Math 211 2018-10-30:

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\y\\-x+2y\end{pmatrix}.$$ Last time: checked additivity.

$$T\left(k\begin{pmatrix}a\\b\end{pmatrix}\right) = T\begin{pmatrix}ka\\kb\end{pmatrix} = \begin{pmatrix}2ka\\kb\\-ka+2kb\end{pmatrix} = k\begin{pmatrix}2a\\b\\-a+2b\end{pmatrix} = kT\begin{pmatrix}a\\b\end{pmatrix}.$$

So T is in fact linear.

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}xy\\x+y\end{pmatrix}$$

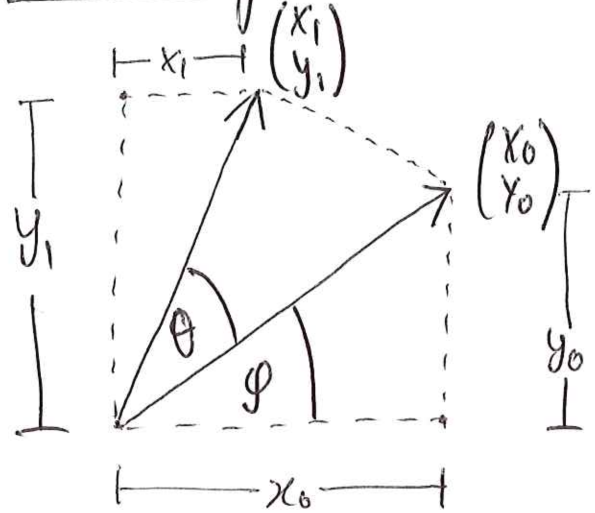
$$T\left(k\begin{pmatrix}a\\b\end{pmatrix}\right) = T\begin{pmatrix}ka\\kb\end{pmatrix} = \begin{pmatrix}kakb\\ka+kb\end{pmatrix} = k\begin{pmatrix}kab\\a+b\end{pmatrix} \neq kT\begin{pmatrix}a\\b\end{pmatrix}.$$

So this T isn't linear.

Matrix rot. by θ.

If A is a matrix then $A(x+y) = Ax+Ay$ and $A(kx) = kAx$.

Rotation by θ is linear



Let $R = \sqrt{x_0^2 + y_0^2}$

So $x_0 = R\cos\varphi$

$y_0 = R\sin\varphi.$

$x_1 = R\cos(\theta+\varphi)$

$y_1 = R\sin(\theta+\varphi)$

So $x_1 = R\cos(\theta+\varphi) = R\left[\cos\theta\cos\varphi - \sin\theta\sin\varphi\right]$

$= x_0\cos\theta - y_0\sin\theta$

$y_1 = R\sin(\theta+\varphi) = R\left[\sin\theta\cos\varphi + \sin\varphi\cos\theta\right].$

$= x_0\sin\theta + y_0\cos\theta.$

$$T\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0\cos\theta - y_0\sin\theta \\ x_0\sin\theta + y_0\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

So rotation by θ is linear.

Find $T\begin{pmatrix}-7\\3\\-9\end{pmatrix}$ given that T is linear,

$$T\begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}4\\4\\0\\-2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}4\\0\\5\end{pmatrix} = \begin{pmatrix}4\\5\\-1\\5\end{pmatrix}$$

Recall $T(a\vec{x}+b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$.

$$\underbrace{T(a\vec{x}+b\vec{y})}_{=\,T(a\vec{x})+T(b\vec{y})} = \underbrace{aT(\vec{x})+bT(\vec{y})}_{=\,T(a\vec{x})+T(b\vec{y})}$$

So need to find $a,b\in\mathbb{R}$ such that

$$a\begin{pmatrix}1\\3\\1\end{pmatrix} + b\begin{pmatrix}4\\0\\5\end{pmatrix} = \begin{pmatrix}-7\\3\\-9\end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c}1&4&-7\\3&0&3\\1&5&-9\end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c}1&4&-7\\0&-12&24\\0&1&-2\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&4&-7\\0&1&-2\\0&1&-2\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&4&-7\\0&1&-2\\0&0&0\end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c}1&0&1\\0&1&-2\\0&0&0\end{array}\right) \Rightarrow \quad a=1 \text{ and } b=-2.$$

$$\begin{pmatrix}1\\3\\1\end{pmatrix}-2\begin{pmatrix}4\\0\\5\end{pmatrix}=\begin{pmatrix}-7\\3\\-9\end{pmatrix} \Rightarrow T\begin{pmatrix}-7\\3\\-9\end{pmatrix}=T\left(\begin{pmatrix}1\\3\\1\end{pmatrix}-2\begin{pmatrix}4\\0\\5\end{pmatrix}\right)$$

$$=T\begin{pmatrix}1\\3\\1\end{pmatrix}-2T\begin{pmatrix}4\\0\\5\end{pmatrix}$$

$$=\begin{pmatrix}4\\4\\0\\-2\end{pmatrix}-2\begin{pmatrix}4\\5\\-1\\5\end{pmatrix}=\begin{pmatrix}-4\\-6\\2\\-12\end{pmatrix}.$$

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① $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y\\x-y\end{pmatrix}$, ② $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$ & $T\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

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① $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$, $[T]=\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}$.

② We need to find $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$. Use $T(ax+by)=aTx+bTy$

I.e. $a\begin{pmatrix}1\\1\end{pmatrix}+b\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$. $\Rightarrow a=1,\ b=1$

So $T\begin{pmatrix}1\\0\end{pmatrix}=T\left(\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}\right)=T\begin{pmatrix}1\\1\end{pmatrix}+T\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}.$

$$c\begin{pmatrix}1\\1\end{pmatrix}+d\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \quad c=0,\ d=-1$$

So $T\begin{pmatrix}0\\1\end{pmatrix}=T\left(-\begin{pmatrix}0\\-1\end{pmatrix}\right)=-T\begin{pmatrix}0\\-1\end{pmatrix}=-\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}-3\\-2\end{pmatrix}.$

$$[T]=\begin{pmatrix}4 & -3\\4 & -2\end{pmatrix}.$$