MATH211: Linear Methods I

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Lecture on Thursday 25th October, 2018

Misc

Examples

Linear transformations

Matrices redux

Last time

- Midterm review
- Cross product
- Applications to distance, area and volume

Misc

Triangle inequality

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$\|u + v\| \leq \|u\| + \|v\|$$

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$|\|u\| - \|v\|| \leq \|u - v\|$$

The Lagrange identity

The following identity follows directly from the Pythagorean theorem.

Theorem

If u and v are in $\mathbb{R}^3$ then

$$\|u\|^2\|v\|^2 = \|u \times v\|^2 + (u \cdot v)^2$$

Example

Example

Find equations for the lines through

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at the two points distance three from $(1, 2, 0)$.

Examples

Example

Find the area of the triangle having vertices

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example

Find the volume of the parallelepiped determined by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Examples

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.

Examples

Example

Find the shortest distance between the following plane and line.

$$X_1 : x - y + 2z = 2, L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Example

Find the shortest distance between the following two planes.

$$X_1 : x - y + 2z = 5, X_2 : 2x - 2y + 4z = 4$$

Examples

Example

Find the distance between the following two lines.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Linear transformations

Functions between Euclidean spaces

Definition

A *function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ written*

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

is a rule that assigns to every x in $\mathbb{R}^n$ an element y in $\mathbb{R}^m$.

- If y is the element assigned to x we write $y = f(x)$ and
$$f : x \mapsto f(x)$$
- Two different elements x_0 and x_1 of $\mathbb{R}^n$ may be assigned the same element y of $\mathbb{R}^m$. I.e:
$$f(x_0) = y = f(x_1)$$
- We cannot assign two different elements y_0 and y_1 of $\mathbb{R}^m$ to a single element x of $\mathbb{R}^n$.

Examples

Example

$$f(x) = x + 1 \text{ and } f(x) = x^2 + 3x - 5 \text{ and } f(x) = 5$$

are all functions $\mathbb{R}^1 \to \mathbb{R}^1$

Example

$$f(x, y) = \begin{bmatrix} x \\ x + y \\ xy \end{bmatrix}$$

is a function $f : \mathbb{R}^2 \to \mathbb{R}^3$

Examples

Example

If $f(x) = x^2$ then both $+1$ and -1 are assigned the value $+1$. I.e.

$$f(-1) = 1 = f(+1)$$

Example (Non-example)

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ +1 & \text{if } x > -1 \end{cases}$$

is not a function because the numbers between -1 and $+1$ are assigned two different numbers.

Linear transformations

Definition

A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function satisfying:

- (Additivity) $T(x + y) = T(x) + T(y)$
- (Scalar multiplication) $T(kx) = kT(x)$

Some consequences:-

- $T(0) = 0$
- $T(-x) = -T(x)$
- $T\left(\sum_{i=1}^n k_i v_i\right) = \sum_{i=1}^n k_i T(v_i)$

Examples

Example

Is the following a linear transformation?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix}$$

Example

Is the following a linear transformation?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$$

Examples

Example

Is rotation about the origin in $\mathbb{R}^2$ through an angle θ a linear transformation?

Example

Is reflection in the x-axis a linear transformation?

Example

Is $f(x) = x + 1$ a linear transformation?

Matrices redux

Linear transformations determined by action on basis

If T is linear then

$$T\left(\sum_{i=0}^{n} k_i v_i\right) = \sum_{i=0}^{n} k_i T(v_i)$$

But for all v in $\mathbb{R}^n$ we can write v as a linear combination of the standard basis vectors:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=0}^{n} v_i e_i = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots v_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Linear transformation determined by action on basis

So:

$$T(v) = T\left(\sum_{i=0}^{n} v_i e_i\right) = \sum_{i=0}^{n} v_i T(e_i)$$

Slogan: A linear transformation is completely determined by its action on the standard basis vectors.

More explicitly: we only need to know

$$T(e_1), T(e_2), \ldots, T(e_n)$$

to work out the whole transformation!

Matrices redux

If $T : \mathbb{R}^n \to \mathbb{R}^m$ then

$$T(e_1), T(e_2), \ldots, T(e_n)$$

are in $\mathbb{R}^m$. And so the transformation is completely determined by $n \times m$ scalars...

Matrices redux

Definition

The *matrix* $[T]$ *of a linear transformation* T is the matrix that has columns given by the images of the standard basis vectors:

$$[T] = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$$

(It turns out that the definition of matrix product is forced on us after this choice if we want it to match the composition of functions.)

Examples

Example

Find

$$T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$$

if T is linear and

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \text{ and } T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Examples

Example

Find

$$T\begin{bmatrix}1\\3\\-2\\-4\end{bmatrix}$$

if T is linear and

$$T\begin{bmatrix}1\\1\\0\\-2\end{bmatrix}=\begin{bmatrix}2\\3\\-1\end{bmatrix} \text{ and } T\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}=\begin{bmatrix}5\\0\\1\end{bmatrix}$$

Examples

Example

Find the matrix of the linear transformation

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

Example

Find the matrix of the linear transformation satisfying

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Examples

Example

For the following linear transformations S and T what is $S \circ T$, $T \circ S$ and what are the corresponding matrices?

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Example

What is the inverse transformation and corresponding matrix for

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

Examples

Example

What is the matrix of the linear transformation for counter-clockwise rotation about the origin by an angle of θ?

Example

What is the matrix of the linear transformation for reflection in the x-axis?

Example

What is the matrix of the linear transformation that is $v \times (-)$ for some vector $v \in \mathbb{R}^3$?

Examples

Example

What is the matrix of the linear transformation for reflection in the line $y = mx$?

Example

What is the matrix of the linear transformation consisting of first reflects in the x-axis and then rotates through an angle of $\frac{\pi}{2}$?

Example

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y-axis.

Summary of linear transformations

- Recall interpretation of matrix action on a vector
- Definition of linear transformation; examples and non-examples
- The matrix of a linear transformation; completely determined by action on basis
- Linear transformation preserves 0, - and linear combinations
- Composition of functions, linear transformation and relationship to matrices.
- Inverse functions and relationship to inverse matrices.
- Determinant of rotation and reflection. Composite of rotations/reflections with rotations/reflections