MATH211: Linear Methods I

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Thursday 13$^{\text{th}}$ September, 2018

Lecture on Thursday 13th September, 2018

Last time

Non-commutativity

(i,j)-entry proofs

Identity and inverse matrices

Inversion algorithm

Last time

- ▶ Matrix addition and scalar multiplication

- ▶ Relationship to systems of equations

- ▶ Matrix composition (product)

- ▶ Non-commutativity of matrix composition

- ▶ (Two sets of notes for 11th September)

Non-commutativity

Last time
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Non-commutativity
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(i,j)-entry proofs
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Identity and inverse matrices
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Inversion algorithm
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Review previous examples

Review previous examples with original matrices printed.

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Non-commutativity
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(i,j)-entry proofs
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Identity and inverse matrices
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Inversion algorithm
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(i,j)-entry proofs

Entries of product matrix

Definition
If A and B are matrices such that

$$\text{No. of rows}(A) = m = \text{No. of columns}(B)$$

then

$$(BA)_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

$$= b_{i1} a_{1j} + b_{i2} a_{2j} + b_{i3} a_{3j} + \cdots + b_{im} a_{mj}$$

Picture

$$(BA)_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + b_{i3} a_{3j} + \cdots + b_{im} a_{mj}$$

is the dot product of the ith row of B with the jth column of A:

$$
\begin{bmatrix}
b_{11} & b_{12} & \ldots & b_{1m} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
\ldots & \ldots & \ldots & \ldots \\
b_{i1} & b_{i2} & \ldots & b_{im} \\
\ldots & \ldots & \ldots & \ldots \\
b_{p1} & b_{p2} & \ldots & b_{pm}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1m} \\
a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2m} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nm}
\end{bmatrix}
$$

Examples of entries of matrix product

Example

Find the $(2, 3)$-entry of

$$
\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}
\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}
$$

Example

Find the $(3, 1)$-entry of

$$
\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 1 \\ 5 & 4 & 9 \end{bmatrix}
\begin{bmatrix} 12 & 1 & 2 \\ 4 & 0 & 4 \\ -2 & 0 & 0 \end{bmatrix}
$$

General examples of (i,j)-entry

In the following examples assume that the matrices are of the appropriate sizes so that the products involved make sense.

Example

Find the (i, j)-entry of

$$(CB)A$$

Example

Find the (i, j)-entry of

$$C(A + B)$$

Questions?

Transpose of a matrix

Definition
If A is *any* matrix the *transpose* A^T of A has entries

$$(A^T)_{ij} = A_{ji}$$

Example

$$\begin{bmatrix} 5 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 0 \\ -3 & 2 \end{bmatrix}$$

In short: the columns of A^T are the rows of A.

Symmetric matrices

Definition
A matrix A is *symmetric* iff $A = A^T$.
In terms of entries: $A_{ij} = A_{ji}$.

Example

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 17 \\ -2 & 17 & 0 \end{bmatrix}$$

is symmetric but

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 5 \\ -2 & 17 & 0 \end{bmatrix}$$

is not.

Examples of (i,j)-entry proofs

Example

Prove $C(B + A) = CB + CA$.

Example

Prove $(BA)^T = A^T B^T$.

Example

Prove $(B + A)^T = B^T + A^T$.

Example

Prove that if A and B are symmetric then $A^T + 2B$ is symmetric.

Example

Prove $C(BA) = (CB)A$. (Called matrix associativity.)

Last time
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Non-commutativity
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(i,j)-entry proofs
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Identity and inverse matrices
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Inversion algorithm
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Identity and inverse matrices

Identity matrix

Definition
The $n \times n$ *identity matrix* is:

$$I_n = \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{array} \right]$$

In terms of entries:

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Product with identity

Whenever the following products make sense:

$$BI_m = B \qquad \text{and} \qquad I_m A = A$$

Indeed for the first:

$$
\begin{aligned}
(BI_m)_{ij} &= \sum_{k=1}^{m} B_{ik} I_{kj} \\
&= \sum_{k \neq j} B_{jk} I_{kj} + B_{ij} I_{jj} \\
&= \sum_{k \neq j} 0 + B_{ij} = B_{ij}
\end{aligned}
$$

Inverse matrix

Definition

If A is an $n \times n$ matrix then the *inverse matrix A^{-1} of A* is a matrix such that

$$AA^{-1} = I_n \qquad \text{and} \qquad A^{-1}A = I_n$$

Some warnings:

▶ It doesn't make sense to ask for an inverse of a non-square matrix. (Think in terms of linear transformations.)

▶ Not all square matrices have inverses.

Examples of non-existence

Example

No matrix that only has zero entries has an inverse.

Example

The following matrix does not have an inverse

$$\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right]$$

In fact a $n \times n$ matrix has an inverse iff $rank(A) = n$.

Uniqueness of inverse

Lemma

If a matrix A has an inverse then it is unique.

Questions?

Motivation from linear combinations

Inverse of a 2x2 matrix

Examples