MATH211: Linear Methods I

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Thursday 18th October, 2018

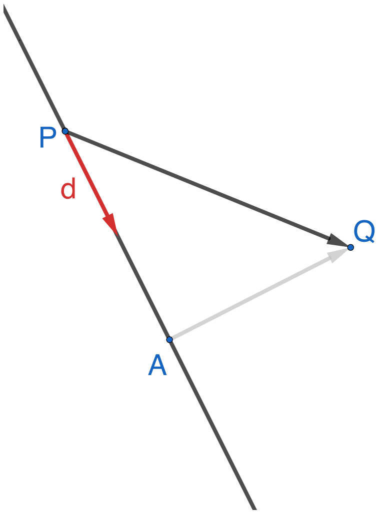
Lecture on Thursday 18th October, 2018

Last time

- Orthogonality
- Projections
- Closest points

Closest points

Line given by parametric equations



First we project onto the direction of the line:

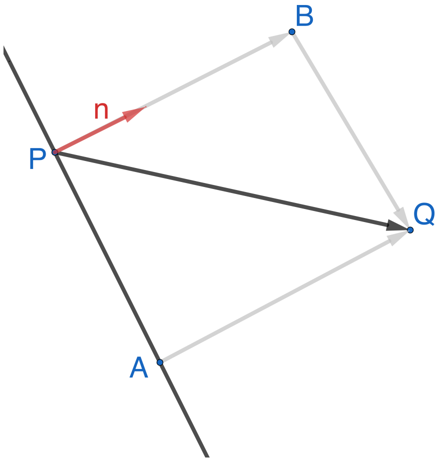
$$\overrightarrow{PA} = \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

and then we can find A:

$$A = P + \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

(Use same technique for line in $\mathbb{R}^n$ also.)

Line given by normal



In this picture

$$\overrightarrow{PB} \| \overrightarrow{AQ} \text{ and } \overrightarrow{PA} \| \overrightarrow{BQ}$$

First we project onto the normal:

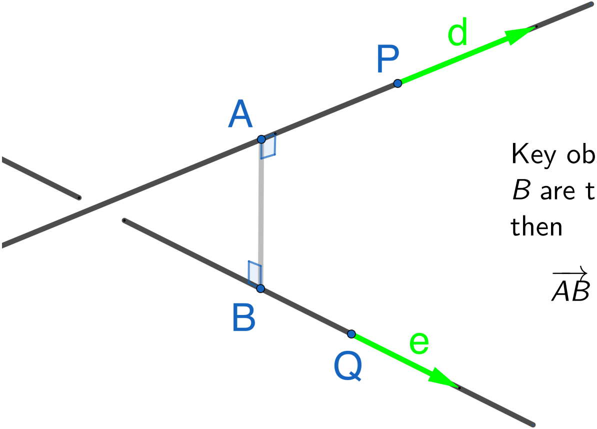
$$\overrightarrow{PB} = \frac{n}{\|n\|} \cdot (Q - P) \frac{n}{\|n\|}$$

and then we can find A:

$$A = Q - \frac{n}{\|n\|} \cdot (Q - P) \frac{n}{\|n\|}$$

(Use same technique for $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ also.)

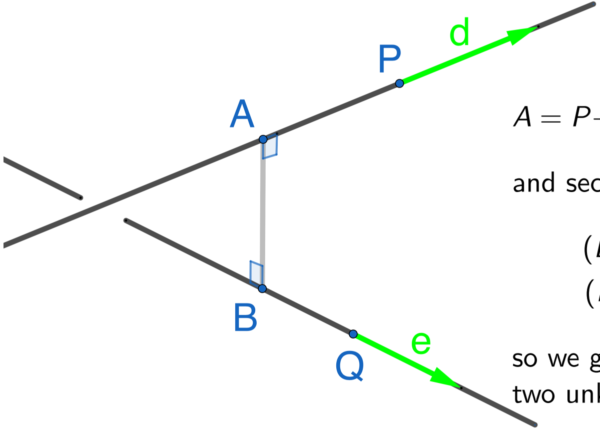
Closest points on skew lines



Key observation: if A and B are the closest points then

$$\overrightarrow{AB} \perp d \text{ and } \overrightarrow{AB} \perp e$$

Closest points on skew lines



Firstly:

$A = P + sd$ and $B = Q + te$

and secondly

$$(B - A) \cdot d = 0$$
$$(B - A) \cdot e = 0$$

so we get two equations in two unknowns s and t.

Example

Example

Find the closest point to

$$Q = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

on the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Example

Example

Find the closest point to

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

on the line with equation $4x + 3y = -3$.

Example

Find the closest point to

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

on the plane with equation $5x + y + z = -1$.

Examples

Example

Given the two lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

find points A on the first line and B on the second line such that the distance A to B is minimised.

Cross product

Motivation for cross product

The *cross product* $u \times v$ measures how non-parallel u and v are.

$$u \| v \iff \exists k.\ ku = v$$

which in $\mathbb{R}^3$ means

$$\begin{aligned} ku_1 &= v_1 \\ ku_2 &= v_2 \\ ku_3 &= v_3 \end{aligned} \qquad \implies \qquad \begin{aligned} u_1 v_2 - u_2 v_1 &= 0 \\ u_2 v_3 - u_3 v_2 &= 0 \\ u_3 v_1 - u_1 v_3 &= 0 \end{aligned}$$

Definition of cross product

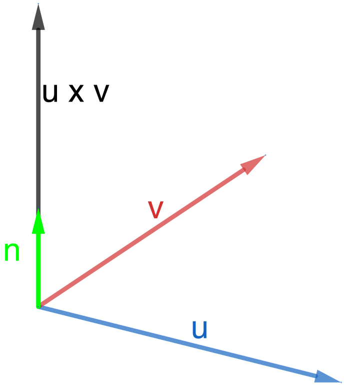
Definition

The *cross product* $u \times v$ *of* u *with* v is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Note that the the order has changed.

Trigonometric definition of cross product



Alternatively:

$$u \times v = \|u\|\|v\| sin\theta n$$

where n is the unit vector given by the 'right hand rule'.

Properties of the cross product

Theorem

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

1. $\vec{u} \times \vec{v}$ *is a vector.*
2. $\vec{u} \times \vec{v}$ *is orthogonal to both* $\vec{u}$ *and* $\vec{v}$.
3. $\vec{u} \times \vec{0} = \vec{0}$ *and* $\vec{0} \times \vec{u} = \vec{0}$.
4. $\vec{u} \times \vec{u} = \vec{0}$.
5. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ *for any scalar* k.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
8. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

Example

Example

Find $u \times v$ where

$$u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Example

Find all vectors orthogonal to

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Examples

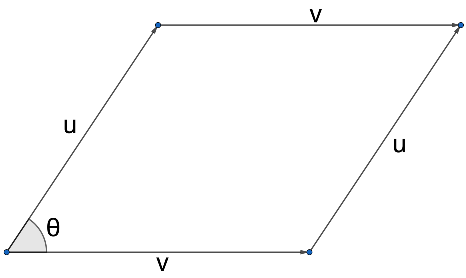
Example

Use a normal vector to give an equation for the plane passing through

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Applications

Area of a parallelogram



The area of the parallelogram is:

$$\|u\|\|v\|sin\theta = \|u \times v\|$$

Triple scalar (box) product

Definition

If u, v and w are vectors in $\mathbb{R}^3$ then the *triple scalar product (or box product) of u, v and w* is

$$u \cdot (v \times w)$$

Note that the order of the vectors matters. In fact

Lemma

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

which shows that it is the signed volume of the parallelepiped generated by u, v and w.

Examples

Example

Find the area of the triangle having vertices

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

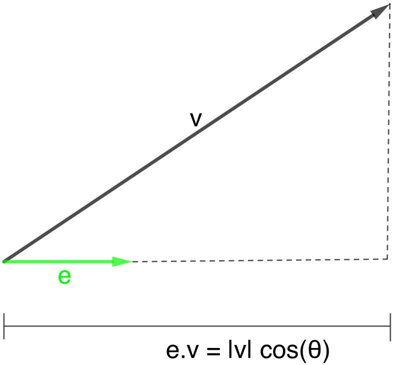
Example

Find the volume of the parallelepiped determined by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Misc

Cauchy-Schwarz inequality



If e is a unit vector then

$$|e \cdot v| \leq \|v\|$$

More generally:

$$|u \cdot v| \leq \|u\|\|v\|$$

which is the *Cauchy-Schwarz inequality*.

Triangle inequality

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$\|u+v\| \leq \|u\| + \|v\|$$

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$|\|u\| - \|v\|| \leq \|u-v\|$$

The Lagrange identity

Theorem

If u and v are in $\mathbb{R}^3$ then

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$$

Example

Example

Find equations for the lines through

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at the two points distance three from $(1, 2, 0)$.

Examples

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.