MATH211: Linear Methods I
Lecture 2

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Tuesday 11th September, 2018

Lecture 2

Reduced row echelon form

Examples

Rank

Examples

Reduced row echelon form

Row echelon form

A matrix is in *row echelon form* iff:-

- All rows consisting entirely of zeros are at the bottom.
- The first nonzero entry in each nonzero row is a 1 (called the **leading 1** for that row).
- Each leading 1 is to the right of all leading 1's in rows above it.

For instance:

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Reduced row echelon form

A matrix is in *reduced row echelon form* iff:-

- It is a row-echelon matrix.
- Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Advantages of reduced row echelon form

- Row echelon form $\leftrightarrow$ ready for back-substitution.
- Reduced row echelon form $\leftrightarrow$ read off solutions.

Another advantage of reduced row echelon form is that:

Theorem

Reduced row echelon form is unique.

Reading off solutions

- The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  is in reduced row echelon form.

- If there are any rows of the form $(0\ 0 \ldots 0|1)$ then we know immediately that there are no solutions.

Identify leading 1s and parameter blocks

- The leading 1s are shown in green

$$\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  and the other non-zero numbers are shown in blue and red.

- Variables associated to the leading 1s are the *leading variables*.

- In this case x_1, x_2 and x_4 are leading and x_3 and x_5 are not.

The non-leading variables can take any value so we assign parameters to them.

In this case:

$$x_3 = s$$
$$x_5 = t$$

and then we solve for the leading variables in terms of these parameters:

$$x_1 = 1 - 5s - t$$
$$x_2 = -s - 2t$$
$$x_4 = -4t$$

Questions?

Examples

Linear equations in one variable

Find all real numbers x such that

$$ax = b$$

where a and b are real numbers.

Linear equation in two variables

Find all real numbers x and y such that both

$$\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= 0 \end{aligned}$$

hold.

Elementary row operations

But what operations are we allowed to do on the rows?

The following operations will not change the solutions:-

1. Swap two rows.
2. Add a multiple of one row to another row.
3. Multiply a row by a *non-zero* scalar.

See pictures on Jupyter notebook.

Questions?

Inconsistent equations

Find all real numbers x and y such that both

$$\begin{aligned} x + 2y &= 1 \\ 5x + 10y &= 42 \end{aligned}$$

hold.

Infinitely many solutions

Find all real numbers x and y such that both

$$\begin{aligned} 3x + 12y &= 18 \\ 4x + 16y &= 24 \end{aligned}$$

hold.

Questions?

Rank

Definition of rank

Definition

The *rank* of a matrix is the number of leading 1s in its reduced row echelon form.

Example

Therefore the rank of

$$\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is 3.

Find all x, y and z such that

$$\begin{aligned} x + y + 2z &= -1 \\ 2x + y + 3z &= 0 \\ 0x + -2y + 1z &= 2 \end{aligned}$$

hold.

Find all x, y and z such that

$$\begin{aligned} x + 2y + 3z &= 4 \\ 5x + 6y + 7z &= 8 \\ 3x + 4y + 5z &= 1 \end{aligned}$$

hold.

Find all x, y and z such that

$$\begin{aligned} 6x + 4y + 2z &= 4 \\ 3x + 2y + 1z &= 2 \\ 9x + 6y + 3z &= 6 \end{aligned}$$

hold.

Questions?

Example

test