MATH211: Linear Methods I

Matthew Burke

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Elementary matrices

Examples

Some theory

Other matrix equations

Last time

- Matrix inversion algorithm
- Examples
- Inverses of products and transposes

Some applications of inverse matrices

- Stochastic systems represented by matrix A
  - a_{ij} denotes the probability of transition from state i to state j
  - sometimes after many iterations these systems
    - reach a 'steady state'
    - or become constrained to a certain set of states
  - being invertible implies that all states are always possible

- The inverse function theorem
  - if a function has invertible derivative at a point
  - then it is invertible in some neighbourhood of the point
  - the domain and the codomain are 'locally the same'

Elementary matrices

Elementary matrices

Recall the elementary row operations:

1. Swap two rows.
2. Multiply a row by a non-zero scalar.
3. Add a multiple of one row to another.

Each corresponds to multiplying on the left

$$A \mapsto EA$$

by a different *elementary matrix* E.

Multiply a row by a non-zero scalar

The matrix that corresponds to multiplying the second row by 3 is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{3}a_{21} & \frac{1}{3}a_{22} & \frac{1}{3}a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to multiplying the second row by $\frac{1}{3}$.

Swap two rows

The matrix that swaps the second and third rows is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and is its own inverse.

Adding a multiple of another row

The matrix that corresponds to subtract 3 times the first row from the second is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-3a_{11} & a_{22}-3a_{12} & a_{23}-3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+3a_{11} & a_{22}+3a_{12} & a_{23}+3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to adding 3 times the first row to the second.

How to find the corresponding matrix?

If we are given an elementary row operation, how do we find the corresponding matrix? Let the transformation act on the identity matrix.

Questions?

Examples

Example

Example

If

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

then find elementary matrices such that $C = FEA$.

Example

Write

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

as a product of elementary matrices.

Examples

Example

Write

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

as a product of elementary matrices.

Questions?

Some theory

Row reduction as factorisation

Is multiplying by sequence of elementary matrices. Interpret in terms of augmented matrices.

Alternative characterisation of invertible matrices

Theorem

Let A be an $n \times n$ matrix, and let X, B be $n \times 1$ vectors. The following conditions are equivalent.

1. *A is invertible.*
2. *The rank of A is n.*
3. *The reduced row echelon form of A is I_n.*
4. *$AX = 0$ has only the trivial solution, $X = 0$.*
5. *A can be transformed to I_n by elementary row operations.*
6. *The system $AX = B$ has a unique solution X for any choice of B.*
7. *There exists an $n \times n$ matrix C with the property that $CA = I_n$.*
8. *There exists an $n \times n$ matrix C with the property that $AC = I_n$.*

Uniqueness of row echelon form

Sketch of proof.

Questions?

Other matrix equations

Examples

Example

Find A such that

$$\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$$

Example

Find A such that

$$(3I - A^T)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Example

Prove that if $A^3 = 4I$ then A is invertible.

Examples

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Questions?