MATH211: Linear Methods I

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Lecture on Thursday 1st November, 2018

Matrices

Composition

Inverses

Complex numbers

Last time

- Linear transformations
- Action on compositions
- Matrix of a linear transformation

Matrices

Matrices and linear transformations

Example

If A is an $m \times n$ matrix then the function $x \mapsto Ax$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then the *matrix* $[T]$ *of* T has columns given by the images of the standard basis vectors:

$$[T] = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$$

Examples

Example

What is the matrix of the linear transformation for counter-clockwise rotation about the origin by an angle of θ?

Example

What is the matrix of the linear transformation for reflection in the line through that origin at an angle θ from the x-axis?

Example

What is the matrix of the linear transformation that is $v \times (-)$ for some vector $v \in \mathbb{R}^3$?

Composition

Composition of functions

Definition

If we have functions

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \text{ and } \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

then the *composite function* $g \circ f$ is the function defined by

$$(g \circ f)(x) = g(f(x))$$

- I.e. first apply f to x and then apply g to the result.
- Note that $\mathbb{R}^n \xrightarrow{g \circ f} \mathbb{R}^p$.

Composite of linear transformations

Example

The composite of two linear transformations is again linear.

Proof.

Suppose that $S : \mathbb{R}^m \to \mathbb{R}^p$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations. Then

$$\begin{aligned}(S \circ T)(x+y) &= S(T(x+y)) = S(T(x)+T(y)) \\ &= S(T(x)) + S(T(y)) = (S \circ T)(x) + (S \circ T)(y)\end{aligned}$$

and

$$\begin{aligned}(S \circ T)(kx) &= S(T(kx)) = S(kT(x)) \\ &= kS(T(x)) = k(S \circ T)(x)\end{aligned}$$

□

Matrix of composite transformation

We know that the matrix of the composite $T \circ S$ is:

$$\begin{bmatrix} T(S(e_1)) & T(S(e_2)) & \dots & T(S(e_n)) \end{bmatrix}$$

where $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^p$.

Theorem

The matrix of a composite is the composite of the matrices:

$$[T \circ S] = [T][S]$$

Proof.

Note that the terms $S(e_i)$ are the columns of $[S]$. $\square$

Determinant of rotations and reflections

Example (Determinant of rotation)

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

Example (Determinant of reflection)

$$\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -\sin^2\theta - \cos^2\theta = -1$$

Composites of rotations and reflections

In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

  where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$

Examples

Example

What is the matrix of the linear transformation that is the composite of reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$?

Example

What is the matrix of the linear transformation that is the composite of reflection in the line $y = -x$ followed by reflection in the y-axis.

Example

If

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

then find $S \circ T$, $T \circ S$ and the matrices $[S \circ T]$ and $[T \circ S]$.

Inverses

Inverse functions

Definition (Identity transformation)

The *identity transformation on* $\mathbb{R}^n$ is the function $\mathbb{R}^n \to \mathbb{R}^n$ defined by $x \mapsto x$.

Definition

If $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations then S is inverse to T iff

$$(S \circ T) = id = (T \circ S)$$

and we write $S = T^{-1}$ and $T = S^{-1}$.

Theorem

If $[S]$ *is the matrix of* S *and* $[T]$ *is the matrix of* T *then:*

$$S^{-1} = T \iff [S]^{-1} = [T]$$

Examples

Example

If

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

find T^{-1} and $[T^{-1}]$.

Example

In general, what is the inverse of the linear transformation

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Complex numbers

Progression of thought about complex numbers

> *...the whole matter seems to rest on sophistry rather than truth...*

(Rafael Bombelli 1526-1572. Early innovator in use of complex numbers)

> *The shortest path between two truths in the real domain passes through the complex domain.*

(Jacques Hadamard 1865-1963)

Algebraic motivation - extending the number system

God made the integers; all else is the work of man.

(Leopold Kronecker)

- Using only positive integers $\{1, 2, 3, \dots\}$
  - we cannot find solution a to $x + 1 = 0$
- Using only integers $\{\dots, -2, -1, 0, 1, 2 \dots\}$
  - we cannot find a solution to $3x + 2 = 0$
- Using only fractions $\{\frac{a}{b}$ for $b \neq 0\}$
  - we cannot find a solution to $x^2 - 2 = 0$
- Using only decimals
  - we cannot find a solution to $x^2 + 1 = 0$...

Complex numbers

Definition

- The *imaginary unit*, denoted i, is defined to be a number with the property that $i^2 = -1$.
- A *pure imaginary* number has the form bi where $b \in \mathbb{R}$, $b \neq 0$.
- A *complex number* is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

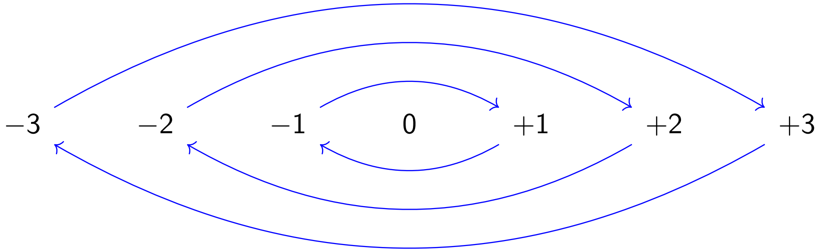
Definition

If $z = a + bi$ then:

- a is called the *real part* of z.
- b is called the *imaginary part* of z.

Graphical interpretation

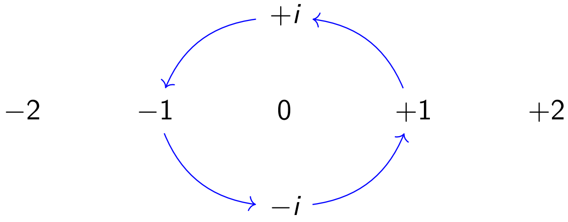
Consider the function $\mathbb{R} \to \mathbb{R}$ that multiplies by -1:



which shows us that $-1 \times (-1 \times x) = x$.

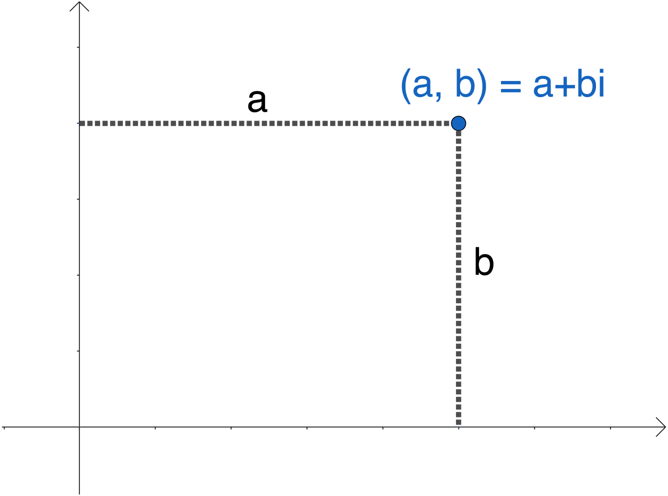
Graphical interpretation

So how do we get a transformation that squares to -1?

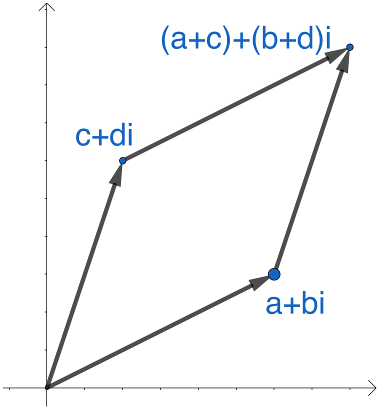


Graphical interpretation

So we draw the complex number $a + bi$ as the point (a, b) in $\mathbb{R}^2$:



Addition of complex numbers



Complex addition is defined component-wise:

$$(a+ib)+(c+id) = (a+c)+(b+d)i$$

and therefore has the same graphical interpretation as the addition of vectors in $\mathbb{R}^2$.

Examples

Example

Evaluate

- $(-3+6i)+(5-i)$.
- $(4-7i)+(6-2i)$.
- $(-3+6i)-(5-i)$.
- $(4-7i)-(6-2i)$.

Example

$$(2-3i)(-3+4i)$$

Historical curiosity

Apparently Bombelli describes complex arithmetic thus:

> *"Plus by plus of minus, makes plus of minus. Minus by plus of minus, makes minus of minus. Plus by minus of minus, makes minus of minus. Minus by minus of minus, makes plus of minus. Plus of minus by plus of minus, makes minus. Plus of minus by minus of minus, makes plus. Minus of minus by plus of minus, makes plus. Minus of minus by minus of minus makes minus."*

according to Crossley in 'The Emergence of Number'. A hint from Crossley is: the last two lines would become

$$-\sqrt{-a}\cdot\sqrt{-b} = \sqrt{ab} \text{ and } (-\sqrt{-a})\cdot(-\sqrt{-b}) = -\sqrt{ab}$$

in modern notation.

Summary

- Graphical interpretation.
- Addition: algebra and geometry.
- Multiplication: algebra and geometry.
- Conjugate: algebra and geometry.

Examples

Example

Find all complex numbers z such that $z^2 = -3 + 4i$.

Example

- If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$, i.e., $\overline{3 + 4i} = 3 - 4i$.
- $\overline{-2 + 5i} = -2 - 5i$.
- $\bar{i} = -i$.
- $\overline{7} = 7$.

Clearing denominators

Example (Exemplar)

Write

$$\frac{a+bi}{c+di}$$

in the form $e+fi$ for real numbers a, b, c, d, e, and f.

- $\frac{1}{i}$
- $\frac{2-i}{3+4i}$
- $\frac{1-2i}{-2+5i}$