MATH211: Linear Methods I

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Lecture on Thursday 4th October, 2018

Last time

- Cofactor expansion along first row
- Cofactor expansion along arbitrary row or column
- Adjugate matrix

Adjugates and inverses

Adjugate matrix

Definition (Matrix of cofactors)

$$
\begin{aligned}
cof(A)_{ij} &= cof_{ij}(A) \\
&= (-1)^{i+j} minor_{ij}(A) \\
&= (-1)^{i+j} \left| A(i|j) \right|
\end{aligned}
$$

Definition (Adjugate matrix)

$$
\begin{aligned}
adj(A)_{ij} &= \left(cof(A)_{ij}\right)^T \\
&= cof_{ji}(A) \\
&= (-1)^{j+i} \left| A(j,i) \right|
\end{aligned}
$$

Theorems

Theorem (Adjugate is almost inverse)

$$A \cdot adj(A) = adj(A) \cdot A = det(A) I$$

So if $det(A)$ is not 0 then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

This is not an efficient method to calculate the inverse.

Example

Example

Find the inverse for

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$$

using the adjugate matrix formula.

Questions?

Cramer's rule

Using the inverse to solve a system

Recall that if A is invertible then

$$Ax = b \implies x = A^{-1}b$$

and we can use the formula above to calculate $A^{-1}b\ldots$

Using adjugate to solve system

$$
\begin{aligned}
x_i &= (A^{-1}b)_{ij} \\
&= \sum_{k=1}^{n} (A^{-1})_{ik} b_k \\
&= \sum_{k=1}^{n} \frac{1}{det(A)} adj(A)_{ik} b_k \\
&= \frac{1}{det(A)} \sum_{k=1}^{n} b_k (-1)^{i+k} \left| A(k,i) \right| \\
&= \frac{1}{det(A)} det(A_i)
\end{aligned}
$$

where A_i results from replacing the ith column of A with b.

Cramer's rule

If A is invertible then the solutions to

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

are

$$x_i = \frac{det(A_i)}{det(A)}$$

This method of solving systems is usually computationally inefficient.

Example

Example

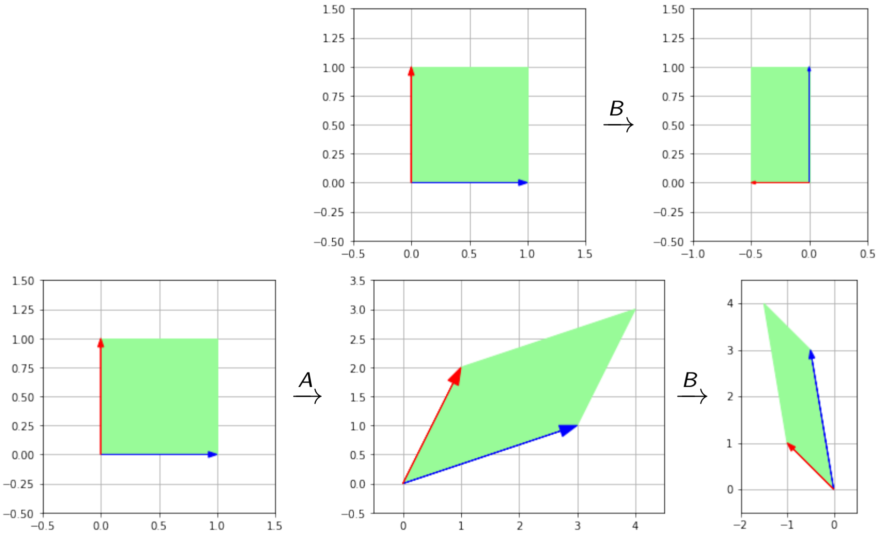
Find x_2 such that

$$\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Questions?

Common determinants

Determinant of product (picture)



Determinants and elementary matrices

Recall:-

- Multiplying a row/column by a scalar k
  - multiplies the determinant by k.
- Swapping two rows/columns
  - multiplies the determinant by -1.
- Adding a multiple of one row/column to another
  - has no effect on the determinant.

Furthermore each operation can be carried out by multiplying on the left by the corresponding elementary matrix:

$$A \mapsto EA$$

Determinants and elementary matrices

Therefore:

- if E multiplies a row/column by a scalar k
  - $det(EA) = k \cdot det(A)$
  - $det(E) = k$
- if E swaps two rows/columns
  - $det(EA) = -det(A)$
  - $det(E) = -1$
- if E adds a multiple of one row/column to another
  - $det(EA) = det(A)$
  - $det(E) = 1$

Determinant of product (algebra)

Lemma (Determinant of product of elementary matrices)

If E and F are elementary matrices then

$$det(EF) = det(E)det(F)$$

Theorem (Determinant of product)

$$det(BA) = det(B)det(A)$$

Proof.

Non-essential exercise. (Divide into cases based on whether A and B are invertible.) $\square$

Determinant of transpose

Theorem (Determinant of transpose)

$$det(A^T) = det(A)$$

Proof.

The row operations that take A to triangular form have the same effect on the determinant as the column operations that take A^T to triangular form. $\square$

Determinant of inverse and adjugate

Theorem (Determinant of inverse)

If A is invertible then

$$det(A^{-1}) = \frac{1}{det(A)}$$

Proof.

Check for elementary matrices. Then express A in terms of elementary matrices. $\square$

Theorem (Determinant of adjugate)

For any square matrix A

$$det(adj(A)) = (det(A))^{n-1}$$

Examples

Example

Find

$$\left| \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right|$$

Example

Find

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 8 \end{bmatrix} \right|$$

Examples

Example

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3 B(A^{-1}))$.

Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Questions?

Polynomial interpolation

Example

Example

Given data points $(0,1)$, $(1,2)$, $(2,5)$ and $(3,10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

General theory

If $x_1, x_2, \ldots, x_n$ are all distinct then

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & x_n^3 & \ldots & x^n \end{vmatrix}$$

is non-zero.

Questions?