COMS21103: Linear Programming

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**New

December 17, 2013

What is Linear Programming?

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- A linear function... e.g. $5 + 3x + 4y$

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In general, linear Programming is optimising a linear function subject to a set of linear inequalities

- Optimising a function is finding the *minimum* or *maximum* value of a function
- A linear function... e.g. $5 + 3x + 4y$
- A linear inequality... e.g. $5x + 3y \leq 10$, $3 + 5y \geq 4z$

Linear Programming - Example

Linear Programs arise in a variety of practical applications...

Example

A publisher has orders for 400 copies of a certain text from Bristol (b) and 600 copies from Leeds(l). The company has 700 copies in a warehouse in Birmingham (B) and 800 copies in a warehouse in London (L). It costs £5 to ship a text from Birmingham to Bristol, but it costs £4 to ship it to Leeds. It costs £10 to ship a text from London to Bristol, but it costs £8 to ship it from London to Leeds. How many copies should the company ship from each warehouse to Bristol and Leeds to fill the order at the least cost?

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- What do you want to optimise?

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- What do you want to optimise?
  - cost

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- What do you want to optimise?
  - cost
  - *minimise*

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- What is the linear function?

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- What is the linear function?
  - Define variables
    x: # of items shipped from B to b
    y: # of items shipped from L to b
    w: # of items shipped from B to l
    z: # of items shipped from L to l

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- What is the linear function?
  - Define variables
    x: # of items shipped from B to b
    y: # of items shipped from L to b
    w: # of items shipped from B to l
    z: # of items shipped from L to l
  - $f(x, y, w, z) = 5x + 10y + 4w + 8z$

Linear Programming - Example

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- Subject to what linear inequalities?

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- Subject to what linear inequalities?
  - Required shipments
    $x + w \geq 400$
    $y + z \geq 600$

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- Subject to what linear inequalities?
  - Required shipments
    $x + w \geq 400$
    $y + z \geq 600$
  - Available Stock
    $x + y \leq 700$
    $w + z \leq 800$

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- Subject to what linear inequalities?
  - Required shipments
    $x + w \geq 400$
    $y + z \geq 600$
  - Available Stock
    $x + y \leq 700$
    $w + z \leq 800$
  - Non-negative Inequalities
    $x \geq 0$, $y \geq 0$, $w \geq 0$, $z \geq 0$

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Linear Program - Non-standard form

minimise $5x + 10y + 4w + 8z$

subject to
$$x + w \geq 400$$
$$y + z \geq 600$$
$$x + y \leq 700$$
$$w + z \leq 800$$
$$x \geq 0,\ y \geq 0,\ w \geq 0,\ z \geq 0$$

Linear Programming - Example

Linear Program - Matrix Representation

$$\begin{array}{ll} \textbf{minimise} & 5x + 10y + 4w + 8z \\ \textit{subject to} & 1x + 0y + 1w + 0z \geq 400 \\ & 0x + 1y + 0w + 1z \geq 600 \\ & 1x + 1y + 0w + 0z \leq 700 \\ & 0x + 0y + 1w + 1z \leq 800 \\ & x \geq 0,\ y \geq 0,\ w \geq 0,\ z \geq 0 \end{array}$$

Linear Programming - Example

Linear Program - Matrix Representation

minimise $5x + 10y + 4w + 8z$

subject to
$$-1x + 0y - 1w + 0z \leq -400$$
$$0x - 1y + 0w - 1z \leq -600$$
$$1x + 1y + 0w + 0z \leq 700$$
$$0x + 0y + 1w + 1z \leq 800$$
$$x \geq 0,\ y \geq 0,\ w \geq 0,\ z \geq 0$$

Linear Programming - Example

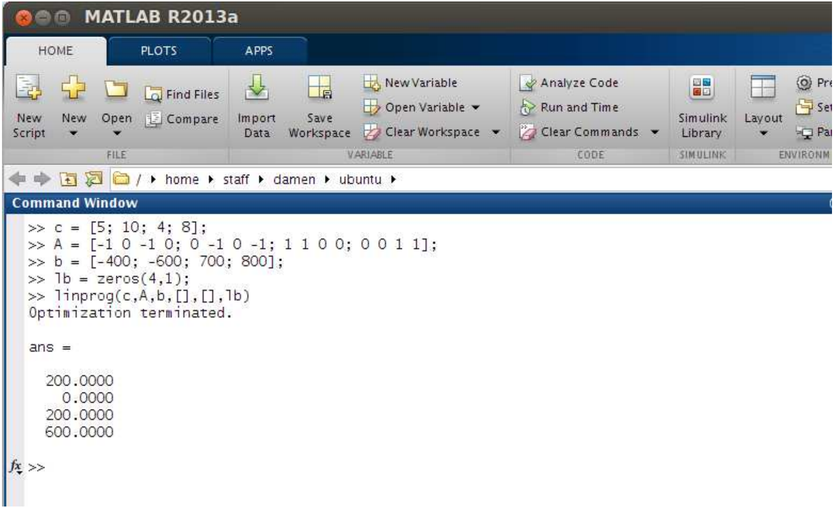
Linear Program - Matrix Representation

$$\mathbf{c} = \begin{bmatrix} 5 \\ 10 \\ 4 \\ 8 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -400 \\ -600 \\ 700 \\ 800 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ w \\ z \end{bmatrix}$$

$$\boxed{\begin{array}{ll} \textbf{minimise} & \mathbf{c}^T\mathbf{x} \\ \textit{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}}$$

Linear Programming - Example

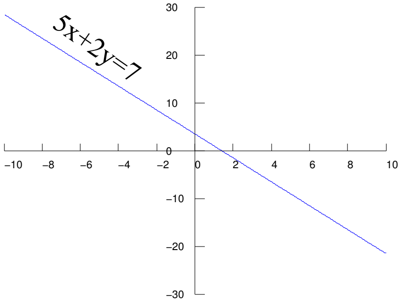
Can be solved using a linear solver



Linear Programming

In two dimensions...

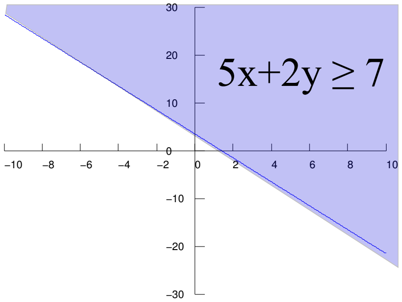
- A linear equation defines a line in space $5x + 2y = 7$



Linear Programming

In two dimensions...

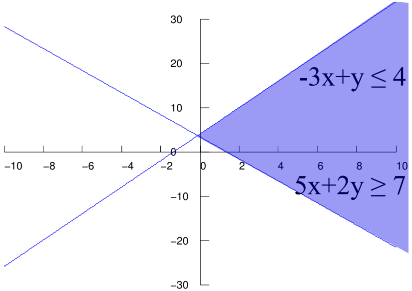
- A linear **inequality** defines a half-space $5x + 2y \geq 7$



Linear Programming

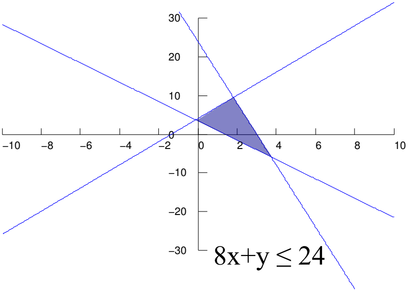
In two dimensions...

- Multiple linear inequalities constrain the space of solutions
- Setting the varialbes *x* and *y* to values that satisfy all constraints results in a **feasible** solution
- Setting the varialbes *x* and *y* to values that fail to satisfy any constraint results in an **infeasible** solution
- The shaded area represents the space of **feasible** solutions



Linear Programming

- When the set of inequalities represent a convex hull in space, the feasible region is said to be **bounded**



Linear Programming

In two dimensions...

- If the linear program has no feasible solution, the linear program is said to be **infeasible**
  e.g.
  $5x + 2y \geq 7$
  $x \leq 0$
  $y \leq 0$

- If a linear program has some feasible solutions but does not have a finite optimal objective value, it is said to be **unbounded**

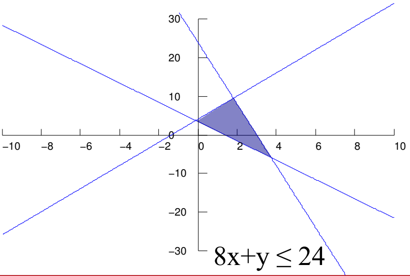
Linear Programming

In two dimensions...

- For the linear program

$$\begin{array}{ll} \textbf{minimise} & x - y \\ \textit{subject to} & 5x + 2y \geq 7 \\ & -3x + y \leq 4 \\ & 8x + y \leq 24 \end{array}$$

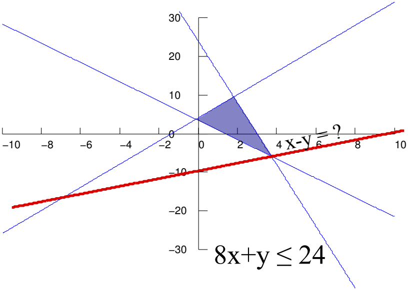
- The search is for a feasible solution that minimises the **objective function** $x - y$



Linear Programming

In two dimensions...

- The objective function takes different values within the feasible region



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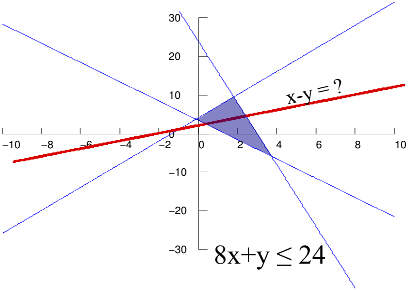

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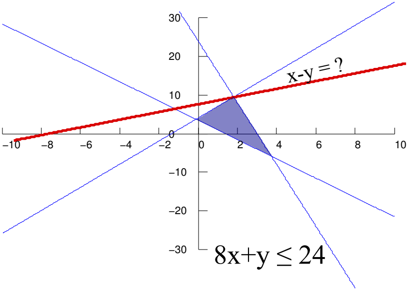

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- It is no accident that the optimal solution to the linear program occurs at a vertex of the feasible solution.

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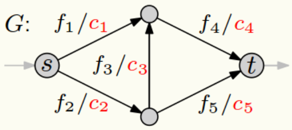
- It is no accident that the optimal solution to the linear program occurs at a vertex of the feasible solution.
- The optimal value must be at the boundary of the feasible region
- The intersection is thus either a single vertex or a line segment (that contains two vertices)
- Eureka... calculate the objective function at all vertices!
- But... we cannot easily graph linear programs in 3+ dimensions

Max-Flow as a Linear Program

Max-flow problem can be formulated as a linear program

Example



Objective function maximise $f_1 + f_2$

Constraints

Ensure the flow is feasible.

$$f_1 \leqslant c_1$$
$$f_2 \leqslant c_2$$
$$f_3 \leqslant c_3$$
$$f_4 \leqslant c_4$$
$$f_5 \leqslant c_5$$

Conservation constraints.

$$f_1 + f_3 - f_4 = 0$$
$$f_2 - f_3 - f_5 = 0$$

$$f_1, f_2, f_3, f_4, f_5 \geqslant 0$$

The Simplex Algorithm

In two dimensions...

- Takes as input a linear program and returns an optimal solution
- It starts at some vertex and performs a sequence of iterations
- In each iteration, it moves along an edge to a neighbouring vertex whose objective value is smaller than the current vertex
- Terminates when it reaches a local optimum
- As the vertex is a convex hull, the local optimum is actually a global optimum

Simplex Algorithm - Standard Form

The standard form to solve a simplex algorithm is:

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, ..m$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, ..m$$

The Simplex Algorithm - Standard Form

A linear function is in non-standard form for the Simplex algorithm if

- The objective function is a minimisation rather than a maximisation
- There might be variables without nonnegativity constraints
- There might be equality rather than inequality constraints
- There might be inequality constraints with an opposite sign

The Simplex Algorithm - Standard Form

To convert to a standard form:

- If the objective function is a minimisation $\rightarrow$ negate the coefficients e.g.

  minimise $-2x + 3y$

  becomes

  maximise $2x - 3y$

The Simplex Algorithm - Standard Form

To convert to a standard form:

- If a variable y does not have a non-negativity constraint $\rightarrow$ change y to $y_1 - y_2$ and add $y_1 \geq 0$, $y_2 \geq 0$
  e.g.

$$\boxed{\begin{array}{ll} \text{maximise} & 2x - 3y \\ \text{subject to} & x + y \leq 7 \\ & x - y \leq 4 \\ & x \geq 0 \end{array}} \rightarrow \boxed{\begin{array}{ll} \text{maximise} & 2x - 3y_1 + 3y_2 \\ \text{subject to} & x + y_1 - y_2 \leq 7 \\ & x - y_1 + y_2 \leq 4 \\ & x, y_1, y_2 \geq 0 \end{array}}$$

The Simplex Algorithm - Standard Form

To convert to a standard form:

- If equality constraint exists $\rightarrow$ replace by two inequalities $\leq b$ and $\geq b$ e.g.

$$\boxed{\begin{array}{ll} \text{maximise} & 2x - 3y \\ \text{subject to} & x + y = 7 \\ & .. \end{array}} \quad \rightarrow \quad \boxed{\begin{array}{ll} \text{maximise} & 2x - 3y \\ \text{subject to} & x + y \leq 7 \\ & x + y \geq 7 \\ & .. \end{array}}$$

The Simplex Algorithm - Standard Form

To convert to a standard form:

- If inequality constraint needs to change sign $\rightarrow$ negate
  e.g.

$$\boxed{\begin{array}{ll} \text{maximise} & 2x - 3y \\ \text{subject to} & x + y \geq 7 \\ & .. \end{array}} \rightarrow \boxed{\begin{array}{ll} \text{maximise} & 2x - 3y \\ \text{subject to} & -x - y \leq -7 \\ & .. \end{array}}$$

The Simplex Algorithm - Standard Form

e.g. convert the following linear program into standard form:

$$
\begin{array}{ll}
\text{minimise} & 2x_1 + 7x_2 + x_3 \\
\text{subject to} & x_1 - x_3 = 7 \\
 & 3x_1 + x_2 \geq 24 \\
 & x_2 \geq 0 \\
 & x_3 \leq 0
\end{array}
$$

The Simplex Algorithm - Slack Form

In addition to being in the standard form, the Simplex algorithm requires the linear program to be in the slack form.

$$
\begin{aligned}
z &= v + \sum_{j=1}^{n} c_j x_j \\
x_i &= b_i - \sum_{j=1}^{n} a_{ij} x_j \quad \text{for } i = 1, 2, .., m \\
& \quad x_i \geq 0 \qquad\qquad\quad \text{for } i = 1, 2, .., m
\end{aligned}
$$

The Simplex Algorithm - Slack Form

- For each inequality $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$
- Introduce a new variable s (called the **slack variable** because it measures the difference between the left and the right hand sides of the equation.)
- Rewrite the inequality $s = b_i - \sum_{j=1}^{n} a_{ij}x_j$
- Add a non-negativity constraint $s \geq 0$

The Simplex Algorithm - Slack Form

e.g. convert the following linear program from standard form to slack form:

$$
\begin{array}{ll}
\text{maximise} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & x_1 + x_2 - x_3 \leq 7 \\
 & -x_1 - x_2 + x_3 \leq -7 \\
 & x_1 - 2x_2 + 2x_3 \leq 4 \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

The Simplex Algorithm - Slack Form

- Step 1: add the slack variables
- Step 2: Replace the objective function value by z

$$
\boxed{
\begin{aligned}
z = \quad & 2x_1 - 3x_2 + 3x_3 \\
x_4 = \quad & 7 - x_1 - x_2 + x_3 \\
x_5 = \quad & -7 - +x_1 + x_2 - x_3 \\
x_6 = \quad & 4 - x_1 + 2x_2 - 2x_3 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
}
$$

The Simplex Algorithm - Slack Form

- Step 1: add the slack variables
- Step 2: Replace the objective function value by z

$$
\boxed{\begin{array}{rl}
z = & 2x_1 - 3x_2 + 3x_3 \\
x_4 = & 7 - x_1 - x_2 + x_3 \\
x_5 = & -7 - +x_1 + x_2 - x_3 \\
x_6 = & 4 - x_1 + 2x_2 - 2x_3 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}}
$$

- The variables on the left hand size of equalities are called **basic variables**
- The variables on the right hand size of equalities are called **nonbasic variables**

The Simplex Algorithm - Basic Solution

- A feasible solution can be found by setting all nonbasic variables (right-hand side variables) to 0
- This is a feasible solution and is a *vertex* in the convex hull because of the non-negativity constraint.
- For the example below, $\mathbf{x} = (0, 0, 0, 30, 24, 36)$ and $z = 0$
- This is referred to as a basic feasible solution

$$
\begin{array}{lllllllllll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & (2) \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 & (3) \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- At each iteration in the simplex algorithm, we select a non-basic variable x_i
- This chosen variable should have a positive coefficient in the objective function
- So that increasing its value would increase the objective function
- Let's choose x_1

$$
\boxed{\begin{array}{llllllllll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & (2) \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 & (3) \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 & (4)
\end{array}}
$$

The Simplex Algorithm - Iteration

- We try to increase x_1 as much as possible without increasing any non-negativity constraint

$$
\boxed{
\begin{array}{lcrcrcrcrl}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & (2) \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 & (3) \\
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\end{array}
}
$$

The Simplex Algorithm - Iteration

- We try to increase x_1 as much as possible without increasing any non-negativity constraint
  - In (2), x_1 could be set to 30 max
  - In (3), x_1 could be set to 12 max
  - In (4), x_1 could be set to 9 max

$$
\boxed{
\begin{array}{lllllllll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & (2) \\
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\end{array}
}
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The Simplex Algorithm - Iteration

- We try to increase x_1 as much as possible without increasing any non-negativity constraint
  - In (2), x_1 could be set to 30 max
  - In (3), x_1 could be set to 12 max
  - In (4), x_1 could be set to 9 max
- Equation (4) is the tightest constraint
- We therefore switch the roles of x_1 and x_6

$$
\boxed{
\begin{array}{lclclclcll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 & (2) \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 & (3) \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 & (4)
\end{array}
}
$$

The Simplex Algorithm - Iteration

- We re-arrange (4) so $x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$

The Simplex Algorithm - Iteration

- We re-arrange (4) so $x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$
- x_1 will be replaced by x_6 on the right-hand side of Equations 1-3

$$
\begin{array}{llllllllll}
z & = & & + & 3(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}) & + & x_2 & + & 2x_3 & (1) \\
x_4 & = & 30 & - & (9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}) & - & x_2 & - & 3x_3 & (2) \\
x_5 & = & 24 & - & 2(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}) & - & 2x_2 & - & 5x_3 & (3) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

$$
\begin{array}{llllllllll}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} & (1) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (2) \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} & (3) \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- After this operation the basic variables become (x_1, x_4 and x_5)

$$
\begin{array}{ccccccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} & (1) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (2) \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} & (3) \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- After this operation the basic variables become (x_1, x_4 and x_5)
- The solution just changes to be (9,0,0,21,6,0) and the objective function increases to 27

$$
\boxed{
\begin{array}{cccccccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} & (1) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (2) \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} & (3) \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} & (4)
\end{array}
}
$$

The Simplex Algorithm - Iteration

- After this operation the basic variables become (x_1, x_4 and x_5)
- The solution just changes to be (9,0,0,21,6,0) and the objective function increases to 27
- This too is a vertex in the convex hull

$$
\begin{array}{cccccccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} & (1) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (2) \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} & (3) \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- This operation is known as a **pivot**, which exchanges the positions of one nonbasic variable (called the **entering variable**) and one basic variable (called the **leaving variable**)
- Next we choose another entering variable (we can choose x_2 or x_3 and let's choose x_3)

$$
\begin{array}{llllllllll}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} & (1) \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} & (2) \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} & (3) \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- We try to increase x_3 as much as possible without increasing any non-negativity constraint

The Simplex Algorithm - Iteration

- We try to increase x_3 as much as possible without increasing any non-negativity constraint
  - In (2), x_3 could be set to 18 max
  - In (3), x_3 could be set to 8.4 max
  - In (4), x_3 could be set to 1.5 max

The Simplex Algorithm - Iteration

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  - In (2), x_3 could be set to 18 max
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  - In (2), x_3 could be set to 18 max
  - In (3), x_3 could be set to 8.4 max
  - In (4), x_3 could be set to 1.5 max
- Equation (4) is the tightest constraint
- We therefore switch the roles of x_3 and x_5
- $x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$

z	=	27	+	$\frac{x_2}{4}$	+	$\frac{x_3}{2}$	-	$\frac{3x_6}{4}$	(1)
x_1	=	9	-	$\frac{x_2}{4}$	-	$\frac{x_3}{2}$	-	$\frac{x_6}{4}$	(2)
x_3	=	$\frac{3}{2}$	-	$\frac{3x_2}{8}$	-	$\frac{x_5}{4}$	+	$\frac{x_6}{8}$	(3)
x_4	=	21	-	$\frac{3x_2}{4}$	-	$\frac{5x_3}{2}$	+	$\frac{x_6}{4}$	(4)

The Simplex Algorithm - Iteration

- The current solution is (33/4, 0, 3/2, 69/4, 0, 0) with the objective value 111/4

$$
\begin{array}{|cccccccccc|}
\hline
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} & (1) \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} & (2) \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} & (3) \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16} & (4) \\
\hline
\end{array}
$$

The Simplex Algorithm - Iteration

- The current solution is (33/4, 0, 3/2, 69/4, 0, 0) with the objective value 111/4
- Next we increase x_2 by substituting it with x_3

$$
\begin{array}{cccccccccl}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} & (1) \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} & (2) \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} & (3) \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16} & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- The current solution is (8,4, 0, 18, 0, 0) with the objective value 28

$$
\begin{array}{lllllllll}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} & (1) \\
x_1 & = & 8 & + & \frac{x_6}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} & (2) \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} & (3) \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- The current solution is (8,4, 0, 18, 0, 0) with the objective value 28
- In the original linear program $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, with the objective value = 28

$$\boxed{\begin{array}{llllllllll}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} & (1) \\
x_1 & = & 8 & + & \frac{x_6}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} & (2) \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} & (3) \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & & (4)
\end{array}}$$

The Simplex Algorithm - Iteration

- The current solution is (8,4, 0, 18, 0, 0) with the objective value 28
- In the original linear program $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, with the objective value = 28
- The slack variables measure how much slack remains within each inequality

$$
\begin{array}{ccccccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} & (1) \\
x_1 & = & 8 & + & \frac{x_6}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} & (2) \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} & (3) \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- The current solution is (8,4, 0, 18, 0, 0) with the objective value 28
- In the original linear program $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, with the objective value = 28
- The slack variables measure how much slack remains within each inequality
- All non-basic variables have a negative coefficient in the objective function

$$
\begin{array}{lclclclcll}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} & (1) \\
x_1 & = & 8 & + & \frac{x_6}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} & (2) \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} & (3) \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & & (4)
\end{array}
$$

The Simplex Algorithm - Iteration

- The current solution is (8,4, 0, 18, 0, 0) with the objective value 28
- In the original linear program $x_1 = 8$, $x_2 = 4$ and $x_3 = 0$, with the objective value = 28
- The slack variables measure how much slack remains within each inequality
- All non-basic variables have a negative coefficient in the objective function
- The vertex with the maximum value is reached — terminate

$$
\begin{array}{lllllllll}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} & (1) \\
x_1 & = & 8 & + & \frac{x_6}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} & (2) \\
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x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & & (4)
\end{array}
$$

The Simplex Algorithm

(N,B,A,b,c,v) = INITIALISE-SIMPLEX (A,b,c);

- convert to slack form (N: nonbasic variable, B: basic variable)

The Simplex Algorithm

(N,B,A,b,c,v) = INITIALISE-SIMPLEX (A,b,c);
while *some index* $j \in N$ *has* $c_j > 0$ **do**

end

- convert to slack form (N: nonbasic variable, B: basic variable)
- find a nonbasic variable that can increase the objective value

The Simplex Algorithm

(N,B,A,b,c,v) = INITIALISE-SIMPLEX (A,b,c);
while *some index* $j \in N$ *has* $c_j > 0$ **do**
  for *each* $i \in B$ **do**
    $\Delta_i = b_i / a_{ie}$;
  end

end

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  end
  choose $l \in B$ that minimises Δ_l;

end

- convert to slack form (N: nonbasic variable, B: basic variable)
- find a nonbasic variable that can increase the objective value
- find the tight basic variable

The Simplex Algorithm

(N,B,A,b,c,v) = INITIALISE-SIMPLEX (A,b,c);
while *some index* $j \in N$ *has* $c_j > 0$ **do**
  for *each* $i \in B$ **do**
    $\Delta_i = b_i / a_{ie}$;
  end
  choose $l \in B$ that minimises Δ_l;
  if $\Delta_l == \inf$ **then**
    return “unbounded”
  else
    (N,B,A,b,c,v) = PIVOT(N,B,A,b,c,v,l,e)
  end
end

- convert to slack form (N: nonbasic variable, B: basic variable)
- find a nonbasic variable that can increase the objective value
- find the tight basic variable
- replace the nonbasic with the basic variable

The Simplex Algorithm

```
(N,B,A,b,c,v) = INITIALISE-SIMPLEX (A,b,c);
while some index j ∈ N has c_j > 0 do
    for each i ∈ B do
        Δ_i = b_i / a_ie;
    end
    choose l ∈ B that minimises Δ_l;
    if Δ_l == inf then
        return "unbounded"
    else
        (N,B,A,b,c,v) = PIVOT(N,B,A,b,c,v,l,e)
    end
end
for i=1..n do
    if i ∈ B then
        x̄_i = b_i
    else
        x̄_i = 0
    end
end
return (x̄_1, x̄_2, .., x̄_n)
```

- convert to slack form (N: nonbasic variable, B: basic variable)
- find a nonbasic variable that can increase the objective value
- find the tight basic variable
- replace the nonbasic with the basic variable
- find the values of the original variables
- return the values of original variables

The Simplex Algorithm

Lemma

Given a linear program (A,b,c), suppose that the call to INITIALISE-SIMPLEX returns a slack form for which the basic solution is feasible, then if SIMPLEX returns a solution, it is a feasible solution to the linear program. If it returns "unbounded", the linear program is unbounded.

The Simplex Algorithm

Proof.

- Because each basic variable x_i is nonnegative, the basic solution sets x_i to b_i
- If $b_i \geq 0$ for all $i \in B$ then the basic solution is feasible.
- In the PIVOT operation, the slack variable is equivalent to the initial slack form
- After the PIVOT operation, $b_i \geq 0$ for each x_i in b_i
- The basic variable remains non-negative, and is thus feasible

$\square$

Example

Solve the following linear program using the Simplex Algorithm

$$\begin{array}{ll} \text{maximise} & 18x_1 + 12.5x_2 \\ \text{subject to} & x_1 + x_2 \leq 20 \\ & x_1 \leq 12 \\ & x_2 \leq 16 \\ & x_1, x_2 \geq 0 \end{array}$$

Integer Program

- An integer program is encountered when the optimum output is expected as integer values
- The book shipment example at the first few slides is an integer program
- This is because the output is not meaningful unless it's in integers

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- This is referred to as **linear program relaxation of an integer program**
- Even though for the solution was actually given as integers (200,0,200,600), solving a linear program cannot guarantee that the solution is in integers
- One way around this is to round numbers
- This can cause serious problems sometimes - more complex methods can be used

Example

A farmer owns a 200-acre farm and can plant any combination of two crops I and II. Crop I requires 1 man-day of labour and £10 of capital for each acre planted, while crop II requires 4 man-days of labour and £20 of capital for each acre planted. Crop I produces £40 of net revenue per acre and crop II produces £60. The farmer has £2200 of capital and 320 man-days of labour for the year. What is the optimal planting strategy?

Formulate this problem as a

- linear program
- linear program in matrix form
- linear program in standard form
- linear program in slack form - and solve it using the SIMPLEX algorithm

Further Reading

- Introduction to Algorithms
  T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 27 – Linear Programming