# MATH211: Linear Methods I

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# Lecture on Thursday 4th October, 2018

Adjugates and inverses

Cramer's rule

Common determinants

Polynomial interpolation

# Last time

- Cofactor expansion along first row
- Cofactor expansion along arbitrary row or column
- Adjugate matrix

# Adjugates and inverses

# Adjugate matrix

## Definition (Matrix of cofactors)

$$
\begin{aligned}
cof(A)_{ij} &= cof_{ij}(A) \\
&= (-1)^{i+j} minor_{ij}(A) \\
&= (-1)^{i+j} \left| A(i|j) \right|
\end{aligned}
$$

## Definition (Adjugate matrix)

$$
\begin{aligned}
adj(A)_{ij} &= \left(cof(A)_{ij}\right)^T \\
&= cof_{ji}(A) \\
&= (-1)^{j+i} \left| A(j,i) \right|
\end{aligned}
$$

# Theorems

### Theorem (Adjugate is almost inverse)

$$A \cdot adj(A) = adj(A) \cdot A = det(A) I$$

*Both the left and right hand side of this equation are matrices.*

So if $det(A)$ is not 0 then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

In dimension higher than 2 or 3 this is very inefficient.

# Example

### Example

Find the inverse for

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

using the adjugate matrix formula.

Questions?

# Cramer's rule

# Using the inverse to solve a system

Recall that if $A$ is invertible then

$$Ax = b \implies x = A^{-1}b$$

and we can use the formula above to calculate $A^{-1}b\ldots$

# Using adjugate to solve system

$$
\begin{aligned}
x_i &= (A^{-1}b)_{ij} \\
&= \sum_{k=1}^{n} (A^{-1})_{ik} b_k \\
&= \sum_{k=1}^{n} \frac{1}{det(A)} adj(A)_{ik} b_k \\
&= \frac{1}{det(A)} \sum_{k=1}^{n} b_k (-1)^{i+k} \left| A(k,i) \right| \\
&= \frac{1}{det(A)} det(A_i)
\end{aligned}
$$

where $A_i$ results from replacing the $i$th column of $A$ with $b$.

# Cramer's rule

If $A$ is invertible then the solutions to

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

are

$$x_i = \frac{det(A_i)}{det(A)}$$

Again this is not efficient computationally.

# Example

## Example

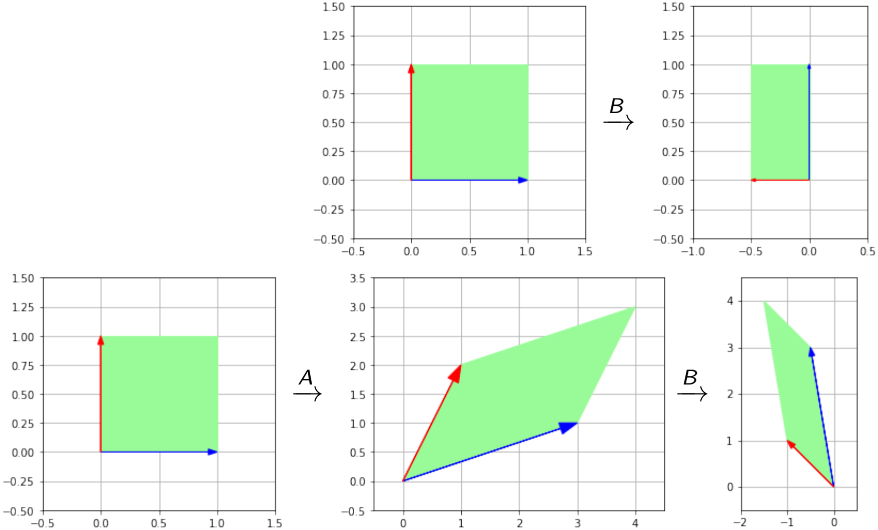
Find $x_2$ such that

$$\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Questions?

# Common determinants

# Determinant of product (picture)



# Determinants and elementary matrices

The effects of the elementary operations:-

- Multiplying a row/column by a scalar $k$
  - multiplies the determinant by $k$.
- Swapping two rows/columns
  - multiplies the determinant by $-1$.
- Adding a multiple of one row/column to another
  - has no effect on the determinant.

can be carried out by multiplying on the left by the corresponding elementary matrix:

$$A \mapsto EA$$

# Determinants and elementary matrices

Therefore:

- if $E$ multiplies a row/column by a scalar $k$
  - $det(EA) = k \cdot det(A)$
  - $det(E) = k$
- if $E$ swaps two rows/columns
  - $det(EA) = -det(A)$
  - $det(E) = -1$
- if $E$ adds a multiple of one row/column to another
  - $det(EA) = det(A)$
  - $det(E) = 1$

# Determinant of product (algebra)

**Lemma (Determinant of product of elementary matrices)**

*If $E$ and $F$ are elementary matrices then*

$$det(EF) = det(E)det(F)$$

**Theorem (Determinant of product)**

$$det(BA) = det(B)det(A)$$

# Determinant of inverse and adjugate

### Theorem (Determinant of inverse)

*If $A$ is invertible then*

$$det(A^{-1}) = \frac{1}{det(A)}$$

### Theorem (Determinant of adjugate)

*For any square matrix $A$*

$$det(adj(A)) = (det(A))^{n-1}$$

# Examples

**Example**

Find

$$\left| \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right|$$

**Example**

Find

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 8 \end{bmatrix} \right|$$

# Examples

## Example

Suppose $A$, $B$ and $C$ are $4 \times 4$ matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3 B(A^{-1}))$.

## Example

Suppose $A$ is a $3 \times 3$ matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Questions?

# Polynomial interpolation

# Example

## Example

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of $y$ corresponding to $x = \frac{3}{2}$.

Questions?