Math211 - 2018-09-20

Find an inverse for $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Last time we solved

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \text{ is the first col of inverse.}$$

We then could have solved

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \text{ is the second col of inverse.}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

$R_2 \leftarrow R_2 - 3R_1$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$R_2 \leftarrow -\frac{1}{2} R_2$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right)$$

$R_1 \leftarrow R_1 - 2R_2$

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

The inverse is $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

Find the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$R_1 \leftarrow \frac{1}{a} R_1$ (so assume $a \neq 0$.)

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$R_2 \leftarrow R_2 - c R_1$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - \frac{cb}{a} & \frac{-c}{a} & 1 \end{array}\right)$$

$$R_2 \leftarrow \left(d - \frac{cb}{a}\right)^{-1} R_2 = \left(\frac{ad-bc}{a}\right)^{-1} R_2 = \frac{a}{ad-bc} R_2.$$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$R_1 \leftarrow R_1 - \frac{b}{a} R_2$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{b}{a}\left(\frac{-c}{ad-bc}\right) & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

Now $\frac{1}{a} - \frac{b}{a}\left(\frac{-c}{ad-bc}\right) = \frac{ad-bc+bc}{a(ad-bc)}$

$$= \frac{d}{ad-bc}$$

So the inverse is:

$$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Find the inverse of $\begin{pmatrix} 5 & 4 \\ 10 & 8 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 10 & 8 & 0 & 1 \end{array}\right)$$

$R_1 \leftarrow \frac{1}{5} R_1$

$$\left(\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 10 & 8 & 0 & 1 \end{array}\right)$$

$R_2 \leftarrow R_2 - 10R_1$

$$\left(\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

So the inverse does not exist.

The formula won't work because $ad - bc = 5 \times 8 - 4 \times 10 = 0$.

Find the inverse of $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right).$$

$R_2 \leftarrow R_2 - 2R_1 ; R_3 \leftarrow R_3 - R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$R_2 \leftarrow \frac{1}{3} R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$R_3 \leftarrow -R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$R_1 \leftarrow R_1 - 3R_3 ; R_2 \leftarrow R_2 + \frac{2}{3} R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \Rightarrow \text{the inverse is } \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1/3 & -2/3 \\ 1 & 0 & -1 \end{pmatrix}.$$

Prove $\boxed{(BA)^T = A^T B^T}$

$$\left((BA)^T\right)_{ij} = (BA)_{ji} = \sum_k B_{jk} A_{ki}$$

$$= \sum_k A_{ki} B_{jk} = \sum_k (A^T)_{ik} (B^T)_{kj}$$

$$= \left((A^T)(B^T)\right)_{ij}.$$

$$A^T\left((A^{-1})^T\right) = (A^{-1}A)^T = I^T = I.$$