MCV4U : Calculus and Vectors
MCV4UR : Advanced Placement Calculus and Vectors

Assignment #3

Reference Declaration

Complete the Reference Declaration section below in order for your assigment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the FT, RRT, MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

Family Name: Wong
First Name: Max

Declared References: Geogebra for visualizing
Wolframalpha for **checking** answers
This formum page for projection: https://tex.stackexchange.com/questions/239047/code-for-the-notation-of-vector-projections

---

Instructions

1. Organize and express complete, effective and concise responses to each problem.
2. Use appropriate mathematical conventions and notation wherever possible.
3. Provide logical reasoning for your arguments and cite any relevant theorems.
4. The quality of your communication will be heavily weighted.

Evaluation

- Students will demonstrate an understanding of derivative rules and use them to solve problems.

Criteria	Level 1	Level 2	Level 3	Level 4
Understanding of Mathematical Concepts	Demonstrates limited understanding	Demonstrates some understanding	Demonstrates considerable understanding	Demonstrates thorough understanding of concepts
Selecting Tools and Strategies	Selects and applies appropriate tools and strategies, with major errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies, with minor errors, omissions, or mis-sequencing	Selects and applies appropriate tools and strategies accurately, and in a logical sequence	Selects and applies appropriate and efficient tools and strategies accurately to create mathematically elegant solutions
Reasoning and Proving	Inconsistently or erroneously employs logic to develop and defend statements	Statements are developed and defended with some omissions or leaps in logic	Frequently develops and defends statements with reasonable logical justification	Consistently develops and defends statements with sophisticated and/or complete logical justification
Communicating	Expresses and organizes mathematical thinking with limited effectiveness	Expresses and organizes mathematical thinking with some effectiveness	Expresses and organizes mathematical thinking with considerable effectiveness	Expresses and organizes mathematical thinking with a high degree of effectiveness

1. Determine whether the following statements are True (T) or False (F).

   (a) If $\vec{u} \cdot \vec{v} = 0$ then either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

   (b) $(\vec{u} \cdot \vec{v})\vec{u} \times (\vec{u} \times \vec{v})$ is a meaningful expression.

   (c) Force and velocity are scalar quantities.

   (d) If $|\vec{a}| = 1$ then $\vec{a}$ is a unit vector.

   (e) If $\vec{a} = \vec{b} \times \vec{c}$ and $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, then $\vec{b}$ and $\vec{c}$ are collinear.

   (f) $\left\{\vec{i}, \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right), \vec{i} \times \vec{j}\right\}$ is a spanning set of $\mathbb{R}^3$.

   (g) Given the standard basis vectors $\vec{i}, \vec{j}, \vec{k} \in \mathbb{R}^3$, we can express $\vec{k}$ as a linear combination of $\vec{i}$ and $\vec{j}$.

   (h) If $\vec{a}$ and $\vec{b}$ are opposite vectors and $|\vec{a}| = n$ then $|\vec{b}| = -n$.

   (i) $\forall \vec{a}, \vec{b} \in \mathbb{R}^3$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.

   (j) $\exists \vec{a}, \vec{b} \in \mathbb{R}^3$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.

   (k) If $\forall m, n \in \mathbb{R}$, $|m\vec{a}| = |n\vec{a}|$ then $\vec{a} = \vec{0}$.

   (l) Given $\vec{u}, \vec{v} \in \mathbb{R}^3$, if $|\vec{u}| = |\vec{v}| = 1$ then $|\vec{u} \times \vec{v}| = 1$.

   (m) $\forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

   (n) $\forall \vec{x}, \vec{y} \in \mathbb{R}^3$, $|\vec{x} \times \vec{y}| = |\vec{y} \times \vec{x}|$.

   (o) Given two lines in $\mathbb{R}^3$, ℓ_1 and ℓ_2, either $\ell_1 \parallel \ell_2$ or ℓ_1 intersects ℓ_2 at some point in $\mathbb{R}^3$.

   (p) If $\vec{x}, \vec{y} \in \mathbb{R}^3$ then $\text{Span}\{\vec{x}, \vec{y}\}$ is a plane in $\mathbb{R}^3$.

   (q) If $\{\vec{a}, \vec{b}\}$ spans $\mathbb{R}^2$ then $\{\vec{a}, \vec{a} + \vec{b}\}$ spans $\mathbb{R}^2$.

   (r) If $\vec{p}$ is a scalar multiple of $\vec{q}$ then $\vec{x} = t\vec{p} + \vec{q}$, $t \in \mathbb{R}$, is a line through the origin.

   (s) Given $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^4$, if $\vec{a} \cdot \vec{b} = 0$ and $\vec{b} \cdot \vec{c} = 0$ then $\vec{a} \cdot \vec{c} = 0$.

   (t) If $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ is a solution to a system of linear equations then $\left(\begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}\right)$ is also a solution.

True-False Answers

(a) F
(b) T
(c) F
(d) T
(e) F
(f) T
(g) F
(h) F
(i) F
(j) T
(k) F
(l) F
(m) F
(n) T
(o) F
(p) T
(q) T
(r) F
(s) F
(t) T

2. An **orthogonal spanning set of** $\mathbb{R}^3$ is a set of 3 non-zero vectors, all of which are orthogonal to each other and span $\mathbb{R}^3$. Given $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, **determine** vectors $\vec{b}$ and $\vec{c}$ such that $\{\vec{a}, \vec{b}, \vec{c}\}$ is an orthogonal spanning set of $\mathbb{R}^3$. **Express** the vector $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ as a linear combination of the vectors in this spanning set.

Solution to question 2:
Since we are looking for a specific solution, we need to get a vector in a form where we can pick 2 arbitrary values and find the third value. Then we can use the 2 known vectors to get the values of the third. Let $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$.

Since $\vec{a} and \vec{b}$ are orthogonal, their dot product must be equal to zero:

$$\begin{aligned} 0 &= \vec{a} \cdot \vec{b} \\ 0 &= (1)(b_1) + (-1)(b_2) + 2(b_3) && \text{Find Dot Product} \\ 0 &= b_1 - b_2 + 2b_3 && \text{Simplify} \end{aligned}$$

Now choose 2 arbitrary values. I like "42" so let $b_1 = 4$ and let $b_2 = 2$.

$$\begin{aligned} 0 &= b_1 - b_2 + 2b_3 \\ 0 &= 4 - 2 + 2b_3 && \text{Sub in } b_1 \text{ and } b_2 \\ -1 &= b_3 && \text{Simplify} \end{aligned}$$

Now that we have $\vec{a}$ and $\vec{b}$, we can find the cross product which, since $\vec{a}$ and $\vec{b}$ are orthogonal, will be $\vec{c}$ which is also orthogonal to $\vec{a}$ and $\vec{b}$:

$$\begin{bmatrix} i & j & k \\ 1 & -1 & 2 \\ 4 & 2 & -1 \end{bmatrix} = \vec{i} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \vec{j} \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} + \vec{k} \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} = \vec{i}(1-4) - \vec{j}(-1-8) + \vec{k}(2+4) = -3\vec{i} + 9\vec{j} + 6\vec{k}$$

$$\boxed{\therefore \text{The 3 vectors are orthogonal on } R^3 : \vec{a} = (1, -1, 2), \vec{b} = (4, 2, -1), \vec{c} = (-3, 9, 6)}$$

Solution continues on next page

We know that a linear combination of $\vec{a}, \vec{b}, \vec{c}$ to make $\vec{u}$ would look something like this, where $p, q, r \in \mathbb{R}$ are constants:

$$p\vec{a} + q\vec{b} + r\vec{c} = \vec{u}$$

$$p(1, -1, 2) + q(4, 2, -1) + r(-3, 9, 6) = (1, 0, 1)$$

Substitute in a, b, c, u

We can now form 3 equations using the components. Getting these equations into a matrix and rearranging into REF form will help us solve for p, q, r:

$$\begin{cases} p + 4q - 3r & = 1 \\ -p + 2q + 9r & = 0 \\ 2p + -1q + 6r & = 1 \end{cases}$$

Form into a matrix:

$$\left(\begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ -1 & 2 & 9 & 0 \\ 2 & -1 & 6 & 1 \end{array}\right)$$

Get bottom 2 elements in column 1 to be zero using replacement operation

$$\equiv \left(\begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & 9 & -12 & 1 \end{array}\right) \quad R_2' = R_1 + R_2, R_3' = 2R_1 - R_3$$

Get Row 3 Column 2 element to become zero by using replacement operation:

$$\equiv \left(\begin{array}{ccc|c} 1 & 4 & -3 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & 0 & 42 & 1 \end{array}\right) \quad R_3' = 3R_2 - 2R_3$$

From this we get the three following equations. From this, we can solve the 3 variable algebra question:

$$\begin{cases} p + 4q - 3r & = 1 \\ 0p + 6q + 6r & = 1 \\ 0p + 0q + 42r & = 1 \end{cases}$$

Solution continues on next page

We can see immedietly that $r = \dfrac{1}{42}$. Now we can solve for q.

$$\begin{aligned}
6q + 6r &= 1 \\
6q + (6)\frac{1}{42} &= 1 && \text{Sub in } r = \frac{1}{42} \\
6q + \frac{6}{42} &= 1 && \text{Simplify} \\
6q + \frac{1}{7} &= 1 \\
6q &= \frac{6}{7} && \text{Subtract 1/7 from both sides} \\
q &= \frac{1}{7} && \text{Divide 6 from both sides}
\end{aligned}$$

Next, with r and q, solve for p.

$$\begin{aligned}
p + 4q - 3r &= 1 \\
p + (4)\frac{1}{7} - (3)\frac{1}{42} &= 1 && \text{Sub in r and q} \\
p + \frac{4}{7} - \frac{3}{42} &= 1 && \text{Simplify} \\
p + \frac{4}{7} - \frac{1}{14} &= 1 \\
p + \frac{8}{14} - \frac{1}{14} &= 1 && \text{Get same denominator} \\
p + \frac{7}{14} &= 1 && \text{Simplify} \\
p + \frac{1}{2} &= 1 \\
p &= \frac{1}{2} && \text{Subtract 1/2 from both sides}
\end{aligned}$$

$$\boxed{\therefore \text{ The linear combination expressed as u is: } \vec{u} = \frac{1}{2}\vec{a} + \frac{1}{7}\vec{b} + \frac{1}{42}\vec{c}}$$

3. In a methane molecule (CH_4), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are located in space at $A(0,0,0), B(1,1,0), C(1,0,1)$ and $D(0,1,1)$ and that the carbon atom is located at $E\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. **Determine** the *bond angle*, the angle from some hydrogen atom to the carbon atom, to another hydrogen atom.

Solution to question 3:
The molecule is currently being cented at E which is offset from the origin. To do our calculations, we need to center everything. I chose A, B and E, to do my calculation but any 2 points + the midle point will be sufficient. Once centered, we can use the projection formula with angles to determine the angle.

First subtract (-1/2, -1/2, -1/2) from all three points to center at origin.

$$\begin{cases} A &= \left(0-\dfrac{1}{2}, 0-\dfrac{1}{2}, 0-\dfrac{1}{2}\right) = \left(-\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right) \\ B &= \left(1-\dfrac{1}{2}, 1-\dfrac{1}{2}, 0-\dfrac{1}{2}\right) = \left(\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right) \\ E &= \left(\dfrac{1}{2}-\dfrac{1}{2}, \dfrac{1}{2}-\dfrac{1}{2}, \dfrac{1}{2}-\dfrac{1}{2}\right) = (0,0,0) \end{cases}$$

Now use the following formula derived from the projection formula: $\theta = \arccos\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

$$\theta = \arccos\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} \qquad \text{Angle Projection Formula}$$

$$= \arccos\frac{-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}\cdot-\frac{1}{2}}{|\vec{A}||\vec{B}|} \qquad \text{Perform dot product}$$

$$= \arccos\frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{|\vec{A}||\vec{B}|} \qquad \text{Simplify}$$

$$= \arccos\frac{-\frac{1}{4}}{|\vec{A}||\vec{B}|}$$

$$= \arccos\frac{-\frac{1}{4}}{\sqrt{\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}\cdot\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}} \qquad \text{Find scalar of A and B}$$

$$\theta = \arccos \frac{-\frac{1}{4}}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \cdot \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}} \qquad \text{From previous page}$$

$$= \arccos \frac{-\frac{1}{4}}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \cdot \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \qquad \text{Simplify}$$

$$= \arccos \frac{-\frac{1}{4}}{\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{4}}}$$

$$= \arccos \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= \arccos -\frac{1}{4} \cdot \frac{4}{3} \qquad \text{Flip fraction, simplify}$$

$$= \arccos \left(-\frac{1}{3}\right)$$

$$\theta = 109.47^\circ \qquad \text{Perform arccos}$$

$$\boxed{\therefore \text{ The angle formed by any 2 carbon atoms is around } 109.47^\circ}$$

4. If θ is the acute angle between $\vec{a}$ and $\vec{b}$ then **prove** $proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b}) \cos^2(\theta)$.

Solution to question 4:
First let's take the LHS, expand it out and see what we get.

$$LHS = proj_{\mathbf{a}}\mathbf{b} \cdot proj_{\mathbf{b}}\mathbf{a}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{a}|}\vec{a} \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{b}||\vec{b}|}\vec{b} \qquad \text{We know that } proj_{\mathbf{x}}\mathbf{y} = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{x}|}\mathbf{x}$$

$$= \frac{(\vec{a} \cdot \vec{b})^2}{|\vec{a}||\vec{a}||\vec{b}||\vec{b}|}(\vec{a} \cdot \vec{b}) \qquad \text{Combine Fraction, using thm 3.1 and algebraic manipuulation}$$

$$= \frac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2}(\vec{a} \cdot \vec{b}) \qquad \text{Simplify}$$

We know that $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. Therefore $\cos^2\theta = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)^2 = \dfrac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2}$.

$$LHS = \frac{(\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2}(\vec{a} \cdot \vec{b})$$

$$LHS = \cos^2\theta(\vec{a} \cdot \vec{b}) = RHS \qquad \text{Substitute identity in}$$

$$\boxed{\text{Therefore the statement } proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b}) \cos^2(\theta) \text{ is true.}}$$

5. **Determine** the equation of a line through the point $A(1, 2, -3)$ and lying within the plane $x - 2y + 2z + 9 = 0$.

Solution to question 5:
Right now we are restricted to the plane, so in a way we are solving for a line in R^2.

We already know 1 point on the plane $x - 2y + 2z + 9 = 0$: A. We can check that A is on the plane by substituting in the values and seeing if it matches the right side: 0.

$$LHS = x - 2y + 2z + 9 = (1) - 2(2) + 2(-3) + 9 = 1 - 4 - 6 + 9 = 0 = RHS$$

Now, we simply need a second point on the plane to find the line (since we are looking for a singular example). Let x and y be 2 arbitrary values: 4 and 2 respectively. We can use substitution to find z using the plane equation.

$$\begin{aligned} x - 2y + 2z + 9 &= 0 && \text{Plane equation} \\ 4 - 2(2) + 2z + 9 &= 0 && textSubinx = 4, y = 2 \\ 2z + 9 &= 0 && \text{Solve algebra} \\ 2z &= -9 \\ z &= -\frac{9}{2} \end{aligned}$$

Now we have our second point, $B = \left(4, 2, -\frac{9}{2}\right)$, we can solve for our direction vector by subtracting B from A:

$$B - A = \left(4 - 1, 2 - 2, -\frac{9}{2} + 3\right) = \left(3, 0, -\frac{3}{2}\right)$$

Since we know that the equation of the line, in vector form is:
r = (point on plane) + t(direction vector) where $t \in \mathbb{R}$ and r is the resulting vector

$$\boxed{\therefore r = (1, 2, -3) + t\left(3, 0, -\frac{3}{2}\right) \text{ is a line on the plane } x - 2y + 2z + 9 = 0 \text{ which intersects A}}$$

6. **Determine** and **classify** the nature of the intersection between the following planes:

$$\Pi_1 : x - 5y + 2z = 10$$

$$\Pi_2 : x + 7y - 2z + 6 = 0$$

$$\Pi_3 : 8x + 5y + z = 20$$

Solution to question 8:
We need to get the equations into a REF form. Evaluating in this way may give us clues. First rewrite the equations as a matrix.

$$\left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 1 & 7 & -2 & -6 \\ 8 & 5 & 1 & 20 \end{array}\right)$$

Next, use interchange to swap the second tha third rows:

$$\equiv \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 8 & 5 & 1 & 20 \\ 1 & 7 & -2 & -6 \end{array}\right) \quad R_2' = R_3, R_3' = R_2$$

Next, use replacement operation to get the bottom half of column 1 as zeroes:

$$\equiv \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -45 & 15 & 60 \\ 0 & -12 & 4 & 16 \end{array}\right) \quad R_2' = 84_1 - R_2, R_3' = R_1 - R_3$$

Now get the element in column 2, row 3 as a zero using the replacement operation:

$$\equiv \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -45 & 15 & 60 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = 4R_2 - 15R_3$$

This results in 2 linear equations:

$$\begin{cases} x - 5y + 2Z = 10 \\ 0X - 45Y + 15Z = 60 \end{cases}$$

Taking the two equations, we can simplify by dividing the second by 15:

$$\begin{cases} x - 5y + 2Z = 10 \\ -3Y + Z = 4 \end{cases}$$

Solution continues on the next page

Now, let z be t. Now we can rewrite the second equation from the top.

$$-3Y + t = 4$$

$$-3Y = 4 - t \text{ Subtract t from both sides}$$

$$Y = -\frac{4-t}{3} \text{ Divide 3 from both sides}$$

Now that we know y, and z, we can solve the third equation.

$$x - 5y + 2Z = 10$$

$$x + 5\left(\frac{4-t}{3}\right) + 2t = 10 \qquad \text{Sub y and z in}$$

$$x + \left(\frac{20-5t}{3}\right) + 2t = 10 \qquad \text{Simplify}$$

$$3x + 20 - 5t + 6t = 30 \qquad \text{Multiply all by 3}$$

$$3x = 30 - 20 + 5t - 6t \qquad \text{Subtract 20-5t+6t from both sides}$$

$$3x = 10 - t \qquad \text{Simplify}$$

$$x = \frac{10-t}{3} \qquad \text{Divide all by 3}$$

Therefore, we know that $(x, y, z) = \left(\frac{10-t}{3}, -\frac{4-t}{3}, t\right)$. Since t is z which is zero: $(x, y, z) = \left(\frac{10}{3}, -\frac{4}{3}, 0\right)$. Since we also know, from the original equation, another point: (1, -5, 2):

$$\vec{r} = t(1, -5, 2) + \left(\frac{10}{3}, -\frac{4}{3}, 0\right)$$

We can observe that the equation above is a line. The original equations also seem to be linear (since no exponents above 1) and also there does not seem to be any contradictions when evaluating. Also, none of the equations seems to be collinear (No multiples of each other). Since a line was found:

$$\boxed{\therefore \text{ I think the equations intersect infinitely along a line (consistent)}}$$