Math 211 2018-11-29:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Q: Diagonalise $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

First find eigenvalues.

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

Next find the eigenspaces:

E_{1+i}: $\left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & i & 0 \\ -i & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\left.\begin{array}{l} y = s \\ x = -is \end{array}\right\} \quad E_{1+i} = s\begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{geom. mult.} = 1$$

E_{1-i}: $\left(\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\left.\begin{array}{l} y = s \\ x = is \end{array}\right\} \quad E_{1-i} = s\begin{pmatrix} i \\ 1 \end{pmatrix}. \quad \text{geom. mult} = 1.$$

Since for all λ_i geom(λ) = alg(λ)

$\Rightarrow$ is diagonalisable with $P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$

and $D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$

<u>Q:</u> Diagonalise $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

First find eigenvalues:

$0 = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$ so $\lambda = 1$ is an eigenvalue with algebraic multiplicity $= 2$

Next find eigenspaces:

$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \left.\begin{array}{l} x = s \\ y = 0 \end{array}\right\} E_1 = s\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Since the geom. mult. < alg. mult. $\Rightarrow$ not diagonalisable.

Q: Diagonalise $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

First find the eigenvalues

$$0 = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = -(1-\lambda)^2(\lambda+3)$$

So the eigenvalues are:

$\lambda = 1$ with alg. mult. $= 2$

$\lambda = -3$ with ~~geo~~ alg. mult. $= 1$

Next find the eigenspaces.

E_1: $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \end{array}\right)$ $\begin{array}{l} x = s \\ y = t \\ z = 0 \end{array}$

$$E_1 = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{geom. mult.} = 2.$$

E_{-3}: $\left(\begin{array}{ccc|c} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

$z = s$
$x = -s/4$ $\quad E_{-3} = s\begin{pmatrix}-1/4\\0\\1\end{pmatrix} = \bar{s}\begin{pmatrix}1\\0\\-4\end{pmatrix}$
$y = 0$

geom. mult. $= 1$

So is diagonalisable with $P = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\6 & 0 & -4\end{pmatrix}$.

and $D = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -3\end{pmatrix}$

$$\begin{pmatrix}5 & 0\\0 & 2\end{pmatrix}^9 = \begin{pmatrix}5^9 & 0\\0 & 2^9\end{pmatrix}$$

$A = \begin{pmatrix}2 & 1 & 0\\0 & 1 & 0\\-1 & -1 & 1\end{pmatrix}$ Find A^{50}.

First find the eigenvalues.

$$0 = \begin{vmatrix}2-\lambda & 1 & 0\\0 & 1-\lambda & 0\\-1 & -1 & 1-\lambda\end{vmatrix} = -\begin{vmatrix}0 & 1-\lambda & 0\\2-\lambda & 1 & 0\\-1 & -1 & 1-\lambda\end{vmatrix}$$

$$= - \begin{vmatrix} -(1-\lambda)(2-\lambda) & 0 & 0 \\ 2-\lambda & 1 & 0 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

$\lambda = 1$ has alg. mult $= 2$

$\lambda = 2$ has alg. mult $= 1$.

Next find the eigenspaces.

$\underline{E_1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \end{array}\right)$$

$y = s \quad z = t \quad x = -s$

$$E_1 = \begin{pmatrix} -s \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\underline{E_2}$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right). \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} z = s \\ x = -s \\ y = 0 \end{array}\right\} \quad E_2 = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

<u>is</u> diagonalisable with $P = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now $A^{50} = (PDP^{-1})^{50} = PD^{50}P^{-1}$

First $D^{50} = \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find P^{-1}

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$A^{50} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2^{50} & -1 & 0 \\ 0 & 1 & 0 \\ 2^{50} & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{50} & 2^{50}-1 & 0 \\ 0 & 1 & 0 \\ -2^{50}+1 & -2^{50}+1 & 1 \end{pmatrix}.$$