

# MATH211: Linear Methods I

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Tuesday 2<sup>nd</sup> October, 2018

## Lecture on Tuesday 2<sup>nd</sup> October, 2018

First row expansion

Arbitrary expansion

Inversion formula

## Last time

- ▶ Determinants as area/volume etc..
- ▶ Computing determinants using row operations.
- ▶ The cofactor expansion (see again this time).

## First row expansion

# Two dimensional determinant

Recall that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

$$\begin{vmatrix} 10 & 1 \\ 9 & 5 \end{vmatrix} = 50 - 9 = 41$$

Example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

# Three dimensional determinants

## Definition

The *cofactor expansion along row 1* is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

## Example

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{vmatrix} &= 1 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\ &= 1 \cdot (24 - 2) - 1 \cdot (12 - 5) + 3 \cdot (4 - 20) \\ &= 22 - 7 - 48 = -33 \end{aligned}$$

# Examples

## Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

## Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

## Expansions in general

### Definition (Non-standard)

If  $A$  is a matrix write  $A(i|j)$  for the matrix obtained from  $A$  by deleting the  $i$ th row and the  $j$ th column.

### Definition

The *cofactor expansion of  $A$  along the first row* is

$$|A| = \sum_{k=1}^n a_{1k} (-1)^{1+k} |A(1|k)|$$

which when expanded is:

$$|A| = a_{11} |A(1|1)| - a_{12} |A(1|2)| + a_{13} |A(1|3)| - \cdots + (-1)^{1+n} a_{1n} |A(1|n)|$$



## Auxiliary definitions

### Definition (Minor)

$$\text{minor}_{ij}(A) = |A(i|j)|$$

### Definition (Cofactor)

$$\begin{aligned}\text{cof}_{ij}(A) &= (-1)^{i+j} \text{minor}_{ij}(A) \\ &= (-1)^{i+j} |A(i|j)|\end{aligned}$$

### Lemma (Expansion along row 1)

$$|A| = \sum_{k=1}^n a_{1k} \text{cof}_{1k}(A)$$

# Examples

## Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

## Example

Find

$$\begin{vmatrix} 2 & 5 & 4 & 3 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

Last time

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First row expansion

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Arbitrary expansion

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Inversion formula

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Questions?

Arbitrary expansion

## General cofactor expansion

In fact we may use the cofactors of arbitrary rows and columns:

Lemma (Expansion along arbitrary row or column)

$$\begin{aligned}|A| &= \sum_{k=1}^n a_{ik} (-1)^{i+k} |A(i|k)| \\ &= \sum_{k=1}^n a_{kj} (-1)^{k+j} |A(k|j)|\end{aligned}$$

So we can choose the row or column which is most convenient.

# Examples

## Example

Find

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

by expanding along row 3 or column 1.

## Example

Find

$$\begin{vmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$$

by expanding along column 2.

# Example

## Example

Find

$$\begin{vmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{vmatrix}$$

by choosing the correct column...

Last time

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First row expansion

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Arbitrary expansion

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Inversion formula

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Questions?



Inversion formula

## Duplicating rows

Suppose that an  $n \times n$  matrix has  $i$ th row equal to  $j$ th row:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then we know that  $|A| = 0$ . (Why?)

# Adjugate matrix

If we expand

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

along the  $j$ th row then we get:

$$\begin{aligned} 0 &= |A| = \sum_{k=1}^n a_{jk} (-1)^{j+k} |A(j|k)| \\ &= \sum_{k=1}^n a_{ik} (-1)^{j+k} |A(j|k)| \end{aligned}$$

# The adjugate matrix

Now the last term looks like a matrix product:

$$\sum_{k=1}^n a_{ik} (-1)^{j+k} |A(j|k)| = (A(\text{adj}(A)))_{ij}$$

Definition (Adjugate matrix)

$$\begin{aligned} \text{adj}(A)_{ij} &= (-1)^{i+j} |A(j|i)| \\ &= \text{cof}_{ji}(A) \end{aligned}$$

So  $\text{adj}(A)$  is the transpose of the matrix of cofactors  $\text{cof}_{ij}(A)$ .

# A theorem

## Theorem

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$$

## Proof.

*Just compute the  $ij$ -entries:*

$$\begin{aligned}(A \cdot \text{adj}(A))_{ij} &= \sum_{k=1}^n a_{ik} \text{adj}(A)_{kj} \\ &= \sum_{k=1}^n a_{ik} (-1)^{j+k} |A(j|k)|\end{aligned}$$



## A theorem continued...

### Theorem

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$$

### Proof.

...so on the diagonal (when  $i = j$ ) the entry is

$$\sum_{k=1}^n a_{ik}(-1)^{i+k} |A(i|k)| = \det(A)$$

and off the diagonal

$$\sum_{k=1}^n a_{ik}(-1)^{j+k} |A(j|k)| = 0$$

as required.



# The inversion formula

So we know that

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$$

and therefore if  $\det(A) \neq 0$  then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

**Warning:** for  $n > 2$  this is a very inefficient way to compute an inverse.

# Examples

## Example

Compute the inverse of the following matrices:-

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$$



Questions?