MATH211: Linear Methods I

Matthew Burke

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Lecture on Tuesday 27th November, 2018

Quadratic formula

Spectral theory

Eigenvalues

Eigenspaces

Diagonalisation

Last time

- Multiplication using polar form.
- Complex roots
- Quadratic formula

Quadratic formula

Examples

Example

Solve $z^2 - 14z + 58 = 0$.

Example

Find a real quadratic with $5 - 2i$ as a root. What is the other root?

Example

Solve $z^2 - (3 - 2i)z + (5 - i) = 0$.

Spectral theory

Simplifying matrix actions

If

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

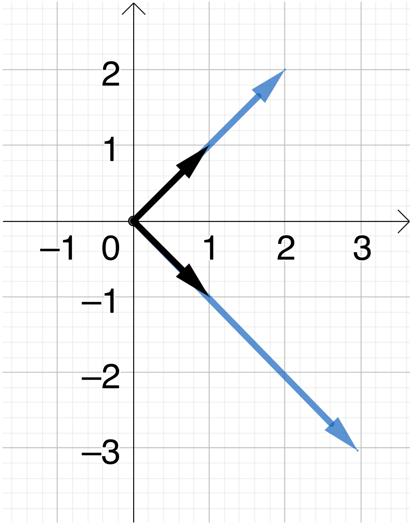
then A acts on some vectors in a simplified way:

$$Av_1 = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2v_1$$

and

$$Av_2 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4v_2$$

Picture



Since

$$Av_1 = 2v_1$$

and

$$Av_2 = 4v_2$$

Simplifying matrix actions

If

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

then

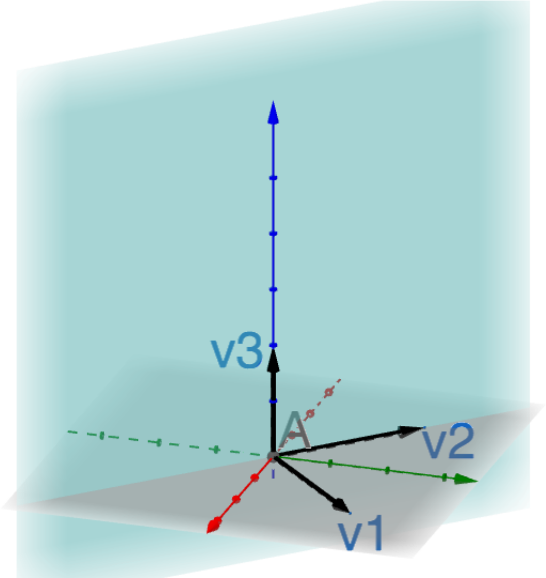
$$Av_1 = A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$$

$$Av_2 = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_3$$

and

$$Av_3 = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -v_2$$

Picture



Blue plane is preserved.

Eigenvectors and eigenvalues

Definition

If A is a square matrix, v is a non-zero vector and

$$Av = \lambda v$$

then A has an *eigenvector* v *with eigenvalue* λ.

Idea: If vector x is a linear combination of eigenvectors of A then the action of A on x is easy:

$$A(x) = A\left(a_1 v_{\lambda_1} + a_2 v_{\lambda_2}\right) = a_1 A(v_{\lambda_1}) + a_2 A(v_{\lambda_2}) = a_1 \lambda_1 v_{\lambda_1} + a_2 \lambda_2 v_{\lambda_2}$$

Examples

Example

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with eigenvalues 3 and 5 respectively.

Example

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 2 and 4 respectively.

Eigenvalues

Finding eigenvalues

Since

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

and $v \neq 0$ we need to find all λ such that

$$A - \lambda I$$

is *not* invertible.

Finding eigenvalues

Definition (Characteristic polynomial)

The *characteristic polynomial of a matrix A* is

$$\chi_A(\lambda) = det(A - \lambda I)$$

- The eigenvalues are the roots of the characteristic polynomial.
- The number of times a root is repeated is called the *algebraic multiplicity* of the root. E.g. if

$$\chi_A(\lambda) = (\lambda - 4)^3(\lambda + 3)(\lambda - 1)^2$$

then 4 has algebraic multiplicity 3, -3 has algebraic multiplicity 1 and 1 has algebraic multiplicity 2.

Examples

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Eigenspaces

Finding eigenvectors

Definition

The *eigenspace* E_λ *associated to* λ is the set of solutions to:

$$(A - \lambda \cdot I)x = 0$$

- After doing row reduction we will get infinitely many solutions.
- The no. of parameters is called the *geometric multiplicity* of λ.
- A set of vectors spanning the solution space are called *basic eigenvectors*.
- E.g. if the solutions are of the form:

$$s\vec{a} + t\vec{b}$$

  then the geometric multiplicity is 2 and the basic eigenvectors are $\vec{a}$ and $\vec{b}$.

Examples

Example

Find the eigenvectors of

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Example

Find the eigenvectors of

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Diagonalisation

Motivation

If $x \in \mathbb{R}^n$ is a linear combination of eigenvectors of A then:

$$A(x) = A\left(\sum_{i=1}^{n} a_i v_{\lambda_i}\right) = \sum_{i=1}^{n} a_i A(v_{\lambda_i}) = \sum_{i=1}^{n} a_i \lambda_i v_{\lambda_i}$$

So if *every* vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A the the *entire* matrix action simplifies.

Definition

An $n \times n$ matrix A is *diagonalisable* iff every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors.

Motivation

Suppose we want to find Av.

$$\begin{tikzcd}
v = \sum_{i=1}^n v_i e_i \arrow[d, mapsto, "P^{-1}"] & Av = \sum_{i=1}^n (Av)_i e_i \\
\sum_{i=1}^n a_i v_{\lambda_i} \arrow[r, mapsto, "\text{Apply } A"] & \sum_{i=1}^n a_i \lambda_i v_{\lambda_i} \arrow[u, mapsto, "P"]
\end{tikzcd}$$

- where P writes a linear combination of eigenvectors as a linear combination of standard basis vectors
- we say that the bottom horizontal action is *diagonal*

Diagonalisation

Definition

The matrix P *diagonalises* A iff

$$P^{-1}AP$$

is a diagonal matrix.

Question: How do we find the matrix P for a matrix A?

Answer: If v_{λ_1}, $v_{\lambda_2} \dots v_{\lambda_n}$ are eigenvectors such that every $x \in \mathbb{R}^n$ is a linear combination of the v_{λ_i} then the matrix

$$P = [v_{\lambda_1} v_{\lambda_2} \dots v_{\lambda_n}]$$

with eigenvectors as the columns is a diagonalising matrix for A.

When is a matrix diagonalisable?

Theorem

The following are equivalent for a matrix A:

1. *there is a P such that PAP^{-1} is diagonal*
2. *there are n eigenvectors such that any $x \in \mathbb{R}^n$ is a linear combination of these eigenvectors*
3. *there are eigenvectors $v_1 \ldots v_n$ such that $[v_1 v_2 \ldots v_n]$ is invertible*
4. *for every eigenvalue λ the geometric multiplicity is equal to the algebraic multiplicity*

Lemma

If the characteristic polynomial of A has n distinct eigenvalues then A is diagonalisable by 4.

Summary

1. Find the eigenvalues by solving $|A - \lambda \cdot I|$.
   - The no. of $(\lambda - \lambda_i)$ factors is the *algebraic multiplicity* $alg(\lambda_i)$.
2. Find the eigenspaces E_{λ_i} by solving $(A - \lambda_i \cdot I) = 0$.
   - Only need to do this for the eigenvalues λ_i found in (1).
   - The no. of parameters is the *geometric multiplicity* $geom(\lambda_i)$.
3.
   - If for all λ_i found in (1) we have $geom(\lambda_i) = alg(\lambda_i)$ then the diagonalising matrix is $P = [v_{\lambda_1} v_{\lambda_2} \dots v_{\lambda_n}]$ where the v_{λ_i} are the basic eigenvectors.
   - If for any of the λ_i found in (1) has $geom(\lambda_i) < alg(\lambda_i)$ then the matrix is not diagonalisable.

Examples

Example

If possible diagonalise

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Example

If possible diagonalise

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Examples

Example

If possible diagonalise

$$\begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$$

Example

If possible diagonalise

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$