Worked Examples

Work in progress

December 7, 2018

Solutions are collected at the end.

Exercise 1

Suppose that the internet consists of three pages P_1, P_2 and P_3. Suppose further that:-

- there are two links from P_1 to P_2
- there is one link from P_1 to P_3
- there is one link from P_2 to P_1
- there is one link from P_2 to P_3
- there is one link from P_3 to P_2

Then calculate the (relative) PageRanks for P_1, P_2 and P_3.

Exercise 2

Find the distance α between the lines

$$L_1 : \vec{x} = \begin{bmatrix} -6 \\ -7 \\ 7 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = P + sd$$

and

$$L_2 : \vec{x} = \begin{bmatrix} -7 \\ -14 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = Q + te$$

and find a points A on L_1 and B on L_2 such that $dist(A, B) = \alpha$.

Exercise 3

Suppose that the sequence x_0, x_1, x_2 ... is defined by $x_0 = 2$, $x_1 = 1 + i$ and $x_{k+2} = (1 + i)x_{k+1} - ix_k$. Find a formula for x_k.

Exercise 4

A tennis club organises its games over the course of a year as follows:-

- At the beginning of the year there is a qualifying tournament.
- Those who finish in the top half of the qualifying tournament enter league A and those who do not enter league B.
- Players in both leagues play matches weekly.
- Adam joins at the beginning of one year:-
  - Adam has probability $\frac{2}{3}$ of finishing in the top half of the qualifying tournament.
  - In league A Adam has a $\frac{2}{5}$ chance of winning each weekly match.
  - In league B Adam has a $\frac{4}{5}$ chance of winning each weekly match.

Model Adam's progress as a Markov chain and find all steady state vectors.

Exercise 5

Find equations for the lines through

$$Q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

that meet the line

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = P + sd$$

at the two points at distance 3 from

$$P = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Exercise 6

Find the scalar equation for the plane passing through the point

$$Q = \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$$

and containing the line

$$\vec{x} = \begin{bmatrix} 9 \\ -6 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix} = P + td$$

Exercise 7

Find a steady state vector for

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Exercise 8

If possible diagonalise

$$\begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$$

Answer of exercise 1

The transition matrix is:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

So we get the steady state vector by solving $(A - I)v = 0$:

$$\left[\begin{array}{ccc|c} -1 & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & -1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 2 & -3 & 3 & 0 \\ 1 & \frac{3}{2} & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{-3}{2} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{-3}{4} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 \end{array}\right]$$

so $z = s$, $x = \frac{3s}{4}$ and $y = \frac{3s}{2}$. So the eigenspace associated to the eigenvalue 1 is:

$$E_1 = s \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix} = \bar{s} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

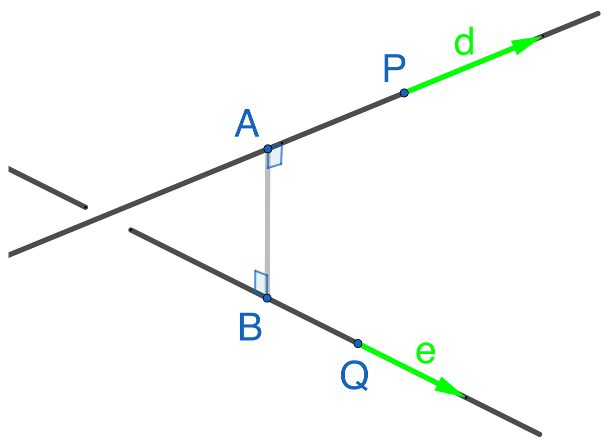
and so the steady state vector (containing the PageRanks) is

$$\frac{1}{13}\begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

because we need to choose $\bar{s}$ so that the sum of the entries is 1.

Answer of exercise 2

First draw the picture:



where we want to find A and B. Since $\vec{AB}$ is orthogonal to both d and e:

$$(B - A) \cdot d = 0 \text{ and } (B - A) \cdot e = 0 \tag{1}$$

First we can express $B - A$ in terms of unknowns s and t:

$$B - A = \begin{bmatrix} -7 + 2t \\ -14 + 3t \\ -t \end{bmatrix} - \begin{bmatrix} -6 + 2s \\ -7 + s \\ 7 - s \end{bmatrix} = \begin{bmatrix} 2t - 2s - 1 \\ 3t - s - 7 \\ -t + s - 7 \end{bmatrix}$$

because A is on L_1 and B is on L_2. Then 1 gives firstly:

$$\begin{aligned} 0 &= \begin{bmatrix} 2t - 2s - 1 \\ 3t - s - 7 \\ -t + s - 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= 4t - 4s - 2 + 3t - s - 7 + t - s + 7 \\ &= 8t - 6s - 2 \end{aligned}$$

and secondly

$$
\begin{aligned}
0 &= \begin{bmatrix} 2t-2s-1 \\ 3t-s-7 \\ -t+s-7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\
&= 4t-4s-2+9t-3s-21+t-s+7 \\
&= 14t-8s-16
\end{aligned}
$$

Next find t and s by solving

$$\left[\begin{array}{cc|c} 8 & -6 & 2 \\ 14 & -8 & 16 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{1}{4} \\ 7 & -4 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{1}{4} \\ 0 & \frac{5}{4} & \frac{25}{4} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{1}{4} \\ 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array}\right]$$

so $t = 4$ and $s = 5$. Therefore

$$A = P + sd = \begin{bmatrix} -6 \\ -7 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

and

$$B = Q + te = \begin{bmatrix} -7 \\ -14 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

so the distance between the lines is:

$$dist(A,B) = \sqrt{3^2 + 0^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$$

Answer of exercise 3

Reformulate as a two dimensional discrete linear dynamical system. Let:

$$v_k = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$$

and so

$$v_{k+1} = \begin{bmatrix} x_{k+2} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 1+i & -i \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = Av_k$$

and

$$v_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

First find the eigenvalues:

$$\begin{aligned} det \begin{bmatrix} 1+i-\lambda & -i \\ 1 & -\lambda \end{bmatrix} &= -\lambda(1+i-\lambda)+i \\ &= \lambda^2-(1+i)\lambda+i \\ &= (\lambda-1)(\lambda-i) \end{aligned}$$

The eigenspace associated to the eigenvalue 1:

$$\left[\begin{array}{cc|c} i & -i & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \end{array}\right]$$

so $y = s$ and $x = s$ and so the eigenspace is

$$E_1 = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace associated to the eigenvalue i:

$$\left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \end{array}\right]$$

so $y = s$ and $x = is$ and the eigenspace is:

$$E_i = s\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Next write the initial vector in terms of the eigenvectors:

$$v_0 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix}$$

to deduce that

$$\begin{aligned} A^k v_0 &= A^k \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \\ &= A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^k \begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i^k \begin{bmatrix} i \\ 1 \end{bmatrix} \end{aligned}$$

therefore $x_k = 1 + i^k$.

Answer of exercise 4

Let the states be:-

- State 1: Before the qualifying tournament.
- State 2: Adam is in league A and won the previous game.
- State 3: Adam is in league A and lost the previous game.
- State 4: Adam is in league B and won the previous game.
- State 5: Adam is in league B and lost the previous game.

The transition matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{5} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{3}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{5} & \frac{4}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

(Notice how the Markov chain splits into two 2×2 blocks after the first iteration.) To find the steady state vector we solve:

$$\left[\begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{-3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{5} & \frac{4}{5} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{5} & \frac{-4}{5} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{-4}{5} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{-4}{5} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right]$$

so $x_3 = s$, $x_5 = t$, $x_1 = 0$, $x_2 = \frac{2s}{3}$ and $x_4 = 4t$. Therefore the eigenspace associated to eigenvalue 1 is:

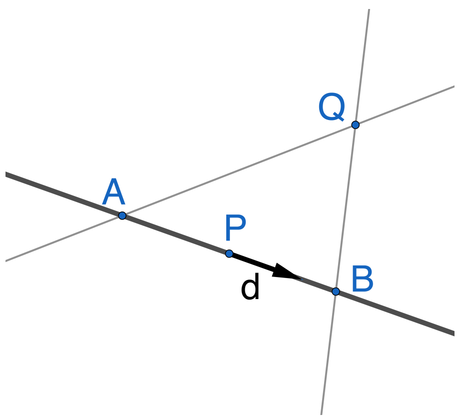
$$E_1 = \begin{bmatrix} 0 \\ \frac{2s}{3} \\ s \\ 4t \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

and so there are two steady state vectors:

$$\frac{1}{5}\begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} \tag{2}$$

Answer of exercise 5

First draw the picture:



We want to find the line through A and Q and the line through B and Q. Now the vector $\frac{d}{\|d\|}$ has unit length so the vector $3\frac{d}{\|d\|}$ has length 3. So we can find A and B:

$$\begin{aligned} A &= P + 3\frac{d}{\|d\|} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3\frac{1}{\sqrt{4+1+4}}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}
B &= P - 3\frac{d}{\|d\|} \\
&= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 3\frac{1}{\sqrt{4+1+4}}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}
\end{aligned}$$

and so the line through Q and A is

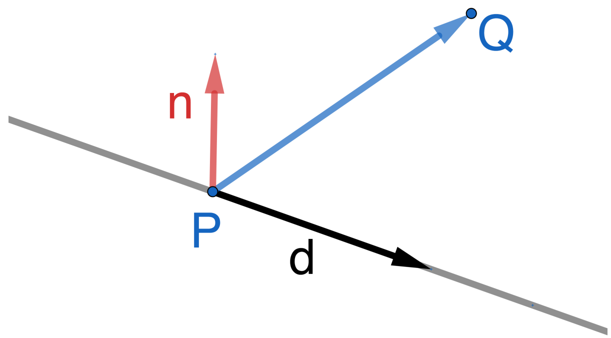
$$\begin{aligned}
\vec{x} &= Q + s(A - Q) \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

and the line through Q and B is

$$\begin{aligned}
\vec{x} &= Q + t(B - Q) \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}
\end{aligned}$$

Answer of exercise 6

First draw the picture:



where the vector n is any vector orthogonal to both d and $\vec{QP}$. To find one such n take the cross-product of d and $\vec{QP}$:

$$n = d \times (Q - P) = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix} \times \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8-5 \\ -20+4 \\ -2-16 \end{bmatrix} = \begin{bmatrix} -13 \\ -16 \\ -18 \end{bmatrix}$$

so the equation for the plane is

$$\begin{aligned} 0 = n \cdot (x - P) &= \begin{bmatrix} -13 \\ -16 \\ -18 \end{bmatrix} \cdot \begin{bmatrix} x-9 \\ y+6 \\ z+1 \end{bmatrix} \\ &= -13x + 117 - 16y - 96 - 18z - 18 \end{aligned}$$

i.e.

$$13x + 16y + 18z = 3$$

Answer of exercise 7

We need to solve $(A - I)x = 0$:

$$\left[\begin{array}{ccc|c} \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{-1}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{-1}{4} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{-3}{4} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array}\right]$$

so $z = s$, $x = \frac{3s}{4}$ and $y = \frac{s}{2}$.

$$E_1 = s \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} = \bar{s} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

therefore the steady state vector is

$$\frac{1}{9} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

(i.e. choose $\bar{s}$ such that the entries sum to 1.)

Answer of exercise 8

First find the eigenvalues:

$$\begin{aligned}
det \begin{bmatrix} 3\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 1 & -5 & 5-\lambda \end{bmatrix} &= -det \begin{bmatrix} 1 & -5 & 5-\lambda \\ 1 & -2-\lambda & 2 \\ 3\lambda & -4 & 2 \end{bmatrix} \\
&= -det \begin{bmatrix} 1 & -5 & 5-\lambda \\ 0 & 3-\lambda & \lambda-3 \\ 0 & 11-5\lambda & 2-(3-\lambda)(5-\lambda) \end{bmatrix} \\
&= (\lambda-3) det \begin{bmatrix} 1 & -5 & 5-\lambda \\ 0 & 1 & -1 \\ 0 & 11-5\lambda & 2-(3-\lambda)(5-\lambda) \end{bmatrix} \\
&= (\lambda-3) det \begin{bmatrix} 1 & -5 & 5-\lambda \\ 0 & 1 & -1 \\ 0 & 0 & 2-(3-\lambda)(5-\lambda)+11-5\lambda \end{bmatrix} \\
&= (\lambda-3)\left(2-(15-8\lambda+\lambda^2)+11-5\lambda\right) \\
&= -(\lambda-3)(\lambda^2-3\lambda+2) \\
&= (\lambda-3)(\lambda-1)(\lambda-2)
\end{aligned}$$

The eigenspace associated to 1:

$$\left[\begin{array}{ccc|c} 2 & -4 & 2 & 0 \\ 1 & -3 & 2 & 0 \\ 1 & -5 & 4 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \\ 1 & -5 & 4 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

and so the eigenspace is

$$E_1 = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The eigenspace associated to 2:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 1 & -4 & 2 & 0 \\ 1 & -5 & 3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 1 & -5 & 3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

and so the eigenspace is:

$$E_2 = s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The eigenspace associated to 3:

$$\left[\begin{array}{ccc|c} 0 & -4 & 2 & 0 \\ 1 & -5 & 2 & 0 \\ 1 & -5 & 2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \end{array}\right]$$

so the eigenspace is

$$E_3 = s \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Now for all of the eigenvalues (1, 2 and 3) the algebraic multiplicities are equal to the geometric multiplicities the matrix is diagonalisable. The diagonalising matrix is

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

with corresponding diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$