MATH211: Linear Methods I

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Lecture on Tuesday 27th November, 2018

Last time

- Multiplication using polar form.
- Complex roots
- Quadratic formula

Quadratic formula

Examples

Example
Solve $z^2 - 14z + 58 = 0$.

Example
Find a real quadratic with $5 - 2i$ as a root. What is the other root?

Example
Solve $z^2 - (3 - 2i)z + (5 - i) = 0$.

Spectral theory

Simplifying matrix actions

If

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

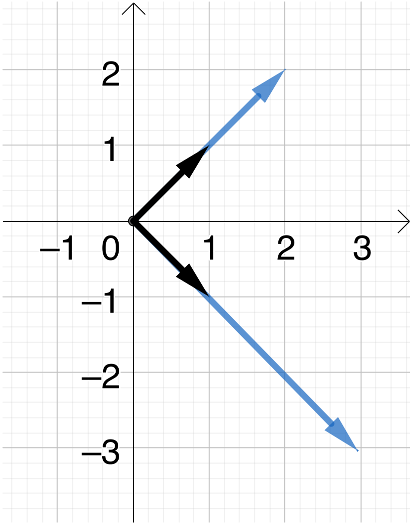
then A acts on some vectors in a simplified way:

$$Av_1 = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2v_1$$

and

$$Av_2 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} 4v_2$$

Picture



Since

$$Av_1 = 2v_1$$

and

$$Av_2 = 4v_2$$

Simplifying matrix actions

If

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

then

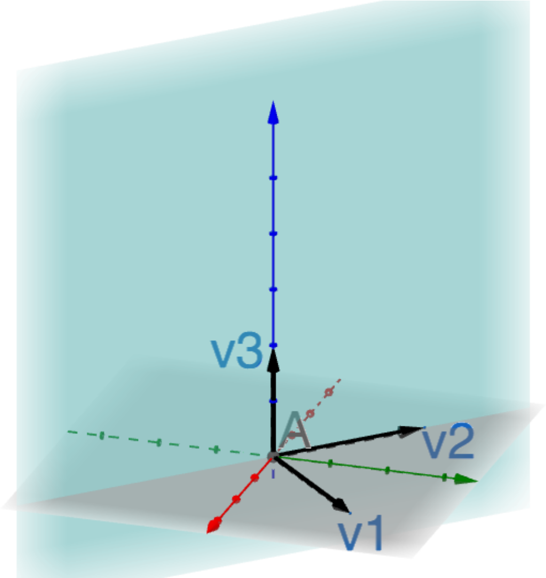
$$Av_1 = A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$$

$$Av_2 = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_3$$

and

$$Av_3 = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -v_2$$

Picture



Blue plane is preserved.

Eigenvectors and eigenvalues

Definition

If A is a square matrix, v is a non-zero vector and

$$Av = \lambda v$$

then A has an *eigenvector* v *with eigenvalue* λ.

Idea: If vector x is a linear combination of eigenvectors of A then the action of A on x is easy:

$$A(x) = A\left(a_1 v_{\lambda_1} + a_2 v_{\lambda_2}\right) = a_1 A(v_{\lambda_1}) + a_2 A(v_{\lambda_2}) = a_1 \lambda_1 v_{\lambda_1} + a_2 \lambda_2 v_{\lambda_2}$$

Examples

Example

$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with eigenvalues 3 and 5 respectively.

Example

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 2 and 4 respectively.

Eigenvalues

Finding eigenvalues

Since

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

and $v \neq 0$ we need to find all λ such that

$$A - \lambda I$$

is *not* invertible.

Finding eigenvalues

Definition (Characteristic polynomial)

The *characteristic polynomial of a matrix A* is

$$\chi_A(\lambda) = det(A - \lambda I)$$

- The eigenvalues are the roots of the characteristic polynomial.
- The number of times a root is repeated is called the *algebraic multiplicity* of the root. E.g. if

$$\chi_A(\lambda) = (\lambda - 4)^3(\lambda + 3)(\lambda - 1)^2$$

then 4 has algebraic multiplicity 3, -3 has algebraic multiplicity 1 and 1 has algebraic multiplicity 2.

Examples

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Eigenvectors

Finding eigenvectors

Suppose that we know λ is an eigenvalue for A.

Then we find the eigenvectors associated to λ by solving:

$$(A - \lambda \cdot I)x = 0$$

- After doing row reduction we will get infinitely many solutions.
- The no. of parameters is called the *geometric multiplicity* of λ.

Examples

Example

Find the eigenvectors of

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Example

Find the eigenvectors of

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Diagonalisation

Motivation

If $x \in \mathbb{R}^n$ is a linear combination of eigenvectors of A then:

$$A(x) = A\left(\sum_{i=1}^{n} a_i v_{\lambda_i}\right) = \sum_{i=1}^{n} a_i A(v_{\lambda_i}) = \sum_{i=1}^{n} a_i \lambda_i v_{\lambda_i}$$

So if *every* vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A the the *entire* matrix action simplifies.

Definition

An $n \times n$ matrix A is *diagonalisable* iff every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors.