# COMS21103: Maximum Flow

Dima Damen
Dima.Damen@bristol.ac.uk

Bristol University, Department of Computer Science
Bristol BS8 1UB, UK
**New

December 3, 2013

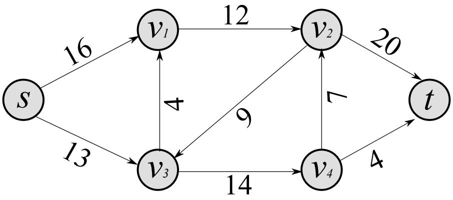
# Flow Networks

- You've previously looked at directed graphs, in order to find the shortest path from one point to another
- Directed graphs can also be interpreted as **flow networks**
- In a flow network, exactly one node is referred to as a **source**, and one node as a **sink**.
- The aim is to push the maximum amount of *material* from the source to the sink.

# Flow Network - Definitions

A *flow network* $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v) \geq 0$. We distringuish two

vertices in a flow network: a ***source*** $s$ and a ***sink*** $t$.

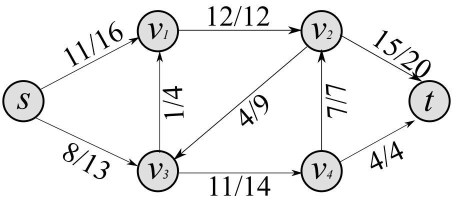



** edges are removed from visualisation when $c(u, v) = 0$

# Flow Network - Definitions

A *flow f* in the *network G* is a real-valued function $f : V \times V \rightarrow \mathbb{R}$ that satisfies two conditions

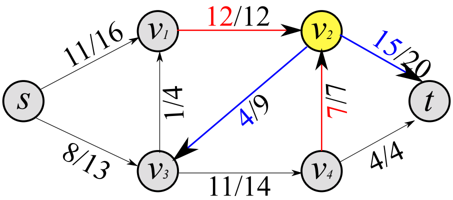
- $\forall u, v \in V \quad \Rightarrow \quad 0 \leq f(u,v) \leq c(u,v)$
- $\forall u \in V - \{s,t\} \quad \Rightarrow \quad \sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u)$



# Flow Network - Definitions

A *flow f* in the *network G* is a real-valued function $f : V \times V \rightarrow \mathbb{R}$ that satisfies two conditions

- $\forall u, v \in V \quad \Rightarrow \quad 0 \leq f(u, v) \leq c(u, v)$
- $\forall u \in V - \{s, t\} \quad \Rightarrow \quad \sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$



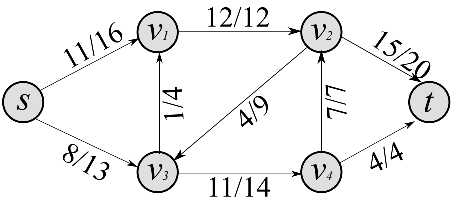
# Flow Network - Definitions

The **value** $|f|$ of the flow function $f$ is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

is the total flow out of the source minus the flow into the source

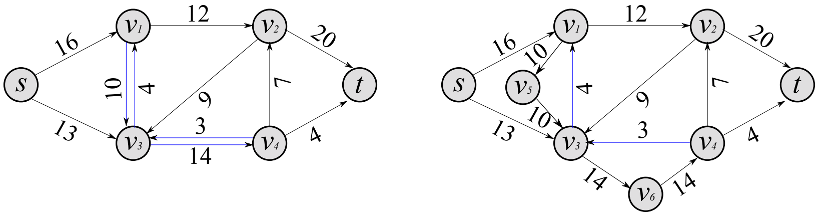
For the flow $f$ below, $|f| = f(s, v_1) + f(s, v_3) = 19$



# Max-Flow Problem

In the ***maximum-flow problem***, we are given a flow network $G$ with source $s$ and sink $t$, and we wish to find a flow of maximum value.

Even though the number of problems you can think of, with one source, one sink and non-antiparallel edges are limited. It is easy to re-formulate these cases as a flow network.

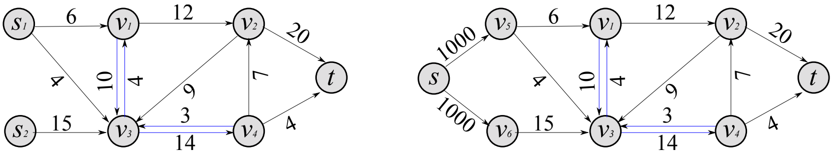



Dima Damen
Dima.Damen@bristol.ac.uk

# Max-Flow Problem

In the ***maximum-flow problem***, we are given a flow network $G$ with source $s$ and sink $t$, and we wish to find a flow of maximum value.

Even though the number of problems you can think of, with one source, one sink and non-antiparallel edges are limited. It is easy to re-formulate these cases as a flow network.



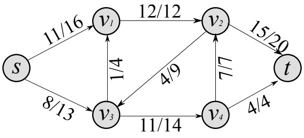
# Ford-Fulkerson Method - Definitions

Given a flow network $G = (V, E)$ and a flow $f$, the ***residual network*** of $G$ induced by $f$ is $G_f = (V, E_f)$ where

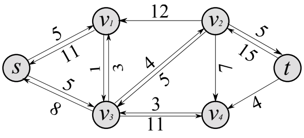
$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

where

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & if (u, v) \in E \\ f(v, u) & if (v, u) \in E \\ 0 & otherwise \end{cases}$$




$f$




$G_f$

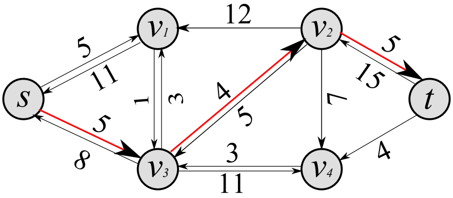
Dima Damen
Dima.Damen@bristol.ac.uk

# Ford-Fulkerson Method - Definitions

An ***augmenting path*** $p$ is a simple path from $s$ to $t$ in the residual network $G_f$.

The ***residual capacity*** of the augmenting path $p$ is given by

$$c_f(p) = \min\{c_f(u, v) : (u, v) is on p\}$$




In the example above, $c_f(p) = \min\{5, 4, 5\} = 4$

# Ford-Fulkerson Method - Definitions

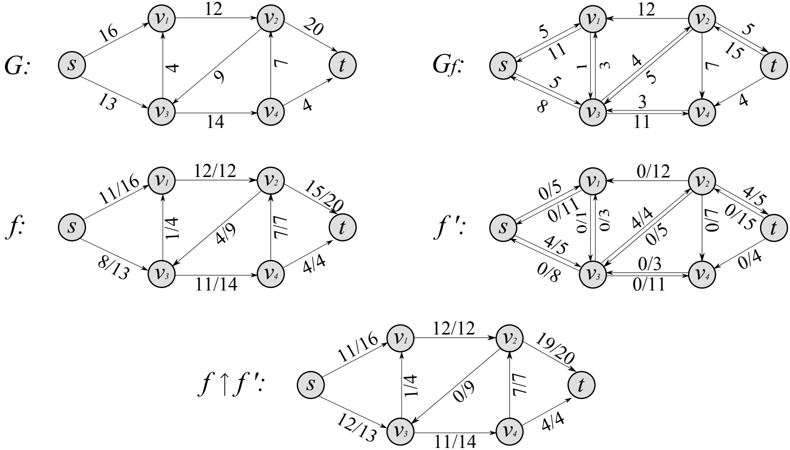
If $f$ is a flow in $G$ and $f'$ is a flow in the corresponding residual network $G_f$, then $f \uparrow f'$ is the ***augmentation*** of flow $f$ by $f'$ defined as:

$f \uparrow f' : V \times V \Rightarrow \mathbb{R}$ such that

$$f \uparrow f' = f(u,v) + f'(u,v) - f'(v,u)$$

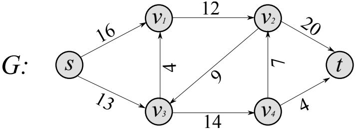
# Ford-Fulkerson Method - Definitions

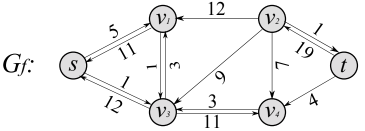
Augmentation example:

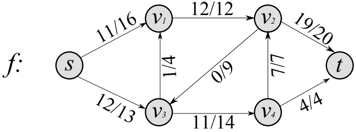


# Ford-Fulkerson Method - Definitions

Augmentation example - continued:



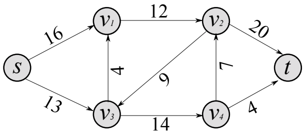



# Ford-Fulkerson Method - Basic Algorithm

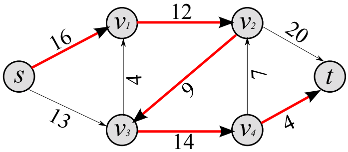
FORD-FULKERSON-METHOD($G$, $s$, $t$)
**begin**
    initialise flow $f$ to 0
    **while** *there exists an augmentation path $p$ in the residual network $G_f$* **do**
        augment flow $f$ along $p$
    **end**
**end**

# Ford-Fulkerson Method - Example

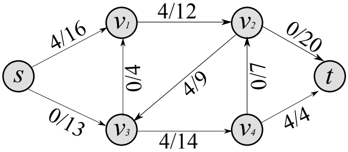
$G$:



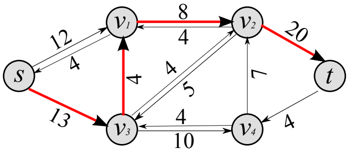

(1) $G_f$:



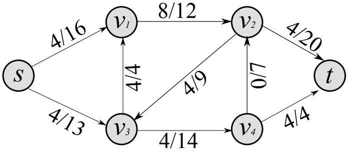

$f$:




(2) $G_f$:

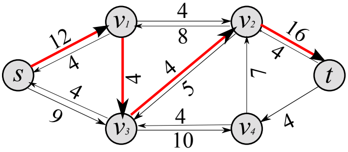



$f$:

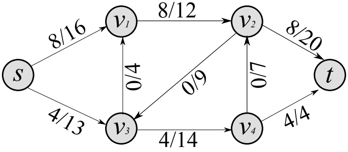


# Ford-Fulkerson Method - Example

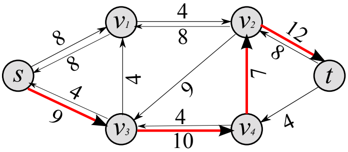
(3) $G_f$:



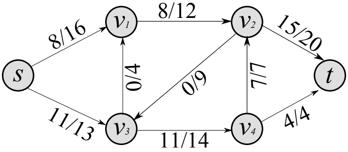

$f$:




(4) $G_f$:

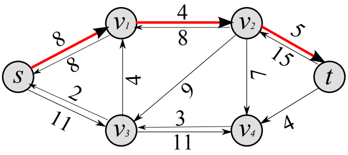



$f$:

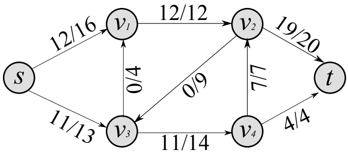


# Ford-Fulkerson Method - Example

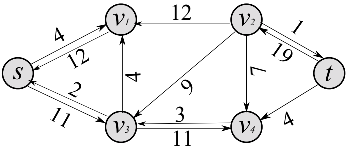
(5) $G_f$:




$f$:



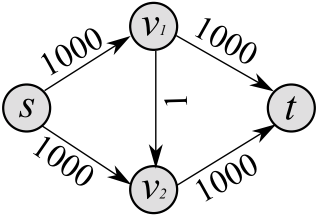

(6) $G_f$:



# Basic Algorithm - Time Analysis

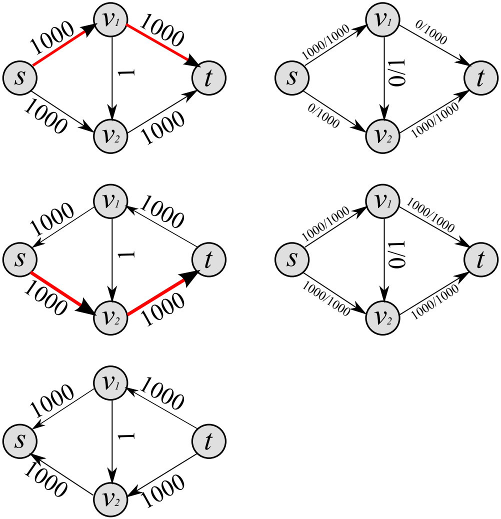
- running time depends on the order of selecting augmenting paths $p$.
- if we choose poorly ...

Ex.



# Basic Algorithm - Time Analysis

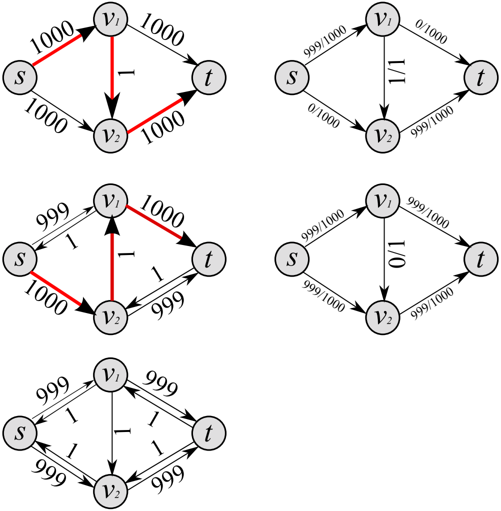
Good Choice:




Dima Damen
Dima.Damen@bristol.ac.uk

# Basic Algorithm - Time Analysis

Bad Choice:



# Basic Algorithm - Time Analysis

- running time depends on the order of selecting augmenting paths $p$.
- if we choose poorly...
- To prove the time bound:
  - each iteration takes $E$ time, where $E$ is the number of edges in the network
  - the flow network increases by at least one unit in each iteration.
  - For the max flow $f^*$, the maximum value $|f^*|$ denotes a bound in the number of iterations
- The total running time is thus $O(|f^*|E)$

# Edmonds-Karp Algorithm

- Choose the augmenting path as the *shortest path* from $s$ to $t$ at each step
- Let $\delta_f(u, v)$ be the shortest-path distance from $u$ to $v$ in $G_f$, where each edge has a unit distance.

### Lemma

*For a network $G = (V, E)$ with source $s$ and sink $t$, then for all vertices $v \in V - \{s, t\}$, the shortest path $\delta_f(s, v)$ in the residual network $G_f$ increases monotonically with each flow augmentation.*

### Lemma

*If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source $s$ and sink $t$, the total number of augmentations performed by the algorithm is $O(VE)$.*

# Edmonds-Karp Algorithm

## Proof.

- Each augmentation path has at least one critical edge that disappears from the residual network.
- If the edge $(u, v)$ disappears, it cannot reappear until the flow from $u$ to $v$ is decreased.

$$\begin{aligned}
\delta_{f'}(s,u) &= \delta_{f'}(s,v) + 1 \\
&\geq \delta_f(s,v) + 1 \quad \text{from previous lemma} \\
&= \delta_f(s,u) + 2
\end{aligned}$$

$\square$

# Edmonds-Karp Algorithm

## Proof.

continued..

- From the time $(u, v)$ becomes critical, to the time it next becomes critical, the distance from $s$ to $u$ increases by at least 2
- Until $u$ becomes unreachable from the source, if ever, its distance would be at most $|V| - 2$
- After the first time, it becomes critical at most $(|V| - 2)/2$ times.
- There are at most $O(E)$ pairs of vertices in the residual network.
- Total number of critical edges is $O(VE)$

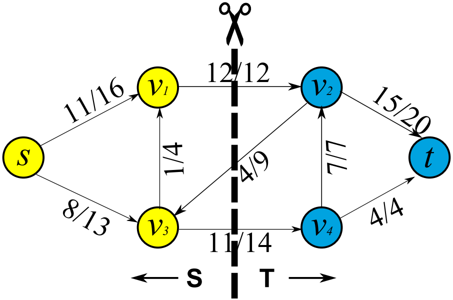
$\square$

# Edmonds-Karp Algorithm

- Total number of critical edges, and thus number of iterations, is $O(VE)$
- Each iteration runs in $O(E)$ times
- Total running time of the Edmonds-Karp algorithm is $O(VE^2)$

# Cuts of Flow Networks - Definition

A ***cut*** $(S, T)$ of flow network $G = (V, E)$ is a partition of vertices $V$ into $S$ and $T = V - S$ such that $s \in S$ and $t \in T$.



# Cuts of Flow Networks - Definition

If $f$ is a flow, then the ***net flow*** $f(S, T)$ across the cut is defined to be

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

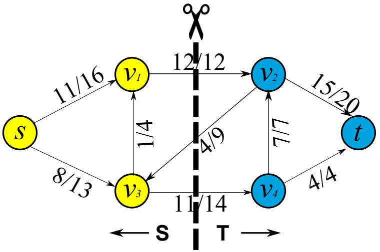
The ***capacity*** of the cut is

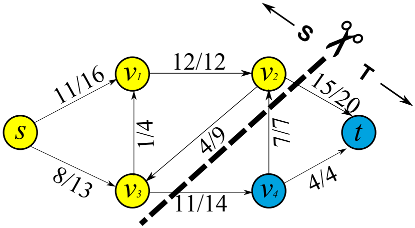
$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

The ***minimum cut*** of a network is a cut whose capacity is minimum over all cuts of the network.

# Cuts of Flow Networks - Definition

Calculate the *net flow* and the *capacity* of these two cuts...




*What do you conclude?*

# Cuts of Flow Networks - Definition

### Lemma

*Let $f$ be a flow in a flow network $G$ with source $s$ and sink $t$, and let $(S, T)$ be any cut of $G$, then $f(S, T) = |f|$*

# Cuts of Flow Networks - Definition

## Proof.

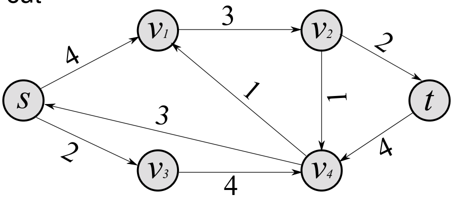
$$
\begin{aligned}
|f| &= \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s) & (1)\\
&= \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s) + \sum_{u\in S-\{s\}} \left(\sum_{v\in V} f(u,v) - \sum_{v\in V} f(v,u)\right) & (2)\\
&= \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s) + \sum_{u\in S-\{s\}} \sum_{v\in V} f(u,v) - \sum_{u\in S-\{s\}} \sum_{v\in V} f(v,u) & (3)\\
&= \sum_{v\in V} \left(f(s,v) + \sum_{u\in S-\{s\}} f(u,v)\right) - \sum_{v\in V} \left(f(v,s) + \sum_{u\in S-\{s\}} f(v,u)\right) & (4)\\
&= \sum_{v\in V} \sum_{u\in S} f(u,v) - \sum_{v\in V} \sum_{u\in S} f(v,u) & (5)\\
&= \sum_{v\in S} \sum_{u\in S} f(u,v) + \sum_{v\in T} \sum_{u\in S} f(u,v) - \sum_{v\in S} \sum_{u\in S} f(v,u) - \sum_{v\in T} \sum_{u\in S} f(v,u) & (6)\\
&= \sum_{v\in T} \sum_{u\in S} f(u,v) - \sum_{v\in T} \sum_{u\in S} f(v,u) & (7)\\
&= f(S,T) & (8)
\end{aligned}
$$

# Max Flow - Min Cut

## Theorem

*The value of the maximum flow is equal to the capacity of the minimum cut*

Find the minimum cut

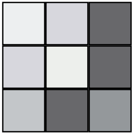


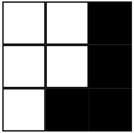
# Application - Image Segmentation

Max-Flow Min-Cut has been used for image segmentation.

# Application - Image Segmentation

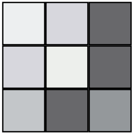
For an image of $3 \times 3$ pixels, you aim to label each pixel as *foreground* or *background*

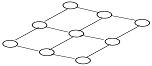


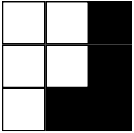


# Application - Image Segmentation

Each pixel is represented by a single node, neighbouring pixels are connected by an edge

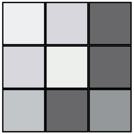


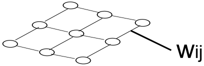


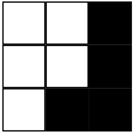


# Application - Image Segmentation

The edge weight is usually represented by the difference in colour (/intensity) between neighbouring pixels

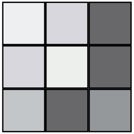


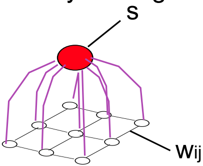


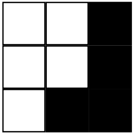



# Application - Image Segmentation

This is transformed to a network flow by adding a *source*.

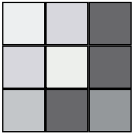


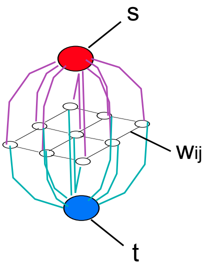


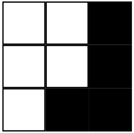



# Application - Image Segmentation

... and a *sink*.

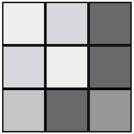


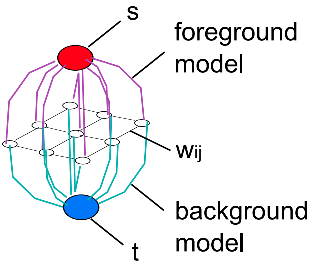


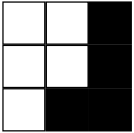



# Application - Image Segmentation

The flow from the source is represented by the foreground model and to the sink is represented by the background model

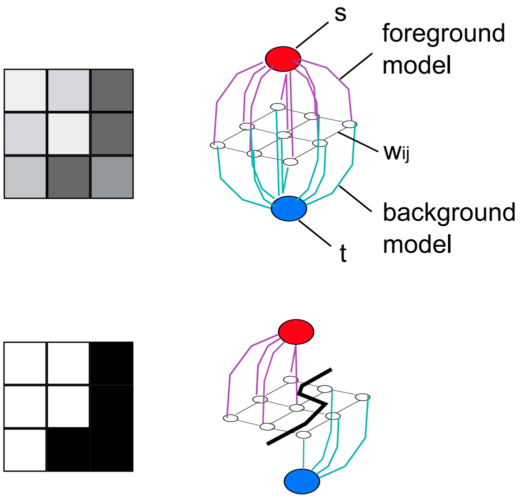




# Application - Image Segmentation

The max-flow min-cut would result in an image segmentation



# Further Reading

- Introduction to Algorithms
  T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 26 – Maximum Flow
- GrabCut: Interactive foreground extraction using iterated graph cuts
  C. Rother, V. Kolmogorov, and A. Blake.
  ACM Trans. Graph., 2004