MATH211: Linear Methods I

Matthew Burke

Thursday 20th September, 2018

Lecture on Thursday 20th September, 2018

Last time

- (i,j)-entry proofs
- Identity matrices
- Inverse matrices
- Small examples of inverses

Matrix inversion algorithm

Standard basis vectors

Definition

The standard basis vectors in n-space are:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

I.e. the columns of I_n.

Aim of matrix inversion

Given B find A such that $BA = I_n$:

$$\begin{bmatrix} B\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} B\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots B\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I.e. find linear combinations of the columns of B that give each of the standard basis vectors.

The algorithm

For the first standard basis vector we solve:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftrightarrow \left[\begin{array}{cccc|c} b_{11} & b_{12} & \dots & b_{1m} & 1 \\ b_{21} & b_{22} & \dots & b_{2m} & 0 \\ \dots & \dots & \dots & \dots & 0 \\ b_{p1} & b_{p2} & \dots & b_{pm} & 0 \end{array}\right]$$

and we can group all together as:

$$\left[\begin{array}{cccc|cccc} b_{11} & b_{12} & \dots & b_{1m} & 1 & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & b_{2m} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pm} & 0 & 0 & \dots & 1 \end{array}\right]$$

Reduction

Row reduce

$$\left[\begin{array}{cccc|cccc} b_{11} & b_{12} & \dots & b_{1m} & 1 & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & b_{2m} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pm} & 0 & 0 & \dots & 1 \end{array}\right]$$

to

$$[R|A] = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & \star & a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & \star & a_{21} & a_{22} & \dots & a_{2n} \\ 0 & 0 & \dots & \star & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \star & a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right]$$

where R is in reduced row echelon form.

Finishing off

$$[R|A] = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & \star & a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & \star & a_{21} & a_{22} & \dots & a_{2n} \\ 0 & 0 & \dots & \star & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \star & a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right]$$

If R is:-

the identity matrix then B is invertible with inverse A

anything else then B is not invertible

Relationship to solutions of systems

Recall that a system of equations is equivalently a matrix equation:

$$Ax = b$$

so if A is invertible

$$x = A^{-1}Ax = A^{-1}b$$

is the *unique solution* to $Ax = b$.

This method is rarely the most efficient thing to do.

Questions?

Examples

Low dimensional examples

Example

Find the inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

Find the inverse for the general 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Examples

Example

Find the inverse for

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$$

Examples

Example

Find the inverse for

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

Questions?

Common Inverses

Inverse of transposes and products

Example

Prove that $(BA)^T = A^T B^T$.

Example

Using the previous example prove that

$$\begin{aligned} \text{if } & A \text{ has inverse } A^{-1} \\ \text{then } & A^T \text{ has inverse } (A^{-1})^T. \end{aligned}$$

Example

Prove that

$$\begin{aligned} \text{if } & A \text{ has inverse } A^{-1} \text{ and } B \text{ has inverse } B^{-1} \\ \text{then } & BA \text{ has inverse } A^{-1}B^{-1}. \end{aligned}$$

Elementary matrices

Recall the elementary row operations:

1. Swap two rows.
2. Multiply a row by a non-zero scalar.
3. Add a multiple of one row to another.

Each corresponds to multiplying on the left

$$A \mapsto EA$$

by a different *elementary matrix* E.

Multiply a row by a non-zero scalar

The matrix that corresponds to multiplying the second row by 3 is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{3}a_{21} & \frac{1}{3}a_{22} & \frac{1}{3}a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to multiplying the second row by $\frac{1}{3}$.

Swap two rows

The matrix that swaps the second and third rows is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and is its own inverse.

Adding a multiple of another row

The matrix that corresponds to subtract 3 times the first row from the second is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-3a_{11} & a_{22}-3a_{12} & a_{23}-3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+3a_{11} & a_{22}+3a_{12} & a_{23}+3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to adding 3 times the first row to the second.

Questions?

Other matrix equations

Examples

Example

Find A such that

$$\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$$

Example

Find A such that

$$(3I - A^T)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Example

Prove that if $A^3 = 4I$ then A is invertible.

Examples

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Questions?