

MHF4U : Advanced Functions

Assignment #3

Reference Declaration Complete the Reference Declaration section below in order for your assignment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with peers or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT (Factor Theorem), RRT (Rational Root Theorem), etc.

Note: Your submitted work must be **your original work**.

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Declared References:

1. Prove the identity $\sin(x) \cos(x) = 1 - \frac{\sin^2(x)}{1 + \cot(x)} - \frac{\cos^2(x)}{1 + \tan(x)}$. Remember to use proper form for proving trigonometric identities by indicating the identities you are using.

Solution to Question 1: We can prove the identity by getting one side of the equal sign to be equivalent to the other side.

Proof.

$$RHS = 1 - \frac{\sin^2(x)}{1 + \cot(x)} - \frac{\cos^2(x)}{1 + \tan(x)} \quad \text{RHS: Right Hand Side}$$

$$= 1 - \frac{\sin^2(x)}{1 + \frac{1}{\tan(x)}} - \frac{\cos^2(x)}{1 + \tan(x)} \quad \cot(x) = \frac{1}{\tan(x)}$$

$$= 1 - \frac{\sin^2(x)}{1 + \frac{1}{\frac{\sin(x)}{\cos(x)}}} - \frac{\cos^2(x)}{1 + \frac{\sin(x)}{\cos(x)}} \quad \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= 1 - \frac{\sin^2(x)}{1 + \frac{\cos(x)}{\sin(x)}} - \frac{\cos^2(x)}{1 + \frac{\sin(x)}{\cos(x)}} \quad \text{Simplify}$$

$$= 1 - \frac{\sin^2(x)}{\frac{\sin(x)}{\sin(x)} + \frac{\cos(x)}{\sin(x)}} - \frac{\cos^2(x)}{1 + \frac{\sin(x)}{\cos(x)}} \quad 1 = \frac{\sin(x)}{\sin(x)}$$

$$= 1 - \frac{\sin^2(x)}{\frac{\sin(x)}{\sin(x)} + \frac{\cos(x)}{\sin(x)}} - \frac{\cos^2(x)}{\frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)}} \quad 1 = \frac{\cos(x)}{\cos(x)}$$

$$= 1 - \frac{\sin^2(x)}{\frac{\sin(x) + \cos(x)}{\sin(x)}} - \frac{\cos^2(x)}{\frac{\sin(x) + \cos(x)}{\cos(x)}} \quad \text{Simplify}$$

$$= 1 - \sin^2(x) \div \frac{\sin(x) + \cos(x)}{\sin(x)} - \cos^2(x) \div \frac{\sin(x) + \cos(x)}{\cos(x)} \quad \frac{a}{b} = a \div b$$

$$= 1 - \sin^2(x) \times \frac{\sin(x)}{\sin(x) + \cos(x)} - \cos^2(x) \times \frac{\cos(x)}{\sin(x) + \cos(x)} \quad \text{Convert } \div \text{ to } \times \text{ with reciprocal}$$

$$= 1 - \frac{\sin^3(x)}{\sin(x) + \cos(x)} - \frac{\cos^3(x)}{\sin(x) + \cos(x)} \quad \text{Simplify}$$

$$\begin{aligned}
RHS &= \frac{\sin(x) + \cos(x)}{\sin(x) + \cos(x)} - \frac{\sin^3(x)}{\sin(x) + \cos(x)} - \frac{\cos^3(x)}{\sin(x) + \cos(x)} & 1 &= \frac{\sin(x) + \cos(x)}{\sin(x) + \cos(x)} \\
&= \frac{\sin(x) + \cos(x) - \sin^3(x) - \cos^3(x)}{\sin(x) + \cos(x)} & \text{Simplify} \\
&= \frac{\sin(x) + \cos(x) - (\sin(x))(\sin^2(x)) - (\cos(x))(\cos^2(x))}{\sin(x) + \cos(x)} & a^3 &= a \times a^2 \\
&= \frac{\sin(x) + \cos(x) - (\sin(x))(1 - \cos^2(x)) - (\cos(x))(\cos^2(x))}{\sin(x) + \cos(x)} & \sin^2(x) &= 1 - \cos^2(x) \\
&= \frac{\sin(x) + \cos(x) - (\sin(x))(1 - \cos^2(x)) - (\cos(x))(1 - \sin^2(x))}{\sin(x) + \cos(x)} & \cos^2(x) &= 1 - \sin^2(x) \\
&= \frac{\sin(x) + \cos(x) - (\sin(x) - \cos^2(x)\sin(x)) - (\cos(x) - \sin^2(x)\cos(x))}{\sin(x) + \cos(x)} & \text{Expand} \\
&= \frac{\sin(x) + \cos(x) - \sin(x) + \cos^2(x)\sin(x) - \cos(x) + \sin^2(x)\cos(x)}{\sin(x) + \cos(x)} & \text{Break brackets} \\
&= \frac{\cos^2(x)\sin(x) + \sin^2(x)\cos(x)}{\sin(x) + \cos(x)} & \text{Simplify} \\
&= \frac{(\sin(x)\cos(x))(\sin(x) + \cos(x))}{\sin(x) + \cos(x)} & \text{Factor } \sin(x)\cos(x) \\
&= \sin(x)\cos(x) & \text{Simplify} \\
&= RHS & \square
\end{aligned}$$

$$\therefore \sin(x)\cos(x) = 1 - \frac{\sin^2(x)}{1 + \cot(x)} - \frac{\cos^2(x)}{1 + \tan(x)}$$

2. Determine equivalent expressions for both of $\sin(4x)$ and $\cos(4x)$ entirely in terms of x . You do not need to justify your steps.

Expand and simplify $\sin(4x)$ using identities:

$$\sin(4x) = ?$$

$$\sin(4x) = \sin(2x + 2x)$$

$$2x + 2x = 4x$$

$$= 2 \sin(2x) \cos(2x)$$

Double Angle Identity for sine where angle is $4x$

$$= 2(2 \sin(x) \cos(x)) \cos(2x)$$

Double Angle Identity for sine where angle is $2x$

$$= 2(2 \sin(x) \cos(x))(\cos^2(x) - \sin^2(x))$$

Double Angle Identity for cosine where angle is $2x$

$$= 4 \sin(x) \cos(x)(\cos^2(x) - \sin^2(x))$$

Expand

$$\therefore \sin(4x) = \boxed{4 \sin(x) \cos^3(x) - 4 \sin^3(x) \cos(x)}$$

Expand and simplify $\cos(4x)$ using identities:

$$\cos(4x) = ?$$

$$\cos(4x) = \cos(2x + 2x)$$

$$2x + 2x = 4x$$

$$= \cos^2(2x) - \sin^2(2x)$$

Double Angle Identity for cosine where angle is $4x$

$$= (\cos^2(x) - \sin^2(x))^2 - \sin^2(2x)$$

Double Angle Identity for cosine where angle is $2x$

$$= (\cos^2(x) - \sin^2(x))^2 - (2 \sin(x) \cos(x))^2$$

Double Angle Identity for sine where angle is $2x$

$$= \cos^4(x) - 2 \sin(x)^2 \cos(x)^2 + \sin^4(x) - 4 \sin(x)^2 \cos(x)^2$$

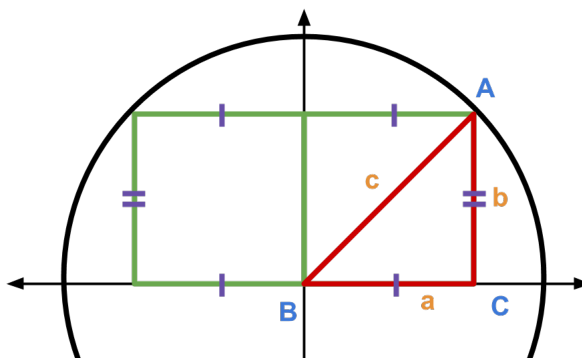
Apply exponents

$$\therefore \cos(4x) = \boxed{\cos^4(x) + \sin^4(x) - 6 \sin(x)^2 \cos(x)^2}$$

Simplify

3. A circle is centered at the origin with a radius of 25. Inside this circle is an inscribed rectangle with its bottom edge on the x-axis and the two vertices on the opposite side on the circumference of the circle,. Represent the area, A , of the rectangle:
- In terms of x , the distance between the y-axis and the right leg of the rectangle.
 - In terms of θ , where θ is an angle in standard position with its terminal arm on the upper right vertex of the rectangle.

Solution for question 3 (a): First, let's get some labels. Take the inscribed rectangle and label the origin B, the top right corner A and the bottom corner C. ABC form a right triangle. Label the appropriate sides with their corresponding angles.



We know that C or the radius is 25 units. We also know that (a) asks us to express the area with respect to "a" or the distance between the y-axis and the right leg of the rectangle.

From Pythagorean Theorem, we can also define b with respect to a and c:

$$a^2 + b^2 = c^2$$

$$b^2 = c^2 - a^2$$

Subtract a^2 from both sides

$$b = \sqrt{c^2 - a^2}$$

Square root a^2 from both sides

When we look at the diagram, we can see that the big rectangle is split in half by the y-axis. We can assume that both halves are of the same dimensions and thus area because both halves only touch the circle at 1 point and since they the circle is centered around the origin and both halves share the same horizontal lines (and thus same height), using pythagorean theorem their x-distance from the origin must also be the same (same length/base).

If we solve the area of one half, we can multiply it by 2 to get the full area. The area of a rectangle is $base \times height$, \therefore the area of the parent rectangle is $A = 2 \times base \times height$ where A is the are, b is base and h is height. We know the base and height as a and b so let's substitute them in.

$$A = 2bh$$

$$A = 2 \times a \times b$$

base = a, height = b

$$= 2 \times a \times \sqrt{c^2 - a^2}$$

$$b = \sqrt{c^2 - a^2}$$

$$= 2 \times a \times \sqrt{25^2 - a^2}$$

c = radius = 25

$$A = 2 \times a \times \sqrt{625 - a^2}$$

Simplify

\therefore The area of the rectangle in relation to a is $2a\sqrt{625 - a^2}$

Solution for question 3 (b):

Taking (a)'s solution, we can biggy back on the previously done work. We know that Area is:

$$2a\sqrt{625 - a^2} \quad (1)$$

If we can find the value of a in terms of θ , we can substitute this value into (a)'s solution and determine the area.

Since B is the target angle, I will now refer the angle B as θ . Let's try and define $\cos(\theta)$ by using the labeling system from (a). Consider the following:

$$\cos(\theta) = \frac{adj}{hyp} = \frac{a}{c} \quad (2)$$

We want to isolate "a" within (2). If we multiply the whole equation by c, we can cancel out the denominator in $\frac{a}{c}$.

$$\cos(\theta) = \frac{a}{c}$$

$$c \times \cos(\theta) = \frac{a}{c} \times c$$

Multiply all by c

$$c \cos(\theta) = a$$

Simplify

$$25 \cos(\theta) = a$$

Since $c = 25$

Now that we know the value of a , substitute this back into (1), the solution of (a):

$$\begin{aligned}
 A &= 2a\sqrt{625 - a^2} \\
 &= 2(25 \cos(\theta))\sqrt{625 - (25 \cos(\theta))^2} && a = 25 \cos(\theta) \\
 &= 50 \cos(\theta)\sqrt{625 - 625 \cos^2(\theta)} && \text{Simplify} \\
 &= 50 \cos(\theta)\sqrt{625(1 - \cos^2(\theta))} && \text{Factor 625} \\
 &= 50 \cos(\theta)\sqrt{625}\sqrt{1 - \cos^2(\theta)} && \sqrt{ab} = \sqrt{a}\sqrt{b} \\
 &= 50 \cos(\theta)25\sqrt{1 - \cos^2(\theta)} && \text{Simplify} \\
 A &= 1250 \cos(\theta)\sqrt{1 - \cos^2(\theta)}
 \end{aligned}$$

$\therefore \text{ The area in relation to } \theta \text{ is } 1250 \cos(\theta)\sqrt{1 - \cos^2(\theta)}$

4. Solve $2\sin^2(x) + \sin(2x) = 2$ ($x \in \mathbb{R}$). Be sure to effectively communicate the multiple solutions.

To solve question 4, we need to factor the function:

$$2\sin^2(x) + \sin(2x) = 2$$

$$2(1 - \cos^2(x)) + \sin(2x) = 2$$

$$\sin^2(x) = 1 - \cos^2(x)$$

$$2(1 - \cos^2(x)) + 2\sin(x)\cos(x) = 2$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$2 - 2\cos^2(x) + 2\sin(x)\cos(x) = 2$$

Expand

$$2 - 2 - 2\cos^2(x) + 2\sin(x)\cos(x) = 0$$

Subtract 2 from both sides

$$-2\cos^2(x) + 2\sin(x)\cos(x) = 0$$

Simplify

$$-2\cos(x)(\cos(x) - \sin(x)) = 0$$

Factor $-2\cos(x)$

From this, we know that $\cos(x)$ is equal to 0 and $\sin(x)$. With this, we can solve for the value of x .

When $\cos(x) = 0$

$x = \boxed{\frac{\pi}{2}}$. Cosine is also positive in quadrant 4 meaning that there is a related acute angle in Q4:

$$x = 2\pi - \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$$

When $\cos(x) = \sin(x)$ $\cos(x) = \sin(x)$ can also be expressed alternatively as $\tan(x) = 1$ with the following procedure:

$$\cos(x) = \sin(x)$$

$$\frac{\cos(x)}{\sin(x)} = 1$$

Divide both sides by $\sin(x)$

$$\tan(x) = 1$$

Quotient identity

We know that $\tan(x) = 1$ when x is $\boxed{\frac{\pi}{4}}$. Since tangent is positive in quadrant 4, the related acute angle is in Q3.

$$x = \pi + \frac{\pi}{4} = \boxed{\frac{5\pi}{4}}$$

$$\therefore x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2} \text{ if } x \in [0, 2\pi]$$

but the question states that $x \in \mathbb{R}$. We must create a pattern that extends infinitely. We know that this pattern repeats at every period or 2π .

$$\therefore x = \frac{\pi}{4} + 2\pi n, \frac{\pi}{2} + 2\pi n, \frac{5\pi}{4} + 2\pi n, \frac{3\pi}{2} + 2\pi n \text{ where } k \in \mathbb{Z}$$

5. Go to the website of the World Meteorological Organization and select a city from the map (right). Once you select a city, click on Climatological Information which is a tab above the map. Locate and record the data for mean max. temperature and mean min. temperature into a table. Using the data, create models for the relationship between the day of the year and both the high and low mean respectively. Finally, test the accuracy of each model by using it to determine the temperature for a single day, each.

