# MATH211: Linear Methods I

Matthew Burke

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# Lecture on Tuesday 20th November, 2018

# Summary of previous work

- $i^2 = -1$
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- graphical interpretation

$$a + bi \leftrightarrow (a, b)$$
$$\mathbb{C} \leftrightarrow \mathbb{R}^2$$

- polar form

$$a + ib \leftrightarrow Re^{i\theta}$$

where $R$ is the *modulus* and $\theta$ the *angle*.

# Examples

## Example

Convert the following to polar form:-

- $-2+2\sqrt{3}i$
- $3i$
- $-1-i$
- $\sqrt{3}+3i$

## Example

Convert the following to standard form:-

- $2e^{\frac{2\pi i}{3}}$
- $3e^{-i\pi}$
- $2e^{\frac{3i\pi}{4}}$

# Multiplication

# Multiplication using polar form

If $z = Re^{i\theta}$ and $w = Qe^{i\phi}$ then

$$\begin{aligned} zw &= Re^{i\theta} Qe^{i\phi} \\ &= (RQ)e^{i(\theta+\phi)} \end{aligned}$$

**Slogan:** To multiply two complex numbers we multiply the moduli and add the angles.

## Theorem (De Moivre Theorem)

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

# Questions?

Questions?

# Examples

**Example**

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

**Example**

Express $(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{17}$ in the form $a+bi$.

# Complex roots

# Principal square roots

Now we know that:

**Slogan:** To multiply two complex numbers we multiply the moduli and add the angles.

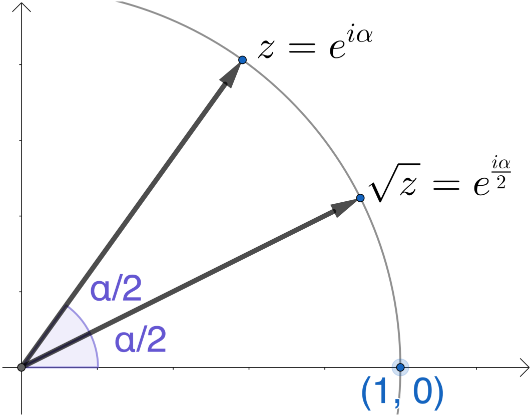
So we could try reversing this:

**Slogan:** To find the principal square root we take the square root of the modulus and halve the angle.

- The answer will depend on the angle measuring convention.

# Picture

If $z = e^{i\alpha}$ has unit modulus:



# Non-principal roots

Let $z = R \cdot e^{i\alpha}$ and take the principal square root:

$$\sqrt{z} = \sqrt{R \cdot e^{i\alpha}} = \sqrt{R} \cdot e^{\frac{i\alpha}{2}}$$

Then all of

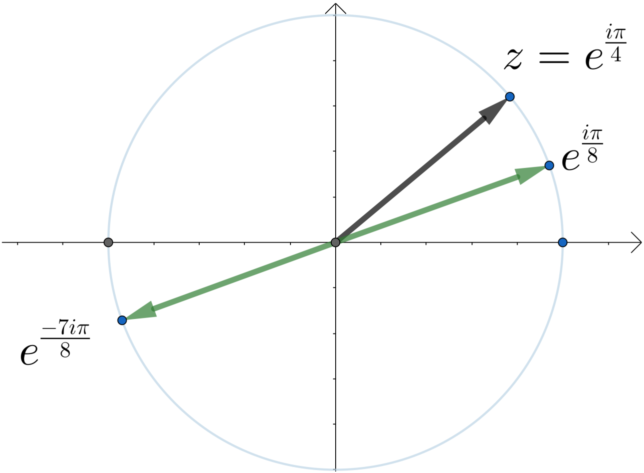
$$\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + k \cdot \pi\right)}$$

are *also* a square roots of $z$:

$$\left(\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + \pi\right)}\right)^2 = R \cdot e^{i\alpha + 2k\pi} = R \cdot e^{i\alpha} = z$$

# Picture

The two roots of $z = e^{\frac{i\pi}{4}}$ are $e^{\frac{i\pi}{8}}$ and $e^{\frac{-7i\pi}{8}}$:



# Nth roots

Similarly there is a principal $n$-th root:

$$\sqrt[n]{z} = \sqrt[n]{R \cdot e^{i\alpha}} = \sqrt[n]{R} \cdot e^{\frac{i\alpha}{n}}$$

and all numbers of the form

$$\sqrt[n]{R} e^{\frac{i\alpha + 2k\pi}{n}}$$

are also $n$-th roots of $z$ for $k \in \mathbb{N}$.

# Square roots

The solutions to

$$z^2 = 25$$

are

$$z = +5 \text{ and } z = -5$$

# Square roots

The solutions to

$$z^2 = i$$

are

$$z = e^{\frac{\pi}{4}} \text{ and } z = e^{\frac{-3\pi i}{4}}$$

# Square roots

In general if $w = Re^{i\theta}$ then all of

$$\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}$$

will be square roots of $w$ because

$$\left(\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}\right)^2 = Re^{i(\theta+2k\pi)} = Re^{i\theta}$$

and then we need to find the values of

$$\frac{\theta}{2} + k\pi$$

that are between $-\pi$ and $+\pi$.

# Questions?

Questions?

# Examples

## Example

Find all solutions to the following equations:

- $z^2 = 25$.
- Find all solutions to $z^2 = -1$
- Find all solutions to $z^2 = e^{\frac{i\pi}{4}}$
- Find all solutions to $z^3 = i$. (Write in Cartesian form.)
- Find all solutions to $z^4 = 2(\sqrt{3}i - 1)$. (Write in Cartesian form.)

## Example

Find all sixth roots of unity.

# Quadratic formula

# Quadratic formula

If $a$, $b$ and $c$ are *complex numbers* then the equation

$$az^2 + bz + c = 0$$

has two solutions given by

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- We assume $a \neq 0$.
- The formula is unchanged $a$, $b$ and $c$ real numbers.
- We always get a solution.

# Galois theory for quadratics

If $u$ and $v$ are the roots of $f(x) = z^2 + bz + c$ then

$$(z-u)(z-v) = z^2 - (u+v)z + uv$$

and so

$$\begin{aligned} b &= -u - v \\ c &= uv \end{aligned}$$

- If $f$ is a *real* quadratic we need both $u + v$ and $uv$ to be real.
- This implies that $u = \overline{v}$ .

# Examples

### Example

Solve $z^2 - 14z + 58 = 0$.

### Example

Find a real quadratic with $5 - 2i$ as a root. What is the other root?

### Example

Find a real quadratic with $-3 + 4i$ as a root. What is the other root?

### Example

Solve $z^2 - (3 - 2i)z + (5 - i) = 0$.

### Example

Find the roots of $z^2 - 3iz + (-3 + i) = 0$.

## Example

Verify that $4 - i$ is a root of $z^2 - (2 - 3i) - (10 + 6i) = 0$. Find the other root.