MATH211: Linear Methods I

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Thursday 20th September, 2018

Lecture on Thursday 20th September, 2018

Matrix inversion algorithm

Examples

Properties of inverses

Other matrix equations

Last time

- ▶ (i,j)-entry proofs

- ▶ Identity matrices

- ▶ Inverse matrices

- ▶ Small examples of inverses

Last time
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Matrix inversion algorithm
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Examples
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Properties of inverses
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Other matrix equations
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Matrix inversion algorithm

Aim of matrix inversion

Given B find A such that $BA = I_n$:

$$\left[B \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \ B \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots B \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I.e. find linear combinations of the columns of B that give each of the standard basis vectors.

The algorithm

For the first standard basis vector we solve:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftrightarrow \left[\begin{array}{cccc|c} b_{11} & b_{12} & \dots & b_{1m} & 1 \\ b_{21} & b_{22} & \dots & b_{2m} & 0 \\ \dots & \dots & \dots & \dots & 0 \\ b_{p1} & b_{p2} & \dots & b_{pm} & 0 \end{array} \right]$$

and we can group all together as:

$$\left[\begin{array}{cccc|cccc} b_{11} & b_{12} & \dots & b_{1m} & 1 & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & b_{2m} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pm} & 0 & 0 & \dots & 1 \end{array} \right]$$

Reduction

Row reduce

$$\left[\begin{array}{cccc|cccc} b_{11} & b_{12} & \ldots & b_{1m} & 1 & 0 & \ldots & 0 \\ b_{21} & b_{22} & \ldots & b_{2m} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{p1} & b_{p2} & \ldots & b_{pm} & 0 & 0 & \ldots & 1 \end{array}\right]$$

to

$$[R|A] = \left[\begin{array}{cccc|cccc} 1 & 0 & \ldots & \star & a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & 1 & \ldots & \star & a_{21} & a_{22} & \ldots & a_{2n} \\ 0 & 0 & \ldots & \star & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \star & a_{m1} & a_{m2} & \ldots & a_{mn} \end{array}\right]$$

where R is in reduced row echelon form.

Finishing off

$$[R|A] = \begin{bmatrix} 1 & 0 & \ldots & \star & a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & 1 & \ldots & \star & a_{21} & a_{22} & \ldots & a_{2n} \\ 0 & 0 & \ldots & \star & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \star & a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

If R is:-

the identity matrix then B is invertible with inverse A

anything else then B is not invertible

Relationship to solutions of systems

Recall that a system of equations is equivalently a matrix equation:

$$Ax = b$$

so if A is invertible

$$x = A^{-1}Ax = A^{-1}b$$

is the *unique solution* to $Ax = b$.

This method is rarely the most efficient thing to do.

Questions?

Examples

Low dimensional examples

Example

Find the inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

Find the inverse for the general 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Examples

Example

Find the inverse for

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Questions?

Properties of inverses

Inverse of transposes and products

Example
Prove that $(BA)^T = A^T B^T$.

Example
Using the previous example prove that

$$\text{if } A \text{ has inverse } A^{-1}$$
$$\text{then } A^T \text{ has inverse } (A^{-1})^T.$$

Example
Prove that

$$\text{if } A \text{ has inverse } A^{-1} \text{ and } B \text{ has inverse } B^{-1}$$
$$\text{then } BA \text{ has inverse } A^{-1} B^{-1}.$$

Inverses of products

Alternative characterisation of invertible matrices

For square matrices $BA = I$ is the same as $AB = I$...

Other matrix equations

Example

Questions?

Questions?