Modular Exponentiation

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Problem 1

Answer: $47^{69} \bmod 143 = 125$

Problem 2

Answer: $15^{15} \bmod 14 = 15$

We begin by reducing the exponent. The first step is to compute the totient function for 14:

$$\varphi(14) = \varphi(2 \cdot 7) = (2-1)(7-1) = 6. \tag{1}$$

Then, since 14 and 15 are relatively prime, we can write

$$15^{15} \equiv 15^{3} \pmod{14}. \tag{2}$$

Next, to reduce the third power of 15, we calculate the square of this number modulo 14:

$$15^{2} \equiv 15 \cdot 15 \equiv 1 \cdot 1 \equiv 1 \pmod{14}, \tag{3}$$

where we've used the fact that 15 is congruent to 1 mod 14. Hence:

$$15^{15} \equiv 15 \cdot 15^{2} \equiv 15 \pmod{14}. \tag{4}$$

We could reduce the above power one more time. However, we stop here since 15 is one of the options.

Problem 3

Answer: The number of positive integers less than and relatively prime to N.

Problem 4

Answer: $\varphi(p) = p - 1$

Problem 5

Answer: $\varphi(2717) = 2160$

First, notice that 2717 can be factored as follows: $2717 = 11 \cdot 13 \cdot 19$. Then, by using the multiplicative property of φ, we can write

$$\varphi(2717) = \varphi(11 \cdot 13 \cdot 19) = (11-1)(13-1)(19-1) = 2160. \tag{5}$$

Problem 6

Answer: $\varphi(3^4) = 54$

The formula for the totient of the k-th power of a prime p is

$$\varphi(p^k) = p^{k-1}(p-1). \tag{6}$$

Applying this result for $p = 3$ and $k = 4$ yields

$$\varphi(3^4) = 3^3(3-1) = 27 \cdot 2 = 54. \tag{7}$$

Problem 7

Answer: If the two numbers are relatively prime.

Problem 8

Answer: Nothing reliable.

Problem 9

Answer: It may or may not exist to other bases.

Problem 10

Answer: A primitive root generates all numbers relatively prime to the modulus.