MATH211: Linear Methods I

Matthew Burke

Tuesday 18$^{\text{th}}$ September, 2018

Lecture on Tuesday 18$^{\text{th}}$ September, 2018

Non-commutativity

(i,j)-entry proofs

Identity and inverse matrices

Inversion examples

Last time

- Matrix addition and scalar multiplication
- Relationship to systems of equations
- Matrix composition (product)
- Non-commutativity of matrix composition
- (Two sets of notes for 11th September)

Non-commutativity

Non-commutativity

1. AB exists but BA does not.
2. Both AB and BA exist but are of different sizes.
3. Both AB and BA exist and are the same size, but they are different.
4. $AB = BA$ (it does happen sometimes!)

(i,j)-entry proofs

Entries of product matrix

Definition

If A and B are matrices such that

$$\text{No. of rows}(A) = m = \text{No. of columns}(B)$$

then

$$\begin{aligned}(BA)_{ij} &= \sum_{k=1}^{m} b_{ik}a_{kj} \\ &= b_{i1}a_{1j} + b_{i2}a_{2j} + b_{i3}a_{3j} + \cdots + b_{im}a_{mj}\end{aligned}$$

Picture

$$(BA)_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj} = b_{i1} a_{1j} + b_{i2} a_{2j} + b_{i3} a_{3j} + \cdots + b_{im} a_{mj}$$

is the dot product of the ith row of B with the jth column of A:

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{i1} & b_{i2} & \dots & b_{im} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nm} \end{bmatrix}$$

Examples of entries of matrix product

Example

Find the $(2,3)$-entry of

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

Find the $(3,1)$-entry of

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 1 \\ 5 & 4 & 9 \end{bmatrix} \begin{bmatrix} 12 & 1 & 2 \\ 4 & 0 & 4 \\ -2 & 0 & 0 \end{bmatrix}$$

General examples of (i,j)-entry

In the following examples assume that the matrices are of the appropriate sizes so that the products involved make sense.

Example

Find the (i,j)-entry of

$$(CB)A$$

Example

Find the (i,j)-entry of

$$C(A+B)$$

Questions?

Transpose of a matrix

Definition

If A is *any* matrix the *transpose* A^T *of* A has entries

$$(A^T)_{ij} = A_{ji}$$

Example

$$\begin{bmatrix} 5 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 0 \\ -3 & 2 \end{bmatrix}$$

In short: the columns of A^T are the rows of A.

Symmetric matrices

Definition

A matrix A is *symmetric* iff $A = A^T$.
In terms of entries: $A_{ij} = A_{ji}$.

Example

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 17 \\ -2 & 17 & 0 \end{bmatrix}$$

is symmetric but

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & -3 & 5 \\ -2 & 17 & 0 \end{bmatrix}$$

is not.

Examples of (i,j)-entry proofs

Example

Prove $C(B+A) = CB + CA$.

Example

Prove $(BA)^T = A^T B^T$.

Example

Prove $(B+A)^T = B^T + A^T$.

Example

Prove that if A and B are symmetric then $A^T + 2B$ is symmetric.

Example

Prove $C(BA) = (CB)A$. (Called matrix associativity.)

Identity and inverse matrices

Identity matrix

Definition

The $n \times n$ *identity matrix* is:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

In terms of entries:

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Product with identity

Whenever the following products make sense:

$$BI_m = B \qquad \text{and} \qquad I_m A = A$$

Indeed for the first:

$$\begin{aligned}(BI_m)_{ij} &= \sum_{k=1}^{m} B_{ik} I_{kj} \\ &= \sum_{k \neq j} B_{jk} I_{kj} + B_{ij} I_{jj} \\ &= \sum_{k \neq j} 0 + B_{ij} = B_{ij}\end{aligned}$$

Inverse matrix

Definition

If A is an $n \times n$ matrix then an *inverse matrix* A^{-1} *of* A is any matrix such that

$$AA^{-1} = I_n \qquad \text{and} \qquad A^{-1}A = I_n$$

Some remarks:

- It doesn't make sense to ask for an inverse of a non-square matrix. (Think in terms of linear transformations.)
- Not all square matrices have inverses.
- If A^{-1} is the inverse of A then A is the inverse of A^{-1}.

Examples of non-existence

Example

No matrix that only has zero entries has an inverse.

Example

The following matrix does not have an inverse

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In fact a $n \times n$ matrix has an inverse iff $rank(A) = n$.

Uniqueness of inverse

Lemma

If a matrix A has an inverse then it is unique.

Questions?

Motivation from linear combinations

Lets try to find an inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We want a matrix A such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Express the standard basis vectors in terms of the columns.

Examples

Example

Find the inverse for

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

Find the inverse for the general 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Examples

Example

Find the inverse for

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$$

Example

Find the inverse for

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$