Multiple Qubit Quantum Gates

Marcio Woitek

Problem 1

Answer:
▷ Apply a Hadamard operation to the first qubit and a Pauli-X (quantum not) gate to the second qubit.

Apply a Pauli-X gate to the first qubit and a Hadamard gate to the second qubit:
The initial state is $|00\rangle = |0\rangle \otimes |0\rangle$. This option suggests applying the operator $U = X \otimes H$ to this state vector. Let's check if this works:

$$U|00\rangle = (X \otimes H)(|0\rangle \otimes |0\rangle) = X|0\rangle \otimes H|0\rangle. \tag{1}$$

The Pauli gate transforms $|0\rangle$ as follows:

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle. \tag{2}$$

Moreover, we already know that the result of applying H to $|0\rangle$ is

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{3}$$

Hence:

$$U|00\rangle = |1\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle). \tag{4}$$

This doesn't match our desired final state. Therefore, this option is **wrong**.

Apply a Hadamard operation to the first qubit and a Pauli-X (quantum not) gate to the second qubit:
The initial state is $|00\rangle = |0\rangle \otimes |0\rangle$. This option suggests applying the operator $U = H \otimes X$ to this state vector. Let's check if this works:

$$\begin{aligned} U|00\rangle &= (H \otimes X)(|0\rangle \otimes |0\rangle) \\ &= H|0\rangle \otimes X|0\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle). \end{aligned} \tag{5}$$

This is exactly what we wanted. Therefore, this option is **correct**. Also notice that this result eliminates the option "There is no unitary transformation that can achieve this." After all, we've just shown that U yields the desired final state.

The remaining option cannot be right, since it requires copying the initial quantum state.

Problem 2

Answer:
▷ Apply the operator $H \otimes I \otimes H$ where H stands for the single qubit Hadamard gate while I is the single qubit identity operator.
▷ Apply a Hadamard gate to the first and third qubits but leave the second qubit unaltered.

Apply the operator $H \otimes I \otimes H$ where H stands for the single qubit Hadamard gate while I is the single qubit identity operator:

In this case, the initial state is given by

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle). \tag{6}$$

We need to apply $U = H \otimes I \otimes H$ to this state vector. This means computing the following vector:

$$U|\psi\rangle = \frac{1}{\sqrt{8}}(U|000\rangle + U|001\rangle + U|010\rangle + U|011\rangle + U|100\rangle + U|101\rangle + U|110\rangle + U|111\rangle). \tag{7}$$

For the sake of clarity, we're going to compute each of $U|000\rangle, \ldots, U|111\rangle$ separately:

$$\begin{aligned} U|000\rangle &= (H \otimes I \otimes H)(|0\rangle \otimes |0\rangle \otimes |0\rangle) \\ &= H|0\rangle \otimes I|0\rangle \otimes H|0\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \left[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \frac{1}{2}(|000\rangle + |100\rangle + |001\rangle + |101\rangle) \end{aligned} \tag{8}$$

$$\begin{aligned} U|001\rangle &= (H \otimes I \otimes H)(|0\rangle \otimes |0\rangle \otimes |1\rangle) \\ &= H|0\rangle \otimes I|0\rangle \otimes H|1\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\ &= \left[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\ &= \frac{1}{2}(|000\rangle + |100\rangle - |001\rangle - |101\rangle) \end{aligned} \tag{9}$$

$$\begin{aligned} U|010\rangle &= (H \otimes I \otimes H)(|0\rangle \otimes |1\rangle \otimes |0\rangle) \\ &= H|0\rangle \otimes I|1\rangle \otimes H|0\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes |1\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \left[\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \frac{1}{2}(|010\rangle + |110\rangle + |011\rangle + |111\rangle) \end{aligned} \tag{10}$$

$$\begin{aligned} U|011\rangle &= (H \otimes I \otimes H)(|0\rangle \otimes |1\rangle \otimes |1\rangle) \\ &= H|0\rangle \otimes I|1\rangle \otimes H|1\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes |1\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\ &= \left[\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\ &= \frac{1}{2}(|010\rangle + |110\rangle - |011\rangle - |111\rangle) \end{aligned} \tag{11}$$

$$\begin{aligned} U|100\rangle &= (H \otimes I \otimes H)(|1\rangle \otimes |0\rangle \otimes |0\rangle) \\ &= H|1\rangle \otimes I|0\rangle \otimes H|0\rangle \\ &= \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \otimes |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \left[\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\ &= \frac{1}{2}(|000\rangle - |100\rangle + |001\rangle - |101\rangle) \end{aligned} \tag{12}$$

$$
\begin{aligned}
U|101\rangle &= (H \otimes I \otimes H)(|1\rangle \otimes |0\rangle \otimes |1\rangle) \\
&= H|1\rangle \otimes I|0\rangle \otimes H|1\rangle \\
&= \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \otimes |0\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\
&= \left[\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\
&= \frac{1}{2}(|000\rangle - |100\rangle - |001\rangle + |101\rangle)
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
U|110\rangle &= (H \otimes I \otimes H)(|1\rangle \otimes |1\rangle \otimes |0\rangle) \\
&= H|1\rangle \otimes I|1\rangle \otimes H|0\rangle \\
&= \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \otimes |1\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\
&= \left[\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \\
&= \frac{1}{2}(|010\rangle - |110\rangle + |011\rangle - |111\rangle)
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
U|111\rangle &= (H \otimes I \otimes H)(|1\rangle \otimes |1\rangle \otimes |1\rangle) \\
&= H|1\rangle \otimes I|1\rangle \otimes H|1\rangle \\
&= \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \otimes |1\rangle \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\
&= \left[\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] \\
&= \frac{1}{2}(|010\rangle - |110\rangle - |011\rangle + |111\rangle)
\end{aligned} \tag{15}
$$

The next step is to add all these results. To make this calculation easier, first we'll compute the sums of consecutive terms:

$$
\begin{aligned}
U|000\rangle + U|001\rangle &= |000\rangle + |100\rangle \\
U|010\rangle + U|011\rangle &= |010\rangle + |110\rangle \\
U|100\rangle + U|101\rangle &= |000\rangle - |100\rangle \\
U|110\rangle + U|111\rangle &= |010\rangle - |110\rangle
\end{aligned} \tag{16}
$$

With these equations, now it's simple to determine the state $U|\psi\rangle$:

$$
U|\psi\rangle = \frac{1}{2\sqrt{2}} \cdot 2(|000\rangle + |010\rangle) = \frac{1}{\sqrt{2}}(|000\rangle + |010\rangle). \tag{17}
$$

This is the desired final state. Therefore, this option is **correct**. Notice that we've also eliminated the option "There is no way to do this with a unitary operation."

Apply a Hadamard gate to the first and third qubits but leave the second qubit unaltered:
This option says the same as the one we've just shown is right. Therefore, this option is also **correct**.

The remaining option is just silly. It's obviously **wrong**. We cannot simply "zap components".

Problem 3

Answer: Apply a controlled-X (quantum not) operation to the second qubit with the first qubit as the control qubit.

Justification:
The initial state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}. \tag{18}$$

The matrix representation of the operation described above can be written as

$$U = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}. \tag{19}$$

Then the transformed state can be computed as follows:

$$U|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{20}$$

This is exactly what we wanted. Therefore, this is the correct option.

Problem 4

Answer: $\frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$

First, recall that the phase gate with phase $\frac{\pi}{2}$ is

$$P\left(\frac{\pi}{2}\right) = \begin{pmatrix}1&0\\0&i\end{pmatrix}. \tag{21}$$

Then the controlled gate described in the problem statement can be expressed as

$$U = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&i\end{pmatrix}. \tag{22}$$

To obtain the desired result, all we need to do is to apply this operator to the state vector $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

$$U|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&i\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\i\end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{23}$$