

# MATH211: Linear Methods I

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Thursday 4<sup>th</sup> October, 2018

## Lecture on Thursday 4<sup>th</sup> October, 2018

Adjugates and inverses

Cramer's rule

Common determinants

Polynomial interpolation

# Last time

- ▶ Cofactor expansion along first row
- ▶ Cofactor expansion along arbitrary row or column
- ▶ Adjugate matrix

Last time  
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Adjugates and inverses  
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Cramer's rule  
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Common determinants  
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Polynomial interpolation  
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## Adjugates and inverses

# Adjugate matrix

## Definition (Matrix of cofactors)

$$\begin{aligned} \text{cof}(A)_{ij} &= \text{cof}_{ij}(A) \\ &= (-1)^{i+j} \text{minor}_{ij}(A) \\ &= (-1)^{i+j} |A(i|j)| \end{aligned}$$

## Definition (Adjugate matrix)

$$\begin{aligned} \text{adj}(A)_{ij} &= (\text{cof}(A)_{ij})^T \\ &= \text{cof}_{ji}(A) \\ &= (-1)^{j+i} |A(j, i)| \end{aligned}$$

# Theorems

## Theorem (Adjugate is almost inverse)

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$$

So if  $\det(A)$  is not 0 then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

This is not an efficient method to calculate the inverse.

## Example

### Example

Find the inverse for

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

using the adjugate matrix formula.

Questions?



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## Cramer's rule

## Using the inverse to solve a system

Recall that if  $A$  is invertible then

$$Ax = b \implies x = A^{-1}b$$

and we can use the formula above to calculate  $A^{-1}b \dots$

## Using adjugate to solve system

$$\begin{aligned}
 x_i &= (A^{-1}b)_i \\
 &= \sum_{k=1}^n (A^{-1})_{ik} b_k \\
 &= \sum_{k=1}^n \frac{1}{\det(A)} \operatorname{adj}(A)_{ik} b_k \\
 &= \frac{1}{\det(A)} \sum_{k=1}^n b_k (-1)^{i+k} |A(k, i)| \\
 &= \frac{1}{\det(A)} \det(A_i)
 \end{aligned}$$

where  $A_i$  results from replacing the  $i$ th column of  $A$  with  $b$ .

## Cramer's rule

If  $A$  is invertible then the solutions to

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

are

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Again this is not efficient computationally.

# Example

## Example

Find  $x_2$  such that

$$\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Questions?

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Adjugates and inverses  
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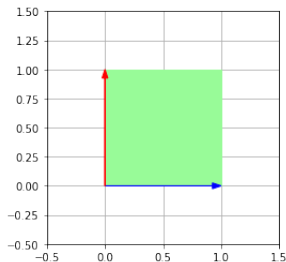
Cramer's rule  
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Common determinants  
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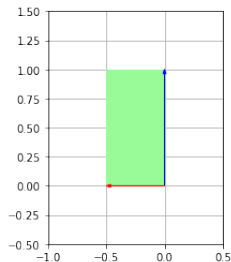
Polynomial interpolation  
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## Common determinants

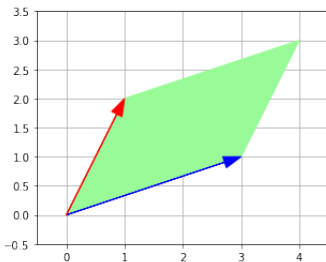
# Determinant of product (picture)



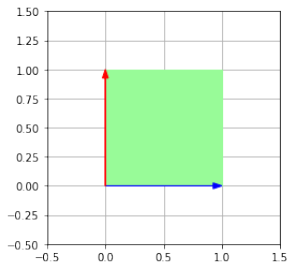
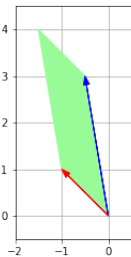
$B \rightarrow$



$A \rightarrow$



$B \rightarrow$





# Determinants and elementary matrices

The effects of the elementary operations:-

- ▶ Multiplying a row/column by a scalar  $k$ 
  - ▶ multiplies the determinant by  $k$ .
- ▶ Swapping two rows/columns
  - ▶ multiplies the determinant by  $-1$ .
- ▶ Adding a multiple of one row/column to another
  - ▶ has no effect on the determinant.

can be carried out by multiplying on the left by the corresponding elementary matrix:

$$A \mapsto EA$$

# Determinants and elementary matrices

Therefore:

- ▶ if  $E$  multiplies a row/column by a scalar  $k$ 
  - ▶  $\det(EA) = k \cdot \det(A)$
  - ▶  $\det(E) = k$
- ▶ if  $E$  swaps two rows/columns
  - ▶  $\det(EA) = -\det(A)$
  - ▶  $\det(E) = -1$
- ▶ if  $E$  adds a multiple of one row/column to another
  - ▶  $\det(EA) = \det(A)$
  - ▶  $\det(E) = 1$

## Determinant of product (algebra)

### Lemma (Determinant of product of elementary matrices)

*If  $E$  and  $F$  are elementary matrices then*

$$\det(EF) = \det(E)\det(F)$$

### Theorem (Determinant of product)

$$\det(BA) = \det(B)\det(A)$$

## Determinant of inverse and adjugate

### Theorem (Determinant of inverse)

*If  $A$  is invertible then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

### Theorem (Determinant of adjugate)

*For any square matrix  $A$*

$$\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$$

# Examples

## Example

Find

$$\left| \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right|$$

## Example

Find

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 8 \end{bmatrix} \right|$$

# Examples

## Example

Suppose  $A$ ,  $B$  and  $C$  are  $4 \times 4$  matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find  $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$ .

## Example

Suppose  $A$  is a  $3 \times 3$  matrix. Find  $\det A$  and  $\det B$  if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Questions?

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Polynomial interpolation  
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## Polynomial interpolation



## Example

### Example

Given data points  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 5)$  and  $(3, 10)$ , find an interpolating polynomial  $p(x)$  of degree at most three, and then estimate the value of  $y$  corresponding to  $x = \frac{3}{2}$ .

Questions?