MATH211: Linear Methods I

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Thursday 8th November, 2018

Lecture on Thursday 8th November, 2018

Last time

- Complex numbers
- Addition of complex numbers
- Multiplication of complex numbers
- Division of complex numbers

Graphical interpretation

Graphical interpretation

Consider the function $\mathbb{R} \to \mathbb{R}$ that multiplies by -1:

$$\ldots \quad -x \quad \ldots \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad \ldots \quad x \quad \ldots$$

which shows us that $-1 \times (-1 \times x) = x$.

(Imagine this as a rotation rather than reflection.)

Graphical interpretation

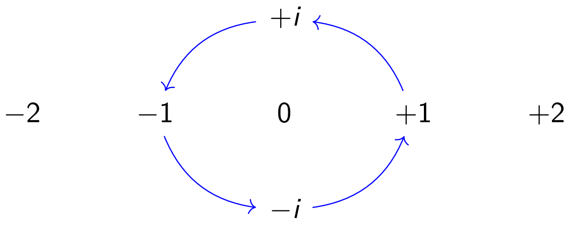
So how do we get a transformation that squares to -1?

$$+i$$

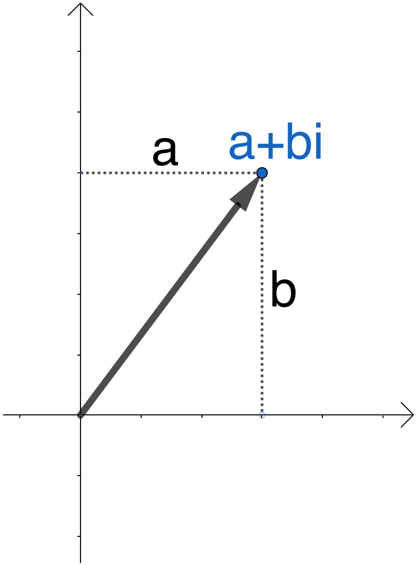
$$-2 \qquad -1 \qquad 0 \qquad +1 \qquad +2$$

Graphical interpretation

So how do we get a transformation that squares to -1?



Graphical interpretation



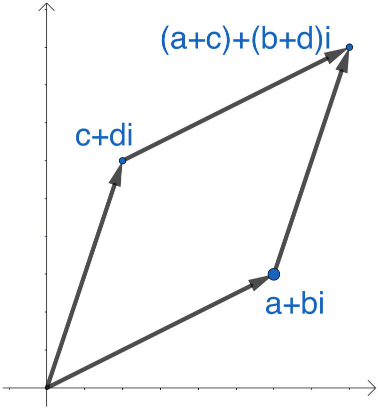
We draw the complex number $z = a + bi$ as the point (a, b) in $\mathbb{R}^2$.

Definition

The *modulus of z* is its length in $\mathbb{R}^2$:

$$|z| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

Addition of complex numbers



Complex addition is defined component-wise:

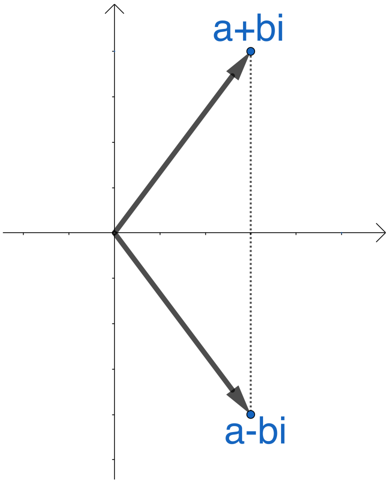
$$(a+ib)+(c+id) = (a+c)+(b+d)i$$

and therefore has the same graphical interpretation as the addition of vectors in $\mathbb{R}^2$.

Graphical interpretation of multiplication

Next section.

Conjugate



Definition (Conjugate)

If

$$z = a + bi$$

then

$$\bar{z} = a - bi$$

Properties of conjugate

Let z and w be complex numbers.

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
- $\overline{(zw)} = \overline{z}\ \overline{w}$.
- $\overline{(\overline{z})} = z$.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
- z is real if and only if $\overline{z} = z$.

If $z = a + bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Questions?

Questions?

Examples

Example

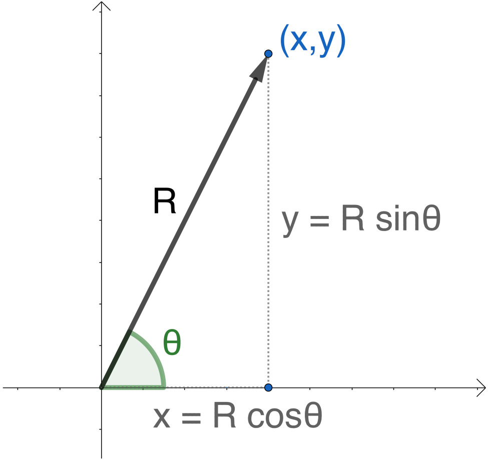
- $|-3+4i|$
- $|3-2i|$
- $|i|$

Example

- $\overline{3+4i}$
- $\overline{-2+5i}$
- $\bar{i}$
- $\overline{7}$

Polar form

Polar co-ordinates

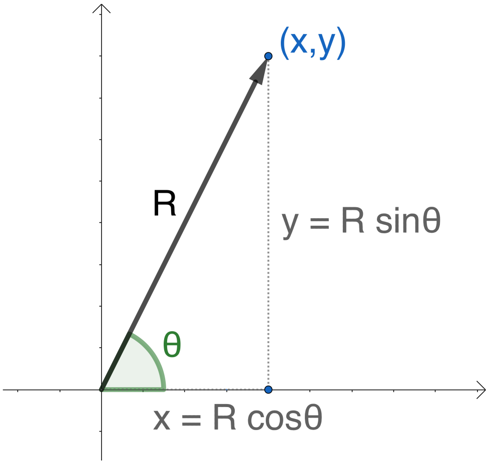


We can describe a point in the plane in terms of an angle and magnitude. By convention we let θ range between

$$-\pi < \theta \leq +\pi$$

and if $R = 0$ we say the angle is undefined.

Converting from polar to Cartesian



If we have R and θ then:

$$x = R\cos\theta$$
$$y = R\sin\theta$$

(and if $R = 0$ then $x = 0$ and $y = 0$).

Examples

Example

- $(1, 0)$ has $R = 1$ and $\theta = 0$.
- $(0, 1)$ has $R = 1$ and $\theta = \frac{\pi}{2}$.
- $(0, -1)$ has $R = 1$ and $\theta = \frac{-\pi}{2}$.
- $(1, 1)$ has $R = \sqrt{2}$ and $\theta = \frac{\pi}{4}$.
- $(1, \sqrt{3})$ has $R = 2$ and $\theta = \frac{\pi}{3}$.

Converting from Cartesian to polar

If we are given Cartesian co-ordinates (x, y) then

$$R = \sqrt{x^2 + y^2}$$

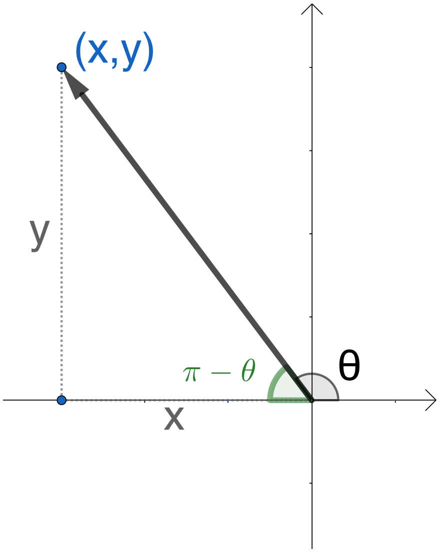
and we would like to use

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

but unfortunately it only works when $x > 0$.

- If $x < 0$ we should draw a picture.
- What we do depends on what quadrant the point is in.
- In general work out in terms of an acute angle.

Top left



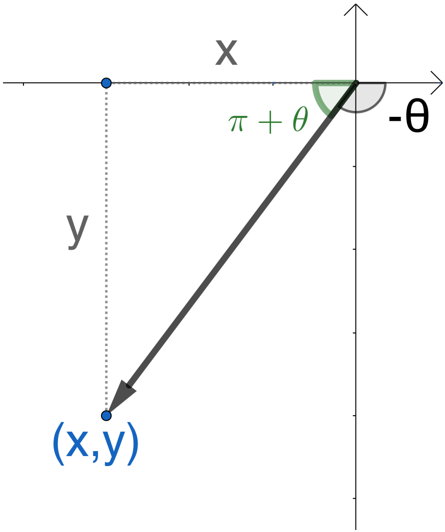
If $x < 0$ and $y > 0$ then

$$\begin{aligned}\pi - \theta &= \tan^{-1}\left(\frac{y}{|x|}\right)\\ &= \tan^{-1}\left(\frac{y}{-x}\right)\\ &= -\tan^{-1}\left(\frac{y}{x}\right)\end{aligned}$$

and so

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Bottom left



If $x < 0$ and $y < 0$ then

$$\pi + \theta = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$
$$= \tan^{-1}\left(\frac{-y}{-x}\right)$$
$$= \tan^{-1}\left(\frac{y}{x}\right)$$

and so

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) - \pi$$

Special cases

- If $x = 0$ and $y > 0$ then $\theta = \frac{\pi}{2}$.
- If $x = 0$ and $y < 0$ then $\theta = \frac{-\pi}{2}$.
- If $x = 0$ and $y = 0$ then θ is undefined.
- If $x < 0$ and $y = 0$ then $\theta = \pi$.

Relationship to complex numbers

Converting between polar and Cartesian forms gives an alternative representation of a complex number:

$$x + iy \leftrightarrow R\cos\theta + R\sin\theta i$$

Theorem (Euler formula)

$$e^{\theta i} = \cos\theta + \sin\theta i$$

and so every complex number can be written as:

$$x + iy \leftrightarrow e^{\theta i}$$

the LHS is called *standard form* and the RHS *polar form*.

Questions?

Questions?

Examples

Example

Convert the following to polar form:-

- $-2+2\sqrt{3}i$
- $3i$
- $-1-i$
- $\sqrt{3}+3i$

Example

Convert the following to standard form:-

- $2e^{\frac{2\pi i}{3}}$
- $3e^{-i\pi}$
- $2e^{\frac{3i\pi}{4}}$

Multiplication

Graphical interpretation of complex numbers

If we multiply $z = Re^{i\theta}$ and $w = Qe^{i\phi}$:

$$zw = Re^{i\theta} Qe^{i\phi} = (RQ)e^{i(\theta+\phi)}$$

Slogan: To multiply two complex numbers we multiply the moduli and add the angles.

Theorem (De Moivre Theorem)

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Questions?

Questions?

Examples

Example

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

Example

Express $(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{17}$ in the form $a+bi$.

Complex roots

Principal roots

If $w = Re^{i\theta}$ then one solution to the equation

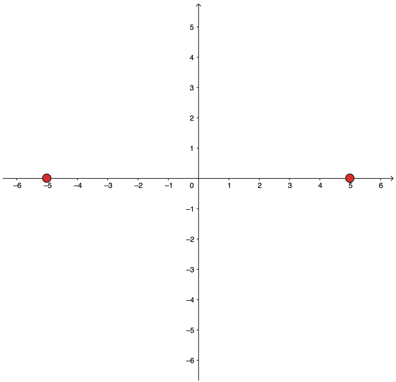
$$z^n = w$$

is

$$z = \sqrt[n]{R} \cdot e^{\frac{i\theta}{n}}$$

although there will in fact be $n - 1$ more solutions.

Square roots



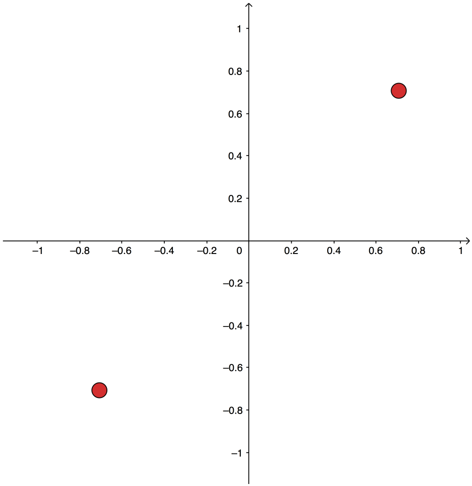
The solutions to

$$z^2 = 25$$

are

$$z = +5 \text{ and } z = -5$$

Square roots



The solutions to

$$z^2 = i$$

are

$$z = e^{\frac{\pi}{4}} \text{ and } z = e^{\frac{-3\pi i}{4}}$$

Square roots

In general if $w = Re^{i\theta}$ then all of

$$\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}$$

will be square roots of w because

$$\left(\sqrt{R}e^{i\left(\frac{\theta}{2}+k\pi\right)}\right)^2 = Re^{i(\theta+2k\pi)} = Re^{i\theta}$$

and then we need to find the values of

$$\frac{\theta}{2} + k\pi$$

that are between $-\pi$ and $+\pi$.

Questions?

Questions?

Examples

Example

Find all solutions to the following equations:

- $z^2 = 25$.
- Find all solutions to $z^2 = -1$
- Find all solutions to $z^2 = e^{\frac{i\pi}{4}}$
- Find all solutions to $z^3 = i$. (Write in Cartesian form.)
- Find all solutions to $z^4 = 2(\sqrt{3}i - 1)$. (Write in Cartesian form.)

Example

Find all sixth roots of unity.

Summary

- Polar form
- Euler's Identity
- Converting between polar and Cartesian form
- Using polar form for multiplication (De Moivre) - slogan
- Roots of complex numbers using de Moivre
- Build up first with square roots, then cube roots and finally 4th roots. (Picture at each turn.)
- General nth case.
- Picture of sixth roots of unity.