Math 211 2018-12-04:

$A \sim B$ so we can find an invertible P such that

$$B = PAP^{-1}$$

$$tr(B) = tr(PAP^{-1}) = \sum_{i=1}^{n} (PAP^{-1})_{ii}$$

$$= \sum_{i} \sum_{j,k} P_{ij} A_{jk} (P^{-1})_{ki}$$

$$= \sum_{i,j,k} (P^{-1})_{ki} P_{ij} A_{jk} = \sum_{j,k} (P^{-1}P)_{kj} A_{jk}$$

$$= \sum_{j,k} I_{kj} A_{jk} = \sum_{k} (IA)_{kk} = \sum_{k} A_{kk} = tr(A).$$

<u>E.g.</u> $X_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$.

First eigenvalues:

$$0 = \begin{vmatrix} 2-\lambda & 0 \\ 3 & -1-\lambda \end{vmatrix} = (\lambda-2)(\lambda+1)$$

$\underline{E_2}$: $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & -3 & 0 \end{array}\right) \rightsquigarrow (1 \; -1 \mid 0)$ $y=s$ $x=s$

$$E_2 = s\begin{pmatrix}1\\1\end{pmatrix}.$$

$\underline{E_{-1}}$: $\left(\begin{array}{cc|c} 3 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right) \rightsquigarrow (1 \; 0 \mid 0)$ $y=s$ $x=0$

$$E_{-1} = s\begin{pmatrix}0\\1\end{pmatrix}.$$

Now $x_0 = \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}$

$$X_k = A^k x_0 = A^k\left(\begin{pmatrix}1\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}\right) = A^k\begin{pmatrix}1\\1\end{pmatrix} - 2A^k\begin{pmatrix}0\\1\end{pmatrix}.$$

$$= 2^k\begin{pmatrix}1\\1\end{pmatrix} - 2(-1)^k\begin{pmatrix}0\\1\end{pmatrix}$$

$\underline{\text{E.g.}}$ $x_0 = \begin{pmatrix}100\\40\end{pmatrix}$ $A = \begin{pmatrix}1/2 & 1/4\\2 & 0\end{pmatrix}.$

First eigenvalues.

$$0 = \begin{vmatrix}1/2-\lambda & 1/4\\2 & -\lambda\end{vmatrix} = -\lambda\left(\tfrac{1}{2}-\lambda\right) - \tfrac{1}{2} = \lambda^2 - 1/2\,\lambda - 1/2$$

$$= (\lambda + 1/2)(\lambda - 1)$$

E_1: $\left(\begin{array}{cc|c} -1/2 & 1/4 & 0 \\ 2 & -1 & 0 \end{array}\right) \rightsquigarrow (1 \ \ -1/2 \,|\, 0)$ $y = s$ $x = s/2$

$$E_1 = s\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \bar{s}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$E_{-\frac{1}{2}}$: $\left(\begin{array}{cc|c} 1 & 1/4 & 0 \\ 2 & 1/2 & 0 \end{array}\right) \rightarrow (1 \ \ 1/4 \,|\, 0)$ $y = s$ $x = -s/4$

$$E_{-\frac{1}{2}} = s\begin{pmatrix} -1/4 \\ 1 \end{pmatrix} = \bar{s}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$x_0 = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 100 \\ 40 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 1 & 1 & 100 \\ 2 & -4 & 40 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 100 \\ 0 & -6 & -160 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 100 \\ 0 & 3 & 80 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 100 \\ 0 & 1 & 80/3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 220/3 \\ 0 & 1 & 80/3 \end{array}\right) \quad a = \frac{220}{3} \quad b = 80/3.$$

$$x_K = A^K x_0 = A^K\left[\frac{220}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{80}{3}\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right] = \frac{220}{3}A^k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{80}{3}A^k\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$= \frac{220}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{80}{3}\left(\frac{-1}{2}\right)^K\begin{pmatrix} 1 \\ -4 \end{pmatrix} \longrightarrow \frac{220}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ as } k \to \infty.$$

E.g.

$$\begin{pmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 4/6 \\ 2/6 \end{pmatrix}$$

E.g. $x_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix}$

$$X_3 = A^3 x_0 = A^2 \begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = A^2 \begin{pmatrix} 1/8 \\ 1/2 + 3/8 \end{pmatrix} =$$

$$= A \begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix} \begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix} = \frac{1}{8} A \begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = \frac{1}{8} A \begin{pmatrix} 7/4 \\ 1 + \frac{21}{4} \end{pmatrix}$$

$$= \frac{1}{8} A \begin{pmatrix} 7/4 \\ 25/4 \end{pmatrix} = \frac{1}{32} \begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix} \begin{pmatrix} 7 \\ 25 \end{pmatrix} = \frac{1}{32} \begin{pmatrix} 25/4 \\ 7 + 75/4 \end{pmatrix}$$

$$= \frac{1}{32} \begin{pmatrix} 25/4 \\ 103/4 \end{pmatrix} = \frac{1}{128} \begin{pmatrix} 25 \\ 103 \end{pmatrix}.$$

Steady state for $\begin{pmatrix} 0 & 1/4 \\ 1 & 3/4 \end{pmatrix}$.

Solve $(A-I)x=0$

$\left(\begin{array}{cc|c} -1 & 1/4 & 0 \\ 1 & -1/4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1/4 & 0 \end{array}\right)$ $y=s$
$x = s/4$

$E_1 = s\begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \bar{s}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

So choose $\bar{s} = \frac{1}{5}$ so steady state vector is $\frac{1}{5}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.