MATH211: Linear Methods I

Matthew Burke

Tuesday 2nd October, 2018

Lecture on Tuesday 2nd October, 2018

Last time

- Determinants as area/volume etc..
- Computing determinants using row operations.
- The cofactor expansion (see again this time).

First row expansion

Two dimensional determinant

Recall that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

$$\begin{vmatrix} 10 & 1 \\ 9 & 5 \end{vmatrix} = 50 - 9 = 41$$

Example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

Three dimensional determinants

Definition

The *cofactor expansion along row 1 is*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix}$$

$$= 1 \cdot (24 - 2) - 1 \cdot (12 - 5) + 3 \cdot (4 - 20)$$

$$= 22 - 7 - 48 = -33$$

Examples

Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Expansions in general

Definition (Non-standard)

If A is a matrix write $A(i|j)$ for the matrix obtained from A by deleting the ith row and the jth column.

Definition

The *cofactor expansion of A along the first row is*

$$|A| = \sum_{k=1}^{n} a_{1k}(-1)^{1+k}\,|A(1|k)|$$

which when expanded is:

$$|A| = a_{11}\,|A(1|1)| - a_{12}\,|A(1|2)| + a_{13}\,|A(1|3)| - \cdots + (-1)^{1+n}\,|A(1|n)|$$

Auxiliary definitions

Definition (Minor)

$$minor_{ij}(A) = |A(i|j)|$$

Definition (Cofactor)

$$\begin{aligned} cof_{ij}(A) &= (-1)^{i+j} minor_{ij}(A) \\ &= (-1)^{i+j} |A(i|j)| \end{aligned}$$

Lemma (Expansion along row 1)

$$|A| = \sum_{k=1}^{n} a_{1k} cof_{1k}(A)$$

Examples

Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

Example

Find

$$\begin{vmatrix} 2 & 5 & 4 & 3 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

Questions?

Arbitrary expansion

General cofactor expansion

In fact we may use the cofactors of arbitrary rows and columns:

Lemma (Expansion along arbitrary row or column)

$$
\begin{aligned}
|A| &= \sum_{k=1}^{n} a_{ik}(-1)^{i+k}\,|A(i|k)| \\
&= \sum_{k=1}^{n} a_{kj}(-1)^{k+j}\,|A(k|j)|
\end{aligned}
$$

So we can choose the row or column which is most convenient.

Examples

Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

by expanding along row 3 or column 1.

Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

by expanding along column 2.

Example

Example

Find

$$\begin{vmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{vmatrix}$$

by choosing the correct column. . .

Questions?

Inversion formula

Duplicating rows

Suppose that an $n \times n$ matrix has ith row equal to jth row:

$$|A| = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then we know that $|A| = 0$. (Why?)

Adjugate matrix

If we expand

$$|A| = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

along the jth row then we get:

$$\begin{aligned} 0 = |A| &= \sum_{k=1}^{n} a_{jk}(-1)^{j+k}\,|A(j|k)| \\ &= \sum_{k=1}^{n} a_{ik}(-1)^{j+k}\,|A(j|k)| \end{aligned}$$

The adjugate matrix

Now the last term looks like a matrix product:

$$\sum_{k=1}^{n} a_{ik}(-1)^{j+k}\,|A(j|k)| = (A(adj(A)))_{ij}$$

Definition (Adjugate matrix)

$$\begin{aligned} adj(A)_{ij} &= (-1)^{i+j}\,|A(j|i)| \\ &= cof_{ji}(A) \end{aligned}$$

So $adj(A)$ is the transpose of the matrix of cofactors $cof_{ij}(A)$.

A theorem

Theorem

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

Proof.

Just compute the ij-entries:

$$\begin{aligned}(A \cdot adj(A))_{ij} &= \sum_{k=1}^{n} a_{ik} adj(A)_{kj} \\ &= \sum_{k=1}^{n} a_{ik} (-1)^{j+k} \left| A(j|k) \right|\end{aligned}$$

$\square$

A theorem continued...

Theorem

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

Proof.

...so on the diagonal (when $i = j$) the entry is

$$\sum_{k=1}^{n} a_{ik}(-1)^{i+k} |A(i|k)| = det(A)$$

and off the diagonal

$$\sum_{k=1}^{n} a_{ik}(-1)^{j+k} |A(j|k)| = 0$$

as required. □

The inversion formula

So we know that

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

and therefore if $det(A) \neq 0$ then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Warning: for $n > 2$ this is a very inefficient way to compute an inverse.

Examples

Example

Compute the inverse of the following matrices:-

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$$

Questions?