MATH211: Linear Methods I

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Tuesday 16$^{\text{th}}$ October, 2018

Lecture on Tuesday 16th October, 2018

Orthogonality

Projections

Closest points

Misc

Last time

- Lines and planes
- Scalar product, length and distance
- Orthogonality

Orthogonality

Orthogonality

Definition

Two vectors u and v are *orthogonal* iff

$$u \cdot v = 0$$

Note that a zero vector is orthogonal to any other vector.

Orthogonal complement

Definition (Slightly non-standard)

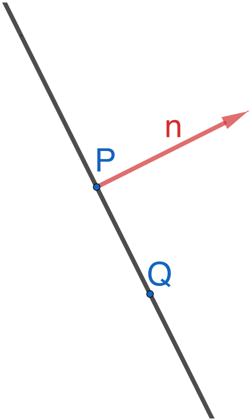
If v is a vector then the *orthogonal complement* $v^\perp$ of v is the set of all vectors u such that $u \cdot v = 0$.

Therefore

- If $v \neq 0 \in \mathbb{R}^1$ then $v^\perp = 0$.
- If $v \neq 0 \in \mathbb{R}^2$ then $v^\perp$ is the line perpendicular to v.
- If $v \neq 0 \in \mathbb{R}^3$ then $v^\perp$ is a plane.

In general if $v \neq 0 \in \mathbb{R}^n$ then $v^\perp$ is a $(n-1)$-dimensional subspace.

Lines via normal vector

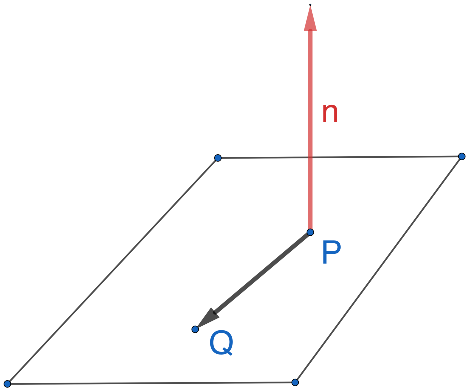


- Let P and n be vectors in $\mathbb{R}^2$.
- The set of points Q such that $\overrightarrow{PQ}$ is orthogonal to n form a line.
- Therefore the solutions to

$$n \cdot (x - P) = 0$$

form a line through P and orthogonal to n.

Planes via normal vector



- Let P and n be vectors in $\mathbb{R}^3$.
- The set of points Q such that $\overrightarrow{PQ}$ is orthogonal to n form a plane.
- Therefore the solutions to

$$n \cdot (x - P) = 0$$

form a plane through P and orthogonal to n.

Examples

Example

Find all vectors orthogonal to both

$$\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Example

Are the following the vertices of a right angled triangle?

$$\begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -6 \end{bmatrix}$$

Example

Example

Find an equation of the plane containing

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

orthogonal to

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

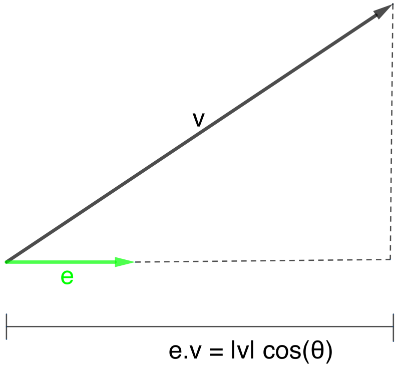
Example

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Questions?

Projections

Projection onto a vector

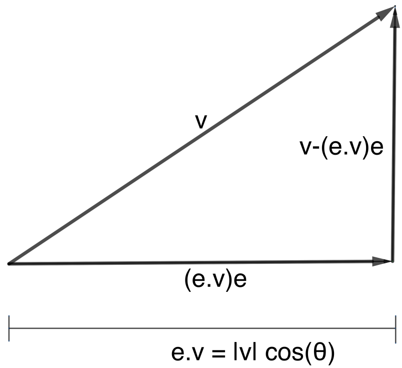


If e is a unit vector then

$$e \cdot v = |e||v| \cos\theta = |v| \cos\theta$$

and so $e \cdot v$ is the length of v projected onto the line in direction e.

Projection onto a vector



Therefore we can write

$$v = (e \cdot v)e + (v - e(e \cdot v))$$

where $(e \cdot v)e$ is parallel to e and $(v - e(e \cdot v))$ is orthogonal to e.

Examples

Example

Let

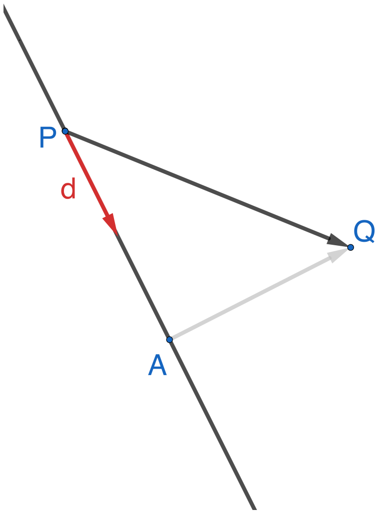
$$v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } u = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Find vectors v_0 and v_1 such that $v = v_0 + v_1$, v_0 is orthogonal to u and v_1 is parallel to u.

Questions?

Closest points

Line given by parametric equations



First we project onto the direction of the line:

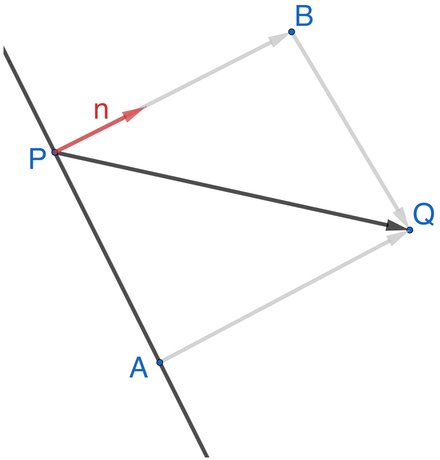
$$\overrightarrow{PA} = \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

and then we can find A:

$$A = P + \frac{d}{\|d\|} \cdot (Q - P) \frac{d}{\|d\|}$$

(Use same technique for line in $\mathbb{R}^n$ also.)

Line given by normal



In this picture

$$\overrightarrow{PB} \| \overrightarrow{AQ} \text{ and } \overrightarrow{PA} \| \overrightarrow{BQ}$$

First we project onto the normal:

$$\overrightarrow{PB} = \frac{n}{\|n\|} \cdot (Q - P) \frac{n}{\|n\|}$$

and then we can find A:

$$A = Q - \frac{n}{\|n\|} \cdot (Q - P) \frac{n}{\|n\|}$$

(Use same technique for $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ also.)

Example

Example

Find the closest point to

$$Q = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

on the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Example

Example

Find the closest point to

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

on the line with equation $4x + 3y = -2$.

Example

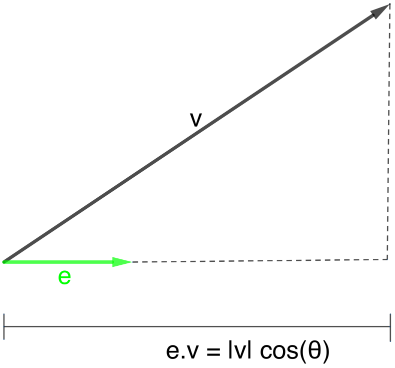
Find the closest point to

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

on the plane with equation $5x + y + z = -1$.

Misc

Cauchy-Schwarz inequality



If e is a unit vector then

$$|e \cdot v| \leq \|v\|$$

More generally:

$$|u \cdot v| \leq \|u\| \|v\|$$

which is the *Cauchy-Schwarz inequality*.

Triangle inequality

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$\|u + v\| \leq \|u\| + \|v\|$$

Theorem

If u and v are vectors in $\mathbb{R}^n$ then

$$|\|u\| - \|v\|| \leq \|u - v\|$$

Example

Example

Find equations for the lines through

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at the two points distance three from $(1, 2, 0)$.

Examples

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.