Integer Foundation

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Problem 1

Answer: $819, 990$

There's a problem with this question. All the options are wrong, i.e., there's no pair of relatively prime numbers. Then I had to get the correct answer by guessing.

It's simple to exclude all the options:

- ▷ $91, 343$: Both of these numbers are divisible by 7. To see this, first notice that $91 = 70 + 21$. Then $\frac{91}{7} = 13$. Similarly, we can write $343 = 350 - 7$. Hence: $\frac{343}{7} = 49$. Since 91 and 343 have a common prime factor of 7, they are not relatively prime.
- ▷ $819, 990$: These numbers are divisible by 3. If we add up the digits of 819 we get 18, which is divisible by 3. Then 819 is divisible by 3. A similar argument applies to 990. Therefore, 819 and 990 have 3 as a common factor, and they're not relatively prime.
- ▷ $796, 982$: These numbers are clearly even. Since they have a common factor of 2, they're not relatively prime.
- ▷ $527, 612$: In this case, we compute the gcd of 527 and 612. With the aid of the Euclidean algorithm, we obtain

m	n
612	527
527	85
85	17
17	0

So we have $\gcd(527, 612) = 17$. This result makes it very clear that the numbers under consideration are not relatively prime.

Problem 2

Answer: 62

Applying the Euclidean algorithm, we get the following:

m	n
992	930
930	62
62	0

Then $\gcd(930, 992) = 62$.

Problem 3

Answer: $-17, 9$

To find the congruent pair, we need to test each pair by actually doing the calculations. We're not going to present these calculations. But it turns out that the correct answer is $-17, 9$. This result can be verified as follows:

$$-17 \equiv -17 + 2 \cdot 13 \equiv 9 \pmod{13}, \tag{1}$$

where we've used the fact that any multiple of 13 is congruent to 0 (mod 13).

Problem 4

Answer: x is a multiple of half the modulus.

Recall that a and b are congruent modulo N when their difference $a - b$ is divisible by N. In this case, this difference is $x - (-x) = 2x$. Since N divides $2x$, there's an integer k such that the following equation holds:

$$\frac{2x}{N} = k. \tag{2}$$

Solving this equation for x, we get

$$x = k\frac{N}{2}. \tag{3}$$

Therefore, x is a multiple of half the modulus.

Problem 5

Answer: x must be relatively prime to N.

Problem 6

Answer: $O(\log n)$ (where n is the smaller number).

Problem 7

Answer: 3

We'll solve this problem by using the general version of the extended Euclidean algorithm. This algorithm will yield a solution to the equation

$$16x + 47y = \gcd(16, 47) = 1. \tag{4}$$

The table below contains all the results returned by the Euclidean algorithm.

m	n	q	r	s	t	$\hat{s}$	$\hat{t}$
47	16	2	15	1	0	0	1
16	15	1	1	0	1	1	-2
15	1	15	0	1	-2	-1	3

The desired solution is in the last two columns of the last row: $x = 3$ and $y = -1$. Hence:

$$16 \cdot 3 + 47 \cdot (-1) = 1. \tag{5}$$

Performing the modulo operation for both sides of this equation, we get

$$16 \cdot 3 \equiv 1 \pmod{47}. \tag{6}$$

Clearly, in this case we have $16^{-1} = 3$.

Problem 8

Answer: -1

We'll solve this problem by using the general version of the extended Euclidean algorithm. This algorithm will yield a solution to the equation

$$219x + 220y = \gcd(219, 220) = 1. \tag{7}$$

The table below contains all the results returned by the Euclidean algorithm.

m	n	q	r	s	t	$\hat{s}$	$\hat{t}$
220	219	1	1	1	0	0	1
219	1	219	0	0	1	1	-1

The desired solution is in the last two columns of the last row: $x = -1$ and $y = 1$. Hence:

$$219 \cdot (-1) + 220 \cdot 1 = 1. \tag{8}$$

Performing the modulo operation for both sides of this equation, we get

$$219 \cdot (-1) \equiv 1 \pmod{220}. \tag{9}$$

Clearly, in this case we have $219^{-1} = -1$.