MCV4U : Calculus and Vectors
MCV4UR : Advanced Placement Calculus and Vectors

Assignment #2

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT, RRT, MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

Family Name:
First Name:

Declared References:

Instructions

1. Organize and express complete, effective and concise responses to each problem.
2. Use appropriate mathematical conventions and notation wherever possible.
3. Provide logical reasoning for your arguments and cite any relevant theorems.
4. The quality of your communication will be heavily weighted.

Evaluation

- Students will demonstrate an understanding of derivative rules and use them to solve problems.

Criteria	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Understanding of Mathematical Concepts	Demonstrates limited understanding	Demonstrates some understanding	Demonstrates considerable understanding	Demonstrates thorough understanding of concepts
Selecting Tools and Strategies	Selects and applies appropriate tools and strategies, with major errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies, with minor errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies accurately, and in a logical sequence	Selects and applies appropriate and efficient tools and strategies accurately to create mathematically elegant solutions
Reasoning and Proving	Inconsistently or erroneously employs logic to develop and defend statements	Statements are developed and defended with some omissions or leaps in logic	Frequently develops and defends statements with reasonable logical justification	Consistently develops and defends statements with sophisticated and/or complete logical justification
Communicating	Expresses and organizes mathematical thinking with limited effectiveness	Expresses and organizes mathematical thinking with some effectiveness	Expresses and organizes mathematical thinking with considerable effectiveness	Expresses and organizes mathematical thinking with a high degree of effectiveness

Instructions: Answer True or False to the following statements.

1. If f and g are differentiable then $\frac{d}{dx}(f+g) = f' + g'$.
2. If f and g are differentiable then $\frac{d}{dx}(fg) = f'g + 2f'g' + fg'$.
3. If f and g are differentiable then $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$.
4. If f is differentiable then $\frac{d}{dx}(\sqrt{f(x)}) = \frac{f'(x)}{2\sqrt{x}}$.
5. The derivative of a polynomial function is a polynomial function.
6. The derivative of a rational function is a rational function.
7. The derivative of an exponential function is an exponential function.
8. The derivative of a sinusoidal function is a sinusoidal function.
9. The derivative of a logarithmic function is a logarithmic function.
10. If $f(x) = (x^p - 1)^q$ where $p, q \in \mathbb{Z}^+$ then $-1 \leq f^{(pq+1)}(x) \leq +1$.
11. If $f'(c) = 0$, then f has a local maximum or minimum at $x = c$.
12. If f has an absolute minimum at $x = c$ then $f'(c) = 0$.
13. If f and g are increasing on an interval I then $f + g$ is also increasing on I.
14. If f and g are increasing on an interval I then fg is also increasing on I.
15. If f is periodic then f' is periodic.
16. If f is even then f' is odd.
17. There exists a function g such that $g(x) > 0$, $g'(x) < 0$, and $g''(x) > 0$ for all $x \in \mathbb{R}$.
18. There exists a function h such that $h(x) < 0$, $h'(x) < 0$, and $h''(x) > 0$ for all $x \in \mathbb{R}$.
19. If $f(x) = \sin(x)$ then the least value of $n \in \mathbb{Z}^+$ such that $f^{(n)}(x) = f(x)$ is $n = 4$.
20. If $f'(x) = g'(x)$ for $0 < x < 1$ then $f(x) = g(x)$ for $0 < x < 1$.

True-False Answers

1. true
2. false
3. true
4. false
5. true
6. true
7. false
8. false
9. false
10. true
11. false
12. false
13. true
14. true
15. true
16. true
17. true
18. false
19. true
20. false

Fill in your responses to the True-False questions above. Be sure to use only capital T or F.

21. **Prove** that the tangent lines to the graph of the function $y = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}, a \neq 0$, at any two points with x-coordinates p and q, where $p, q \in \mathbb{R}$, must intersect at a point whose x-coordinate is $\dfrac{p+q}{2}$.

Solution to Question 21:
What we need to do is solve for the equations of the 2 tangent lines (linear equations), then solve their intercept. Consider the constant k. This is a stand-in for p or q. Pretty much $k \in \mathbb{R}$. Let's try and solve for the slope first. We know that the slope comes from the derivative evaluated at a certain point, like k.

$y = ax^2 + bx + c$	base quadratic function
$y' = 2ax + b$	Get derivative
$y' = 2ak + b$	Evaluate at x=k

Now that we have our slope, we can substitute the slope into the standard linear equation $y = mx+i$. where m represents the slope and i (because b is already used) represents the y-intercept. Next, we need to determine the y-intercept. We can find i by evaluating the linear equation at a point (x,y) and isolate for i. First, we need to find a point. Given the $x = k$, we can say the following:

$y = ax^2 + bx + c$	base quadratic function
$y = ak^2 + bk + c$	Evaluate at x=k

With the coordinate $(k, ak^2 + bk + c)$, which is where the tangent line meets the quadratic equation, we can now solve for i:

$y = mx + i$	standard linear equation formula
$y = (2ak + b)x + i$	Substitute slope in $m = 2ak + b$
$ak^2 + bk + c = (2ak + b)k + i$	Substitute coordinate in $(x, y) = (k, ak^2 + bk + c)$
$ak^2 + bk + c = 2ak^2 + bk + i$	Expand
$ak^2 + bk + c - 2ak^2 - bk = i$	Simplify
$-ak^2 + c = i$	Simplify

Therefore, now that we have the y-intercept, we can substitute this back into the linear equation. We now have $y = (2ak + b)x - ak^2 + c$. Since k was a fill in for p and q, why dont we re-evaluate the linear equation found using p and q.

$$\begin{cases} y = (2aq + b)x - aq^2 + c \quad \textbf{(1)} \quad k = p \\ y = (2aq + b)x - aq^2 + c \quad \textbf{(2)} \quad k = q \end{cases}$$

Now solve by substituting one into the other using y. Isolating for x will result in the x-coordinate of the intercept.

$$(2ap + b)x - ap^2 + c = (2aq + b)x - aq^2 + c \qquad \text{Sub (1) into (2) as y}$$

$$2apx + bx - ap^2 + c = 2aqx + bx - aq^2 + c \qquad \text{Expand}$$

$$2apx - ap^2 = 2aqx - aq^2 \qquad \text{Simplify}$$

$$2px - p^2 = 2qx - q^2 \qquad \text{Divide all by a}$$

$$2px - 2qx = -q^2 + p^2 \qquad \text{Subtract } 2qx \text{ and } p^2 \text{ from both sides}$$

$$2x(p - q) = -q^2 + p^2 \qquad \text{factor 2x}$$

$$x = \frac{-q^2 + p^2}{2(p - q)} \qquad \text{Divide all by 2(p-q)}$$

$$x = \frac{p^2 - q^2}{2(p - q)} \qquad \text{Rearrange numerator}$$

$$x = \frac{(p + q)(p - q)}{2(p - q)} \qquad \text{Quadtratic identity} p^2 - q^2 = (p + q)(p - q)$$

$$x = \frac{p + q}{2} \qquad \text{Simplify} \quad \square$$

$\boxed{\text{Therefore the original statement from the question is correct.}}$ If 2 tangent lines are taken from $y = ax^2 + bx + c$ at x = q and p, the tangents will intersect at x = p+q/2.

22. **Determine** the derivative of the function $f(x) = \ln\left(\dfrac{xe^{-x} + \sin(4x)}{1 + 6\sqrt{2x}}\right)$. Go one step at a time and justify each step. Do *not* simplify your answer.

Solution to Question 22:

$$f(x) = \ln\left(\frac{xe^{-x} + \sin(4x)}{1 + 6\sqrt{2x}}\right)$$

$$f'(x) = \frac{1}{\frac{xe^{-x} + \sin(4x)}{1 + 6\sqrt{2x}}} \times \frac{d}{dx}\left(\frac{xe^{-x} + \sin(4x)}{1 + 6\sqrt{2x}}\right) \qquad \text{Chain Rule}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{d}{dx}\left(\frac{xe^{-x} + \sin(4x)}{1 + 6\sqrt{2x}}\right) \qquad \text{Simplify}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\frac{d}{dx}(xe^{-x} + \sin(4x))(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))\frac{d}{dx}(1 + 6\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Quotient Rule}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + \frac{d}{dx}\sin(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))(\frac{d}{dx}1 + \frac{d}{dx}6\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Distrubute derivative}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + \frac{d}{dx}\sin(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))(\frac{d}{dx}6\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Apply derivative to 1}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + \frac{d}{dx}4x \times \cos(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))(\frac{d}{dx}6\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Chain Rule to sin(4x)}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + 4\cos(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))(\frac{d}{dx}6\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Power Rule, Simplify}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + 4\cos(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))(6\frac{d}{dx}\sqrt{2x})}{(1 + 6\sqrt{2x})^2} \qquad \text{Constant Multiple Rule}$$

$$= \frac{1 + 6\sqrt{2x}}{xe^{-x} + \sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x} + 4\cos(4x)\right)(1 + 6\sqrt{2x}) - (xe^{-x} + \sin(4x))\left(6\frac{d}{dx}2x \times -\frac{1}{\sqrt{2x}}\right)}{(1 + 6\sqrt{2x})^2} \qquad \text{Chain Rule}$$

$$f'(x) = \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x}+4\cos(4x)\right)(1+6\sqrt{2x}) - (xe^{-x}+\sin(4x))\left(6\times 2\times -\frac{1}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Power Rule}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{\left(\frac{d}{dx}xe^{-x}+4\cos(4x)\right)(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Simplify}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{\left(\left(\frac{d}{dx}x\times e^{-x}+x\times\frac{d}{dx}e^{-x}\right)+4\cos(4x)\right)(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Product Rule}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{\left(\left(1\times e^{-x}+x\times\frac{d}{dx}e^{-x}\right)+4\cos(4x)\right)(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Power Rule}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{\left(\left(1\times e^{-x}+x\times\frac{d}{dx}-x\times e^{-x}\right)+4\cos(4x)\right)(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Chain Rule}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{((1\times e^{-x}+x\times -1\times e^{-x})+4\cos(4x))(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Power Rule}$$

$$= \frac{1+6\sqrt{2x}}{xe^{-x}+\sin(4x)} \times \frac{((e^{-x}-xe^{-x})+4\cos(4x))(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})^2} \qquad \text{Simplify}$$

$$= \frac{1}{xe^{-x}+\sin(4x)} \times \frac{((e^{-x}-xe^{-x})+4\cos(4x))(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})}$$

$$f'(x) = \frac{((e^{-x}-xe^{-x})+4\cos(4x))(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})(xe^{-x}+\sin(4x))}$$

$$\boxed{\therefore f'(x) = \frac{((e^{-x}-xe^{-x})+4\cos(4x))(1+6\sqrt{2x}) + (xe^{-x}+\sin(4x))\left(\frac{12}{\sqrt{2x}}\right)}{(1+6\sqrt{2x})(xe^{-x}+\sin(4x))}}$$

23. **Prove** that if $f(x)$ is a cubic polynomial function with zeros a, b, c then the x-coordinate of the inflection point of $f(x)$ is $\frac{a+b+c}{3}$.

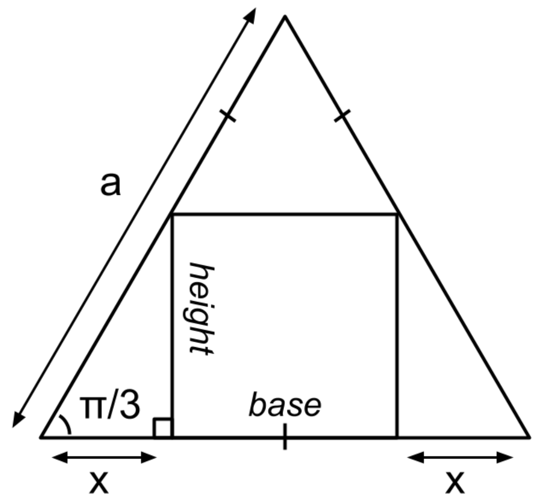
Solution to Question 23:
First off, we can find the inflection points when the second derivative has a zero. We can start by determining f(x) before finding the second derivative. Since we know the cubic function has zeroes at a, b, c, the function must be $f(x) = (x-a)(x-b)(x-c)$. Now we can expand, simplify and then solve for the first and second derivative.

$f(x) = (x-a)(x-b)(x-c)$	Base Cubic Function
$f(x) = (x^2 - ax - bx + ab)(x-c)$	Expand, multiply brackets
$f(x) = x^3 - ax^2 - bx^2 + abx - cx^2 + acx + bcx - abc$	
$f(x) = x^3 - x^2(a+b+c) + x(ab+bc+ac) - abc$	Factor $-x^2$ and x
$f'(x) = 3x^2 - 2x(a+b+c) + ab + bc + ac - 0$	Apply Power Rule, get 1st derivative
$f''(x) = 6x - 2(a+b+c) + 0$	Apply Power Rule again, get 2nd derivative
$0 = 6x - 2(a+b+c)$	Solve for x with f”(x) = 0 to find inflection
$2(a+b+c) = 6x$	Add 2(a+b+c) to both sides
$\frac{2(a+b+c)}{6} = x$	Divide al by 6
$\frac{a+b+c}{3} = x$	Simplify

$\boxed{\therefore\text{, the x-coord for the inflection is (a+b+c)/3}}$

24. Determine the maximum area of a rectangle inscribed within an equilateral triangle that has a side length of a where $a \in \mathbb{R}^+$.

Solution to Question 24:
First off, let's draw a diagram from the information we know. Since the triangle is equilateral, all sides of the triangle are of the length a and every internal angle is pi/3 radians (60 degrees).



Now if we form an equation that represents the area of the rectangle before taking the derivative of this equation, we can find a global maximum. Let x be the distance between the leftmost corner of the diagram and the leftmost side of the rectangle. Since the triangle is equilateral, on the rightmost side there is another x. If we take the total base of the triangle, a, and subtract 2x out, we get the base of the rectangle.

$$base = a - 2x \quad \mathbf{(1)}$$

For the height of the rectangle, first consider the 30-60-90 triangle where the 60 degrees is labeled in the diagram. Since we know the angle and the base of the triangle is x, we can use tan(x) to find the height.

$$tan(\theta) = \frac{opp}{adj}$$

$$tan\left(\frac{\pi}{3}\right) = \frac{height}{x} \qquad \theta = \frac{\pi}{3}, adj = x, opp = height$$

$$x \times tan\left(\frac{\pi}{3}\right) = height \qquad \text{multiply x on both sides}$$

$$\sqrt{3}x = height \qquad \mathbf{(2)} \text{ By special triangle, } tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Now that we know the base and height of the rectangle, we can calculate create an equation to express the area before finding the derivative and the global maximum. Let A be area.

$A = base \times height$	Area of a rectangle
$A = (a - 2x)(\sqrt{3}x)$	Sub (1) and (2) in as base and height
$A = \sqrt{3}ax - 2\sqrt{3}x^2$	Expand
$A' = \sqrt{3}a - 4\sqrt{3}x$	**(3)** Use Power Rule, get derivative
$0 = \sqrt{3}a - 4\sqrt{3}x$	Solve for A' = 0 for critical points
$4\sqrt{3}x = \sqrt{3}a$	Add $4\sqrt{3}x$ to both sides
$4x = a$	Divide $\sqrt{3}$ from both sides
$x = \frac{a}{4}$	Divide 4 from both sides

Now that we have a critial number, we need to test to see if it the value really is the absolute maximum. Apply product rule to (3) again to get the second derivative for a second derivative test.

$A' = \sqrt{3}a - 4\sqrt{3}x$	**(3)**
$A'' = -4\sqrt{3}$	Apply product rule

The second derivative is always negative. This means at x = a/4, according to the second derivative test, the point will be the relative maximum. Also thinking logically for a moment, the area equation is a negative quadtratic meaning the vertex (which we found) will be the highest point.

$\boxed{\therefore\text{, The maximum area is achieved at x = a/4}}$