Pre-Read Notes

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1 Chapter 1: Limits and Continuity

1.1 Tangent Lines

- Limit binds many concepts of Calculus together
- With $\frac{\delta y}{\delta x}$, you can approximate slopes
- But with curves, it no longer works well
- The close you zoom in, say $\rightarrow \infty$ zoom creates a straigh-ish line who's slope can now be approximated
- Recall calculating distance between 2 points is $\sqrt{(\delta y)^2 + (\delta x)^2}$
- Adding succesive line segments will improve accuracy

1.2 Limits

- Given a function f(x) = L where x gets close to a value a, this would be defined as "L is the limit of f(x) as x appraoches a"
- $\lim_{x\to a} f(x) = L$
- Restriction at x=a
- $\lim_{x\to a} f(x) = L$ Exists if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

1.3 Computation of Limits

THEOREM 3.1

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and let c be any constant. The following then apply:

(i) $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$,

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$ and

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ $\left(\text{if } \lim_{x \to a} g(x) \neq 0\right)$.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a,

$$\lim_{x \to a} p(x) = p(a).$$

THEOREM 3.3

Suppose that $\lim_{x \to a} f(x) = L$ and n is any positive integer. Then,

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L},$$

where for n even, we must assume that $L > 0$.

THEOREM 3.4

For any real number a, we have

(i) $\lim_{x \to a} \sin x = \sin a$,

(ii) $\lim_{x \to a} \cos x = \cos a$,

(iii) $\lim_{x \to a} e^x = e^a$ and

(iv) $\lim_{x \to a} \ln x = \ln a$, for $a > 0$.

(v) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a$, for $-1 < a < 1$,

(vi) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(vii) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a$, for $-\infty < a < \infty$ anc

(viii) if p is a polynomial and $\lim_{x \to p(a)} f(x) = L$, then $\lim_{x \to a} f(p(x)) = L$.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x \to a} g(x) = L, \text{ also.}$$

1.4 Continuity and Its Consequences

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THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and e^x are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x > 0$, when n is even. We also have that $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

THEOREM 4.2

Suppose that f and g are continuous at $x = a$. Then all of the following are true:

(i) $(f \pm g)$ is continuous at $x = a$,
(ii) $(f \cdot g)$ is continuous at $x = a$ and
(iii) (f/g) is continuous at $x = a$ if $g(a) \ne 0$.

THEOREM 4.3

Suppose that $\lim_{x \to a} g(x) = L$ and f is continuous at L. Then,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b), we say that f is **continuous on (a, b).** Following Figure 1.27, we say that f is **continuous on the closed interval $[a, b]$,** if f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous.** (That is, when we don't specify an interval, we mean continuous everywhere.)

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a *zero* of f.)

1.5 Limits Involving Infinity: Asymptotes

- We can express Asymptotes as Limits
- Ex: Reciprocal function

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$