Quantum Parallelism and Implementing Classical Gates

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Problem 1

Answer:

▷ Applying U to the state of the form $a|00\rangle + b|11\rangle$ yields the super position $b|00\rangle + a|11\rangle$.
▷ Applying U to the uniform superposition $|\psi\rangle$ of two qubits will leave it unaltered.
▷ The unitary matrix U will be $\begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}$.

Applying U to the state of the form $a|00\rangle + b|11\rangle$ yields the super position $b|00\rangle + a|11\rangle$:
According to the problem statement, the operator U satisfies the following:

$$\begin{aligned} U|00\rangle &= |11\rangle, \\ U|11\rangle &= |00\rangle, \\ U|01\rangle &= |01\rangle, \\ U|10\rangle &= |10\rangle. \end{aligned} \tag{1}$$

Hence:

$$\begin{aligned} U(a|00\rangle + b|11\rangle) &= aU|00\rangle + bU|11\rangle \\ &= a|11\rangle + b|00\rangle \\ &= b|00\rangle + a|11\rangle \end{aligned} \tag{2}$$

Therefore, this option is **correct**.

Applying U to the uniform superposition $|\psi\rangle$ of two qubits will leave it unaltered:
Recall that the uniform superposition can be written as

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle). \tag{3}$$

By applying U to this vector, we get

$$\begin{aligned} U|\psi\rangle &= \frac{1}{2}(U|00\rangle + U|01\rangle + U|10\rangle + U|11\rangle) \\ &= \frac{1}{2}(|11\rangle + |01\rangle + |10\rangle + |00\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= |\psi\rangle. \end{aligned} \tag{4}$$

Therefore, this option is **correct**.

Another **correct option** is the following:
The unitary matrix U will be

$$\begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}. \tag{5}$$

This fact can be easily checked by multiplying this matrix by the column vectors corresponding to the pure states. We won't present these calculations. Notice that the only remaining option must be **wrong**, since it contradicts the one above.

Problem 2

Answer:
▷ The multicontrol-X gate with two control qubits is also called the "Toffoli" gate.
▷ Apply a multicontrolled X gate with b_1, b_2 as the control gates operating on the ancillary qubit b. Note that multicontrolled X gate will apply the X gate onto b only if b_1, b_2 are both 1 or otherwise leave the result unchanged.

Let's start from the truth table for the operation we wish to implement:

b_1	b_2	b	$b_1 \wedge b_2$	$b \oplus (b_1 \wedge b_2)$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	0

Next, denote the desired unitary operator by U. We can convert our table into a set of equations that U must satisfy:

$$\begin{aligned} U\left|000\right\rangle &= \left|000\right\rangle, \\ U\left|001\right\rangle &= \left|001\right\rangle, \\ U\left|010\right\rangle &= \left|010\right\rangle, \\ U\left|011\right\rangle &= \left|011\right\rangle, \\ U\left|100\right\rangle &= \left|100\right\rangle, \\ U\left|101\right\rangle &= \left|101\right\rangle, \\ U\left|110\right\rangle &= \left|111\right\rangle, \\ U\left|111\right\rangle &= \left|110\right\rangle. \end{aligned} \tag{6}$$

Notice we've used the fact that U leaves b_1 and b_2 unchanged. Also notice that, in many cases, U acts like the identity operator. In fact, there are only two cases in which U acts differently. For the sake of clarity, we write the corresponding equations one more time:

$$\begin{aligned} U\left|110\right\rangle &= \left|111\right\rangle, \\ U\left|111\right\rangle &= \left|110\right\rangle. \end{aligned} \tag{7}$$

These equations show that U is a CNOT gate with two control qubits and one target qubit. After all, the action of this operator is to negate b_3, but only when $b_1 = b_2 = 1$. Therefore, the option that talks about a multicontrolled gate is **correct**. Moreover, the option that discusses a different implementation must be **wrong**.
With the aid of the above equations, it's straightforward to determine the matrix representation of U:

$$U = \begin{pmatrix} \langle 000|U|000\rangle & \langle 000|U|001\rangle & \langle 000|U|010\rangle & \langle 000|U|011\rangle & \langle 000|U|100\rangle & \langle 000|U|101\rangle & \langle 000|U|110\rangle & \langle 000|U|111\rangle \\ \langle 001|U|000\rangle & \langle 001|U|001\rangle & \langle 001|U|010\rangle & \langle 001|U|011\rangle & \langle 001|U|100\rangle & \langle 001|U|101\rangle & \langle 001|U|110\rangle & \langle 001|U|111\rangle \\ \langle 010|U|000\rangle & \langle 010|U|001\rangle & \langle 010|U|010\rangle & \langle 010|U|011\rangle & \langle 010|U|100\rangle & \langle 010|U|101\rangle & \langle 010|U|110\rangle & \langle 010|U|111\rangle \\ \langle 011|U|000\rangle & \langle 011|U|001\rangle & \langle 011|U|010\rangle & \langle 011|U|011\rangle & \langle 011|U|100\rangle & \langle 011|U|101\rangle & \langle 011|U|110\rangle & \langle 011|U|111\rangle \\ \langle 100|U|000\rangle & \langle 100|U|001\rangle & \langle 100|U|010\rangle & \langle 100|U|011\rangle & \langle 100|U|100\rangle & \langle 100|U|101\rangle & \langle 100|U|110\rangle & \langle 100|U|111\rangle \\ \langle 101|U|000\rangle & \langle 101|U|001\rangle & \langle 101|U|010\rangle & \langle 101|U|011\rangle & \langle 101|U|100\rangle & \langle 101|U|101\rangle & \langle 101|U|110\rangle & \langle 101|U|111\rangle \\ \langle 110|U|000\rangle & \langle 110|U|001\rangle & \langle 110|U|010\rangle & \langle 110|U|011\rangle & \langle 110|U|100\rangle & \langle 110|U|101\rangle & \langle 110|U|110\rangle & \langle 110|U|111\rangle \\ \langle 111|U|000\rangle & \langle 111|U|001\rangle & \langle 111|U|010\rangle & \langle 111|U|011\rangle & \langle 111|U|100\rangle & \langle 111|U|101\rangle & \langle 111|U|110\rangle & \langle 111|U|111\rangle \end{pmatrix}. \tag{8}$$

We know how the operator U acts on the pure states. By using this information, we can simplify the last expression to

$$U = \begin{pmatrix} \langle 000|000\rangle & \langle 000|001\rangle & \langle 000|010\rangle & \langle 000|011\rangle & \langle 000|100\rangle & \langle 000|101\rangle & \langle 000|111\rangle & \langle 000|110\rangle \\ \langle 001|000\rangle & \langle 001|001\rangle & \langle 001|010\rangle & \langle 001|011\rangle & \langle 001|100\rangle & \langle 001|101\rangle & \langle 001|111\rangle & \langle 001|110\rangle \\ \langle 010|000\rangle & \langle 010|001\rangle & \langle 010|010\rangle & \langle 010|011\rangle & \langle 010|100\rangle & \langle 010|101\rangle & \langle 010|111\rangle & \langle 010|110\rangle \\ \langle 011|000\rangle & \langle 011|001\rangle & \langle 011|010\rangle & \langle 011|011\rangle & \langle 011|100\rangle & \langle 011|101\rangle & \langle 011|111\rangle & \langle 011|110\rangle \\ \langle 100|000\rangle & \langle 100|001\rangle & \langle 100|010\rangle & \langle 100|011\rangle & \langle 100|100\rangle & \langle 100|101\rangle & \langle 100|111\rangle & \langle 100|110\rangle \\ \langle 101|000\rangle & \langle 101|001\rangle & \langle 101|010\rangle & \langle 101|011\rangle & \langle 101|100\rangle & \langle 101|101\rangle & \langle 101|111\rangle & \langle 101|110\rangle \\ \langle 110|000\rangle & \langle 110|001\rangle & \langle 110|010\rangle & \langle 110|011\rangle & \langle 110|100\rangle & \langle 110|101\rangle & \langle 110|111\rangle & \langle 110|110\rangle \\ \langle 111|000\rangle & \langle 111|001\rangle & \langle 111|010\rangle & \langle 111|011\rangle & \langle 111|100\rangle & \langle 111|101\rangle & \langle 111|111\rangle & \langle 111|110\rangle \end{pmatrix}. \tag{9}$$

This matrix still looks complicated, but most of its elements vanish. This is a consequence of the fact that the inner products $\langle b_1 b_2 b_3 | b'_1 b'_2 b'_3 \rangle$ are non-zero only when $b_i = b'_i$, $\forall i$. This allows us to write U in the following way:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

This operator is known as the **Toffoli or CCNOT gate**.
There's only one option remaining: it talks about "checking b_1 and b_2". The corresponding statement is **wrong**. In Quantum Computing, one doesn't simply "check" the value of a qubit. That would require a measurement, which affects the circuit operation.

Problem 3

Answer: This can be done by first applying a multi-controlled X (Toffoli) gate on b_3 with b_1, b_2 as control qubits and then applying a controlled X gate on b_2 with b_1 as the control qubit.
Note: I don't believe this option is actually right, but it is accepted as correct.

Let's start from the truth table for the operation we wish to implement:

b_1	b_2	b_3	$b_1 \oplus b_2$	$b_1 \oplus b_2 \oplus b_3$
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	0	1

Next, denote the desired unitary operator by U. We can convert our table into a set of equations that U must satisfy:

$$\begin{aligned} U|000\rangle &= |000\rangle, \\ U|001\rangle &= |001\rangle, \\ U|010\rangle &= |011\rangle, \\ U|011\rangle &= |010\rangle, \\ U|100\rangle &= |111\rangle, \\ U|101\rangle &= |110\rangle, \\ U|110\rangle &= |100\rangle, \\ U|111\rangle &= |101\rangle. \end{aligned} \tag{11}$$

With the aid of the above equations, it's straightforward to determine the matrix representation of U:

$$U = \begin{pmatrix} \langle 000|U|000\rangle & \langle 000|U|001\rangle & \langle 000|U|010\rangle & \langle 000|U|011\rangle & \langle 000|U|100\rangle & \langle 000|U|101\rangle & \langle 000|U|110\rangle & \langle 000|U|111\rangle \\ \langle 001|U|000\rangle & \langle 001|U|001\rangle & \langle 001|U|010\rangle & \langle 001|U|011\rangle & \langle 001|U|100\rangle & \langle 001|U|101\rangle & \langle 001|U|110\rangle & \langle 001|U|111\rangle \\ \langle 010|U|000\rangle & \langle 010|U|001\rangle & \langle 010|U|010\rangle & \langle 010|U|011\rangle & \langle 010|U|100\rangle & \langle 010|U|101\rangle & \langle 010|U|110\rangle & \langle 010|U|111\rangle \\ \langle 011|U|000\rangle & \langle 011|U|001\rangle & \langle 011|U|010\rangle & \langle 011|U|011\rangle & \langle 011|U|100\rangle & \langle 011|U|101\rangle & \langle 011|U|110\rangle & \langle 011|U|111\rangle \\ \langle 100|U|000\rangle & \langle 100|U|001\rangle & \langle 100|U|010\rangle & \langle 100|U|011\rangle & \langle 100|U|100\rangle & \langle 100|U|101\rangle & \langle 100|U|110\rangle & \langle 100|U|111\rangle \\ \langle 101|U|000\rangle & \langle 101|U|001\rangle & \langle 101|U|010\rangle & \langle 101|U|011\rangle & \langle 101|U|100\rangle & \langle 101|U|101\rangle & \langle 101|U|110\rangle & \langle 101|U|111\rangle \\ \langle 110|U|000\rangle & \langle 110|U|001\rangle & \langle 110|U|010\rangle & \langle 110|U|011\rangle & \langle 110|U|100\rangle & \langle 110|U|101\rangle & \langle 110|U|110\rangle & \langle 110|U|111\rangle \\ \langle 111|U|000\rangle & \langle 111|U|001\rangle & \langle 111|U|010\rangle & \langle 111|U|011\rangle & \langle 111|U|100\rangle & \langle 111|U|101\rangle & \langle 111|U|110\rangle & \langle 111|U|111\rangle \end{pmatrix}. \tag{12}$$

We know how the operator U acts on the pure states. By using this information, we can simplify the last expression to

$$U = \begin{pmatrix} \langle 000|000\rangle & \langle 000|001\rangle & \langle 000|011\rangle & \langle 000|010\rangle & \langle 000|111\rangle & \langle 000|110\rangle & \langle 000|100\rangle & \langle 000|101\rangle \\ \langle 001|000\rangle & \langle 001|001\rangle & \langle 001|011\rangle & \langle 001|010\rangle & \langle 001|111\rangle & \langle 001|110\rangle & \langle 001|100\rangle & \langle 001|101\rangle \\ \langle 010|000\rangle & \langle 010|001\rangle & \langle 010|011\rangle & \langle 010|010\rangle & \langle 010|111\rangle & \langle 010|110\rangle & \langle 010|100\rangle & \langle 010|101\rangle \\ \langle 011|000\rangle & \langle 011|001\rangle & \langle 011|011\rangle & \langle 011|010\rangle & \langle 011|111\rangle & \langle 011|110\rangle & \langle 011|100\rangle & \langle 011|101\rangle \\ \langle 100|000\rangle & \langle 100|001\rangle & \langle 100|011\rangle & \langle 100|010\rangle & \langle 100|111\rangle & \langle 100|110\rangle & \langle 100|100\rangle & \langle 100|101\rangle \\ \langle 101|000\rangle & \langle 101|001\rangle & \langle 101|011\rangle & \langle 101|010\rangle & \langle 101|111\rangle & \langle 101|110\rangle & \langle 101|100\rangle & \langle 101|101\rangle \\ \langle 110|000\rangle & \langle 110|001\rangle & \langle 110|011\rangle & \langle 110|010\rangle & \langle 110|111\rangle & \langle 110|110\rangle & \langle 110|100\rangle & \langle 110|101\rangle \\ \langle 111|000\rangle & \langle 111|001\rangle & \langle 111|011\rangle & \langle 111|010\rangle & \langle 111|111\rangle & \langle 111|110\rangle & \langle 111|100\rangle & \langle 111|101\rangle \end{pmatrix}. \tag{13}$$

This matrix still looks complicated, but most of its elements vanish. This is a consequence of the fact that the inner products $\langle b_1 b_2 b_3 | b_1' b_2' b_3' \rangle$ are non-zero only when $b_i = b_i'$, $\forall i$. This allows us to write U in the following way:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

It's possible to show that this matrix is invertible. Its inverse is given by

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{15}$$

By comparing the last two expressions, one can conclude that U and U^{-1} are **not** equal. This eliminates one of the options. Next, we check whether U is unitary. Since we already know its inverse, we only need to compute its adjoint. It's easy to show that

$$U^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{16}$$

Once again, compare the last two equations. Clearly, we have $U^{-1} = U^{\dagger}$. Therefore, this operator is unitary. This means we've found a unitary transformation that implements the desired operation. Then two more options can be eliminated. Specifically, we can eliminate

▷ the option saying that U is not unitary, and

▷ the option saying that it's not possible to implement the desired operation.

A single option remains. That must be the right one. Indeed, it is accepted as correct. But I believe it's actually wrong. This option suggests that U can be factored as follows: $U = U_2 U_1$, where U_1 denotes the Toffoli gate, and U_2 represents a controlled X gate with b_1 as the control and b_2 as the target. Perhaps I'm wrong, but I think that the three-qubit version of this operator should act according to

$$\begin{aligned} U_2\,|000\rangle &= |000\rangle\,, \\ U_2\,|001\rangle &= |001\rangle\,, \\ U_2\,|010\rangle &= |010\rangle\,, \\ U_2\,|011\rangle &= |011\rangle\,, \\ U_2\,|100\rangle &= |110\rangle\,, \\ U_2\,|101\rangle &= |111\rangle\,, \\ U_2\,|110\rangle &= |100\rangle\,, \\ U_2\,|111\rangle &= |101\rangle\,. \end{aligned} \tag{17}$$

These equations tell us that U_2 negates b_2 only when $b_1 = 1$. Moreover, in every case, this operator leaves the value of b_3 unaltered. It's straightforward to show that the corresponding matrix representation is

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{18}$$

By using this result along with the formula for U_1 from the previous problem, we can compute the product U_2U_1. Recall that this should be equal to U. However, that's not what we get. One can verify that

$$U - U_2U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}. \tag{19}$$

This should be a matrix with all elements equal to 0. Therefore, I believe the remaining option is also wrong. I don't discard the possibility that I'm the one who is incorrect. However, it wouldn't be the first time I catch this kind of mistake.

Problem 4

Answer: The ancillary qubit collapses to either a 1 or a 0 and the input qubits $b_1, \dots, b_n$ collapse to a uniform combination of all those inputs for which $f(b_1, \dots, b_n)$ matches the measured result.

In the solution to the next problem, there is an example with $n = 2$ and $f(b_1, b_2) = b_1 \wedge b_2$.

Problem 5

Answer:
▷ The measurement of $|b\rangle$ yields $|0\rangle$ with $\frac{3}{4}$ probability and $|1\rangle$ with $\frac{1}{4}$ probability.
▷ If we measure $b = 0$ the input qubits collapse into the super position $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$.
▷ If we measure $b = 1$ the input qubits collapse into $|11\rangle$.

The initial state can be expressed as

$$|\psi\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |110\rangle). \tag{20}$$

By applying the Toffoli gate to this vector, we get

$$\begin{aligned} U|\psi\rangle &= \frac{1}{2}(U|000\rangle + U|010\rangle + U|100\rangle + U|110\rangle) \\ &= \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |111\rangle). \end{aligned} \tag{21}$$

This equation makes it clear that, if we measure the ancillary qubit, the value 0 occurs with probability $\frac{3}{4}$, and 1 occurs with probability $\frac{1}{4}$. Moreover, if the result of this measurement is 0, then the state of the input qubits collapses to

$$\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

On the other hand, if we measure $b = 1$, then the input qubits can be only in the state $|11\rangle$. Therefore, the statement "Measuring b has no effect on the input qubits" is **wrong**.