Chapter 6 - Two-level (2^k) Factorial Designs

6.1. The 2^k Factorial Design

- Two-level factorial: special case of general factorial
  - We have k factors. All factors at two levels
  - "-" and "+"denote the low and high levels of a factor, respectively
  - 2^k factorial designs provide smallest number of runs with which the joint effects of k factors can be studied
  - Total number of observations in complete experiment

$$2 \times 2 \times \cdots \times 2 \times n = n2^k$$

- Very useful as factor "screening" experiments
  - Screening is done early stages of experimental work when many factors investigated but only some are likely important

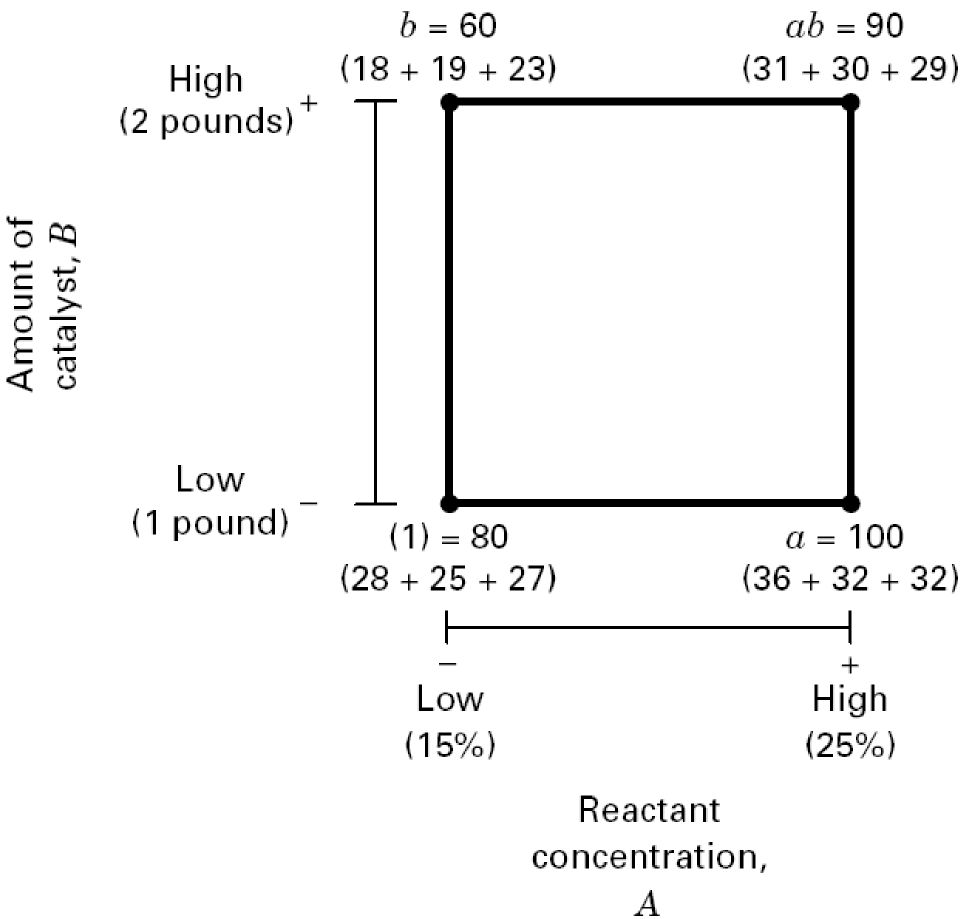
6.1. The 2^k Factorial Design - Assumptions

- Response approximately linear
  - each factor tested at two levels: only linear effects are estimable
  - In screening experiments when we are just starting to study the process this is a reasonable assumption
  - Section 6.8. for testing significance of curvature
- Runs are completely randomized
  - Chapter 7: Blocking in 2^k design
- Normal and constant variance errors
  - Section 6.6. for variance stabilizing transformations

Background: Binary Orthogonal Vectors

- Used in all your cell phones (from the towers to the cell phones)
- All digits are either -1 or +1
- Orthogonal if "dot-product" = 0
- Examples
  - -1, +1, -1, +1
  - +1, +1, -1, -1
  - -1, +1, +1, -1
  - +1, +1, +1, +1
  - -1, -1, -1, -1
  - -1, -1, -1, +1
  - +1, +1, -1, +1

6.2. The 2^2 design



- Four combinations: corners of a square
- Geometrical view is helpful to determine expressions for the effect estimates
  - change in response produced by a change in the level of factor (average over levels of all other factors)
- Treatment combinations represented as: *(1), a, b, ab*
  - Also to represent the totals of the response observations taken at the treatment combinations: *(1), a, b, ab*

Estimation of Factor Effects from Contrasts

- For two-level designs the effect estimates are determined from contrasts

$$\boxed{Effect = \frac{Contrast}{n2^{k-1}}}$$

- Contrast: a linear function of the treatment combinations in which coefficients sum to zero

$$\text{Contrast}_A = ab + a - b - (1)$$
$$\text{Contrast}_B = ab + b - a - (1)$$
$$\text{Contrast}_{AB} = ab + (1) - a - b$$

- For $k=2$, and for Factor A effect

$$A = \frac{Contrast_A}{2n} = \frac{ab + a - b - (1)}{2n}$$

Contrasts to obtain the SS and ANOVA

- For two-level designs contrasts can be used to determine sum of squares

$$\boxed{SS_{Effect} = \frac{(contrast)^2}{n2^k}}$$

- For $k=2$ and factor A

$$SS_A = \frac{\left(ab + a - b - (1)\right)^2}{4n}$$

Finding the contrast coefficients of 2^k designs

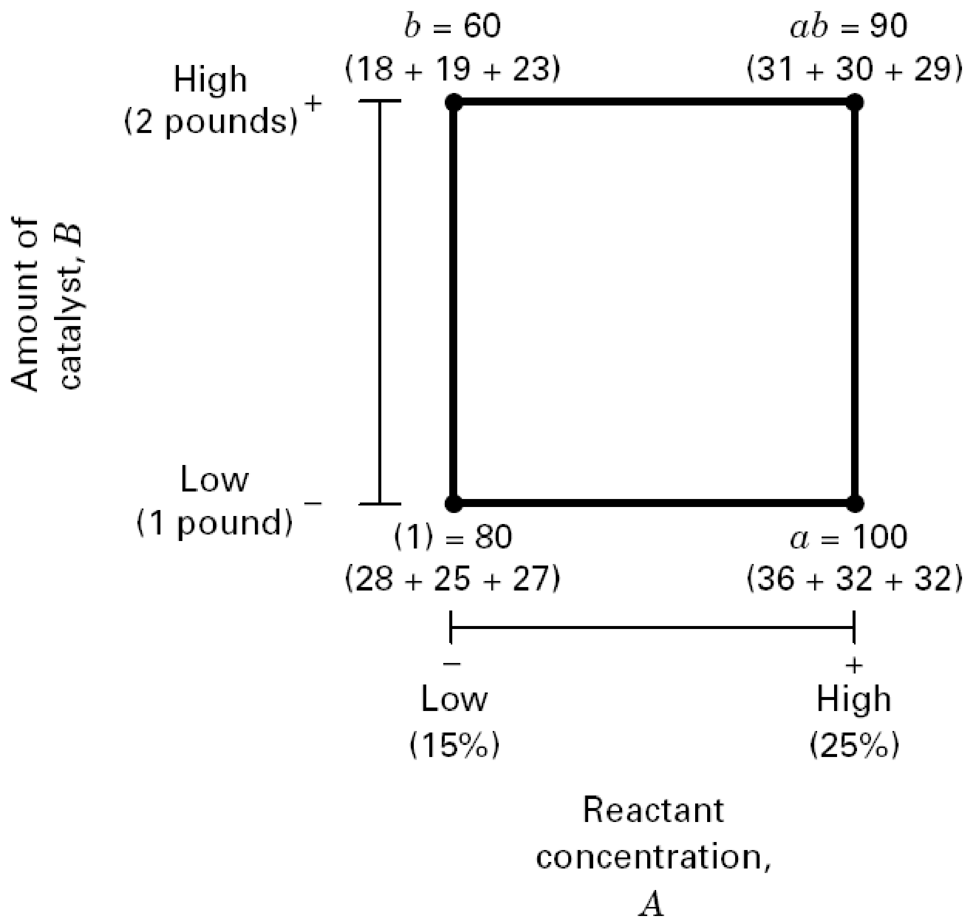
- Get contrast coefficients from table of +/- signs
  - Treatment combinations in standard order

$$(1), a, b, ab$$

Treatment	A	B	AB
(1)	-	-	+
a	+	-	-
b	-	+	-
ab	+	+	+

$$Contrast_A = ab - b + a - (1)$$
$$Contrast_B = ab + b - a - (1)$$
$$Contrast_{AB} = ab - b - a + (1)$$

Estimation of Factor Effects (ex. 1)



- Effect of A at high level of B ($A\ at\ B^+$)

$$\frac{ab-b}{n}$$

- Effect of A at low level of B ($A\ at\ B^-$)

$$\frac{a-(1)}{n}$$

- Main effect of A

$$A=\frac{A\ at\ B^+ + A\ at\ B^-}{2}$$

$$A=\frac{1}{2n}[ab-b+a-(1)]$$

- Interaction effect AB:

$$AB=\frac{A\ at\ B^+ - A\ at\ B^-}{2}$$

$$AB=\frac{1}{2n}[ab-b-a+(1)]$$

Contrasts to obtain the SS and ANOVA

- Total sum of squares (SS_T) has *d.f. = 4n-1*

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}^2 - y_{...}^2/N$$

- Each of SS_A, SS_B and SS_{AB} has *d.f. =1*

- Error sum of squares (SSE) has *d.f. = 4(n-1)*

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$

Effects of chemical process example

- Effect of A

$$A = [ab + a - b - (1)]/2n$$
$$= [90+100-60-80]/6 = 50/6 = 8.33$$

  - Increasing reactant concentration from 15% to 25% increases yield by 8.33%

- Effect of B

$$B = [ab - a + b - (1)]/2n$$
$$= [90-100+60-80]/6 = -30/6 = -5.00$$

- Effect of AB

$$AB = [ab - a - b + (1)]/2n$$
$$= [90-100-60+80]/6 = 10/6 = 1.67$$

ANOVA for chemical process example

$$SS_A = \frac{\left(ab+a-b-(1)\right)^2}{4n} = (50)^2/(4(3)) = 208.33$$

$$SS_B = (-30)^2/(4(3)) = 75.00$$

$$SS_{AB} = (10)^2/(4(3)) = 8.33$$

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\dots}^2}{N} = 323.00$$

$$SS_E = 323.00 - 208.33 - 75 - 8.33 = 31.34$$

ANOVA for chemical process example

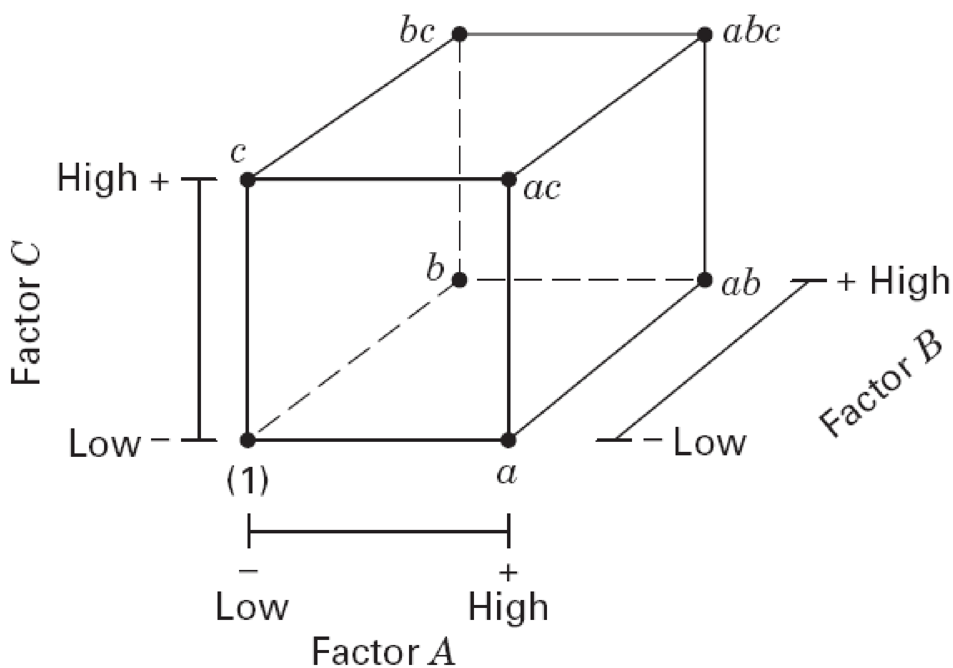
- To test all effects, use $F_{0.05,1,8} = 5.32$
- A and B are significant. AB is not significant

■ **TABLE 6.1**

Analysis of Variance for the Experiment in Figure 6.1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_q	P-Value
A	208.33	1	208.33	53.15	0.0001
B	75.00	1	75.00	19.13	0.0024
AB	8.33	1	8.33	2.13	0.1826
Error	31.34	8	3.92		
Total	323.00	11			

6.3. The 2^3 Factorial Design



(a) Geometric view

Standard order

Run	Factor A	Factor B	Factor C
1	–	–	–
2	+	–	–
3	–	+	–
4	+	+	–
5	–	–	+
6	+	–	+
7	–	+	+
8	+	+	+

(b) Design matrix

Three factors, A, B and C. All tested at 2 levels.
2^3 design: eight treatment combinations can be displayed geometrically as a cube (a) or using the + and – notation as in a table (b)

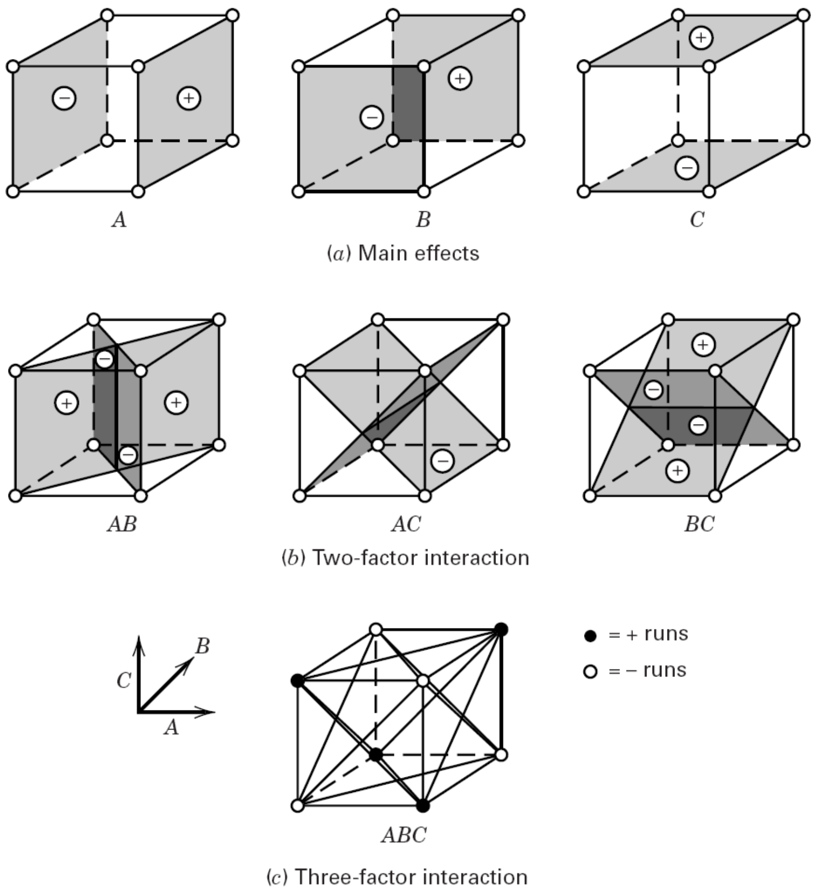
6.3. The 2^3 Factorial Design

- Write design matrix using the treatment combinations
  - Similar to the 2^2 design: presence of a (lower case) letter indicates factor level is high, absence indicates factor level is low

A	B	C	Treatment
-	-	-	(1)
+	-	-	a
-	+	-	b
+	+	-	ab
-	-	+	c
+	-	+	ac
-	+	+	bc
+	+	+	abc

Effects in The 2^3 Factorial Design

■ FIGURE 6.5 Geometric presentation of contrasts corresponding to the main effects and interactions in the 2^3 design



$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$

$$C = \bar{y}_{C^+} - \bar{y}_{C^-}$$

etc, etc, ...

Effect of any factor: change in *y* produced by a change in the factor averaged over levels of all other factors

Effects in The 2^3 Factorial Design

Main effect of A: Average of four runs at high level minus average of four runs at low level

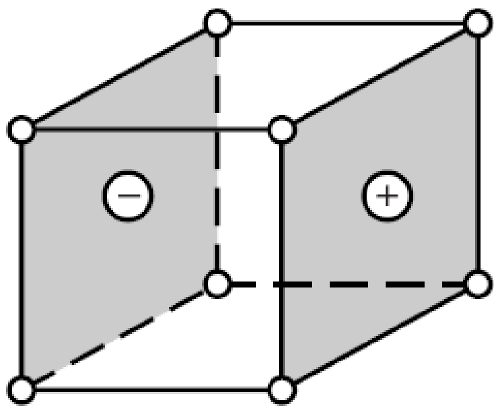
$$A = \bar{y}_{A^+} - \bar{y}_{A^+}$$

$$A = \frac{1}{4n}[a + ab + ac + abc - ((1) + b + c + bc)]$$

Alternatively: Average of A effect at 4 different B and C values

$$A = \frac{A\ at\ B^-C^- + A\ at\ B^+C^- + A\ at\ B^-C^+ + A\ at\ B^+C^+}{4}$$

$$A = \frac{1}{4n}[a - (1) + ab - b + ac - c + abc - bc)]$$



A

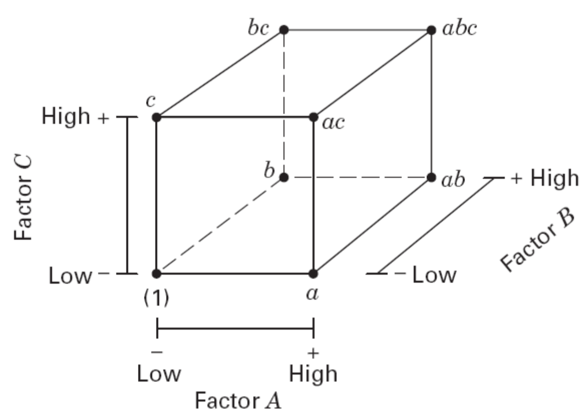


Table of – and + signs to find contrast coefficients in the 2^3 Factorial Design (pg. 244)

■ **TABLE 6.3**

Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

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Treatment Combination	Factorial Effect							
	I	*A*	*B*	*AB*	*C*	*AC*	*BC*	*ABC*
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

$$A = \frac{Contrast_A}{4n}$$
$$= \frac{[-(1) + a - b + ab - c + ac - bc + abc]}{4n}$$

Table of – and + signs to find contrast coefficients in the 2^3 Factorial Design (pg. 244)

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Treatment Combination	Factorial Effect							
	I	*A*	*B*	*AB*	*C*	*AC*	*BC*	*ABC*
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

$$A = \frac{Contrast_A}{4n}$$

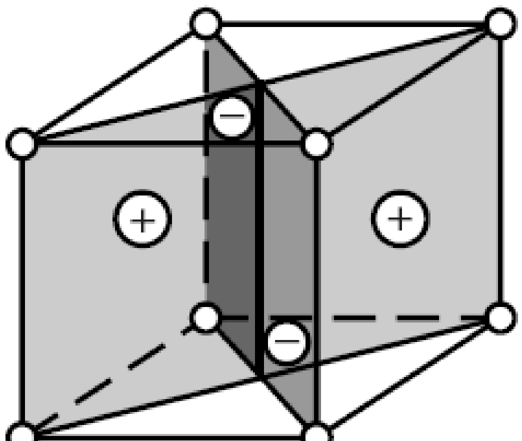
$$= \frac{[-(1) + a - b + ab - c + ac - bc + abc]}{4n}$$

Divide by 4n: because we average over 4 combinations of B and C

Effects in The 2^3 Factorial Design

AB interaction: difference between the average responses on two diagonal planes of the cube

$$AB = \frac{1}{4n}\left[abc + ab + c + (1) - (bc + b + ac + a)\right]$$



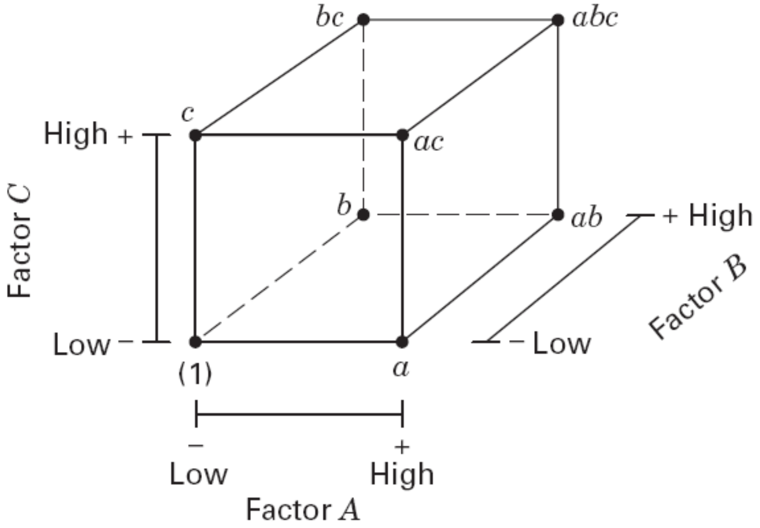


Table of – and + signs for calculating effects in the 2^3 Factorial Design (pg. 244)

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Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

$$AB = \frac{Contrast_{AB}}{4n}$$

$$= \frac{[(1) - a - b + ab + c - ac - bc + abc]}{4n}$$

General formula for 2^k design

- Estimate of an effect

$$AB \ldots K = \frac{contrast_{AB \ldots K}}{n2^{k-1}}$$

- Sum of squares of an effect

$$SS_{AB \ldots K} = \frac{(contrast_{AB \ldots K})^2}{n2^k}$$

- Note: Find the contrast: using table of – and + signs may be impractical for large k
  - Use Eq. 6.21 (p. 255)

6.4. The General 2^k Factorial Design

- There will be k main effects, and

$$\binom{k}{2} \text{ two-factor interactions}$$

$$\binom{k}{3} \text{ three-factor interactions}$$

$$\vdots$$

$$1 \; k-\text{factor interaction}$$

- Complete model has $2^k - 1$ effects

ANOVA table for a 2^k design with n replicates

TABLE 6.9
Analysis of Variance for a 2^k Design

Source of Variation	Sum of Squares	Degrees of Freedom
k main effects		
A	SS_A	1
B	SS_B	1
$\vdots$	$\vdots$	$\vdots$
K	SS_K	1
$\binom{k}{2}$ two-factor interactions		
AB	SS_{AB}	1
AC	SS_{AC}	1
$\vdots$	$\vdots$	$\vdots$
JK	SS_{JK}	1
$\binom{k}{3}$ three-factor interactions		
ABC	SS_{ABC}	1
ABD	SS_{ABD}	1
$\vdots$	$\vdots$	$\vdots$
IJK	SS_{IJK}	1
$\vdots$	$\vdots$	$\vdots$
$\binom{k}{k}$ k-factor interaction		
$ABC \cdots K$	$SS_{ABC \cdots K}$	1
Error	SS_E	$2^k(n-1)$
Total	SS_T	$n2^k - 1$

- If no replications (n=0) then the full model will have error d.f. = 0
  - We cannot estimate error variance or test for significance of the treatment effects
    - MSE=SSE/dfE
    - F_0 = SSA/MSE

Recall: General formula for 2^k design

- Estimate of an effect

$$AB \dots K = \frac{contrast_{AB \dots K}}{n2^{k-1}}$$

- Sum of squares of an effect

$$\boxed{SS_{AB \dots K} = \frac{(contrast_{AB \dots K})^2}{n2^k}}$$

- Note: Find the contrast: using table of – and + signs may be impractical for large k
  - Use Eq. 6.21

6.5 Unreplicated 2^k Factorial Designs

- These are 2^k factorial designs with one observation at each corner of the "cube" ($n=1$)
- Lack of replication causes potential problems in statistical testing
  - Replication admits an estimate of "pure error" (internal estimate of error)
  - With no replication, fitting full model results in zero degrees of freedom for error
  - With only one observation at each corner, if response is highly variable then misleading conclusions may result from experiment (fitting the model to noise)

What about replicants

- 2^k is already a really big number…
- You could:
  - ❑Ignore higher order effects
  - ❑Do a normplot to find out which things you care about, remove the others and call them noise…
  - ❑Increase the distance between factor levels
  - ❑Increase Type-1 errors

Estimate error variance in unreplicated 2^k

- An approach to analyze unreplicated design
  - Sparsity of effects principle: systems are typically dominated by main effects or low order interactions
  - Assume certain high-order interactions negligible and pool their mean squares to form the error mean square
- How to determine which high order interactions are negligible?

Normal probability plot of estimates of effects

- The effects that are negligible are normally distributed with mean 0 and variance σ^2
  - if normal distribution is adequate then the plotted data will fall on a straight line
  - Negligible effects will fall on a straight line, significant effects (with nonzero means) will fall far from a straight line
- Form a preliminary model based on NPP by including the nonzero effects. Combine negligible effects to estimate error variance.

Example 6.2. Unreplicated 2^4 Design

- A chemical product is produced in a pressure vessel. A factorial experiment is carried out in the pilot plant to study the factors thought to influence the filtration rate of this product

- The factors are A = temperature, B = pressure, C = formaldehyde concentration, D= stirring rate. Response y= filtration rate
- 2^4 factorial. Single replicate: 16 runs.
- Goal: obtain increased filtration rate. Current rate = 75 gal/h

The Resin Plant Experiment

■ **TABLE 6.10**
Pilot Plant Filtration Rate Experiment

Run Number	Factor *A*	*B*	*C*	*D*	Run Label	Filtration Rate (gal/h)
1	−	−	−	−	(1)	45
2	+	−	−	−	*a*	71
3	−	+	−	−	*b*	48
4	+	+	−	−	*ab*	65
5	−	−	+	−	*c*	68
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	−	*abc*	65
9	−	−	−	+	*d*	43
10	+	−	−	+	*ad*	100
11	−	+	−	+	*bd*	45
12	+	+	−	+	*abd*	104
13	−	−	+	+	*cd*	75
14	+	−	+	+	*acd*	86
15	−	+	+	+	*bcd*	70
16	+	+	+	+	*abcd*	96

Design matrix shown in standard order. Experiments are made in random order (CRD)

Table of +/- signs

■ **TABLE 6.11**

Contrast Constants for the 2^4 Design

	A	*B*	*AB*	*C*	*AC*	*BC*	*ABC*	*D*	*AD*	*BD*	*ABD*	*CD*	*ACD*	*BCD*	*ABCD*
(1)	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
a	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
b	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
ab	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
c	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
ac	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
bc	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
abc	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
d	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
ad	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
bd	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
abd	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
cd	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
acd	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
bcd	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Estimates of effects, sum of squares of effects

- Use the table of +/- signs to find the contrasts and effect estimates
  - estimate 15 effects (m.e., 2 f.i., 3f.i., 4 f.i) and their SS.

$$Effect = \frac{Contrast}{2^{k-1}}, SS = \frac{Contrast^2}{2^k}$$

- Example

$$D \;=\; contrast_D/2^3$$

$$=\; (-y1 - \cdots - y8 + y9 + \cdots + y16)/8$$
$$= 117/8 \;=\; 14.625$$
$$SSD \;=\; contrast_D^2\;/2^4$$

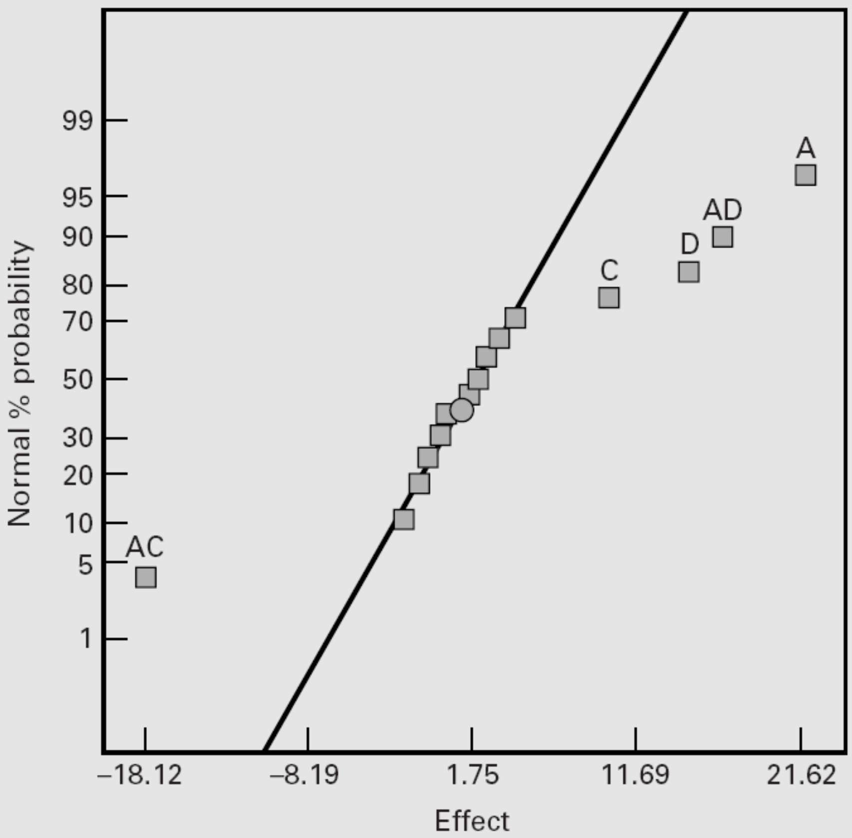
$$=\; 1172\;/16 \;=\; 855.563$$

- Repeat for A,B,C,...etc.,

Estimates of effects, sum of squares of effects and NPP of effects

■ TABLE 6.12
Factor Effect Estimates and Sums of Squares for the 2^4 Factorial in Example 6.2

Model Term	Effect Estimate	Sum of Squares	Percent Contribution
A	21.625	1870.56	32.6397
B	3.125	39.0625	0.681608
C	9.875	390.062	6.80626
D	14.625	855.563	14.9288
AB	0.125	0.0625	0.00109057
AC	−18.125	1314.06	22.9293
AD	16.625	1105.56	19.2911
BC	2.375	22.5625	0.393696
BD	−0.375	0.5625	0.00981515
CD	−1.125	5.0625	0.0883363
ABC	1.875	14.0625	0.245379
ABD	4.125	68.0625	1.18763
ACD	−1.625	10.5625	0.184307
BCD	−2.625	27.5625	0.480942
ABCD	1.375	7.5625	0.131959

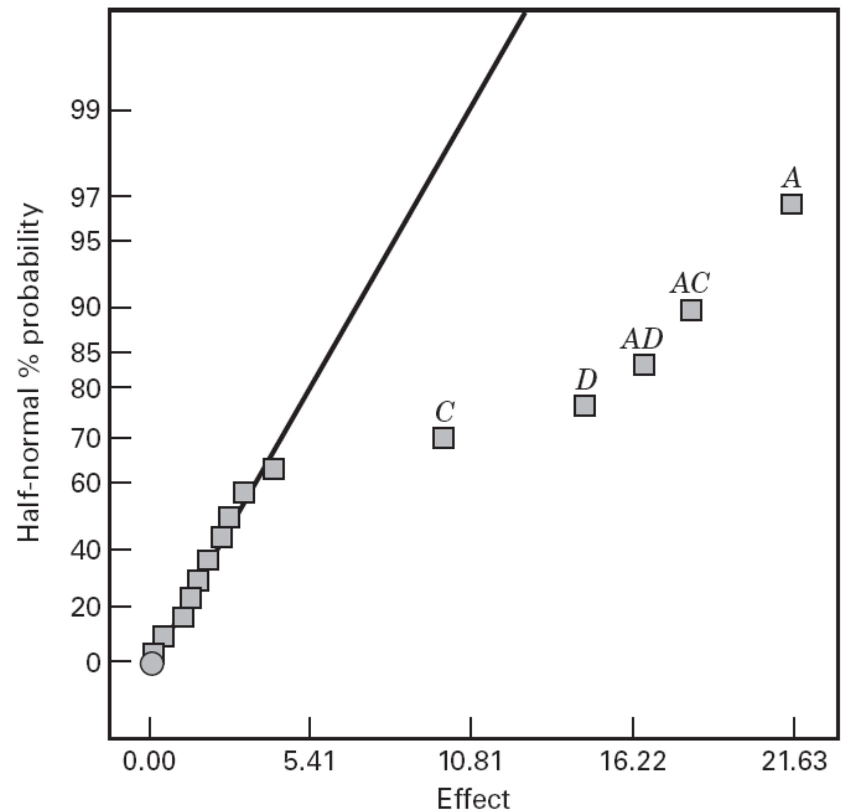


■ FIGURE 6.11 **Normal probability plot of the effects for the 2^4 factorial in Example 6.2**

Effects that lie along line negligible: labels not shown
Effects that lie far from line are large: A,C,D, AC, AD

Half-Normal Probability Plot of Effects

- An alternative to the NPP of factor effects is the half-normal probability plot
  - Absolute value of effect estimates against their cumulative normal probabilities
  - Straight line always passes from the origin and should pass close to the 50th percentile of data
  - Makes it easier than NPP to rank the effects since only the magnitudes of effects are considered



- From the NPP of effect estimates following effects appear to be non-negligible

A, C, D, AC, AD

- Fit a model that contains only these terms. Drop the insiginificant terms, which will be used to form the error SS

B, AB, BC, BD, CD, ABC, ABD, ACD, BCD, ABCD

Work for today

- Develop our own +/- chart
- Analyze dof for 2^3 w/out replicates or blocking
- Analyze dof for 2^3 w/ blocking and replicates
- Analyze dof for 2^3 w/out replicates and w/ blocking (*confounding*)
  - Random blocking design
  - Blocking aligned with a effect
- Analyze dof for 2^3 w/replicates and w/ blocking (smaller than 2^3)
  - Designing blocking to enable testing of all effects

Fractional Factorial Designs

- As the number of factors becomes large enough to be interesting, size of designs grows very quickly
  - information on main effects and low order interactions can be obtained by running a fraction of the complete factorial (assuming certain high-order interactions are negligible)

Fractional Factorial Designs

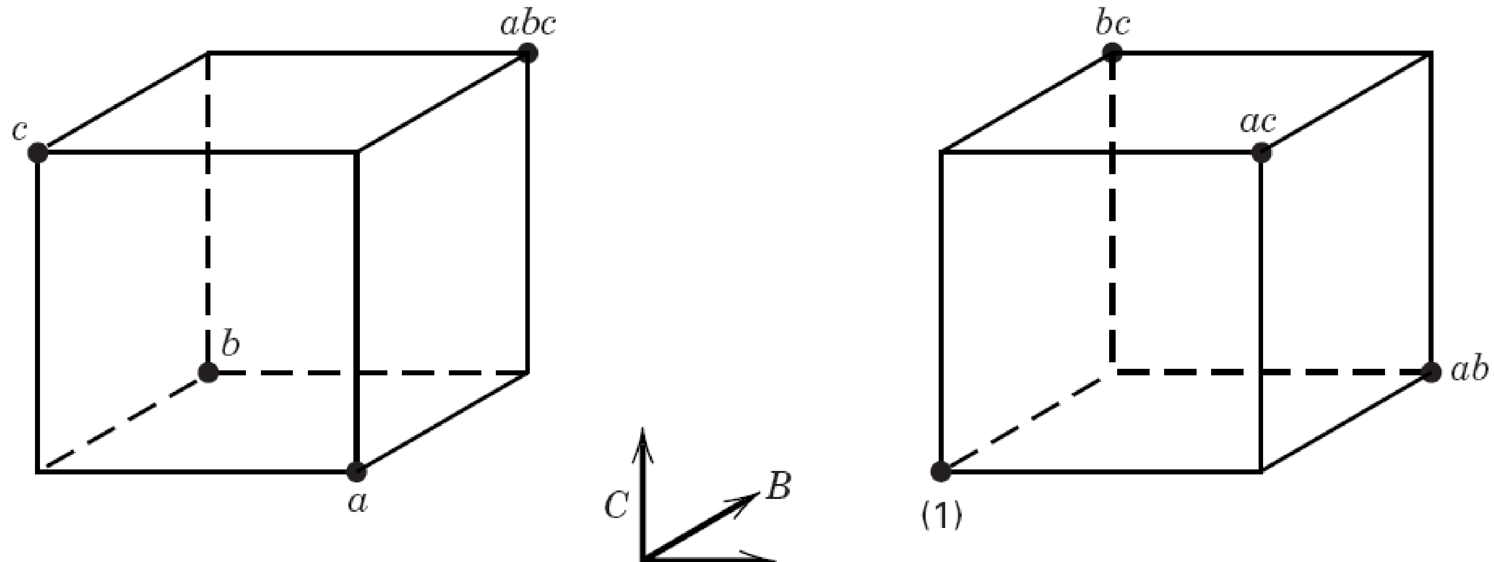
- As the number of factors becomes large enough to be interesting, size of designs grows very quickly
  - information on main effects and low order interactions can be obtained by running a fraction of the complete factorial (assuming certain high-order interactions are negligible)
- Emphasis is on factor screening; efficiently identify the factors with large effects
  - There may be many variables (often because we don’t know much about the system)
  - Factors identified as important are to be investigated more thoroughly in follow up experiments
- Almost always run as unreplicated, but often with center points

Why do Fractional Factorial Designs Work?

- The sparsity of effects principle
  - There may be lots of factors, but few are important
  - System is dominated by main effects, low-order interactions
- The projection property
  - Every fractional factorial contains full factorials in fewer factors
- Sequential experimentation
  - Can add runs to a fractional factorial to resolve difficulties (or ambiguities) in interpretation

8.2. The One-Half Fraction of the 2^k

- One half of 2^k. Referred to as a 2^{k-1}
- Example: the $2^{3\text{-}1}$



(*a*) The principal fraction, $I = +ABC$

(*b*) The alternate fraction, $I = -ABC$

■ **FIGURE 8.1** The two one-half fractions of the 2^3 design

Sufficient to conduct either one of these two fractional experiments

One-Half Fraction 2^{3-1}

- The fraction is formed by selecting combinations with "+" or "-" in *ABC* column
  - *ABC* is the *Generator* of this particular fraction
- If we choose *a, b, c* and *abc* as the combinations to run
  - *I = ABC* is the *Defining Relation* of the design (Principal Fraction)
  - Defining relation: set of all columns equal to identity column *I*.

■ TABLE 8.1
Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

One-Half Fraction 2^{3-1}

- The fraction is formed by selecting combinations with "+" or "-" in *ABC* column
  - *ABC* is the <u>*Generator*</u> of this particular fraction
- If we choose *a, b, c* and *abc* as the combinations to run
  - *I = ABC* is the <u>*Defining Relation*</u> of the design (Principal Fraction)
  - Defining relation: set of all columns equal to identity column *I*.

TABLE 8.1
Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect I	A	B	C	AB	AC	BC	ABC
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

Principal fraction: $I = ABC$

Construction of a One-half Fraction

- Example: one-half fraction of a 2^3, using *I=ABC* as the defining relation (principal fraction)
  - ❑Write down *basic design* on *A* and *B* (For 2^k, basic design is full factorial on *k-1* factors)
  - ❑Find levels of *C* from defining relation *I=ABC or C=AB*

TABLE 8.2
The Two One-Half Fractions of the 2^3 Design

	Full 2^2 Factorial (Basic Design)		2_{III}^{3-1}, $I = ABC$			2_{III}^{3-1}, $I = -ABC$		
Run	A	B	A	B	$C = AB$	A	B	$C = -AB$
1	−	−	−	−	+	−	−	−
2	+	−	+	−	−	+	−	+
3	−	+	−	+	−	−	+	+
4	+	+	+	+	+	+	+	−

One-Half Fraction 2^{3-1}

- Two possible fractions and their defining relations:
  - $I = ABC$ principle fraction
  - $I = -ABC$ alternate fraction
- If no restrictions on treatment combinations specified, by default only the ***principle fraction*** is conducted

■ **TABLE 8.1**

Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	A	B	C	AB	AC	BC	ABC
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

Principal fraction: $I = ABC$

Principal one-half fraction 2^{3-1}

- Estimates of main effects A, B, and C from the principal fraction 2^{3-1}

$$[A] = \frac{1}{2}(a - b - c + abc)$$

$$[B] = \frac{1}{2}(-a + b - c + abc)$$

$$[C] = \frac{1}{2}(-a - b + c + abc)$$

Principal one-half fraction 2^{3-1}

- Estimates of main effects A, B, and C from the principal fraction 2^{3-1}

$$[A] = \frac{1}{2}(a - b - c + abc)$$
$$[B] = \frac{1}{2}(-a + b - c + abc)$$
$$[C] = \frac{1}{2}(-a - b + c + abc)$$

- Estimates of interaction effects AB, AC, and BC from the principal fraction 2^{3-1}

$$[AB] = \frac{1}{2}(-a - b + c + abc)$$
$$[AC] = \frac{1}{2}(-a + b - c + abc)$$
$$[BC] = \frac{1}{2}(a - b - c + abc)$$

- Notation $[.]$ is used to indicate the "linear combinations associated with effect estimates"

Principal one-half fraction 2^{3-1}

- Linear combination for estimating main effect A is same as that used for estimating the interaction effect BC.
  - Thus, [A]=[BC], or A and BC are <u>*aliases*</u>
  - impossible to differentiate between effect of A and effect of BC
- We indicate this by the notation

$$[A] \rightarrow A+BC$$

  - When we estimate A we are really estimating A+BC
- This phenomena is called <u>*aliasing*</u> and it occurs in all fractional designs

Principal one-half fraction 2^{3-1}

- Aliases of 2^{3-1} with defining relation $I = ABC$

[*A*] = 1/2 (*a* – *b* – *c* + *abc*) and [*BC*] = 1/2 (*a* – *b* – *c* + *abc*) *A* aliased with *BC*

[*B*] = 1/2 (-*a* + *b* – *c* + *abc*) and [*AC*] = 1/2 (-*a* + *b* – *c* + *abc*) *B* aliased with *AC*

[*C*] = 1/2 (-*a* – *b* + *c* + *abc*) and [*AB*] = 1/2 (-*a* – *b* + *c* + *abc*) *C* aliased with *AB*

- Aliases are found from columns of the Table of +/ - signs

■ **TABLE 8.1**
Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	–	–	–	–	+	+
b	+	–	+	–	–	+	–	+
c	+	–	–	+	+	–	–	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	–	+	–	–	–
ac	+	+	–	+	–	+	–	–
bc	+	–	+	+	–	–	+	–
(1)	+	–	–	–	+	+	+	–

[B]=[AC], thus, B and AC are <u>*aliases*</u>

Principal one-half fraction 2^{3-1}

- Complete alias structure of the fraction is

$$[A] \rightarrow A+BC$$

$$[B] \rightarrow B+AC$$

$$[C] \rightarrow C+AB$$

  - Main effects aliased with 2 factor interactions (*m.e. = 2 f.i.*)
  - A Resolution III design (more on this later)

- Complete alias structure is found from *I = ABC* (defining relation)

$$A(I = ABC) \text{ or } A = A^2BC \text{ or } [A] = [BC]$$

$$B(I = ABC) \text{ or } [B] = [AC]$$

$$C(I = ABC) \text{ or } [C] = [AB]$$

- Complete alias structure of a fractional consists of $2^{k-p} - 1$ "alias relations" or "alias chains"
  - Ex: For $k = 3$ and $p = 1$, there are $2^{3-1} - 1 = 3$ alias chains

Alternate one-half fraction 2^{3-1}

$[A]' = 1/2\ (ab + ac - bc - (1))$ — *A* aliased with -*BC*

$[B]' = 1/2\ (ab - ac + bc - (1))$ — *B* aliased with -*AC*

$[C]' = 1/2\ (-ab + ac + bc - (1))$ — *C* aliased with -*AB*

Alternative fraction:

■ **TABLE 8.1**
Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

Alternate one-half fraction 2^{3-1}

- *I = -ABC* is the defining relation
- Implies slightly different alias structure:

$$[A]' = -[BC]'$$
$$[B]' = -[AC]'$$
$$[C]' = -[AB]'$$

- or

$$[A]' \rightarrow A - BC$$
$$[B]' \rightarrow B - AC$$
$$[C]' \rightarrow C - AB$$

Backup slides (Chapter 7)

Blocking & Confounding in the 2^k Design

- Blocking is a technique for dealing with controllable nuisance variables
  - Useful when it is not possible to conduct the entire set of experimental combinations under homogeneous conditions
  - Examples: time periods, batches of raw material, operators, etc. – homogeneous experimental units
  - Form set of homogeneous units as "Blocks" and perform the statistical analyses with the knowledge of which runs belongs to which block
- We have seen blocking in general factorial designs
  - Chapter 4 (single factor) and Chapter 5.6 (multi-factors)
- Chapter 7: blocking in 2^k factorial designs
  - Replicated designs (complete block)
  - Unreplicated designs (incomplete block)

7.2 Blocking a replicated 2^k factorial

- n replicates of 2^k design; each replicate is a block
  - same scenario of Chapters 4 and 5.6
  - Complete block design: each block contains all treatment combinations
- Each block is not large enough to run all $n2^k$ combinations but large enough to run 2^k combinations
  - All 2^k combinations are run in one of the n blocks
  - Runs within the block are randomized

7.2 Blocking a replicated 2^k factorial

- 2^k replicated n times. If each replication is a block → $b = n$
- Effect and SS of factors are found as if there is no blocking: Chapter 6 formulas for 2^k designs

$$\boxed{Effect = \frac{contrast}{n2^{k-1}}, \qquad SS_{Effect} = \frac{(contrast)^2}{n2^k}}$$

- Ex: For $k = 2$

$$A = \frac{1}{2n}[ab + a - b - (1)]$$

$$B = \frac{1}{2n}[ab + b - a - (1)]$$

$$AB = \frac{1}{2n}[ab + (1) - a - b]$$

NOTE: These formulas work for both replicated *(n>1)* and unreplicated *(n=1)* experiments

- Find SS_A, SS_B, SS_{AB}

7.3. Confounding 2^k factorial designs

- Unless k is small it is usually infeasible to accommodate all 2^k treatments in the same block
  - Assign a fraction of the 2^k treatment combinations to each block
  - each block does not contain all treatment combinations: incomplete block design
  - a complete factorial is arranged in blocks such that $N_b < 2^k$ (i.e. block size smaller than # treatment combinations)

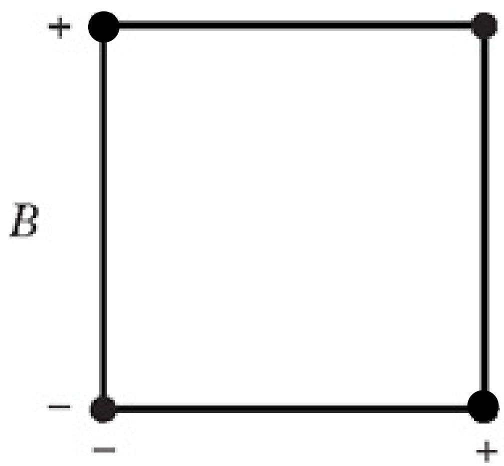
- Confounding: Information about certain treatment effects (usually high order interactions) will be indistinguishable from or confounded with blocks

7.3. Confounding 2^k factorial designs

- The special structure of 2^k factorial designs allows one to choose which effects are confounded with incomplete blocks
- Construct and analyze a 2^k factorial in 2^p incomplete blocks, where $p < k$
  - For $p = 1,2,3$, the design is divided into $2, 4, 8$ blocks, etc.
  - Each block contains $2^k/2^p$ or 2^{k-p} treatment combinations
- Blocking arrangement
  - Number of observations in each block $N_b = 2^{k-p}$
  - Number of blocks 2^p
  - Total number of observations $N = 2^k$

7.4. Confounding 2^k factorial in Two Blocks

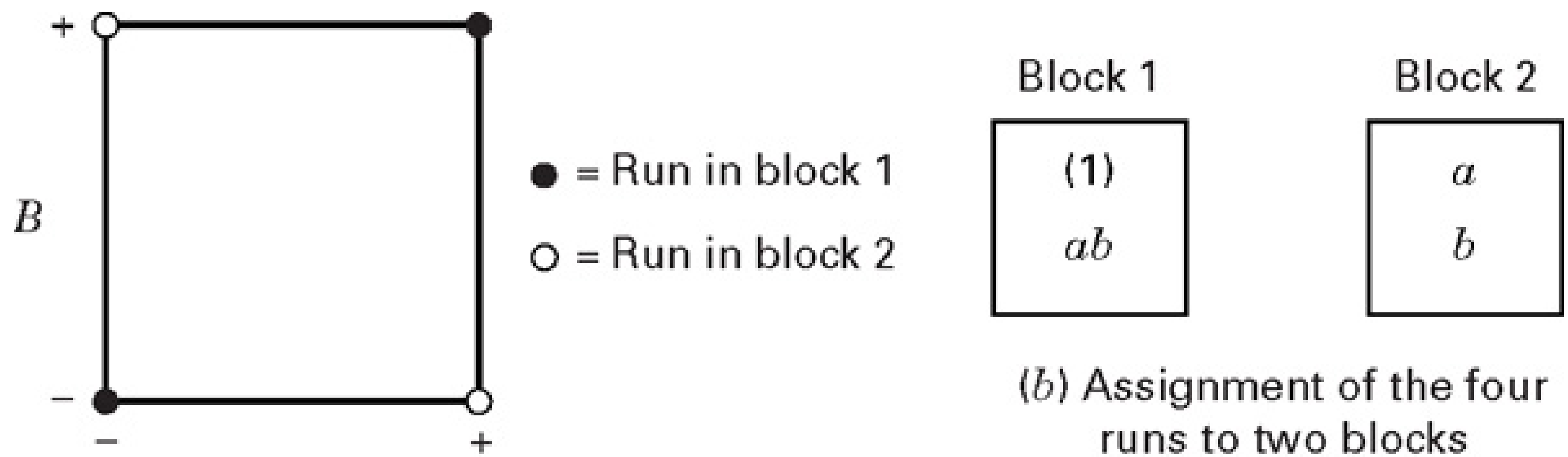
- Confound a 2^2 into two blocks
  - Each batch of raw material is big enough for two treatment combinations→ Need two batches to run all 4 treatment combinations
  - Divide 2^2 treatment combinations in 2 blocks. Each batch is a block.
  - An example blocking arrangement:



(a) Geometric view

7.4. Confounding 2^k factorial in Two Blocks

- Confound a 2^2 into two blocks
  - Each batch of raw material is big enough for two treatment combinations→ Need two batches to run all 4 treatment combinations
  - Divide 2^2 treatment combinations in 2 blocks. Each batch is a block.
  - An example blocking arrangement:



(a) Geometric view

(b) Assignment of the four runs to two blocks

Confound a 2^2 into two blocks

- Example blocking arrangement
  - *a* and *b* to Block 2
  - *(1)* and *ab* to Block 1
- Table of +/- signs:
  - treatments that have a "-" sign on AB assigned to Block 1
  - treatments that have a "+" sign on AB assigned to Block 2
- Estimate the effects as if no blocking had occurred

$$A = \frac{1}{2}[ab + a - b - (1)]$$

$$B = \frac{1}{2}[ab + b - a - (1)]$$

$$AB = \frac{1}{2}[ab + (1) - a - b]$$

TABLE 7.3

Table of Plus and Minus Signs for the 2^2 Design

Treatment Combination	Factorial Effect				
	I	A	B	AB	Block
(1)	+	−	−	+	1
a	+	+	−	−	2
b	+	−	+	−	2
ab	+	+	+	+	1