MATH211: Linear Methods I

Matthew Burke

Wednesday 31st October, 2018

Lecture on Wednesday 31^{st} October, 2018

Last time

- Linear transformations
- Action on compositions
- Matrix of a linear transformation

Matrices

Matrices and linear transformations

Example

If A is an $m \times n$ matrix then the function $x \mapsto Ax$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then the *matrix* $[T]$ *of* T has columns given by the images of the standard basis vectors:

$$[T] = \left[\begin{array}{cccc} T(e_1) & T(e_2) & \dots & T(e_n) \end{array} \right]$$

Examples

Example

What is the matrix of the linear transformation for counter-clockwise rotation about the origin by an angle of θ?

Example

What is the matrix of the linear transformation for reflection in the line through that origin at an angle θ from the x-axis?

Example

What is the matrix of the linear transformation that is $v \times (-)$ for some vector $v \in \mathbb{R}^3$?

Composition

Composition of functions

Definition

If we have functions

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \text{ and } \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

then the *composite function* $g \circ f$ is the function defined by

$$(g \circ f)(x) = g(f(x))$$

- I.e. first apply f to x and then apply g to the result.
- Note that $\mathbb{R}^n \xrightarrow{g \circ f} \mathbb{R}^p$.

Composite of linear transformations

Example

The composite of two linear transformations is again linear.

Proof.

Suppose that $S : \mathbb{R}^m \to \mathbb{R}^p$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations. Then

$$\begin{aligned}(S \circ T)(x + y) &= S(T(x + y)) = S(T(x) + T(y)) \\ &= S(T(x)) + S(T(y)) = (S \circ T)(x) + (S \circ T)(y)\end{aligned}$$

and

$$\begin{aligned}(S \circ T)(kx) &= S(T(kx)) = S(kT(x)) \\ &= kS(T(x)) = k(S \circ T)(x)\end{aligned}$$

$\square$

Matrix of composite transformation

We know that the matrix of the composite $T \circ S$ is:

$$\begin{bmatrix} T(S(e_1)) & T(S(e_2)) & \ldots & T(S(e_n)) \end{bmatrix}$$

where $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^p$.

Theorem

The matrix of a composite is the composite of the matrices:

$$[T \circ S] = [T][S]$$

Proof.

Note that the terms $S(e_i)$ are the columns of $[S]$. □

Determinant of rotations and reflections

Example (Determinant of rotation)

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

Example (Determinant of reflection)

$$\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -\sin^2\theta - \cos^2\theta = -1$$

Composites of rotations and reflections

In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

  where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$

Examples

Example

What is the matrix of the linear transformation that is the composite of reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$?

Example

What is the matrix of the linear transformation that is the composite of reflection in the line $y = -x$ followed by reflection in the y-axis.

Example

If

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

then find $S \circ T$, $T \circ S$ and the matrices $[S \circ T]$ and $[T \circ S]$.

Inverses

Inverse functions

Definition (Identity transformation)

The *identity transformation on* $\mathbb{R}^n$ is the function $\mathbb{R}^n \to \mathbb{R}^n$ defined by $x \mapsto x$.

Definition

If $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations then S is inverse to T iff

$$(S \circ T) = id = (T \circ S)$$

and we write $S = T^{-1}$ and $T = S^{-1}$.

Theorem

If $[S]$ is the matrix of S and $[T]$ is the matrix of T then:

$$S^{-1} = T \iff [S]^{-1} = [T]$$

Examples

Example

If

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

find T^{-1} and $[T^{-1}]$.

Example

In general, what is the inverse of the linear transformation

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Complex numbers

Early difficulties...

> *.. subtle as they are useless ..*

(Girolamo Cardano - discoverer of complex numbers)

> *..the whole matter seems to rest on sophistry rather than truth*

(Rafael Bombelli - early innovator in use of complex numbers)

> *The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being.*

(Leibniz)

More recently...

> *It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one.*

(Jacques Hadamard)

> *I tell you, with complex numbers you can do anything.*

(John Derbyshire - mathematics writer)

> *Indeed, nowadays no electrical engineer could get along without complex numbers, and neither could anyone working in aerodynamics or fluid dynamics.*

(Keith Devlin - British mathematician)

Algebraic motivation - extending the number system

God made the integers; all else is the work of man.

(Leopold Kronecker)

- Using only positive integers $\{1, 2, 3, \dots\}$
  - we cannot find solution a to $x + 1 = 0$
- Using only integers $\{\dots, -2, -1, 0, 1, 2 \dots\}$
  - we cannot find a solution to $3x + 2 = 0$
- Using only fractions $\{\frac{a}{b}$ for $b \neq 0\}$
  - we cannot find a solution to $x^2 - 2 = 0$
- Using only decimals
  - we cannot find a solution to $x^2 + 1 = 0$...

Complex numbers

Definition

- The *imaginary unit*, denoted i, is defined to be a number with the property that $i^2 = -1$.
- A *pure imaginary* number has the form bi where $b \in \mathbb{R}$, $b \neq 0$.
- A *complex number* is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

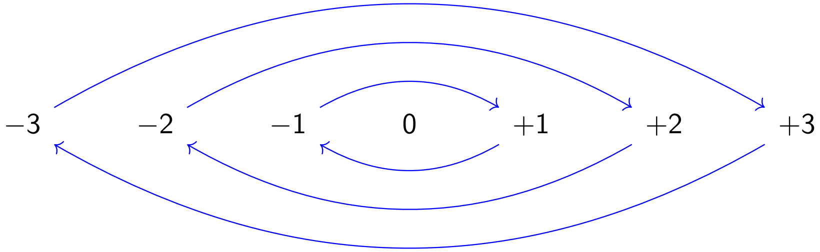
Definition

If $z = a + bi$ then:

- a is called the *real part* of z.
- b is called the *imaginary part* of z.

Graphical interpretation

Consider the function $\mathbb{R} \to \mathbb{R}$ that multiplies by -1:



which shows us that $-1 \times (-1 \times x) = x$.

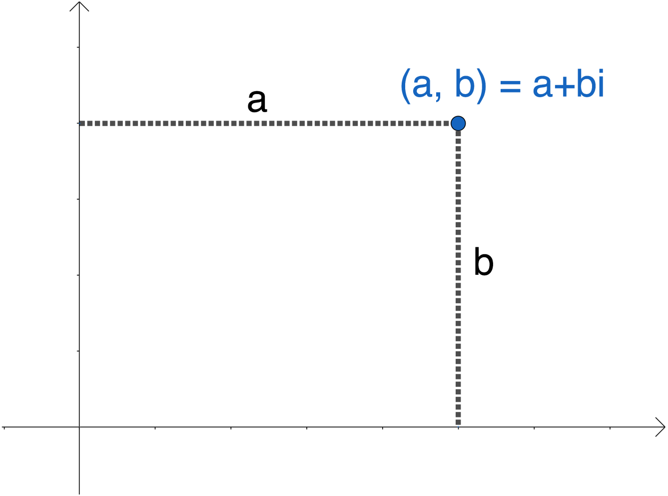
Graphical interpretation

So how do we get a transformation that squares to -1?

$$\begin{tikzcd}
 & & +i \arrow[dl, bend right] & & \\
-2 & -1 \arrow[dr, bend right] & 0 & +1 \arrow[ul, bend right] & +2 \\
 & & -i \arrow[ur, bend right] & &
\end{tikzcd}$$

Graphical interpretation

So we draw the complex number $a + bi$ as the point (a, b) in $\mathbb{R}^2$:



Summary

- Graphical interpretation.
- Addition: algebra and geometry.
- Multiplication: algebra and geometry.
- Conjugate: algebra and geometry.

Examples

Example

Evaluate

- $(-3+6i)+(5-i)$.
- $(4-7i)+(6-2i)$.
- $(-3+6i)-(5-i)$.
- $(4-7i)-(6-2i)$.

Example

$$(2-3i)(-3+4i)$$

Examples

Example

Find all complex numbers z such that $z^2 = -3 + 4i$.

Example

- If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$, i.e., $\overline{3 + 4i} = 3 - 4i$.
- $\overline{-2 + 5i} = -2 - 5i$.
- $\bar{i} = -i$.
- $\overline{7} = 7$.

Clearing denominators

Example (Exemplar)

Write

$$\frac{a+bi}{c+di}$$

in the form $e+fi$ for real numbers a, b, c, d, e, and f.

- $\frac{1}{i}$
- $\frac{2-i}{3+4i}$
- $\frac{1-2i}{-2+5i}$