MATH211: Linear Methods I

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Lecture on Thursday 22nd November, 2018

Last time

- Converting to polar form.
- Converting from polar form.
- Multiplication using polar form.

Examples

Examples

Example

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

Example

Express $(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{17}$ in the form $a+bi$.

Complex roots

Principal square roots

Now we know that:

Slogan: To multiply two complex numbers we multiply the moduli and add the angles.

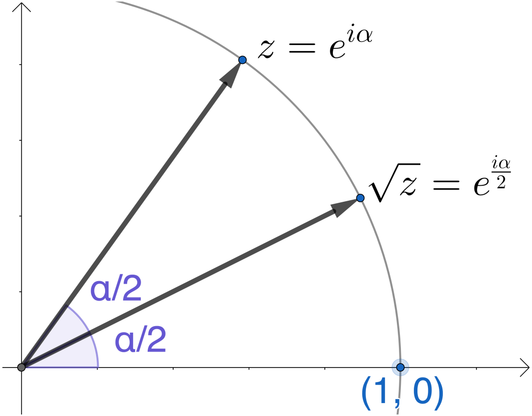
So we could try reversing this:

Slogan: To find the principal square root we take the square root of the modulus and halve the angle.

- (The answer will depend on the angle measuring convention.)

Picture

If $z = e^{i\alpha}$ has unit modulus:



Non-principal square roots

Let $z = R \cdot e^{i\alpha}$ and take the principal square root:

$$\sqrt{z} = \sqrt{R \cdot e^{i\alpha}} = \sqrt{R} \cdot e^{\frac{i\alpha}{2}}$$

Then all of numbers of the form

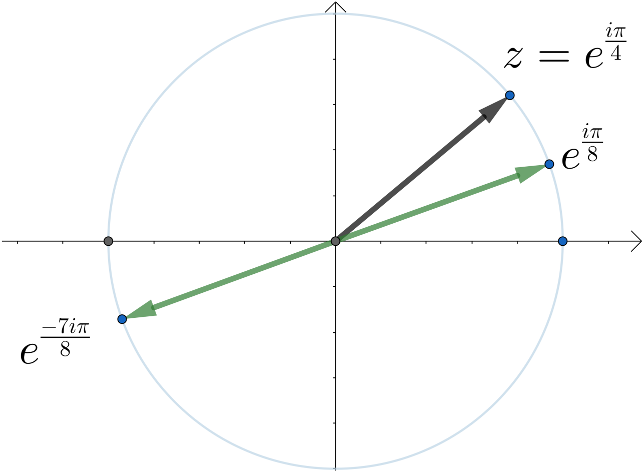
$$\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + ki \cdot \pi\right)}$$

for $k \in \mathbb{Z}$ are *also* a square roots of z:

$$\left(\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + ki\pi\right)}\right)^2 = R \cdot e^{i\alpha + 2ki\pi} = R \cdot e^{i\alpha} = z$$

Example

The two square roots of $z = e^{\frac{i\pi}{4}}$ are $e^{\frac{i\pi}{8}}$ and $e^{\frac{-7i\pi}{8}}$:



N-th roots

Similarly there is a principal n-th root:

$$\sqrt[n]{z} = \sqrt[n]{R \cdot e^{i\alpha}} = \sqrt[n]{R} \cdot e^{\frac{i\alpha}{n}}$$

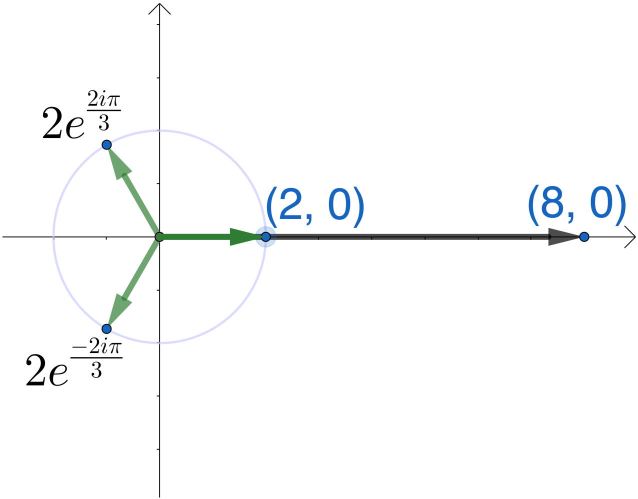
and all numbers of the form

$$\sqrt[n]{R} e^{\frac{i\alpha + 2ki\pi}{n}}$$

are also n-th roots of z for $k \in \mathbb{Z}$.

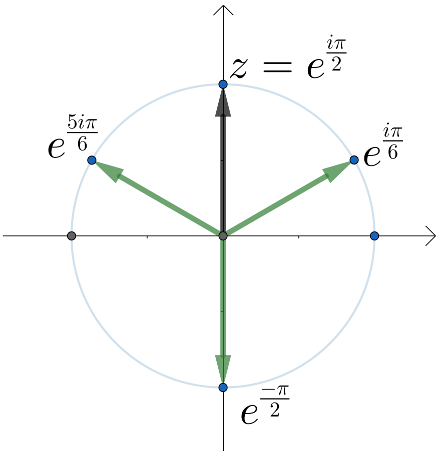
Cube roots

The cube roots of 8 are 2, $2e^{\frac{2i\pi}{3}}$ and $2e^{\frac{-2i\pi}{3}}$:



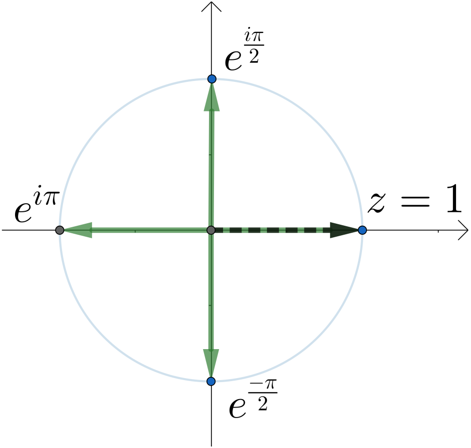
Cube roots

The cube roots of i are $e^{\frac{i\pi}{6}}$, $-i$ and $e^{\frac{5i\pi}{6}}$:



4-th roots

The fourth roots of unity are: 1, i, -1 and $-i$:



Questions?

Questions?

Examples

Example

Find all solutions to the following equations:

- $z^2 = 25$.
- Find all solutions to $z^2 = -1$
- Find all solutions to $z^2 = e^{\frac{i\pi}{4}}$
- Find all solutions to $z^3 = -1$.
- Find all solutions to $z^4 = 2(\sqrt{3}i - 1)$.

Example

Find all sixth roots of unity.

Quadratic formula

Quadratic formula

If a, b and c are *complex numbers* then the equation

$$az^2 + bz + c = 0$$

has two solutions given by

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- We assume $a \neq 0$.
- If we use complex numbers we always get two solutions.

Galois theory for quadratics

If u and v are the roots of $f(x) = z^2 + bz + c$ then

$$(z-u)(z-v) = z^2 - (u+v)z + uv$$

and so

$$\begin{aligned} b &= -u - v \\ c &= uv \end{aligned}$$

- If b and c are real then either
  - u and v are real or
  - $u = \overline{v}$.

Examples

Example

Solve $z^2 - 14z + 58 = 0$.

Example

Find a real quadratic with $5 - 2i$ as a root. What is the other root?

Example

Find a real quadratic with $-3 + 4i$ as a root. What is the other root?

Example

Find the roots of $z^2 - 3iz + (-3 + i) = 0$.

Example

Verify that $4 - i$ is a root of $z^2 - (2 - 3i)z - (10 + 6i) = 0$. Find the other root.

Example

Solve $z^2 - (3 - 2i)z + (5 - i) = 0$.

Eigenvectors

Simplifying matrix actions

If

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

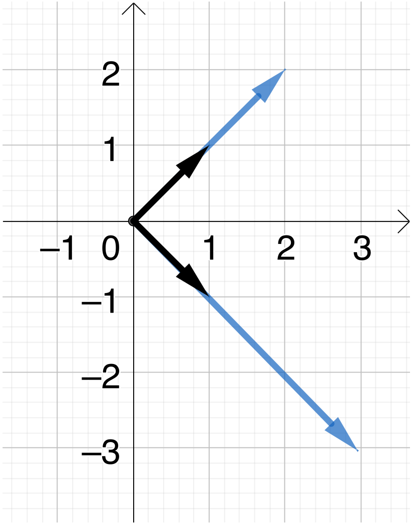
then A acts on some vectors in a simplified way:

$$Av_1 = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2v_1$$

and

$$Av_2 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} 4v_2$$

Picture



Since

$$Av_1 = 2v_1$$

and

$$Av_2 = 4v_2$$

simplifying matrix actions

If

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

then

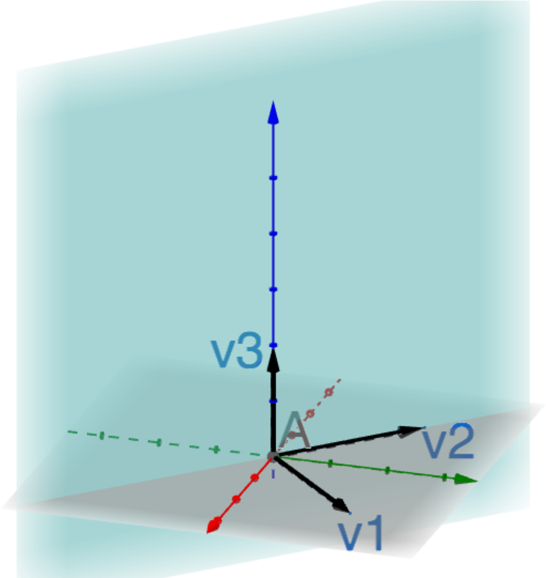
$$Av_1 = A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$$

$$Av_2 = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_3$$

and

$$Av_3 = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -v_2$$

Picture



Blue plane is preserved.

Eigenvectors and eigenvalues

Definition

The matrix A has an *eigenvector* v *with eigenvalue* λ iff $v \neq 0$ and

$$Av = \lambda v$$

where λ could be any complex number.

Examples

Example

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with eigenvalues 3 and 5 respectively.

Example

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 2 and 4 respectively.

Finding eigenvalues

Since

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

and $v \neq 0$ we need to find all λ such that

$$A - \lambda I$$

is *not* invertible.

Definition (Characteristic polynomial)

The *characteristic polynomial of a matrix* A is

$$\chi_A(\lambda) = det(A - \lambda I)$$

So to find the eigenvalues we find the roots of the characteristic polynomial.

Examples

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$