MATH211: Linear Methods I

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Lecture on Tuesday 30th October, 2018

Last time

- Triangle inequality and Lagrange identity.
- More example problems in Euclidean space.
- Linear transformations.

Linear transformations

Linear transformations

Definition

A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function satisfying:

- (Additivity) $T(x + y) = T(x) + T(y)$
- (Scalar multiplication) $T(kx) = kT(x)$

Examples

Example

Is the following a linear transformation?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix}$$

Example

Is the following a linear transformation?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$$

Examples

Example

Is rotation about the origin in $\mathbb{R}^2$ through an angle θ a linear transformation?

Example

Is reflection in the x-axis a linear transformation?

Example

Is $f(x) = x + 1$ a linear transformation?

Action on combinations

Action on combinations

Recall that if T is linear then

$$T\left(\sum_{i=0}^{n} k_i v_i\right) = \sum_{i=0}^{n} k_i T(v_i)$$

so if we know $T(v_i)$ then we can work out the value of T on any linear combination of the v_i.

Examples

Example

Find

$$T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$$

if T is linear and

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \text{ and } T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Examples

Example

Find

$$T \begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}$$

if T is linear and

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

Matrices redux

Linear transformations determined by action on basis

If T is linear then

$$T\left(\sum_{i=0}^{n} k_i v_i\right) = \sum_{i=0}^{n} k_i T(v_i)$$

But for all v in $\mathbb{R}^n$ we can write v as a linear combination of the standard basis vectors:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=0}^{n} v_i e_i = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots v_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Linear transformation determined by action on basis

So:

$$T(v) = T\left(\sum_{i=0}^{n} v_i e_i\right) = \sum_{i=0}^{n} v_i T(e_i)$$

Slogan: A linear transformation is completely determined by its action on the standard basis vectors.

More explicitly: we only need to know

$$T(e_1), T(e_2), \ldots, T(e_n)$$

to work out the whole transformation!

Matrices redux

If $T : \mathbb{R}^n \to \mathbb{R}^m$ then

$$T(e_1), T(e_2), \ldots, T(e_n)$$

are in $\mathbb{R}^m$. And so the transformation is completely determined by $n \times m$ scalars...

Matrices redux

Definition

The *matrix* $[T]$ *of a linear transformation* T is the matrix that has columns given by the images of the standard basis vectors:

$$[T] = \left[\begin{array}{cccc} T(e_1) & T(e_2) & \ldots & T(e_n) \end{array}\right]$$

(It turns out that the definition of matrix product is forced on us after this choice if we want it to match the composition of functions.)

Examples

Example

Find the matrix of the linear transformation

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

Example

Find the matrix of the linear transformation satisfying

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Examples

Example

For the following linear transformations S and T what is $S \circ T$, $T \circ S$ and what are the corresponding matrices?

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Example

What is the inverse transformation and corresponding matrix for

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

Examples

Example

What is the matrix of the linear transformation for counter-clockwise rotation about the origin by an angle of θ?

Example

What is the matrix of the linear transformation for reflection in the x-axis?

Example

What is the matrix of the linear transformation that is $v \times (-)$ for some vector $v \in \mathbb{R}^3$?

Examples

Example

What is the matrix of the linear transformation for reflection in the line $y = mx$?

Composition

Composition of functions

Definition

If we have functions

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \text{ and } \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

then the *composite function* $g \circ f$ is the function defined by

$$(g \circ f)(x) = g(f(x))$$

- I.e. first apply f to x and then apply g to the result.
- Note that $\mathbb{R}^n \xrightarrow{g \circ f} \mathbb{R}^p$.

Example

The composite of two linear transformations is again linear. (Easy exercise.)

The matrix of a composite transformation

We know that the matrix of the composite $T \circ S$ is:

$$\begin{bmatrix} T(S(e_1)) & T(S(e_2)) & \dots & T(S(e_n)) \end{bmatrix}$$

where $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^p$.

Question: What is this in terms of the matrices of T and S?

Theorem

The matrix of a composite is the composite of the matrices:

$$[T \circ S] = [T][S]$$

Proof.

Note that the terms $S(e_i)$ are the columns of $[S]$. $\square$

Examples

Example

What is the matrix of the linear transformation consisting of first reflects in the x-axis and then rotates through an angle of $\frac{\pi}{2}$?

Example

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y-axis.

Summary of linear transformations

- Recall interpretation of matrix action on a vector
- Definition of linear transformation; examples and non-examples
- The matrix of a linear transformation; completely determined by action on basis
- Linear transformation preserves 0, - and linear combinations
- Composition of functions, linear transformation and relationship to matrices.
- Inverse functions and relationship to inverse matrices.
- Determinant of rotation and reflection. Composite of rotations/reflections with rotations/reflections