MATH211: Linear Methods I

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Similar matrices

Dynamical systems

Markov chains

PageRank

Last time

- Diagonalisation
- Matrix powers

Similar matrices

Similar matrices

Definition

Two matrices A and B are *similar* iff there exists an invertible P such that

$$A = PBP^{-1}$$

and we write $A \sim B$.

Example

Every diagonalisable matrix is similar to a diagonal matrix.

Trace of a matrix

Definition

The trace of a matrix is the sum of its diagonal elements:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Theorem

If $A \sim B$ then $tr(A) = tr(B)$.

Dynamical systems

Dynamical systems

Definition

A *dynamical system* consists of a function $\alpha(t)$ that prescribes how the state of the system changes over time.

Definition

A *discrete linear dynamical system* consists of a sequence of vectors

$$x_0, x_1, x_2, \ldots, x_k, \ldots$$

such that $x_{k+1} = Ax_k$ for some matrix A.

Long term behaviour using eigenvectors

If x_0 is a linear combination of the eigenvectors v_{λ_i} of A then

$$x_k = A^k x_0 = A^k \left(\sum_{i=1}^{n} b_i v_{\lambda_i} \right) = \sum_{i=1}^{n} b_i A^k \left(v_{\lambda_i} \right) = \sum_{i=1}^{n} b_i (\lambda_i)^k v_{\lambda_i}$$

and so the long-term behaviour is determined by the limits:

$$\lim_{k \to \infty} (\lambda_i)^k$$

Dominant eigenvalue

Definition

If A is a square matrix then a *dominant eigenvalue* λ_{max} is one for which $|\lambda_{max}| > |\lambda_i|$ for all other eigenvalues λ_i.

If this exists then

$$x_k = \sum_{i=1}^{n} b_i(\lambda_i)^k v_{\lambda_i} \approx b_i(\lambda_{max})^k v_{\lambda_{max}}$$

and we can read off the long term behaviour. E.g.

- if $|\lambda_{max}| < 1$ then the system converges to 0
- if $\lambda_{max} = 1$ then the system converges to $b_i v_{\lambda_{max}}$
- if $\lambda_{max} = -1$ then the system oscillates between $\pm b_i v_{\lambda_{max}}$
- if $|\lambda_{max}| > 1$ then the system diverges

Examples

Example

Find a formula for x_k if $x_{k+1} = Ax_k$,

$$x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$$

Example

Estimate the long term behaviour of the dynamical system with

$$x_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix} \text{ and } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$$

Markov chains

Markov chains

A *Markov chain* consists of:-

- a finite set of *states* $x_1, x_2, \ldots, x_n$
- a repeated *transition interval* at the end of which the system transitions between states
- a *not necessarily deterministic rule* for predicting the probability that the system will transition into a certain state
  - **This probability only depends on the current state.**
  - (Not the entire history of the chain.)

Markov transition matrices

This means that a Markov chain is described by a *transition matrix* A such that

$$
\begin{aligned}
A_{ij} &= \mathbb{P}(X_1 = i | X_0 = j) \\
&= \text{the probability that the next state will be } i \\
&\quad\ \text{given that the current state is } j
\end{aligned}
$$

Therefore:-

- all of the entries are between 0 and 1
  - (I.e. they are probabilities.)
- in any column the sum of the entries is 1
  - (The system must be in one of the states at all times.)

Steady state

Definition

A *steady state vector for a matrix A* is an eigenvector with eigenvalue 1.

(In other words it is a *non-zero* vector v such that $Av = v$.)

Definition

A *steady state vector for a Markov chain* is a steady state vector for the transition matrix of the Markov chain such that the entries sum to 1.

Regular Markov chains

Definition

A matrix A is *regular* iff there exists a positive integer k such that all of the entries of A^k are *strictly positive*.

Theorem

If the transition matrix of a Markov chain is regular then it has a steady state vector.

Example

The converse is false! E.g. the following matrix has two steady state vectors:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Example

Calculate and interpret x_3 given that

$$x_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$$

Example

What is the probability that we are in state 1 after two iterations if

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Examples

Example

Find a steady state vector for

$$\begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$$

Example

Find a steady state vector for

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

PageRank

Simplified Google PageRank

The core part of Google PageRank (mainly due to Larry Page) is calculated recursively:-

- Suppose we have n pages: $P_1, P_2 \ldots P_n$
- denote the PageRank of P_i by $PR(i)$
- then (recursively) define

$$PR(i) = \sum_{j=1}^{n} \frac{PR(j) \cdot L_{ji}}{Out(j)}$$

where L_{ji} is the number of links from P_j to P_i and $Out(j)$ is the total number of links on page j.

At first glance it is not even clear that this is well-defined...

Steady state vector

To work out the PageRank for each site:-

- Form the matrix with entries $A_{ij} = \frac{L_{ji}}{Out(j)}$.
- Find a steady state vector for this system.

PageRank Example

Example (Simplified PageRank)

Suppose that the internet consists of three(!) pages P_1, P_2 and P_3. Suppose further that:-

- there are two links from P_1 to P_2
- there is one link from P_1 to P_3
- there is one link from P_2 to P_1
- there is one link from P_2 to P_3
- there is one link from P_3 to P_2

Then calculate the (relative) PageRanks for P_1, P_2 and P_3.

Example

Example

If possible diagonalise

$$\begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$$