MATH211: Linear Methods I

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Tuesday 9$^{\text{th}}$ October, 2018

Lecture on Tuesday 9th October, 2018

Euclidean space

Equations for lines and planes

Distance and scalar product

Last time

- Adjugates and inverses
- Cramer's rule
- Common determinants
- Polynomial interpolation

Midterm

- On the 26th October.
- Won't need to memorize any proofs.
- Will need to understand the methods used and answer abstract questions about them.

Polynomial interpolation

Example

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

Euclidean space

Column vectors as points

Recall a column vector consists of n numbers in a column:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The set of all column vectors of length n is called $\mathbb{R}^n$.

(So $\mathbb{R}^n$ is n-dimensional space: each component is a dimension.)

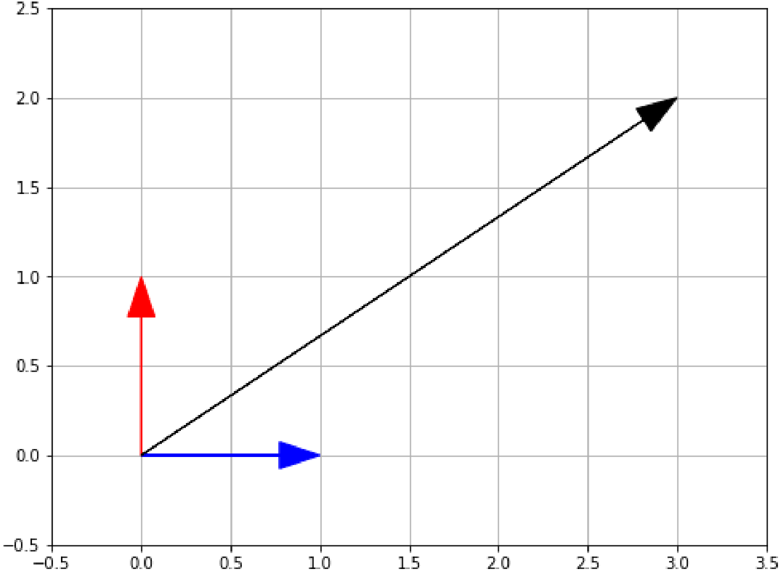
Interpreting a scalar

To interpret a number we need to choose ‘units’.

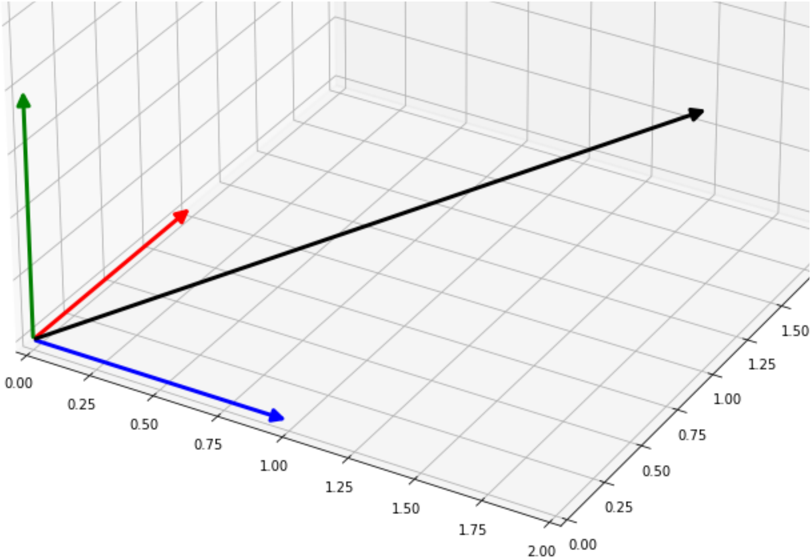
Example (Energy)

- Can measure energy differences.
- Pick an arbitrary zero energy for system.
- Pick an arbitrary ‘unit’ perhaps Joule.

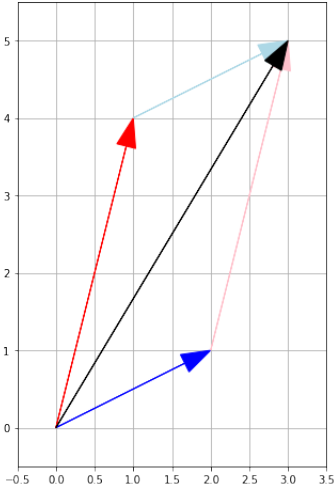
Picture in 2D



Picture in 3D

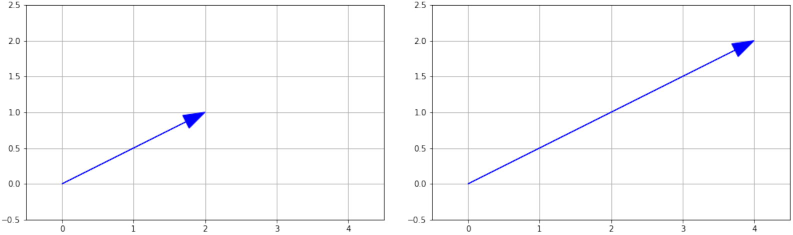


Picture of addition



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_4 \end{bmatrix}$$

Picture of scalar multiplication



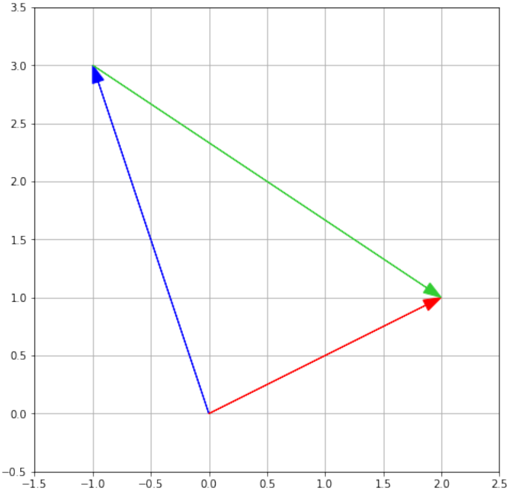
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Definition

Vectors A and B are *parallel* iff there is some scalar k such that

$$B = kA$$

Notation about base of vector

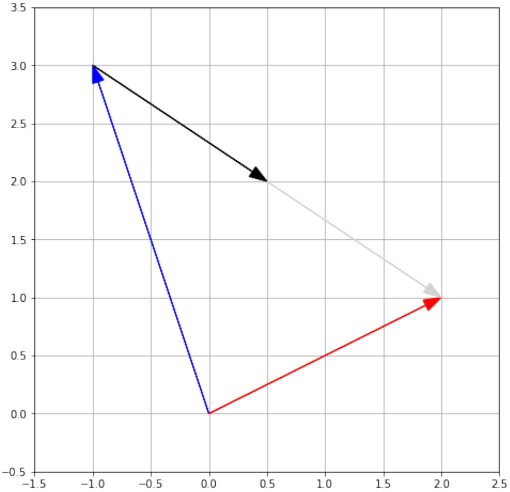


Definition

We write $\overrightarrow{AB}$ to denote the vector starting at A and ending at B. Therefore

$$\overrightarrow{AB} = B - A$$

Bisection



$$
\begin{aligned}
&\mathit{midpoint}(A, B) \\
&= \overrightarrow{0A} + \frac{1}{2}\overrightarrow{AB} \\
&= (A - 0) + \frac{1}{2}(B - A) \\
&= \frac{1}{2}(A + B)
\end{aligned}
$$

Questions?

Examples

Example

Find the point that is midway between

$$\begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$$

Example

Find the two points trisecting the segment between

$$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 8 \\ -6 \\ 2 \end{bmatrix}$$

Examples

Example

Let

$$P = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}, X = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

Is $\overrightarrow{PQ}$ parallel to $\overrightarrow{XY}$? Is $\overrightarrow{PX}$ parallel to $\overrightarrow{QY}$?

Example

The diagonals of a parallelogram bisect each other.

Example

If $ABCD$ is an arbitrary quadrilateral then the midpoints of the four sides form a parallelogram.

Equations for lines and planes

Two dimensions

Given a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

the solution set could be:-

- empty
- a point (a unique solution)
- a line (infinitely many solutions)
- all of $\mathbb{R}^2$ (infinitely many solutions)

. . . and we have to do row reduction to find out which.

Two dimensions

The non-trivial situations are:-

- a unique solution
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$
- a single parameter (say s)
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + s \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Definition

A *parametric (vector) equation* describing a line in $\mathbb{R}^2$ is a vector equation of the form
$$\vec{x} = \vec{c} + s\vec{d}$$

Three dimensions

Given a system of equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3\end{aligned}$$

the solution set could be:-

- empty
- a point (a unique solution)
- a line (infinitely many solutions)
- a plane (infinitely many solutions)
- all of $\mathbb{R}^3$ (infinitely many solutions)

. . . and we have to do row reduction to find out which.

Three dimensions

The non-trivial situations are:-

- a unique solution
- a single parameter (say s)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + s \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- two parameters (say s and t)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + s \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + t \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Parametric forms of 3D lines and planes

Definition

A *parametric (vector) equation* describing a line in $\mathbb{R}^3$ is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d}$$

for $\vec{x}$, $\vec{c}$ and $\vec{d}$ in $\mathbb{R}^3$ and s in $\mathbb{R}$.

Definition

A *parametric (vector) equation* describing a plane in $\mathbb{R}^3$ is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d} + t\vec{e}$$

for $\vec{x}$, $\vec{c}$, $\vec{d}$ and $\vec{e}$ in $\mathbb{R}^3$ and s, t in $\mathbb{R}$.

Properties of parametric form

- Can immediately see what kind of geometric object it is.
- Often easier to write down.
- The parametric form of a line/plane is **not** unique.

The symmetric form of a line in 3D

Sometimes a 'half-way' form of the equation of a line in $\mathbb{R}^3$ is used.

Definition (Symmetric form of equation for line)

$$\frac{x_1 + b_1}{a_1} = \frac{x_2 + b_2}{a_2} = \frac{x_3 + b_3}{a_3}$$

where $a_i \neq 0$.

We can convert this to parametric form immediately by setting all three terms equation to a parameter s:

$$\begin{aligned} x_1 + b_1 &= a_1 s \\ x_2 + b_2 &= a_2 s \\ x_3 + b_3 &= a_3 s \end{aligned}$$

Questions?

Examples

Example

Convert the following line into a parametric form

$$\frac{x-2}{3} = \frac{y-1}{2} = z+3$$

Example

Write down a parametric equation for the line passing through the points

$$\begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$$

Example

Example

Write down a parametric equation for the line passing through

$$\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$$

and parallel to the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Example

Example

For the following two lines find the point of intersection.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Example

Write down a system of equations whose solution set is the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Distance and scalar product

Length of a vector

Definition (Two dimensions)

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$

Definition (Three dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Definition (n dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Distance between two points

Definition (Two dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Definition (Three dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

N dimensions and scalar multiplication

Definition (n dimensions)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Lemma (Scalar multiplication)

If $\vec{v}$ is a non-zero vector and a is a scalar then

$$||a\vec{v}|| = |a|||\vec{v}||$$

Examples

Definition

A *unit vector* is a vector with length equal to 1.

Example

The following vectors are unit vectors:-

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{2}} \end{bmatrix}$$

Examples

Example

Find the distance between

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Example

Find the length of

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Dot product

Definition

If $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$ then

$$\begin{aligned}\vec{x} \cdot \vec{y} &= \sum_{i=1}^{n} x_i y_i \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\end{aligned}$$

And so in fact

$$\|\vec{x}\| = \sqrt{\vec{x}\vec{x}}$$

Examples

Example

If

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

then find $u \cdot v$.

Example

Find the length of

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$$

Questions?