Course Selection under Social Network Effects

Zhenyu Hu, Jussi Keppo, Meixi Wu
NUS Business School & Institute of Operations Research and Analytics,
National University of Singapore, Singapore 119245, bizhuz@nus.edu.sg, keppo@nus.edu.sg, meixi@nus.edu.sg

We consider a course selection model that matches a continuum of students to a finite number of courses. Students' valuations for the courses are subject to their individual-level social network effects. We adopt a pseudo-market allocation mechanism for course selection that takes the students' reported valuations for the courses as input and then outputs the students' bid-price vector for each course and the corresponding allocations. We show the existence of equilibrium prices and allocations under any distribution of students' reported preferences for the courses and show that one such equilibrium can be solved by a linear program. Simulations suggest that the welfare is non-increasing in the strength of the network effects.

1. Allocation Mechanism

Consider an infinite number of students and C courses, denoted by C. The course capacities are expressed by a vector $q = [q_c]_{c \in C} \in \{q \in [0,1]^C : \sum_{c \in C} q_c = 1\}$. Every student submits a report of her perceived utility from the courses, denoted as $u \in \Theta = [0,1]^C$. Let the distribution of all students' reported utilities be F.

We define an allocation as a function $\pi : \Theta \to A$, where $A = [0,1]^C$constitutes the individual allocation outcome space. For any $u \in \Theta$ and $c \in C$, $\pi_c(u|F)$ represents the probability of a student of type θ being allocated to course c. For simplicity, we utilize the notation $\pi_c(u)$.

In our pseudo-market allocation mechanism, we distribute courses based on artificial prices, which are adjusted iteratively to mirror supply and demand dynamics. These prices facilitate informed decision-making and resource allocation. The price space is defined as $\hat{P} = [0, +\infty]^C$. To make the price space compact, we apply the arctangent function, which exhibits positive monotonicity, thus transforming the price space to $P = [0, \pi/2]^C$.

With the assumption that every student is allocated exactly one course, the unit demand is expressed as $\sum_{c\in C}\pi_c = 1$. The allocation simplex for any student is defined as $\Delta^C = \{\pi \in [0,1]^C : \sum_{c\in C}\pi_c = 1\}$.

Every student is allocated a budget, denoted as b, which remains uniform across all students. Thus, the budget correspondence is $B(p) = \{\pi \in [0,1]^C : p\cdot\pi \le b\}$. As the unit demand might be exceeded, for example, during high prices, an external option c_0 is introduced, which has unlimited supply and offers the least utility to every agent: $q_0 = +\infty$ and $u_0 < \min_{u\in[0,1]^C}\min_{c\in C} u_c$. This leads to the formation of the expanded allocation simplex Δ^{C+1}, and for ease of reference, we will denote Δ^{C+1} as Δ.

In the presence of a price vector $p \in P$ corresponding to the courses, every student's expected utility is maximized under the constraints of unit demand and budget, as shown by:

$$\Pi^u(p) \in \arg\max_{\pi}\left\{\sum_{c\in C}\pi_c u_c \middle| \sum_{c\in C}\pi_c = 1, \sum_{c\in C} p_c\pi_c \le b\right\}, \forall u\in\Theta. \tag{1}$$

In addition, the price must satisfy the capacity constraint:

$$\int_{u\in\Theta}\Pi_c^u(p)\,\mathrm{d}F \le q_c, \forall c\in C. \tag{2}$$

The following theorem establishes the existence of equilibrium prices and allocations for any reported utility distribution:

Theorem 1. ***(Existence of Equilibrium Allocation)*** *For any submitted utility distribution F, there exists equilibrium prices and allocations that satisfy (1) and (2).*

To validate this theorem, we establish the properties of individual and aggregate demand correspondences. Under the purview of Berge's Maximum Theorem, these correspondences are demonstrated to be non-empty, upper hemicontinuous, and convex valued. Subsequently, we introduce a price correspondence, which mirrors these properties, and construct an extended correspondence. This sets the stage for the application of Kakutani's Fixed Point Theorem, which forms the basis for asserting the existence of an equilibrium price and allocation in the original economy.

Since there could be multiple equilibria, we need to prescribe a price selection rule. We propose selecting the equilibrium price that maximizes the total welfare of all the students. We aim to select the price vector $p^* \in \mathcal{P}$ from the set of all equilibrium price vectors $\mathcal{P}$ that satisfy the conditions outlined in Theorem 1. This p^* optimizes total student welfare, as expressed in the following equation:

$$p^* = \arg\max_{p \in \mathcal{P}} \int_{u \in \Theta} \sum_{c \in C} \Pi_c^u(p) u_c \mathrm{d}F \tag{3}$$

The individual utility maximization problem (1) can be reformulated to facilitate our analysis:

$$\sum_{c \in C} \Pi_c^u u_c \geq \max_{\pi \in B(p) \cap \Delta} \left\{ \sum_{c \in C} \pi_c u_c \right\}, \forall u \in [0,1]^C. \tag{4}$$

Transitioning to the dual formulation, we have:

$$\exists \mu^u \geq 0 \text{ s.t.} \sum_{c \in C} \Pi_c^u u_c \geq u_c - p_c \mu^u + b\mu^u. \tag{5}$$

To linearize the constraint, let $z_c^u = p_c \mu^u, \forall u \in \Theta, c \in C$. The stipulation of $p \in [0, \pi/2]^C$ corresponds to $z^u \in [0, \mu\pi/2]^C, \forall u \in \Theta$. With $u_c \in [0,1], \forall c \in C$, this constraint simplifies to $z_c^u \leq \mu\pi/2, \forall u \in \Theta, c \in C$.

We can now frame the pseudo-market allocation mechanism as an integrated linear program, as shown in 6.

$$\begin{aligned}
&\max_{\Pi, z, \mu} \int_{u \in \Theta} \sum_{c \in C} \Pi_c^u u_c \mathrm{d}F \\
&\text{s.t.} \\
&\qquad \sum_{c \in C} \Pi_c^u u_c \geq u_c - z_c^u + b\mu^u, \quad && \forall u \in \Theta, c \in C \\
&\qquad \int_{u \in \Theta} \Pi_c^u \mathrm{d}F \leq q_c, && \forall c \in C \\
&\qquad \sum_{c \in C} \Pi_c^u = 1, && \forall u \in \Theta \\
&\qquad \Pi_c^u \geq 0, && \forall u \in \Theta, c \in C \\
&\qquad z_c^u \leq \mu^u \pi/2, && \forall u \in \Theta, c \in C \\
&\qquad z^u \geq u, && \forall u \in \Theta \\
&\qquad \mu \geq \mathbf{0}.
\end{aligned} \tag{6}$$

In this linear program, parameters such as each student's budget b, the space of reported utilities $\Theta = [0,1]^C$, the distribution F of reported utilities, and the capacities $[q_c]_{c\in C}$ are considered exogenous. Let $\mathcal{A}$ represent the set of all possible allocation functions that map each point $u \in \Theta$ to a probability vector in the allocation outcome space Δ^C. The optimal solutions to this problem are denoted as $\Pi(F)$, where $\Pi : \Delta(\Theta) \rightrightarrows \mathcal{A}$ is a correspondence.

We proceed to establish two lemmas, which demonstrate the properties of the objective function and the constraint correspondence.

The objective function $\phi : \mathcal{A} \times \Delta(\Theta) \to [0, \sum_{c\in C} q_c]$ is defined as

$$\phi(\Pi, F) = \int_{u\in\Theta} \sum_{c\in C} \Pi_c^u u_c \mathrm{d}F \tag{7}$$

We define a correspondence $Q : \Delta(\Theta) \rightrightarrows \mathcal{A}$ such that for all $F \in \Delta(\Theta)$, $Q(F)$ denotes the set of all feasible allocations given the distribution of reports.

We further define:

$$\Pi(F) \in \arg\max\{\phi(\Pi, F) : \Pi \in Q(F)\}, \forall F \in \Delta(\Theta) \tag{8}$$

and

$$\phi^*(F) := \max\{\phi(\Pi, F) : \Pi \in Q(F)\}, \forall F \in \Delta(\Theta) \tag{9}$$

Two key lemmas, Lemma 1 and Lemma 2, provide properties of the objective function and constraint correspondence, respectively:

LEMMA 1. *$\phi : \mathcal{A} \times \Delta(\Theta) \to [0, \sum_{c\in C} q_c]$ is continuous on $\mathcal{A} \times \Delta(\Theta)$.*

LEMMA 2. *$Q : \Delta(\Theta) \rightrightarrows \mathcal{A}$ is non-empty, compact-valued, and continuous at any $F \in \Delta(\Theta)$.*

In conclusion, we can state Theorem 2 to demonstrate the characteristics of Π and ϕ^*:

THEOREM 2. *$\Pi : \Delta(\Theta) \rightrightarrows \mathcal{A}$ is compact-valued, convex-valued, upper hemicontinuous, and has a closed graph. Moreover, $\phi^* : \Delta(\Theta) \to [0, \sum_{c\in C} q_c]$ is continuous at F.*

2. Reporting Game with Herding Behavior

Consider that each student's unique preferences for classes C follow a continuous measure F_t over preference space $\Theta = [0,1]^C$.

In this setting, a potential preference vector is denoted as $\theta = [\theta_c]_{c\in C} \in \Theta'$, where each component stands for the preference for a class. Note that our valuation space is multi-dimensional. For any measurable subset $A \subset \Theta$, $F_t(A)$ signifies the mass of students having preferences in A. The total measure $F_t(\Theta) = 1$. The distribution F_t is common knowledge, while the exact preferences of each student are private knowledge.

A student's utility comprises two factors: idiosyncratic preferences and the proportion of students assigned to the same group. Given a belief that the distribution of reported utilities of students is characterized by the cumulative distribution function (CDF) F, a student of type θ has the expected utility from reporting u as

$$\mathbb{E}U^\theta(u|F) = \sum_{c\in C} \Pi_c^u(F) \left((1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';F)\mathrm{d}F \right), \forall u \in \Theta, \tag{10}$$

where $\gamma \in [0,1]$ measures the network effects' strength. Therefore, the optimal strategy for a student is to report

$$u^\theta(F) \in \arg\max_{u\in\Theta} \mathbb{E}U^\theta(u|F). \tag{11}$$

Subsequently, we define the best response correspondence $\Psi : \Delta(\Theta) \rightrightarrows \Delta(\Theta)$ as

$$\Psi(F) := \left\{ F_{BR} \in \Delta(\Theta) : F_{BR}(A) = \int_{\theta\in\Theta} 1\left\{u^\theta(F) \in A\right\} \mathrm{d}F_t, \forall A \in \mathcal{B}(\Theta), \hat{\Pi} \in \Pi(F) \right\}. \tag{12}$$

Here, $\mathcal{B}(\Theta)$ signifies the Borel sigma-algebra on Θ, and 1 represents the indicator function. $\Psi(F) \in \Delta(\Theta)$ provides an updated report distribution when each student opts for their best response, given that the original distribution is $F \in \Delta(\Theta)$. A distribution F is identified as a Bayesian Nash equilibrium if $F \in \Psi(F)$.

Theorem 3. *There always exists a selection of equilibrium allocation in the optimal solutions to the linear program 6 such that there exists a Bayesian Nash equilibrium for a student of type θ to report their utility as*

$$u(\theta) = \left[(1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';F)\,dF \right]_{c\in C}, \forall \theta \in \Theta, \tag{13}$$

such that $u(\theta)$ follows the distribution F.

3. Reporting Game with Multiple Social Groups

Envision a student network that can be represented as a finite set $G = \{g_1, g_2, \cdots, g_{|G|}\}$ of groups, with a student mass of n^g in each group $g \in G$. This network structure is considered public information. For each student in group $g \in G$, we assume idiosyncratic preferences for various classes are distributed following a continuous measure F_t^g over the preference space $\Theta = [0,1]^C$.

In this setting, a potential preference vector is denoted as $\theta = [\theta_c]_{c\in C} \in \Theta'$, where each component stands for the preference for a class. Note that our valuation space is multi-dimensional. For any measurable subset $A \subset \Theta$, $F_t^g(A)$ signifies the mass of students having preferences in A. Since the total mass for group g is n_g, the total measure $F_t^g(\Theta) = n^g$. The distributions F_t^g's are common knowledge, while the exact preferences of each student are private knowledge.

A student's utility comprises two factors: idiosyncratic preferences and the proportion of students assigned to the same group. Given a belief that the distribution of reported utilities of students is characterized by the cumulative distribution function (CDF) F, a student of type θ in group g has the expected utility from reporting u as

$$\mathbb{E}U^{\theta,g}(u|[F^g]_{g\in G}) = \sum_{c\in C} \Pi_c^u(\bar{F}) \left((1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';\bar{F}) \mathrm{d}F^g \right), \forall u \in \Theta, \tag{14}$$

where $\gamma \in [0,1]$ measures the network effects' strength, and

$$\bar{F} = \sum_{g\in G} n^g F^g \tag{15}$$

represents the distribution of reported utilities of all students. Consequently, the optimal strategy for a student is to report

$$u^{\theta,g}([F^g]_{g\in G}) \in \arg\max_{u\in\Theta} \mathbb{E}U^{\theta,g}(u|[F^g]_{g\in G}). \tag{16}$$

Subsequently, we define the best response correspondence $\Psi : \Delta(\Theta) \rightrightarrows \Delta(\Theta)$ as

$$\Psi(F) := \left\{ F_{BR} \in \Delta(\Theta) : F_{BR}(A) = \int_{\theta\in\Theta} 1\left\{u^\theta(F) \in A\right\} \mathrm{d}F_t, \forall A \in \mathcal{B}(\Theta), \hat{\Pi} \in \Pi(F) \right\}. \tag{17}$$

Here, $\mathcal{B}(\Theta)$ signifies the Borel sigma-algebra on Θ, and 1 represents the indicator function. $\Psi(F) \in \Delta(\Theta)$ provides an updated report distribution when each student opts for their best response, given that the original distribution is $F \in \Delta(\Theta)$. A distribution F is identified as a Bayesian Nash equilibrium if $F \in \Psi(F)$.

Subsequently, we define the best response correspondence $\Psi : \Delta^G(\Theta) \rightrightarrows \Delta^G(\Theta)$ as

$$\Psi\left([F^g]_{g\in G}\right) := \left\{ [F^g_{BR}]_{g\in G} \in \Delta^G(\Theta) : F^g_{BR}(A) = \int_{\theta\in\Theta} 1\left\{u^{\theta,g}([F^g]_{g\in G}) \in A\right\} \mathrm{d}F^g_t, \forall A \in \mathcal{B}(\Theta), \hat{\Pi} \in \Pi(\bar{F}) \right\}. \tag{18}$$

Here, $\mathcal{B}(\Theta)$ signifies the Borel sigma-algebra on Θ, and 1 is the indicator function. $\Psi\left([F^g]_{g\in G}\right) \in \Delta^G(\Theta)$ provides an updated report distribution when each student opts for their best response, given that the original distribution profile is $[F^g]_{g\in G} \in \Delta^G(\Theta)$. A distribution profile $[F^g]_{g\in G}$ is identified as a Bayesian Nash equilibrium if $[F^g]_{g\in G} \in \Psi\left([F^g]_{g\in G}\right)$.

Theorem 4. *There always exists a selection of equilibrium allocation in the optimal solutions to the linear program 6 such that there exists a Bayesian Nash equilibrium for a student type θ in group g to report their utility as*

$$u^g(\theta) = \left[(1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u'; \bar{F})\, dF^g\right]_{c\in C}, \forall \theta \in \Theta, g \in G, \tag{19}$$

such that $u^g(\theta)$ follow F^g for all $g \in G$.

4. Comparative Statics

In this section, we delve deeper into the dynamics of our model and how the interplay of network effects and individual incentives influence the Bayesian Nash equilibrium. We consider a Bayesian Nash equilibrium under a network effect level denoted by $\gamma \in [0,1]$, and let $[F^g]_{g\in G}$ be the distribution profile of reported utilities.

The allocation probabilities corresponding to a given report $u \in \Theta$ and the average distribution of reports $\bar{F}$, as defined in equation 15, are represented as $\Pi\left(u; \bar{F}\right)$. For any student belonging to group g and with a preference of θ, the true utility derived from the allocation mechanism is given by $W^g(\theta) := \sum_{c\in C} \theta_c \Pi_c \left(u^g(\theta); \bar{F}\right)$,where $u^g(\theta)$ is defined in equation 19. We can then compute the aggregate true total welfare, denoted by W, across all students as $W := \sum_{g\in G} \int_{\theta\in\Theta} W^g(\theta)\mathrm{d}F_t^g$.

We want to examine how the degree of network effects, as measured by γ, influences the equilibrium outcomes and the true total welfare.

THEOREM 5. *Assuming the presence of any positive network effect ($\gamma > 0$), the true total societal welfare in the Bayesian Nash equilibrium cannot be higher than the case with no network effects ($\gamma = 0$).*

The above theorem offers a somewhat surprising conclusion. While one might initially presume that network effects, by contributing additional value, would enhance the utility of students, the result illustrates a nuanced outcome due to strategic behavior in reporting preferences. In an effort to maximize their individual utility, students may report preferences that deviate from their true ones. This strategic misreporting can distort the allocation process, resulting in an aggregate allocation that might not be optimal.

This decrease in total welfare underscores the complexity of systems involving network effects and individual strategic behavior. It highlights the critical importance of understanding these dynamics in the design of allocation mechanisms, where careful consideration must be given to the potential trade-offs between network effects and the truthful representation of preferences. The insights from this theorem are instrumental not only for academic understanding but also for practical policy making, helping guide the creation of strategies that balance individual incentives and overall societal welfare.

Define the network effect term as $\lambda_c^g(\gamma) := \int_{u\in\Theta} \Pi_c(u; \bar{F})\mathrm{d}F^g$, where F is the equilibrium distribution of reported utilities.

When there are no capacity constraints, the allocation probabilities are not influenced by the reported utilities of other students. Therefore, the allocation probabilities can be written as $\Pi_c(u;\bar{F}) = \Pi_c(u), \forall c \in C$, where $\Pi_c(u)$ is the allocation probability for class c. $W^g(\theta) := \sum_{c\in C} \theta_c \Pi_c\left(u^g(\theta)\right)$, where $u^g(\theta)$ is defined in equation 19. Moreover, all students are assigned to their most preferred class. Therefore, $W^g(\theta) = \theta_{c^*}$, where $c^* = \arg\max_{c\in C} u_c^g(\theta)$.

Under a given network effect level denoted as γ, the distribution of reported utility is represented as F, the allocation is expressed as $\Pi \equiv [\Pi(u;F)]_{u\in\Theta}$, and the resulting welfare is $W = \sum_{g\in G} \int_{\theta\in\Theta} \sum_{c\in C} \theta_c \Pi_c\left(u^g(\theta);\bar{F}\right) \mathrm{d}F_t^g$. This scenario serves as the initial condition for our analysis.

To examine the dynamic adjustments in equilibrium outcomes as the network effect level transitions from γ to γ', we draw inspiration from the "tatonnement" price adjustment process introduced in Walrasian models of market equilibrium.

As we incrementally raise the network effect level from γ to γ', we commence with an initial point represented by the pair $(F, \Pi(F))$. At this stage, we compute the adjustments of reported utilities denoted as $u_0^g(\theta)$ for each student in group g and with preferences θ. These adjustments are determined by the expression $u_0^g(\theta) = [(1-\gamma)\theta_c + \gamma\lambda_c^g(\gamma)]_{c\in C}$, where $\lambda_c^g(\gamma)$ characterizes the specific adjustment function for class c under network effect level γ. The resulting distribution of these adjusted utilities is denoted as F_0.

Should the disparity between the distributions F_0 and F surpass a predefined threshold, denoted as ϵ, we proceed by employing the allocation outcome from the preceding step. Subsequently, we continue to compute the updated reported utilities, expressed as $u_1^g(\theta)$, which take into account the distribution F_0. That is, we compute $u_1^g(\theta) = \left[(1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';\bar{F}_0)\mathrm{d}F_0^g\right]_{c\in C}, \forall\theta\in\Theta, g\in G$. The distribution of these updated utilities is designated as F_1.

We iterate through these steps until the difference between distributions F_n and F_{n+1} falls below or equals the threshold ϵ. At this point, we determine the distribution F' to be equal to F_n. The corresponding allocation is represented as $\Pi' \equiv [\Pi(u;F')]_{u\in\Theta}$, and the societal welfare under this network effect level γ' is quantified as $W' = \sum_{g\in G} \int_{\theta\in\Theta} \sum_{c\in C} \theta_c \Pi_c\left(u^g(\theta);\bar{F}'\right) \mathrm{d}F'^g_t$. This iterative approach allows us to gain insights into the dynamic adjustments in equilibrium outcomes as network effects evolve.

Lemma 3. *When there are two classes with no capacity constraints, then as network effect level γ increases, for any group $g \in G$, one class becomes more popular and the other class becomes less popular.*

Let $C = \{a, b\}$, assume $\lambda_a^g(0) < \lambda_b^g(0)$. Then $\lambda_a^g(\gamma)$ is increasing in γ and $\lambda_b^g(\gamma)$ is decreasing in γ. Moreover, $\lambda_a^g(1) = 0$ and $\lambda_b^g(1) = 0$.

Lemma 4. *When there are two classes with no capacity constraints, then as network effect level γ increases, the true total welfare decreases.*

Multiple equilibria: If for any γ value, we use the allocation at $\gamma = 0$ as the starting point, then the resulting equilibrium reported utility profiles and allocations can be different. That is, given a fixed $\gamma \in (0,1)$, there could be multiple equilibria. This may violate the monotonicity of the true total welfare with respect to γ. There is an example.

5. Model Identifiability

Now assume that the true preferences θ_c is normally distributed: $\theta_c \sim \mathcal{N}(\mu_c, \sigma_c^2), \forall c \in C$. A principal can observe the distribution profile $[F^g]_{g \in G}$ of reported preferences. In addition, we assume that the allocation mechanism $\Pi(\cdot, \bar{F})$ in equilibrium is also observable. This section describes the extent to which the model parameters, including the network effect level γ, can be estimated from the observable information.

Based on the distribution profile $[F^g]_{g \in G}$ of reported preferences in equilibrium, we have for each $c \in C, g \in G$,

$$\hat{\mu}_c^g := \mathbb{E}\left[u_c^g(\theta)\right] = (1-\gamma)\mu_c + \gamma N_c^g, \tag{20}$$

and

$$\hat{\sigma}_c^{g\,2} := \mathrm{Var}\left[u_c^g(\theta)\right] = (1-\gamma)^2 \sigma_c^2. \tag{21}$$

Our structural model thus exhibits a quadratic relationship between the equilibrium mean and variance:

$$\hat{\sigma}_c^{g\,2} = \frac{\sigma_c^2}{\mu_c^2}\left(\hat{\mu}_c^g - \gamma N_c^g\right)^2, \forall c \in C, g \in G. \tag{22}$$

In practice, however, perfect quadratic relationships between the means and variances of reported preferences are rare. Hence, we include a noise term in the model to account for deviations and unmeasured factors. Specifically, we denote the mean and variance of the reported preferences in equilibrium as m_c^g and v_c^g, respectively. In estimating model parameters, we assume for all $c \in C, g \in G$:

$$\log\left(v_c^g\right) = \log\left[\frac{\sigma_c^2}{\mu_c^2}\left(m_c^g - \gamma N_c^g\right)^2\right] + \epsilon_c^g, \tag{23}$$

where each error term $\epsilon_c^g \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right)$ is independent of all other model variables.

We aim to estimate model parameters $\Xi = \left(\gamma, \mu_c, \sigma_c, \sigma_\epsilon\right)_{c\in C}$ given the distribution profile of reported preferences and allocation. We define $N_c^g := \int_{u'\in\Theta} \Pi_c(u'; \bar{F}) \mathrm{d}F^g$ as the mass of students in group g allocated to class c, observable through $[F^g]_{g\in G}$ and $\Pi(\cdot, \bar{F})$. Identification of the network effect level γ depends on the variance in the student population size across distinct groups allocated to different classes. Specifically, γ is identifiable when there exists at least one class, c, and two distinct groups, $g, h \in G$, such that $N_c^g \neq N_c^h$ is satisfied. Once γ is identified, the observable mean and variance of reported preferences within each class and group facilitate the identification of the mean μ_c and variance σ_c^2 of true preferences associated with each class c.

However, when there are at least three groups with different student population sizes allocated to distinct classes, the network effect level γ becomes overidentified. This overidentification offers greater precision and efficiency in estimates but also brings the risk of potential bias if invalid assumptions are made. To assess the validity of the overidentification, we can use statistical tests such as the Sargan test, the Basmann test, and the J-test, which check whether the conditions are valid and hence the variables are correctly specified.

THEOREM 6. 1. *If there is only one group in total, or if N_c^g is the same for all $g \in G$, then Ξ is not identifiable.*

2. *If there exists a class c and at least two groups $g, h \in G$ such that $N_c^g \neq N_c^h$, then γ is identifiable. Furthermore, if $\gamma \neq 1$, then the full model Ξ is identifiable.*

3. *If there exists a class c and at least three groups $g, h, k \in G$ such that $N_c^i \neq N_c^j, \forall i \neq j$ and $i, j \in \{g, h, k\}$, then γ is overidentified.*

Proofs

EC.1. Proof

Proof of Theorem **??** Define the individual demand correspondence $\psi^\theta : P \to \Delta$ such that a student who reports type θ is perceived to have a demand of

$$\psi^\theta(p) = \arg\max_{\pi \in B(p)\cap\Delta} \sum_{c\in C} \pi_c \theta_c. \tag{EC.1}$$

Note that the set $B(p)\cap\Delta$ is nonempty and compact for all $p \in P$. By Berge's Theorem, $\psi^\theta(p)$ is non-empty, upper hemicontinuous, and convex valued for all $\theta \in \Theta$.

Therefore, the aggregate demand correspondence can be written as $\Psi^D : P \to \Delta$ and

$$\Psi^D(p) = \int_\Theta \psi^\theta(p)\mathrm{d}F. \tag{EC.2}$$

As an integral of the individual demand correspondence, this is also non-empty, upper hemicontinuous, and convex valued for all $\theta \in \Theta$. Given the aggregate demand $\Pi = [\Pi_c]_{c\in C}$, the price is updated according to $\Psi^P : \Delta \to P$,

$$\Psi^P(\Pi) = \arg\max_{p\in P} p + (\Pi - q). \tag{EC.3}$$

This price correspondence is non-empty, upper hemicontinuous, and convex valued for all $\Pi \in A$. Now let us define $\Psi^* : P \times \Delta \to P \times \Delta$ by

$$\Psi^*(p,\Pi) = (\Psi^P(\Pi), \Psi^D(p)) \tag{EC.4}$$

Ψ^* is also non-empty, upper hemicontinuous, and convex valued because it is a product of the correspondences with these properties. Therefore, we can apply Kakutani's theorem which establishes a fixed point $(p^*,\Pi^*) \in \Psi^*(p^*,\Pi^*)$.

In equilibrium, if there is a positive demand for the artificial outside option c_0, then there would be at least one class $c \in C$ that has unfilled seats: $\Pi_c^* \le q_c$. This implies that $p_c^* = 0$. Then for any student who is allocated to this class with positive probability can afford a positive probability share of being allocated to class c and be strictly better off. Therefore, the equilibrium (p^*,Π^*) is an equilibrium price and allocation for the original economy with Δ^C. □

Proof of Lemma 1 Define $\|F\| = \sup_{u\in\Theta}\{|F(u)|\}$, $\|\Pi\| = \sum_{c\in C}\sup_{u\in\Theta}\{|\Pi_c(u)|\}$, $\|(\Pi, F)\| \equiv \sum_{c\in C}\sup_{u\in\Theta}\{|\Pi(u)|\} + \sup_{u\in\Theta}\{|F(u)|\}$.

$$\lim_{n\to\infty}\sum_{c\in C}\sup_{u\in\Theta}\{|\Pi_{n,c}(u)u_c - \Pi_c(u)u_c|\} = 0. \tag{EC.5}$$

Take any $(\Pi, F)\in\mathcal{A}\times\Delta(\Theta)$ and any sequence $(\Pi_n, F_n)\in(\mathcal{A}\times\Delta(\Theta))^\infty$ such that $(\Pi_n, F_n)\to(\Pi, F)$, we want to show: $\phi(\Pi_n, F_n)\to\phi(\Pi, F)$. That is,

$$\lim_{n\to\infty}\left|\int_{u\in\Theta}\sum_{c\in C}\Pi_{n,c}(u)u_c \mathrm{d}F_n(u) - \int_{u\in\Theta}\sum_{c\in C}\Pi_c(u)u_c\mathrm{d}F(u)\right| = 0 \tag{EC.6}$$

Since the sequence (Π_n, F_n) converges uniformly to (Π, F), it also converges pointwise to (Π, F). Moreover, we have $\Pi_{n,c}(u)\geq 0$ for all $c\in C, u\in\Theta$. Since $\sum_{c\in C}\Pi_{n,c}(u)u_c\leq\sum_{c\in C}u_c$, by the dominated convergence theorem, we have for all $\tilde{F}\in\Delta(\Theta)$:

$$\lim_{n\to\infty}\int_{u\in\Theta}\sum_{c\in C}\Pi_{n,c}(u)u_c\mathrm{d}\tilde{F}(u) = \int_{u\in\Theta}\sum_{c\in C}\Pi_c(u)u_c\mathrm{d}\tilde{F}(u). \tag{EC.7}$$

This implies that $\lim_{n\to\infty}|\phi(\Pi_n, F_n) - \phi(\Pi, F_n)| = 0$. Using the triangle inequality, we can obtain:

$$\lim_{n\to\infty}|\phi(\Pi_n, F_n) - \phi(\Pi, F)| \tag{EC.8}$$

$$\leq \lim_{n\to\infty}|\phi(\Pi_n, F_n) - \phi(\Pi, F_n)| + \lim_{n\to\infty}|\phi(\Pi, F_n) - \phi(\Pi, F)| \tag{EC.9}$$

$$= 0 \tag{EC.10}$$

Thus, $\phi(\Pi, F)$ is continuous with respect to both Π and F. $\square$

Proof of Lemma 2 Let us focus on the only constraint that involves f, and define the correspondence $\tilde{Q}:\Delta(\Theta)\rightrightarrows\mathcal{A}$ as

$$\tilde{Q}(F) = \left\{\Pi\in\mathcal{A}: \int_{u\in\Theta}\Pi_c(u)\mathrm{d}F\leq q_c, \forall c\in C\right\}. \tag{EC.11}$$

Take an arbitrary $F\in\Delta(\Theta)$ and sequences $(F_n)\in(\Delta(\Theta))^\infty$ and $(\Pi_n)\in(\mathcal{A})^\infty$ such that $F_n\to F$ and $\Pi_n\in\tilde{Q}(F_n)$ for each n. Since $(\Pi_n)\in\mathcal{A}$ for each n, while $\mathcal{A}$ is a closed and bounded set. By the Heine-Borel Theorem and the Bolzano–Weierstrass theorem, therefore, there exists a subsequence (Π_{n_k})

that converges to some $\Pi \in \mathcal{A}$. But then, by a straightforward continuity argument, $\int_{u\in\Theta} \Pi_c(u)\mathrm{d}F = \lim \int_{u\in\Theta} \Pi_{n_k,c}(u)\mathrm{d}F_{n_k} \le q_c$, that is, $\Pi \in \tilde{Q}(F)$. Since F was arbitrarily chosen in $\Delta(\Theta)$, we may conclude that $\tilde{Q}$ is upper hemicontinuous.

Now let us show that $\tilde{Q}$ is lower hemicontinuous. Take an arbitrary $F \in \Delta(\Theta)$ and any open subset O of $\mathcal{A}$ with $\tilde{Q}(F)\cap O \neq \emptyset$. To derive a contradiction, suppose that for every $n \in \mathbb{N}$, (however large), there exists an F_n within $\frac{1}{n}$-neighborhood of F such that $\tilde{Q}(F_n)\cap O = \emptyset$. Now pick any $\Pi \in \tilde{Q}(F)\cap O$. Since O is open in $\mathcal{A}$, we have $\lambda\Pi \in \tilde{Q}(F)\cap O$ for $\lambda \in (0,1)$ close enough to 1. But, since $F_n \to F$, a straightforward continuity argument yields $\lambda \int_{u\in\Theta} \Pi_{n,c}(u)\mathrm{d}F \le q_c, \forall c \in C$ for n large enough. Then, for any such n, we have $\lambda\Pi \in \tilde{Q}(F_n)$, that is, $\tilde{Q}(F_n)\cap O \neq \emptyset$, a contradiction.

Since $\tilde{Q}$ is both upper hemicontinuous and lower hemicontinuous at F, it is continuous at F. Therefore, Q is continuous as well. □

Proof of Theorem 2 Consider the linear program (6) with the optimal solution $\Pi(F_n)$ for each F_n. Since the linear program is feasible, its solution set is nonempty and compact. By the lemmas 1 and 2, we can apply Berge's Maximum Theorem to conclude that Π is compact-valued, upper hemicontinuous, and closed at F, and ϕ^* is continuous at F. □

Proof of Theorem 3 We first show that given all other students reporting using this strategy, then each student is in the best interest to use the same strategy. Then we prove that there exist distributions F such that the reported u^θ in Bayesian Nash Equilibrium follows F.

If all other students of type $\theta' \in \Theta$ report their utility as

$$u_c^{\theta'}(F) = (1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';F)\mathrm{d}F, \tag{EC.12}$$

then we want to show that a student of type θ has the best response strategy to report $[u_c^\theta(F)]_{c\in C}$.

Let $u_c^* = (1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u';F)\mathrm{d}F, \forall c \in C$, then we can write

$$\mathbb{E}U^\theta(u|F) = \sum_{c\in C} \Pi_c^u(F)u^*. \tag{EC.13}$$

Given all other students' strategy fixed, (6) is reduced to

$$\Pi(u) \in \operatorname*{arg\,max}_{\pi\in B(p)\cap\Delta} \sum_{c\in C} \pi_c u_c, \tag{EC.14}$$

since the integrals that are taken over the continuous measure cannot be affected by an individual u. That is

$$\sum_{c\in C}\Pi_c(u)u_c \geq \max_{\pi\in B(p)\cap\Delta}\left\{\sum_{c\in C}\pi_c u_c\right\}. \tag{EC.15}$$

Specifically, we have

$$\sum_{c\in C}\Pi_c(u)u_c \geq \sum_{c\in C}\Pi_c(u')u_c, \forall u'\in\Theta. \tag{EC.16}$$

Therefore,

$$\mathbb{E}U^\theta(u^*|F) \geq \mathbb{E}U^\theta(u|F), \forall u\in\Theta. \tag{EC.17}$$

Now we establish the best response correspondence $\Psi:\Delta(\Theta)\rightrightarrows\Delta(\Theta)$ as

$$\Psi\left(F\right) := \left\{F_{BR}\in\Delta(\Theta): F_{BR}(A) = \int_{\theta\in\Theta} 1\left\{\psi\left(\theta;F,\hat{\Pi}\right)\in A\right\}\mathrm{d}F_t, \forall A\in\mathcal{B}(\Theta), \hat{\Pi}\in\Pi(F)\right\}, \tag{EC.18}$$

where $\psi\left(\theta;F,\hat{\Pi}\right) = \left[(1-\gamma)\theta_c + \gamma\int_{u'\in\Theta}\hat{\Pi}_c(u')\mathrm{d}F\right]_{c\in C}$, $\mathcal{B}(\Theta)$ represents the Borel sigma-algebra on Θ, and 1 is the indicator function. $\Psi\left(F\right)\in\Delta(\Theta)$ describes the updated report distribution when each student chooses their best response, given that the original distribution is $F\in\Delta(\Theta)$. A distribution F establishes a Bayesian Nash equilibrium if $F\in\Psi\left(F\right)$.

Take any $F\in\Delta(\Theta)$, then for any $F_1,F_2\in\Psi\left(F\right)$, we can find $\hat{\Pi}_1,\hat{\Pi}_2\in\Pi(F)$ such that $F_1(A) = \int_{\theta\in\Theta} 1\left\{\psi\left(\theta;F,\hat{\Pi}_1\right)\in A\right\}\mathrm{d}F_t$ and $F_2(A) = \int_{\theta\in\Theta} 1\left\{\psi\left(\theta;F,\hat{\Pi}_2\right)\in A\right\}\mathrm{d}F_t, \forall A\in\mathcal{B}(\Theta)$. For any $\lambda\in[0,1]$, define $F_3(A) := \int_{\theta\in\Theta} 1\left\{\psi\left(\theta;F,\hat{\Pi}_3\right)\in A\right\}\mathrm{d}F_t, \forall A\in\mathcal{B}(\Theta)$. Since F_t is continuous and monotonically increasing, by Intermediate Value Theorem, there exists a $\lambda^*\in[0,1]$ and $\hat{\Pi}_3(\theta) := \lambda^*\hat{\Pi}_1(\theta) + (1-\lambda^*)\hat{\Pi}_2(\theta)$, which is another optimal solution to 6 as $\hat{\Pi}_1,\hat{\Pi}_2\in\Pi(F)$, such that

$$F_3(A) = \int_{\theta\in\Theta}\lambda 1\left\{\psi\left(\theta;F,\hat{\Pi}_1\right)\in A\right\} + (1-\lambda)1\left\{\psi\left(\theta;F,\hat{\Pi}_2\right)\in A\right\}\mathrm{d}F_t, \forall A\in\mathcal{B}(\Theta). \tag{EC.19}$$

That is, $F_3\in\Psi\left(F\right)$. Therefore, $\Psi\left(F\right)$ is convex for each $F\in\Delta(\Theta)$, and $\Psi:\Delta(\Theta)\rightrightarrows\Delta(\Theta)$ is convex-valued.

The closed graph property of Π implies that take any $F\in\Delta(\Theta)$, then for any convergent sequences $(F_n)\in\Delta^\infty(\Theta)$ and $(\mathbf{\Pi}_n)\in\mathcal{A}^\infty$ with $F_n\to F$ and $\mathbf{\Pi}_n\to\mathbf{\Pi}$, we have $\mathbf{\Pi}\in\Pi(F)$ whenever $\mathbf{\Pi}_n\in\Pi(F_n)$ for each $n=1,2,\cdots$.

Take any $(F_n) \in \Delta^\infty(\Theta)$ and $(Y_n) \in \Delta^\infty(\Theta)$ such that $F_n \to F$, $Y_n \in \Psi(F_n)$ for each n, and $Y_n \to Y$. Whenever $Y_n \in \Psi(F_n)$, we have $\mathbf{\Pi}_n \in \Pi(F_n)$ such that $Y_n(A) = \int_{\theta\in\Theta} 1\{\psi(\theta; F_n, \mathbf{\Pi}_n) \in A\} \mathrm{d}F_t, \forall A \in \mathcal{B}(\Theta)$. Since Π is upper hemicontinuous, there exists a subsequence of $(\mathbf{\Pi}_n)$ that converges to a point in $\Pi(F)$. Let us denote that point as $\mathbf{\Pi}$. Therefore, there exists a subsequence of (Y_n) that converges to $Y^* \in \Psi(F)$, where $Y^*(A) = \int_{\theta\in\Theta} 1\{\psi(\theta; F_n, \mathbf{\Pi}) \in A\} \mathrm{d}F_t, \forall A \in \mathcal{B}(\Theta)$. Therefore, $Y = Y^* \in \Psi(F)$.

Therefore, $\Psi : \Delta(\Theta) \rightrightarrows \Delta(\Theta)$ is a convex-valued self-correspondence on a nonempty compact and convex set and has a closed graph. By the Glicksberg-Fan fixed point theorem, Ψ has a fixed point, that is, there exists a $F \in \Delta(\Theta)$with $F \in \Psi(F)$. $\square$

Proof of Theorem 4 We first show that given all other students reporting using this strategy, then each student is in the best interest to use the same strategy. Then we prove that there exists a distribution profile $[F_g]_{g\in G}$ such that the reported $u^{\theta,g}$ in Bayesian Nash Equilibrium follows F_g for all $g \in G$.

If all other students of type $\theta' \in \Theta$ in group $g' \in G$ report their utility as

$$u_c^{\theta',g'}\left([F_g]_{g\in G}\right) = (1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u'; \bar{F}) \mathrm{d}F^{g'}, \tag{EC.20}$$

then we want to show that a student of type θ in group g has the best response strategy to report $\left[u_c^{\theta,g}\left([F_g]_{g\in G}\right)\right]_{c\in C}$.

Let $u_c^* = (1-\gamma)\theta_c + \gamma \int_{u'\in\Theta} \Pi_c(u'; \bar{F}) \mathrm{d}F^g, \forall c \in C$, then we can write

$$\mathbb{E}U^{\theta,g}\left(u | [F_g]_{g\in G}\right) = \sum_{c\in C} \Pi_c^u\left([F_g]_{g\in G}\right) u^*. \tag{EC.21}$$

Given all other students' strategy fixed, (6) is reduced to

$$\Pi(u) \in \underset{\pi\in B(p)\cap\Delta}{\arg\max} \sum_{c\in C} \pi_c u_c, \tag{EC.22}$$

since the integrals that are taken over the continuous measure cannot be affected by an individual u. That is

$$\sum_{c\in C} \Pi_c(u) u_c \geq \max_{\pi\in B(p)\cap\Delta} \left\{\sum_{c\in C} \pi_c u_c\right\}. \tag{EC.23}$$

Specifically, we have

$$\sum_{c\in C}\Pi_c(u)u_c \geq \sum_{c\in C}\Pi_c(u')u_c, \forall u' \in \Theta. \tag{EC.24}$$

Therefore,

$$\mathbb{E}U^{\theta,g}\left(u^*|\,[F_g]_{g\in G}\right) \geq \mathbb{E}U^{\theta,g}\left(u|\,[F_g]_{g\in G}\right), \forall u \in \Theta. \tag{EC.25}$$

Similar to the proof of Theorem 3, we have that $\Psi : \Delta^G(\Theta) \rightrightarrows \Delta^G(\Theta)$ is a convex-valued self-correspondence on a nonempty compact and convex set and has a closed graph. By the Glicksberg-Fan fixed point theorem, Ψ has a fixed point, that is, there exists a $[F_g]_{g\in G} \in \Delta^G(\Theta)$ with $[F_g]_{g\in G} \in \Psi([F_g]_{g\in G})$. $\square$

Proof of Lemma 5 Consider any Bayesian Nash equilibrium under the network effect level $\gamma = 0$, the distribution profile of reported utilities is F_t, and the corresponding allocation probabilities given a report $u \in \Theta$ is $\Pi(u; F_t)$. For any student with preference θ in group g, their true utility from the allocation mechanism is defined as $W_0^g(\theta) := \sum_{c\in C}\theta_c \Pi_c(\theta; F_t)$. Aggregating this over all the students, the true total welfare is $W_0 := \sum_{g\in G}\int_{\theta\in\Theta} W_0^g(\theta)\mathrm{d}F_t^g$.

Given any $\bar{F} \in \Delta(\Theta)$, By Theorem 4,

$$\mathbb{E}U^{\theta,g}(u^g(\theta)|[F^g]_{g\in G}) \geq \mathbb{E}U^{\theta,g}(u|[F^g]_{g\in G}) = \sum_{c\in C}\Pi_c(u;\bar{F})u^g(\theta), \forall u \in \Theta. \tag{EC.26}$$

When $\gamma = 0$, we have $\sum_{c\in C}\theta_c\Pi_c(\theta;\hat{F}) \geq \sum_{c\in C}\theta_c\Pi_c(u;\hat{F}), \forall u \in \Theta, \hat{F} \in \Delta(\Theta)$. Therefore, $\sum_{c\in C}\theta_c\Pi_c(\theta;\bar{F}) \geq \sum_{c\in C}\theta_c\Pi_c\left(u^g(\theta);\bar{F}\right)$. Morevoer, since $\Pi(\theta;F_t) \in \arg\max\{\phi(\Pi,F_t) : \Pi \in Q(F_t)\}$ we have $\phi(\Pi(\theta;F_t),F_t) \geq \phi(\Pi',F_t), \forall \Pi' \in Q(F_t)$.

Let $\hat{\Pi}^g(\theta) := \Pi\left(u^g(\theta);\bar{F}\right)$ and let $\hat{\Pi}(\theta)$ be such that $\int_{\theta\in\Theta}\hat{\Pi}_c(\theta)\mathrm{d}F_t = \int_{\theta\in\Theta}\sum_{g\in G}\hat{\Pi}_c^g(\theta)\mathrm{d}F_t^g, \forall c \in C$, then we have $\hat{\Pi}(\theta) \in Q(F_t)$ since $\int_{\theta\in\Theta}\hat{\Pi}_c(\theta)\mathrm{d}F_t \leq q_c \forall c \in C$.

Therefore, we have

$$W_0 := \int_{\theta\in\Theta}\sum_{c\in C}\Pi_c(\theta;F_t)\theta_c\mathrm{d}F_t \tag{EC.27}$$

$$= \int_{\theta\in\Theta}\sum_{c\in C}\sum_{g\in G}\Pi_c(\theta;F_t)\theta_c\mathrm{d}F_t^g \tag{EC.28}$$

$$\geq \int_{\theta\in\Theta}\sum_{c\in C}\sum_{g\in G}\hat{\Pi}_c^g(\theta)\theta_c\mathrm{d}F_t^g \tag{EC.29}$$

$$= \int_{\theta\in\Theta}\sum_{c\in C}\sum_{g\in G}\Pi_c\left(u^g(\theta);\bar{F}\right)\theta_c\mathrm{d}F_t^g. \tag{EC.30}$$

That is, $W \le W_0$. $\square$

Proof of Lemma 3 Suppose the initial group allocation when $\gamma = 0$ is $\lambda^g(0)$. When γ increases from 0 to $\gamma_0 > 0$, in the first iteration, students update their reported utilities based on $\lambda^g(0)$: $u_0^g(\theta;\gamma_0) = [(1-\gamma_0)\theta_c + \gamma_0\lambda_c^g(0)]_{c\in C}$. For any student with type θ, denote her favorite class as $*$: $\theta_* > \theta_c, \forall c \in C\setminus\{*\}$. She will switch to a different class $c' \in C\setminus\{*\}$ if and only if $u_{0,c'}^g(\theta;\gamma_0) > u_{0,c}^g(\theta;\gamma_0), \forall c \in C\setminus\{c'\}$. Specifically, $u_{0,c'}^g(\theta;\gamma_0) > u_{0,*}^g(\theta;\gamma_0)$ is equivalent to $\lambda_{c'}^g(0) > \lambda_*^g(0) + \frac{1-\gamma_0}{\gamma_0}(\theta_* - \theta_{c'})$.

When there are two classes $C = \{a,b\}$, assume $\lambda_a^g(0) < \lambda_b^g(0)$. Let us denote the set of students in group g who switch from class a to b in this iteration by $T_{a\to b}^g(0) = \left\{\theta\in\Theta : 0 < \theta_a - \theta_b < \frac{\gamma_0}{1-\gamma_0}\left(\lambda_b^g(0) - \lambda_a^g(0)\right)\right\}$, and those who switch from class b to a by $T_{b\to a}^g(0) = \left\{\theta\in\Theta : 0 < \theta_b - \theta_a < \frac{\gamma_0}{1-\gamma_0}\left(\lambda_a^g(0) - \lambda_b^g(0)\right)\right\}$. Note that since $\lambda_a^g(0) < \lambda_b^g(0)$, $T_{b\to a}^g(0) = \emptyset$. Given the updated student reports, the allocation outcome for group g becomes $\lambda_{iter=1}$, which can be characterized as $\lambda_{iter=1,a}^g(0) = \lambda_a^g(0) - |T_{a\to b}^g(0)|$ and $\lambda_{iter=1,b}^g(0) = \lambda_b^g(0) + |T_{a\to b}^g(0)|$. Obviously, $\lambda_{iter=1,a}^g(0) < \lambda_a^g(0)$, $\lambda_{iter=1,b}^g(0) > \lambda_b^g(0)$, and $\lambda_{iter=1,a}^g(0) < \lambda_{iter=1,b}^g(0)$.

Iterating the above process until the equilibrium is reached, we have $\lambda_a^g(\gamma_0) < \lambda_a^g(0)$, $\lambda_b^g(\gamma_0) > \lambda_b^g(0)$, and $\lambda_a^g(\gamma_0) < \lambda_b^g(\gamma_0)$.

Now, for all γ_0, γ_1 such that $0 \le \gamma_0 < \gamma_1 \le 1$, consider when γ increases from γ_0 to γ_1. In the first iteration, students update their reported utilities based on $\lambda^g(\gamma_0)$: $u_0^g(\theta;\gamma_1) = [(1-\gamma_1)\theta_c + \gamma_1\lambda_c^g(\gamma_0)]_{c\in C}$. Let us denote the set of students in group g who switch from class a to b in this iteration by $T_{a\to b}^g(\gamma_0) = \left\{\theta\in\Theta : \frac{\gamma_0}{1-\gamma_0}\left(\lambda_b^g(\gamma_0) - \lambda_a^g(\gamma_0)\right) < \theta_a - \theta_b < \frac{\gamma_1}{1-\gamma_1}\left(\lambda_b^g(\gamma_0) - \lambda_a^g(\gamma_0)\right)\right\}$. Since $\lambda_a^g(\gamma_0) < \lambda_b^g(\gamma_0)$, $T_{b\to a}^g(\gamma_0) = \emptyset$. Following a similar analysis in the previous step when γ increases from 0 to γ_0, we have $\lambda_a^g(\gamma_1) < \lambda_a^g(\gamma_0)$, $\lambda_b^g(\gamma_1) > \lambda_b^g(\gamma_0)$, and $\lambda_a^g(\gamma_1) < \lambda_b^g(\gamma_1)$.

Specifically, when $\gamma = 1$, we have $\lambda_a^g(1) = 0$ and $\lambda_b^g(1) = 0$. $\square$

Proof of Lemma 4 For any student (θ, g), if she doesn't switch to any other class, then she does not affect the total welfare, so we focus on the scenarios when she switches to a different class.

Assume $\theta_a > \theta_b$, then the only possible switch is from a to b. Suppose not, that is, she switches from b to a, then it must be true that $(1-\gamma_1)\theta_a + \gamma_1\lambda_a^g(\gamma_1) < (1-\gamma_1)\theta_b + \gamma_1\lambda_b^g(\gamma_1)$

and $(1-\gamma_2)\theta_a + \gamma_2\lambda_a^g(\gamma_2) > (1-\gamma_2)\theta_b + \gamma_2\lambda_b^g(\gamma_2)$. Therefore, we have $\lambda_a^g(\gamma_2) - \lambda_a^g(\gamma_1) > \lambda_b^g(\gamma_2) - \lambda_b^g(\gamma_1) + \left(\frac{1-\gamma_1}{\gamma_1} - \frac{1-\gamma_2}{\gamma_2}\right)(\theta_a - \theta_b)$. Since $\lambda_b^g(\gamma_2) - \lambda_b^g(\gamma_1) = -\left(\lambda_a^g(\gamma_2) - \lambda_a^g(\gamma_1)\right)$, $\lambda_a^g(\gamma_2) - \lambda_a^g(\gamma_1) > \frac{1}{2}\left(\frac{1-\gamma_1}{\gamma_1} - \frac{1-\gamma_2}{\gamma_2}\right)(\theta_a - \theta_b) > 0$.

Since $\theta_a > \theta_b$, she is allocated to a when $\gamma = 0$. There must exist some $\gamma_0 \in (0, \gamma_1]$ at which point she switches from a to b. Then we have $(1-\gamma_0)\theta_a + \gamma_0\lambda_a^g(\gamma_0) > (1-\gamma_0)\theta_b + \gamma_0\lambda_b^g(\gamma_0)$ and $(1-\gamma_1)\theta_a + \gamma_1\lambda_a^g(\gamma_1) < (1-\gamma_1)\theta_b + \gamma_1\lambda_b^g(\gamma_1)$. Therefore, $\lambda_b^g(\gamma_1) - \lambda_b^g(\gamma_0) > \lambda_a^g(\gamma_1) - \lambda_a^g(\gamma_0) + \left(\frac{1-\gamma_1}{\gamma_1} - \frac{1-\gamma_0}{\gamma_0}\right)(\theta_a - \theta_b) > 0$. This implies that when γ increases from γ_0 to γ_1 and then to γ_2, λ_b^g increases and then decreases, λ_a^g decreases and then increases. This violates the monotonicity of $\lambda_c^g(\gamma)$ for all $c \in C$. □

Proof of Theorem 6 LEMMA EC.1. *$\zeta = (\gamma, \mu_c, \sigma_c)_{c\in C}$ is identifiable for Law$(m_c^g, \hat{\sigma}_c^{g2})$. Moreover, ζ uniquely determines Law$(m_c^g, \hat{\sigma}_c^{g2})$.*

Proof of Lemma EC.1 Suppose $(\gamma, \mu_c, \sigma_c,)_{c\in C}$ is a set of model parameters, and $(\gamma', \mu_c', \sigma_c',)_{c\in C}$ is another set of model parameters that generate the same distribution in the observations. Thus, the two sets of model parameters must satisfy the following:

$$(1-\gamma)\mu_c + \gamma N_c^g = (1-\gamma')\mu_c' + \gamma' N_c^g, \qquad \forall c \in C, g \in G \text{ and} \tag{EC.31}$$

$$(1-\gamma)^2\sigma_c^2 = (1-\gamma')^2{\sigma_c'}^2, \qquad \forall c \in C. \tag{EC.32}$$

Take the difference of the first set of equations with any two distinct groups $g, h \in G$, we have $\gamma\left(N_c^g - N_c^h\right) = \gamma'\left(N_c^g - N_c^h\right)$. For any $g, h \in G$ with $N_c^g - N_c^h \neq 0$, this implies that $\gamma = \gamma'$ and (EC.32) can be equivalently written as

$$(1-\gamma)\mu_c = (1-\gamma)\mu_c' \text{ and} \tag{EC.33}$$

$$(1-\gamma)^2\sigma_c^2 = (1-\gamma)^2{\sigma_c'}^2, \forall c \in C. \tag{EC.34}$$

Therefore, if $\gamma \neq 1$, it must also be true that $\mu_c = \mu_c'$ and $\sigma_c = \sigma_c', \forall c \in C$. □

LEMMA EC.2. *Consider an arbitrary ζ that is identifiable for Law$(m_c^g, \hat{\sigma}_c^{g2})$. Then (ζ, σ_ϵ) is identifiable for Law$(m_c^g, \log v_c^g)$.*

Proof of Lemma EC.2 Consider the moment generating function of $Law(m_c^g, \log v_c^g)$:

$$\mathbb{E}\left[e^{t_1 m_c^g + t_2 \log v_c^g}\right] = e^{\frac{1}{2}t_2^2\sigma_\epsilon^2}\mathbb{E}\left[e^{t_1 m_c^g}(\hat{\sigma}_c^{g2})^{t_2}\right]. \tag{EC.35}$$

If there are two sets of parameters (ζ, σ_ϵ) and $(\zeta', \sigma'_\epsilon)$, such that their $Law(m_c^g, \log v_c^g)$'s are the same, then their moment generating functions must also be the same. We claim that $\log\mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right]/t^2 \to 0$ as $t\to\infty$. Before we prove this claim, suppose it is true and consider its' implications. Based on this growth condition, letting $t_1 = 0$ and taking the logarithm, the cumulant generating function (log-moment generating function) of $\log v_c^g$ has a quadratic growth in t_2. Given that the$Law(m_c^g, \log v_c^g)$'s are the same under the two sets of parameters (ζ, σ_ϵ) and $(\zeta', \sigma'_\epsilon)$, we know the coefficients of the quadratic term must be the same, that is $\sigma_\epsilon = \sigma'_\epsilon$. Given that $\mathbb{E}\left[e^{t_1 m_c^g + t_2 \log\hat{\sigma}_c^{g2}}\right] = \mathbb{E}\left[e^{t_1 m_c^g}(\hat{\sigma}_c^{g2})^{t_2}\right]$, the two moment generating functions for $Law(m_c^g, \hat{\sigma}_c^{g2})$ must be the same. According to our assumption, ζ is identifiable for $Law(m_c^g, \hat{\sigma}_c^{g2})$. Therefore, it is also identifiable for $Law(m_c^g, \log\hat{\sigma}_c^{g2})$, and $\zeta = \zeta'$. Now, all that remains is to prove our claim. Suppose first that $t > 0$. Then,

$$d^t \le \mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right] = \mathbb{E}\left[\left(C_1(\epsilon+k)^2 + d\right)^t\right] \le d^t\mathbb{E}\left[\left(C_2(\epsilon^2+k^2)+1\right)^t\right] \le 2^{t-1}d^t\mathbb{E}\left[C_2^t\epsilon^{2t} + C_3^t\right] \le C_4^t\Gamma\left(\frac{2t+1}{2}\right) + C_5^t, \tag{EC.36}$$

where $\epsilon\sim\mathcal{N}(0,1)$. Therefore,

$$t\log d \le \log\mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right] \le t\log\left(C_4\vee C_5\right) + \log\left(1+\Gamma\left(\frac{2t+1}{2}\right)\right), \tag{EC.37}$$

where $\Gamma(\cdot)$ is the gamma function. The upper bound is

$$t\log\left(C_4\vee C_5\right) + \log\left(1+\Gamma\left(\frac{2t+1}{2}\right)\right) = C't + \log\Gamma\left(\frac{2t+1}{2}\right) + o(1) \tag{EC.38}$$

as $t\to\infty$. Given that the log-gamma function $\log\Gamma(z)\sim z\log z - z - \frac{1}{2}\log\frac{z}{2\pi}$ as $z\to\infty$, we have that the upper bound is

$$(C'-1)t + (t+\frac{1}{2})\log(t+\frac{1}{2}) - \frac{1}{2}\log\frac{2t+1}{4\pi} + o(1), \tag{EC.39}$$

hence $\log\mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right]/t^2\to 0$ as $t\to+\infty$. Finally, suppose that $t<0$. Given that $(x+1)^t$ is a convex function in $x>0$, we have that

$$d^t(1+C\mathbb{E}[(\epsilon+k)^2])^t \le \mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right] = d^t\mathbb{E}\left[\left(C(\epsilon+k)^2+1\right)^t\right] \le d^t. \tag{EC.40}$$

Therefore $\log\mathbb{E}\left[\left(\hat{\sigma}_c^{g2}\right)^t\right]/t^2\to 0$ as $t\to-\infty$. $\square$

Combining Lemmas EC.1 and EC.2, we know that $\Xi = (\zeta, \sigma_\epsilon)$ is identifiable if there exists a class c and at least two groups $g, h \in G$ such that $N_c^g \neq N_c^h$, and $\gamma \neq 1$. $\square$