MATH211: Linear Methods I

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Lecture on Tuesday 6th November, 2018

Examples

Complex numbers

Plus and times

Graphical interpretation

Last time

- Matrices and linear transformations
- Composition of linear transformations
- Inverse of linear transformations

Examples

Examples

Example

If

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

then find $S \circ T$, $T \circ S$ and the matrices $[S \circ T]$ and $[T \circ S]$.

Example

If

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

find T^{-1} and $[T^{-1}]$.

Complex numbers

Progression of thought about complex numbers

> *...the whole matter seems to rest on sophistry rather than truth...*

(Rafael Bombelli 1526-1572. Early innovator in use of complex numbers)

> *The shortest path between two truths in the real domain passes through the complex domain.*

(Jacques Hadamard 1865-1963)

Algebraic motivation - extending the number system

Suppose that we only knew about the natural numbers:

$$\mathbb{N} := \{0, 1, 2, 3, \dots\}.$$

Then we wouldn't have a solution to the equation:

$$x + 1 = 0$$

Solution:

- Add a new symbol -1 such that

$$(-1) + 1 = 0$$

- Figure out addition and multiplication for this new symbol.
- (So in particular forced to add $-k$ for all $k \in \mathbb{N}$.)

Algebraic motivation - extending the number system

God made the integers; all else is the work of man.

(Leopold Kronecker 1823-1891)

- Using only positive integers $\{1, 2, 3, \dots\}$
  - we cannot find a solution to $x + 1 = 0$.
- Using only integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
  - we cannot find a solution to $3x + 2 = 0$.
- Using only fractions $\{\frac{a}{b}$ where $b \neq 0\}$
  - we cannot find a solution to $x^2 - 2 = 0$.
- Using only real numbers (decimals)
  - we cannot find a solution to $x^2 + 1 = 0$...

Algebraic motivation - extending the number system

Suppose that we only knew about real numbers.
Then we wouldn't have a solution to the equation:

$$x^2 + 1 = 0$$

Solution:

- Add a new symbol i such that

$$i^2 = -1$$

- Figure out addition and multiplication for this new symbol.
  - (Later in this lecture.)
- It turns out that we are forced to add all numbers of the form:

$$a + ib$$

  where a and b are real numbers.

Complex numbers

Definition

- The *imaginary unit*, denoted i, is defined to be a number with the property that $i^2 = -1$.
- A *pure imaginary* number has the form bi where $b \in \mathbb{R}$, $b \neq 0$.
- A *complex number* is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

Definition

If $z = a + bi$ then:

- a is called the *real part* of z.
- b is called the *imaginary part* of z.

Questions

Questions?

Plus and times

Motivation

Treat i as a number that satisfies the usual laws. E.g.:-

- $a(b+c) = ab + ac$
- $a + b = b + a$
- $ab = ba$ ...etc ...

as well as

$$i^2 = -1$$

and see what happens.

Example

$$\begin{aligned} 1 + i + i^2 + i^3 + i^4 + i^5 &= 1 + i - 1 - i + 1 + i \\ &= 1 + i \end{aligned}$$

which cannot be simplified any further.
(Because we only know $i^2 = -1$.)

Addition

Example

We would want addition to satisfy:-

- $i + i = 2i$
- $i - i = 0$
- $(1 + i) + 2 = 3 + i$

Definition

If a, b, c and d are real numbers then:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication

Example

We would want multiplication to satisfy:-

- $i \cdot i = -1$
- $2 \cdot i = 2i$
- $(1+i) \cdot i = i + i \cdot i = -1 + i$

Definition

If a, b, c and d are real numbers then

$$(a+bi) \cdot (c+di) = ac + adi + bci - db = (ac - bd) + (ad + bc)i$$

Examples

Example

Evaluate

- $(-3+6i)+(5-i)$.
- $(4-7i)+(6-2i)$.
- $(-3+6i)-(5-i)$.
- $(4-7i)-(6-2i)$.

Example

$$(2-3i)(-3+4i)$$

Example

Find all complex numbers z such that $z^2=-3+4i$.

Equations

Definition

If a, b, c and d are real numbers then

$$a + bi = c + di$$

if and only if $a = c$ and $b = d$.

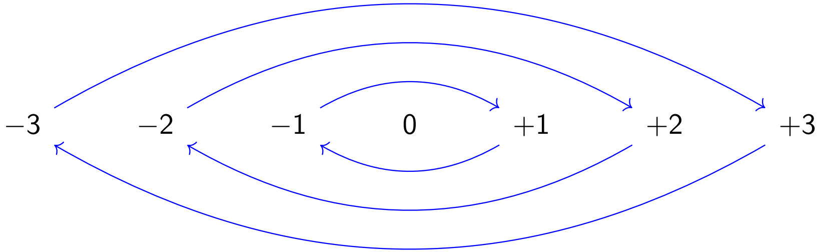
Questions?

Questions?

Graphical interpretation

Graphical interpretation

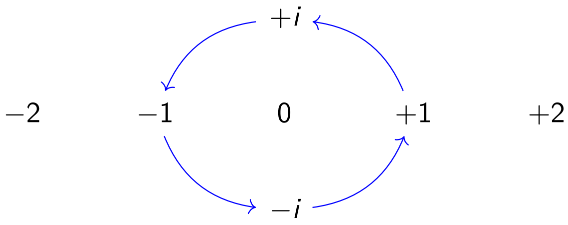
Consider the function $\mathbb{R} \to \mathbb{R}$ that multiplies by -1:



which shows us that $-1 \times (-1 \times x) = x$.

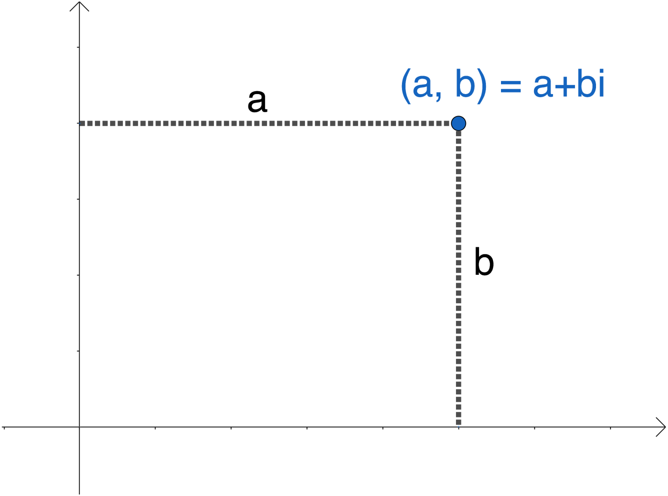
Graphical interpretation

So how do we get a transformation that squares to -1?

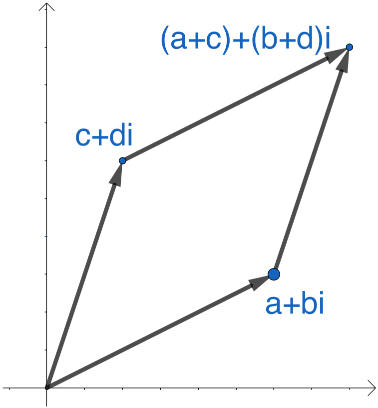


Graphical interpretation

So we draw the complex number $a + bi$ as the point (a, b) in $\mathbb{R}^2$:



Addition of complex numbers



Complex addition is defined component-wise:

$$(a+ib)+(c+id) = (a+c)+(b+d)i$$

and therefore has the same graphical interpretation as the addition of vectors in $\mathbb{R}^2$.

Historical curiosity

Apparently Bombelli describes complex arithmetic thus:

> *"Plus by plus of minus, makes plus of minus. Minus by plus of minus, makes minus of minus. Plus by minus of minus, makes minus of minus. Minus by minus of minus, makes plus of minus. Plus of minus by plus of minus, makes minus. Plus of minus by minus of minus, makes plus. Minus of minus by plus of minus, makes plus. Minus of minus by minus of minus makes minus."*

according to Crossley in 'The Emergence of Number'. A hint from Crossley is: the last two lines would become

$$-\sqrt{-a}\cdot\sqrt{-b}=\sqrt{ab} \text{ and } (-\sqrt{-a})\cdot(-\sqrt{-b})=-\sqrt{ab}$$

in modern notation.

Summary

- Graphical interpretation.
- Addition: algebra and geometry.
- Multiplication: algebra and geometry.
- Conjugate: algebra and geometry.

Clearing denominators

Example

- If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$, i.e., $\overline{3 + 4i} = 3 - 4i$.
- $\overline{-2 + 5i} = -2 - 5i$.
- $\bar{i} = -i$.
- $\overline{7} = 7$.

Example (Exemplar)

Write

$$\frac{a + bi}{c + di}$$

in the form $e + fi$ for real numbers a, b, c, d, e, and f.

- $\frac{1}{i}$
- $\frac{2-i}{3+4i}$
- $\frac{1-2i}{-2+5i}$