MATH211: Linear Methods I

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Lecture on Tuesday 20^{th} November, 2018

Summary of previous work

- $i^2 = -1$
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- graphical interpretation

$$a + bi \leftrightarrow (a, b)$$
$$\mathbb{C} \leftrightarrow \mathbb{R}^2$$

- polar form

$$a + ib \leftrightarrow Re^{i\theta}$$

  where R is the *modulus* and θ the *angle*.

Examples

Example

Convert the following to polar form:-

- $-2+2\sqrt{3}i$
- $3i$
- $-1-i$
- $\sqrt{3}+3i$

Example

Convert the following to standard form:-

- $2e^{\frac{2\pi i}{3}}$
- $3e^{-i\pi}$
- $2e^{\frac{3i\pi}{4}}$

Multiplication

Multiplication using polar form

If $z = Re^{i\theta}$ and $w = Qe^{i\phi}$ then

$$\begin{aligned} zw &= Re^{i\theta}Qe^{i\phi} \\ &= (RQ)e^{i(\theta+\phi)} \end{aligned}$$

Slogan: To multiply two complex numbers we multiply the moduli and add the angles.

Theorem (De Moivre Theorem)

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Questions?

Questions?

Examples

Example

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

Example

Express $(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{17}$ in the form $a+bi$.

Complex roots

Principal square roots

Now we know that:

Slogan: To multiply two complex numbers we multiply the moduli and add the angles.

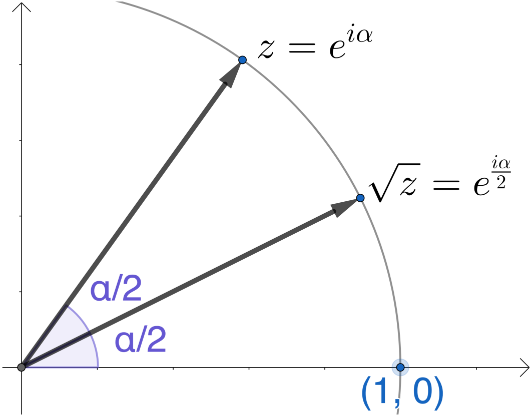
So we could try reversing this:

Slogan: To find the principal square root we take the square root of the modulus and halve the angle.

- (The answer will depend on the angle measuring convention.)

Picture

If $z = e^{i\alpha}$ has unit modulus:



Non-principal square roots

Let $z = R \cdot e^{i\alpha}$ and take the principal square root:

$$\sqrt{z} = \sqrt{R \cdot e^{i\alpha}} = \sqrt{R} \cdot e^{\frac{i\alpha}{2}}$$

Then all of numbers of the form

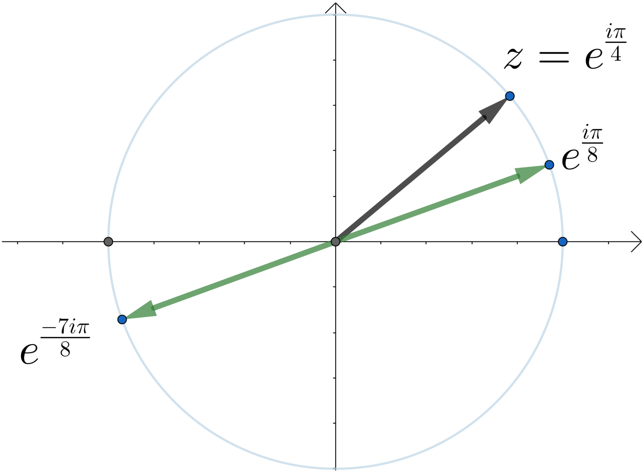
$$\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + ki \cdot \pi\right)}$$

for $k \in \mathbb{Z}$ are *also* a square roots of z:

$$\left(\sqrt{R} \cdot e^{\left(\frac{i\alpha}{2} + ki\pi\right)}\right)^2 = R \cdot e^{i\alpha + 2ki\pi} = R \cdot e^{i\alpha} = z$$

Example

The two square roots of $z = e^{\frac{i\pi}{4}}$ are $e^{\frac{i\pi}{8}}$ and $e^{\frac{-7i\pi}{8}}$:



N-th roots

Similarly there is a principal n-th root:

$$\sqrt[n]{z} = \sqrt[n]{R \cdot e^{i\alpha}} = \sqrt[n]{R} \cdot e^{\frac{i\alpha}{n}}$$

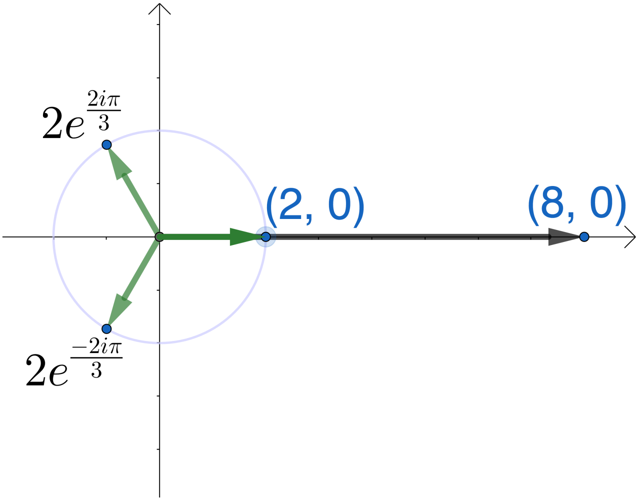
and all numbers of the form

$$\sqrt[n]{R} e^{\frac{i\alpha + 2ki\pi}{n}}$$

are also n-th roots of z for $k \in \mathbb{N}$.

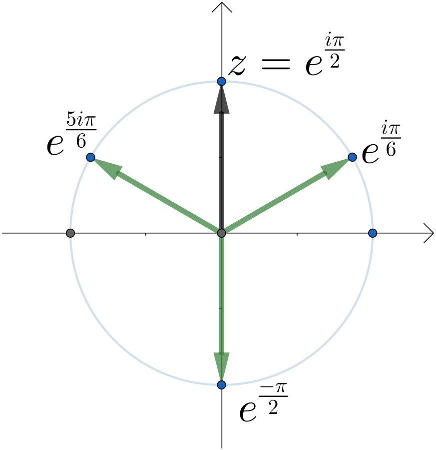
Cube roots

The cube roots of 8 are 2, $2e^{\frac{2i\pi}{3}}$ and $2e^{\frac{-2i\pi}{3}}$:



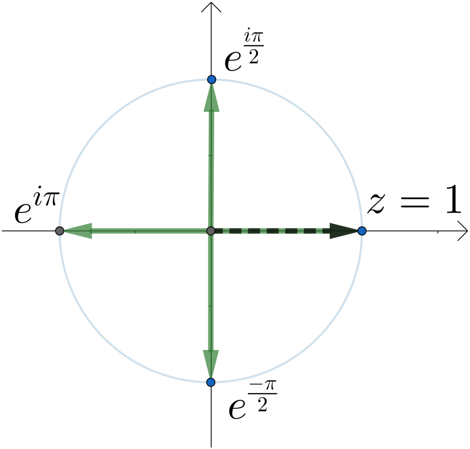
Cube roots

The cube roots of i are $e^{\frac{i\pi}{6}}$, $-i$ and $e^{\frac{5i\pi}{6}}$:



4-th roots

The fourth roots of unity are: 1, i, -1 and $-i$:



Questions?

Questions?

Examples

Example

Find all solutions to the following equations:

- $z^2 = 25$.
- Find all solutions to $z^2 = -1$
- Find all solutions to $z^2 = e^{\frac{i\pi}{4}}$
- Find all solutions to $z^3 = -1$.
- Find all solutions to $z^4 = 2(\sqrt{3}i - 1)$.

Example

Find all sixth roots of unity.

Quadratic formula

Quadratic formula

If a, b and c are *complex numbers* then the equation

$$az^2 + bz + c = 0$$

has two solutions given by

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- We assume $a \neq 0$.
- The formula is unchanged a, b and c real numbers.
- We always get a solution.

Galois theory for quadratics

If u and v are the roots of $f(x) = z^2 + bz + c$ then

$$(z - u)(z - v) = z^2 - (u + v)z + uv$$

and so

$$\begin{aligned} b &= -u - v \\ c &= uv \end{aligned}$$

- If f is a *real* quadratic we need both $u + v$ and uv to be real.
- This implies that $u = \overline{v}$.

Examples

Example

Solve $z^2 - 14z + 58 = 0$.

Example

Find a real quadratic with $5 - 2i$ as a root. What is the other root?

Example

Find a real quadratic with $-3 + 4i$ as a root. What is the other root?

Example

Find the roots of $z^2 - 3iz + (-3 + i) = 0$.

Example

Verify that $4 - i$ is a root of $z^2 - (2 - 3i) - (10 + 6i) = 0$. Find the other root.

Example

Solve $z^2 - (3 - 2i)z + (5 - i) = 0$.