MATH211: Linear Methods I

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Summary of previous work

- $i^2 = -1$
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- graphical interpretation

$$a + bi \leftrightarrow (a, b)$$

$$\mathbb{C} \leftrightarrow \mathbb{R}^2$$

- polar form

$$a + ib \leftrightarrow Re^{i\theta}$$

where R is the *modulus* and θ the *angle*.

Examples

Example

Convert the following to polar form:-

- $-2+2\sqrt{3}i$
- $3i$
- $-1-i$
- $\sqrt{3}+3i$

Example

Convert the following to standard form:-

- $2e^{\frac{2\pi i}{3}}$
- $3e^{-i\pi}$
- $2e^{\frac{3i\pi}{4}}$

Multiplication

Multiplication using polar form

If $z = Re^{i\theta}$ and $w = Qe^{i\phi}$ then

$$\begin{aligned} zw &= Re^{i\theta}Qe^{i\phi} \\ &= (RQ)e^{i(\theta+\phi)} \end{aligned}$$

Slogan: To multiply two complex numbers we multiply the moduli and add the angles.

Powers using polar form

Theorem (De Moivre Theorem)

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Questions?

Questions?

Examples

Example

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

Example

Express $(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{17}$ in the form $a+bi$.

Complex roots

Principal roots

If $w = Re^{i\theta}$ then one solution to the equation

$$z^n = w$$

is

$$z = \sqrt[n]{R} \cdot e^{\frac{i\theta}{n}}$$

although there will in fact be $n - 1$ more solutions.

Square roots

The solutions to

$$z^2 = 25$$

are

$$z = +5 \text{ and } z = -5$$

Square roots

The solutions to

$$z^2 = i$$

are

$$z = e^{\frac{\pi}{4}} \text{ and } z = e^{\frac{-3\pi i}{4}}$$

Square roots

In general if $w = Re^{i\theta}$ then all of

$$\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}$$

will be square roots of w because

$$\left(\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}\right)^2 = Re^{i(\theta+2k\pi)} = Re^{i\theta}$$

and then we need to find the values of

$$\frac{\theta}{2} + k\pi$$

that are between $-\pi$ and $+\pi$.

Questions?

Questions?

Examples

Example

Find all solutions to the following equations:

- $z^2 = 25.$
- Find all solutions to $z^2 = -1$
- Find all solutions to $z^2 = e^{\frac{i\pi}{4}}$
- Find all solutions to $z^3 = i$. (Write in Cartesian form.)
- Find all solutions to $z^4 = 2(\sqrt{3}i - 1)$. (Write in Cartesian form.)

Example

Find all sixth roots of unity.