MATH211: Linear Methods I

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Lecture on Tuesday 9th October, 2018

Lines and planes

Scalar product

Orthogonality

Projections

Last time

- Euclidean space
- Addition scalar multiplication etc..
- Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

Lines and planes

Two dimensions

Definition

A *parametric (vector) equation* describing a line in $\mathbb{R}^2$ is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d}$$

for $\vec{x}$, $\vec{c}$ and $\vec{d}$ in $\mathbb{R}^2$ and s in $\mathbb{R}$.

Definition

A *parametric (vector) equation* describing a line in $\mathbb{R}^3$ is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d}$$

for $\vec{x}$, $\vec{c}$ and $\vec{d}$ in $\mathbb{R}^3$ and s in $\mathbb{R}$.

Parametric forms of 3D lines and planes

Definition

A *parametric (vector) equation* describing a plane in $\mathbb{R}^3$ is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d} + t\vec{e}$$

for $\vec{x}$, $\vec{c}$, $\vec{d}$ and $\vec{e}$ in $\mathbb{R}^3$ and s, t in $\mathbb{R}$.

Definition (Symmetric form of equation for line)

$$\frac{x_1 + b_1}{a_1} = \frac{x_2 + b_2}{a_2} = \frac{x_3 + b_3}{a_3}$$

where $a_i \neq 0$.

Converting between various forms

$$\begin{tikzcd}
 & \text{System of equations} \arrow[dr, "\text{Row reduction}"] & \\
\text{Symmetric form} \arrow[ur, "\text{Split and rearrange}"] & & \text{Parametric form} \arrow[ll, "\text{Eliminate parameter}"]
\end{tikzcd}$$

Conversion example

Example

Write down all three forms for the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Example

Example

Write down a parametric equation for the line passing through

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and parallel to the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Example

Example

Write down the equation for the plane passing through the following three points.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Example

For the following two lines find the point of intersection.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Scalar product

Length of a vector

Definition (Two dimensions)

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$

Definition (Three dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Definition (n dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Distance between two points

Definition (Two dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Definition (Three dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

N dimensions and scalar multiplication

Definition (n dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Lemma (Scalar multiplication)

If $\vec{v}$ is a non-zero vector and a is a scalar then

$$\|a\vec{v}\| = |a|\|\vec{v}\|$$

Examples

Definition

A *unit vector* is a vector with length equal to 1.

Example

The following vectors are unit vectors:-

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{2}} \end{bmatrix}$$

Examples

Example

Find the distance between

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Example

Find the length of

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Scalar/dot product

Definition

If $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$ then

$$\begin{aligned}\vec{x}\cdot\vec{y} &= \sum_{k=1}^{n} x_k y_k \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\end{aligned}$$

And so in fact

$$\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$$

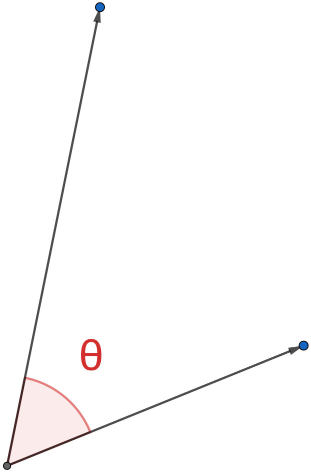
Properties of the scalar product

Theorem

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v}$ *is a real number*
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3. $\vec{u} \cdot \vec{0} = 0$
4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
5. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
6. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
   $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$

Scalar product in terms of angle



Definition (Scalar product in terms of angle)

$$\vec{x} \cdot \vec{y} = |x||y|cos(\theta)$$

Examples

Example

If

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

then find $u \cdot v$.

Example

Find the angle between

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Examples

Example

Find the angle between

$$\begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

Questions?

Orthogonality

Orthogonality

Definition

Two vectors u and v are *orthogonal* iff

$$u \cdot v = 0$$

Note that a zero vector is orthogonal to any other vector.

Orthogonal complement

Definition (Slightly non-standard)

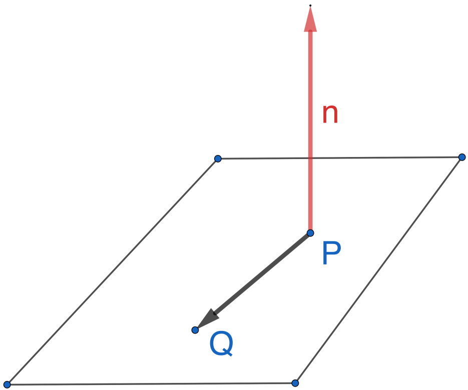
If v is a vector then the *orthogonal complement* $v^\perp$ of v is the set of all vectors u such that $u \cdot v = 0$.

Therefore

- If $v \neq 0 \in \mathbb{R}^1$ then $v^\perp = 0$.
- If $v \neq 0 \in \mathbb{R}^2$ then $v^\perp$ is the line perpendicular to v.
- If $v \neq 0 \in \mathbb{R}^3$ then $v^\perp$ is a plane.

In general if $v \neq 0 \in \mathbb{R}^n$ then $v^\perp$ is a $(n-1)$-dimensional subspace.

Planes via normal vector



- Let P be a point in the plane and n a normal vector to the plane.
- For every point Q in the plane the vector $\overrightarrow{PQ}$ is orthogonal to n.
- Therefore the solutions to

$$n \cdot (x - P) = 0$$

form a plane.

Examples

Example

Find all vectors orthogonal to both

$$\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Example

Are the following the vertices of a right angled triangle?

$$\begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -6 \end{bmatrix}$$

Example

Example

Find an equation of the plane containing

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

orthogonal to

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

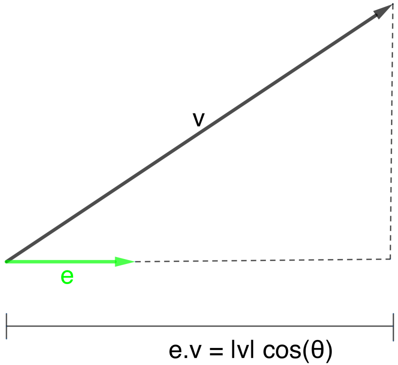
Example

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Questions?

Projections

Projection onto a vector

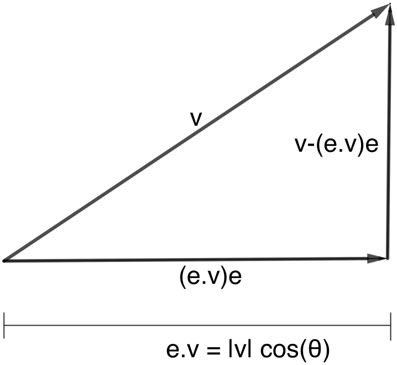


If e is a unit vector then

$e \cdot v = |e||v| \cos\theta = |v| \cos\theta$

and so $e \cdot v$ is the length of v projected onto the line in direction e.

Projection onto a vector



Therefore we can write

$$v = (e \cdot v)e + (v - e(e \cdot v))$$

where $(e \cdot v)e$ is parallel to e and $(v - e(e \cdot v))$ is orthogonal to e.

Examples

Example

Let

$$v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } u = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Find vectors v_0 and v_1 such that $v = v_0 + v_1$, v_0 is orthogonal to u and v_1 is parallel to u.

Example

Example

If

$$p = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

find the closest point to p on the following line:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Example

Example

Find the shortest distance from the point

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

to the plane with equation $5x + y + z = -1$.

Questions?