MATH211: Linear Methods I
Lecture 2

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Lecture 2

Last time

- Linear equations in one variable
- Linear equations in two variables
  - graphical interpretation
- What elementary row operations are allowed?
  - graphical interpretation
- Linear equations in three variables

Reduced row echelon form

Row echelon form

A matrix is in *row echelon form* iff:-

- All rows consisting entirely of zeros are at the bottom.
- The first nonzero entry in each nonzero row is a 1 (called the **leading 1** for that row).
- Each leading 1 is to the right of all leading 1's in rows above it.

For instance:

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Reduced row echelon form

A matrix is in *reduced row echelon form* iff:-

- It is a row-echelon matrix.
- Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Advantages of reduced row echelon form

- Row echelon form $\leftrightarrow$ ready for back-substitution.
- Reduced row echelon form $\leftrightarrow$ read off solutions.

Another advantage of reduced row echelon form is that:

Theorem
Reduced row echelon form is unique.

Reading off solutions

- The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  is in reduced row echelon form.

- If there are any rows of the form $(0\ 0 \ldots 0|1)$ then we know immediately that there are no solutions.

Identify leading 1s and parameter blocks

- The leading 1s are shown in green

$$\left[\begin{array}{ccccc|c} \color{green}{1} & 0 & \color{blue}{5} & 0 & \color{red}{1} & 1 \\ 0 & \color{green}{1} & \color{blue}{1} & 0 & \color{red}{2} & 0 \\ 0 & 0 & 0 & \color{green}{1} & \color{red}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  and the other non-zero numbers are shown in blue and red.

- Variables associated to the leading 1s are the *leading variables*.

- In this case x_1, x_2 and x_4 are leading and x_3 and x_5 are not.

The non-leading variables can take any value so we assign parameters to them.

In this case:

$$x_3 = s$$
$$x_5 = t$$

and then we solve for the leading variables in terms of these parameters:

$$x_1 = 1 - 5s - t$$
$$x_2 = -s - 2t$$
$$x_4 = -4t$$

Questions?

Examples

Worked example

Solve

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & -8 & 4 \\ -3 & 6 & -4 & -9 & 3 & -1 \\ -1 & 2 & -2 & -4 & -3 & 3 \\ 1 & -2 & 1 & 3 & -1 & 1 \end{array}\right]$$

Infinitely many solutions

Solve

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 9 & 1 \\ 5 & 6 & 7 & 8 & 5 \\ 6 & 18 & 24 & 54 & 12 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Inconsistent

Solve

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 5 & 6 & 7 & 8 & 5 \\ 6 & 9 & 8 & 10 & 10 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Unique solution

Solve

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 8 & 5 \\ 11 & 10 & 10 & 12 & 6 \\ 1 & 1 & 1 & 4 & 1 \end{array}\right]$$

Questions?

Terminology

Gaussian elimination

An *elementary row operation* is one of:-

- swap two rows
- multiply a row by a non-zero scalar
- add a multiple of one row to another

The method of *Gaussian elimination* consists of:-

- using elementary row operations to reduce a matrix to reduced row echelon form
- make the non-leading variables parameters
- read off the solutions

Definition of rank

Definition

The *rank* of a matrix is the number of leading 1s in its reduced row echelon form.

Example

Therefore the rank of the following matrix is 3.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is a property of a matrix without augmentation.

Relationship to solution

For a *consistent* system in n variables of rank r:

- If $n = r$ there is a unique solution.
- If $r < n$ there are infinitely many solutions.

Homogeneous systems

A system of equations

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ a_{21} & a_{22} & \dots & a_{2m} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} & b_n \end{array}\right]$$

is *homogeneous* iff for all i we have $b_i = 0$. I.e. one that looks like:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & 0 \\ a_{21} & a_{22} & \dots & a_{2m} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} & 0 \end{array}\right]$$

Linear combinations

Definition

A *linear combination* of the vectors v_1, v_2, ..., v_n is a sum of the form

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$

where the a_i are scalars.

Example linear combinations

Example

Suppose that we found that the solution to a system of equations was

$$\begin{aligned} x_1 &= 1 - 5s - t \\ x_2 &= -s - 2t \\ x_4 &= -4t \end{aligned}$$

in parameters s and t. Then the solutions are all linear combinations of the form

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$$

Questions?

Reading off example

Consider the system:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 6 & 8 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

What is the rank of the corresponding matrix? What is the number of parameters? Is the system homogeneous?

Worked example

Consider the system

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

What is the rank of the corresponding matrix? What is the number of parameters? Is the system homogeneous?

Computer example

Consider the system

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 2 \\ 0 & 0 & 5 & 6 \\ 1 & 5 & 1 & -5 \end{array}\right]$$

What is the rank of the corresponding matrix? What is the number of parameters? Is the system homogeneous?

Computer example

Consider the system

$$\left[\begin{array}{rrr|r} 9 & -1 & 0 & -24 \\ -1 & 7 & -4 & 17 \\ 0 & -4 & 8 & -14 \end{array}\right]$$

What is the rank of the corresponding matrix? What is the number of parameters? Is the system homogeneous?

Questions?