

# MATH211: Linear Methods I

Matthew Burke

Thursday 11<sup>th</sup> October, 2018

# Lecture on Thursday 11<sup>th</sup> October, 2018

Lines and planes

Scalar product

Orthogonality

Projections

# Last time

- ▶ Euclidean space
- ▶ Addition scalar multiplication etc..
- ▶ Lines and planes in  $\mathbb{R}^2$  and  $\mathbb{R}^3$

## Lines and planes

## Two dimensions

### Definition

A *parametric (vector) equation* describing a line in  $\mathbb{R}^2$  is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d}$$

for  $\vec{x}$ ,  $\vec{c}$  and  $\vec{d}$  in  $\mathbb{R}^2$  and  $s$  in  $\mathbb{R}$ .

### Definition

A *parametric (vector) equation* describing a line in  $\mathbb{R}^3$  is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d}$$

for  $\vec{x}$ ,  $\vec{c}$  and  $\vec{d}$  in  $\mathbb{R}^3$  and  $s$  in  $\mathbb{R}$ .

# Parametric forms of 3D lines and planes

## Definition

A *parametric (vector) equation* describing a plane in  $\mathbb{R}^3$  is a vector equation of the form

$$\vec{x} = \vec{c} + s\vec{d} + t\vec{e}$$

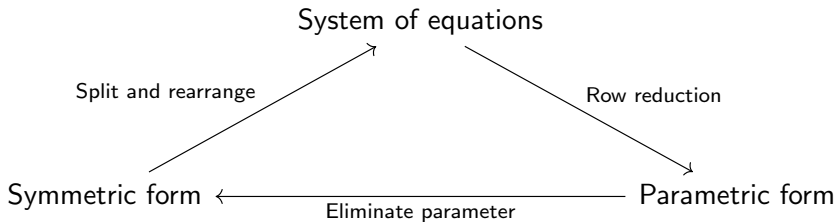
for  $\vec{x}$ ,  $\vec{c}$ ,  $\vec{d}$  and  $\vec{e}$  in  $\mathbb{R}^3$  and  $s$ ,  $t$  in  $\mathbb{R}$ .

## Definition (Symmetric form of equation for line)

$$\frac{x_1 + b_1}{a_1} = \frac{x_2 + b_2}{a_2} = \frac{x_3 + b_3}{a_3}$$

where  $a_i \neq 0$ .

## Converting between various forms



# Conversion example

## Example

Write down all three forms for the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$



# Example

## Example

Write down a parametric equation for the line passing through

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and parallel to the line

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

# Example

## Example

Write down the equation for the plane passing through the following three points.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

## Example

For the following two lines find the point of intersection.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

## Scalar product

# Length of a vector

## Definition (Two dimensions)

$$\text{If } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ then } \|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$$

## Definition (Three dimensions)

$$\text{If } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ then } \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

## Definition (n dimensions)

$$\text{If } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ then } \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

## Distance between two points

### Definition (Two dimensions)

If  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$  then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

### Definition (Three dimensions)

If  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$  then

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

# N dimensions and scalar multiplication

## Definition (n dimensions)

$$\text{If } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then}$$

$$d(\vec{x}, \vec{y}) = d(0, \vec{y} - \vec{x}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

## Lemma (Scalar multiplication)

*If  $\vec{v}$  is a non-zero vector and  $a$  is a scalar then*

$$\|a\vec{v}\| = |a|\|\vec{v}\|$$

# Examples

## Definition

A *unit vector* is a vector with length equal to 1.

## Example

The following vectors are unit vectors:-

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{2}} \end{bmatrix}$$

# Examples

## Example

Find the distance between

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

## Example

Find the length of

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$



# Scalar/dot product

## Definition

If  $\vec{x}$  and  $\vec{y}$  are in  $\mathbb{R}^n$  then

$$\begin{aligned}\vec{x} \cdot \vec{y} &= \sum_{k=1}^n x_k y_k \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\end{aligned}$$

And so in fact

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

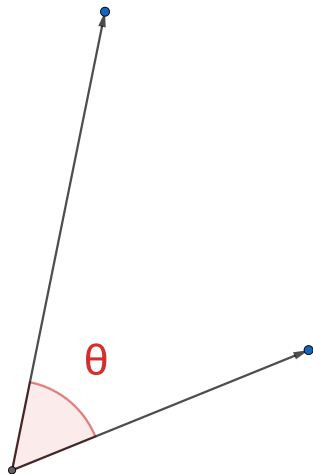
# Properties of the scalar product

## Theorem

Let  $\vec{u}, \vec{v}, \vec{w}$  be vectors in  $\mathbb{R}^3$  (or  $\mathbb{R}^2$ ) and let  $k \in \mathbb{R}$ .

1.  $\vec{u} \cdot \vec{v}$  is a real number
2.  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3.  $\vec{u} \cdot \vec{0} = 0$
4.  $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
5.  $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
6.  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$   
 $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$

# Scalar product in terms of angle



Definition (Scalar product in terms of angle)

$$\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos(\theta)$$

# Examples

## Example

If

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

then find  $u \cdot v$ .

## Example

Find the angle between

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

# Examples

## Example

Find the angle between

$$\begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

Last time  
○

Lines and planes  
○○○○○○○

Scalar product  
○○○○○○○○○○○●

Orthogonality  
○○○○○○○

Projections  
○○○○○○○

Questions?

# Orthogonality

# Orthogonality

## Definition

Two vectors  $u$  and  $v$  are *orthogonal* iff

$$u \cdot v = 0$$

Note that a zero vector is orthogonal to any other vector.



# Orthogonal complement

## Definition (Slightly non-standard)

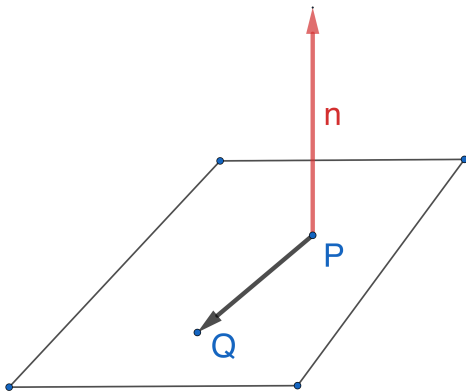
If  $v$  is a vector then the *orthogonal complement*  $v^\perp$  of  $v$  is the set of all vectors  $u$  such that  $u \cdot v = 0$ .

Therefore

- ▶ If  $v \neq 0 \in \mathbb{R}^1$  then  $v^\perp = 0$ .
- ▶ If  $v \neq 0 \in \mathbb{R}^2$  then  $v^\perp$  is the line perpendicular to  $v$ .
- ▶ If  $v \neq 0 \in \mathbb{R}^3$  then  $v^\perp$  is a plane.

In general if  $v \neq 0 \in \mathbb{R}^n$  then  $v^\perp$  is a  $(n-1)$ -dimensional subspace.

# Planes via normal vector



- ▶ Let  $P$  be a point in the plane and  $n$  a normal vector to the plane.
- ▶ For every point  $Q$  in the plane the vector  $\overrightarrow{PQ}$  is orthogonal to  $n$ .
- ▶ Therefore the solutions to

$$n \cdot (x - P) = 0$$

form a plane.

# Examples

## Example

Find all vectors orthogonal to both

$$\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

## Example

Are the following the vertices of a right angled triangle?

$$\begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -6 \end{bmatrix}$$

## Example

### Example

Find an equation of the plane containing

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

orthogonal to

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

### Example

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Last time  
○

Lines and planes  
○○○○○○○

Scalar product  
○○○○○○○○○○○○○

Orthogonality  
○○○○○○●

Projections  
○○○○○○○

Questions?

Last time  
○

Lines and planes  
○○○○○○○

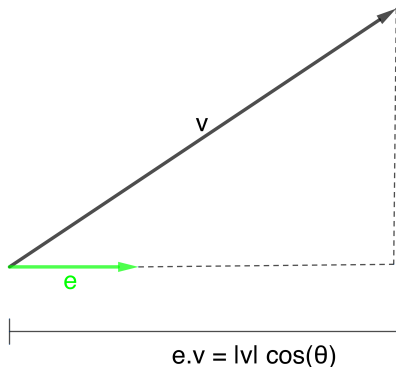
Scalar product  
○○○○○○○○○○○○○

Orthogonality  
○○○○○○○

Projections  
●○○○○○○

# Projections

# Projection onto a vector

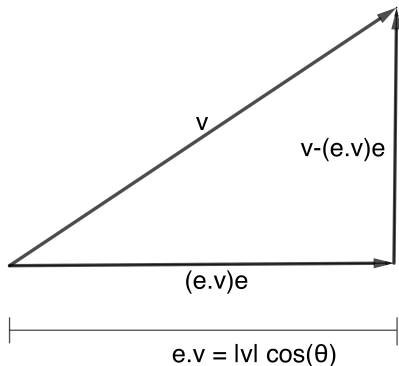


If  $e$  is a unit vector then

$$e \cdot v = |e||v|\cos\theta = |v|\cos\theta$$

and so  $e \cdot v$  is the length  
of  $v$  projected onto the  
line in direction  $e$ .

# Projection onto a vector



Therefore we can write

$$v = (e \cdot v)e + (v - e(e \cdot v))$$

where  $(e \cdot v)e$  is parallel to  $e$  and  $(v - e(e \cdot v))$  is orthogonal to  $e$ .



# Examples

## Example

Let

$$v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Find vectors  $v_0$  and  $v_1$  such that  $v = v_0 + v_1$ ,  $v_0$  is orthogonal to  $u$  and  $v_1$  is parallel to  $u$ .

# Example

## Example

If

$$p = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

find the closest point to  $p$  on the following line:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

# Example

## Example

Find the shortest distance from the point

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

to the plane with equation  $5x + y + z = -1$ .

Last time  
○

Lines and planes  
○○○○○○○

Scalar product  
○○○○○○○○○○○○○

Orthogonality  
○○○○○○○

Projections  
○○○○○○●

Questions?