MATH211: Linear Methods I

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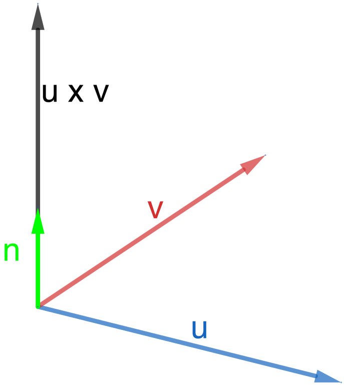
Lecture on Tuesday 23rd October, 2018

Last time

- Closest points
- Examples
- Cross product

Cross product

Trigonometric definition of cross product



Alternatively:

$$u \times v = ||u||||v||sin\theta n$$

where n is a unit vector orthogonal to both v and u.

Aide-memoire:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Cross product on standard basis vectors

If

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then

$$\begin{aligned} e_1 \times e_1 &= 0 & e_1 \times e_2 &= e_3 & e_1 \times e_3 &= -e_2 \\ e_2 \times e_1 &= -e_3 & e_2 \times e_2 &= 0 & e_2 \times e_3 &= e_1 \\ e_3 \times e_1 &= e_2 & e_3 \times e_2 &= -e_1 & e_3 \times e_3 &= 0 \end{aligned}$$

Algebraic definition of cross product

So in general:

$$u \times v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 e_1 + u_2 e_2 + u_3 e_3) \times (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

$$\begin{aligned} =& u_1 v_1 (e_1 \times e_1) && + u_1 v_2 (e_1 \times e_2) && + u_1 v_3 (e_1 \times e_3) \\ &+ u_2 v_1 (e_2 \times e_1) && + u_2 v_2 (e_2 \times e_2) && + u_2 v_3 (e_2 \times e_3) \\ &+ u_3 v_1 (e_3 \times e_1) && + u_3 v_2 (e_3 \times e_2) && + u_3 v_3 (e_3 \times e_3) \end{aligned}$$

$$\begin{aligned} =& 0 && + u_1 v_2 e_3 && - u_1 v_3 e_2 \\ &- u_2 v_1 e_3 && + 0 && + u_2 v_3 e_1 \\ &+ u_3 v_1 e_2 && - u_3 v_2 e_1 && + 0 \end{aligned}$$

Algebraic definition

Definition

The *cross product $u \times v$ of u with v* is

$$u \times v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

The following heuristic doesn't make sense formally but may be an easier way to remember.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Properties of the cross product

Theorem

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

1. $\vec{u} \times \vec{0} = \vec{0}$ *and* $\vec{0} \times \vec{u} = \vec{0}$.
2. $\vec{u} \times \vec{u} = \vec{0}$.
3. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
4. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ *for any scalar k.*
5. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
6. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

Example

Example

Find $u \times v$ where

$$u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Example

Find all vectors orthogonal to

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Examples

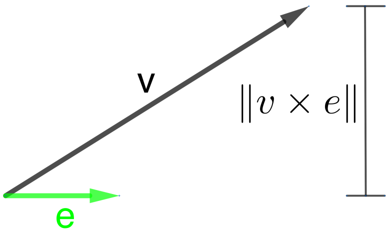
Example

Give a non-parametric equation for the plane passing through

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Applications

Cross product and projection

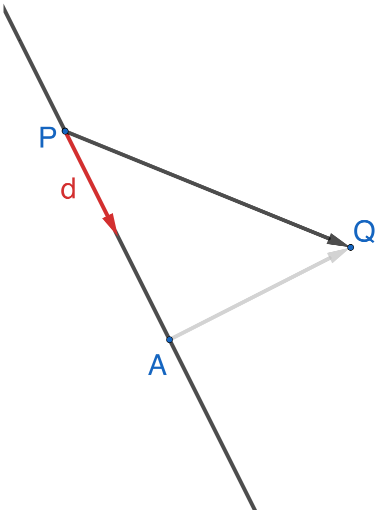


If e is a unit vector then

$$\|v \times e\| = \|v\|\|e\| sin\theta = \|v\| sin\theta$$

which is the distance from the tip of v to the line in direction e.

Line given by parametric equations

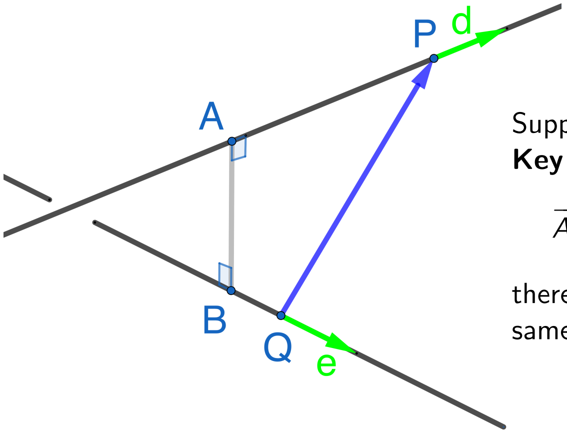


If we only require the distance from Q to the line *and are in* $\mathbb{R}^3$:

$$d(A, Q) = \left\| \frac{d}{\|d\|} \times \overrightarrow{PQ} \right\| = \frac{\|d \times (Q - P)\|}{\|d\|}$$

(Technique only works in $\mathbb{R}^3$.)

Closest points on skew lines

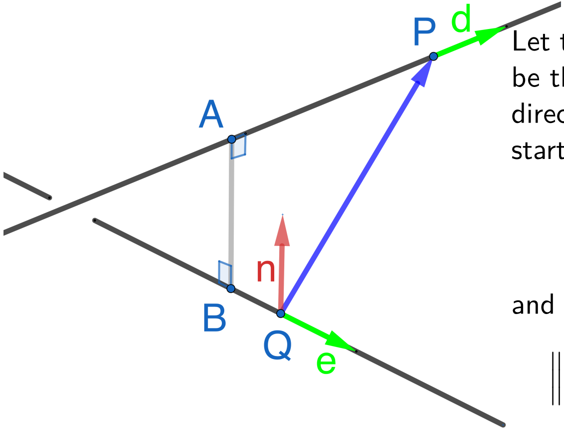


Suppose we are in $\mathbb{R}^3$.
Key observation: since

$$\overrightarrow{AB} \perp d \text{ and } \overrightarrow{AB} \perp e$$

therefore $d \times e$ is in the same direction as $\overrightarrow{AB}$.

Closest points on skew lines



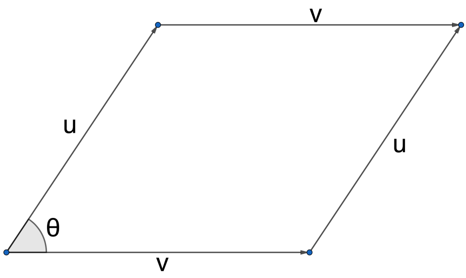
Let the n be the red vector be the unit vector in direction $\overrightarrow{AB}$ translated to start at Q.

$$n = .\frac{d \times e}{\|d \times e\|}$$

and so

$$\left\|\overrightarrow{AB}\right\| = n \cdot .\frac{d \times e}{\|d \times e\|}$$

Area of a parallelogram



The area of the parallelogram is:

$$\|u\|\|v\|sin\theta = \|u \times v\|$$

Triple scalar (box) product

Definition

If u, v and w are vectors in $\mathbb{R}^3$ then the *triple scalar product (or box product) of u, v and w* is

$$u \cdot (v \times w)$$

Note that the order of the vectors matters. In fact

Lemma

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

which shows that it is the signed volume of the parallelepiped generated by u, v and w.

Examples

Example

Find the area of the triangle having vertices

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example

Find the volume of the parallelepiped determined by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Misc

Triangle inequality

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$\|u+v\| \leq \|u\| + \|v\|$$

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$|\|u\| - \|v\|| \leq \|u-v\|$$

The Lagrange identity

Theorem

If u and v are in $\mathbb{R}^3$ then

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

Example

Example

Find equations for the lines through

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at the two points distance three from $(1, 2, 0)$.

Examples

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.

Examples

Example

Find the shortest distance between the following plane and line.

$$P_1 : x - y + 2z = 2, L_1 : \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Example

Find the shortest distance between the following two planes.

$$P_1 : x - y + 2z = 5, P_2 : 2x - 2y + 4z = 4$$

Examples

Example

Find the distance between the following two lines.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$