MHF4U : Advanced Functions

Assignment #2

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT (Factor Theorem), RRT (Rational Root Theorem), etc.

Note: Your submitted work must be **your original work**.

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Declared References:

Used this forum answer to number the an element (Question 1):
stackexchange align* but show one equation number at the end
Used YouTube tutorial to do long division with polynomials (Question 1 and 2):

Used Desmos for graphs

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1. Given the volume of a rectangular box is $V(x) = x^3 + 6x^2 + 11x + 6$, if the length of the box is $x + 3$ cm, and its width is $x + 2$ cm then **determine** the height of the box. Also, **determine** the domain and range of the function V.

Solution to Question 1:

Considering v(x) has a degree of 3. We are also given the length and width, $x + 3$ and $x + 2$ respectively. This means that:

$$(x + 3)(x + 2)(x + a) = v(x)$$

Where $a \in \mathbb{R}$. We know that there are 3 factors because of the degree. Since:

$$v(x) = x^3 + 6x^2 + 11x + 6$$

$$\therefore (x + 3)(x + 2)(x + a) = x^3 + 6x^2 + 11x + 6 \quad \text{Combining the previously listed equations} \quad \textbf{(1)}$$

Dividing both sides by $(x + 3)(x + 2)$:

$$\begin{aligned}
(x + 3)(x + 2)(x + a) &= x^3 + 6x^2 + 11x + 6 \\
(x + a) &= \frac{x^3 + 6x^2 + 11x + 6}{(x + 3)(x + 2)} \\
(x + a) &= \frac{x^3 + 6x^2 + 11x + 6}{x^2 + 5x + 6} && \text{Expanding denominator bracket}
\end{aligned}$$

Now, using long division, find the final missing value factor, $(x + a)$:

$$\begin{array}{r|l}
 & \phantom{x^3 + 6x^2 + {}} x + 1 \\
\hline
x^2 + 5x + 6 & x^3 + 6x^2 + 11x + 6 \\
 & -x^3 - 5x^2 - 6x \\
\hline
 & \phantom{-x^3 - {}} x^2 + 5x + 6 \\
 & - x^2 - 5x - 6 \\
\hline
 & \phantom{x^3 + 6x^2 + 11x + {}} 0
\end{array}$$

$$\boxed{\therefore \text{the height of the box is x+1}}$$

Second part of solution on next page

Also since this relationship is of degree 3, it is a cubic function. Cubic functions have should the properties of a domain and range of all real whole numbers but since the dimensions of a of a 3D shape (Domain) cannot be negative in reality, the domain can only be equal to or greater than zero. When the domain is at zero, or f(0), we can calculate the minimum range value.

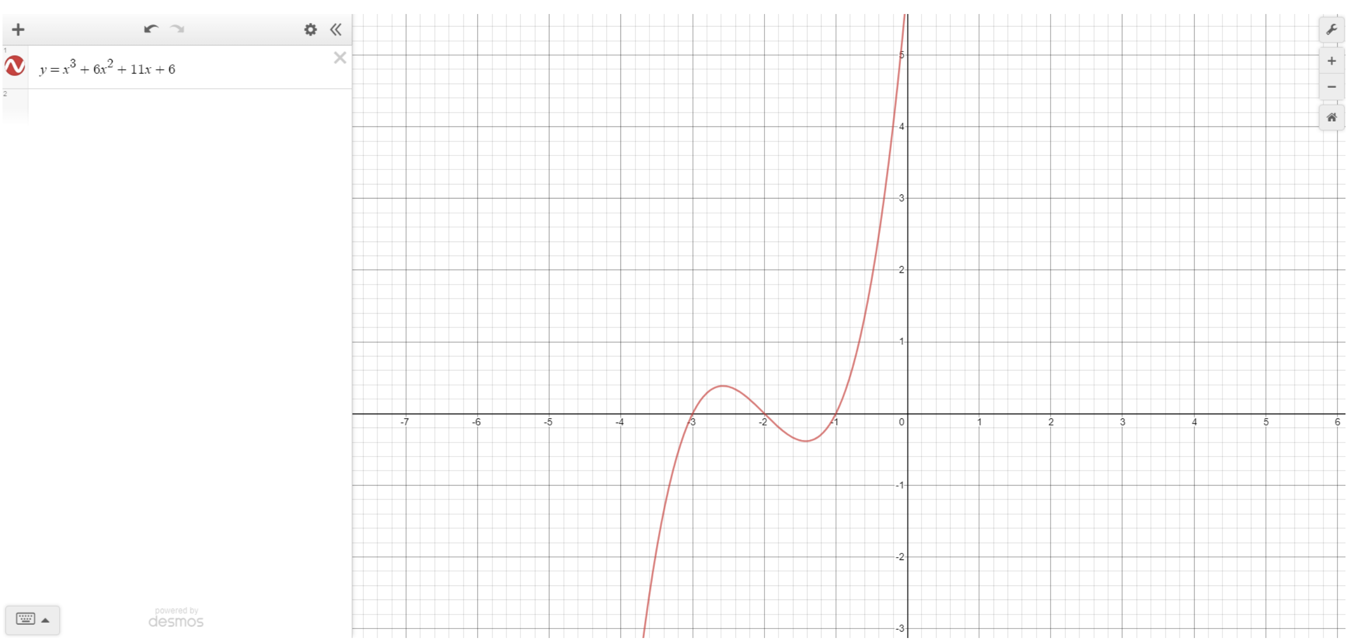
$$f(0) = 0^3 + 6 \times 0^2 + 11 \times 0 + 6 = 6$$

$$\boxed{\therefore D : \{x \in \mathbb{R} | x \geq 0\}, R : \{y \in \mathbb{R} | y \geq 6\}}$$

Check work for question 1 by graphing and using a logical check: From before, we stated that $(x + 3)(x + 2)(x + a) = x^3 + 6x^2 + 11x + 6$. Now that we know the value of a (1), redo the operation to check the work.

$$\begin{aligned}
x^3 + 6x^2 + 11x + 6 &= (x + 3)(x + 2)(x + a) && \text{From (1)} \\
x^3 + 6x^2 + 11x + 6 &= (x + 3)(x + 2)(x + 1) && a = 1 \\
&= (x^2 + 5x + 6)(x + 1) && \text{multiply first 2 factors and simplify} \\
&= x^3 + 6x^2 + 11x + 6 && \text{expand and simplify} \\
LHS &= RHS && \therefore \text{height is correct}
\end{aligned}$$

From the graph, we can also see that domain and ranges of the given function visually matches those determined by the solution.



2. Consider the quartic polynomial function $f(x) = x^4 - 5x^3 + x^2 + 21x - 18$. Given that there is a local minimum on $f(x)$ at $(-1, -32)$, use a logical argument to **prove** that this must be the absolute minimum of $f(x)$ without graphing the function.

Recall that $x = c$, where $c \in \mathbb{R}$, corresponds to a *local minimum* of a function $f(x)$ on an interval $I = (a, b)$ which contains c provided that for every value of x on the interval I we have that $f(c) \leq f(x)$.

Solution for question 2:

Proof. If we can create a sign table from this function we can better visualize the relaitonship and better prove the local minimum. First factor f(x) by finding the values of x where f(x) = 0. We know that factors are typically between -6 and 6. From the Rational Root Theorum we can also narrow down our search by using factors of the last term divided by the factors of the first term, plus minus. Possible candidates include x= -6, -3, -2, -1, 1, 2, 3 adnd 6.From experimentations:

$$f(1) = 0$$

$$f(3) = 0$$

$$f(-2) = 0$$

To find the 4th missing factor, divide f(x) by the current 3 known factors found:

$$\text{missing factor} = \frac{x^4 - 5x^3 + x^2 + 21x - 18}{(x-1)(x-3)(x+2)}$$

$$= \frac{x^4 - 5x^3 + x^2 + 21x - 18}{(x^2 - 4x + 3)(x+2)} \qquad \text{expand and simplify denominator}$$

$$= \frac{x^4 - 5x^3 + x^2 + 21x - 18}{(x^3 + 2x^2 - 4x^2 - 8x + 3x + 6}$$

$$= \frac{x^4 - 5x^3 + x^2 + 21x - 18}{(x^3 - 2x^2 - 5x + 6} \qquad \text{simplified}$$

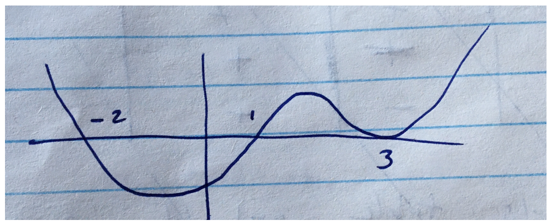
Now use long division

$$\begin{array}{r} x \quad - 3 \\ x^3 - 2x^2 - 5x + 6 \overline{) \; x^4 - 5x^3 \;\; + x^2 + 21x - 18} \\ - x^4 + 2x^3 + 5x^2 \;\; - 6x \qquad\quad \\ \hline - 3x^3 + 6x^2 + 15x - 18 \\ 3x^3 - 6x^2 - 15x + 18 \\ \hline 0 \end{array}$$

$\therefore$ in factor form $f(x) = (x-3)^2(x-1)(x+2)$. Since f(x) is positive (no negative vertical reflection) and utilizing the x intercepts given by the factor form:

	($-\infty$, -2)	(-2, 1)	(1, 3)	(3, ∞)
x + 2	$-$	+	+	+
x - 1	-	-	+	+
x - 3	-	-	-	+
x - 3	-	-	-	+
f(x)	+	-	+	+

Using the intervals from the sign table above and the known propterties of linear and quadratic factors, we can create a rough diagram:



We can agree that the minimum points have the lowest values, and theoretically with a positive quadric function there are two possible negative areas where the minimum is likely to reside within (since the negative interval sections are the lowest points). On the diagram these spots are the areas where rate of change goes from negative to positive (angled down proceeded by an angle up).

Lets check these two candidates. One of the possible minimums, where x=3, the line "skims" the x-axis. This means that this segment does not actually drop into the negative parts of the cartesian plane. This means that the only area that goes negative, at interval (-2, 1), is where the local minimum must reside. Since the x value x=-1 of the coordinate (-1, -32) is inside the previously identified interval, we can conclude that this coordinate is likely the local minimum. □

There is a possibility that 2 seperate local minimums can reside within the negative interval but there must be a turn point (local maximum) that goes from up to down within the interval. The properties do not support this idea because of the following logic:

Properties: We know that the function degree is 3 meaning that there are 3 and only 3 turning points. We also know that the relationship "skims" the x-axis only at the farthest right of the three intercepts: at x=3, since only the x-3 factor is squared and $3 > 1 > -2$.

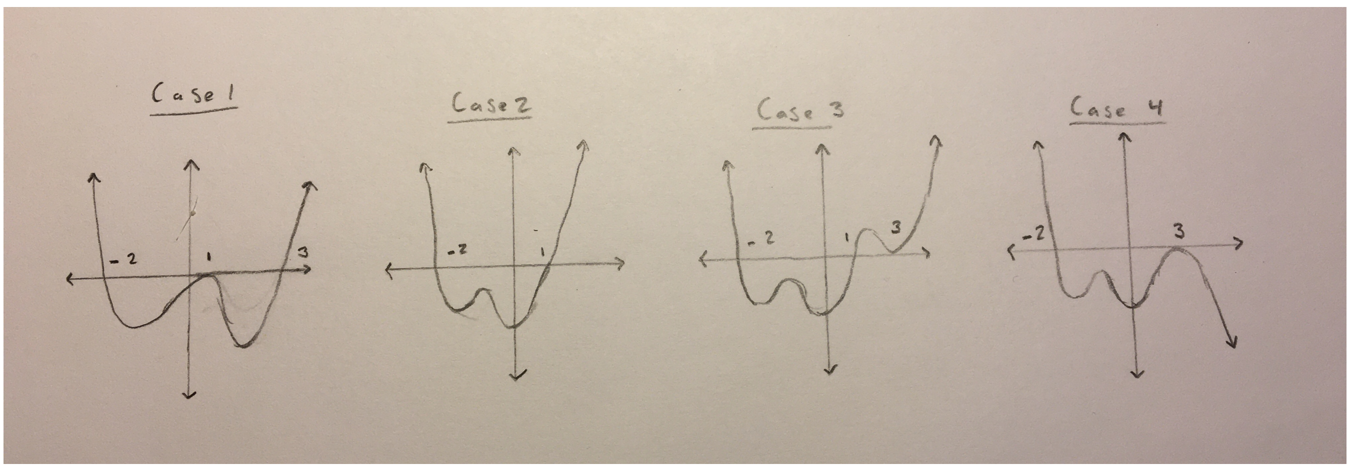
Remember, the local maximum must be less than or equal to zero to indicate a second local minimum

Case 1: if it skims the x-axis (y=0), the intercept that "skims" is in the middle at x=1, violating the second property.

Case 2: If the local maximum stays negative, not touching the x-axis, and then goes to the positive, there would be only 2 turning points with no indication of a quadratic factor, violating the first second property.

Case 3: If the local maximum stays negative and then goes to the positive before skiming the x-axis in the positive interval, like the real function, there wold be too many turning points, violating the first property.

Case 4: If the local maximum stays negative and then skims the x-axis in the negative interval before infinitly going negative, there would be too few turning points violating the first property.



From left to right, Case 1 to 4. All violate at least one property.

$\boxed{\therefore \text{2 local minimums within the negative interval is not possible}}$

3. Recall that a prime number is defined as an integer $n > 1$ such that its only positive divisors are 1 and n. Let $\mathbb{P}$ represent the set of prime numbers. Suppose that you know that when you divide $g(x) = x^3 + 2x^2 + cx + d$ by $x - 2$ you obtain a remainder of 14, **determine** the specific values of c and d given that $c \in \mathbb{Z}$ and $d \in \mathbb{P}$.

Solution to question 3: Since we know that $g(x)$ divided by $x - 2$ gives a remainder of 14, We

can compare try the division and compare the resulting polynomial remainder with the real value of 14. Since we know a factor is ax $x = 2$ (from $x - 2$) leads to a remainder or y value of 14, we can state that $g(2) = 14$.

$$
\begin{aligned}
g(x) &= x^3 + 2x^2 + cx + d && \\
g(2) &= 2^3 + 2 \times 2^2 + c \times 2 + d && \text{Substitute x as 2} \\
g(2) &= 8 + 8 + 2c + d && \text{Simplify} \\
g(2) &= 16 + 2c + d && \\
14 &= 16 + 2c + d && g(2) = 14 \\
-2 &= 2c + d && \text{Subtract 16 from both sides}
\end{aligned}
$$

Now that we have the equation $-2 = 2c + d$ and we know that $d \in \mathbb{P}$ and $c \in \mathbb{Z}$, consider the following.

In the equation $-2 = 2c + d$, c is multiplied by 2. This means that no matter the value of c, if it is an integer, the value of 2c must be even. 2c and d are added together to also create an even number: -2. When are two numbers are added together to an even number both numbers must be even or both must be odd. This means if 2c is even, d must also be even. The only prime number that exists is 2 since all other numbers that are even are divisible by 2. Therefore d must be 2. From this and the equation, we can determine the value of c.

$$
\begin{aligned}
-2 &= 2c + d && \\
-2 &= 2c + 2 && \text{Substitute d as 2} \\
-2 - 2 &= 2c && \text{subtract 2 to both sides} \\
-4 &= 2c && \text{Simplify} \\
\frac{-4}{2} &= c && \text{divide both sides by 2} \\
-2 &= c &&
\end{aligned}
$$

$\boxed{\therefore \text{ the value of c and d are -2 and 2 respectively.}}$

Check Answer for Question 3 by substituting the know values of c and d back into the original equation as g(2) = 14:

Proof.

$$g(x) = x^3 + 2x^2 + cx + d \qquad \text{Original equation}$$

$$g(x) = x^3 + 2x^2 - 2x + 2 \qquad \text{Since } c = -2, d = 2$$

$$g(2) = 2^3 + 2 \times 2^2 - 2 \times 2 + 2 \qquad \text{Substitute x as 2}$$

$$14 = 2^3 + 2 \times 2^2 - 2 \times 2 + 2 \qquad g(2) = 14$$

$$14 = 8 + 8 - 4 + 2$$

$$14 = 14$$

$$LHS = RHS$$

□

$\therefore$ the values of c and d found are correct

4. Solve $\dfrac{7}{x+2}+\dfrac{5}{x-2}=\dfrac{10x-2}{x^2-4}$.

Solution to question 4:

To solve isolate all values to one side of the equal sign before simplifying. Then, when fully simplified, take the numerator and solve for x.

$$\frac{7}{x+2}+\frac{5}{x-2}=\frac{10x-2}{x^2-4}$$

$$\frac{7}{x+2}+\frac{5}{x-2}-\frac{10x-2}{x^2-4}=0 \qquad \text{subtract } \frac{10x-2}{x^2-4} \text{ from both sides}$$

$$\frac{7}{x+2}+\frac{5}{x-2}-\frac{10x-2}{(x+2)(x-2))}=0 \qquad x^2-4=(x+2)(x-2)$$

$$\frac{7(x-2)}{(x+2)(x-2)}+\frac{5(x+2)}{(x-2)(x+2)}-\frac{10x-2}{(x+2)(x-2))}=0 \qquad \text{Get common denominators}$$

$$\frac{7(x-2)+5(x+2)-(10x-2)}{(x+2)(x-2)}=0 \qquad \text{Add all fractions together}$$

$$\frac{7x-14+5x+10-10x+2}{(x+2)(x-2)}=0 \qquad \text{Expand and Simplify}$$

$$\frac{2x-2}{(x+2)(x-2)}=0$$

Now take the numerator and solve for x:

$$\frac{2x-2}{(x+2)(x-2)}=0$$

$$2x-2=0$$

$$2x=2 \qquad \text{subtract 2 from both sides}$$

$$\therefore x=1 \qquad \text{divide all by 2}$$

$$\boxed{\therefore \text{the solution to } \frac{7}{x+2}+\frac{5}{x-2}=\frac{10x-2}{x^2-4} \text{ is } x=1}$$

Proof is on next page

prove question 4 by substituting x=1 back into the original equation:

Proof.

$$LHS = \frac{7}{x+2} + \frac{5}{x-2}$$

$$= \frac{7}{1+2} + \frac{5}{1-2} \qquad \text{Substitute x as 1}$$

$$= \frac{7}{3} + \frac{5}{-1} \qquad \text{Simplify}$$

$$= \frac{7}{3} + \frac{-15}{3}$$

$$LHS = -\frac{8}{3}$$

$$RHS = \frac{10x-2}{x^2-4}$$

$$= \frac{10 \times 1 - 2}{1^2 - 4} \qquad \text{Substitute x as 1}$$

$$= \frac{10-2}{1-4} \qquad \text{Simplify}$$

$$RHS = -\frac{8}{3}$$

$$RHS = LHS$$

□

$\therefore$ the answer is correct.

5. Solve $\left|\dfrac{x-4}{x+5}\right| \leq 4$.

Solution to question 5:

Consider the following identity of an absolute value inequality: $|x| \leq c \iff -c \leq x \leq c$. Lets apply this to the given equation where c is 4 and x is the absolute value rational function.

$$\left|\frac{x-4}{x+5}\right| \leq 4$$

$$-4 \leq \frac{x-4}{x+5} \leq 4 \qquad \text{applying identity}$$

Now solve the inequality found as if it were composed of two seperate inequalities with the rational function being repeated twice:

$$-4 \leq \frac{x-4}{x+5} \qquad (1)$$

$$\frac{x-4}{x+5} \leq 4 \qquad (2)$$

Take (1), $-4 \leq \dfrac{x-4}{x+5}$ and solve the inequality.

$$-4 \leq \frac{x-4}{x+5} \qquad (1)$$

$$0 \leq \frac{x-4}{x+5} + 4 \qquad \text{add 4 to both sides}$$

$$0 \leq \frac{x-4}{x+5} + \frac{(4)(x+5)}{x+5} \qquad \text{Get same denominator}$$

$$0 \leq \frac{x-4}{x+5} + \frac{(4x+20}{x+5} \qquad \text{Expand}$$

$$0 \leq \frac{x-4+4x+20}{x+5} \qquad \text{Add fractions}$$

$$0 \leq \frac{5x+16}{x+5} \qquad \text{Simplify}$$

To solve the rational inequality, we create a sign table to determine the appropriate intervals.

	$(-\infty, -5)$	$\left(-5, -\frac{16}{5}\right)$	$\left(-\frac{16}{5}, \infty\right)$
5x+16	-	-	+
x+5	-	+	+
f(x)	+	-	+

Since we are searching for values greater than or equal to zero, look for the intervals where the value is positiive or zero. We can see from the sign table that, when including the values at zero:

$$x \in (-\infty, -5], \left[-\frac{16}{5}, \infty\right) \qquad (3)$$

Take (2), $\frac{x-4}{x+5} \leq 4$ and solve the inequality.

$$\frac{x-4}{x+5} \leq 4 \qquad \text{(2)}$$

$$\frac{x-4}{x+5} - 4 \leq 0 \qquad \text{subtract 4 to both sides}$$

$$\frac{x-4}{x+5} - \frac{(4)(x+5)}{x+5} \leq 0 \qquad \text{Get same denominator}$$

$$\frac{x-4}{x+5} - \frac{4x+20}{x+5} \leq 0 \qquad \text{Expand}$$

$$\frac{x-4-(4x+20)}{x+5} \leq 0 \qquad \text{Combibe fractions}$$

$$\frac{x-4-4x-20}{x+5} \leq 0 \qquad \text{expand}$$

$$\frac{-3x-24}{x+5} \leq 0 \qquad \text{Simplify}$$

$$\frac{3x+24}{x+5} \geq 0 \qquad \text{Multiply by -1 to all, flip inequality sign}$$

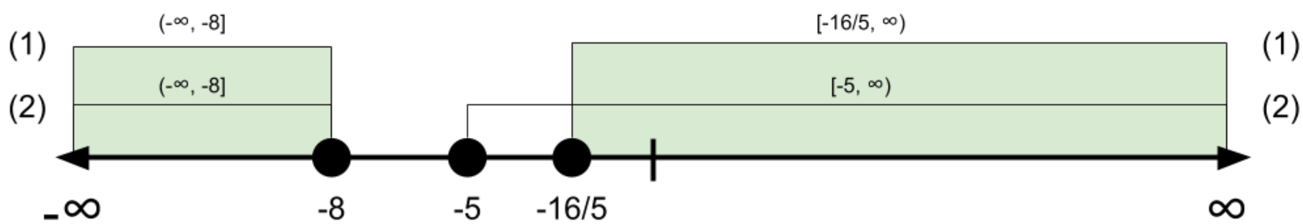
To solve the rational inequality, we create a sign table to determine the appropriate intervals.

	$(-\infty, -8)$	$(-8, -5)$	$(-5, \infty)$
3x+24	-	+	+
x+5	-	-	+
f(x)	+	-	+

Since we are searching for values greater than or equal to zero, look for the intervals where the value is positiive or zero. We can see from the sign table that, when including the values at zero:

$$x \in (-\infty, -8], [-5, \infty) \qquad (4)$$

Now let's combine the intervals found from (1) and (2). We can do this by plotting the intervals on a line diagram and look for the intervals where (3) and (4) overlap. In essence we need to find intervals that satisfy both inequalities, which is why we look for overlap.

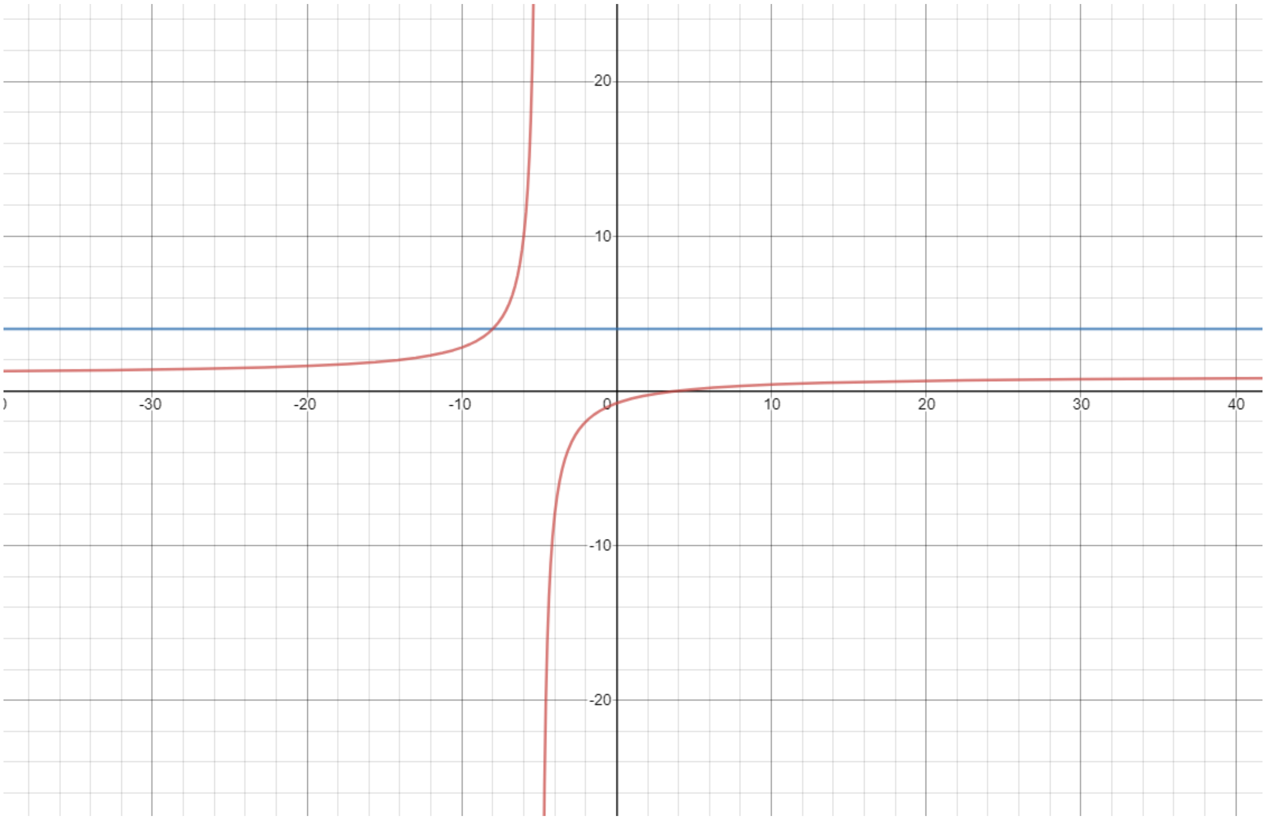


From the line diagram, we can see that the 4 intervals overlap at 2 areas, highlighted by the green.

$$\boxed{\therefore \text{ the intervals that satisfy the inequality} \left|\frac{x-4}{x+5}\right| \leq 4 \text{ are } (-\infty, -8] \cup \left[-\frac{16}{5}, \infty\right).}$$

Proof is on the next page

Prove the intervals found in question 5 with a graph:



From the graph we can see that the intervals found seem to intervals on the graph that exist below or equal to y=4.