MATH211: Linear Methods I

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Lecture on Tuesday 6th November, 2018

Examples

Complex numbers

Plus and times

Graphical interpretation

Last time

▶ Matrices and linear transformations

▶ Composition of linear transformations

▶ Inverse of linear transformations

Examples

Examples

Example
If
$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$
then find $S \circ T$, $T \circ S$ and the matrices $[S \circ T]$ and $[T \circ S]$.

Example
If
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$
find T^{-1} and $[T^{-1}]$.

Complex numbers

Progression of thought about complex numbers

> *...the whole matter seems to rest on sophistry rather than truth...*

(Rafael Bombelli 1526-1572. Early innovator in use of complex numbers)

> *The shortest path between two truths in the real domain passes through the complex domain.*

(Jacques Hadamard 1865-1963)

Algebraic motivation - extending the number system

Suppose that we only knew about the natural numbers:

$$\mathbb{N} := \{0, 1, 2, 3, \dots\}.$$

Then we wouldn't have a solution to the equation:

$$x + 1 = 0$$

Solution:

▶ Add a new symbol -1 such that

$$(-1) + 1 = 0$$

▶ Figure out addition and multiplication for this new symbol.

▶ (So in particular forced to add $-k$ for all $k \in \mathbb{N}$.)

Algebraic motivation - extending the number system

God made the integers; all else is the work of man.

(Leopold Kronecker 1823-1891)

- ▶ Using only positive integers $\{1, 2, 3, \dots\}$
  - ▶ we cannot find a solution to $x + 1 = 0$.

- ▶ Using only integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
  - ▶ we cannot find a solution to $3x + 2 = 0$.

- ▶ Using only fractions $\{\frac{a}{b}$ where $b \neq 0\}$
  - ▶ we cannot find a solution to $x^2 - 2 = 0$.

- ▶ Using only real numbers (decimals)
  - ▶ we cannot find a solution to $x^2 + 1 = 0$…

Algebraic motivation - extending the number system

Suppose that we only knew about real numbers.
Then we wouldn't have a solution to the equation:

$$x^2 + 1 = 0$$

Solution:

▶ Add a new symbol i such that

$$i^2 = -1$$

▶ Figure out addition and multiplication for this new symbol.
  ▶ (Later in this lecture.)

▶ It turns out that we are forced to add all numbers of the form:

$$a + ib$$

where a and b are real numbers.

Complex numbers

Definition

▶ The *imaginary unit*, denoted i, is defined to be a number with the property that $i^2 = -1$.

▶ A *pure imaginary* number has the form bi where $b \in \mathbb{R}$, $b \neq 0$.

▶ A *complex number* is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

Definition

If $z = a + bi$ then:

▶ a is called the *real part* of z.

▶ b is called the *imaginary part* of z.

Questions

Questions?

Plus and times

Motivation

Treat i as a number that satisfies the usual laws. E.g.:-

- $a(b + c) = ab + ac$
- $a + b = b + a$
- $ab = ba$ ...etc ...

as well as

$$i^2 = -1$$

and see what happens.

Example

$$1 + i + i^2 + i^3 + i^4 + i^5 = 1 + i - 1 - i + 1 + i$$
$$= 1 + i$$

which cannot be simplified any further.
(Because we only know $i^2 = -1$.)

Addition

Example

We would want addition to satisfy:-

- $i + i = 2i$
- $i - i = 0$
- $(1 + i) + 2 = 3 + i$

Definition

If a, b, c and d are real numbers then:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication

Example

We would want multiplication to satisfy:-

- $i \cdot i = -1$
- $2 \cdot i = 2i$
- $(1 + i) \cdot i = i + i \cdot i = -1 + i$

Definition

If a, b, c and d are real numbers then

$$(a + bi) \cdot (c + di) = ac + adi + bci - db = (ac - bd) + (ad + bc)i$$

Equations

Definition

If a, b, c and d are real numbers then

$$a + bi = c + di$$

if and only if $a = c$ and $b = d$.

Examples

Example

Evaluate

- $(-3 + 6i) + (5 - i)$.
- $(4 - 7i) + (6 - 2i)$.
- $(-3 + 6i) - (5 - i)$.
- $(4 - 7i) - (6 - 2i)$.

Example

$$(2 - 3i)(-3 + 4i)$$

Example

Find all complex numbers z such that $z^2 = -3 + 4i$.

Clearing denominators

Example (Exemplar)

Write

$$\frac{a + bi}{c + di}$$

in the form $e + fi$ for real numbers a, b, c, d, e, and f.

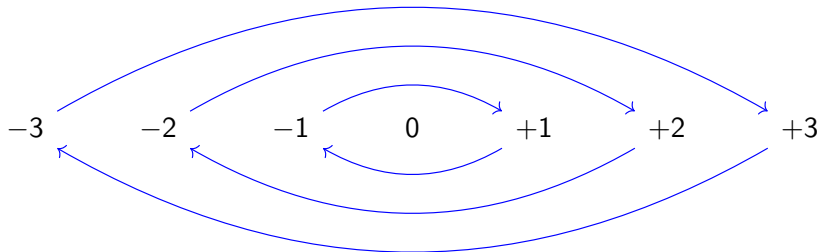- ▶ $\frac{1}{i}$
- ▶ $\frac{2-i}{3+4i}$
- ▶ $\frac{1-2i}{-2+5i}$

Questions?

Questions?

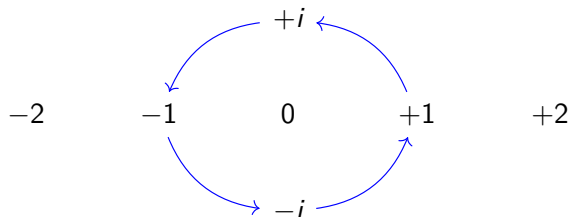# Graphical interpretation

Graphical interpretation

Consider the function $\mathbb{R} \to \mathbb{R}$ that multiplies by -1:

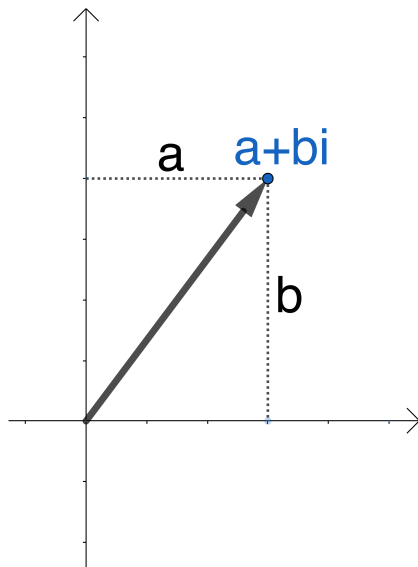$$-3 \qquad -2 \qquad -1 \qquad 0 \qquad +1 \qquad +2 \qquad +3$$

which shows us that $-1 \times (-1 \times x) = x$.

Graphical interpretation

So how do we get a transformation that squares to -1?
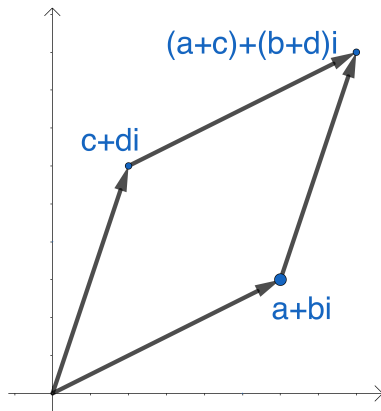
Graphical interpretation

We draw the complex number $z = a + bi$ as the point (a, b) in $\mathbb{R}^2$.

Definition

The *modulus of z* is its length in $\mathbb{R}^2$:

$$|z| = \left| \left| \begin{bmatrix} a \\ b \end{bmatrix} \right| \right| = \sqrt{a^2 + b^2}$$

Addition of complex numbers
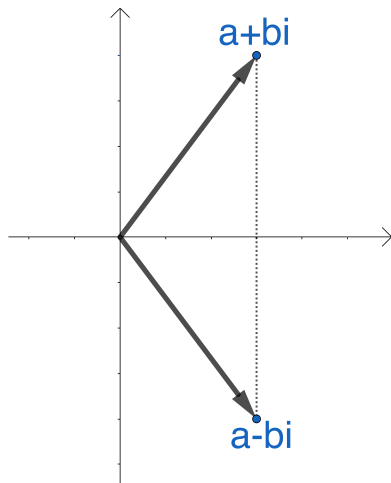

Complex addition is defined component-wise:

$$(a+ib)+(c+id) = (a+c)+(b+d)i$$

and therefore has the same graphical interpretation as the addition of vectors in $\mathbb{R}^2$.

Graphical interpretation of multiplication

Next time.

Conjugate

Definition (Conjugate)

If
$$z = a + bi$$
then
$$\bar{z} = a - bi$$

Properties of conjugate

Let z and w be complex numbers.

► $\overline{z \pm w} = \bar{z} \pm \bar{w}$.

► $\overline{(zw)} = \bar{z}\,\bar{w}$.

► $\overline{(\bar{z})} = z$.

► $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.

► z is real if and only if $\bar{z} = z$.

If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Examples

Example

- ► $|-3+4i|$
- ► $|3-2i|$
- ► $|i|$

Example

- ► $\overline{3+4i}$
- ► $\overline{-2+5i}$
- ► $\bar{i}$
- ► $\bar{7}$