Math 211 2018-11-08:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

$$|-3+4i| = \left\| \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\| = \sqrt{9+16} = 5$$

$$|3-2i| = \sqrt{3^2+(-2)^2} = \sqrt{13}$$

$$|i| = \sqrt{0^2+1^2} = 1$$

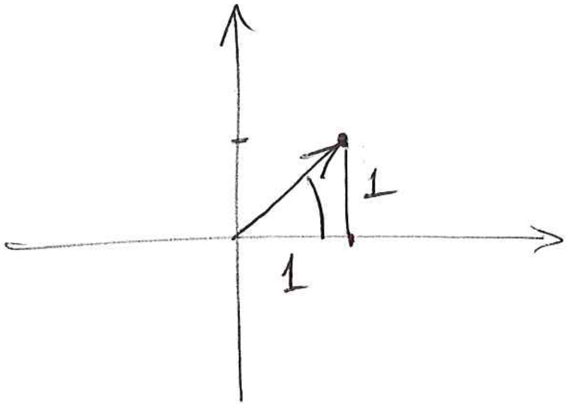
$$\overline{3+4i} = 3-4i$$

$$\overline{-2+5i} = -2-5i$$

$$\overline{i} = -i$$

$$\overline{7} = 7.$$

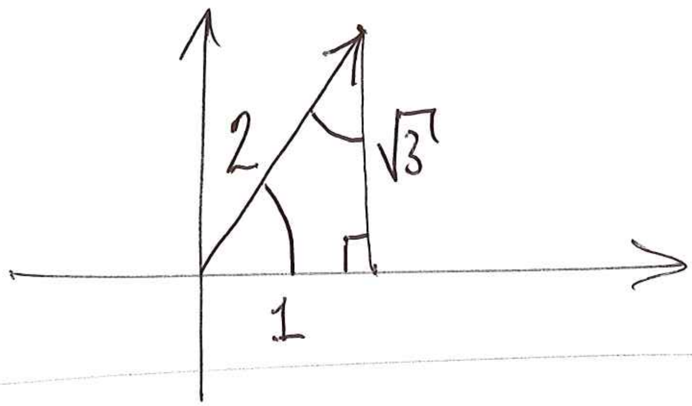
$(1,1)$:



$\theta = \frac{\pi}{4}$

$R = \sqrt{2}$

$(1, \sqrt{3})$:

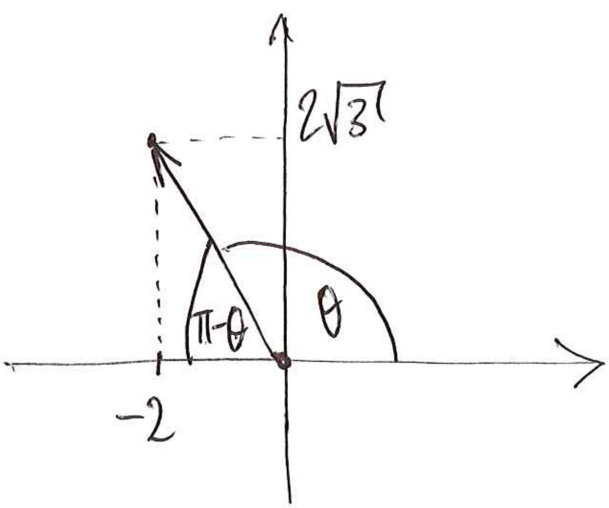


$\theta = \frac{\pi}{3}$

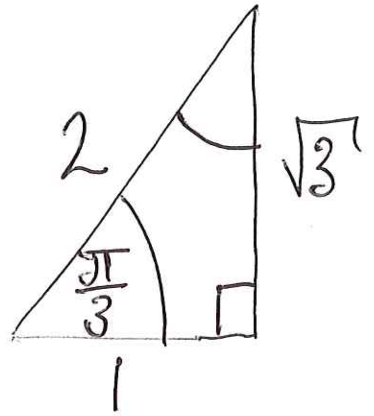
$R = 2$.

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

$-2 + 2\sqrt{3}i$: $R = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$



$$\pi - \theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \tan^{-1}(\sqrt{3})$$



$$\therefore \quad \pi - \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}.$$

$$-2 + 2\sqrt{3}i = 4e^{\frac{2\pi i}{3}}$$

Lemma: The function $f(t) = e^{-it}(\cos(t) + i\sin(t))$ is constant in t. (Where t is a real variable.)

Proof: We show that the derivative is zero for all t:

$$
\begin{aligned}
f'(t) &= -ie^{-it}(\cos(t) + i\sin(t)) + e^{-it}(-\sin(t) + i\cos(t)) \\
&= e^{-it}(-i\cos(t) + \sin(t) - \sin(t) + i\cos(t)) \\
&= 0.
\end{aligned}
$$

$\square$

Theorem: (Euler's formula)

$$e^{it} = \cos(t) + i\sin(t)$$

Proof: Since $f(t) = e^{-it}(\cos(t) + i\sin(t))$ is constant in t we see:

$$f(t) = f(0) = e^{0}(\cos(0) + i\sin(0)) = 1.$$

Therefore

$$1 = e^{-it}(\cos(t) + i\sin(t))$$

$$\Rightarrow e^{it} = \cos(t) + i\sin(t) \text{ as required.}$$

$\square$