MATH211: Linear Methods I

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Thursday 27$^{\text{th}}$ September, 2018

Lecture on Thursday 27$^{\text{th}}$ September, 2018

Last time

- Elementary matrices
- Elementary matrices as theoretical tool
- Matrix equations

Determinants

Two dimensional determinants

The standard basis vectors get sent to the columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

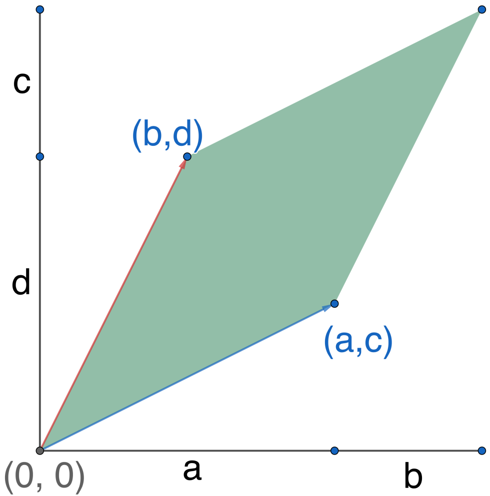
Definition

The *determinant*

$$det(A) = |A|$$

of any square matrix A is the *signed area* spanned by the columns of the matrix.

Picture of two dimensional determinant



$$
\begin{aligned}
Area &= (a+b)(c+d) \\
&\quad - bc - bc \\
&\quad - \frac{1}{2}ac - \frac{1}{2}bd \\
&\quad - \frac{1}{2}ac - \frac{1}{2}bd \\
&= ac + ad + bc + bd \\
&\quad - 2bc - ac - bd \\
&= ad - bc
\end{aligned}
$$

Examples

Example

All identity matrices I_n have determinant equal to 1.

Example

All matrices with repeated columns have determinant equal to 0.

Example

In fact a matrix is invertible iff its determinant is **not** equal to 0.

Effect of elementary row/column operations

The effect of the elementary row/column operations on $det(A)$ are:-

- Multiplying a row/column by a scalar k
  - multiplies the determinant by k.
- Swapping two rows/columns
  - multiplies the determinant by -1.
- Adding a multiple of one row/column to another
  - has no effect on the determinant.

Pictures on Jupyter notebook.

Determinants of diagonal matrix and scalar multiple

Example (Diagonal matrices)

$$det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \lambda_1\lambda_2\lambda_3\lambda_4$$

Example (Scalar multiple)

If A is an $n \times n$ matrix then

$$det(kA) = k^n det(A)$$

Determinant of triangular matrix

Example

Triangular matrix

$$det \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & a_{14} \\ 0 & \lambda_2 & a_{23} & a_{24} \\ 0 & 0 & \lambda_3 & a_{34} \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ a_{21} & \lambda_2 & 0 & 0 \\ a_{31} & a_{32} & \lambda_3 & 0 \\ a_{41} & a_{42} & a_{43} & \lambda_4 \end{bmatrix}$$

$$= det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$$= \lambda_1 \lambda_2 \lambda_3 \lambda_4$$

Calculating the determinant using row/column operations

- Start with a square matrix A.
- Perform row/column operations on A to convert into triangular form.
  - (Keep track of these because they can change the determinant!)
  - If we ever get a zero row/column then the determinant is 0.
- Calculate the determinant of the resulting triangular matrix.

Questions?

Examples

Examples

Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

Example

Find

$$\begin{vmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

Examples

Example

Find

$$\begin{vmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{vmatrix}$$

Example

If

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4 \text{ find } \begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix}$$

Examples

Example

Find

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix}$$

Questions?

Traditional determinants

Minors and cofactors

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The ij^{th} minor of A, denoted as $minor\,(A)_{ij}$, is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Examples

The following example is corrected from lectures.

Example

The (1,2)-minor of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

is

$$\begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} = 12 - 5 = 7$$

Example

The following example is corrected from lectures.

Example

The (2,2)-minor of

$$\begin{bmatrix} 1 & 5 & 2 & 5 \\ 2 & 6 & 7 & 1 \\ 9 & 9 & 8 & 2 \\ 0 & 8 & 6 & 0 \end{bmatrix}$$

is

$$\begin{vmatrix} 1 & 2 & 5 \\ 9 & 8 & 2 \\ 0 & 6 & 0 \end{vmatrix} = \cdots = 258$$

(Please see written notes.)

Cofactors

Definition

The ij^{th} *cofactor* of A is

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} minor\,(A)_{ij}$$

Examples

The following examples are corrected from lectures.

Example

The (1,2)-cofactor of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

is $(-1)^{1+2} minor_{12}(A) = -7$.

Example

The (2,2)-cofactor of

$$\begin{bmatrix} 1 & 5 & 2 & 5 \\ 2 & 6 & 7 & 1 \\ 9 & 9 & 8 & 2 \\ 0 & 8 & 6 & 0 \end{bmatrix} \text{ is } (-1)^{2+2} minor_{22}(A) = 258$$

Recursive definition of determinant

Definition (Cofactor expansion along row i)

$$\begin{aligned}\det A &= \sum_{k=1}^{n} a_{ik}\mathrm{cof}(A)_{ik} \\ &= a_{i1}\mathrm{cof}(A)_{i1} + a_{i2}\mathrm{cof}(A)_{i2} + a_{i3}\mathrm{cof}(A)_{i3} + \cdots + a_{in}\mathrm{cof}(A)_{in}\end{aligned}$$

Definition (Cofactor expansion along column j)

$$\begin{aligned}\det A &= \sum_{k=1}^{n} a_{kj}\mathrm{cof}(A)_{kj} \\ &= a_{1j}\mathrm{cof}(A)_{1j} + a_{2j}\mathrm{cof}(A)_{2j} + a_{3j}\mathrm{cof}(A)_{3j} + \cdots + a_{nj}\mathrm{cof}(A)_{nj}\end{aligned}$$

Three dimensions

Example (Row 1 expansion)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example (Row 2 expansion)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Questions?

Examples

Examples

Example

Find

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{vmatrix}$$

Examples

Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

Example

Find

$$\begin{vmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{vmatrix}$$

Examples

Example

Find

$$det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Example

Find

$$det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Questions?