MCV4U : Calculus and Vectors
MCV4UR : Advanced Placement Calculus and Vectors

Assignment #1

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT, RRT, MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

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Declared References:

- Used Desmos to visualize y-intercept equation in Q14.

Instructions

1. Organize and express complete, effective and concise responses to each problem.
2. Use appropriate mathematical conventions and notation wherever possible.
3. Provide logical reasoning for your arguments and cite any relevant theorems.
4. The quality of your communication will be heavily weighted.

Evaluation

- Students will demonstrate an understanding of limits and apply them to solve problems.

Criteria	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Understanding of Mathematical Concepts	Demonstrates limited understanding	Demonstrates some understanding	Demonstrates considerable understanding	Demonstrates thorough understanding of concepts
Selecting Tools and Strategies	Selects and applies appropriate tools and strategies, with major errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies, with minor errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies accurately, and in a logical sequence	Selects and applies appropriate and efficient tools and strategies accurately to create mathematically elegant solutions
Reasoning and Proving	Inconsistently or erroneously employs logic to develop and defend statements	Statements are developed and defended with some omissions or leaps in logic	Frequently develops and defends statements with reasonable logical justification	Consistently develops and defends statements with sophisticated and/or complete logical justification
Communicating	Expresses and organizes mathematical thinking with limited effectiveness	Expresses and organizes mathematical thinking with some effectiveness	Expresses and organizes mathematical thinking with considerable effectiveness	Expresses and organizes mathematical thinking with a high degree of effectiveness

1. The sum $3 - \frac{3}{10} + \frac{3}{100} - \frac{3}{1000} + \cdots + \frac{(-1)^k 3}{10^k} + \cdots$ is closest to:
   (a) $\frac{2}{e}$
   (b) 2
   (c) e
   (d) e^2

2. The value of $\lim_{x \to a} \frac{x-a}{\sqrt{x} - \sqrt{a}}$ is:
   (a) $2a$
   (b) $2\sqrt{a}$
   (c) $\sqrt{2a}$
   (d) a

3. If $\lim_{x \to c} f(x) = L$ where $L \in \mathbb{R}$ then which of the following must be true?
   (a) f is defined at $x = c$
   (b) f is continuous at $x = c$
   (c) $f(c) = L$
   (d) None of the above

4. Given f is continuous on $[-2, 6]$, if $f(-2) = 7$ and $f(6) = -1$ then the Intermediate Value Theorem guarantees that:
   (a) $f(0) = 0$
   (b) $f(c) = 2$ for at least one value of $c \in [-2, 6]$
   (c) $f(c) = 0$ for at least one value of $c \in [-1, 7]$
   (d) $\forall x \in [-2, 6], -1 \leq f(x) \leq 7$

5. The value of $\lim_{x \to 0} \frac{\sin(4x)}{2x}$ is:
   (a) 0
   (b) 0.5
   (c) 1
   (d) 2

6. The value of $\lim_{x \to -\infty} \frac{\sqrt{8x^2 - 4x}}{x + 2}$ is:
   (a) $-\infty$
   (b) $-2\sqrt{2}$
   (c) 4
   (d) $2\sqrt{2}$

7. The value of k such that $f(x) = \begin{cases} x^2 + 2 & : x \leq -1 \\ kx + 4 & : x > -1 \end{cases}$ is continuous on $\mathbb{R}$ is:

   (a) -3
   (b) -1
   (c) 1
   (d) 3

8. For what value of a is the function $f(x) = \dfrac{x^2 - 3x}{x - a}$ continuous on $\mathbb{R}$?

   (a) -3
   (b) -1
   (c) 3
   (d) There does not exist a value of a such that f is continuous on $\mathbb{R}$.

9. The value of $\lim\limits_{x \to 0} \dfrac{e^{ax} \sin(bx) \tan(cx)}{x^2}$, where $a, b, c \in \mathbb{R} \setminus \{0\}$ is:

   (a) $\frac{a}{bc}$
   (b) bc
   (c) $\frac{ab}{c}$
   (d) ab

10. The value of $\lim\limits_{x \to 0} x^3 \sin\left(\dfrac{1}{x}\right)$ is:

   (a) -1
   (b) 0
   (c) 1
   (d) nonexistent

Multiple Choice Answers

1. C
2. B
3. D
4. B
5. D
6. B
7. C
8. D
9. B
10. B

Fill in your responses to the multiple choice questions above. Be sure to use capital letters.

11. If $f(x) = 3x^3 - 2x + 1$ then **determine** $f'(x)$ from *First Principles* by using the definition of the derivative.

Solution To Question 11:

Given the function $f(x) = 3x^2 - 2x + 1$, we can solve for $f'(x)$ using the first principal definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{First Principle Definition}$$

$$f'(x) = \lim_{h\to 0} \frac{(3(x+h)^3 - 2(x+h) + 1) - (3x^3 - 2x + 1)}{h} \qquad \text{Implement f(x) with x+h and x}$$

$$f'(x) = \lim_{h\to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) - (2x + 2h) + 1 - 3x^3 + 2x - 1}{h} \qquad \text{Expand and Simplify}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^3 + 9x^2h + 9xh^2 + 3h^3 - 2x - 2h + 1 - 3x^3 + 2x - 1}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{9x^2h + 9xh^2 + 3h^3 - 2h}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h(9x^2 + 9xh + 3h^2 - 2)}{h} \qquad \text{Factor h}$$

$$f'(x) = \lim_{h\to 0} 9x^2 + 9xh + 3h^2 - 2 \qquad \text{Simplify, Cancel out h}$$

$$f'(x) = 9x^2 + 9x(0) + 3(0)^2 - 2 \qquad \text{Apply Limit}$$

$$f'(x) = 9x^2 - 2$$

$$\boxed{\text{Therefore, using the first principles we find that the derivative of f(x) is } 9x^2 - 2.}$$

12. **Evaluate** $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$ or assert that it does not exist and provide evidence to support your conclusion.

Solution To Question 12:
When you look at the function within the limit, the first thing that might come to mind is the square root being subtracted to something. If we used 1 rule in order to rationalize, we can elimite the square root and unlock the possibility for futher simplification.

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$$

$$=\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{\sqrt{1+t}}{t\sqrt{1+t}}\right) \qquad \text{Crossmultiply, Get Same Denominators}$$

$$=\lim_{t\to 0}\left(\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right) \qquad \text{Simplify, Add Fractions Together}$$

$$=\lim_{t\to 0}\left(\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\right) \qquad \text{Use 1 Rule, Rationalize}$$

$$=\lim_{t\to 0}\left(\frac{1-(1+t)}{(t\sqrt{1+t})(1+\sqrt{1+t})}\right) \qquad \text{Simplify, } (a+b)(a-b)=a^2-b^2$$

$$=\lim_{t\to 0}\left(\frac{-t}{(t\sqrt{1+t})(1+\sqrt{1+t})}\right) \qquad \text{Simplify}$$

$$=\lim_{t\to 0}\left(\frac{-1}{(\sqrt{1+t})(1+\sqrt{1+t})}\right) \qquad \text{Cancel t out}$$

$$=\frac{-1}{(\sqrt{1+0})(1+\sqrt{1+0})} \qquad \text{Apply Limit}$$

$$=\frac{-1}{(1)(2)} \qquad \text{Simplify}$$

$$=-\frac{1}{2}$$

$$\boxed{\text{Therefore, the limit, when evaluated, is } -\frac{1}{2}}$$

13. **Evaluate** $\lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x}$ where $c \in \mathbb{R}^*$.

Solution To Question 13:
Just like the previous question, we can see that there is some sort of root (cubed this time) which is being subtracted. If we could rationaize this, possibly with a cubic identity, we could potentially gain opportunities to simplify. Given the cubic identity $(a-b)(a^2+ab+b^2) = a^3 - b^3$, we can possibly use this to eliminate the cube root.

$$\lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x}$$

$$= \lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x} \cdot \frac{(\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2}{(\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2} \qquad \text{Use 1 rule \& cubic identity, rationalize}$$

$$= \lim_{x\to 0} \frac{(\sqrt[3]{1+cx}-1)((\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2)}{(x)((\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2)} \qquad \text{Simplify}$$

$$= \lim_{x\to 0} \frac{(\sqrt[3]{1+cx})^3 - 1^3}{(x)((\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2)} \qquad \text{Apply Cubic identity where } a = \sqrt[3]{1+cx}, b = 1$$

$$= \lim_{x\to 0} \frac{1 + cx - 1}{(x)((\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2)} \qquad \text{Simplify}$$

$$= \lim_{x\to 0} \frac{cx}{(x)((\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2)}$$

$$= \lim_{x\to 0} \frac{c}{(\sqrt[3]{1+cx})^2 + 1(\sqrt[3]{1+cx}) + 1^2} \qquad \text{Cancel out x}$$

$$= \frac{c}{(\sqrt[3]{1+c(0)})^2 + 1(\sqrt[3]{1+c(0)}) + 1^2} \qquad \text{Apply Limit}$$

$$= \frac{c}{(1)^2 + 1(1) + 1} \qquad \text{Simplify}$$

$$= \frac{c}{3}$$

$$\boxed{\text{Therefore, when evaluated the value of the limit is } \frac{c}{3} \text{ For } c \in \mathbb{R}^*}$$

14. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, **determine** the limiting position of Q.

Solution To Question 14:

Let's first look at the three important points that tell us information about Q: O, P and the midpoint. O is at (0, 0). Considering the function $f(x) = x^2$, we can say that if P is some point of the porabola, P is $(x, f(x)) = (x, x^2)$. The midpoint (M) is halfway between O and P or: $(x/2, x^2/2)$. First, start by getting the slope of line OP.

$$\begin{aligned} slope &= \frac{\Delta y}{\Delta x} && \text{Standard slope formula} \\ &= \frac{x^2 - 0}{x - 0} && \text{Substitute points in} \\ &= x \end{aligned}$$

Since we know the slope of the perpundicular line is the negative reciprocal of the slope of the original line, if the slope of the original is x, the the negative reciprocal is $-1/x$. Just to make sure not to confuse variables later on (confusing x from the slope and x from the linear equation), let $h = x$. The perpundicular line's slope is $-1/h$. From this we can form a rough linear equation and then substitute the midpoint, which is the only known point on the perpindicular line to find the y-intercept. We know that a linear function is formed from $y = mx + b$ where Where m is slope, b is y-intercept. Since $h = x$ we can also define the midpoint in terms of h: $(x/2, x^2/2) = (h/2, h^2/2)$

$$\begin{aligned} y &= mx + b && \text{Where m is slope, b is y-int} \\ y &= -\frac{1}{h}x + b && \text{Sub slope in} \\ \frac{h^2}{2} &= -\frac{1}{h} \cdot \frac{h}{2} + b && \text{Sub in midpoint: } \left(\frac{h}{2}, \frac{h^2}{2}\right) \text{ as (x,y)} \\ \frac{h^2}{2} &= -\frac{1}{2} + b && \text{Simplify} \\ \frac{h^2}{2} + \frac{1}{2} &= b && \text{Add } \frac{1}{2} \text{ to both sides} \\ \frac{h^2 + 1}{2} &= b && \text{Simplify} \\ \frac{x^2 + 1}{2} &= b && \text{Note: x=h} \end{aligned}$$

Now we have the y-intercept or b. The b value, since it is defined by x, can represent the specific point the that the line intercepts the y-axis at a specific x. In essence, it represents the y-component of Q (since the x-component of Q is always 0). Notice how b is quadratic-like. This shows that as point P approaches O, the value Q gets smaller and smaller until reaching a limiting position. If we take the limit of x approaching 0 for b, we can find this limiting value.

$$
\begin{aligned}
Q_{\text{y at x=0}} &= \lim_{x \to 0} \frac{x^2+1}{2} && Q_y = b = \frac{x^2+1}{2} \\
&= \frac{0^2+1}{2} && \text{Apply limit} \\
&= \frac{1}{2}
\end{aligned}
$$

$$\boxed{\therefore \text{ as the point P approaches point O, the point Q decreases until reaching a limiting value of } \frac{1}{2}}$$