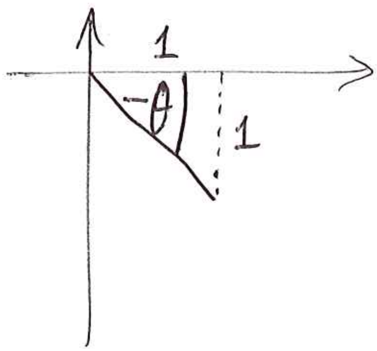
Math211 2018-11-22:

Q: $(1-i)^6(\sqrt{3}+i)^3 = ?$

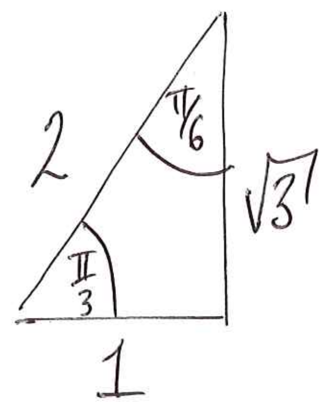
Polar form of $1-i$: $R=\sqrt{1+1}=\sqrt{2}$

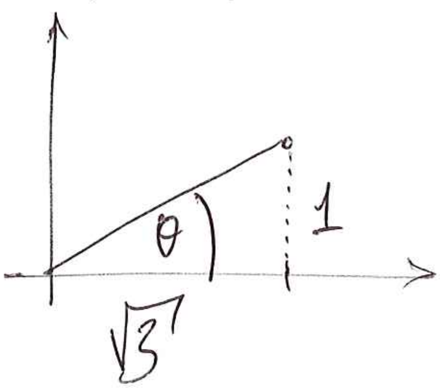


$-\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4 \Rightarrow \theta = \frac{-\pi}{4}.$

So: $1-i = \sqrt{2}\, e^{-i\pi/4}.$

Polar form of $\sqrt{3}+i$: $R=\sqrt{3+1}=2$





$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

So $\sqrt{3}+i = 2e^{i\pi/6}.$

Overall: $(1-i)^6(\sqrt{3}+i)^3 = \left(\sqrt{2}\,e^{-i\pi/4}\right)^6\left(2e^{i\pi/6}\right)^3$

$= \sqrt{2}^6 \cdot e^{\frac{-3i\pi}{2}} \cdot 2^3 \cdot e^{\frac{i\pi}{2}} = 64\, e^{-i\pi} = -64$

$$z^2 = 25 = 25e^{i0}$$

$$z_1 = 5e^{i0/2} = 5$$

$$z_2 = 5e^{\frac{i0}{2}+i\pi} = 5e^{i\pi} = -5$$

$$z^2 = -1 = e^{i\pi}$$

$$z_1 = e^{i\pi/2} \ (= i)$$

$$z_2 = e^{i\pi/2+i\pi} = e^{\frac{3i\pi}{2}} = e^{\frac{-i\pi}{2}} \ (= -i).$$

$$z^2 = e^{i\pi/4}$$

$$z_1 = e^{i\pi/8}$$

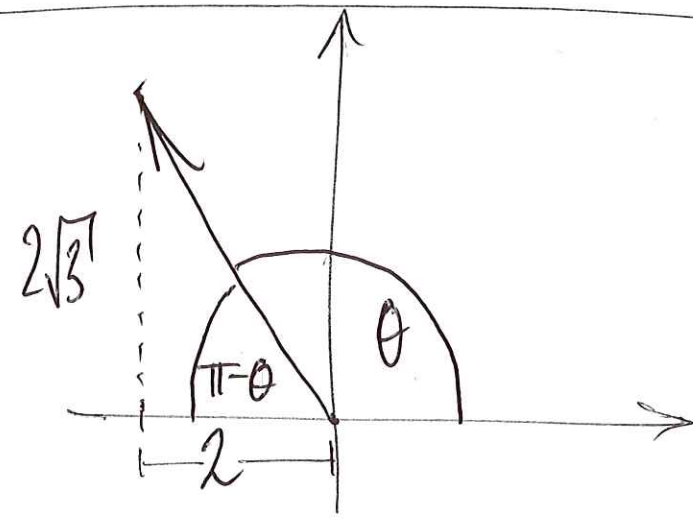
$$z_2 = e^{\frac{i\pi}{8}-i\pi} = e^{\frac{-7i\pi}{8}}.$$

$$z^3 = -1 = e^{i\pi}$$

$$z_1 = e^{\frac{i\pi}{3}}, \quad z_2 = e^{\frac{i\pi}{3} + \frac{2i\pi}{3}} = e^{i\pi} (= -1).$$

$$z_3 = e^{\frac{i\pi}{3} - \frac{2i\pi}{3}} = e^{\frac{-i\pi}{3}}.$$

$$z^4 = -2 + 2\sqrt{3}\,i$$



<u>Polar form for $-2+2\sqrt{3}\,i$:</u>

$$R = \sqrt{4+12} = 4$$

$$\pi - \theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}.$$

$$\text{So } -2+2\sqrt{3}\,i = 4e^{\frac{2i\pi}{3}}$$

$$z_1 = \sqrt[4]{4}\, e^{i\pi/6} = \sqrt{2}\, e^{i\pi/6}.$$

$$z_2 = \sqrt{2}\, e^{i\pi/6 + \frac{i\pi}{2}} = \sqrt{2}\, e^{\frac{i\pi}{6} + \frac{3i\pi}{6}} = \sqrt{2}\, e^{\frac{4i\pi}{6}} = \sqrt{2}\, e^{\frac{2i\pi}{3}}.$$

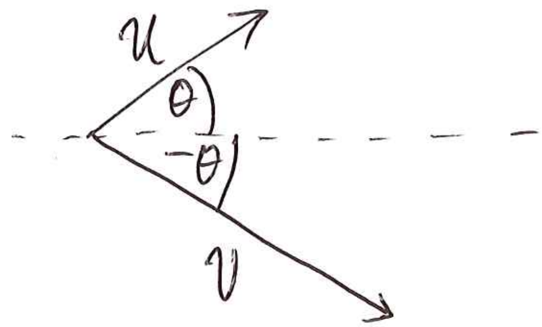
$$z_3 = \sqrt{2}\,e^{\frac{i\pi}{6}-\frac{\pi i}{2}} = \sqrt{2}\,e^{\frac{-i\pi}{3}}$$

$$z_4 = \sqrt{2}\,e^{\frac{-5i\pi}{6}}$$

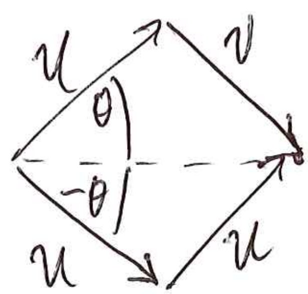
If $u+v$ real AND uv real then either

(1) u and v real or

(2) uv real $\Rightarrow$



$v+u$ real $\Rightarrow$



$\Rightarrow u = \bar{v}$.

Solve $z^2 - 3iz + (-3+i) = 0$

$$z_{1,2} = \frac{3i \pm \sqrt{(-3i)^2 - 4(-3+i)}}{2}$$

$$= \frac{3i \pm \sqrt{-9+12-4i}}{2} = \frac{3i \pm \sqrt{3-4i}}{2}$$

So find a, b such that $(a+bi)^2 = 3-4i$.

i.e. $a^2 - b^2 + 2abi = 3-4i$

$\Leftrightarrow$ $a^2 - b^2 = 3$ and $2ab = -4$ if $b \neq 0$ then

$$\Rightarrow a = \frac{-2}{b} \Rightarrow \frac{4}{b^2} - b^2 = 3 \Rightarrow 0 = b^4 + 3b^2 - 4$$
$$= (b^2 - 1)(b^2 + 4)$$

so $b = \pm 1$ $a = \mp 2$ overall

$$z_{1,2} = \frac{3i \pm (2-i)}{2} \; ; \; z_1 = \frac{2i+2}{2} = i+1$$

$$z_2 = \frac{4i+(-2)}{2} = 2i-1.$$