

# MATH211: Linear Methods I

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# Lecture on Thursday 27<sup>th</sup> September, 2018

Determinants

Examples

Traditional determinants

Examples

# Last time

- ▶ Elementary matrices
- ▶ Elementary matrices as theoretical tool
- ▶ Matrix equations

Last time  
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Determinants  
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Examples  
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Traditional determinants  
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Examples  
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## Determinants

## Two dimensional determinants

The standard basis vectors get sent to the columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

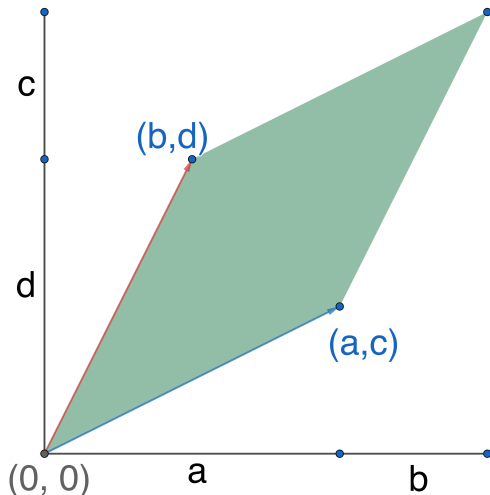
### Definition

The *determinant*

$$\det(A) = |A|$$

of any square matrix  $A$  is the *signed area* spanned by the columns of the matrix.

# Picture of two dimensional determinant



$$\begin{aligned} \text{Area} &= (a+b)(c+d) \\ &\quad - bc - bc \\ &\quad - \frac{1}{2}ac - \frac{1}{2}bd \\ &\quad - \frac{1}{2}ac - \frac{1}{2}bd \\ &= ac + ad + bc + bd \\ &\quad - 2bc - ac - bd \\ &= ad - bc \end{aligned}$$

# Examples

## Example

All identity matrices  $I_n$  have determinant equal to 1.

## Example

All matrices with repeated columns have determinant equal to 0.

## Example

In fact a matrix is invertible iff its determinant is **not** equal to 0.

# Effect of elementary row/column operations

The effect of the elementary row/column operations on  $\det(A)$  are:-

- ▶ Multiplying a row/column by a scalar  $k$ 
  - ▶ multiplies the determinant by  $k$ .
- ▶ Swapping two rows/columns
  - ▶ multiplies the determinant by  $-1$ .
- ▶ Adding a multiple of one row/column to another
  - ▶ has no effect on the determinant.

Pictures on Jupyter notebook.



# Determinants of diagonal matrix and scalar multiple

## Example (Diagonal matrices)

$$\det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \lambda_1 \lambda_2 \lambda_3 \lambda_4$$

## Example (Scalar multiple)

If  $A$  is an  $n \times n$  matrix then

$$\det(kA) = k^n \det(A)$$

# Determinant of triangular matrix

## Example

Triangular matrix

$$\begin{aligned} \det \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & a_{14} \\ 0 & \lambda_2 & a_{23} & a_{24} \\ 0 & 0 & \lambda_3 & a_{34} \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} &= \det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ a_{21} & \lambda_2 & 0 & 0 \\ a_{31} & a_{32} & \lambda_3 & 0 \\ a_{41} & a_{42} & a_{43} & \lambda_4 \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \\ &= \lambda_1 \lambda_2 \lambda_3 \lambda_4 \end{aligned}$$

# Calculating the determinant using row/column operations

- ▶ Start with a square matrix  $A$ .
- ▶ Perform row/column operations on  $A$  to convert into triangular form.
  - ▶ (Keep track of these because they can change the determinant!)
  - ▶ If we ever get a zero row/column then the determinant is 0.
- ▶ Calculate the determinant of the resulting triangular matrix.

Questions?

## Examples

# Examples

## Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

## Example

Find

$$\begin{vmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

# Examples

## Example

Find

$$\begin{vmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{vmatrix}$$

## Example

If

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4 \text{ find } \begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix}$$

# Examples

## Example

Find

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix}$$



Questions?

## Traditional determinants

# Minors and cofactors

## Definition

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. The  $ij^{th}$  minor of  $A$ , denoted as  $\text{minor}(A)_{ij}$ , is the determinant of the  $n - 1 \times n - 1$  matrix which results from deleting the  $i^{th}$  row and the  $j^{th}$  column of  $A$ .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

# Examples

## Example

Find the (1,2)-minor of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

## Example

Find the (2,2)-minor of

$$\begin{bmatrix} 1 & 5 & 2 & 5 \\ 2 & 6 & 7 & 1 \\ 9 & 9 & 8 & 2 \\ 0 & 8 & 6 & 0 \end{bmatrix}$$

# Cofactors

## Definition

The  $ij^{th}$  *cofactor* of  $A$  is

$$\text{cof}(A)_{ij} = (-1)^{i+j} \text{minor}(A)_{ij}$$

# Examples

## Example

Find the (1,2)-cofactor of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

## Example

Find the (2,2)-cofactor of

$$\begin{bmatrix} 1 & 5 & 2 & 5 \\ 2 & 6 & 7 & 1 \\ 9 & 9 & 8 & 2 \\ 0 & 8 & 6 & 0 \end{bmatrix}$$

# Recursive definition of determinant

## Definition (Cofactor expansion along row $i$ )

$$\begin{aligned}\det A &= \sum_{k=1}^n a_{ik} \operatorname{cof}(A)_{ik} \\ &= a_{i1} \operatorname{cof}(A)_{i1} + a_{i2} \operatorname{cof}(A)_{i2} + a_{i3} \operatorname{cof}(A)_{i3} + \cdots + a_{in} \operatorname{cof}(A)_{in}\end{aligned}$$

## Definition (Cofactor expansion along column $j$ )

$$\begin{aligned}\det A &= \sum_{k=1}^n a_{kj} \operatorname{cof}(A)_{kj} \\ &= a_{1j} \operatorname{cof}(A)_{1j} + a_{2j} \operatorname{cof}(A)_{2j} + a_{3j} \operatorname{cof}(A)_{3j} + \cdots + a_{nj} \operatorname{cof}(A)_{nj}\end{aligned}$$

# Three dimensions

## Example (Row 1 expansion)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

## Example (Row 2 expansion)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$



Questions?

## Examples

# Examples

## Example

Find

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{vmatrix}$$

# Examples

## Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

## Example

Find

$$\begin{vmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{vmatrix}$$

# Examples

## Example

Find

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

## Example

Find

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Questions?