Final examination question 25

Work in progress

December 20, 2018

Exercise 1

Consider the dynamical system $v_{k+1} = Av_k$ where

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} x_k \\ y_k \end{bmatrix} \text{ and } v_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

It turns out that A is diagonalisable, that is $P^{-1}AP = D$ is diagonal, where

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{3} \end{bmatrix}$$

What is the value of x_k?

Answer of exercise 1

We are given that $A = PDP^{-1}$ so

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue 1 and

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue $\frac{-1}{3}$. First write the initial vector as a linear combination of the eigenvectors:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

I.e. solve

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 4 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

So $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Therefore

$$\begin{aligned} v_k &= A^k v_0 \\ &= A^k \left[\frac{1}{2}\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \frac{1}{2}\left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right] \\ &= \frac{1}{2}A^k\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \frac{1}{2}A^k\left[\begin{array}{c} -3 \\ 1 \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \frac{1}{2}\left(\frac{-1}{3}\right)^k\left[\begin{array}{c} -3 \\ 1 \end{array}\right] \\ &= \left[\begin{array}{c} x_k \\ y_k \end{array}\right] \end{aligned}$$

and so

$$x_k = \frac{1}{2}\left(1 - 3\left(\frac{-1}{3}\right)^k\right)$$