MCV4UR : Advanced Placement Calculus and Vectors

Assignment #2

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

Family Name:
First Name:

Declared References:

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1. Given $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, show the derivation of $\cosh^{-1}(x)$ as a function of x.

2. Assume that $f(x)$ is continuous on the interval $J = [3, 5]$, $f(3) = 2$ and that $f'(x) = \dfrac{1}{1+x^3}$ on J.

   (a) Determine the maximum and minimum values of $f'(x)$ on J.

   (b) Prove that $\dfrac{127}{63} \leq f(5) \leq \dfrac{29}{14}$.

3. Consider a function $f(x)$.

   (a) Show that if $f(x)$ is differentiable on $\mathbb{R}$ and that $\forall x \in \mathbb{R}\, f'(x) = 0$, then $\forall x \in \mathbb{R}\, f(x) = f(0)$.

   (b) Assume that $f(x)$ is such that $\forall x \in \mathbb{R}\, f'(x) = f(x)$. Show that there exists a constant $C \in \mathbb{R}$ such that $f(x) = Ce^x$.

   (Hint: Let $g(x) = \dfrac{f(x)}{e^x}$ and show that it is a constant function.)

4. Given $f(x) = \dfrac{1}{x}$, prove $\forall n \in \mathbb{Z}^+$ that $f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}}$.

5. Hailstones originate at an altitude of about 3000 metres, although this varies. As they fall, air resistance slows down the hailstones considerably. In one model of air resistance, the speed (in metres per second) of a hailstone of mass m as a function of time t (in seconds) is given by $v(t) = \frac{mg}{k}(1 - e^{\frac{-kt}{m}})$ where $g \approx 9.8$ (in metres per second squared) is the acceleration due to gravity and k is a constant that depends on the size of the hailstone and the conditions of the air.

   (a) Determine the acceleration function $a(t)$ of the hailstone as a function of time.

   (b) Determine $\lim_{t\to\infty} v(t)$. What does this say about the speed of the hailstone?

   (c) Determine $\lim_{t\to\infty} a(t)$. What does this say about the acceleration of the hailstone?

6. Prove that if $\sqrt{x+y} - \sqrt{x-y} = 1$ then $\frac{dy}{dx}$ can be expressed as a function of y.

7. Consider the three circles defined as $C_1 : x^2 + y^2 = 1$, $C_2 : (x-4)^2 + (y+1)^2 = 4$ and $C_3 : (x+5)^2 + (y-1)^2 = 9$. An *external tangent* of two circles is a line that is tangent to both circles but does not pass between them. A pair of circles will have two external tangents.

   (a) Determine the external tangents of C_1 and C_2.

   (b) Determine the external tangents of C_1 and C_3.

   (c) Determine the external tangents of C_2 and C_3.

   (d) Determine the points of intersection for each system of external tangents for each pair of circles.

   (e) Show that the points of intersection are collinear.

   (f) Conjecture, with evidence, whether you believe that the intersection points of these systems of external tangents for three circles will always be collinear or not. Cite any sources you use in your investigation.