

MATH211: Linear Methods I

Matthew Burke

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Lecture on Tuesday 23rd October, 2018

Cross product

Applications

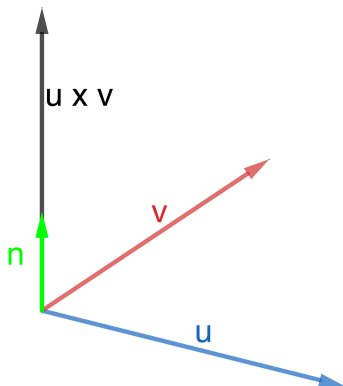
Misc

Last time

- ▶ Closest points
- ▶ Examples
- ▶ Cross product

Cross product

Trigonometric definition of cross product



Alternatively:

$$u \times v = \|u\| \|v\| \sin \theta n$$

where n is a unit vector orthogonal to both v and u .

Aide-memoire:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Cross product on standard basis vectors

If

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then

$$e_1 \times e_1 = 0$$

$$e_1 \times e_2 = e_3$$

$$e_1 \times e_3 = -e_2$$

$$e_2 \times e_1 = -e_3$$

$$e_2 \times e_2 = 0$$

$$e_2 \times e_3 = e_1$$

$$e_3 \times e_1 = e_2$$

$$e_3 \times e_2 = -e_1$$

$$e_3 \times e_3 = 0$$

Algebraic definition of cross product

So in general:

$$u \times v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 e_1 + u_2 e_2 + u_3 e_3) \times (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

$$\begin{aligned} &= u_1 v_1 (e_1 \times e_1) & + u_1 v_2 (e_1 \times e_2) & + u_1 v_3 (e_1 \times e_3) \\ &+ u_2 v_1 (e_2 \times e_1) & + u_2 v_2 (e_2 \times e_2) & + u_2 v_3 (e_2 \times e_3) \\ &+ u_3 v_1 (e_3 \times e_1) & + u_3 v_2 (e_3 \times e_2) & + u_3 v_3 (e_3 \times e_3) \end{aligned}$$

$$\begin{aligned} &= 0 & + u_1 v_2 e_3 & - u_1 v_3 e_2 \\ &- u_2 v_1 e_3 & + 0 & + u_2 v_3 e_1 \\ &+ u_3 v_1 e_2 & - u_3 v_2 e_1 & + 0 \end{aligned}$$

Algebraic definition

Definition

The *cross product* $u \times v$ of u with v is

$$u \times v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

The following heuristic doesn't make sense formally but may be an easier way to remember.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Properties of the cross product

Theorem

Let \vec{u}, \vec{v} and \vec{w} be in \mathbb{R}^3 .

1. $\vec{u} \times \vec{0} = \vec{0}$ and $\vec{0} \times \vec{u} = \vec{0}$.
2. $\vec{u} \times \vec{u} = \vec{0}$.
3. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
4. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar k .
5. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
6. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

Example

Example

Find $u \times v$ where

$$u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Example

Find all vectors orthogonal to

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Examples

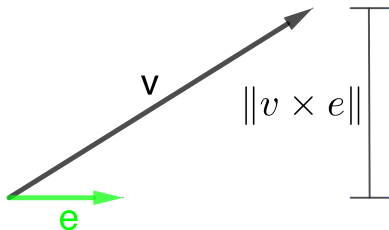
Example

Use a normal vector to give an equation for the plane passing through

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Applications

Cross product and projection



If e is a unit vector then

$$\|v \times e\| = \|v\| \|e\| \sin \theta = \|v\| \sin \theta$$

which is the distance from the tip of v to the line in direction e .

Line given by parametric equations

First we project onto the direction of the line:

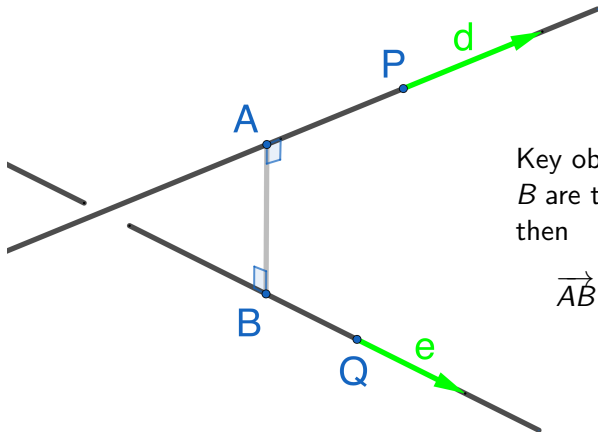
$$\overrightarrow{PA} = \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

and then we can find A :

$$A = P + \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

(Use same technique for line in \mathbb{R}^n also.)

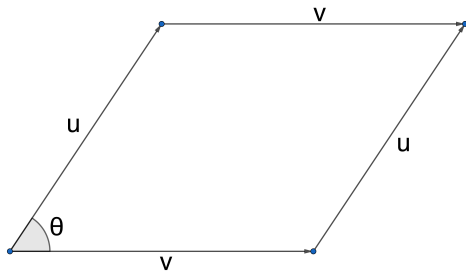
Closest points on skew lines



Key observation: if A and B are the closest points then

$$\overrightarrow{AB} \perp d \text{ and } \overrightarrow{AB} \perp e$$

Area of a parallelogram



The area of the parallelogram is:

$$\|u\| \|v\| \sin \theta = \|u \times v\|$$

Triple scalar (box) product

Definition

If u , v and w are vectors in \mathbb{R}^3 then the *triple scalar product* (or *box product*) of u , v and w is

$$u \cdot (v \times w)$$

Note that the order of the vectors matters. In fact

Lemma

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

which shows that it is the signed volume of the parallelepiped generated by u , v and w .

Examples

Example

Find the area of the triangle having vertices

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example

Find the volume of the parallelepiped determined by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Last time
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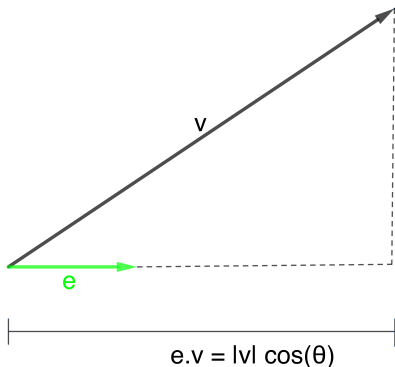
Cross product
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Misc
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Misc

Cauchy-Schwarz inequality



If e is a unit vector then

$$|e \cdot v| \leq \|v\|$$

More generally:

$$|u \cdot v| \leq \|u\| \|v\|$$

which is the *Cauchy-Schwarz inequality*.

Triangle inequality

Theorem

If u and v are vectors in \mathbb{R}^n then

$$\|u + v\| \leq \|u\| + \|v\|$$

Theorem

If u and v are vectors in \mathbb{R}^n then

$$|\|u\| - \|v\|| \leq \|u - v\|$$

The Lagrange identity

Theorem

If u and v are in \mathbb{R}^3 then

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

Example

Example

Find equations for the lines through

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at the two points distance three from $(1, 2, 0)$.

Examples

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.