MATH211: Linear Methods I

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Lecture on Thursday 13th September, 2018

Last time

- Reduced row echelon form
- Rank
- Homogeneous systems
- Linear combinations
- Examples

Fundamentals of matrices

Definition

Definition

An $m \times n$ *matrix* is a rectangular array of numbers with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]$$

We call the entry in the ith row and jth column a_{ij}.

Special matrices

- If $n = m$ we say the matrix is *square*. E.g. if $n = m = 2$:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- If $m = 1$ we have a row vector: $X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$

- If $n = 1$ we have a column vector: $Y = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$

Matrix addition and subtraction

To add two matrices we add the corresponding entries. E.g.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 5 \end{bmatrix}$$

In general if $A = A_{ij}$ and $B = B_{ij}$ are matrices then

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

Similarly for matrix subtraction, we subtract the corresponding entries;

$$(A - B)_{ij} = A_{ij} - B_{ij}$$

Scalar multiplication

If $A = A_{ij}$ is a matrix and k is a number then the matrix kA has entries

$$(kA)_{ij} = k \cdot A_{ij}$$

I.e. we multiply every entry of the matrix by k. E.g.

$$5 \cdot \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 10 & 15 \end{bmatrix}$$

Questions?

Systems of equations

Matrix acting on vector

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Then define the action of A on a vector of length n as:

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This does not make sense if vector does not have length n!

Slogan

The vector

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is the linear combination of the columns of A weighted by the x_i.

Matrix acting on vector

Example

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 8 \\ 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

Matrix form of system of linear equations

Since

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

we can write a system of m linear equations in n variables as

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where A is an $m \times n$ matrix. Examples follow..

Example of matrix form

Example

Find the matrix equation for the system

$$\begin{aligned} x + 3y &= 9 \\ 2x + 4y &= 3 \end{aligned}$$

Example of linear combinations

Example

Express

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a linear combination of

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \text{ and } a_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Questions?

Composition

Matrix product

Let $A = a_{ij}$ be an $m \times n$ matrix and $B = b_{jk}$ be an $p \times m$ matrix.

Definition

The *matrix product BA* is the matrix with columns

$$\left[\begin{array}{cccc} B\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & B\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \dots & B\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{array}\right]$$

In other words we apply the matrix B to each of the columns of A.

When does this make sense?

Recall that

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

doesn't make sense if A doesn't have precisely n columns.

Therefore the matrix product BA only makes sense iff

$$\text{Number of rows}(A) = \text{Number of columns}(B)$$

Examples

Example

Find BA where

$$B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

The matrix product AB where

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

does not make sense.

Questions?

Non-commutativity

Warning

In general

$$AB \neq BA$$

In other words the order matters.

(Explanation in terms of linear transformations.)

Examples

Four different examples on Jupyter of matrices A and B such that:

1. AB exists but BA does not.
2. Both AB and BA exist but are of different sizes.
3. Both AB and BA exist and are the same size, but they are different.
4. $AB = BA$ (it does happen sometimes!)

Questions?