MATH211: Linear Methods I

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Last time

- Matrix inversion algorithm
- Examples
- Inverses of products and transposes

Some applications of inverse matrices

- Stochastic systems represented by matrix A
  - a_{ij} denotes the probability of transition from state i to state j
  - sometimes after many iterations these systems
    - reach a ‘steady state’
    - or become constrained to a certain set of states
  - being invertible implies that all states are always possible
- The inverse function theorem
  - if a function has invertible derivative at a point
  - then it is invertible in some neighbourhood of the point
  - the domain and the codomain are ‘locally the same’

Elementary matrices

Elementary matrices

Recall the elementary row operations:

1. Swap two rows.
2. Multiply a row by a non-zero scalar.
3. Add a multiple of one row to another.

Each corresponds to multiplying on the left

$$A \mapsto EA$$

by a different *elementary matrix* E.

Multiply a row by a non-zero scalar

The matrix that corresponds to multiplying the second row by 3 is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{3}a_{21} & \frac{1}{3}a_{22} & \frac{1}{3}a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to multiplying the second row by $\frac{1}{3}$.

Swap two rows

The matrix that swaps the second and third rows is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and is its own inverse.

Adding a multiple of another row

The matrix that corresponds to subtract 3 times the first row from the second is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-3a_{11} & a_{22}-3a_{12} & a_{23}-3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and has inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+3a_{11} & a_{22}+3a_{12} & a_{23}+3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

that corresponds to adding 3 times the first row to the second.

How to find the corresponding matrix?

Suppose we have an elementary row operation

$$A \mapsto e(A)$$

We know that there is some E such that

$$e(A) = EA$$

therefore to find E we can let $A = I$:

$$e(I) = EI = E$$

In words: apply the row operation to the identity matrix.

Questions?

Examples

Examples

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Example

Compute

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Example

Example

If

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

then find elementary matrices such that $C = FEA$.

Example

Write

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

as a product of elementary matrices.

Examples

Example

Write

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

as a product of elementary matrices.

Example

Find a matrix U such that

$$U \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

is in reduced row echelon form.

Questions?

Some theory

Row reduction as factorisation

- If we row reduce

$$A \mapsto R$$

then there is a sequence of elementary matrices E_i such that

$$E_n \dots E_2 E_1 A = R$$

- In augmented matrix notation

$$[A|I] \mapsto [R|E_n \dots E_2 E_1]$$

because the row operations that reduce $A \mapsto R$ are precisely those that take $I \mapsto E_n \dots E_2 E_1$.

Invertibility of square matrices

Theorem (The inversion algorithm works)

If B is a square matrix and $[B|I]$ row reduces as

$$[B|I] \mapsto [I|A]$$

then $BA = I$ and $AB = I$.

Proof.

If $[B|I] \mapsto [I|A]$ then there is a sequence of elementary matrices E_i such that

$$I = E_n \dots E_2 E_1 B$$

and

$$A = E_n \dots E_2 E_1 I$$

so $AB = I$ and $BA = (E_n \dots E_2 E_1)^{-1}(E_n \dots E_2 E_1) = I$ because the elementary matrices E_i are invertible. $\square$

Uniqueness of reduced row echelon form

Theorem

If R and S are reduced row echelon matrices and there is a sequence of elementary row operations converting R into S then $R = S$.

(So the reduced row echelon form is unique.)

Alternative characterisation of invertible matrices

Theorem

Let A be an $n \times n$ matrix, and let X, B be $n \times 1$ vectors. The following conditions are equivalent.

1. *A is invertible.*
2. *The rank of A is n.*
3. *The reduced row echelon form of A is I_n.*
4. *$AX = 0$ has only the trivial solution, $X = 0$.*
5. *A can be transformed to I_n by elementary row operations.*
6. *The system $AX = B$ has a unique solution X for any choice of B.*
7. *There exists an $n \times n$ matrix C with the property that $CA = I_n$.*
8. *There exists an $n \times n$ matrix C with the property that $AC = I_n$.*

Questions?

Other matrix equations

Examples

Example

Find A such that

$$\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$$

Example

Find A such that

$$(3I - A^T)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Example

Prove that if $A^3 = 4I$ then A is invertible.

Examples

Example

Which of the following equations is true for arbitrary matrices A and B?

1. $(A - B)^2 = A^2 - 2AB + B^2$
2. $(A + B)(A - B) = A^2 - B^2$
3. $A^2B^2 = A(AB)B$

Example

Example

For what values of a does the system

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -2 & -3 & 2 & 4 \\ a & -5 & 1 & 16 \end{array}\right]$$

have

- infinitely many solutions
- a unique solution
- no solutions.

Questions?