

MATH211: Linear Methods I

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Similar matrices

Dynamical systems

Markov chains

PageRank

Last time

▶ Diagonalisation

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Last time
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Similar matrices
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Similar matrices

Similar matrices

Definition

Two matrices A and B are *similar* iff there exists an invertible P such that

$$A = PBP^{-1}$$

and we write $A \sim B$.

Example

Every diagonalisable matrix is similar to a diagonal matrix.

Trace of a matrix

Definition

The trace of a matrix is the sum of its diagonal elements:

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

Theorem

If $A \sim B$ then $\text{tr}(A) = \text{tr}(B)$.

Dynamical systems

Dynamical systems

Definition

A *dynamical system* consists of a function $\alpha(t)$ that prescribes how the state of the system changes over time.

Definition

A *discrete linear dynamical system* consists of a sequence of vectors

$$x_0, x_1, x_2, \dots, x_k, \dots$$

such that $x_{k+1} = Ax_k$ for some matrix A .

Long term behaviour using eigenvectors

If x_0 is a linear combination of the eigenvectors v_{λ_i} of A then

$$x_k = A^k x_0 = A^k \left(\sum_{i=1}^n b_i v_{\lambda_i} \right) = \sum_{i=1}^n b_i A^k (v_{\lambda_i}) = \sum_{i=1}^n b_i (\lambda_i)^k v_{\lambda_i}$$

and so the long-term behaviour is determined by the limits:

$$\lim_{k \rightarrow \infty} (\lambda_i)^k$$

Dominant eigenvalue

Definition

If A is a square matrix then a *dominant eigenvalue* λ_{max} is one for which $|\lambda_{max}| > |\lambda_i|$ for all other eigenvalues λ_i .

If this exists then

$$x_k = \sum_{i=1}^n b_i (\lambda_i)^k v_{\lambda_i} \approx b_i (\lambda_{max})^k v_{\lambda_{max}}$$

and we can read off the long term behaviour. E.g.

- ▶ if $|\lambda_{max}| < 1$ then the system converges to 0
- ▶ if $|\lambda_{max}| = 1$ then the system converges to $b_i v_{\lambda_{max}}$
- ▶ if $|\lambda_{max}| = -1$ then the system oscillates between $\pm b_i v_{\lambda_{max}}$
- ▶ if $|\lambda_{max}| > 1$ then the system diverges

Examples

Example

Find a formula for x_k if $x_{k+1} = Ax_k$,

$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$$

Example

Estimate the long term behaviour of the dynamical system with

$$x_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix} \text{ and } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$$

Markov chains

Markov chains

A *Markov chain* consists of:-

- ▶ a finite set of *states* x_1, x_2, \dots, x_n
- ▶ a repeated *transition interval* at the end of which the system transitions between states
- ▶ a *not necessarily deterministic rule* for predicting the probability that the system will transition into a certain state
 - ▶ **This probability only depends on the current state.**
 - ▶ (Not the entire history of the chain.)

Markov transition matrices

This means that a Markov chain is described by a *transition matrix* A such that

$$\begin{aligned} A_{ij} &= \mathbb{P}(X_1 = j | X_0 = i) \\ &= \text{the probability that the next state will be } j \\ &\quad \text{given that the current state is } i \end{aligned}$$

Therefore:-

- ▶ all of the entries are between 0 and 1
 - ▶ (I.e. they are probabilities.)
- ▶ in any column the sum of the entries is 1
 - ▶ (The system must be in one of the states at all times.)

Steady state

Definition

A *steady state vector* for a matrix A is an eigenvector with eigenvalue 1.

(In other words it is a *non-zero* vector v such that $Av = v$.)

Definition

A *steady state vector* for a *Markov chain* is a steady state vector for the transition matrix of the Markov chain such that the entries sum to 1.

Regular Markov chains

Definition

A matrix A is *regular* iff there exists a positive integer k such that all of the entries of A^k are *strictly positive*.

Theorem

If the transition matrix of a Markov chain is regular then it has a steady state vector.

Example

The converse is false! E.g. the following matrix has two steady state vectors:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Example

Calculate and interpret x_3 given that

$$x_0 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$$

Example

What is the probability that we are in state 1 after two iterations if

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Examples

Example

Find a steady state vector for

$$\begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$$

Example

Find a steady state vector for

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

PageRank

Simplified Google PageRank

The core part of Google PageRank (mainly due to Larry Page) is calculated recursively:-

- ▶ Suppose we have n pages: $P_1, P_2 \dots P_n$
- ▶ denote the PageRank of P_i by $PR(i)$
- ▶ then (recursively) define

$$PR(i) = \sum_{j=1}^n \frac{PR(j) \cdot L_{ji}}{Out(j)}$$

where L_{ji} is the number of links from P_j to P_i and $Out(j)$ is the total number of links on page j .

At first glance it is not even clear that this is well-defined. . .

Steady state vector

To work out the PageRank for each site:-

- ▶ Form the matrix with entries $A_{ij} = \frac{L_{ji}}{Out(j)}$.
- ▶ Find a steady state vector for this system.

PageRank Example

Example (Simplified PageRank)

Suppose that the internet consists of three(!) pages P_1 , P_2 and P_3 . Suppose further that:-

- ▶ there are two links from P_1 to P_2
- ▶ there is one link from P_1 to P_3
- ▶ there is one link from P_2 to P_1
- ▶ there is one link from P_2 to P_3
- ▶ there is one link from P_3 to P_2

Then calculate the (relative) PageRanks for P_1 , P_2 and P_3 .

Example

Example

If possible diagonalise

$$\begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$$