MATH211: Linear Methods I

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Lecture on Tuesday 18th September, 2018

Last time

- Matrix addition and scalar multiplication
- Relationship to systems of equations
- Matrix composition (product)
- Non-commutativity of matrix composition
- (Two sets of notes for 11th September)

Non-commutativity

Review previous examples

In general $AB \neq BA$.

(i,j)-entry proofs

Entries of product matrix

Definition

If A and B are matrices such that

$$\text{No. of rows}(A) = m = \text{No. of columns}(B)$$

then

$$\begin{aligned}(BA)_{ij} &= \sum_{k=1}^{m} b_{ik}a_{kj} \\ &= b_{i1}a_{1j} + b_{i2}a_{2j} + b_{i3}a_{3j} + \cdots + b_{im}a_{mj}\end{aligned}$$

Picture

$$(BA)_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + b_{i3} a_{3j} + \cdots + b_{im} a_{mj}$$

is the dot product of the ith row of B with the jth column of A:

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{i1} & b_{i2} & \dots & b_{im} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nm} \end{bmatrix}$$

Examples of entries of matrix product

Example

Find the $(2,3)$-entry of

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

Find the $(3,1)$-entry of

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 1 \\ 5 & 4 & 9 \end{bmatrix} \begin{bmatrix} 12 & 1 & 2 \\ 4 & 0 & 4 \\ -2 & 0 & 0 \end{bmatrix}$$

General examples of (i,j)-entry

In the following examples assume that the matrices are of the appropriate sizes so that the products involved make sense.

Example

Find the (i,j)-entry of

$$(CB)A$$

Example

Find the (i,j)-entry of

$$C(A+B)$$

Questions?

Transpose of a matrix

Definition

If A is *any* matrix the *transpose* A^T *of* A has entries

$$(A^T)_{ij} = A_{ji}$$

Example

$$\begin{bmatrix} 5 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 0 \\ -3 & 2 \end{bmatrix}$$

In short: the columns of A^T are the rows of A.

Symmetric matrices

Definition

A matrix A is *symmetric* iff $A = A^T$.
In terms of entries: $A_{ij} = A_{ji}$.

Example

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 17 \\ -2 & 17 & 0 \end{bmatrix}$$

is symmetric but

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 5 \\ -2 & 17 & 0 \end{bmatrix}$$

is not.

Examples of (i,j)-entry proofs

Example

Prove $C(B + A) = CB + CA$.

Example

Prove $(BA)^T = A^T B^T$.

Example

Prove $(B + A)^T = B^T + A^T$.

Example

Prove that if A and B are symmetric then $A^T + 2B$ is symmetric.

Example

Prove $C(BA) = (CB)A$. (Called matrix associativity.)

Identity and inverse matrices

Identity matrix

Definition

The $n \times n$ *identity matrix* is:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

In terms of entries:

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Product with identity

Whenever the following products make sense:

$$BI_m = B \qquad \text{and} \qquad I_m A = A$$

Indeed for the first:

$$\begin{aligned}(BI_m)_{ij} &= \sum_{k=1}^{m} B_{ik} I_{kj} \\ &= \sum_{k \neq j} B_{jk} I_{kj} + B_{ij} I_{jj} \\ &= \sum_{k \neq j} 0 + B_{ij} = B_{ij}\end{aligned}$$

Inverse matrix

Definition

If A is an $n \times n$ matrix then an *inverse matrix* A^{-1} *of* A is any matrix such that

$$AA^{-1} = I_n \qquad \text{and} \qquad A^{-1}A = I_n$$

Some remarks:

- It doesn't make sense to ask for an inverse of a non-square matrix. (Think in terms of linear transformations.)
- Not all square matrices have inverses.
- If A^{-1} is the inverse of A then A is the inverse of A^{-1}.

Examples of non-existence

Example

No matrix that only has zero entries has an inverse.

Example

The following matrix does not have an inverse

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In fact a $n \times n$ matrix has an inverse iff $rank(A) = n$.

Uniqueness of inverse

Lemma

If a matrix A has an inverse then it is unique.

Questions?

Motivation from linear combinations

Lets try to find an inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We want a matrix A such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} & \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \quad \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Express the standard basis vectors in terms of the columns.

Examples

Example

Find the inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

Find the inverse for the general 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Examples

Example

Find the inverse for

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$