MATH211: Linear Methods I

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Tuesday 2nd October, 2018

Lecture on Tuesday 2nd October, 2018

First row expansion

Any row expansion

Inversion formula

Last time

► Determinants as area/volume etc..

► Computing determinants using row operations.

► The cofactor expansion (see again this time).

Last time
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First row expansion
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Any row expansion
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Inversion formula
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First row expansion

Last time
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First row expansion
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Any row expansion
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Inversion formula
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Two dimensional determinant

Recall that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

$$\begin{vmatrix} 10 & 1 \\ 9 & 5 \end{vmatrix} = 50 - 9 = 41$$

Example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

Three dimensional determinants

Definition

The *cofactor expansion along row 1 is*

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11}
\begin{vmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{vmatrix}
- a_{12}
\begin{vmatrix}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{vmatrix}
+ a_{13}
\begin{vmatrix}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{vmatrix}
$$

Example

$$
\begin{vmatrix}
1 & 1 & 3 \\
2 & 4 & 1 \\
5 & 2 & 6
\end{vmatrix}
= 1
\begin{vmatrix}
4 & 1 \\
2 & 6
\end{vmatrix}
- 1
\begin{vmatrix}
2 & 1 \\
5 & 6
\end{vmatrix}
+ 3
\begin{vmatrix}
2 & 4 \\
5 & 2
\end{vmatrix}
$$

$$
= 1 \cdot (24 - 2) - 1 \cdot (12 - 5) + 3 \cdot (4 - 20)
$$

$$
= 22 - 7 - 48 = -33
$$

Examples

Example
Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

Example
Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Expansions in general

Definition (Non-standard)
If A is a matrix write $A(i|j)$ for the matrix obtained from A by deleting the ith row and the jth column.

Definition
The *cofactor expansion of A along the first row is*

$$|A| = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} |A(1|k)|$$

which when expanded is:

$$|A| = a_{11} |A(1|1)| - a_{12} |A(1|2)| + a_{13} |A(1|3)| - \cdots + (-1)^{1+n} |A(1|n)|$$

Auxiliary definitions

Definition (Minor)

$$minor_{ij}(A) = |A(i|j)|$$

Definition (Cofactor)

$$cof_{ij}(A) = (-1)^{i+j} minor_{ij}(A)$$
$$= (-1)^{i+j} |A(i|j)|$$

Lemma (Expansion along row 1)

$$|A| = \sum_{k=1}^{n} a_{1k} cof_{1k}(A)$$

Examples

Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

Example

Find

$$\begin{vmatrix} 2 & 5 & 4 & 3 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

Questions?

Last time

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Any row expansion

General cofactor expansion

In fact we may use the cofactors of arbitrary rows and columns:

Lemma (Expansion along arbitrary row or column)

$$|A| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |A(i|k)|$$
$$= \sum_{k=1}^{n} a_{kj}(-1)^{k+j} |A(k|j)|$$

So we can choose the row or column which is most convenient.

Examples

Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

by expanding along row 3 or column 1.

Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

by expanding along column 2.

Example

Example

Find

$$\begin{vmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{vmatrix}$$

by choosing the correct column...

Questions?

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Inversion formula

Duplicating rows

Suppose that an $n \times n$ matrix has ith row equal to jth row:

$$|A| = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

then we know that $|A| = 0$. (Why?)

Adjugate matrix

If we expand

$$|A| = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

along the jth row then we get:

$$0 = |A| = \sum_{k=1}^{n} a_{jk}(-1)^{j+k} |A(j|k)|$$

$$= \sum_{k=1}^{n} a_{ik}(-1)^{j+k} |A(j|k)|$$

The adjugate matrix

Now the last term looks like a matrix product:

$$\sum_{k=1}^{n} a_{ik}(-1)^{j+k} |A(j|k)| = (A(adj(A)))_{ij}$$

Definition (Adjugate matrix)

$$adj(A)_{ij} = (-1)^{i+j} |A(j|i)|$$
$$= cof_{ji}(A)$$

So $adj(A)$ is the transpose of the matrix of cofactors $cof_{ij}(A)$.

A theorem

Theorem

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

Proof.
Just compute the ij-entries:

$$(A \cdot adj(A))_{ij} = \sum_{k=1}^{n} a_{ik} \, adj(A)_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} (-1)^{j+k} \, |A(j|k)|$$

□

A theorem continued...

Theorem

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

Proof.

...so on the diagonal (when $i = j$) the entry is

$$\sum_{k=1}^{n} a_{ik}(-1)^{i+k} |A(i|k)| = det(A)$$

and off the diagonal

$$\sum_{k=1}^{n} a_{ik}(-1)^{j+k} |A(j|k)| = 0$$

as required. □

The inversion formula

So we know that

$$A \cdot adj(A) = adj(A) \cdot A = det(A)I$$

and therefore if $det(A) \neq 0$ then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Warning: for $n > 2$ this is a very inefficient way to compute an inverse.

Examples

Example

Compute the inverse of the following matrices:-

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$$

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Questions?