STABLE BUNDLES OVER RIEMANN SURFACES

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Contents

Keywords: Moduli Space, Stable Bundle, Gauge Theory, GIT, Stability.

1. Introduction

We fix a compact Riemann surface Σ.

2. Holomorphic structures and $U(n)$ connections

2.1. **Complex vector bundle over a Riemann surface.**

Proposition 2.1. *The equivalent complex vector bundle over Σ is totally determined by its degree and rank, in another word, two vectoe bundle with the same degree and rank is isomorphic.*

This is the direct corollary of the following teo lemmas:

Lemma 2.2. *Two line bundle over Σ are isomorphic iff they have the same degree, and there exists line bundle of any given degree.*

Proof. This is the canonical Abel-Jacobi theorem, note that we can even make it in the category of holomorphic line bundles. □

Lemma 2.3. *Let $E \to \Sigma$ be a vector bundle with $n = rank(E) > dim\Sigma = m$, then E have a non-vanishing section.*

Proof. We can perturb an arbitray section s into an non-vanishing one:
First, we may assume there are only finite many zeros by the compactness of X;
Next, locally we may assume near every zero p, we can find local sections $s_1, \cdots, s_m, s_{m+1}, \cdots, s_n$ such that $im(s|U) \subset span\{s_1, \cdots, s_m\}$ by implictic function theorem: locally s is $\mathbb{R}^m \to \mathbb{R}^n . m < n$;
Secondly, we can perturb s to $s_{m+1}, \cdots, s_n$ directions near p, note that this operation doesn't add zeros. □

Remark. Splitting of short exact sequence doesn't always exists in the catergory of holomorphic bundles.
As we have show, because the structure of line bundle over a Riemann surface is quite easy and the dimension of a riemann surface is 1, there is only an orbit fixing the rank and degree, so we may fix a complex bundle of tupe (r, d) and consider the classification of complex structures on it, which equivalently gives the classification of holomorphic bundles of type (r, d).

2.2. **Holomorphic structures and $U(n)$ connections.**
First we review some basic concepts in gauge theory:
In the catergory of G (principal) bundles, then we may consider the morphisms:

Definition 2.4. *A morphism between two G bundles P_1, P_2 is a bundle map which is G-quivariant, that is:*

$$\forall p \in P_1, g \in G, \varphi_2 f(p_1) = \varphi_2 f(p_1 g), f(pg) = f(p)g$$

Fix a G bundle $P \to \Sigma$, then we can consider the automorphism group, say $\mathscr{G}(P) = Aut(P)$.

Proposition 2.5. *Consider the adjoint action of G on G, and the associated bundle $AdP = P \times_G G$,then*

$$\mathscr{G}(P) = Aut(P) \cong \Gamma AdP$$

Proposition 2.6. *Consider the co-adjoint action of G on $\mathfrak{g}$, and the associated bundle $adP = P \times_G \mathfrak{g}$, we denote the (affine) space of G connections on P by $\mathscr{A}(P)$, then:*

$$\mathscr{A}(P) \cong \Omega^1(\Sigma, adP)$$

Definition 2.7. *The action of gauge group $\mathscr{G}(P)$ on the space of G connections is given by (locally, one can check it's global):*

$$u(A) = A - d_A u u^{-1}$$

Now we turn to our topic, as we have shown in the last section, we just need to fix a complex bundle E of type (n, d) and study holomorphic structures on it.

Natrually we may think about the space of holomorphic structures on E, denoted $\mathscr{C}(E)$, which has an affine structure associated with $\Omega^{0,1}(E)$, and the group of bundle automorphism, say $Aut(E)$, locally determined by a left action of $GL(n, \mathbb{C})$, so the moduli of equivalent holomorphic bundles is $\mathscr{C}/Aut(E)$.

It turns out it's better to consider the space of $U(n)$ connections since we have Yang Mills functional on this space, which is the gauge theoritical point of view, but the first observation is:

Proposition 2.8. *On E, there is an affine-linear isomorphism between the space of unitary connections, say $\mathscr{A}$, to the space of holomorphic structures on it, say $\mathscr{C}$.*

Proof. Given a unitary connection A, the operator $d_A : \Omega^0(E) \to \Omega^1(E)$ defines $\overline{\partial}_A : \Omega^0(E) \to \Omega^{0,1}(E)$, which determines a holomorphic structure on E in the sense that a (smooth) seciton s of E is holomorphic if and only if $\overline{\partial}_A s = 0$.
Conversely, a holomorphic structure on E defines an operator $D'' : \Omega^0(E) \to \Omega^{0,1}(E)$. We first fix a hermitian structure on E and then by checking locally, we find an unique unitary connection $d_A : \Omega^0(E) \to \Omega^1(E)$ such that $d_A^{0,1} = D''$. □

Remark. In this sence, we can view $Aut(E)$ as the complexification $\mathscr{G}^{\mathbb{C}}$ of $\mathscr{G}$, and extend the gauge group action of $\mathscr{G}$ to $\mathscr{G}^{\mathbb{C}}$ (See [2], P2):

$$g(A) = A - (\overline{\partial}_A g)g^{-1} + ((\overline{\partial}_a g)g^{-1})^*, g \in \mathscr{G}^{\mathbb{C}}, A \in \mathscr{A}$$

Corollary 2.9. *The space of equivalent class of holomorphic bundle of type (n,d), is given by the orbit space:*

$$\mathscr{A}/\mathscr{G}^{\mathbb{C}}$$

3. Stable bundles and Harder-Narasimhan theorem

3.1. (Semi)stable bundle and Harder-Narasimhan filtration.

Theorem 3.1. *(Harder-Narasimhan)*
Any holomorphic bundle admits a canonical Harder-Narasimhan filtration:

$$0 = F_0 \subset F_1 \subset F_2 \cdots \subset F_r = E$$

with $D_i = F_i/F_{i-1}$ semi-stable and $\mu(D_1) > \mu(D_2) > \cdots \mu(D_r)$.

Remark. Fix the Rie surface and fix type (n,d), then if n,d are coprime, semistable and stable are equivalant.

Proposition 3.2. *(Seshadri P102)*
stable $\Rightarrow$ simple ($End(E) = \mathbb{C}^*$)$\Rightarrow$ indecomposible.
semistable doesn't imply simple or indecomposible.

Remark. A polystable vector bundle over a smooth curve is direct sum of stable bundles with the same slope.

3.2. **$\Gamma_{\mathbb{R}}$ and Yang Mills connection.** Let Γ be the free group generated by the generators of $\pi_1(\Sigma)$, denoted $A_1, B_1, \cdots, A_g, B_g$. We use an element J which commutes with A_i, B_i to get a center expansion of $\pi_1(M)$, i.e:

$$1 \to \mathbb{Z}J \to \Gamma \to \pi_1(M) \to 1$$

we may extending Γ to $\Gamma_{\mathbb{R}}$ by extending the centre to $\mathbb{R}$ in the sense that:

$$1 \to \mathbb{R}J \to \Gamma_{\mathbb{R}} \to \pi_1(M) \to 1$$

To understand $\Gamma_{\mathbb{R}}$ we need to introduce the concept of path group. For the following discussions we refer to [3]

Definition 3.3. *(path group) We denote by $\Phi_0(M)$ the subset of $\Phi(M)$ contains of contractible loops and by $\Phi_\omega(M)$ the subset of $\Phi_0(M)$ with zero area in the interior.*

Proposition 3.4. *For $g \geqslant 1$, we get $\Gamma_{\mathbb{R}} \cong \Phi(M)/\Phi_\omega(M)$*

The relationship between path group and connections are related by holomony representation:

Proposition 3.5. *Fixing a lie group K. There is a one-to-one correspondence between the equivalent classes of a pair $(K \to P \to M, \mathcal{A})$ of K-bundle and a connection on it, and the conjugate classes of group homomorphism: $\Phi(M) \to K$.*

Finally we relate the Yang Mills connections to group homomorphism through similar perspective:

Proposition 3.6. *([1], P39)*
There is a one-to-one correspondence between the equivalent classes of a pair $(U(n) \to P \to M, \mathcal{A})$ of $U(n)$-bundle and a Yang Mills connection on it, and the conjugate classes of group homomorphism: $\Phi(M)/\Phi_\omega(M) \to U(n)$.

3.3. **Harder Narasimhan theorem.**

Theorem 3.7. *(Narasimhan & Seshadri)*
A holomorphic bundles of rank n is stable if and only if it arises from an irreducible representation $\rho : \Gamma_{\mathbb{R}} \to U(n)$, moreover isomorphic bundles correspond to equivalent irreducible representations.

Remark. In Narasimhan & Seshadris original paper (Ref1,2), the statement is a little different, here we use Atiyah-Botts statement.

In the lauguage of gauge theory, we can say

Corollary 3.8. *([1], P49)*
Let $\mathscr{N} \subset \mathscr{A}$ denote the set of connections giving the minimum for the Yang Mills functional, they are $\mathscr{G}$ equivalent to those representations $\rho : \Gamma_{\mathbb{R}}$ with $\rho(\mathbb{R})$ central. Let $\mathscr{N}^s$ be those given by irreducible representations (irreducible implies central, See Ref1, P43), then under identification of $\mathscr{A}$ with $\mathscr{C}$, we have $\mathscr{N}^s \subset \mathscr{C}^s$ and the induced map of quotient spaces:

$$\mathscr{N}//\mathscr{G} = \mathscr{N}^s/\mathscr{G} \to \mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$$

is a homeomorphism, in another word, the equivalent class of stable bundle of type (n, d) is $\mathscr{N}^s/\mathscr{G}$.

4. An analoge to Kempf Ness theorem

4.1. **The Kempf Ness theorem.**

The Kempf-Ness theorem appears in the study of geometric invariant theory. Consider a reductive group G action of a projective variety $X \subset \mathbb{P}(V)$ through a linear representation. We assume that the real part of G, say K, maps through $U(V)$. On the algebraic side there is a natural way to define a quotient variety $X//G$, which is locally the spectrum of invariant part of function ring. What's more as an topological space $X//G$ is the geometric quotient of the semistable points $X^{ss} \subset X$. On the other hand one can consider the canonical moment map induced by $U(V)$ action on $\mathbb{P}(V)$, and in symplectic geometry the symplectic quotient $\mu^{-1}(0)//K$ is considered. The Kempf Ness theorem tells us that these two quotients are homeomorphic to each other.

Theorem 4.1. *(Kempf-Ness theorem)*
Let $G = K^{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ and suppose its maximal compact subgroup

K is connected and acts symplectically on X (where the restriction of the Fubini Study form on $\mathbb{P}^n$ is used to give X its symplectic structure). Let $\mu : X \to \mathfrak{K}^*$ denote the associated moment map; then:
i) $G\mu^{-1}(0) = X^{ps}$.
ii) If $x \in X$ is polystable, then its orbit $G \cdot x$ meets $\mu^{-1}(0)$ in a single K-orbit.
iii) $x \in X$ is semistable if and only if its orbit closure $\overline{G \cdot x}$ meets $\mu^{-1}(0)$.
iv)The inclusion $\mu^{-1}(0) \subset X^{ss}$ induce a homeomorphism $\mu^{-1}(0)/K \to X//G$.

4.2. **Infinite dimensional Kempf Ness theorem.** From the last subsection we see the moduli space of stable bundle is given by: $\mathscr{N}//\mathscr{G}$, we can explain the isomorphism

$$\mathscr{N}//\mathscr{G} = \mathscr{N}^s/\mathscr{G} \to \mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$$

as a infinitely version of Kempf Ness theorem by understranding the curvature as a moment map:
Giving a principal bundle $P \to M$, consider the affine space of connections $\mathscr{A} = \Omega^1(M, ad(P))$:
Using an invariant inner product on $\mathfrak{g}$ we can construct an skew product on $\mathscr{A}$, and the compatible complex structure is given by Hodge star $*$ (see Ref1, 94):

$$(\theta, \varphi) = \int_M (\theta \wedge *\varphi)$$

The gauge group $\mathscr{G} = \Gamma(AdP) = Aut(P)$ acts on it, with the Lie algebra $\Gamma(adP) = \Omega^0(M, adP)$.
Note that $\Omega^2(M, adP)$ is dual to $\Omega^0(M, adP)$, $\forall \varphi \in \Omega^2(M, adP)$ we may consider φ as an element of Lie algebra, and we define:

$$F_\varphi(A) = (F_A, \varphi), \forall A \in \mathscr{A}, \forall \varphi \in \Omega^2(M, adP)$$

We want to show F_φ is a Hamiltonian function, in fact we can show, for $\forall \psi \in \Omega^1(M, ad(P))$,

$$(df_\varphi, \psi) = \int (d_A\varphi) \wedge \psi = \ell_{X_\varphi}\omega$$

(Since $\mathscr{A}$ is affine, the vector field determind by φ is φ itself),
So we have show that: the action of $\mathscr{G}$ is hamiltonian and the hamiltonian function is F_φ,
By definition, the moment map μ is determind by $(\mu(A), \varphi) = F_\varphi(A)$, but the definition of F_φ is exactly $F_\varphi(A) = (F_A, \varphi)$, so:
The moment map is given by $A \to F_A$!!!
Remeber (3.8) the moduli space is given by $\mathscr{N}//\mathscr{G}$, and $\mathscr{N}$ contains those connections that minimise the moment map (curvature), we take this minimum as c, then $\mathscr{N} = F_A^{-1}(c)$, and the homeomorphism:

$$\mathscr{N}//\mathscr{G} = \mathscr{N}^s/\mathscr{G} \to \mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$$

can be translate into :

$$F_A^{-1}(c)//\mathscr{G} = \mathscr{N}^s/\mathscr{G} \cong \mathscr{C}^s/\mathscr{G}^{\mathbb{C}} = \mathscr{C}//\mathscr{G}^{\mathbb{C}}$$

4.3. **Donaldson's proof of Harder- Narasimhan theorem.**

Theorem 4.2. *(Donaldson, [2])*
An indecomposable holomorphic bundle E over a Riemann surface X is stable if and only if there is a unitary connection on E having constant central curvature $*F = 2\pi i\mu(E)$. *Such a connection is unique up to isomorphism.*

Proof. (Sketch) Given a Hermitian vector bundle E, we have a surjective map from the space of unitary connections, say $\mathcal{A}$, to the space of holomorphic map.
The gauge group G of unitary automorphism acts on $\mathcal{A}$, and the action extends to the complexification $\mathcal{A}$ C, two unitary bundles gives the same holomorphic structure iff they are in the same $\mathcal{A}$ orbit.
Given a holomorphic bundle $\mathcal{E}$ we write $\mathcal{O}(\mathcal{E})$ for the orbit of connections of the appropriate C^∞ bundle.
Consider the functional

$$J(A) = N(\frac{*F}{2\pi i} + \mu \cdot 1)$$

where N : $\Gamma(\Omega^0(End(E))) \to \mathbb{R}$ is defined as:

$$N(s) = (\int_X v(s)^2)^{\frac{1}{2}}$$

with the trace norm: $v(M) = Tr(M * M)^{1/2} = \sum |\lambda_i|$, F is the curvature and $\mu = \mu(E)$.
The existence of $inf_J|_{0(\mathcal{E})}$ in $\mathcal{O}(\mathcal{E})$ is relative to the stability condition. □

Remark. Such corresponding theorem also appears in at least two other cases, one is the Hitchin-Kobayashi corresponding, the other is Yau-Tian-Donaldson theorem, in this philosophy, one can always find a stability condition corresponds to the existence for a good metric(connection).

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