Y//0 = Spec(R_0)$*, this circumstance is a little bit special, since* R_0 *is in fact the invariant part of the diagonal action of* $\mathbb{Z}_n$ *on* $\mathbb{C}[x_1, \cdots, x_n]$*, hence* $Y//0 = \mathbb{C}_n/\mathbb{Z}_n$*;*

$R_- \cong R_0[x_1, \cdots, x_n]$*, and* $Y//- = ProjR_-$ *is the gluing of* $Spec(R_+[1/x_i]_{(0)})$*, which is a subvariety of* $Y_0 \times \mathbb{P}^n$*, and can be considered as the total spae of* $\mathcal{O}_{\mathbb{P}^{n-1}}(-n) = \omega_{\mathbb{P}^{n-1}}$*, and* f_- *is the natural projection map;*

$R_+ \cong R_0[y], degy = n$*, so* $Y//+ = Proj(R_0[y]) = SpecR_0 = \mathbb{C}^n/\mathbb{Z}_n$*,* f_+ *is the identity map.*

Next, we consider the general case in, using the same argument in Eg3.5, we get:

$Y//0 = SpecR(0)$ *is the weighted cone over*

$$\mathbb{P}(\omega_1, \cdots, \omega_p) \times \mathbb{P}(\omega_{p+1}, \cdots, \omega_{p+q}) \subset \mathbb{P}(\omega_1\omega_{p+1}, \cdots, \omega_p\omega_{p+q})$$

$R_+ = R_0[y_1, \cdots, y_q]$*,* $Y//+$ *is a subvariety of* $Y//0 \times \mathbb{P}(\omega_{p+1}, \cdots, \omega_{p+q})$*, and* f_+ *is isomorphism at all point but 0, where the fiber is the* $\mathbb{P}(\omega_{p+1}, \cdots, \omega_{p+q})$*;*

$R_- = R_0[x_1, \cdots, x_p]$*,* $Y//-$ *is a subvariety of* $\mathbb{P}(\omega_1, \cdots, \omega_p) \times Y//0$*, and* f_+ *is isomorphism at all point but 0, where the fiber is the* $\mathbb{P}(\omega_1, \cdots, \omega_p)$*;*

Take the weighted blow up of $Y//0$ *at 0, one get the variety over* $Y//-$ *and* $Y//+$*, and the exceptional set is the product*

$$\mathbb{P}(\omega_1, \cdots, \omega_p) \times \mathbb{P}(\omega_{p+1}, \cdots, \omega_{p+q})$$

Example 3.8. *The general case is just product with a affine space of spaces in 3.7.*

$\mathbb{C}^*$ *acts on a affine variety.*

Example 3.9. $X = Spec\mathbb{C}[a^2, ab, b^2, c, d]/ < ad - bc >$*, where* a, b, c, d *are of degree 1,* $\mathbb{C}^*$ *acting with weights* $1, -1, 1, -1$ *respectively,*

$R_0 = \mathbb{C}[ab, cd, a^2d^2, b^2c^2]/ < ad - bc >= \mathbb{C}[ab, cd]$*, hence:*

$$Y//0 = Spec(\mathbb{C}[ab, cd]) = \mathbb{C}^2$$

$R_- = R_0[z^2b^2, zd]$, and

$$Y//- = Proj R_- = Proj\mathbb{C}[u, v, zu, zv]$$

this is the blow up $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ at the zero.

By symmetry, $Y//_+$ is also $\widetilde{\mathbb{C}^2}$.

Torus action on a vector space.

Example 3.10. *$G = (\mathbb{C}^*)^2 \curvearrowright \mathbb{A}^{p+q+1}$ by*

$$(s,t) \circ (x_1, \cdots, x_p, y_1, \cdots, y_q, z) = (sx_1, \cdots, sx_p, ty_1, \cdots, ty_q, stz)$$

since $\chi(G) \cong \mathbb{Z} \oplus \mathbb{Z}$, fir every $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$, we define $\chi_{(a,b)}$ the corresponding character.
Now we use the consequences in 3.2 to compute the git quotients w.r.t different characters:

$$X//_\chi G = Proj(\oplus_{m \geqslant 0} R^{G_\chi}_{(m)})$$

$X^{ss}(\chi_{1,0}) = \mathbb{C}^{p+q+1} - \{x_1 = \cdots, x_p = 0\}$
$X^{ss}(\chi_{2,1}) = \mathbb{C}^{p+q+1} - \{x_1 = \cdots = x_p = 0\} - \{y_1 = \cdots, y_q = z = 0\}$
$X^{ss}(\chi_{1,1}) = \mathbb{C}^{p+q+1} - \{x_1 = \cdots = x_p = z = 0\} - \{y_1 = \cdots, y_q = z = 0\}$

The corr. quotients are:

$Y(\chi_{1,0}) = Proj(\mathbb{C}[x_1, \cdots, x_n]) = \mathbb{P}^{n-1}$
$Y(\chi_{2,1}) = Proj(\mathbb{C}[x_i^2 y_j, x_i z]) = \mathbb{P}(\mathcal{O}^{\oplus q}_{\mathbb{P}^{p-1}} \oplus \mathcal{O}_{\mathbb{P}^{p-1}}(-1))$
$Y(\chi_{1,1}) = Proj(\mathbb{C}[x_i y_j][z]) = Spec(\mathbb{C}[x_i y_j]) = Cone(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \subset \mathbb{P}^{p+q-1})$

4. Torus action on a projective space

4.1. **The weight polytope and its decomposition.** In this part we consider torus $T = (\mathbb{C}^*)^n$ (linear) action on a projective $X = P(V)$ via a linear representation: $\rho : T \to GL(V)$. Such discussion appears in [3].

In this case $NS^T(\mathbb{P}(V)) \cong Pic(\mathbb{P}(V)) \times \mathfrak{X}(T) \cong \mathbb{Z}^{n+1}$, each T-linearized ample line bundle is of the form $\mathcal{O}(n) \otimes \mathcal{O}(\chi)$, where $\mathcal{O}(n)$ is the canonical bundle with the natrual extension of T action, and $\mathcal{O}(\chi)$ is the trivial bundle with the T action through χ. To be explicit, for

$$f(x) \in H^0(\mathcal{O}(n) \otimes \mathcal{O}(\chi))$$

$$\vec{t} \circ (f(x)) = \chi(\vec{t}) f(\vec{t}^{-1} x)$$

Definition 4.1. *We denote by E (resp. $E_\mathbb{Q}$) the vector space $\mathfrak{X}(T) \otimes \mathbb{R}$ (resp. $\mathfrak{X}(T) \otimes \mathbb{Q}$).*
Let

$$V = \oplus_{\chi \in \mathfrak{X}(T)} V_\chi$$

be the weight space decomposition of V, where $V_\chi = \{v \in V, \vec{t}v = \chi(\vec{t})v\}$, also we define the subset of weights

$$\Pi = \{\chi, V_\chi \neq 0\}$$

For any $x \in X$, soppose $\tilde{x} \in V$ represent x in $P(V)$. Through the decomposition we get $\tilde{x} = \sum x_\chi$, and define:

$$\Pi(x) = \{\chi \in \Pi, x_\chi \neq 0\}$$

We can get more information from the T action:

Definition 4.2. *We call*

$$\mathscr{C} = Conv(\Pi) \subset E$$

the weight polytope of this action. For $\forall p \in E$, let

$$\overline{F}(p) = \cap_{S\subset\Pi, p\in ConvS} ConvS \subset E$$

this is a convex set and we denote $F(p)$ by its relative interior;
*We call a subset F of $\mathscr{C}$ a **face**, if $F = F(p)$ for some p.*
For any p, we denote

$$X^{ss}(p) = \{x \in X, p \in Conv(\Pi(x))\}$$

$$X^{s}(p) = \{x \in X, p \in Conv(\Pi(x))^o\}$$

Remark. Take care of the notion of interior and relative interior. Interior means the union of the subset of $\overline{F}$ which is open in E, while the relative one means the interior of $\overline{F}$ in the affine space it generates.

Proposition 4.3. *$X^{ss}(p)$ and $X^s(p)$ depend only on the face of p, so we define X^{ss} and X^s for any faces.*

Using the criterion 2.11 we can check that:

Proposition 4.4. *For any face F, if we choose any $p = \chi/n \in F \cap E_{\mathbb{Q}}$ (note that the vertexes of faces are integer number, so $F \cap E_{\mathbb{Q}} \neq \emptyset$), and consider the line bundle*

$$\mathcal{O}(n) \otimes \mathcal{O}(\chi)$$

then

$$x \in X^{ss}(F) \Leftrightarrow x \in X^{ss}(\mathcal{O}(n) \otimes \mathcal{O}(\chi))$$

$$x \in X^{s}(F) \Leftrightarrow x \in X^{s}(\mathcal{O}(n) \otimes \mathcal{O}(\chi))$$

4.2. **Relationship with symplectic quotients (in preparation).**

4.3. **Cross a codimension 1 wall.**

Proposition 4.5. *([3]) Let F and F' be distinct faces such that $F \subset \overline{F'}$, then:*
(1)$X^s(F) \subset X^s(F') \subset X^{ss}(F') \subset X^{ss}(F)$;
(2)Denote the corresponding GIT quotient by $Y(F)(Y(F'))$, then there exists $\pi_{F,F'}$ such that the following diagram computes:

$$\begin{tikzcd} X^{ss}(F') \arrow[r, "i_{F',F}"] \arrow[d, "\pi_{F'}"'] & X^{ss}(F) \arrow[d, "\pi_F"] \\ Y(F') \arrow[r, "\pi_{F,F'}"] & Y(F) \end{tikzcd}$$

(3)If F is not contained in the boundary of $\mathscr{C}$, then $\pi_{F,F'}$ is birational.

Now consider the case where F belongs to the boundary of $\mathscr{C}$, and the vertexes are all integer points, then there exists a 1PS $\lambda \in \mathfrak{X}(T)$ and an integer a, such that:

$$< \lambda, p >> a, \forall p \in \mathscr{C} \setminus < F >; < \lambda, p >= a, \forall p \in F$$

Now for any x, we get $\mathbb{C}^* \to X : t \mapsto \lambda(t)x$, which extends to $\mathbb{C} \to X$ (by completeness of X).

Proposition 4.6. *([3])With the notions above, and let $X^F = Fix(\lambda) \cap X^{ss}(F)$, then we get:*
(1) $X^{ss}(F) = \{x \in X, \lim_{t\to 0} \lambda(t)x \in X^F\}$;
(2) Let

$$p : X^{ss}(F) \to X^F : x \to \lim_{t\to 0} \lambda(t)x$$

and $r = \pi_F|_{X_F}$, then:

$$\pi_F = r \circ p$$

What's more, r is a geometric quotient.
(3)If X is smooth, then X^F is also smooth, moreover p is a locally trivial fibration in affine spaces.
In special, if moreover F is of codimension 1, then $F \subset \overline{F_+}$ for an open face F_+, in this case:
(4) $X^{ss}(F_+) = X^{ss}(F) \setminus X^F$;
(5) π_{F,F_+} is a locally trivial fibration for the étale topology, with fibers being the quotients by λ the sets

$$\{x \in X^{ss}(F) \setminus X^F, lim_{t\to 0}\lambda(t)x = y\}$$

for $y \in X^F$. If X is smooth, then the fibers are weighted projective spaces.

Proof. (1) The $\subset$ part is easy, for the $\supset$ part, we assume

$$\lim_{t\to 0} \lambda(t)x = y \in X^F$$

Then $\pi(y) \subset \pi(x)$ and hence $F \subset Conv(\pi(y)) \subset Conv(\pi(x))$, which implies $x \in X^{ss}(F)$.
(2)It suffices to show r is a geometric quotient, in fact, we can show X^F is the stable points of T action on a subvariety. We choose those $\chi_i \in \Pi$ such that $\chi_i \in <F>$, and let

$$V' = \oplus_{\chi_i \in <F>} V_{\chi_i}$$

Then $X' = X \cap \mathbb{P}(V')$ is the subvariety, which is fixed under λ and invariant under T, now $X^F = X^{ss} \cap X^\lambda$ is the semistable point of F, restricted to X'. What's more, since $\forall x \in X', \pi(x) \subset <F>$, X^F is in fact the set of stable points, this finish the proof;
(3)This is the Bialynicki-Birula Decomposition theorem;
(4)Since $X^{ss}(F_+) \subset X^{ss}(F)$, we use (1) to get the conclusion. Note that F is codim 1, for $x \in X^{ss}(F)$, $x \in X^{ss}(F_+)$ if and only if $F \subsetneq Conv(\pi(x))$, that is, $x \notin X^F$;
(5)We use the topological criterion. Note that any element of $X^{ss}(F)$ have limit in X^F, it suffice to consider the inverse image of an element $y \in X^F$, since F is of codimension 1, y has 1 dimensional isotropy group, that is, λ, so clearly

$$\pi^{-1}_{F',F}(y) = \{x \in X^{ss}(F) \setminus X^F, lim_{t\to 0}\lambda(t)x = y\}/\lambda \cong \mathbb{C}^r/\lambda$$

which is a weighted projective space $\mathbb{P}(W)$, and the weight W is determined by λ. $\square$

Corollary 4.7. *Suppose that X is smooth, F belongs to the boundary of $\mathscr{C}$ and $F' \neq F, F \subset \overline{F'}$, then $\pi_{F,F'}$ is not birational.*

Now we consider the case of two faces having closure intersects on a codim-1 wall, that is, we denote this structure by F_-, F, F_+. Let λ and X^F be as before.

Definition 4.8. *With the notions before, we set:*
(1) $X_+ = \{x \in X | \lim_{t\to 0} \lambda(t)x \in X^F\}$;
(2) $X_- = \{x \in X | \lim_{t\to \infty} \lambda(t)x \in X^F\}$;
(3) $\dot{X}_+ = X_+ \setminus X^F$;
(4) $\dot{X}_- = X_- \setminus X^F$;
(5) We denote $Y_-(Y_+)$ *be the image of* $\dot{X}_-(\dot{X}_+) \subset X^{ss}(F_-)$ *in* $Y(F_-)(Y(F_+))$, *and* Y^F *be the image of* X^F *in* $Y(F)$.

Proposition 4.9. *([3])With the notions above, one get:*
(1) $X^s(F_-) \setminus X^s(F) = \dot{X}_-, X^{ss}(F) \setminus X^{ss}F_- = X_+$;
(1)' $X^s(F_+) \setminus X^s(F) = \dot{X}_+, X^{ss}(F) \setminus X^{ss}F_+ = X_-$;
(2) $\pi_{F_-,F} : Y(F_-) \to Y(F)$ *induces an isom from* $Y(F_-) \setminus Y_-$ *to* $Y(F) \setminus Y^F$;
(2)' $\pi_{F_+,F} : Y(F_+) \to Y(F)$ *induces an isom from* $Y(F_+) \setminus Y_+$ *to* $Y(F) \setminus Y^F$;

(3) There is a commutative diagram:

$$\begin{tikzcd}
\dot{X}_- \arrow[r, "x\to\lim_{t\to\infty}\lambda(t)x"] \arrow[d, "\pi_{F_-}"'] & X^F \arrow[d, "\pi_F"] & \dot{X}_+ \arrow[l, "x\to\lim_{t\to 0}\lambda(t)x"'] \arrow[d, "F_+"] \\
Y_- \arrow[r, "\pi_{F_-,F}"'] & Y^F & Y_+ \arrow[l, "\pi_{F_+,F}"]
\end{tikzcd}$$

Proof. (1)$x \in X^s(F_-) \setminus X^s(F)$, iff $F_- \subset Conv(\Pi(x))^o, F \subset \partial Conv(\Pi(x))$, not that $codim < F >= 1$, hence $< F >$ separates E into two parts. Therefore $F \subset \partial Conv(\Pi(x))$ implies $\emptyset \neq Conv(\Pi(x))^o \subset \{\chi \in E, \chi \cdot \lambda < a\}$. Based on this observation, we get $X^s(F_-) \setminus X^s(F) = \dot{X}_-$.
$x \in X^{ss}(F) \setminus X^{ss}F_-$, if and only if $F \subset Conv(\Pi(x))$ and $F_- \nsubseteq Conv(\Pi(x))$, note that $codim < F >= 1$, this is equivalent to say, $F \subset Conv(\Pi(x))$ and $\forall \chi \in \Pi(x), \chi \cdot \lambda \geqslant a$, then by the definition of X_+, we get $X^{ss}(F) \setminus X^{ss}F_- = X_+$.
(2) By (1) we get $X^s(F_-) \setminus \dot{X}_- = X^s(F)$, so π_F and π_{F_-} agree on $X^s(F_-) \setminus \dot{X}_-$. □

4.4. **A stratified structure in general wall crossing (in preparation).** Now consider two faces $G \prec F$ in the interior, and the corresponding map $\pi_{F,G} : X^{ss}(F)//T \to X^{ss}(G)//T$, from the last part we know what happens when G is a codimension 1 face. In the general case we will get a stratified map, on each strata $i_{F,G}$ is a fibration tower with fiber weighted projective space. The idea comes from [9], where symplectic quotient rather than GIT quotient is considered (but we know they tell the same thing!). In that paper, the stratification is used to compute intersection cohomology.

Fist for any subpolytope N and $v \in \mathcal{C}$, we define:

$$\mathcal{U}_v(N) = \{x, q \in Conv(\Pi(x)) \subset N\}$$

Let the set of all codimension 1 subpolytopes containg G being $\{N_1, \cdots, N_l\}$, since G is in the interior, G and $RelInt(\cap N_i)$ span the same affine space.

For any $I \subset \{1, \cdots, l\}$, let

$$N_I = \cap_{i \in I} N_i$$

We take $q \in G$ and $p \in F$ and define $\mathcal{U}_q(N_I)//T := S_I$. Now the stratification can be defined inductively as follows:

$C_{[1,\cdots,l]} = S_1 \cap \cdots \cap S_l$;

$C_{[1,\cdots,\hat{i},\cdots,l]} = S_1 \cap \cdots \cap \hat{S}_i \cap \cdots \cap S_l - C_{[1,\cdots,l]}$;

$C_{[1,\cdots,\hat{i},\cdots,\hat{j},\cdots,l]} = S_1 \cap \cdots \hat{S}_i \cap \cdots \cap \hat{S}_i \cap \cdots \cap S_l -$ the union of previous stratas;

$\cdots$

$C_{[i]} = S_i -$the union of previous stratas;

$C_\emptyset = X^{ss}(q)//T - \cup S_i$;

then by our construction,

$$X^{ss}(q)//T \cong C_\emptyset \cup \coprod_{I\subset\{1,\cdots,l\}} C_I$$

Now consider $\pi_{F,G} : X^{ss}(F)//T \to X^{ss}(G)//T$, the result is as follows:

Theorem 4.10. *([9, Chap 5 Theorem3]*
Suppose F, G are two interior faces such that $G \subset \overline{F}$, and denote the set of all codim 1 subpolytopes containg G by $\{N_1, \cdots, N_l\}$, then there is a canonical stratification $X^{ss}(G)//T = \cup_{I\subset\{1,\cdots,l\}} C_I$, such that reducing to each strata, the morphism induced by inclusion $\pi_{F,G} : \pi^{-1}_{F,G} C_I \to C_I$ is a fibration tower. What's more, the fibers are all weighted projective spaces.

5. Torus action on a linear space

5.1. **Relationship with toric variety.** Consider a torus acts linearly on a linear space, the first result is that in this case the quotients would always be a toric variety:

Proposition 5.1. *Any GIT quotient of the form $\mathbb{C}^n//_\chi(\mathbb{C}^*)^r$ is a toric variety.*

Proof. ([5, Theorem12.1])
First let the action $(\mathbb{C}^*)^r \curvearrowright \mathbb{C}^n$ by:

$$\vec{t}(z_1, \cdots, z_n) = (\vec{t}^{\vec{a_1}} z_1, \cdots, \vec{t}^{\vec{a_n}} z_n)$$

Let

$$A = \begin{pmatrix} \vec{a_1} \\ \cdots \\ \vec{a_n} \end{pmatrix}_{n\times r}$$

given any $\overline{\omega} \in \mathbb{Z}^r$ and denote the corresponding character by $\chi_{\overline{\omega}}$, then consider:

$$\mathbb{C}^n//_{\chi_{\overline{\omega}}}(\mathbb{C}^*)^r$$

Now for $Z^{\vec{m}} = z_1^{m_1} \cdots z_n^{m_n}$, check that:

$$Z^{\vec{m}} \in (\mathbb{C}[z_1, \cdots, z_n]^{\chi_{\overline{\omega}}})_{(d)}$$

iff $\vec{m} \cdot A = d\overline{\omega}$, that is to say

$$(\vec{m}, d) \begin{pmatrix} A \\ -\overline{\omega} \end{pmatrix} = \vec{0} (\vec{m} \geqslant 0, d \geqslant 0) \tag{5.1}$$

Now consider the rational convex polyhedral σ corr to the system 5.1, then there is a fact:
Fact1: σ is a finitely generated submonid of $\mathbb{Z}^{n+1}$.
Now let S be the set of $\vec{m}$ such that $(\vec{m}, d) \in \sigma \cap \mathbb{Z}^{n+1}$ for some $d \geqslant 0$,

and consider $\mathbb{C}[S] \subset \mathbb{C}[\mathbb{Z}^n]$ via $\vec{m} \to Z^{\vec{m}}$, it is f.g $\mathbb{C}$ algebra by Fact1, and obviously:

$$\mathbb{C}^n//_{\chi_{\varpi}}(\mathbb{C}^*)^r \cong Proj(\mathbb{C}[S])$$

Next we study how $Proj(\mathbb{C}[S])$ is glued by affine charts:
Now assume $\mathbb{C}[S]_{>0}$ is generated by

$$Z^{\vec{m_1}}, \cdots, Z^{\vec{m_s}}$$

For each $j \in \{1, \cdots, s\}$, let

$$I_j = \{i, m_{ji} \neq 0\}$$

and for each $I \subset \{1, \cdots, n\}$, let

$$Z_I = \Pi_{i \in I} z_i$$

then $D(Z^{\vec{m_j}}) = D(Z_{I_j})$, hence

$$(\mathbb{C}^n)^{ss}_{\chi_{\varpi}} = \cup_{j=1}^s D(Z_{I_j})$$

Now consider

$$A : \mathbb{Z}^n \to \mathbb{Z}^r$$

the kernel M is a free $\mathbb{Z}$ mod of rank $l = n - rank(A)$, denote the standard basis of $\mathbb{Z}^n$ by e_i, and consider $i : M \to \mathbb{Z}^n$, for each $j \in \{1, \cdots, n\}$, let

$$\sigma_j = Conv\{i^* e_i^* | i \notin I_j\} \subset \mathbb{R}(M) \cong \mathbb{R}^l$$

Let

$$R_j = \mathcal{O}(D(Z_{I_j}))^{(\mathbb{C}^*)^r}$$

Fact2 $R_j \cong \mathbb{C}[\hat{\sigma}_j \cap M]$.

Now let $\Sigma = \{\sigma_i\}_{i \in \{1, \cdots, s\}}$ then Σ gives a fan structure and

$$\mathbb{C}^n//_{\chi_{\varpi}}(\mathbb{C}^*)^r \cong X_\Sigma$$

□

We finish this part with a result of Cox:

Theorem 5.2. *[4, Theorem5.1.11 and Theorem14.1.9]*
Any projective torus variety can be realized as a GIT quotient of affine space by a torus action with respect to some character.

5.2. **The weight cone and its decomposition.** We identify $\chi_*(T)$ with $\mathbb{Z}^r$, any $x \in V$, let:

$$\Pi(x) = \{\omega \in M, \exists f \in R_{\omega, f(x) \neq 0}\}$$

it generates the weight cone:

$$\omega_T(x) = Cone(\Pi(x))$$

and also a sublattice $M_T(x)$.

Every $\omega \in \mathbb{Z}^r$ determins a character χ^ω and hence induces a linearization:

$$t(x, z) = (tx, \chi^\omega(t)z)$$

The following lemma describes the semistable points:

Lemma 5.3. *[2, Lemma2.7]*

$$X^{ss}(\omega) = \bigcup_{f \in R_{n,w} n \in \mathbb{Z}_{>0}} X_f = \{x \in X, \omega \in \omega_T(x)\}$$

In particular, $X^{ss}(\omega) \neq \emptyset$ iff $\omega \in \Omega_T(X) \cap M$.

Proof. The first equation is by definition, for the second one, assume $x = x_{\chi_1} + \cdots x_{\chi_r}$, then $x \in X^{ss}(\omega)$ if and only if exists $l_1, \cdots, l_r \geqslant 0$, such that the monomial satisfies $f = (e_1^*)^{l_1} \cdots (e_r^*)^{l_r}$,

$$f(\vec{t}x) = \vec{t}^{l_1\chi_1 + \cdots l_r\chi_r} f(x) = \vec{t}^{n\omega} f(x), \exists n > 0$$

That is to say $\omega \in \omega_T(x)$. □

Now for $\omega \in \Omega_T(X) \cap M$, we define the **GIT cone**

$$\overline{\sigma}_T(\omega) = \bigcap_{\omega \in \omega_T(y)} \omega_T(y)$$

We set $\sigma_T(\omega) = RelInt\overline{\sigma}_T(\omega)$. Note that $\omega \in \omega_T(x)$ implies $\sigma_T(\omega) \subset \omega_T(x)$, so $X^{ss}(\omega')$ is constant for any $\omega' \in \sigma_T(\omega)$.

And we define:

$$\Sigma_T(X) = \{\sigma_T(\omega); \omega \in \omega_T(X) \cap M\}$$

Proposition 5.4. *([2, Proposition 2.9])*

(1)$X^{ss}(\omega_1) \subset X^{ss}(\omega_2)$ iff $\sigma_T(\omega_1) \supset \sigma_T(\omega_2)$;

(2)$X^{ss}(\omega_1) = X^{ss}(\omega_2)$ iff $\sigma_T(\omega_1) = \sigma_T(\omega_2)$.

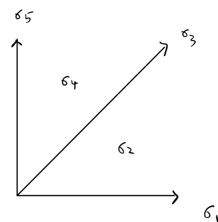
Now $X^{ss}(\sigma)$ for any $\sigma \in \Sigma_T(X)$ is well defined.

Theorem 5.5. *([2, Theorem 2.11]) $\Sigma_T(X)$ admits a quasifan structrue in its natrual way.*

5.3. **Cross a codimension 1 wall.** First we return to the example 3.10:

Example 5.6. *Continue what we discussed in 3.10.*

The decomposition of the weight cone is as the picture:



We have compute the semistable points $X^{ss}(\sigma_1) = X^{ss}(\chi_{1,0}), X^{ss}(\sigma_2) = X^{ss}(\chi_{1,1}), X^{ss}(\sigma_3) = X^{ss}(\chi_{0,1})$ *and the correspoding quotients.*

Now first look at the inclusion $X^{ss}(\sigma_2) \to X^{ss}(\sigma_1)$*, then consider the 1-PS* $\lambda_{0,1}$*, we get:*

$$X^{\sigma_1} = X^{ss}(\sigma_1) \cap Fix(\lambda_{0,1})$$

and a projection:

$$p_{\sigma_1} : X^{ss}(\sigma_1) \to X^{\sigma_1} : x \mapsto \lim_{t \to 0} \lambda_{0,1}(t)x$$

also we check that

$$X^{ss}(\sigma_2) = X^{ss}(\sigma_1) - X^{\sigma_1}$$

the fiber of p_{σ_1} *is a* $q+1$ *dimensional vector space, in fact, we find* π_{σ_2,σ_1} *is exactly the natrual projection* $Y(\sigma_2) \to Y(\sigma_1)$

Now consider $X^{ss}(\sigma_2) \to X^{ss}(\sigma_3)$*, use the 1-PS* $\lambda_{1,-1}$ *one can get*

$$X^{\sigma_3} = \{x = 0 = y\} \cap X^{ss}(\sigma_3)$$

and

$X_+ = \{\lim_{t \to 0} \lambda_{1,-1}(t)(x, y, z) \in X^{\sigma_{\sigma_3}}\} = \{y = 0\} \cap X^{ss}(\sigma_3)$

$X_- = \{\lim_{t \to +\infty} \lambda_{1,-1}(t)(x, y, z) \in X^{\sigma_{\sigma_3}}\} = \{x = 0\} \cap X^{ss}(\sigma_3)$

We find

$$X^{ss}(\sigma_3) - X^{ss}(\sigma_2) = X_-, X^{ss}(\sigma_4) - X^{ss}(\sigma_2) = X_+$$

Now consider the fibers of the projections, we find i_{σ_3,σ_2} *is stratified, set* $X_+^\bullet = X_+ \setminus X^{\sigma_3}$*, we find*

$$i^{-1}_{\sigma_3,\sigma_2}(X^{ss}(\sigma_3)) = i^{-1}_{\sigma_3,\sigma_2}(X_+ \coprod (X^{ss}(\sigma_3) - X^{\sigma_3})) = X_+^\bullet \coprod (X^{ss}(\sigma_2) - X_+^\bullet)$$

restricting to $(X^{ss}(\sigma_2) - X_+^\bullet)$ i_{σ_3,σ_2} *is a isom, and* $p_{\sigma_3} i_{\sigma_3,\sigma_2} X_+^\bullet \to X^{\sigma_3}$ *is a fiberation with fiber* $\mathbb{C}^p - \{0\}$

so the morphism on quotients is stratified, with exceptional set $Y_+ = \pi_{\sigma_3}(X_+^{\bullet})$*, the fiber will be* $\mathbb{P}^{p-1}$*.*

We can use a diagram to represent the morphism:

$$\begin{tikzcd}
Y_+ \arrow[r, hook] \arrow[d, "\mathbb{P}^{p-1}"', "\text{fiber}"] & Y(\sigma_2) \arrow[d] & Y(\sigma_2) \setminus Y_+ \arrow[l, hook'] \arrow[d, "\cong"] \\
Y^{\sigma_3} \arrow[r, hook] & Y(\sigma_3) & Y(\sigma_3) \setminus Y^{\sigma_3} \arrow[l, hook']
\end{tikzcd}$$

5.4. **A stratified structure in general wall crossing (in preparation).**

References

[1] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983.

[2] Florian Berchtold and Jrgen Hausen. Git-equivalence beyond the ample cone. *Michigan Mathematical Journal*, 54(3):483–516, 2006.

[3] Michel BRION and Claudio PROCESI. Action of a torus in a projective variety. 1990.

[4] David A Cox, John B Little, and Henry K Schenck. *Toric varieties*, volume 124. American Mathematical Soc., 2011.

[5] Igor Dolgachev. *Lectures on invariant theory*. Number 296. Cambridge University Press, 2003.

[6] Igor V Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques*, 87(1):5–51, 1998.

[7] Simon K Donaldson. A new proof of a theorem of narasimhan and seshadri. *Journal of Differential Geometry*, 18(2):269–277, 1983.

[8] Huijun Fan, Tyler Jarvis, and Yongbin Ruan. A mathematical theory of the gauged linear sigma model. *Geometry & Topology*, 22(1):235–303, 2017.

[9] Yi Hu. The geometry and topology of quotient varieties of torus actions. *Duke Mathematical Journal*, 68(1):151–184, 1992.

[10] Yi Hu. Geometric invariant theory and birational geometry. *arXiv preprint math/0502462*, 2005.

[11] Yi Hu and Sean Keel. Mori dream spaces and git. *arXiv preprint math/0004017*, 2000.

[12] George Kempf and Linda Ness. *The length of vectors in representation spaces*, pages 233–243. Springer, 1979.

[13] Young-Hoon Kiem. Intersection cohomology of quotients of nonsingular varieties. *arXiv preprint math/0101254*, 2001.

[14] Frances Kirwan. Rational intersection cohomology of quotient varieties. *Inventiones mathematicae*, 86(3):471–505, 1986.

[15] Frances Clare Kirwan, John N Mather, and Phillip Griffiths. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31. Princeton university press, 1984.

[16] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[17] Masayoshi Nagata and M Pavaman Murthy. *Lectures on the fourteenth problem of Hilbert*, volume 31. Tata Institute of fundamental research Bombay, 1965.

[18] Michael Thaddeus. Geometric invariant theory and flips. *Journal of the American Mathematical Society*, 9(3):691–723, 1996.

[19] Edward Witten. Phases of n = 2 theories in two dimensions. *Nuclear Physics B*, 403:159, 1993.