

# Worked Examples

Work in progress

December 7, 2018

## Contents

### Exercise 1

Suppose that the internet consists of three pages  $P_1$ ,  $P_2$  and  $P_3$ . Suppose further that:-

- there are two links from  $P_1$  to  $P_2$
- there is one link from  $P_1$  to  $P_3$
- there is one link from  $P_2$  to  $P_1$
- there is one link from  $P_2$  to  $P_3$
- there is one link from  $P_3$  to  $P_2$

Then calculate the (relative) PageRanks for  $P_1$ ,  $P_2$  and  $P_3$ .

### Answer of exercise 1

The transition matrix is:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

So we get the steady state vector by solving  $(A - I)v = 0$ :

$$\left[ \begin{array}{ccc|c} -1 & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & -1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 2 & -3 & 3 & 0 \\ 1 & \frac{3}{2} & -3 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 2 & -3 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{-3}{2} & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & \frac{-3}{4} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 \end{array} \right]$$

so  $z = s$ ,  $x = \frac{3s}{4}$  and  $y = \frac{3s}{2}$ . So the eigenspace associated to the eigenvalue 1 is:

$$E_1 = s \left[ \begin{array}{c} \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{array} \right] = \bar{s} \left[ \begin{array}{c} 3 \\ 6 \\ 4 \end{array} \right]$$

and so the steady state vector (containing the PageRanks) is

$$\frac{1}{13} \left[ \begin{array}{c} 3 \\ 6 \\ 4 \end{array} \right]$$

because we need to choose  $\bar{s}$  so that the sum of the entries is 1.

## Exercise 2

Find the distance  $\alpha$  between the lines

$$L_1 : \vec{x} = \begin{bmatrix} -6 \\ -7 \\ 7 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = P + sd$$

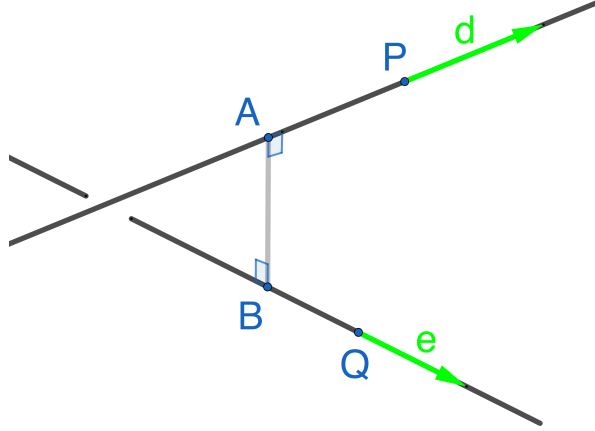
and

$$L_2 : \vec{x} = \begin{bmatrix} -7 \\ -14 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = Q + te$$

and find a points  $A$  on  $L_1$  and  $B$  on  $L_2$  such that  $dist(A, B) = \alpha$ .

### Answer of exercise 2

First draw the picture:



where we want to find  $A$  and  $B$ . Since  $\vec{AB}$  is orthogonal to both  $d$  and  $e$ :

$$(B - A) \cdot d = 0 \text{ and } (B - A) \cdot e = 0 \quad (1)$$

First we can express  $B - A$  in terms of unknowns  $s$  and  $t$ :

$$B - A = \begin{bmatrix} -7 + 2t \\ -14 + 3t \\ -t \end{bmatrix} - \begin{bmatrix} -6 + 2s \\ -7 + s \\ 7 - s \end{bmatrix} = \begin{bmatrix} 2t - 2s - 1 \\ 3t - s - 7 \\ -t + s - 7 \end{bmatrix}$$

because  $A$  is on  $L_1$  and  $B$  is on  $L_2$ . Then 1 gives firstly:

$$\begin{aligned} 0 &= \begin{bmatrix} 2t - 2s - 1 \\ 3t - s - 7 \\ -t + s - 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= 4t - 4s - 2 + 3t - s - 7 + t - s + 7 \\ &= 8t - 6s - 2 \end{aligned}$$

and secondly

$$\begin{aligned} 0 &= \begin{bmatrix} 2t - 2s - 1 \\ 3t - s - 7 \\ -t + s - 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\ &= 4t - 4s - 2 + 9t - 3s - 21 + t - s + 7 \\ &= 14t - 8s - 16 \end{aligned}$$

Next find  $t$  and  $s$  by solving

$$\left[ \begin{array}{cc|c} 8 & -6 & 2 \\ 14 & -8 & 16 \end{array} \right]$$

$$\begin{bmatrix} 1 & \frac{-3}{4} & \left| \frac{1}{4} \right. \\ 7 & -4 & \left. 8 \right] \\ \left[ \begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{1}{4} \\ 0 & \frac{5}{4} & \frac{25}{4} \end{array} \right] \\ \left[ \begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{1}{4} \\ 0 & 1 & 5 \end{array} \right] \\ \left[ \begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array} \right]$$

so  $t = 4$  and  $s = 5$ . Therefore

$$A = P + sd = \begin{bmatrix} -6 \\ -7 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

and

$$B = Q + te = \begin{bmatrix} -7 \\ -14 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

so the distance between the lines is:

$$\text{dist}(A, B) = \sqrt{3^2 + 0^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$$

### Exercise 3

Suppose that the sequence  $x_0, x_1, x_2 \dots$  is defined by  $x_0 = 2, x_1 = 1 + i$  and  $x_{k+2} = (1 + i)x_{k+1} - ix_k$ . Find a formula for  $x_k$ .

#### Answer of exercise 3

Reformulate as a two dimensional discrete linear dynamical system. Let:

$$v_k = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$$

and so

$$v_{k+1} = \begin{bmatrix} x_{k+2} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 1+i & -i \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = Av_k$$

and

$$v_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

First find the eigenvalues:

$$\begin{aligned} \det \begin{bmatrix} 1+i-\lambda & -i \\ 1 & -\lambda \end{bmatrix} &= -\lambda(1+i-\lambda) + i \\ &= \lambda^2 - (1+i)\lambda + i \\ &= (\lambda-1)(\lambda-i) \end{aligned}$$

The eigenspace associated to the eigenvalue 1:

$$\begin{bmatrix} i & -i & 0 \\ 1 & -1 & 0 \end{bmatrix} \\ \rightarrow \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

so  $y = s$  and  $x = s$  and so the eigenspace is

$$E_1 = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace associated to the eigenvalue  $i$ :

$$\begin{bmatrix} 1 & -i & 0 \\ 1 & -i & 0 \end{bmatrix} \\ \rightarrow \begin{bmatrix} 1 & -i & 0 \end{bmatrix}$$

so  $y = s$  and  $x = is$  and the eigenspace is:

$$E_i = s \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Next write the initial vector in terms of the eigenvectors:

$$v_0 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix}$$

to deduce that

$$\begin{aligned} A^k v_0 &= A^k \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \\ &= A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^k \begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i^k \begin{bmatrix} i \\ 1 \end{bmatrix} \end{aligned}$$

therefore  $x_k = 1 + i^k$ .

#### Exercise 4

A tennis club organises its games over the course of a year as follows:-

- At the beginning of the year there is a qualifying tournament.
- Those who finish in the top half of the qualifying tournament enter league  $A$  and those who do not enter league  $B$ .
- Players in both leagues play matches weekly.

- Adam joins at the beginning of one year:-
  - Adam has probability  $\frac{2}{3}$  of finishing in the top half of the qualifying tournament.
  - In league  $A$  Adam has a  $\frac{2}{5}$  chance of winning each weekly match.
  - In league  $B$  Adam has a  $\frac{4}{5}$  chance of winning each weekly match.

Model Adam's progress as a Markov chain and find all steady state vectors.

**Answer of exercise 4**

Let the states be:-

- State 1: Before the qualifying tournament.
- State 2: Adam is in league A and won the previous game.
- State 3: Adam is in league A and lost the previous game.
- State 4: Adam is in league B and won the previous game.
- State 5: Adam is in league B and lost the previous game.

The transition matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{5} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{3}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{5} & \frac{4}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

(Notice how the Markov chain splits into two parts after the first iteration.) To find the steady state vector we solve:

$$\begin{aligned} & \left[ \begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{5} & -\frac{4}{5} & 0 \end{array} \right] \\ & \rightarrow \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & -\frac{4}{5} & 0 \end{array} \right] \\ & \rightarrow \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & -\frac{4}{5} & 0 \end{array} \right] \\ & \rightarrow \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array} \right] \end{aligned}$$

so  $x_3 = s$ ,  $x_5 = t$ ,  $x_1 = 0$ ,  $x_2 = \frac{2s}{3}$  and  $x_4 = 4t$ . Therefore the eigenspace associated to eigenvalue 1 is:

$$E_1 = \begin{bmatrix} 0 \\ \frac{2s}{3} \\ s \\ 4t \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

and so there are two steady state vectors:

$$\frac{1}{5} \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{5} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} \quad (2)$$

### Exercise 5

Find equations for the lines through

$$Q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

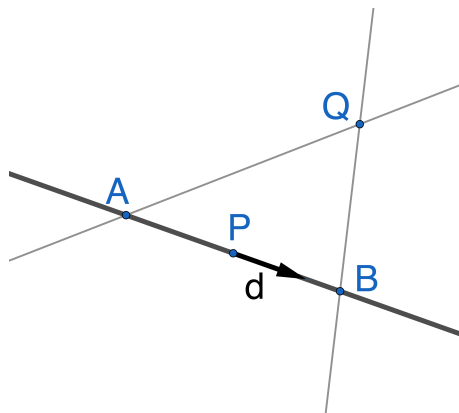
$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = P + sd$$

at the two points at distance 3 from

$$P = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

### Answer of exercise 5

First draw the picture:



We want to find the line through  $A$  and  $Q$  and the line through  $B$  and  $Q$ . Now the vector  $\frac{d}{\|d\|}$  has unit length so the vector  $3\frac{d}{\|d\|}$  has length 3. So we can find  $A$  and  $B$ :

$$\begin{aligned} A &= P + 3\frac{d}{\|d\|} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3\frac{1}{\sqrt{4+1+4}} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} B &= P - 3\frac{d}{\|d\|} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 3\frac{1}{\sqrt{4+1+4}} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \end{aligned}$$



and so the line through  $Q$  and  $A$  is

$$\begin{aligned}\vec{x} &= Q + s(A - Q) \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\end{aligned}$$

and the line through  $Q$  and  $B$  is

$$\begin{aligned}\vec{x} &= Q + t(B - Q) \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}\end{aligned}$$

### Exercise 6

Find the scalar equation for the plane passing through the point

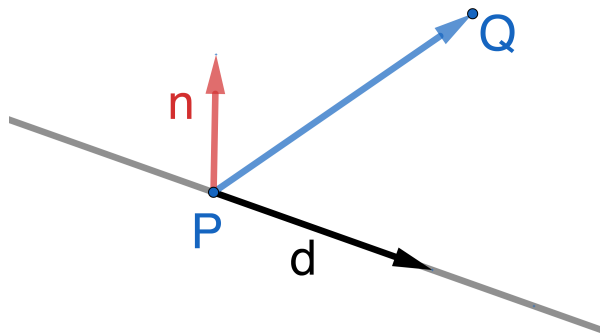
$$Q = \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$$

and containing the line

$$\vec{x} = \begin{bmatrix} 9 \\ -6 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix} = P + td$$

### Answer of exercise 6

First draw the picture:



where the vector  $n$  is any vector orthogonal to both  $d$  and  $\vec{QP}$ . To find one such  $n$  take the cross-product of  $d$  and  $\vec{QP}$ :

$$n = d \times (Q - P) = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} \times \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 - 5 \\ -20 + 4 \\ -2 - 8 \end{bmatrix} = \begin{bmatrix} -9 \\ -16 \\ -10 \end{bmatrix}$$

so the equation for the plane is

$$\begin{aligned} 0 = n \cdot (x - P) &= \begin{bmatrix} -9 \\ -16 \\ -10 \end{bmatrix} \cdot \begin{bmatrix} x - 9 \\ y + 6 \\ z + 1 \end{bmatrix} \\ &= -9x + 81 - 16y - 96 - 10z - 10 \\ &= -9x - 16y - 10z - 25 \end{aligned}$$

i.e.

$$9x + 16y + 10z = -25$$

### Exercise 7

Find a steady state vector for

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

### Answer of exercise 7

We need to solve:

$$\begin{aligned} &\left[ \begin{array}{ccc|c} \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{-1}{2} & 0 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{-1}{4} & 0 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & \frac{-3}{4} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array} \right] \end{aligned}$$

so  $z = s$ ,  $x = \frac{3s}{4}$  and  $y = \frac{s}{2}$ .

$$E_1 = s \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} = \bar{s} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

therefore the steady state vector is

$$\frac{1}{9} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

(i.e. choose  $\bar{s}$  such that the entries sum to 1.)