BASIC GEOMETRIC INVARIANT THEORY

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Contents

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1. Thoughts of GIT

1.1. **Reductive Group Action.**

Proposition 1.1. *([3] Prop 2.36)*
The bounday of an orbit $\overline{G \cdot x} - G \cdot x$ is a union of orbits of strictly smaller dimensions. Inparticular, each orbit closure contains a closed orbit (of minimal dimension). 1

Proposition 1.2. *([3] Lemma 3.8)*
Let G be an affine algebraic group acting on a variety X. Then the dimension of the stabiliser subgroup viewed as a function $dimG_{\cdot}X \to \mathbb{N}$ is upper semi-continuous, that is for every n, the set

$$\{x \in X : dimG_x \geqslant n\}$$

is closed in X. Equivalently,

$$\{x \in X : dimG \cdot x \leqslant n\}$$

is closed in X for all n.

Proposition 1.3. *([3] Lemma 3.8)*
Suppose G is a gemetrically reductive and acts on affine variety X. If W_1 and W_2 are disjoint G invariant closed subsets of X, then there is an invariant function $f \in A(X)^G$ which seperates these sets, i.e.

$$f(W_1) = 0, f(W_2) = 1$$

Lemma 1.4. *(Newstead, moduli problems, lemma 3.4.1)*
Suppose G is a reductive group acting ratioanlly on a finitely generated k algebra R, then:
Let J be an ideal in R, invariant under G, If $f \in (R/J)^G$, then $f^t \in R^G/(J \cap R^G)$ for some positive integer t.

1.2. **Algebraic Geometry Side.**

Throughout we fix an algebraically closed field k.

1.2.1. *Categorical Quotient and Good Quotient.*

Definition 1.5. *(categorical quotient) A categorical quotient for the action of G on X is a G-invariant morphism $\varphi : X \to Y$ of varieties which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through φ so that there exists a unique morphism h:$Y \to Z$ such that $f = \varphi \circ h$.*

Remark. As φ is continuous and constant on orbits, it is also constant on orbit closures. Hence, a categorical quotient is an orbit space only if the action of G on X is closed; that is, all the orbits G(x) are closed. GIT deals with problems occurring when we have non-closed orbits.

Let G be an affine algebraic group acting on a variety X over k. The group G acts on the k-algebra $\mathcal{O}_X$(X) of regular functions on X by

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

and we denote the subalgebra of invariant functions by

$$\mathcal{O}_X(X)^G := f \in \mathcal{O}_X(X) : g \cdot f = g, \forall g \in G$$

Similarly if U $\subset$ X is a subset which is invariant under the action of G (that is, g·u $\in$ U,$\forall$u $\in$ U,g $\in$ G), then G acts on $\mathcal{O}_X$(U) and we write $\mathcal{O}_X(U)^G$ for the subalgebra of invariant functions.

Definition 1.6. *(good quotient) A morphism φ:X $\to$Y is a good quotient for the action of G on X if*
i) φ is constant on orbits.
ii) φ is surjective.
iii) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X$ $(\varphi^{-1}(U))$is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.
iv) If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W\ )$ is closed in Y .
v) If W1 and W2 are disjoint G-invariant closed subsets, then φ(W1) and φ(W2) are disjoint.
vi) φ is affine (i.e. the preimage of every affine open is affine).

Remark. We note that the two conditions iv) and v) together are equivalent to: v) If W1 and W2 are disjoint G-invariant closed subsets, then the closures of φ(W1) and φ(W2) are disjoint.

Proposition 1.7. *If $\varphi : X \to Y$ is a good quotient for the action of G on X, then it is a categorical quotient.*

Corollary 1.8. *Hence a good quotient is unique up to isom.*

Corollary 1.9. *([3] Cor 2.39)*
Let $\varphi : X \to Y$ be a good quotient, then: (1) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ iff $\varphi(x_1) = \varphi(x_2)$;
(2) For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed (all orbits are closed), then φ is a geometric quotient.

Proof. (1)Note that under zariski topology every point in a variety is closed, hence $\varphi^{-1}(\varphi(x))$ is closed and $\overline{Gx} \subset \varphi^{-1}(\varphi(x))$, that is $\Rightarrow$); Note also that φ is a good quotient, then $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$ implies their image is distinct, that is $\varphi(x) \neq \varphi(y)$, which proves $\Leftarrow$.
(2)Since disjoint closed sets has disjoint image, we could only have one closed orbit, the existence of closed orbit is guaranteed by **??** $\square$

1.2.2. *Affine GIT Quotients.*

Acknowledgements. Remember for an affine AG in $k[x_1, \cdots, x_n]$, k is alg. closed, there are several corres.:
(1) (Radical) ideal of $k[x_1, \cdots, x_n]$ and algebraic set of k^n (Hilbert's Nullstellensatz);
(2) Affine varieties and f.g k algebras: $X \to A(X)$, inversely $A = k[x_1, \cdots, x_n]/I \to Z(I)$;
(3) Ring of regular functions and the coordinate ring: $\mathcal{O}_X(X) \cong A(X)$ (See 52,Thm 1.3.2);
(4) Affine varieties and its max spectral, $x \in X \in \mathfrak{m}_x \in Max(A(X))$ (See Atiyah CG, Ex1.2.7).

In this case X is a affine variety and $\mathcal{O}_X$ becomes A(X), which is finitely generated, if the subalgebra of G invariant regular functions on X, noted $A(X)^G$, is also finitely generated, then the categorical quotient is the morphism $\varphi :$ X $\to Y := Spec(A(X)^G)$, induced by $A(X)^G \hookrightarrow A(X)$, however, $A(X)^G$ is not always f.g. (the Hilberts 14th problem).

Theorem 1.10. *(Nagata) Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. Then the G-invariant subalgebra A^G is finitely generated.*

Remark. The double slash notation X//G used for the GIT quotient is a reminder that this quotient is not necessarily an orbit space and so it may identify some orbits. In nice cases, the GIT quotient is also a geometric quotient and in this case we shall often write X/G instead to emphasise the fact that it is an orbit space.

Theorem 1.11. *. Let G be a reductive group acting on an affine variety X. Then the affine GIT quotient $\varphi : X \to Y := X//G$ is a good quotient and in particular Y is an affine variety. Moreover if the action of G is closed(orbits are all closed) on X, then it is a geometric quotient.*

Proof. (Sketch)
Nagata thm tells us $A(X)^G$ is f.g, so it defines an affine variety Y, that is vi);
If $f \in A(X)^G$, then f is constant on any Gx, note that $\mathfrak{m}_x$ contain those f s.t $f(x) = 0$, so $i^*(\mathfrak{m}_x) = \mathfrak{m}_x \cap A(X)^G = i^*(\mathfrak{m}_y), \forall y \in Gx$, this poves i);
For any $y \in Y$, we choose generators $f_1, \cdots, f_m$ of $\mathfrak{m}_y$ and G being reductive is used to show:

$$(\sum f_i A(X)) \cap A(X)^G = \sum f_i A(X)^G$$

in special $\sum f_i A(X)$ is a proper subset of $A(X)$ hence contains in a $\mathfrak{m}_x$ for some x, then $\varphi(x) = y$ and we prove ii);
iii) can be checked locally: just consider open set of the form $Y_f = \{y \in Y, f(y) \neq 0\}, f \in A(X)^G$ since they give an open cover, now:

$$\mathcal{O}_X(\varphi^{-1}(U))^G = \mathcal{O}_X(X_f)^G = (A(X)_f)^G = (A(X)^G)_f = \mathcal{O}_Y(U)$$

Here we use the fact that localization commutes with taking G invariant; In the proof of iv) and v), reductivity still makes sense. □

As we saw above, when a reductive group G acts on an affine variety X in general a geometric quotient (i.e. orbit space) does not exist because in general the action is not closed. For finite groups G, every good quotient is a geometric quotient as the action of a finite group is always closed (every orbit is a finite number of points which is a closed subset). For general G, we ask if there is an open subset of X for which there is a geometric quotient.

Definition 1.12. *We say $x \in X$ is stable if its orbit is closed in X and $dimGx = 0$ (or equivalently, $dimG(x) = dimG$). We let X^s denote the set of stable points.*

Proposition 1.13. *Suppose a reductive group G acts on an affine variety X and let $\varphi : X \to Y$ be the associated good quotient. Then $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$ is also open. Moreover, $\varphi : X^s \to Y^s$) is a geometric quotient.*

1.2.3. *Projective GIT Quotients.* This part mainly comes from Refn3.

Acknowledgements. Here we give some properties of projective varieties, one can find their proof in GTM52,I.3.4 and II.2.5
Let X be an irr. proj. variety over k, $R(X) = k[x_0, \cdots, x_n]/I(X)$, here $I(X)$ is the homogeneous ideal, then:
(1) $\mathcal{O}_X(X) = k$;
(2) There is a 1-1 corrs. between X and homo. maximal ideals in $R(X)$ **which does not contain** $R(X)_+$ via

$$p \to \mathfrak{m}_p$$

(3) For $f \in R(X)_+$, we define $X_f = \{x \in X, f(x) \neq 0\}$, $(R(X)_f)_0$ being the degree 0 piece of local ring $R(X)_f$, then

$$X_f \cong Spec(R(X)_f)_0$$

and

$$\mathcal{O}_X(X_f) \cong (R(X)_f)_0$$

(4) The opensets X_f for homogeneous $f \in R(X)_+$ form a basis for the Zariski topology of X.

In this case X is a projective variety and $R(X) = k[x_0, \cdots, x_n]/I(X)$, I(X) is the homogeneous ideal of X.

Definition 1.14. *For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we let $X//G$ denote the projective variety $Proj(R^G)$ associated to the finitely generated graded k-algebra R^G of G-invariant functions* *where $R = R(X)$ is the homogeneous coordinate ring of X. The inclusion $R^G \to R$ defines a rational map*

$$\varphi : X \dashrightarrow X//G$$

which is undefined on the null cone

$$N_{R^G}(X) := \{x \in X : f(x) = 0, \forall f \in R^G_+\}.$$

We define the semistable locus $X^{ss} := X - N_{R^G}(X)$to be the complement to the nullcone.

Remark. For every $f \in R(X)^G_+$ we see X_f as $Spec(R(X)_f)_0$, then the natrual embedding $R(X)^G \to R(X)$ induce $\varphi|_{X_f} : X_f \to Y_f$, note that in Spec it is just restriction of $\mathfrak{m}_p$ so φ is well defined on X^{ss};
Note that $Y = \cup_{f \in R^G_+} Y_f$, so $\varphi^{-1}Y = X^{ss}$ and φ is not defined on N_{R^G};
Another way to think about this is that, for $x \in N_{R^G}$, $R^G_+ \subset \mathfrak{m}_x$ so $\mathfrak{m}_x$ doesn't comes from any element of Y where $R(Y) = R(X)^G$, one may also understand this picture in the following example.

Example 1.15. *Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ by*

$$(t, [x_0, \cdots, x_n]) \to (t^{-1}x_0, tx_1, \cdots, tx_n)$$

Then

$$R(X) = k[x_0, \cdots, x_n], R(X)^G = k[x_0x_1, \cdots, x_0x_n] = k[y_1, \cdots, y_n]$$

then $Y \cong \mathbb{P}^{n-1}$ and φ is :

$$[x_0, \cdots, x_n] \to [x_0x_1, \cdots, x_0x_n]$$

which is not defined on

$$N_{R^G}(X) = \{[x_0, \cdots, x_n], x_0 = 0 \, or \, x_1 = \cdots = x_n = 0\}$$

Acknowledgements. To construct φ explicitly we have to using gluing construction of algebraic variety.
Let $\{X_i\}_{i \in I}$ be a finite subset $U_{ij} \subset X_i$ for each $j \in I$, the gluing data is a choice of an open affine subset $U_{ij} \subset X_i$ for each $j \in I$, and an isom $\phi_{ij} : U_{ij} \to U_{ji}$ for each pair (i, j) satiesfying:
(1) $U_{ii} = X_i, \phi_{ii} = id_{X_i}$;
(2) $\forall i, j, k, \phi_{ji}(U_{ij} \cap U_{ik}) \subset U_{jk}$ and

$$(\phi_{kj} \circ \phi_{ji})|_{U_{ij} \cap U_{ik}} = \varphi_{ki}|_{U_{ij} \cap U_{ik}}$$

Then we can use these data to construct a ringed space $(X, \mathcal{O}_X)$, such a space is called an abstruct alg. variety. An abstruct alg. variety is isom. to a quasi proj. algebraic variety iff it admit an ample line bundle.

Proposition 1.16. *The projective GIT quotient for the linear action of G on $X \subset \mathbb{P}^n$ is the morphism φ:X^{ss} $\to$$X//G$.*

Proof. Firstly we give the explicit structure of ϕ:
We choose a basis $\{f_1, \cdots, f_r\}$ for $R(X)^G$, then Y_{f_j} give a top. basis of Y, note that :

$$\mathcal{O}_X(X_{f_i})^G = (R(X)_{f_j})_0^G = ((R(X)_{f_j}^G)_0 = \mathcal{O}_Y(Y_{f_j})$$

we get a good catergorical quotient $\varphi_i : X_{f_i} \to Y_{f_i}$ as is shown in affine case, use the isom. we can see f_i/f_j as an reguler function ϕ_{ij} in $\mathcal{O}_Y(Y_{f_j})$, consider the open set $D(\phi_{ij}) \subset \mathcal{O}(Y_j)$, then:

$$\varphi^{-1}(D(\phi)_{ij})) = \varphi^{-1}(D(\phi_{ji})) = X_{f_i} \cap X_{f_j}$$

Using $\mathcal{O}(Y_{f_i}) \cong \mathcal{O}(X_{f_i})^G$ we have:

$$\mathcal{O}(\mathcal{D}_{ji}) = \mathcal{O}((Y_{f_i})_{\phi ji}) \cong \mathcal{O}(X_{f_i})^G \cap \mathcal{O}(X_{f_j})^G = \mathcal{O}_X(X_{f_i} \cap X_{f_j})^G$$

As is shown in affine case,

$$\varphi_i|_{X_{f_i} \cap X_{f_j}} : X_{f_i} \cap X_{f_j} \to \mathcal{D}_{ji}, \varphi_j|_{X_{f_i} \cap X_{f_j}} : X_{f_i} \cap X_{f_j} \to \mathcal{D}_{ij}$$

are all categorical quotient of $X_{f_i} \cap X_{f_j}$, by universal property of quotient, we get

$$\phi_{ij} : D(\phi_{ij}) \to D(\phi_{ji})$$

by uniqueness, $\{Y_{f_i}, \phi_{ij}\}$ satisfies the condition of gluing, so we can glue together Y_{f_j} and φ_i to obtain a morphism $\varphi : X^{ss} \to Y$ □

Definition 1.17. *Consider a linear action of a reductive group G on a closed subvariety $X \subset P$. Then a point $x \in X$ is:*
(1) semistable if there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $f(x) \neq 0$.
(2) stable if $dimG_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ that $x \in X_f$ and the action of G on X_f is closed.
(3) unstable if it is not semistable.
We denote the set of stable points by X^s and the set of semistable points by X^{ss} .

Theorem 1.18. *For a linear action of a reductive group G on a closed subvariety $X \subset \mathbb{P}^n$, we have:*
i) The GIT quotient $\varphi : X^{ss} \to Y := X//G$ is a good quotient and a categorical quotient. Moreover, Y is a projective variety.
ii) $G \cdot x_1 \cap G \cdot x_2 \cap X^{ss} \neq 0$ if and only if $\varphi(x_1) = \varphi(x_2)$.
iii)There is an open subset $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and $\varphi : X^s \to Y^s$ is a geometric quotient.

Remark. The nicest case is when $X^{ss} = X^s$ and so Y is a projective quotient which is also an orbit space.

Remark. Note that $\pi : X^{ss} \to Y$ is a categorical quotient, but it doesn't mean X^{ss} is uniquely determined. Variation of GIT theory comes from this.

Definition 1.19. *A semistable point x is said to be polystable if its orbit is closed in X^{ss}. We say two semistable points are S-equivalent if their orbit closures meet in X^{ss}.*

Proposition 1.20. *Every stable point is polystable.*

Lemma 1.21. *Let x be a semistable point; then its orbit closure $G \cdot x$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.*

Corollary 1.22. *Let x and x be semistable points; then $\varphi(x) = \varphi(x)$ if and only if x and x' are S-equivalent. Moreover, there is a bijection in the sense of sets*

$$X//G \cong X^{ps}/G$$

where $X^{ps} \subset X^{ss}$ is the set of polystable points.

In general, it is not so easy to compute the semistable points for an action. Often one uses the HilbertMumford criterion, which is a numerical criterion, to determine semistability. This follows from the following topological criterion for semistability, which we will use later on in the proof of the KempfNess Theorem.

Proposition 1.23. *(Hoskin, Kempf Ness, Prop 2.7)*
(Topological criterion for semistability). Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For $x \in \mathbb{P}^n$, choose a non-zero lift $\widetilde{x}$ in the affine cone $\widetilde{X} \in \mathbb{A}^{n+1}$. Then the following statements hold.
(1) x is semistable if and only if $0 \notin \overline{G \cdot \widetilde{x}}$.
(2) x is polystable if and only if $G \cdot \widetilde{x}$ is closed.
(3) x is stable if and only if $G \cdot \widetilde{x}$ is closed and has dimension equal to the dimension of G.

1.2.4. *GIT for Polarised Projecticve Varieties.* Thi part manily comes from Refn 1,3,5.

Definition 1.24. *(ample and very ample) A line bundle is called basepoint-free if it has enough sections to give a morphism to projective space. Explitícly, if s_i are sections such that $\forall p \in X, \exists k\, s.t\, s_k(p) \neq 0$, then we write this morphism as:*

$$s : X \to H^0(X, L), x \mapsto [s_0(x), \cdots, s_n(x)]$$

A line bundle is semi-ample if some positive power of it is basepoint-free; semi-ampleness is a kind of "nonnegativity". More strongly, a line bundle on X is very ample if it has enough sections to give a closed immersion (or "embedding") of X into projective space. A line bundle is ample if some positive power is very ample.

Remark. Note that $\mathcal{O}(1)$ on a projective space $\mathbb{P}(V)$ is fiberwisely linear functions in the coordinates x_i, so the $s^*(\mathcal{O}(1)) = L$.

Acknowledgements.
(1)(Danial, CG, Prop2.4.1) For $k \geqslant 0$ the space $H^0(\mathbb{P}^n, \mathcal{O}(k))$ is canonically isomorphic to the space $\mathbb{C}$ of all homogeneous polynomials of degree k.
(2)There is a bijective corr. between the set of polorized proj. scheme and graded k-algebra f.g. in degree 1 via:

$$(ProjR, \mathcal{O}(1)) \leftarrow R$$
$$(X, L) \rightarrow R(X, L) = \oplus_{r \geqslant 0} H^0(X, L^{\otimes r})$$

Proof. (1) The transition function of $\mathcal{O}(k)$ is $g_{\alpha\beta} = (\frac{z_\alpha}{z_\beta})^k$, so a section of $\mathcal{O}(k)$ in the nbhd U_α satiesfies:

$$z_\alpha^k F_\alpha(\frac{z_0}{z_\alpha}, \cdots, \frac{\hat{z_\alpha}}{z_\alpha}, \cdots, \frac{z_n}{z_\alpha}) = z_\beta^k F_\beta(\frac{z_0}{z_\beta}, \cdots, \frac{\hat{z_\beta}}{z_\beta}, \cdots, \frac{z_n}{z_\beta})$$

So a degree k homogeneous polynomial F gives a section of $\mathcal{O}(k)$, conversely, we first choose a section deg k homo function, say G, then any section F, $s = \frac{F}{G}$ gives a meromorphic function on $\mathbb{P}^n$, which is a mero. function on $\mathbb{C}^{n+1} - \{0\}$ of degree 0, now consider $H = sG$, it is a holomorphic function on $\mathbb{C}^{n+1} - \{0\}$ and is homogeneous of degree k, since $G(\lambda z) = \lambda^k G(z)$, G is bounded near zero, so it can be extended to a holomorphic function H' on $\mathbb{C}^{n+1}$, use a power series expansion of H' we conclude that H' is in fact a homo. poly. Now by our construction the section F is induced by H'. □

An abstract projective variety does not come with a specified embedding in projective space, however, an ample line bundle L on X determineds an bedding of X into a projective space. In order to construct a GIT quotient of an abstract projective variety X we need the extra data of a lift of the G-action to a line bundle on X; such a choice is called a linearisation of the action.

Definition 1.25. *(Linearizations of actions)*
Let $\varphi : L \rightarrow X$ be a line bundle on X, a linearisation of the action of G w.r.t. L is an action of G on L such that:
1) For all $g \in G$ and $l \in L$, we have $\varphi(g \cdot l) = g \cdot \varphi(l)$
2) For all $x \in X$ and $g \in G$ the map of fibres $\sigma_x(g) : L_x \rightarrow L_{g \cdot x}$ is a linear map;
3) For all $x \in X$ and $g, g' \in G$, $\sigma_x(gg') = \sigma_{g'(x)}(g)\sigma_x(g')$, that is, the G action on L is equivariant.

Remark. For a given embedding $X \rightarrow \mathbb{P}^n$ and a linear action of G, this action natrually has a liearization on $\mathcal{O}(-1)|_X$ and then on $\mathcal{O}(1)|_X$, note that in the corres. in ackn. $\mathcal{O}(1)|_X$ is just the ample line bundle L needed for an abstract variety ,so the purpose to find linearization is just to make the action of G compatable with the embedding of X into $\mathbb{P}^n$ induced by L.

And natrually we have stability and GIT quotient in this sense:

Definition 1.26. *Let G be a reductive group acting on a projective scheme X w.r.t. an ample linearization.*
(1) A point $x \in X$ is semistable w.r.t L if there exists $r > 0$ and an invariant section $\sigma \in R(X, L^{\otimes r})^G$ such that $\sigma(x) \neq 0$. We denote $X^{ss}(L) = X - V(R(X, L))^G_+$.
(2) The proj. GIT quotient of G acting on X w.r.t L is the morphism

$$\varphi : X^{ss}(L) \to X//_L G := Proj R(X, L)^G$$

obtained by resting the above rational map to its domain of definition.

Theorem 1.27. *Let G be a reductive group acting on a projective scheme X w.r.t. an ample linearization.*
The the GIT quotient is a good quotient of the G actoin on $X^{ss}(L)$.

Remark. More generally, Mumford constructs quasi-projective GIT quotients w.r.t a linearization of the action, see Mumford's book named by GIT.

1.2.5. *About Linearization.*

Theorem 1.28. *(Existence of linearization)*

Theorem 1.29. *(Existence of embedding preserving the linearization)*
Let X be a quasi-projective noamal algebraic variety, acted by an irreducible algebraic group G, thrn there exists a G-equivariant embedding via its $X \hookrightarrow \mathbb{P}^n$ linear representation $G \to GL_{n+1}$.

Remark. Abviously the quotient constructed above is related to the chose of linearization, hence a natrual problem is, when does two linearizations give the same quotient, and how many different quotients are there? This problem has been studied, see Refn 4. Roughly speaking, as the article say: "among ample line bundles which give projective geometric quotients there are only nitely many equivalence classes. These classes span certain convex subsets (chambers) in a certain convex cone in Euclidean space; and when we cross a wall separating one chamber from another, the corresponding quotient undergoes a birational transformation which is similar to a Mori ip.'

2. Criterion of (semi)Stability

2.1. Topological Criterion of (semi)stability.

Remark. I think to show an orbit Gx is closed in $X^s s$ it suffices to show it's closed in any X_f such that f is a G homo. poly and $f(x) \neq 0$ but it seems people use a lot of words to avoid this, my reasons is as follows:
1.Since f is G invariat, $G_x \in X_f$;
2.If a subset is totally contained in a larger one: $A \subset B \subset C$, then $\overline{A} \cap B = \overline{A \cap B}$, here the latter closure is taken as induced sub space topology of B. I don't know if I made a mistake here.

Lemma 2.1. *([3] 4.14)*
A point x is stable iff x is semistable and its orbit $G \cdot x$ is closed in X^{ss} and has zero dimensional stabiliser.

Proof. $\Rightarrow$): we need to show $\overline{Gx} \cap X^{ss} = Gx$, take $y \in \overline{Gx} \cap X^{ss}$, since under zariski topology every point in a variety is closed (T1 axiom), then $\varphi(y) = \varphi(x)$ and $y \in \varphi^{-1}(\varphi(x)) \subset \varphi(Y^s) = X^s$. Note that $\varphi : X^s \to Y^s$ is a gemetric quotient and hence the action of on X^s is closed, so $y \in \overline{Gx} \cap X^s = Gx$, that is, $y \in Gx$, i.e $\overline{Gx} \cap X^{ss} = Gx$;
$\Leftarrow$): we want to show Gx is closed in some X_F, choose f such that $x \in X_f$, by property of reductive group action 1.2, the G invariant set

$$z \in X_f, dimG_z < dimG$$

is closed in X_f, now $Z \cap Gx = \emptyset$, they can be seperated by a G invariant function $h \in A(X_f)^G$, here we admit (see 1.4) a conclusion that we can find a homo invariant poly h' such that $h^s = h'/f^r$ for some s and r, then we can consider $X_{fh'}$ and $Z \neq X_{fh'}, Gx \in X_{fh'}$ so every orbit Gy in $X_{fh'}$ is closed. ☐

Corollary 2.2. *Every stable point is polystable.*

Lemma 2.3. *([3] 4.16)*
Let x be a semistable point, then $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but nout stable, then this unique polystable orbit is also not stable.

Proof. Now $\varphi : X^{ss} \to Y$ is a good quotient, so by 1.9 $\overline{Gx}$ contains a unique polystable orbit.
By **??** the unique closed orbit has minimal dimension in $\overline{Gx}$, if $\overline{Gx} \setminus Gx \neq \emptyset$(that is, not stable), then agian by **??** this closed orbit has strictly lower dimension hence is not stable. ☐

Proposition 2.4. *(Hoskin, Kempf Ness, Prop 2.7; Hoskin, GIT and SQ, Prop 5.1)*
(Topological criterion for semistability). Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For $x \in \mathbb{P}^n$, choose a non-zero lift $\widetilde{x}$ in the affine cone $\widetilde{X} \in \mathbb{A}^{n+1}$. Then the following statements hold.
(1) x is semistable if and only if $0 \notin \overline{G \cdot \widetilde{x}}$.
(2) x is polystable if and only if $G \cdot \widetilde{x}$ is closed.
(3) x is stable if and only if $G \cdot \widetilde{x}$ is closed and has dimension equal to the dimension of G.

Proof. (1)x is semistable iff $x \in X_F$ for some G invariant homo. poly. F, note that F can be considered as a G invariant homo. poly. on $\widetilde{X}$, and since F is constant on $G\widetilde{x}$, F is also constant on its closure (Zariski topology on variety is T1!), so F separates 0 and $\widetilde{X}$, that is $\Rightarrow$,on the other hand, if they are disjoint, then we can find a G inv. poly. separating them hence a G

inv. poly. separating them by choice a suitable homo. term, this finish the proof.
(3)Since $G_{\widetilde{x}} \leqslant G_x$, if x is stable, then $G_{\widetilde{x}}$ is also 0 dimensional, it leaves to show $G_{\widetilde{x}}$ is closed in $\widetilde{X} - \{0\}$, now we consider a subset $Z = f^{-1}(f(\widetilde{x}))$ contains $G_{\widetilde{x}}$ and it suffices to show $G_{\widetilde{x}}$ is closed in Z, Z is a closed subset of $\widetilde{X}$ and there is a natrual morphism $\pi : Z \to X_f$ as restriction of quotient map $\widetilde{X} - \{0\} \to X$, note also that π is a finite cover: fixing z, $f(tz) = t^n f(z) = f(\widetilde{x})$ has only n choices of t, so $\pi^{-1}(G \cdot x)$ is n disjoint orbits in C_i Z. Since the $\widetilde{X}$ and X has topological basis of the same type X_g, so taking closure is commutative with taking quotient, so in particularly $\pi\overline{G(\widetilde{x})} = \overline{Gx} = Gx$, we have show points in $\overline{G(\widetilde{x})} - G(\widetilde{x})$ are in the other orbits, but we can use G invariant homo. poly to seperate different points, hence seperates their belonging G orbits, so we get a contradiction and $\overline{G(\widetilde{x})} - G(\widetilde{x})$ is empty. This proves $\Rightarrow$.
To prove $\Leftarrow$, we can still use this method, since $G\widetilde{x}$ is closed in Z, the underlying orbit Gx is also closed in X_f and since in a fiber $\pi^{-1}(y)$ different points belongs to different orbits, the G_x is in fact equal to $G_{\widetilde{x}}$ hence have dimension 0, this is true for any f s.t $f(x) \neq 0$, by 2.1 we show x is stable.
(2) We can judge if an orbit is closed by choosing invariant function f s.t $f(x)$ and to show it's closed in X_f. now just use $\pi : Z \to X_f$. □

2.2. **Numberical criterion: Hilbert Mumford Criterion.**

Let G be an reductive group acting linearly on $X \subset \mathbb{P}^n$. For $x \in \mathbb{P}^n$, choose a non-zero lift $\widetilde{x}$ in the affine cone $\widetilde{X} \in \mathbb{C}^{n+1}$, given any regular homorphism $\lambda\mathbb{G}_m \to G$, we have a induced action on $X \subset \mathbb{P}^n$.

Definition 2.5. *A 1-parameter subgroup (1-PS) of G is a nontrivial group hom $\lambda : \mathbb{G}_m \to G$.*

Proposition 2.6. *every action of $\mathbb{G}_m$ on $\mathbb{C}^{n+1}$ is diagonized, moreover, in appropriate coordinates it acts by*

$$\lambda(t) \cdot v = (t^{m_0} x_0, \cdots, t^{m_n} x_n)$$

now suppose under this representation (choice of coor.) $\widetilde{x} = \sum a_i e_i$, then

$$\lambda(t) \cdot \widetilde{x} = \sum t^{m_i} a_i e_i$$

Definition 2.7. *(Hilbert-Mumford criterion)*

$$\mu(x, \lambda) := -min\{m_i : a_i \neq 0\}$$

and this number is independent of the choice of the lift and the basis e_i.

Remark. There are different definition for μ, and our definition comes from Mumford's book [4], in Dolgachev, Igor's book [1], it is defined as $-min\{m_i : a_i \neq 0\}$.

Use topological criterioin for stability, it's easy to shown that $x \in X$ is (semi)stable iff $\mu(x, \lambda) > 0 (\geqslant 0)$ for all 1-PSs λ of G. The Hilbert-Mumford criterion gives the converse to this statement.

Theorem 2.8. *(Hilbert-Mumford Criterion)*
Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$, then

$$x \in X^{ss} \Leftrightarrow \mu(x, \lambda) \geqslant 0, \forall \lambda$$

$$x \in X^{s} \Leftrightarrow \mu(x, \lambda) > 0, \forall \lambda$$

Remark. So to judge stability in the case of $\mathbb{G}_m$ action, it suffices to compute the weight.

2.3. **Torus Action Case.** This part mainly comes from Dolgachev, Igor's book [1]
Let G be a torus $\mathbb{C}^{*r}$ and L an ample G-linearized line bundle on a projective G-variety X, suppose the ample line bundle induce the embedding into $\mathbb{P}^n$. We denote the representation of G by $\rho : \mathbb{C}^{*r} \to \mathbb{C}^n = P(V)$, since G is abelian, all irr reps are q dimensional, so we may find basis e_i such that:

$$V = \sum_{i=1}^{n} \mathbb{C} e_i = \sum_{i=1}^{n} V_{\chi_i}$$

note also that irr rep of $\mathbb{C}^*$ are all of the form $t \cdot v = t^n v$ for some n, so ρ is determined by r integer valued vectors in $\mathbb{Z}^r$, to be precisely, we have:

$$\rho(t_1, \cdots, t_r)(e_j) = t_1^{m_1^j} \cdots t_r^{m_r^j} e_j := \vec{t}^{\vec{m}_j} e_j$$

for some $\vec{m_j} \in \mathbb{Z}^r$.
Note also that any 1-PS of G is given by:

$$\lambda : \mathbb{C}^* \to \mathbb{C}^{*r} : \lambda(t) = (t^{a_1}, \cdots, t^{a_r})$$

for some $\vec{a} = (a_1, \cdots, a_r)$ in $\mathbb{Z}^r$.
Then the action of this 1-PS is given by:

$$\rho \circ \lambda(t)(\sum_j x_j e_j) = \rho(t^{a_1}, \cdots, t^{a_r})\widetilde{x} = \sum_j t^{\vec{a} \cdot \vec{m}_j} x_j e_j$$

so in this case we can read the stability case using this informations.

Definition 2.9. *for $\widetilde{x} = (\sum_j x_j e_j) \in V$ we define the* ***weight set*** *of $\widetilde{x}$ by*

$$wt(\widetilde{x}) = \{\vec{m}_j, x_j \neq 0\}$$

and the ***weight polytope*** *of $\widetilde{x}$, say $\overline{wt(\widetilde{x})}$ is defined as the convex hull of $wt(x)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$.*
we define

$$\mu^L(x, \lambda_{\vec{a}}) = -min\{\vec{a}\vec{m}_j, x_j \neq 0\} = min_{\chi \in wt(x)} < \vec{a}, \chi >$$

Proposition 2.10. *Let G be a torus $\mathbb{C}^{*r}$ and L an ample G-linearized line bundle on a projective G-variety X, then:*

$$x \in X^{ss}(L) \Leftrightarrow 0 \in \overline{wt(x)}$$

$$x \in X^{s}(L) \Leftrightarrow 0 \in \overline{wt(x)}^{o}$$

Proof. [1] Thm9.2
Note that $\overline{wt(x)}$ us convex, so we may use the seperating thm for convex set, that is, for a convex set A, $x \notin A(A^o)$ iff we have a hyperplane seprating thm. Note also that we can take the normal vector of this hyperplane to be rational valued (hence integer valued) by small pertubation, so:
$0 \notin \overline{wt(x)}(\overline{wt(x)}^{o})$ iff exists $\vec{a} \in \mathbb{Z}^n$, $\overline{wt(x)} \cdot \vec{a} \leqslant 0(< 0)$, iff exists $\vec{a} \in \mathbb{Z}^n, s.t.\vec{a} \cdot \vec{m_j} \leqslant 0(< 0)$ forall $\vec{m_j}(j \neq 0)$. Now use the Hilbert-Mumford criterion. □

2.4. **A criterion for arbitray reductive action.** Now let G be any reductive group acting linearly on a projective variety. Note that any 1-PS of G has its image in a maximal torus T of G, so the information of maximal torus cover enough for searching (semi)stable points.

Proposition 2.11.

$$X^{ss}(L) = \cap_{max.\,tori\,T} X_T^{ss}(L_T)$$

$$X^{s}(L) = \cap_{max.\,tori\,T} X_T^{ss}(L_T)$$

Note that maximal torus are in a single G conj orbit, so we may fix one maximal tori T. suppose $gT'g^{-1} = T$, note that for any 1PS $\lambda(\mathbb{C}^*)$, to judge $\lambda(\mathbb{C}^*)\widetilde{x}$ (T action) is equal to judge the $g\lambda(\mathbb{C}^*)g^{-1}g\widetilde{x}$ (T' action), so:

Proposition 2.12. *Let T be a maximal torus of G, then*

$$x \in X^{ss}(L) \Leftrightarrow \forall g \in G, g \cdot x \in X_T^{ss}(L_T)$$

$$x \in X^{s}(L) \Leftrightarrow \forall g \in G, g \cdot x \in X_T^{s}(L_T)$$

Remark. Using this criterion one can the discuss variation of choice of linearization, and this is the start point of Igor Dolgachev and Yi Hu's papaer [2] which studies the variation of GIT (VGIT), we will see how it works and survey the further development in another article.

3. GIT Quotient and Symplectic Reduction

Definition 3.1. *For a Lie subgroup of $U(n)$, its **complexification** is the subgroup of $GL(n, \mathbb{C})$*

$$K^{\mathbb{C}} = \{exp(i\eta)u | \eta \in \mathfrak{K}, u \in K\}$$

Let $G = K^{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety X $\subset \mathbb{P}^n$ via a representation : G $\to GL_{n+1}(\mathbb{C})$, Then K also acts on $\mathbb{P}^n$ and as K is compact, we can choose coordinates on $\mathbb{P}^n$ so restricts to a unitary representation : K $\to$ U(n + 1) of K; that is, the associated Fubini-Study form ω_{FS} for this choice of coordinates is the preserved by the action of K. Therefore the action of K on both X and $\mathbb{P}^n$ is symplectic.
Since the action of U(n) on $\mathbb{CP}^n$ is Hamiltonian and have moment map, we can restrict these structures to X, in special we get moment map, noted , in this case we can consider the symplectic reduction $\mu^{-1}(0)/K$.
Here we review the basic knowledge of stability condition:

Definition 3.2. *(From Mcduff, P246)*
Let K be a compact Lie group (with Lie algebra $\mathfrak{k}$:= Lie(K)) that acts on a closed Kahler manifold (M, ω, J) by Kahler isometries, and the K-action is Hamiltonian and generated by a moment-map $\mu : M \to \mathfrak{k}$, the stability conditions can be defined as follows. An element $p \in M$ is called:
μ-unstable, if $\overline{G^c(p)} \cap \mu^{-1}(0) = \emptyset$;
μ-semistable, if $\overline{G^c(p)} \cap \mu^{-1}(0) \neq \emptyset$;
μ-polystable, if $G^c(p) \cap \mu^{-1}(0) \neq \emptyset$;
μ-stable, if $G^c(p) \cap \mu^{-1}(0) \neq \emptyset$, and $G^c_p = \{g \in G^c | gp = p\}$ is discrete.
the set of semi-stable points is denoted as M^{ss}, M^s and M^{ps} for stable points and polystable points:

On the other hand, consider action of G on X we have GIT quotient $X//G(= X^{ps}/G)$, using algebraic GIT theory in the latter subsections.
Kempf-Ness theorem tells us these two operations coincides:

Theorem 3.3. *(Kempf-Ness theorem)*
Let $G = K^{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ and suppose its maximal compact subgroup K is connected and acts symplectically on X (where the restriction of the Fubini Study form on $\mathbb{P}^n$ is used to give X its symplectic structure). Let μ : $X \to \mathfrak{K}^*$ denote the associated moment map; then:
i) $G\mu^{-1}(0) = X^{ps}$.
ii) If $x \in X$ is polystable, then its orbit $G \cdot x$ meets $\mu^{-1}(0)$ in a single K-orbit.
iii) $x \in X$ is semistable if and only if its orbit closure $\overline{G \cdot x}$ meets $\mu^{-1}(0)$.

To prove this theorem we need to consider a numberical criterion based on the topology criterion:
Remember that the moment map the action of $U(n)$ on $\mathbb{P}^n$ is given by:

$$\mu([v]) \cdot A = \frac{Tr\overline{v}^t Av}{2i|v|^2} = \frac{1}{2}\omega_{FS,[v]}(Av, v)$$

Here $||v||$ is the norm associated to the Hermitian inner product, we consider the non-negative function:

$$p_v : G \to \mathbb{R}, g \mapsto ||g \cdot v||^2$$

This function has the following properties:

Lemma 3.4. *([3] Lemma10.7)*
Let $v \in \mathbb{C}^{n+1} - \{0\}$ and $p_v : G \to \mathbb{R}$ be as above; then:
(1)p_v is connected on K.
(2)Let e denote the identity of G; then $p_v(g) = p_{g,v}(e)$ and so $d_g p_v = d_e p_{g \cdot v} : \mathfrak{g} \to \mathbb{R}$.
(3)If $A \in \mathfrak{K}$, then $d_e p_v(iA) = -4||v||^2 \mu([v]) \cdot A$ and so $d_e p_v$=0, that is, e is critical point iff $\mu[u] = 0$.
(4)Moreover, $g \in G$ is a critical point of g_v iff $\mu(g \cdot [v]) = 0$.
(5)The second derivatives of p_v are non-negative and so p_v is connex. In particular every critical point of p_v is a minimum.

Lemma 3.5. *The norm function $||-||^2 : \mathbb{C}^{n+1} \to \mathbb{R}$ is proper. Furthermore, if $G \cdot v$ is a closed orbit where $v \in \mathbb{C}^{n+1} - \{0\}$, then p_v achieves its minimum at some $g \in G$ and $\mu([g \cdot v]) = 0$.*

Lemma 3.6. *Let $v \in \mathbb{C}^{n+1} - \{0\}$, if $G \cdot v$ is not closed, then $p_v : G \to \mathbb{R}$ does not attain a minimum.*

Proof. (1)Let $x = [v]$, by the first lemma, $\mu(g \cdot x) = 0$ iff p_v is attain a minimum at g, by the second and third lemma, iff $G \cdot v$ is closed, that is, by the topological cri. 2.4, x is polystable.
(2)Firstly, if x is polystable, then by (1) $\mu(x) = 0$, and since μ is K equivariant, $\mu(K \cdot x) = 0$, it remains to show that if x,y is in a single polystable orbit, then x and y is in a sigle K orbit, we write

$$y = gx, g = kexp(iA), y = [u], x = [v]$$

then p_u and p_v both attain their minimum at e and $p_u(e) = p_v(e) = p_u(g)$ Now since the Hermitian inner product is K-invariat, we have:

$$p_u(g) = ||g \cdot u||^2 = ||exp)(iA) \cdot u||^2 = p_u(exp)(iA)$$

note that p_u is convex by the first lemma, so

$$p_u(exp(itA)) \leqslant p_u(exp(iA)) = p_u(e), t \in [0, 1]$$

but $p_u(e)$ is the unique global minimum of p_u, so $p_u(exp(itA)) = p_u(exp(iA))$

$$0 = \frac{d^2}{dt^2} p_u(exp(itA))|_{t=0} = \frac{\pi}{||u||^2} H(Au, Au)$$

There $Au = 0$ and $exp(itA)u = u + \sum \frac{(it)^n}{n!} A^n u = u$, therefore $v = gu = kexp(itA)u = ku$ and $x = ky$.
(3)By topological criterion 2.3, every semistable orbit Gx, there is an unique closed polystable orbit Gy in $\overline{Gx}$, since a polystable point meets the level set $\mu^{-1}(0)$, $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, conversely, choose $y \in \overline{Gx} \cap \mu^{-1}(0)$, then Gy

is polystable, note that $X^{ps} \subset X^{ss}$, if $x \in X^{us} = X - X^{ss}$ which is a closed set, then $Gx \in X^{us}$ and contradict to the choice of y: y is in the closure of Gx. □

Corollary 3.7. *The inclusion $\mu^{-1}(0) \subset X^{ss}$ induce a homeomorphism $\mu^{-1}(0)/K \to X//G$.*

Proposition 3.8. *The origin is a regular value of if and only if $X^{ss} = X^s$, in this case the GIT quotient is a projective variety which is an orbit which is the orbit space for the action of G on X .*

This part mainly comes from Dolgachev, Igor's book [1]

4. Examples

For an action of a group G on a topological space X, we denote X/G the topological quotient space, namely the orbit space.
Firstly we consider relationship of general X.

Example 4.1. *If X is compact or connected, then so is X/G. The orbit space can have nice geometric properties for certain types of group actions:*

Example 4.2. *If X is hausdorff, X/G need not be hausdorff, for example, consider C on $S^2 = CP^1 : z([a, b]) := [za, b]$*

Example 4.3. *If X is hausdorff, G is a Lie group, and the action is proper, then X/G is hausdorff.*

Example 4.4. *If M is a smooth manifold, G is a Lie group and the action is smooth, free, and proper, then M/G is a smooth manifold and the natural map $\varphi : M \to M/G$ is a smooth submersion.*

Example 4.5. *Soppose M is a symplectic manifold , G is a Lie group and the action is Hamiltonian and admits G-eqivariant moment map . Suppose the action of G on $\mu^{-1}(0)$ is free and proper, then:*
0 is the regular value of μ and $M//G := \mu^{-1}0)/G$ is an symplectic manifold of dimension $dimM - 2dimG$, which is called the Marsden-Weinstein quotient.

Then we consider some spesific cases:

Example 4.6. *(Hoskin, P55)*

Example 4.7. *(Hoskin, P59)*
S^1 acts on $\mathbb{C}^n$.

Example 4.8. *(Mcduff P246)S^1 acts on $\mathbb{P}^1$ by rotation about the origin: $\varphi(t)[u, v] = [e^{2\varphi it}u, v]$*
In this case, the moment map is the height function, the action of complexification $\mathbb{C}^*$ is in the same way: $z[u, v] = [zu, v]$, now we consider the respective points:

If $\mu^{-1}(0)$ is a circle, then $M^s = M^{ss} = M^{ps} = \mathbb{C} \setminus \{0\}$, $M^{us} = \{0, \infty\}$;
If $\mu(0) = 0$, then $M^s = \emptyset, M^{ps} = \{0\} \subset M^{ss} = \mathbb{C}, M^{us} = \{\infty\}$;
If $\mu(\infty) = 0$, then $M^s = \emptyset, M^{ps} = \{\infty\} \subset M^{ss} = S^2 - \{0\}, M^{us} = \{0\}$.

Example 4.9. *(Hoskin P62)S^1 acts on $\mathbb{C}P^n$ by $t[z_0, z_1, \cdots, z_n] = [tz_0, t^{-1}z_1, \cdots, t^{-1}z_n]$. This action maps through $U(n+1)$ by*

$$\varphi(t)([z]) = [\begin{pmatrix} t^{-1} & & & \\ & t & & \\ & & t & \\ & & & t \end{pmatrix} z]$$

so we may use the moment map of $U(n+1)$ on $\mathbb{C}P^n$:

$$< \mu_0[z], \zeta >:= \frac{iz^*\zeta z}{2|z|^2}$$

we see $u(1) = 2\varphi i\mathbb{R}$, then the corresponding value of $2\varphi i$ in $u(n)$ is

$$\zeta = \begin{pmatrix} -2\varphi i & & & \\ & 2\varphi i & & \\ & & 2\varphi i & \\ & & & 2\varphi i \end{pmatrix}$$

so in our case the Hamiltonian function is:

$$H_{2\varphi i}([z]) =< \mu_0[z], \zeta >:= \frac{iz^*\zeta z}{2|z|^2} = -2\varphi\frac{-|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}{2|z|^2}$$

so

$$\mu([z]) = \frac{-|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}{2i|z|^2}$$

$\mu^{-1}(0) = \{[z_0, \cdots, z_n], |z_0|^2 = |z_1|^2 + \cdots |z_n|^2\} = \{[1, z_1, \cdots, z_n], |z_1|^2 + \cdots |z_n|^2 = 1\} \cong S^{2n-1}$
then we can check that:

$$\begin{aligned} M^{ss} = M^{ps} = M^s &= \{[z_0, \cdots, z_n], z_0 \neq 0, (z_1, \cdots, z_n) \neq 0\} \\ &= \{[1, z_1, \cdots, z_n], (z_1, \cdots, z_n) \neq 0\} \cong \mathbb{C}^n - \{0\} \end{aligned}$$

And in this case the Kempt Ness theorem is given by

$$\mu^{-1}(0)/K = S^{2n-1}/S^1 \cong M//G = M^{ps}/G = \mathbb{C}^n - \{0\}/\mathbb{C}^*$$

Example 4.10. *(Mcduff P250)$PSL(2, \mathbb{C})$ is the complexification of $SO(3)$ and the complexification of the standard action of $SO(3)$ on S^2 is the action of $PSL(2, \mathbb{C})$ by Mobious trnsformation.*

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