Homework Template

1.4.1 Problems 54 through 64 illustrate the application of Torricelli's law.
A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

1.4.7 Problems 54 through 64 illustrate the application of Torricelli's law.
A cylindrical tank with length 5 ft and radius 3 ft is situated with its axis horizontal. If a circular bottom hole with a radius of 1 in. is opened and the tank is initially half full of water, how long will it take for the liquid to drain completely?

1.4.22 Problems 29 through 32 explore the connections among general and singular solutions, existence, and uniqueness.
Pollution increase The amount `A(t)` of atmospheric pollutants in a certain mountain valley grows naturally and is tripling every 7.5 years.

1. If the initial amount is 10 pu (pollutant units), write a formula for `A(t)` giving the amount (in pu) present after `t` years.
2. What will be the amount (in pu) of pollutants present in the valley atmosphere after 5 years?
3. If it will be dangerous to stay in the valley when the amount of pollutants reaches 100 pu, how long will this take?

1.4.24 Problems 29 through 32 explore the connections among general and singular solutions, existence, and uniqueness.
Growth of languages There are now about 3300 different human "language families" in the whole world. Assume that all these are derived from a single original language and that a language family develops into 1.5 language families every 6 thousand years. About how long ago was the single original human language spoken?

1.5.3

1 Solution

Division by x^2 gives the linear first-order equation

$$\frac{dy}{dx} + \frac{1}{x}\ \mathrm{y} = \frac{\sin\ x}{x^2}$$

with $P(x) = 1/x$ and $Q(x) = (\sin\ x)/x^2$. With $x_0 = 1$ the integrating factor in (12) is

$$\rho(x) = \exp\left(\int_1^x \frac{1}{t} dt\right) = \exp(\ln\ x) = x,$$

so the desired particular solution is given by
(14)

$$y(x) = \frac{1}{x}\left[y_0 + \int_1^x \frac{\sin\ t}{t}\ dt\right].$$

In accord with Theorem 1, this solution is defined on the whole positive x-axis.
Solve the initial value problem
(13) $x^2 \frac{dy}{dx} + xy = \sin\ x, \quad y(1) = y_0$

1.5.5 Problems 33 through 37 illustrate the application of linear first-order differential equations to mixture problems.
A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

1.5.7 Problems 33 through 37 illustrate the application of linear first-order differential equations to mixture problems.
Two tanks Suppose that in the cascade shown in Fig. 1.5.5, tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min.**(a)** Find the amounts x(t) and y(t) of ethanol in the two tanks at time $t \geqq 0$. **(b)** Find the maximum amount of ethanol ever in tank 2.

1.5.16 Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.
$y' = (1 - y)\ \cos\ x,\ y(\pi) = 2$

1.5.23 Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.
$xy' + (2x-3)y = 4x^4$

1.5.24 Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.
$(x^2+4)y' + 3xy = x$, $y(0) = 1$

1.5.1

2 Solution

Here we have $P(x) \equiv -1$ and $Q(x) = \frac{11}{8}e^{-x/3}$, so the integrating factor is

$$\rho(x) = e^{\int(-1)dx} = e^{-x}.$$

Multiplication of both sides of the given equation by e^{-x} yields
(7)

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{11}{8}e^{-4x/3},$$

which we recognize as

$$\frac{d}{dx}(e^{-x}y) = \frac{11}{8}e^{-4x/3}.$$

Hence integration with respect to x gives

1.5.2

$$e^{-x}y = \int \frac{11}{8}e^{-4x/3}dx = -\frac{33}{32}e^{-4x/3} + C,$$

and multiplication by e^x gives the general solution
(8)

$$y(x) = Ce^x - \frac{33}{32}e^{-x/3}.$$

Substitution of $x = 0$ and $y = -1$ now gives $C = \frac{1}{32}$, so the desired particular solution is

$$y(x) = \frac{1}{32}e^x - \frac{33}{32}e^{-x/3} = \frac{1}{32}(e^x - 33e^{-x/3}).$$

Solve the initial value problem
$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1.$

3 Solution

After division of both sides of the equation by x^2+1, we recognize the result

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

as a first-order linear equation with $P(x) = 3x/(x^2+1)$ and $Q(x) = 6x/(x^2+1)$. Multiplication by

$$\rho(x) = \exp\left(\int \frac{3x}{x2+1}dx\right) = \exp\left(\frac{3}{2}\ln(x^2+1)\right) = (x^2+1)^{3/2}$$

yields

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{1/2}y = 6x(x^2+1)^{1/2},$$

and thus

$$D_x\left[(x^2+1)^{3/2}y\right] = 6x(x^2+1)^{1/2}.$$

Integration then yields

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}dx = 2(x^2+1)^{3/2} + C.$$