Advanced Functions

Assignment #1

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and class notes (such as other texts, discussions with peers or online resources), indicate this information in the space below. If you did not use any aids then explicitly state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT (Factor Theorem), RRT (Rational Root Theorem), etc.

Note: Your submitted work must be **your original work**.

Family Name: WongFirst Name: Max

Declared References:

Used this forum answer to number the an element within align (Question 2):
stackexchange align* but show one equation number at the end

Instructions

1. Organize and express complete, effective and concise responses to each problem.
2. Use appropriate mathematical conventions and notation wherever possible.
3. Provide logical reasoning for your arguments and cite any relevant theorems.
4. Ask your teacher questions if you need any clarification.

Evaluation

D3 Students will compare the characteristics of functions, and solve problems by modelling and reasoning with functions.

Criteria	Level 1	Level 2	Level 3	Level 4
Understanding of Mathematical Concepts	Demonstrates limited understanding	Demonstrates some understanding	Demonstrates considerable understanding	Demonstrates thorough understanding of concepts
Selecting Tools and Strategies	Selects and applies appropriate tools and strategies, with major errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies, with minor errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies accurately, and in a logical sequence	Selects and applies appropriate and efficient tools and strategies accurately to create mathematically elegant solutions
Reasoning and Proving	Inconsistently or erroneously employs logic to develop and defend statements	Statements are developed and defended with some omissions or leaps in logic	Frequently develops and defends statements with reasonable logical justification	Consistently develops and defends statements with sophisticated and/or complete logical justification
Communicating	Expresses and organizes mathematical thinking with limited effectiveness	Expresses and organizes mathematical thinking with some effectiveness	Expresses and organizes mathematical thinking with considerable effectiveness	Expresses and organizes mathematical thinking with a high degree of effectiveness

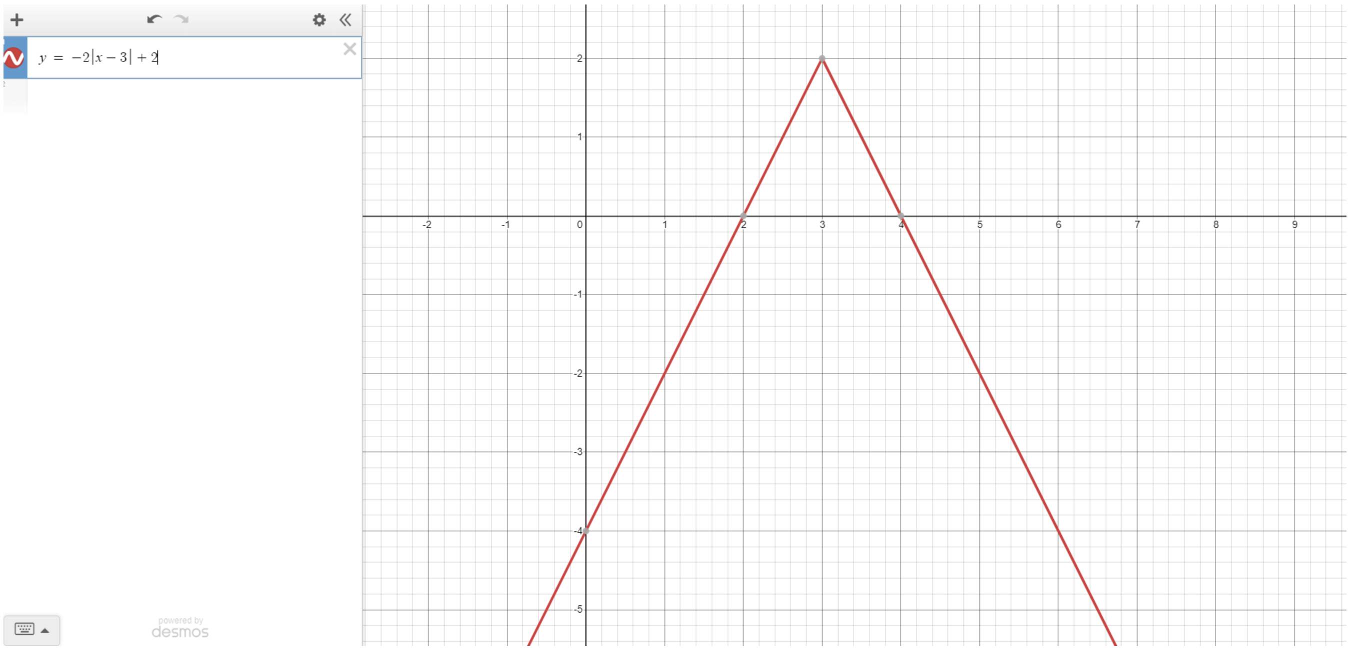
1. **Describe** the characteristics of the function $f(x) = -2|x-3|+2$ by filling in the table given below. **Write** a paragraph briefly explaining how you determined each characteristic.

Characteristic	
domain	$\{x \mid x \in \mathbb{R}\}$
range	$\{y \mid y \in \mathbb{R}, y < 2\}$
zero(s)	x = 4, 2
y-intercept	(0,-4)
interval(s) of increase	none
interval(s) of decrease	$f(x)$ decreasing on $(-\infty, 3)$ $f(x)$ decreasing on $[3, \infty)$
discontinuities	none
symmetry	even symmetry along vertical line $x = 3$
end behaviours	as $x \to \infty, f(x) \to -\infty$ as $x \to -\infty, f(x) \to -\infty$

The domain is all real whole numbers because of the key characteristic of an absolute value relationship. The range for the parent function of an absolute value relationship is all non negative values but since f(x) is reflected vertically and vertically translted 2 up the range becomes all real values below and equal to 2. Zeroes are found by solving for x=0. The y intercept is found by solving for x = 0 or simply using the vertical translation. In the parent function of f(x) there are 2 positive intervals but since the functions is reflected vertically and translated 3 to the right, there are 2 decreasing intervals to the left and right of x=3. This function does not have any restrictions or characteristics that cause discontinuities. Doing the even, odd and neither symmetry test with f(x), -f(x), f(-x) and -f(-x) I determined that there is even symmetry and due to the horizontal translation mentioned before the line of symmetry is at x=3. Referencing the intervals both ends of the function are decreasing.

Check answer by graphing and comparing result to answer given:

From the graph you can see that the answers in the table above match the relatioship described by the graph below.



2. **Solve** both inequalities.

   (a) Solve $|3x - 5| \leq 2$

   (b) Solve $-|-2x - 1| < -4$

Solution for (a):

$$|3x - 5| \leq 2$$

$$\left|3(x - \frac{5}{3})\right| \leq 2 \qquad \text{Factor by 3}$$

$$|3|\left|x - \frac{5}{3}\right| \leq 2 \qquad \text{1st property of absolute value}$$

$$3\left|x - \frac{5}{3}\right| \leq 2 \qquad \text{Since —3— = 3}$$

$$\left|x - \frac{5}{3}\right| \leq \frac{2}{3} \qquad \text{divide by 3} \tag{1}$$

$$-\frac{5}{3} - \frac{2}{3} \leq x \leq -\frac{5}{3} + \frac{2}{3} \qquad k - c \leq x \leq k + c \Longleftrightarrow |x - k| \leq c$$

$$1 \leq x \leq \frac{7}{3} \qquad \text{simplify}$$

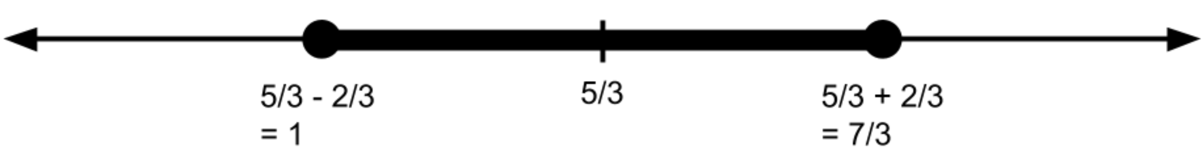
$\therefore$ The solutiont to the inequality$|3x - 5| \leq 2$ is $\boxed{1 \leq x \leq \frac{7}{3}}$

Check answer by starting from (1) to check the key points and use a line diagram to check the solution through logic:

$$\left|x - \frac{5}{3}\right| = \frac{2}{3}$$

$$x - \frac{5}{3} = \pm\frac{2}{3} \qquad |x| = c \longleftrightarrow x = \pm c$$

$$x = \frac{5}{3} \pm \frac{2}{3} \qquad \text{add } \frac{5}{3} \text{ to both sides}$$



Solution for (b):

$$-|-2x-1| < -4 \qquad \text{multiply all by -1, flip inequality sign; 3rd property of inequalities}$$

$$|-2x-1| > 4$$

$$\left|-2(x+\frac{1}{2})\right| > 4 \qquad \text{factor -2}$$

$$|-2|\left|x+\frac{1}{2}\right| > 4 \qquad \text{1st properoty of absolute values}$$

$$2\left|x+\frac{1}{2}\right| > 4 \qquad |-2| = 2$$

$$\left|x+\frac{1}{2}\right| > 2 \qquad \text{divide all by 2} \tag{2}$$

$$x < -\frac{1}{2} - 2 \text{ or } x > -\frac{1}{2} + 2 \qquad |x-k| > c \Longleftrightarrow x < k-c \text{ or } x > k+c$$

$$x < -\frac{5}{2},\ \frac{3}{2} < x \qquad \text{simplify}$$

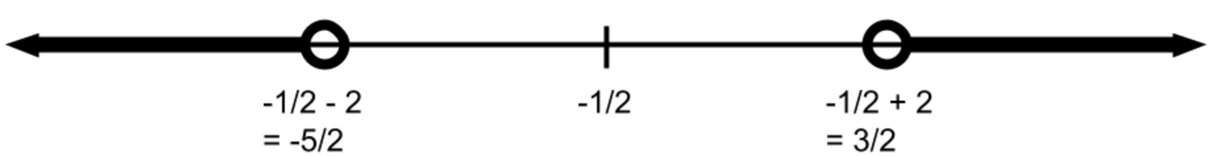
$\therefore$ The solutiont to the inequality $-|-2x-1| < -4$ is $\boxed{-\frac{5}{2},\ \frac{3}{2} < x}$

Check answer by starting from (2) to check the key points and use a line diagram to check the solution through logic:

$$\left|x+\frac{1}{2}\right| = 2$$

$$x+\frac{1}{2} = \pm 2 \qquad |x| = c \longleftrightarrow x = \pm c$$

$$x = -\frac{1}{2} \pm 2 \qquad \text{subtract } \frac{1}{2}$$



3. A 10 foot long stem of bamboo is broken in such a way that its tip touches the ground 3 feet away from the base of the stem. **Determine** the height of the break.

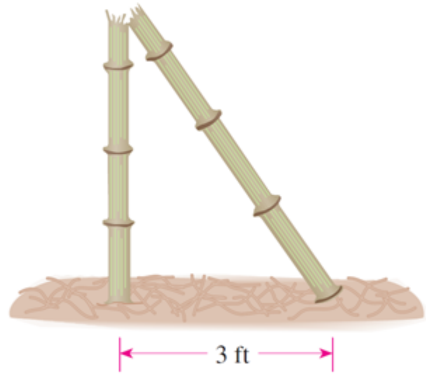
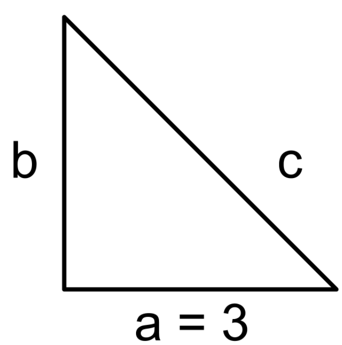


Figure 1: Diagram from Problem 3

Question 3 Solution:

Consider labeling all three sides of the triangle in the diagram given. The hypotenuse is c while the other sides are a and b. We know that a, or the bottom side, is 3 ft long.



From Pythagorean Theorum (PT) we know that $a^2 + b^2 = c^2$ or in this case $3^2 + b^2 = c^2$. We also know from the question that $b + c = 10$. Therefore, with 2 equations we can solve for the unknow values: b and c.

$$c^2 = b^2 + 9 \tag{1}$$

$$10 = b + c \tag{2}$$

Rearrange the second equation $10 = b + c$ to isolate one of the variables to solve by substitution. I am isolating c to solve for b (the height).

$$10 = b + c$$

$$10 - b = c \tag{3}$$ subtract b from all

Continue solution on next page

Now substitute (3) into (1) as c and solve for b:

$$c^2 = b^2 + 9 \qquad (1)$$

$$(10 - b)^2 = b^2 + 9 \qquad \text{(3) into (1) as c}$$

$$100 - 20b + b^2 = b^2 + 9 \qquad \text{expand square}$$

$$100 - 20b = 9 \qquad \text{subtract} b^2 \text{from both sides}$$

$$100 - 9 = 20b \qquad \text{subtract 9 and -20b from both sides}$$

$$91 = 20b \qquad \text{simplify}$$

$$b = \frac{91}{20} \qquad \text{divide 20 from both sides}$$

$\therefore$ the height of the vertical bamboo shoot (break) is $\boxed{\frac{91}{20}\text{ft}}$ or 4.55 ft

Proof. **Check answer by solving for c with (2) and then confirming result with (1)**

$$10 - b = c \qquad \text{From (3)}$$

$$c = 10 - 4.55 \qquad \text{substitute b for 4.55}$$

$$c = 5.45$$

Now substitute b and c values into (1) to prove the equivalent values

$$c^2 = b^2 + 9 \qquad \text{From (1)}$$

$$(5.45)^2 = (4.55)^2 + 9 \qquad \text{substitute the values of b and c}$$

$$29.7025 = 20.7025 + 9 \qquad \text{expand and simplify}$$

$$29.7025 = 29.7025$$

$$\text{LHS} = \text{RHS}$$

□

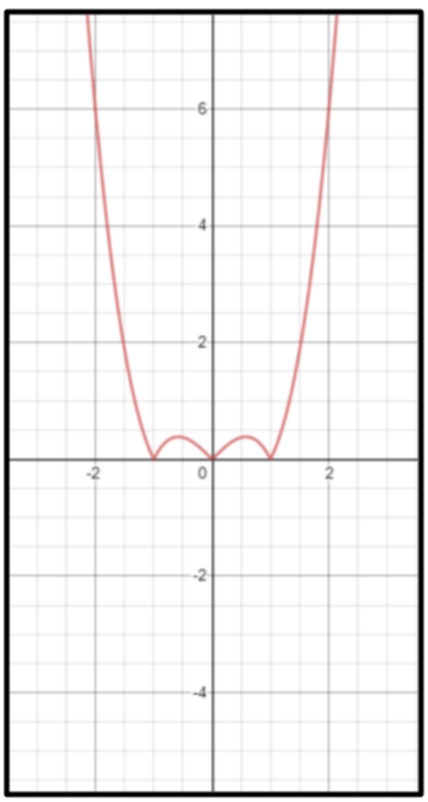
4. **Define** (rewrite) $f(x) = |x^3 - x|$ as a piecewise function not including any expressions involving absolute value.

Solution for Question 4:

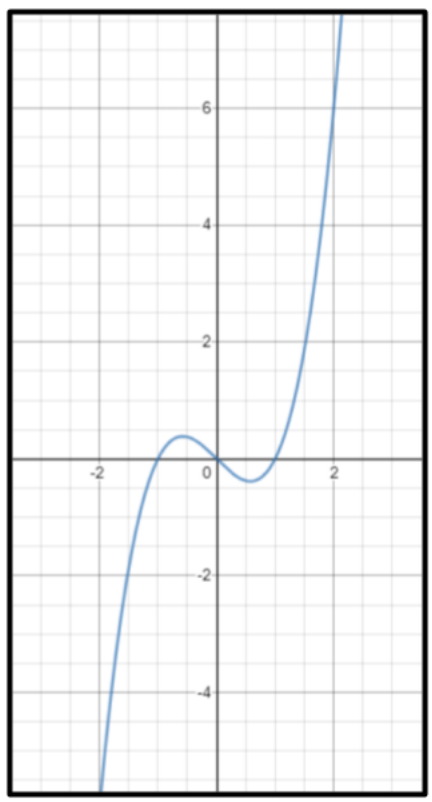
Consider $f(x) = |x^3 - x|$. The x intercepts (f(x) = 0) are found by converting the function to factor from.

$$\begin{aligned} 0 &= |x^3 - x| \\ &= |x(x^2 - 1)| && \text{factor x} \\ &= |x(x + 1)(x - 1)| && (x^2 - a^2) = (x + a)(x - a) \\ &= |x||x + 1||x - 1| && |ab| = |a||b| \text{ 1st property of absolute value} \end{aligned}$$

From the calculations above, the x intercepts are at $x = 0, -1, 1$. Now Consider f(x) and it's non-absolute value equivalent, $g(x) = x^3 - x$.



f(x)



g(x)

The 3 key points show us that there are 4 sections. The diagrams/Graphs shows that 2 of the 4 segments seperated by the x intercepts in g(x) represent negative values. In f(x), the absolute value function, these segments are reflected vertically to ensure they are not negative values.

These segments are at: $x \in (\infty, -1)$ and $x \in (0, 1)$.

(I used $<$ instead of $\leq$ because 0 is not negative, therefore it does not need to be reflected)

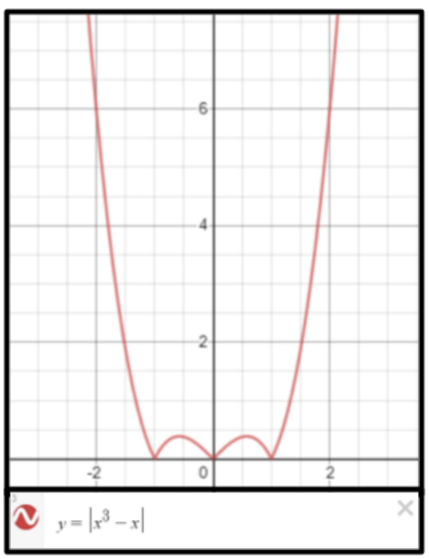
$\therefore$ using the information gathered I wrote the function in piecewise form.

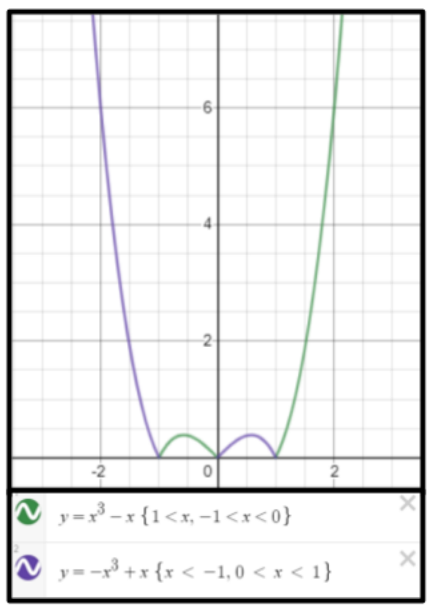
$$f(x) = \begin{cases} x^3 - x & : 1 \leq x, \\ & \quad -1 \leq x \leq 0 \\ -x^3 + x & : x < -1 \\ & \quad 0 < x < 1 \end{cases}$$

We use $x^3 - x$ to express the segments that are positive and not reflected when comparing g(x) to f(x). Those parts of g(x) that are negative are positive in f(x) meaning there is a vertical reflection.

This is represented by $-(x^3 - x)$ where the "-" at the front is the vertical reflection. In a simplified form, this is $-x^3 + x$.

Check answer by graphing original f(x) and piecewise f(x) version:





Both graphs look identical and when placed on the same graph, they cover each other exactly.

5. **Determine** the inverse of the function $g(x) = \dfrac{-2}{x-1} + 4$. **Prove** that g and g^{-1} satisfy the expression $g(g^{-1}(x)) = x$.

Solution for Question 5: First we need to find the inverse of g(x) by swapping the dependant and independant variable

$$g(x) = \frac{-2}{x-1} + 4$$

$$y = \frac{-2}{x-1} + 4 \qquad \text{replace g(x) with y}$$

Now I swap x and y to create the inverse.

$$x = \frac{-2}{y-1} + 4 \qquad \text{x and y have been “swapped”}$$

$$x - 4 = \frac{-2}{y-1} \qquad \text{subtracted 4 from both sides}$$

$$(x-4)(y-1) = -2 \qquad \text{multiply al by y-1}$$

$$xy - x - 4y + 4 = -2 \qquad \text{expand}$$

$$xy - 4y - x + 4 = -2 \qquad \text{Rearrange}$$

$$xy - 4y = -2 - 4 + x \qquad \text{subtract (-x+4) from both sides}$$

$$y(x-4) = -6 + x \qquad \text{factor y on the left side, simplify}$$

$$y = \frac{x-6}{x-4}$$

$\therefore$ the inverse of g(x) is: $\boxed{g^{-1}(x) = \dfrac{x-6}{x-4}}$

Solution continues on next page

Proof. **Now prove that $g(g^{-1}(x)) = x$ with the given function of g(x):**

$$g(x) = \frac{-2}{x-1} + 4$$

$$g^{-1}(x) = \frac{x-6}{x-4}$$

Given that $x = g(g^{-1}(x))$ and the know equations of g(x) and it's inverse:

$$x = g(g^{-1}(x))$$

$$= \frac{-2}{g^{-1}(x) - 1} + 4 \qquad \text{substitute } g(x) \text{ as } \frac{-2}{x-1} + 4$$

$$= \frac{-2}{\frac{x-6}{x-4} - 1} + 4 \qquad \text{substitute } g^{-1}(x) \text{ as } \frac{x-6}{x-4}$$

$$= \frac{-2}{\frac{x-6}{x-4} - \frac{x-4}{x-4}} + 4 \qquad \text{1 rule: } \frac{x-4}{x-4} = 1$$

$$= \frac{-2}{\frac{x-6-x+4}{x-4}} + 4 \qquad \text{simplify}$$

$$= \left(\frac{-2}{\frac{-2}{x-4}}\right) + 4$$

$$= \left(\frac{-2}{1} \times \frac{x-4}{-2}\right) + 4 \qquad \text{divide both sides by}$$

$$= x - 4 + 4 \qquad \text{simplify}$$

$$x = x$$

$$LHS = RHS$$

$\therefore$ Since LHS = RHS, $g(g^{-1}(x)) = x$. $\square$

proof/check on next page

Check my work by using an example point:

I took an arbitrary x value, x=0, and found the coordinate of x=0 on $g(x)$.

$$g(0) = \frac{-2}{0-1} + 4$$

$$= 6$$

The coordinate at x=0 is (0,6). We know that the inverse of a relationship switches the values on the domain and range. This means that if (0,6) is a point on $g(x)$, (6,0) should be a point on $g^{-1}(x)$.

$$g^{-1}(x) = \frac{x-6}{x-4}$$

$$g^{-1}(6) = \frac{6-6}{6-4}$$

$$= \frac{0}{2}$$

$$= 0$$

Therefore, following the logic, the inverse of $g(x)$ found by this solution is in fact the inverse.