

MCV4U : Calculus and Vectors  
MCV4UR : Advanced Placement Calculus and Vectors

Assignment #3

---

**Reference Declaration**

Complete the Reference Declaration section below in order for your assignment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT, RRT, MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

Family Name: Wong

First Name: Max

Declared References: Geogebra for visualizing

Wolframalpha for **checking** answers

This forum page for projection: <https://tex.stackexchange.com/questions/239047/code-for-the-notation-of-vector-projections>

---

## Instructions

1. Organize and express complete, effective and concise responses to each problem.
2. Use appropriate mathematical conventions and notation wherever possible.
3. Provide logical reasoning for your arguments and cite any relevant theorems.
4. The quality of your communication will be heavily weighted.

## Evaluation

- Students will demonstrate an understanding of derivative rules and use them to solve problems.

| Criteria                                      | Level 1   | Level 2   | Level 3  | Level 4  |
|---|---|---|--|--|
| <i>Understanding of Mathematical Concepts</i> | Demonstrates limited understanding  | Demonstrates some understanding   | Demonstrates considerable understanding  | Demonstrates thorough understanding of concepts  |
| <i>Selecting Tools and Strategies</i>         | Selects and applies appropriate tools and strategies, with major errors, omissions, or mis-sequencing | Selects and applies appropriate tools and strategies, with minor errors, omissions, or mis-sequencing | Selects and applies appropriate tools and strategies accurately, and in a logical sequence | Selects and applies appropriate and efficient tools and strategies accurately to create mathematically elegant solutions |
| <i>Reasoning and Proving</i>                  | Inconsistently or erroneously employs logic to develop and defend statements                          | Statements are developed and defended with some omissions or leaps in logic                           | Frequently develops and defends statements with reasonable logical justification           | Consistently develops and defends statements with sophisticated and/or complete logical justification                    |
| <i>Communicating</i>                          | Expresses and organizes mathematical thinking with limited effectiveness                              | Expresses and organizes mathematical thinking with some effectiveness                                 | Expresses and organizes mathematical thinking with considerable effectiveness              | Expresses and organizes mathematical thinking with a high degree of effectiveness  |

1. Determine whether the following statements are True (T) or False (F).

- (a) If  $\vec{u} \cdot \vec{v} = 0$  then either  $\vec{u} = \vec{0}$  or  $\vec{v} = \vec{0}$ .
- (b)  $(\vec{u} \cdot \vec{v})\vec{u} \times (\vec{u} \times \vec{v})$  is a meaningful expression.
- (c) Force and velocity are scalar quantities.
- (d) If  $|\vec{a}| = 1$  then  $\vec{a}$  is a unit vector.
- (e) If  $\vec{a} = \vec{b} \times \vec{c}$  and  $\vec{a}$  is perpendicular to both  $\vec{b}$  and  $\vec{c}$ , then  $\vec{b}$  and  $\vec{c}$  are collinear.
- (f)  $\left\{ \vec{i}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{i} \times \vec{j} \right\}$  is a spanning set of  $\mathbb{R}^3$ .
- (g) Given the standard basis vectors  $\vec{i}, \vec{j}, \vec{k} \in \mathbb{R}^3$ , we can express  $\vec{k}$  as a linear combination of  $\vec{i}$  and  $\vec{j}$ .
- (h) If  $\vec{a}$  and  $\vec{b}$  are opposite vectors and  $|\vec{a}| = n$  then  $|\vec{b}| = -n$ .
- (i)  $\forall \vec{a}, \vec{b} \in \mathbb{R}^3, \vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ .
- (j)  $\exists \vec{a}, \vec{b} \in \mathbb{R}^3, \vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ .
- (k) If  $\forall m, n \in \mathbb{R}, |m\vec{a}| = |n\vec{a}|$  then  $\vec{a} = \vec{0}$ .
- (l) Given  $\vec{u}, \vec{v} \in \mathbb{R}^3$ , if  $|\vec{u}| = |\vec{v}| = 1$  then  $|\vec{u} \times \vec{v}| = 1$ .
- (m)  $\forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3, \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ .
- (n)  $\forall \vec{x}, \vec{y} \in \mathbb{R}^3, |\vec{x} \times \vec{y}| = |\vec{y} \times \vec{x}|$ .
- (o) Given two lines in  $\mathbb{R}^3$ ,  $\ell_1$  and  $\ell_2$ , either  $\ell_1 \parallel \ell_2$  or  $\ell_1$  intersects  $\ell_2$  at some point in  $\mathbb{R}^3$ .
- (p) If  $\vec{x}, \vec{y} \in \mathbb{R}^3$  then  $\text{Span}\{\vec{x}, \vec{y}\}$  is a plane in  $\mathbb{R}^3$ .
- (q) If  $\{\vec{a}, \vec{b}\}$  spans  $\mathbb{R}^2$  then  $\{\vec{a}, \vec{a} + \vec{b}\}$  spans  $\mathbb{R}^2$ .
- (r) If  $\vec{p}$  is a scalar multiple of  $\vec{q}$  then  $\vec{x} = t\vec{p} + \vec{q}, t \in \mathbb{R}$ , is a line through the origin.
- (s) Given  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^4$ , if  $\vec{a} \cdot \vec{b} = 0$  and  $\vec{b} \cdot \vec{c} = 0$  then  $\vec{a} \cdot \vec{c} = 0$ .
- (t) If  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is a solution to a system of linear equations then  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$  is also a solution.

## True-False Answers

- (a) F
- (b) T
- (c) F
- (d) T
- (e) F
- (f) T
- (g) F
- (h) F
- (i) F
- (j) T
- (k) F
- (l) F
- (m) F
- (n) T
- (o) F
- (p) T
- (q) T
- (r) F
- (s) F
- (t) T

2. An **orthogonal spanning set of  $\mathbb{R}^3$**  is a set of 3 non-zero vectors, all of which are orthogonal to each other and span  $\mathbb{R}^3$ . Given  $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ , **determine** vectors  $\vec{b}$  and  $\vec{c}$  such that  $\{\vec{a}, \vec{b}, \vec{c}\}$  is an orthogonal spanning set of  $\mathbb{R}^3$ . **Express** the vector  $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  as a linear combination of the vectors in this spanning set.

**Solution to question 2:**

Since we are looking for a specific solution, we need to get a vector in a form where we can pick 2 arbitrary values and find the third value. Then we can use the 2 known vectors to get the values of the third. Let  $b = (b_1, b_2, b_3)$  and  $c = (c_1, c_2, c_3)$ .

Since  $\vec{a}$  and  $\vec{b}$  are orthogonal, their dot product must be equal to zero:

$$0 = \vec{a} \cdot \vec{b}$$

$$0 = (1)(b_1) + (-1)(b_2) + 2(b_3) \quad \text{Find Dot Product}$$

$$0 = b_1 - b_2 + 2b_3 \quad \text{Simplify}$$

Now choose 2 arbitrary values. I like “42” so let  $b_1 = 4$  and let  $b_2 = 2$ .

$$0 = b_1 - b_2 + 2b_3$$

$$0 = 4 - 2 + 2b_3 \quad \text{Sub in } b_1 \text{ and } b_2$$

$$-1 = b_3 \quad \text{Simplify}$$

Now that we have  $\vec{a}$  and  $\vec{b}$ , we can find the cross product which, since  $\vec{a}$  and  $\vec{b}$  are orthogonal, will be  $\vec{c}$  which is also orthogonal to  $\vec{a}$  and  $\vec{b}$ :

$$\begin{bmatrix} i & j & k \\ 1 & -1 & 2 \\ 4 & 2 & -1 \end{bmatrix} = \vec{i} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \vec{j} \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} + \vec{k} \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} = \vec{i}(1-4) - \vec{j}(-1-8) + \vec{k}(2+4) = -3\vec{i} + 9\vec{j} + 6\vec{k}$$

$\therefore$  The 3 vectors are orthogonal on  $\mathbb{R}^3$  :  $\vec{a} = (1, -1, 2)$ ,  $\vec{b} = (4, 2, -1)$ ,  $\vec{c} = (-3, 9, 6)$

**Solution continues on next page**

We know that a linear combination of  $\vec{a}, \vec{b}, \vec{c}$  to make  $\vec{u}$  would look something like this, where  $p, q, r \in \mathbb{R}$  are constants:

$$p\vec{a} + q\vec{b} + r\vec{c} = \vec{u}$$

$$p(1, -1, 2) + q(4, 2, -1) + r(-3, 9, 6) = (1, 0, 1)$$

*textSubstitute in a, b, c, u*

We can now form 3 equations using the components. Getting these equations into a matrix and rearranging into REF form will help us solve for p, q, r:

$$\begin{cases} p + 4q - 3r &= 1 \\ -p + 2q + 9r &= 0 \\ 2p + -1q + 6r &= 1 \end{cases}$$

Form into a matrix:

$$\left( \begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ -1 & 2 & 9 & 0 \\ 2 & -1 & 6 & 1 \end{array} \right)$$

Get bottom 2 elements in column 1 to be zero using replacement operation

$$\equiv \left( \begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & 9 & -12 & 1 \end{array} \right) \quad R'_2 = R_1 + R_2, R'_3 = 2R_1 - R_3$$

Get Row 3 Column 2 element to become zero by using replacement operation:

$$\equiv \left( \begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & 0 & 42 & 1 \end{array} \right) \quad R'_3 = 3R_2 - 2R_3$$

From this we get the three following equations. From this, we can solve the 3 variable algebra question:

$$\begin{cases} p + 4q - 3r &= 1 \\ 0p + 6q + 6r &= 1 \\ 0p + 0q + 42r &= 1 \end{cases}$$

**Solution continues on next page**

We can see immedietly that  $r = \frac{1}{42}$ . Now we can solve for q.

$$6q + 6r = 1$$

$$6q + (6)\frac{1}{42} = 1$$

$$\text{Sub in } r = \frac{1}{42}$$

$$6q + \frac{6}{42} = 1$$

Simplify

$$6q + \frac{1}{7} = 1$$

$$6q = \frac{6}{7}$$

Subtract 1/7 from both sides

$$q = \frac{1}{7}$$

Divide 6 from both sides

Next, with r and q, solve for p.

$$p + 4q - 3r = 1$$

$$p + (4)\frac{1}{7} - (3)\frac{1}{42} = 1$$

Sub in r and q

$$p + \frac{4}{7} - \frac{3}{42} = 1$$

Simplify

$$p + \frac{4}{7} - \frac{1}{14} = 1$$

$$p + \frac{8}{14} - \frac{1}{14} = 1$$

Get same denominator

$$p + \frac{7}{14} = 1$$

Simplify

$$p + \frac{1}{2} = 1$$

$$p = \frac{1}{2}$$

Subtract 1/2 from both sides

|  |
|--|
| $\therefore \text{The linear combination expressed as u is: } \vec{u} = \frac{1}{2}\vec{a} + \frac{1}{7}\vec{b} + \frac{1}{42}\vec{c}$ |
|--|

3. In a methane molecule ( $CH_4$ ), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are located in space at  $A(0, 0, 0)$ ,  $B(1, 1, 0)$ ,  $C(1, 0, 1)$  and  $D(0, 1, 1)$  and that the carbon atom is located at  $E\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ . **Determine** the *bond angle*, the angle from some hydrogen atom to the carbon atom, to another hydrogen atom.

### Solution to question 3:

The molecule is currently being centered at E which is offset from the origin. To do our calculations, we need to center everything. I chose A, B and E, to do my calculation but any 2 points + the middle point will be sufficient. Once centered, we can use the projection formula with angles to determine the angle.

First subtract  $(-1/2, -1/2, -1/2)$  from all three points to center at origin.

$$\begin{cases} A &= \left(0 - \frac{1}{2}, 0 - \frac{1}{2}, 0 - \frac{1}{2}\right) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \\ B &= \left(1 - \frac{1}{2}, 1 - \frac{1}{2}, 0 - \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \\ E &= \left(\frac{1}{2} - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\right) = (0, 0, 0) \end{cases}$$

Now use the following formula derived from the projection formula:  $\theta = \arccos \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$

$$\theta = \arccos \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} \quad \text{Angle Projection Formula}$$

$$= \arccos \frac{-\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot -\frac{1}{2}}{|\vec{A}||\vec{B}|} \quad \text{Perform dot product}$$

$$= \arccos \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{|\vec{A}||\vec{B}|} \quad \text{Simplify}$$

$$= \arccos \frac{-\frac{1}{4}}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \cdot \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}} \quad \text{Find scalar of A and B}$$



$$\theta = \arccos \frac{-\frac{1}{4}}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \cdot \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}} \quad \text{From previous page}$$

$$= \arccos \frac{-\frac{1}{4}}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \cdot \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \quad \text{Simplify}$$

$$= \arccos \frac{-\frac{1}{4}}{\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{4}}}$$

$$= \arccos \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= \arccos -\frac{1}{4} \cdot \frac{4}{3} \quad \text{Flip fraction, simplify}$$

$$= \arccos \left( -\frac{1}{3} \right)$$

$$\theta = 109.47^\circ \quad \text{Perform arccos}$$

|  |
|--|
| $\therefore$ The angle formed by any 2 carbon atoms is around $109.47^\circ$ |
|--|

4. If  $\theta$  is the acute angle between  $\vec{a}$  and  $\vec{b}$  then **prove**  $proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b}) \cos^2(\theta)$ .

**Solution to question 4:**

First let's take the LHS, expand it out and see what we get.

$$LHS = proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{a}|} \vec{a} \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{b}||\vec{b}|} \vec{b} \quad \text{We know that } proj_{\vec{x}}\vec{y} = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{x}|} \vec{x}$$

$$= \frac{(\vec{a} \cdot \vec{b})^2}{|\vec{a}||\vec{a}||\vec{b}||\vec{b}|} (\vec{a} \cdot \vec{b}) \quad \text{Combine Fraction, using thm 3.1 and algebraic manipulation}$$

$$= \frac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2} (\vec{a} \cdot \vec{b}) \quad \text{Simplify}$$

$$\text{We know that } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}. \text{ Therefore } \cos^2 \theta = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right)^2 = \frac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2}.$$

$$LHS = \frac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2} (\vec{a} \cdot \vec{b})$$

$$LHS = \cos^2 \theta (\vec{a} \cdot \vec{b}) = RHS \quad \text{Substitute identity in}$$

Therefore the statement  $proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b}) \cos^2(\theta)$  is true.

5. **Determine** the equation of a line through the point  $A(1, 2, -3)$  and lying within the plane  $x - 2y + 2z + 9 = 0$ .

**Solution to question 5:**

Right now we are restricted to the plane, so in a way we are solving for a line in  $R^2$ .

We already know 1 point on the plane  $x - 2y + 2z + 9 = 0$ : A. We can check that A is on the plane by substituting in the values and seeing if it matches the right side: 0.

$$LHS = x - 2y + 2z + 9 = (1) - 2(2) + 2(-3) + 9 = 1 - 4 - 6 + 9 = 0 = RHS$$

Now, we simply need a second point on the plane to find the line (since we are looking for a singular example). Let x and y be 2 arbitrary values: 4 and 2 respectively. We can use substitution to find z using the plane equation.

$$x - 2y + 2z + 9 = 0 \quad \text{Plane equation}$$

$$4 - 2(2) + 2z + 9 = 0 \quad \text{Sub in } x = 4, y = 2$$

$$2z + 9 = 0 \quad \text{Solve algebra}$$

$$2z = -9$$

$$z = -\frac{9}{2}$$

Now we have our second point,  $B = \left(4, 2, -\frac{9}{2}\right)$ , we can solve for our direction vector by subtracting B from A:

$$B - A = \left(4 - 1, 2 - 2, -\frac{9}{2} + 3\right) = \left(3, 0, -\frac{3}{2}\right)$$

Since we know that the equation of the line, in vector form is:

$\mathbf{r} = (\text{point on plane}) + t(\text{direction vector})$  where  $t \in \mathbb{R}$  and  $\mathbf{r}$  is the resulting vector

$$\therefore \mathbf{r} = (1, 2, -3) + t\left(3, 0, -\frac{3}{2}\right) \text{ is a line on the plane } x - 2y + 2z + 9 = 0 \text{ which intersects A}$$

6. **Determine** and **classify** the nature of the intersection between the following planes:

$$\Pi_1 : x - 5y + 2z = 10$$

$$\Pi_2 : x + 7y - 2z + 6 = 0$$

$$\Pi_3 : 8x + 5y + z = 20$$

**Solution to question 8:**

We need to get the equations into a REF form. Evaluating in this way may give us clues. First rewrite the equations as a matrix.

$$\left( \begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 1 & 7 & -2 & -6 \\ 8 & 5 & 1 & 20 \end{array} \right)$$

Next, use interchange to swap the second and third rows:

$$\equiv \left( \begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 8 & 5 & 1 & 20 \\ 1 & 7 & -2 & -6 \end{array} \right) \quad R'_2 = R_3, R'_3 = R_2$$

Next, use replacement operation to get the bottom half of column 1 as zeroes:

$$\equiv \left( \begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -45 & 15 & 60 \\ 0 & -12 & 4 & 16 \end{array} \right) \quad R'_2 = 8R_1 - R_2, R'_3 = R_1 - R_3$$

Now get the element in column 2, row 3 as a zero using the replacement operation:

$$\equiv \left( \begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -45 & 15 & 60 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad R'_3 = 4R_2 - 15R_3$$

This results in 2 linear equations:

$$\begin{cases} x - 5y + 2z = 10 \\ 0x - 45y + 15z = 60 \end{cases}$$

We can observe that these equations are linear (since no exponents) and also there does not seem to be any contradiction. Also, none of the equations seems **not** be collinear. Since a single point cannot be found from the equations above:

$\therefore$  I think the equations converge infinitely along a line (consistent)