Geometry of Taylor – Goldstein equation and stability

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Abstract

Taylor–Goldstein equation (TGE) governs the stability of a shear-flow of an inviscid fluid of variable density . Using a canonical class of its transformations it is investigated from a geometrical point of view .

Rayleigh's point of inflection criterion and Fjørtoft's condition of instability of a homogenous shear-flow have been generalized here so that only the profile carrying the point of inflection is modified by the variation of density . This fulfils a persistent expectation in the literature .

A pair of bounds exists such that in any unstable flow the flow-curvature (a function of flow-layers) exceeds the upper bound at some flow-layer and falls below the lower bound at a higher layer . This is the main result proved here .

Bounds are obtained on the growth rate and the wave numbers of unstable modes , in fulfillment of longstanding predictions of Howard . A result of Drazin and Howard on the boundedness of the wave numbers is generalized to TGE .

The results above hold if the local Richardson number does not exceed 1 / 4 , otherwise a weakening of the conditions necessary for instability is seen .

Conditions for the propagation of neutrally stable waves and bounds on the phase speeds of destabilizing waves are obtained . It is also shown that the set of complex wave velocities of normal modes of an arbitrary flow is bounded .

Fundamental solutions of TGE are obtained and their smoothness is examined. Finally sufficient conditions for instability are suggested.

1 introduction

The object of our study here is Taylor-Goldstein equation (TGE). It was discovered by G.I.Taylor and S.Goldstein, independently and simultaneously, in connection with their studies on ‘parallel shear-flow’ of a fluid of variable density (see Taylor 1931; Goldstein 1931). It governs the stability of the flow. TGE is important because of its applications to oceanography and meteorology and has received considerable attention in the literature on hydrodynamic stability (see Drazin & Howard 1966; Drazin & Reid 1981). It describes waves in the ocean and clouds in the sky. Here we explore the fascinating mathematical structure, that this equation is naturally associated with.

In the limit of vanishing density variation TGE collapses into the classical equation of Rayleigh (1880) which has a well developed theory (see Lin 1955) characterized by its elegance, close agreement with experiments and geometrical flavour. The fundamental result of this theory is the celebrated point of inflection criterion of Rayleigh for the instability of a parallel shear-flow.

Theory of shear-flows has many beautiful results but much remains to be understood yet. The literature on the subject (see Drazin & Howard 1966; Drazin & Reid 1981) is full of expectations that under suitable conditions the characteristic features of Rayleigh-theory will carry over to TGE. There also are laments to the effect that the point of inflection loses its significance in the context of TGE. In spite of persistent and valuable attempts starting with Synge (1933), Yih (1957), Drazin (1958) up to S. Friedlander (2001) (see Drazin & Reid 1981 ; Friedlander 2001), these expectations have found limited fulfillment. There also are standing questions. The predictions of Howard (1961) are yet to be settled and it is not known if the well known condition (Richardson number is less than $1/4$ somewhere in the flow) of Howard (1961) and Miles (1961) is sufficient to ensure instability (see Friedlander 2001). Further there is no simple method for solving TGE. Thus there is a need for a detailed investigation.

To understand TGE, we construct a large class of its transformations which transform it into a canonical form resembling Rayleigh's equation and develop a method that yields information on stability via each transformation of the class. The entire information is then focused upon the question of stability and made independent of transformations and coherent. This results in a description of the inner core of TGE. A central role in this description is seen to be played by a characteristic function of the flow, we call the flow-curvature.

This analysis establishes the point of inflection criterion for TGE, generalizes results from Rayleigh-theory, shows that the condition of Howard and Miles does not ensure instability, proves the predictions of Howard (1961), relates instability to the crossing of a pair of bounds in a definite order by the flow-curvature, examines the propagation of neutral and unstable waves and yields fundamental solutions of TGE.

$$\text{Taylor–Goldstein equation is} \qquad w''(z) + A(z)\, w(z) = 0 \qquad (1.1)$$

$$\text{with boundary conditions} \qquad w(z_1) = 0 = w(z_2). \qquad (1.2)$$

Here $z_1 \le z \le z_2$, $w(z)$ is a complex-valued C^2-function on $[z_1, z_2]$, the prime " $'$ " denotes differentiation d/dz and

$$A(z) = -k^2 - \frac{u''(z)}{u(z) - c} + \frac{g\,\beta(z)}{\{u(z) - c\}^2}$$

where $g > 0$ is a constant, $\beta(z) = -\rho'(z)/\rho_o \ge 0$ (unless stated otherwise), the flow velocity $u(z)$ and the fluid density $\rho(z)$ are sufficiently smooth real–valued functions, the average density ρ_o is a positive constant and $c = c_r + i\,c_i$ is a complex constant called the complex wave velocity. $k > 0$ is the wave number, c_i the phase speed and $k\,c_i$ the growth rate of a harmonic perturbation wave (k, c).

The stability of 'parallel shear-flows' is an important problem in classical theory of hydrodynamics. Many eminent investigators have contributed to its understanding (see Friedlander & Howard 1998). TGE governs the stability of a 2-dimensional parallel shear-flow of an inhomogeneous fluid.

We imagine an inviscid and incompressible fluid of variable density flowing in the (x, z)-plane, between the lines $z = z_1$ and $z = z_2$. Here the $z-$axis is vertical, directed upwards and the $x-$axis is horizontal $\hat{\imath}$ being the unit vector along it.

Suppose at the point (x, z) the velocity of the fluid particle is $u(z)\,\hat{\imath}$ and the density is $\rho(z)$. Thus a layer of the fluid characterized by a fixed value of z has a constant velocity and density. It experiences a force due to buoyancy, proportional to the density gradient $\beta(z)$, exerted on it by the neighbouring layers. These incompressible and parallel layers of the fluid slip smoothly on each other and generate a shearing movement in the fluid. This time–independent flow is called a parallel shear-flow and denoted by (u, β).

The (linear) instability of a shear-flow is perceived of in the theory of stability, as the existence of a harmonic perturbation wave, which vanishes at the boundaries, whose amplitude grows exponentially with time and whose propagation is permitted by the linearized form of the equation of motion of the flow— a nonlinear PDE due to Eüler.

Only stable parallel shear-flows are seen to persist in the laboratory or in nature. Governed by the equation of motion, the unstable flows even if they be parallel to start with, soon become time–dependent and nonparallel. This is because small and time–dependent natural perturbations grow with time in these flows. Thus stability is directly related to the question : which parallel flows persist ?

Here we adapt from classical hydrodynamics the well known criterion for the stability of this flow (see Drazin & Reid 1981), as a definition as follows :

A harmonic perturbation wave (k, c) *is called a normal mode of the flow* (u, β) *if equations (1) and (2) have a nontrivial smooth solution* $w(z)$. *A normal mode is called unstable if* $c_i > 0$ *and neutral if* $c_i = 0$. *A flow is called unstable if it has at least one unstable normal mode and stable otherwise.*

The most fundamental result on TGE was conjectured by Taylor in 1931 and proved by Howard (1961) and Miles (1961). Howard transformed TGE to prove that for any unstable flow (u, β), the flow discriminant $\Delta(z)$ is positive for some z in (z_1, z_2), where $\Delta(z) = [\, u'^2(z) - 4\, g\, \beta(z)\,]$. Equivalently if Richardson number $[\, g\, \beta(z) / u'^2(z)\,] \geq 1/4$ everywhere, then the flow is stable.

Synge (1933) proved that $u_{\min} < c_r < u_{\max}$ for any unstable mode (k, c) of a flow (u, β). Howard (1961) went deeper and elegantly proved that in this case the wave velocity c must lie in the upper half plane, inside the semicircle whose diameter is the real interval $[u_{\min}, u_{\max}]$. Kochar & Jain (1979) took a step further and replaced the semicircle by a semi-ellipse inside it with the same base. Howard also proved that the growth rate $k\, c_i$ of unstable modes is bounded above by $\sqrt{\Delta_{\max}}\, / 2$, generalizing a result due to Hoiland (1953) (see Drazin & Howard 1966). Further he predicted that $k\, c_i \to 0$ as $k \to \infty$ (this is proved easily for homogeneous flows) and also that under suitable conditions the wave numbers k of unstable modes of a flow are bounded.

The special case of TGE when $u \equiv 0$ and β takes arbitrary real values was known to Rayleigh. In this case instability is expected due to gravitational overturning if a heavier layer of the fluid lies above a lighter layer. Rayleigh provided the mathematical support for this observation and proved that the static flow $(0, \beta)$ is unstable if and only if $\beta(z) < 0$ for some z.

We now discuss the basic results on homogeneous flows $(u, 0)$. When $\beta \equiv 0$, (1.1) is called Rayleigh's equation. Rayleigh (1880) proved the fundamental result that in any unstable flow, $u''(z)$ must assume both positive and negative values so that the flow-velocity $u(z)$ must have a point of inflection at some point z_s in (z_1, z_2). Fjørtoft (1950) added that in this case $u''(z)\{u(z) - u(z_s)\} < 0$ for some z.

Friedrichs (1942) (see Drazin & Howard 1966) using Sturm- Liouville theory proved the existence of a smooth, neutrally stable mode $(k_s, u(z_s))$ if for some z_s in $[z_1, z_2]$, $u''(z_s) = 0$, $\mathrm{K}(z) = -u''(z)/\{u(z) - u(z_s)\}$ is integrable and $\mathrm{K}(z) > \pi^2/(z_2 - z_1)^2$ whenever $u(z) \neq u(z_s)$. Here

$$k_s^2 = -\min_{\phi} \left\{ \int_{z_1}^{z_2} (\phi'^2 - \mathrm{K}\,\phi^2)\, dz \Bigg/ \int_{z_1}^{z_2} \phi^2\, dz \right\},$$

the minimum being taken over all functions ϕ such that ϕ and ϕ' are square integrable and $\phi(z_1) = 0 = \phi(z_2)$.

Drazin & Howard (1966) proved that if $\mathrm{K}(z)$ is integrable and $\mathrm{K}(z) \geq 0$ whenever $u(z) \neq u(z_s)$, then $k^2 \leq k_s^2$ for any unstable mode (k, c). Their argument also shows that $\mathrm{K}(z) > [\pi^2/(z_2 - z_1)^2 + k^2]$ somewhere. The last statement follows also from independent arguments given by M.B.Banerjee *et. al.* (2000). Thus if $0 \leq \mathrm{K}(z) \leq \pi^2/(z_2 - z_1)^2$ whenever $u(z) \neq u(z_s)$, then the flow is stable.

Heisenberg (1924) and Tollmien (1929) (see Drazin & Reid 1981) obtained neutral wave solutions $(c_i = 0)$ of Rayleigh's equation when $u(z)$ is analytic in some neighbourhood of the point $z = z_c$ in the complex $z-$plane and $u'(z_c) \neq 0$ where $u(z_c) = c$. Their results agree. They show that one of the solutions is analytic near z_c and vanishes at it while the other has a logarithmic branch point there.

The results proved here will be described now. In § 2 and § 3 we mainly examine the stability of flows for which $u'(z)$, $\Delta(z)$ and $\beta(z)$ are nonnegative functions while theorems [3 .15] and [3 .22] deal with other classes of flows .

Theorem [2 .7] is the generalization of the criteria of Rayleigh and Fjørtoft . The only difference from the homogeneous case as above is that the function $u(z)$ is replaced by $u_+(z)$, where $2\,u'_\pm(z) = [\ u'(z) \pm \sqrt{\Delta(z)}\ ]$. It shows that the condition ' $\Delta(z) > 0$ for some z ' of Howard and Miles is not sufficient for instability . It is also shown that for unstable modes the growth rate $k\,c_i \to 0$ as $k \to \infty$, and that if $\beta(z) > 0$ everywhere then k is bounded above . These prove the predictions of Howard (1961) .

Section (3) relates the flow-curvature $\mathrm{T}(z) = u''_+(z)\,/\,2\,u'_-(z)$ to stability . Theorem [3 .2] gives a pair of bounds $k \coth[\,k\,(\,z - z_i\,)\,]$, $i = 1, 2,$ which $\mathrm{T}(z)$ must cross if $(\,k\,,\,c\,)$ is an unstable mode . In particular $\mathrm{T}(z)$ must cross the bounds $1\,/\,(\,z\,-\,z_i\,)$ if the flow is unstable . It is shown in corollary [3 .4] that if $\mathrm{T}(z)$ is bounded above or bounded below then k is bounded . Corollary [3 .5] shows that for a fixed upper or lower bound on T, flows with sufficiently small depth $(\,z_2\,-\,z_1\,)$ are stable .

Theorem [3 .6] sets an order in which the bounds must be crossed by $\mathrm{T}(z)$ in an unstable flow $(\,u\,,\beta\,)$ when $u''(z)$ and $\beta(z)$ have no zeros in common . If $(\,k\,,\,c\,)$ is an unstable mode then for some $t_1 < t_2$, $\mathrm{T}(t_i) = k \coth[\,k\,(\,z - z_i\,)\,]$ for each i . Theorem [3 .7] includes the case when $u''(z)$ and $\beta(z)$ have common zeros .

Corollary [3 .8] shows that if for some z_s in $[\,z_1\,,\,z_2\,]$, T is bounded above when $z \le z_s$ and is bounded below when $z \ge z_s$, then k is bounded for unstable modes $(\,k\,,\,c\,)$. Theorem [3 .13] leads to the same conclusion when T is bounded below in $z \le z_s$, bounded above in $z \ge z_s$, $u''_+(z)$ and $\beta(z)$ have no zeros in common and $u'(z_s) \ne 0$. Theorem [3 .14] refines the result when $u''_+(z)$ and $\beta(z)$ have common zeros .

In § 3.4 stability of flows is examined when $u'(z)$ and $\Delta(z)$ change sign in $[\,z_1\,,\,z_2\,]$ in a prescribed way (see condition B) . Theorem [3 .15] shows that weaker but nontrivial necessary conditions of instability continue to persist showing thereby that some of these flows are stable . For flows not satisfying condition (B) , no conditions seem to be necessary for instability , suggesting that all these flows are unstable .

In § 3.5 a sufficient condition for the propagation of neutral waves ($c_i = 0$) is obtained . Theorems [3 .17] and [3 .18] give necessary conditions for existence of marginally stable modes $(u_{\min} \le c \le u_{\max})$ and internal gravity waves $(c < u_{\min}$ or $c > u_{\max})$ respectively. It is seen that the possibility of propagation of marginally stable neutral waves of arbitrarily large wave numbers remains open even if the flow-curvature be bounded but the set of wave numbers of the internal gravity waves is bounded if either of the functions $u''_{\pm}(z) / 2\, u'_{\mp}(z)$ is bounded.

Upper bounds are obtained in § 3.6 in two different situations, on the phase speed c_i, for unstable flows not satisfying a Rayleigh–Fjørtoft type condition. Theorem [3.19] shows that for a flow satisfying condition (A) either $u_-(z)$ has a point of inflection at z_s with $u''_-(z)\{u(z) - u(z_s)\} > 0$ for some z or $c_i \to 0$ as $\nu_{\max} \to 0$ where $\nu^2(z) = [1 - 4\, g\, \beta(z) / u'^2(z)]$. Theorem [3.22] shows that for a flow with $\beta(z) < 0$ somewhere, either $u_+(z)$ has a point of inflection at z_s with $u''_+(z)\{u(z) - u(z_s)\} < 0$ for some z or $c_i \to 0$ as $\nu_{\max} \to 1$.

Propositions [3.16] and [3.20] together show that the set of complex wave velocities of normal modes of an arbitrary flow is bounded.

In § 4 TGE is converted into a quadratic recursion relationship on a sequence of smooth functions on $[z_1, z_2]$ and solved. For a class of flows Theorem [4.1] gives a pair of linearly independent smooth solutions of TGE in some neighbourhood of the critical layer $z = z_c$ where $u(z_c) = c_r$, when $c_i > 0$ is sufficiently small. Further if $\beta \equiv 0$ corollary [4.3] gets back as a limiting case, the smooth solution of Rayleigh's equation that vanishes at $z = z_c$ when $c_i = 0$, in agreement with results of Heisenberg and Tollmien (see Drazin & Reid 1981).

In § 4.1 sufficient conditions are suggested for the existence of an unstable mode solution to the Taylor-Goldstein boundary value problem.

2 The basic method

In this section stability of monotonic flows with nonnegative discriminant will be discussed. The basic method is developed in § 2.1. It is then used to obtain generalizations of the instability criteria of Rayleigh and Fjørtoft in § 2.2. Sharper use of this method will be made in § 3 to obtain deeper results.

§ 2.1 Canonical transformations of TGE

The information on stability obtained from transformations of TGE is stated in lemma [2.1]. It is then restricted to a canonical class of transformations to obtain propositions [2.3] and [2.6]. These have been developed in sufficient generality in view of their many applications later in the text.

Lemma 2.1 *Let (u, β) be an unstable flow. Suppose f is a complex–valued C^1–function. Let f'' be integrable and $f(z) \neq 0$ for every z in $[z_1, z_2]$ then*

$$\int_{z_1}^{z_2} f^2 |F'|^2 \, dz = \int_{z_1}^{z_2} (f f'' + A f^2) |F|^2 \, dz .$$

Further if $Im\{f^2(z)\} < 0$ for every z in $[z_1, z_2]$, where Im denotes the imaginary part then

$$Im [f(z) f''(z) + A(z) f^2(z)] < 0 \qquad \text{for some } z \text{ in } (z_1, z_2) .$$

Proof: Substituting $F(z) = w(z)/f(z)$ in (1.1) gives

$$[f^2(z) F'(z)]' + [f(z) f''(z) + A(z) f^2(z)] F(z) = 0 \qquad (2.1)$$

valid at all points z where $f''(z)$ exists. On multiplying this with $F^*(z)$ (the complex conjugate of $F(z)$) and integrating one gets

$$\int_{z_1}^{z_2} f^2 |F'|^2 \, dz = \int_{z_1}^{z_2} (f f'' + A f^2) |F|^2 \, dz + \int_{z_1}^{z_2} (f^2 F' F^*)' \, dz .$$

Now $f^2 F' F^* = [w' f - w f'] (w f^{-1})^*$ is continuous and its derivative is integrable in $[z_1, z_2]$. Further $f^2 F' F^*$ vanishes at the boundary points z_1 and z_2 by (1.2). Thus the last integral in the equation above vanishes proving the first statement. The lemma now follows from the imaginary part of this equation.

Let the flow-discriminant $\Delta(z) = [u'^2(z) - 4 g \beta(z)]$. In this and the next section we shall mainly focus upon the class of flows (u, β) satisfying :

Condition (A) **:** (a) $u'(z), \beta(z) \geq 0$ for every z in $[z_1, z_2]$.

(b) $\Delta(z)$ has a smooth extensions to some neighbourhood of $[z_1, z_2]$ in $\mathbb{R}$ satisfying $\Delta(z) \geq 0$ everywhere.

Remark 2.2 *It is easily seen that if* $(u, \beta, k, c_r, c_i; w)$ *is a solution of the equations (1.1) & (1.2) then so is* $(-u, \beta, k, -c_r, c_i; w^*)$. *Thus either both the flows* (u, β) *and* $(-u, \beta)$ *are stable or both are unstable. Therefore any result which does not change on* u *being replaced by* $-u$ *if proved under condition* (A), *would remain valid if the hypothesis :* $u'(z) \geq 0$ *in (a) above is replaced by* $u'(z)$ *is monotonic in* $[z_1, z_2]$. *All the results proved under condition* (A) *here are of this type.*

Suppose the unstable flow (u, β) satisfies condition (A). Let

$$u_{\pm}(z) = u(z_1) + \frac{1}{2}\int_{z_1}^{z} [\, u'(z) \pm \sqrt{\Delta(z)}\,]\, dz .$$

Part (b) of condition (A) ensures that $\sqrt{\Delta(z)}$ is smooth so that $u_{\pm}(z)$ are smooth functions and $2\,u'_{\pm}(z) = [\, u'(z) \pm \sqrt{\Delta(z)}\,] \geq 0$ for every z in $[z_1, z_2]$. Further if $\beta \equiv 0$ then $u_{+}(z) = u(z)$.

Let (k, c) be an unstable mode of the flow (u, β). The function $\{u(z) - c\}$ then takes its values in the 3rd and 4th quadrants.

$$\text{Let} \qquad u(z) - c = |u(z) - c| \, \exp\{\, i\,\theta(z)\,\} \tag{2.2}$$

$$\text{where} \qquad -\pi < \theta(z) < 0 . \tag{2.3}$$

$\theta(z)$ is then a well defined smooth function of z and a smooth branch of $\log\{u(z) - c\}$ on $[z_1, z_2]$ is defined by

$$\log\{u(z) - c\} = \log|u(z) - c| + i\,\theta(z) .$$

On differentiating this equation and taking the imaginary part, one gets

$$\begin{aligned} \theta'(z) &= \frac{c_i u'(z)}{|u(z) - c|^2} \qquad (2.4) \\ &\geq 0 \text{ for every z in } [z_1, z_2] . \end{aligned}$$

Let $h(z)$ be a real–valued, piecewise $-C^1$ function on $[z_1, z_2]$. Let z_o be in $[z_1, z_2]$ and let

$$f(z) = \exp\left[\frac{i\,\theta(z_o)}{2} + \int_{z_o}^{z} \left\{ \frac{u'_{-}(z)}{\{\, u(z) - c\,\}} + h(z) \right\} dz \right] . \tag{2.5}$$

Clearly $f(z)$ is a C^1-function and f'' is integrable on $[z_1, z_2]$. Also $f(z) \neq 0$ for any z in $[z_1, z_2]$.

$$\text{We write} \qquad f(z) = |f(z)| \exp[\, i\,\phi(z)/2\,] \tag{2.6}$$

$$\text{where} \qquad \phi(z) = \theta(z_o) + c_i \int_{z_o}^{z} \frac{2\,u'_-(z)}{|u(z)-c|^2}\,dz \tag{2.7}$$

$$\text{so that} \qquad \phi'(z) = \frac{2\,c_i\,u'_-(z)}{|u(z)-c|^2} \geq 0\,.$$

It follows now from (2.4) and (2.7) respectively that

$$0 \geq \phi'(z) - \theta'(z) = \frac{-c_i\,\sqrt{\Delta(z)}}{|u(z)-c|^2} \quad \text{and} \quad \phi(z_o) = \theta(z_o) \tag{2.8}$$

so that $|\phi(z)-\phi(z_o)| \leq |\theta(z)-\theta(z_o)|$ for every z in $[z_1, z_2]$.

Thus $\phi(z)$ lies between $\theta(z_o)$ and $\theta(z)$. Equations (2.3) and (2.6) now give in turn

$$-\pi < \phi(z) < 0 \tag{2.9}$$

$$\text{and} \qquad Im\left\{f^2(z)\right\} = |f^2(z)|\,\sin\{\phi(z)\} < 0 \quad \text{for every z in } [z_1, z_2]\,. \tag{2.10}$$

It is easily calculated from (2.5) that

$$f(z)f''(z) + A(z)f^2(z) = \left[\,(h'+h^2-k^2) - \frac{\mathrm{U}_h(z)}{\{u(z)-c\}}\,\right] f^2(z) \tag{2.11}$$

where $\mathrm{U}_h(z) = \left[u''_+(z) - 2\,h(z)\,u'_-(z)\right]$. Equation (2.1) then yields

$$\left[f^2(z)\,F'(z)\right]' + \left\{(h'+h^2-k^2) - \frac{\mathrm{U}_h(z)}{\{u(z)-c\}}\right\} f^2(z)\,F(z) = 0\,.$$

On taking the imaginary part of (2.11) and using (2.2) and (2.6) one obtains

$$Im\left[\,ff''+Af^2\,\right] = \left[\,(h'+h^2-k^2)\ \sin\phi - \frac{\mathrm{U}_h\ \sin(\phi-\theta)}{|u-c|}\,\right]|f^2|\,.$$

In view of (2.10) and lemma [2.1] it is proved that :

Proposition 2.3 *Let* (u, β) *be an unstable flow satisfying the condition* (A). *Let* h *be a real–valued , piecewise*$-C^1$ *function on* $[z_1, z_2]$ *and let* $z_1 \le z_o \le z_2$ *then for some* z *in* (z_1, z_2).

$$\frac{\mathrm{U}_h(z) \quad \sin\{\phi(z) - \theta(z)\}}{|u(z) - c| \quad \sin\phi(z)} < \left[h'(z) + h^2(z) - k^2 \right] .$$

Further suppose $\left[h'(z) + h^2(z) - k^2 \right] \le 0$ *everywhere then the inequality above holds at some* z *which also satisfies* $\phi(z) \ne \theta(z)$ *and* $u(z) \ne u(z_o)$.

Remark 2.4 *If* $f(z)$ *and* $\phi_+(z)$ *are defined by replacing* $u'_-(z)$ *by* $u'_+(z)$ *in (2.5) and (2.7) respectively then an argument similar to the one above yields :*

If $-\pi < \phi_+(z) < 0$ *then proposition [2.3] continues to hold provided* $\phi(z)$ *is replaced by* $\phi_+(z)$ *and* $\mathrm{U}_h(z)$ *by* $\left[u''_-(z) - 2\,h(z)\,u'_+(z) \right]$.

It will be seen in §3.6 that violation of the bounds on $\phi_+(z)$ as above implies an upper bound on c_i .

Definition 2.5 *We say that two numbers* $a, b \in \mathbb{R}$, *are sign–equivalent and write* $a \simeq b$ *if* $a = \lambda\, b$ *for some* $\lambda > 0$ *i.e. either both* a *and* b *are positive or both are equal to zero or both are negative.*

Equations (2.3) and (2.9) show that $-\pi < \phi - \theta < \pi$ so that

$$\sin\left\{ \phi(z) - \theta(z) \right\} \simeq \left\{ \phi(z) - \theta(z) \right\} \quad \text{for every z in } [z_1, z_2] .$$

Further (2.8) implies that

$$\left\{ \phi(z) - \theta(z) \right\} \simeq -\left\{ u(z) - u(z_o) \right\} \quad \text{if} \quad \phi(z) \ne \theta(z) .$$

Thus if $\sin\left\{ \phi(z) - \theta(z) \right\} \ne 0$ then

$$\sin\left\{ \phi(z) - \theta(z) \right\} \simeq -\left\{ u(z) - u(z_o) \right\} . \tag{2.12}$$

We now prove :

Proposition 2.6 *Suppose the flow* (u, β) *is unstable and condition* (A) *holds then* $\Delta(z) > 0$ *for some* z. *Let* h *be a real-valued, piecewise*$-C^1$ *function satisfying* $\left[h'(z) + h^2(z) - k^2 \right] \le 0$ *for every* z *in* $[z_1, z_2]$

and let $\mathrm{U}_h(z) \;=\; \left[\; u''_+(z) \,-\, 2\,h(z)\,u'_-(z) \;\right]$ *then*

(a) $\mathrm{U}_h(z)$ *takes both negative and positive values in* $[\,z_1\,,\,z_2\,]$ *so that* $\mathrm{U}_h(z_s) \;=\; o$ *for some* z_s *in* $(\,z_1\,,\,z_2\,)$.

(b) $\mathrm{U}_h(z)\;\{\,u(z) \;-\; u(z_s)\,\} \;<\; 0$ *for some* z *in* $(\,z_1\,,\,z_2\,)$.

Proof: Let z_o be any point in $[\,z_1\,,\,z_2\,]$. Proposition [2.3] shows that $\sin(\,\phi-\theta\,) \;\neq\; 0$ for some z so that $\Delta(z) \;>\; 0$ somewhere in $[\,z_1\,,\,z_2\,]$ by (2.8). Further (2.9) and (2.12) now give

$$\mathrm{U}_h(z)\;[\,u(z)-u(z_o)\,]\;<\;0 \quad \text{for some z in } (\,z_1\,,\,z_2\,)\,.$$

Taking $z_o \;=\; z_1$ (resp. $z_o = z_2$) here, we see that $\mathrm{U}_h(z)$ takes both negative (resp. positive) values because $u(z)$ is monotonic. This proves part (a). Now (b) follows on substituting $z_o = z_s$.

§ 2.2 Generalization of Rayleigh, and Fjørtoft criteria

Parts (1) and (2) of theorem [2.7] generalize the well known instability criteria of Rayleigh and Fjørtoft while parts (3) and (4) prove the predictions of Howard (1961). Boundedness of k^2 will be proved under weaker conditions in § 3. Corollary [2.9] shows that the condition of Howard and Miles is not sufficient for instability.

Taking $h \equiv 0$ in proposition [2.6] gives the parts (1) and (2) below.

Theorem 2.7 *Let the unstable flow* $(\,u\,,\beta\,)$ *satisfy condition* (A) *and let* $(\,k\,,\,c\,)$ *be one of its unstable modes then*

(1) $u''_+(z)$ *takes both negative and positive values so that* $u''_+(z_s) \;=\; 0$ *for some* z_s *in* $(\,z_1\,,\,z_2\,)$,

(2) $u''_+(z)\;[\,u(z)-u(z_s)\,]<0$ *for some* z *in* $(\,z_1\,,\,z_2\,)$,

(3) $k^2\,c_i \;<\; \dfrac{1}{2}\,|\,u''_+\,|_{\max}$ *so that the growth rate* $k\,c_i \;\rightarrow\; 0$ *as* $k \rightarrow \infty$,

(4) *If* $\beta(z) \,>\, 0$ *everywhere in* $[\,z_1\,,\,z_2\,]$, *then,* $k^2 \;<\; [\,\mathrm{T}'(z) + \mathrm{T}^2(z)\,]$ *for some* z *in* $(\,z_1\,,\,z_2\,)$, *where* $\mathrm{T}(z) \;=\; u''_+(z)\,/\,2\,u'_-(z)$.

Proof : Only the parts (3) and (4) remain to be proved. It is an easy consequence of Howard's semicircle–theorem that $u_{\min} < c_r < u_{\max}$, (a result due to Synge) where $u_{\min}$ and $u_{\max}$ denote respectively the minimum and the maximum of $u(z)$ for z in $[z_1, z_2]$. Let z_o be a point satisfying $u(z_o) = c_r$, then (2.2) shows that

$$\exp\{ i\,\theta(z_o) \} = -i .$$

Thus $\theta(z_o) = -\pi/2$ and $\phi(z)$ lies between $-\pi/2$ and $\theta(z)$ by (2.8). Let $\sin x = |\sin\theta|$ and $\sin y = |\sin\phi|$ where $0 \le x \le y \le \pi/2$. Clearly then $\sin(y-x) = |\sin\{\phi-\theta\}|$ because both $\theta(z)$ and $\phi(z)$ are either in $(-\pi, -\pi/2]$ or in $[-\pi/2, 0)$. Now

$$\max_x [\sin(x)\,\sin(y-x)] = \sin^2(y/2) = \frac{1}{2}\sin y\,\tan(y/2) \le \frac{1}{2}\sin y,$$

so that $\quad 2|\sin\theta|\,|\sin(\phi-\theta)| < |\sin\phi| \quad$ for every z in $[z_1, z_2]$.

Proposition [2.3] now shows that

$$2k^2\,|u(z)-c|\,|\sin\theta(z)| < |u_+''(z)| \qquad \text{for some z and so}$$

$$2k^2 c_i < |u_+''|_{\max} \equiv \max_{z\in[z_1, z_2]} |u_+''(z)|.$$

This proves part (3). Part (4) follows from proposition [2.3], on substituting $T(z)$ for $h(z)$ because $\mathrm{U}_T \equiv 0$. This completes the proof.

Remark 2.8 *It is easily seen from the theorem that the criteria of Rayleigh and Fjørtoft remain valid, as stated by them, when the flow-discriminant $\Delta(z)$ is a constant or the Richardson number $R(z) = g\,\beta/u'^2$ is a constant.*

The theorem shows that the flow $(2\,g\,z^2, \; g\,z^2)$ is stable though $\Delta(z)$ is positive everywhere.

Corollary 2.9 *The condition $\Delta(z) > 0$ for some z in $[z_1, z_2]$ is not sufficient to ensure instability.*

Further we have $2\,u_+''(z) = u''(z) + \dfrac{u'(z)\,u''(z) - 2\,g\,\beta'(z)}{\sqrt{\Delta(z)}}$. This gives :

Corollary 2.10 *Suppose $u'(z)$, $\beta(z)$, and $\Delta(z) > 0$ for every z in $[z_1, z_2]$, then*

$$u'(z)\,u''(z) \simeq \beta'(z) \qquad\qquad \textit{for some } z \textit{ in } [z_1, z_2].$$

3 Geometry of Taylor–Goldstein equation

Valuable information on the stability of flows satisfying condition (A) is coded in the propositions [2.3] and [2.6] of the previous section via the arbitrary nature of the point z_o and the function $h(z)$. This information is decoded in this section to bring out into the open the intimate relationship between bounds on the flow curvature and stability.

Theorems [3.2], [3.6] and [3.7] show that $\mathrm{T}(z)$ must cross the bounds $k \coth[k(z-z_i)]$, $i = 1, 2,$ in a definite order if (k, c) is to be an unstable mode. On the other hand corollary [3.8] and theorems [3.13] and [3.14] give different kinds of bounds on $\mathrm{T}(z)$ each of which ensures the boundedness of the wave numbers of the unstable modes.

In § 3.4 the stability of nonmonotonic flows with indefinite discriminant is examined. Theorem [3.15] establishes that the class of flows satisfying condition (B) has stable flows in it.

In § 3.5 necessary conditions are obtained for the propagation of neutral waves. Theorems [3.19] and [3.22] in § 3.6 give upper bounds on c_i as an alternatives to point of inflection type conditions.

§ 3.1 Bounds on the flow-curvature $u''_+(z) / 2\, u'_-(z)$

Definition 3.1 *Let* $\mathscr{T} = \{ z \in [z_1, z_2] \mid u''_+(z) \neq 0,$ *or* $\beta(z) \neq 0 \}$, *and let the flow–curvature* $\mathrm{T} : \mathscr{T} \to [-\infty, \infty]$ *be defined by*

$$\mathrm{T}(z) = [\, u''_+(z) / 2\, u'_-(z) \,] .$$

Clearly T is a continuous function in view of $u'_-(z) \geq 0$. It will now be proved that in any unstable flow $\mathrm{T}(z)$ must cross a pair of bounds.

Let $h(z) = k \coth\big\{k(z-z_1+\epsilon)\big\}$, *where* $\epsilon > 0$ is sufficiently small, then h is a $C^\infty-$ function on $[z_1, z_2]$ and

$$h'(z) + h^2(z) - k^2 = 0 \quad \text{for every} \quad z \quad \text{in} \quad [z_1, z_2] .$$

Applying now proposition [2.6] part(a) one gets

$$\mathrm{T}(z) - k \coth\{k(z - z_1 + \epsilon\} > o \quad \text{for some z in } \mathscr{T} .$$

In the same way taking $h(z) = k \coth\{k(z - z_2 - \epsilon\}$ gives

$$\mathrm{T}(z) - k \coth\{k(z - z_2 - \epsilon\} < o \quad \text{for some z in } \mathscr{T} .$$

Now in the limit as $\epsilon \to 0$ one obtains the following theorem.

Theorem 3.2 *Let the flow* (u, β) *satisfy the condition* (A) *and let* (k, c) *be an unstable mode, then*

$$\limsup_{t \to z} \mathrm{T}(t) \geq k \coth\{k(z - z_1)\} \quad for \ some \ z \ in \ \overline{\mathscr{T}} \quad and$$

$$\liminf_{t \to z} \mathrm{T}(t) \leq k \coth\{k(z - z_2)\} \quad for \ some \ z \ in \ \overline{\mathscr{T}}.$$

***Remark* 3.3** *Suppose* $u_+''(z)$ *and* $\beta(z)$ *have no common zeros in* $[z_1, z_2]$*, then* T *is continuous on* $[z_1, z_2]$ *and the left hand sides of the inequalities in the theorem reduce to* $\mathrm{T}(z)$.

Theorem [3.2] has some interesting consequences. Using the inequality $\coth(t) > 1$ if $t > 0$ one obtains :

Corollary 3.4 *Suppose the hypothesis of theorem [3.2] holds. Let* $\mathrm{T}(z)$ *be bounded above or bounded below, then*

$$k < \min\left[\sup_{z \in \mathscr{T}}\{\mathrm{T}(z)\}, \ -\inf_{z \in \mathscr{T}}\{\mathrm{T}(z)\}\right] < \infty$$

so that the set of wave numbers of unstable modes of (u, β) *is bounded.*

In §3.2 and §3.3 boundedness of k will be proved under weaker conditions. Using $k \coth(kt) > 1/t$ for $t > 0$ one obtains :

Corollary 3.5 *Suppose the hypothesis of corollary [3.4] holds, then*

$$(a) \quad \frac{1}{z - z_1} < \mathrm{T}(z) \quad for \ some \ z \ in \ (z_1, z_2) \quad and$$

$$(b) \quad \frac{1}{z_2 - z} < -\mathrm{T}(z) \quad for \ some \ z \ in \ (z_1, z_2) \quad so \ that$$

$$(c) \quad \frac{1}{z_2 - z_1} < \min\left[\sup_{z \in \mathscr{T}}\{\mathrm{T}(z)\}, \ -\inf_{z \in \mathscr{T}}\{\mathrm{T}(z)\}\right] < \infty.$$

This shows that given a fixed upper or lower bound on $\mathrm{T}(z)$ flows with sufficiently small depth $(z_2 - z_1)$ are stable. Thus we see that instability needs sufficient room to manifest itself.

§ 3.2 The main instability criterion

We are now in a position to prove the main result on instability.

Theorem 3.6 *Let* (u, β) *be an unstable flow satisfying the condition* (A). *Suppose* $u''_+(z)$ *and* $\beta(z)$, *have no common zeros in* $[z_1, z_2]$ *then for some* $t_1,\ t_2$ *in* $[z_1, z_2]$, $t_1 < t_2$,

$$\mathrm{T}(t_1) = k \coth\big[\, k\,(t_1 - z_1)\,\big] > \frac{1}{t_1 - z_1}$$

and

$$\mathrm{T}(t_2) = k \coth\big[\, k\,(t_2 - z_2)\,\big] < \frac{1}{t_2 - z_2} \ .$$

Proof: Remark [3.3] shows that T, in this case is a continuous function into $[-\infty, \infty]$.

$$\textit{Let} \qquad t_1 = \min_{z_1 \le z \le z_2} \big\{\, z \mid \mathrm{T}(z) \ \ge\ k \coth\,[\,k\,(z - z_1)\,]\,\big\}$$

$$\textit{and} \qquad t_2 = \max_{z_1 \le z \le z_2} \big\{\, z \mid \mathrm{T}(z) \ \le\ k \coth\,[\,k\,(z - z_2)\,]\,\big\}.$$

Theorem [3.2] shows that t_1 *and* t_2 are well defined, and clearly

$$t_1 \ne t_2 \quad \text{and} \quad \mathrm{T}(t_i) = k \coth\,[\,k\,(t_i - z_i)\,] \quad \text{for} \quad i = 1, 2\,.$$

We prove now that $t_1 < t_2$. If possible suppose $t_2 < t_1$. Let z_s be any point in (t_2, t_1), and $\mathrm{T}_s = \mathrm{T}(z_s)$. Let $\epsilon > 0$ be a small number and let $h : [z_1, z_2] \to \mathbb{R}$ be defined by

$$h(z) = \begin{cases} k \coth\,[\,k\,(z - z_1 + \epsilon)\,] & \text{if} \quad z \in [\,z_1,\ z_s - \epsilon\,] \ ; \\ l_1(z) & \text{if} \quad z \in [\,z_s - \epsilon,\ z_s\,] \ ; \\ l_2(z) & \text{if} \quad z \in [\,z_s,\ z_s + \epsilon\,] \ ; \\ k \coth\,[\,k\,(z - z_2 - \epsilon)\,] & \text{if} \quad z \in [\,z_s + \epsilon,\ z_2\,] \ ; \end{cases}$$

where the graphs of $l_i(z)$ for $i = 1, 2$ are obtained by joining the point (z_s, T_s) with the points $\big(z_s \mp \epsilon,\ k \coth\,[\,k\,(z_s - z_i)\,]\big)$ respectively, by line segments. When ϵ is sufficiently small it is easily checked that,

1. h is a piecewise $-C^1$ function, and $h(z_s) = \mathrm{T}(z_s)$;
2. $h' + h^2 - k^2 = 0$ if z is not in $[\,z_s - \epsilon,\ z_s + \epsilon\,]$;

3. $k \coth[\, k\,(t_2 - z_2)\,] \;\le\; h(z) \;\le\; k \coth[\, k\,(t_1 - z_1)\,]$ for every z in $[\, z_s - \epsilon\,,\; z_s + \epsilon\,]$;

4. for any fixed $m < 0$, $h'(z) < m$ for every z in $[\, z_s - \epsilon\,,\; z_s + \epsilon\,]$;

5. $h' + h^2 - k^2 \le 0$ for every z in $[\, z_1\,,\, z_2\,]$;

6. $\mathrm{T}'(z) - h'(z) > 0$ for every z in $[\, z_s - \epsilon\,,\; z_s + \epsilon\,]$;

7. $\mathrm{T}(z) < h(z)$ if $z \le z_s - \epsilon$ and $\mathrm{T}(z) > h(z)$ if $z \ge z_s + \epsilon$;

8. $\mathrm{T}(z) = h(z)$ only if $z = z_s$ and $\mathrm{T}(z) - h(z) \simeq (z - z_s)$.

It follows that $\Big[\, \mathrm{T}(z) - h(z)\,\Big]\,\Big[\, u(z) - u(z_s)\,\Big] \ge 0$ for every z in $[\, z_1\,,\, z_2\,]$. This contradicts proposition [2.6]. Thus $t_1 < t_2$, and the proof is complete.

With a little more care one can include the case when $u_+''(z)$ and $\beta(z)$ have common zeros. We state the result and briefly indicate its proof.

Theorem 3.7 *Let $(u\,,\beta)$ be an unstable flow satisfying condition (A), then for some t_1 and t_2 in $\overline{\mathscr{T}}$, $t_1 \le t_2$,*

$$\text{(A)} \quad \limsup_{z \to t_1} \mathrm{T}(z) \;\ge\; k\, \coth\Big[\, k\, (t_1 - z_1)\,\Big] \;>\; \frac{1}{t_1 - z_1} \quad \textit{and}$$

$$\text{(B)} \quad \liminf_{z \to t_2} \mathrm{T}(z) \;\le\; k\, \coth\Big[\, k\, (t_2 - z_2)\,\Big] \;<\; \frac{1}{t_2 - z_2}$$

and one of the following conditions (a) or (b) hold:

(a) $t_1 < t_2$;

(b) $t_1 = t_2 = t_0$ *(say) and one of the following conditions hold:*

(1) $\mathrm{L}_+ \ge k\, \coth\Big[\, k\, (t_1 - z_1)\,\Big]$ where $\mathrm{L}_+ = \limsup_{z \to t_0 - 0} \mathrm{T}(z)$.

(2) $\mathrm{R}_- \le k\, \coth\Big[\, k\, (t_2 - z_2)\,\Big]$ where $\mathrm{R}_- = \liminf_{z \to t_0 + 0} \mathrm{T}(z)$.

(3) $\mathrm{L}_+ > \mathrm{R}_-$.

(4) $\mathrm{L}_+ = \mathrm{R}_- = \mathrm{T}_0$ (say) and $\displaystyle \liminf_{z \to t_0} \frac{\mathrm{T}(z) - \mathrm{T}_0}{z - t_0} = -\infty$.

Proof :

$$\textit{Let} \qquad t_1 = \inf \Big\{\, z \in \mathscr{T} \;\Big|\; \limsup_{t \to z} \mathrm{T}(t) \;\ge\; k\, \coth\, [\, k\, (z - z_1)\,]\,\Big\}$$

and $\quad t_2 = \sup \left\{ z \in \mathscr{T} \mid \liminf_{t \to z} \mathrm{T}(t) \leq k \coth \left[k \, (z - z_2) \right] \right\}$.

It is clear that inequalities (A) and (B) above hold.

Suppose both conditions (a) and (b) do not hold. Clearly $t_2 \leq t_1$. If $t_2 = t_1$ then $k \coth[k\,(t_2 - z_2)] < \mathrm{L}_+ \leq \mathrm{R}_- < k \coth[k\,(t_1 - z_1)]$ and t_1 is not in $\mathscr{T}$. Let $z_s = [t_1 + t_2]/2$ and let

$$\mathrm{T}_s = \begin{cases} \mathrm{T}(z_s) & \text{if} \quad t_2 < t_1 \quad \text{and} \quad z_s \text{ is in } \mathscr{T} \\ (\mathrm{L}_+ + \mathrm{R}_-)/2 & \text{if} \quad t_2 = t_1 \quad \text{or} \quad z_s \text{ is not in } \mathscr{T} . \end{cases}$$

Let the function h be defined as in the proof of theorem [3.6] above. It is not difficult to check, as in the proof of the previous theorem, that for sufficiently small ϵ

$$\left[\mathrm{T}(z) - h(z) \right] \left[u(z) - u(z_s) \right] \geq 0 \qquad \text{for every } z \text{ in } [z_1, z_2] .$$

This contradicts proposition [2.6]. Thus condition (a) or (b) above must hold.

For any z_s, t_1, t_2 in $[z_1, z_2]$, $t_1 \leq t_2$, either t_1 is in $[z_1, z_s]$ or t_2 is in $[z_s, z_2]$. From parts (A) and (B) of the theorem it follows easily that :

Corollary 3.8 *Let* (u, β) *be an unstable flow satisfying condition (A), and let* z_s *be a point in* $[z_1, z_2]$ *. Suppose* $\mathrm{T}(z) \leq \mathrm{T}_0$ *in* $z \leq z_s$ *and* $\mathrm{T}(z) \geq -\mathrm{T}_0$ *in* $z \geq z_s$ *for some* $T_o > 0$, *then* $k < \mathrm{T}_0$ *and the set of wave numbers of unstable modes of* (u, β) *is bounded.*

§ 3.3 Bounds on the wave numbers of unstable modes .

Corollary [3.8] gives conditions on $\mathrm{T}(z)$ so that the unstable modes have bounded wave numbers. In this subsection different sets of conditions will be described that ensure the same. The following lemma generalizes a deft piece of estimation in Drazin & Howard (1966). Let $z_s \in [z_1, z_2]$ and let

$$u_0 = \begin{cases} 2c_r - u(z_s) & \text{if} \quad u(z_1) \leq 2\,c_r - u(z_s) \leq u(z_2) \,; \\ u(z_1) & \text{if} \quad 2\,c_r - u(z_s) \leq u(z_1) \,; \\ u(z_2) & \text{if} \quad 2\,c_r - u(z_s) \geq u(z_2) \,. \end{cases}$$

Lemma 3.9 *Let* $z_s \in [z_1, z_2]$ *and* u_0 *be as above. Let* z_o *be a point such that* $u(z_o) = u_0$. *Let* $\theta(z)$ *and* $\phi(z)$ *be as in proposition [2.3], then* $\eta(z) \leq 1$ *for every* z *in* $[z_1, z_2]$ *where*

$$\eta(z) = \frac{u(z) - u(z_s)}{|u(z) - c|} \cdot \frac{\sin\{\phi(z) - \theta(z)\}}{\sin\phi(z)} .$$

Proof: let z_c be a point such that $u(z_c) = c_r$. For any t , we write u_t *and* θ_t for $u(z_t)$ *and* $\theta(z_t)$. We assume first that $u_s \leq c_r$, then

$$z_s \leq z_c \leq z_o \quad \text{and} \quad \theta_s \leq -\pi/2 \leq \theta_0 \leq -\pi - \theta_s .$$

Case(1) : $z_1 \leq z \leq z_s$, then

$u(z) \leq u_s$ and $0 \leq \phi(z) - \theta(z) \leq \pi + \theta(z) \leq -\theta_s \leq \pi$,

so that $c_i \cot(\phi - \theta) \geq -c_i \cot(\theta) = u_s - c_r$.

Thus

$$\begin{aligned} \eta(z) &= \frac{(u - u_s)}{|u - c| \{ \cos\theta - sin\theta \cot(\phi - \theta) \}} \\ &= \frac{(u - u_s)}{(u - c_r) - c_i \cot(\phi - \theta)} \\ &\leq \frac{(u_s - u)}{(c_r - u) + (u_s - c_r)} = 1 . \end{aligned}$$

Case(2) : $z_s \leq z \leq z_o$, then

$u - u_s \geq 0$, $0 \leq \phi - \theta \leq \pi$ and $\sin\phi \leq 0$ so that $\eta(z) \leq 0$.

Case (3) : $z_o \leq z \leq z_2$, then

$(u - u_s) \geq 0$ *and* $0 \geq \phi - \theta \geq \theta_0 = -\pi - \theta_s \geq -\pi$

and so $c_i \cot(\phi - \theta) \leq c_i \cot(-\pi - \theta_s) = (u_s - c_r)$.

Thus

$$\begin{aligned} \eta(z) &= \frac{u - u_s)}{(u - c_r) - c_i \cot(\phi - \theta)} \\ &\leq \frac{(u - u_s)}{(u - c_r) - (u_s - c_r)} = 1 . \end{aligned}$$

We have proved that if $u_s \leq c_r$ then $\eta(z) \leq 1$ for every z. A similar argument applies when $u_s \geq c_r$ and completes the proof of the lemma.

Proposition 3.10 *Let* (u, β) *be an unstable flow satisfying condition* (A). *Let* h *be a piecewise*$-C^1$ *function on* $[z_1, z_2]$. *Let* z_s *be in* $[z_1, z_2]$

and $\mathrm{U}_h(z_s) = 0$. *Let* $\mathrm{K}_h(z) = -\mathrm{U}_h(z) \,/\, \{ u(z) - u(z_s) \}$ *whenever* $u(z) \neq u(z_s)$ *and suppose* $\mathrm{K}_h(z) \geq 0$. *Let* k^2 *be sufficiently large so that* $\left[h'(z) + h^2(z) - k^2 \right] \leq 0$ *for every* z *in* $[z_1 , z_2]$, *then*

$$k^2 < \sup_z \left\{ h'(z) + h^2(z) + \mathrm{K}_h(z) \mid u(z) \neq u(z_s) \right\}$$

and so the set of wave numbers of the unstable modes is bounded if $\mathrm{K}_h(z)$ *is bounded on the set* $\{ u(z) \neq u(z_s) \}$.

Proof : Let z_o be as in lemma [3.9]. From proposition [2.3] we have

$$\mathrm{K}_h(z) \,.\, \eta(z) < h'(z) + h^2(z) - k^2 \quad \text{for some} \quad z \ \textit{in} \ [z_1 , z_2] .$$

The proposition now follows from lemma [3.9].

***Remark* 3.11** *Suppose* $u'(z_s) \neq 0$, *then* $\mathrm{K}(z)$ *has a continuous extension to* $[z_1 , z_2]$, *and so it is bounded.*

Taking $h \equiv 0$ in this proposition we obtain a generalization of a result due to Drazin & Howard (1966).

Corollary 3.12 *Let* $\mathrm{K}_0(z) = -u_+''(z) \,/\, \{ u(z) - u(z_s) \} \geq 0$, *whenever* $u(z) \neq u(z_s)$, *then* $k^2 < \mathrm{K}_0(z)$ *for some* z.

We use this proposition to obtain the following intrinsic criterion for the boundedness of the wave numbers of unstable modes.

Theorem 3.13 *Let* (u , β) *be an unstable flow satisfying condition* (A). *Suppose* $u_+''(z)$ *and* $\beta(z)$ *have no common zeros. Let* $z_1 \leq z_s \leq z_2$ *and let* $u'(z_s) \neq 0$. *Let* $\mathrm{T}(z) > \mathrm{T}_1$ *if* $z < z_s$ *and* $\mathrm{T}(z) < \mathrm{T}_2$ *if* $z > z_s$ *for some constants* T_1 *and* T_2 , *then the set of wave numbers of the unstable modes is bounded.*

Proof : It is clear from the hypothesis that $\mathrm{T}(z_s)$ is finite. Let $T_s = \mathrm{T}(z_s)$ and let $\epsilon > 0$ be a small number. Let h be the piecewise linear function on $[z_1 , z_2]$, whose graph is obtained by joining the points (z_1 , T_1), $(z_s - \epsilon , \mathrm{T}_1)$, (z_s , T_s), $(z_s + \epsilon , \mathrm{T}_2)$ *and* (z_2 , T_2) .

Clearly $h(z_s) = \mathrm{T}(z_s)$. When ϵ is sufficiently small and $u(z) \neq u(z_s)$, we have

$$\frac{\mathrm{T}(z) - h(z)}{u(z) - u(z_s)} > 0 \qquad \text{so that} \qquad \mathrm{K}(z) = \frac{-\,\mathrm{U}_h(z)}{u(z) - u(z_s)} > 0 .$$

Further by the remark above, $K(z)$ is bounded above. The theorem now follows from proposition [3.10].

With some more effort one can include the case when $u''_+(z)$ and $\beta(z)$ have common zeros. We state the result.

Theorem 3.14 *Let* (u, β) *be an unstable flow satisfying condition* (A). *Let* $z_1 \leq z_s \leq z_2$. *Let* $u'(z_s) \neq 0$. *Let* $T(z)$ *satisfy*

$T(z) > T_1$ *if* $z < z_s$ and $T(z) < T_2$ *if* $z > z_s$
for some constants T_1 and T_2.

(b) $L_- \geq R_+$, where $L_- = \liminf_{z \to z_s - 0} T(z)$ and $R_+ = \limsup_{z \to z_s + 0} T(z)$

(c) If $L_- = R_+ = T_s$ (say), then $\limsup_{z \to z_s} \dfrac{T(z) - T_s}{z - z_s} \neq \infty$.

Then the set of wave numbers of the unstable modes is bounded.

§ 3.4 Flows with indefinite velocity and discriminant.

In this subsection the stability of a class of flows, in which $u'(z)$ and $\Delta(z)$ change sign in $[z_1, z_2]$ in a prescribed way will be discussed. It is shown that nontrivial necessary conditions of instability continue to persist, though in a weaker form, showing thereby that some of these flows are stable. This class of flows is described by the following condition.

Condition(B): A flow (u, β) satisfies this condition if for some s_1, s_2 in $[z_1, z_2]$, $s_1 \leq s_2$,

(1) $\beta(z) \geq 0$ for every z in $[z_1, z_2]$.

(2) $\Delta(z)$ has an extension to some neighbourhood of $[z_1, z_2]$ satisfying $\Delta(z) \leq 0$ if z is in $[s_1, s_2]$ and $\Delta(z) \geq 0$ otherwise.

(3) $u(z)$ is monotonic in each of the intervals, $[z_1, s_1]$ and $[s_2, z_2]$.

Thus $\Delta(s_i) = 0$, for $i = 1, 2$; $u'(z)$ is unrestricted in $[s_1, s_2]$, and its signs in $S_1 = [z_1, s_1]$ and $S_2 = [s_2, z_2]$ could be different. For $i = 1, 2$ let

$$\sigma_i = \begin{cases} 1 & \text{if } u'(z) \geq 0 \text{ for every } z \text{ in } S_i ; \\ -1 & \text{if } u'(z) \leq 0 \text{ for every } z \text{ in } S_i . \end{cases}$$

Let $\quad u_\pm(z) \;=\; u(z_1) \;+\; \dfrac{1}{2}\displaystyle\int_{z_1}^{z} \left[\, u'(z) \,\pm\, p\,(z) \,\right] dz \;,$

$$\textit{where} \quad p\,(z) \;=\; \begin{cases} \sigma_i \sqrt{\Delta(z)} & \text{if} \quad \Delta(z) \,\geq\, 0 \quad \text{and} \;\; z \;\text{is in}\;\; S_i \;; \\ 0 & \text{if} \quad \Delta(z) \,\leq\, 0 \;. \end{cases}$$

This extends the definition of $u''_\pm(z)$ given earlier after remark [2.2]. $u_\pm(z)$ are C^1–functions in $[\,z_1\,,\,z_2\,]$, and have continuous second derivatives except possibly at the points s_1 and s_2. We now prove :

Theorem 3.15 *Let* $(\,u\,,\beta\,)$ *be an unstable flow satisfying condition (B), then*

(a) $(-)^i\,\sigma_i\,u''_+(z) \,<\, 0$ *for some* $i \,=\, 1$ *or* 2 *and some* z *in* S_i .

(b) If $\Delta'(s_1 \,-\, 0) \;\neq\; 0 \;\neq\; \Delta'(s_2 \,+\, 0)$ *then* $u_+(z)$ *has a point of inflection at some point* z_s *outside* $[\,s_1\,,\,s_2\,]$, *satisfying* $\Delta(z_s) \,\leq\, 0$.

(c) If $s_1 \,=\, s_2$, *let one of the conditions* (1) *or* (2) *hold :*

(1) $\quad \mathrm{L}_+ \,<\, \mathrm{R}_- \quad \textit{where} \quad \mathrm{L}_+ \,=\, \limsup\limits_{z\to t_0-0}\; \mathrm{T}(z) \quad \textit{and} \quad \mathrm{R}_- \,=\, \liminf\limits_{z\to t_0+0}\; \mathrm{T}(z)\,,$

(2) $\quad \mathrm{L}_+ \,=\, \mathrm{R}_- \,=\, \mathrm{T}_0 \;\textit{(say)} \qquad \textit{and} \qquad \liminf\limits_{z\to t_0} \dfrac{\mathrm{T}(z) - \mathrm{T}_0}{z - t_0} \;\neq\; -\infty\,,$

then one of the conditions (3) *or* (4) *is satisfied :*

(3) $\quad \limsup\limits_{t\to z}\;\; \mathrm{T}\,(t) \;\geq\; k\,\coth\{\,k\,(\,z\,-\,z_1\,)\} \quad \textit{for some} \;\; z \;\; \textit{in} \;\; [\;z_1\,,\,s_1\;]\,,$

(4) $\quad \liminf\limits_{t\to z}\;\; \mathrm{T}\,(t) \;\leq\; k\,\coth\{\,k\,(\,z\,-\,z_2\,)\} \quad \textit{for some} \;\; z \;\; \textit{in} \;\; [\;s_2\,,\,z_2\;]\,,$

so that if $(-)^{\,i+1}\,\mathrm{T}(z)$ *is bounded above in* S_i *for each* i, *then*

$$k \;<\; \max\;\left[\;\sup_{z\,\in\,S_1}\,\{\;\mathrm{T}\,(z)\;\}\;\;,\quad -\inf_{z\,\in\,S_2}\,\{\;\mathrm{T}\,(z)\;\}\;\right] \;<\; \infty\;.$$

Proof : Let $z_o \;=\; [\,s_1 \,+\, s_2\,]\,/\,2$. Let $\phi(z)$ be as in (2.7), with $u_\pm(z)$ as defined above. Clearly $\phi'(z) \;=\; \theta'(z)$ in $[\,s_1\,,\,s_2\,]$. It follows that $\phi(z) \;=\; \theta(z)$ in $[\,s_1\,,\,s_2\,]$. Further if z is in S_i and $\phi(z) \,\neq\, \theta(z)$, then $[\;\phi(z) \,-\, \theta(z)\;] \;\simeq\; (-)^{i+1}\,\sigma_i$ [see definition 2.5], because $\phi'(z) \,-\, \theta'(z) \;\simeq\; 2\,u'_-(z) \,-\, u'(z) \;\simeq\; -\,\sigma_i$.

It follows from proposition [2.3] and equation (2.9), that for any piecewise–C^1 function $h(z)$, satisfying $[\;h'(z) \,+\, h^2(z) \,-\, k^2\;] \;\leq\; 0$ and for some $i \,=\, 1$ or 2

$$(-)^i\,\sigma_i\,\mathrm{U}_h(z) \;<\; 0 \quad \text{for} \quad \text{some} \;\; z \;\; \text{in} \;\; S_i\,. \tag{3.1}$$

Part (a) follows from this on taking $h \equiv 0$.

Under the conditions of part (b), $\sigma_1 u_+''(z) \rightarrow -\infty$ as $z \rightarrow s_1 - 0$, and $\sigma_2 u_+''(z) \rightarrow \infty$ as $z \rightarrow s_2 + 0$. Now part (b) follows from (a). To prove part (c) we may assume that if $s_1 = s_2$ then
$k \coth[k(s_2 - z_2)] < \mathrm{L}_+ \leq \mathrm{R}_- < k \coth[k(s_1 - z_1)]$. Let

$$\mathrm{T}_0 = \begin{cases} \mathrm{T}(z_o) & \text{if} \quad s_1 < s_2 \quad \text{and} \quad z_o \text{ is in } \mathscr{T} \\ [\,\mathrm{L}_+ + \mathrm{R}_-\,]/2 & \text{if} \quad s_1 = s_2 \quad \text{or} \quad z_o \text{ is not in } \mathscr{T}. \end{cases}$$

Let $\epsilon > 0$ be a small number. Let $h(z)$ be the function defined in the proof of theorem [3.6] with z_o and T_0 replacing z_s and T_s respectively. If possible suppose neither of the conditions (3) and (4) hold. It is clear then that for sufficiently small ϵ,

$$\mathrm{T}(z) - h(z) < 0 \quad \text{in} \quad S_1 \quad \text{and} \quad \mathrm{T}(z) - h(z) > 0 \quad \text{in} \quad S_2.$$

Further $\sigma_i u'(z) \geq 0$ for every z outside $[s_1, s_2]$. It follows that $(-)^i \sigma_i [\,u_+''(z) - 2h(z)u_-'(z)\,] > 0$ for each $i = 1$ and 2 and every z in S_i. This contradicts (3.1) and proves part (c).

§ 3.5 Propagation of neutral modes

A normal mode (k, c) of a flow (u, β) is called a neutral mode if $c_i = 0$. It is called a marginally stable mode if c is in $[u_{\min}, u_{\max}]$. These modes are important. In the space of normal modes of flows where a point looks like (u, β, k, c) satisfying (1.1) and (1.2), it is the marginally stable modes that separate stability from instability.

A neutral mode with c outside $[u_{\min}, u_{\max}]$ is called an internal gravity wave. Lemma [2.1] with $f(z) = \{u(z) - c\}$ then gives

$$\int_{z_1}^{z_2} (u - c)^2 |F'|^2 dz = \int_{z_1}^{z_2} \left[-k^2(u - c)^2 + g\beta(z)\right] |F(z)|^2 dz.$$

This shows as observed in Drazin & Howard (1966) that no internal gravity waves can exist if $\beta(z)$ is negative everywhere. Otherwise using the well known inequality that if a C^1-function $\psi(z)$ satisfies $\psi(z_1) = 0 = \psi(z_2)$ then

$$\int_{z_1}^{z_2} |\psi(z)|^2 dz \leq \frac{(z_2 - z_1)^2}{\pi^2} \int_{z_1}^{z_2} |\psi'(z)|^2 dz, \qquad (3.2)$$

one obtains $|u(z) - c|^2_{\min} < g\beta_{\max}(z_2 - z_1)^2/\pi^2$. It follows that :

Proposition 3.16 *Let* (k, c) *be a neutral mode of a flow* (u, β) *and suppose* $\beta(z) \geq 0$ *for some* z, *then*

$$u_{\min} - \alpha < c < u_{\max} + \alpha \quad \textit{where} \quad \alpha = \sqrt{g\,\beta_{\max}}\ [z_2 - z_1] / \pi .$$

Let $c \in \mathbb{R}$. Let $\beta(z_s) = \beta'(z_s) = 0 = u''(z_s)$ whenever $u(z_s) = c$. Let $\mathrm{K}(z) = -\dfrac{u''(z)}{\{u(z) - c\}} + \dfrac{g\,\beta(z)}{\{u(z) - c\}^2}$, be integrable in $[z_1, z_2]$. This is the case if c is outside $[u_{\min}, u_{\max}]$ otherwise the condition $u'(z_s) \neq 0$ whenever $u(z_s) = c$ is sufficient to ensure this. Suppose $\mathrm{K}(z) > \pi^2 / \{z_2 - z_1\}^2$ whenever $u(z) \neq c$. Equations (1.1) and (1.2) then constitute a regular Sturm-Liouville problem, and so the flow (u, β) admits a neutral mode (k_s, c), where

$$k_s^2 = -\min_{\phi} \left\{ \int_{z_1}^{z_2} (\phi'^2 - \mathrm{K}\,\phi^2)\,dz \Big/ \int_{z_1}^{z_2} \phi^2\,dz \right\} > 0 ,$$

the minimum being taken over all functions ϕ such that ϕ and ϕ' are square integrable and $\phi(z_1) = 0 = \phi(z_2)$. Further if (k, c) is any neutral mode then $k^2 \leq k_s^2$. We now prove :

Theorem 3.17 *Let* (k, c) *be a smooth neutral mode for* (u, β) *satisfying condition (A). For some* $z_s \in [z_1, z_2]$ *let* $u(z_s) = c$, $\beta(z_s) = 0 = u_+''(z_s)$ *and* $u'(z_s) \neq 0$, *then one of the following conditions hold :*

(1) $\limsup_{t\to z} \mathrm{T}(t) \geq k \coth[k(z - z_1)] > \dfrac{1}{z - z_1}$, *for some* $z < z_s$.

(2) $\liminf_{t\to z} \mathrm{T}(t) \leq k \coth[k(z - z_2)] < \dfrac{1}{z - z_2}$, *for some* $z > z_s$.

(3) *Condition (b) of theorem [3.7] holds with* $t_1 = t_2 = t_0 = z_s$.

Proof: $u_-'(z_s) = 0$ because $\beta(z_s) = 0$. Let $h(z)$ be piecewise$-C^1$.

$$\text{Let} \qquad f(z) = \exp\left[\int_{z_s}^{z} \left\{ \frac{u_-'(z)}{\{u(z) - c\}} + h(z) \right\} dz \right] .$$

$f(z)$ is then smooth and positive everywhere. Equation (2.1) then shows that for every z in $[z_1, z_2]$,

$$[f^2(z)\,F'(z)]' + \left[\{h'(z) + h^2(z) - k^2\} - \frac{\mathrm{U}_h(z)}{\{u(z) - c\}} \right] F(z) = 0 .$$

Multiplication by $F^*(z)$ and integration as in lemma [2.1] yields

$$\frac{\mathrm{U}_h(z)}{\{\, u(z) - u(z_s)\,\}} \;<\; \{\, h'(z) + h^2(z) - k^2 \,\} \quad \text{for some} \quad z \in (\, z_1 \,,\, z_2 \,) \;.$$

The theorem follows from this by an argument similar to that used to prove theorem [3.7]. We next prove :

Theorem 3.18 *Let* (u, β) *satisfy condition* (*A*). *Let* (k, c) *be a smooth neutral mode and let* c *be outside* $[\, u_{\min}\,,\, u_{\max}\,]$, *then*

(1) $\dfrac{u''_+(z)}{2\,u'_-(z)} \;\geq\; k \;\coth\left[\; k\;(z - z_1)\;\right] \quad for \;\; some \;\; z \quad if \quad c > u_{\max}\,.$

(2) $\dfrac{u''_+(z)}{2\,u'_-(z)} \;\leq\; k \;\coth\left[\; k\;(z - z_2)\;\right] \quad for \;\; some \;\; z \quad if \quad c < u_{\min}\,.$

(3) *Parts* (1) *and* (2) *hold if* $u_+(z)$ *and* $u_-(z)$ *are interchanged.*

(4) *If either of* $[\; u''_\pm(z) \,/\, 2\,u'_\mp(z)\;]$ *is bounded then* k *is bounded.*

Proof : Let $\quad f(z) \;=\; \exp\left[\; \displaystyle\int_{z_s}^{z} \left\{ \frac{u'_\mp(z)}{\{\, u(z) - c\,\}} \;+\; h(z) \right\} dz \;\right]$ in turn.
Parts (1) and (2) follow on taking the $-ve$ sign above by an argument similar to that used to prove theorem [3.6] while part (3) follows on taking the $+ve$ sign. Part (4) follows from (1), (2) and (3).

§ 3.6 Bounds on the phase speed c_i of unstable modes.

This section deals with two situations where upper bounds on the phase speed c_i are obtained as alternatives to Rayleigh–Fjørtoft type conditions. For flows satisfying condition (A), theorem [3.19] is obtained if $u_-(z)$ is replaced by $u_+(z)$ in (2.7). Theorem [3.22] is about flows in which $\beta(z)$ is negative somewhere. In both cases an upper bound on c_i is obtained if (2.9) does not hold.

Let $\quad A \;=\; \pi \,+\, \theta(z_1) \;=\; \tan^{-1}\;[\; c_i \,/\, (c_r - u_{\min})\;]\;,$

and $\quad B \;=\; -\,\theta(z_2) \;=\; \tan^{-1}\;[\; c_i \,/\, (u_{\max} - c_r)\;]\;.$

Clearly $0 \,<\, A\,,\; B \;\leq\; \pi \,/\, 2\,.$ Let $\quad r \,=\, \frac{1}{2}\,[\, u_{\max} \,-\, u_{\min}\,]$ then $\quad c_i \,\leq\, r\,.$

It follows from (2.4) that

$$\theta(z_2) - \theta(z_1) = \pi - A - B = c_i \int_{z_1}^{z_2} \frac{u'(z)\, dz}{|u - c|^2} . \tag{3.3}$$

Further $$\min\{A, B\} \geq \tan^{-1} \left(\frac{c_i}{2r} \right) \geq \frac{c_i}{r} \tan^{-1} (1/2) \tag{3.4}$$

because $\tan^{-1} x \geq (x / x_o) \tan^{-1} x_o$ if $0 \leq x \leq x_o \leq \pi/2$.

and $$A + B = \tan^{-1} \left\{ \frac{2 r c_i}{r^2 - \left[c_r - \frac{1}{2}(u_{\max} + u_{\min}) \right]^2 - c_i^2} \right\}$$

$$\geq \tan^{-1} (2 c_i / r) \geq \frac{c_i}{r} \tan^{-1} (2) . \tag{3.5}$$

Theorem 3.19 *Let (k, c) be an unstable mode for the flow (u, β) satisfying condition (A). Let $\nu(z) = \Delta(z) / u'(z)$ and $\nu_m = \max_z \nu(z)$, then one of the following conditions (a) or (b) holds*

(a) *$u''_-(z)$ takes both negative and positive values so that $u''_-(z_s) = 0$ for some z_s in (z_1, z_2), and $u''_-(z) [u(z) - u(z_s)] > 0$ for some z in (z_1, z_2).*

(b) $c_i \leq \dfrac{\nu_m \pi r}{[\tan^{-1}(1/2) + \nu_m \tan^{-1}(2)]}$ *so that* $c_i \to 0$ *as* $r \nu_m \to 0$.

Proof: Let $\phi_+(z) = \theta(z_o) + c_i \displaystyle\int_{z_o}^{z} \frac{2 u'_+(z)}{|u(z) - c|^2} dz$ and suppose the condition: $-\pi < \phi_+(z) < 0$ for every z_o, z in $[z_1, z_2]$, holds. In this case it is easily seen that $\sin\{\phi_+(z) - \theta(z)\} \simeq \{u(z) - u(z_o)\}$. Remark [2.4] then leads to the result in proposition [2.3] with $u_+(z)$ and $u_-(z)$ interchanged and ϕ replaced by ϕ_+. The arguments used to prove parts (1) and (2) of theorem [2.7] now prove the part (a) above.

On the other hand suppose for some z_o, z in $[z_1, z_2]$, the condition: $-\pi < \phi_+(z) < 0$ does not hold, then

$$c_i \int_{z_1}^{z_2} \frac{u' (1 + \nu) \, dz}{|u - c|^2} \geq \min(A, B) + \theta(z_2) - \theta(z_1)$$

so that $$\nu_m (\pi - A - B) \geq \min(A, B)$$

because of (3.3). Part (b) now follows from (3.4) and (3.5). This completes the proof of the theorem.

The flows (u, β) with $\beta(z) < 0$ somewhere will be considered now. In this case $\nu_m > 1$. All these flows are believed to be unstable. We now prove that for such a flow the complex wave velocities of unstable modes is bounded.

Proposition 3.20 *Let (k, c) be an unstable mode of a flow (u, β) and $\beta(z) \leq 0$ somewhere. Let $r_o^2 = r^2 + [-g\beta]_{\max}(z_2 - z_1)^2/\pi^2$, then*
(1) $u_{\min} < c_r < u_{\max}$ *and*
(2) $\left[c_r - \frac{1}{2}(u_{\max} + u_{\min}) \right]^2 + c_i^2 \leq r_o^2$.

Proof: Taking $f(z) = \{u(z) - c\}$ in lemma [2.1] and proceeding as in the proof of Howard's semicircle theorem (see Howard 1961), one obtains

$$(1) \qquad \int_{z_1}^{z_2} \{ u(z) - c_r \} \left(|F'|^2 + k^2 |F|^2 \right) dz = 0 \qquad \text{and}$$

$$(2) \qquad \left\{ \left[c_r - \tfrac{1}{2}(u_{\max} + u_{\min}) \right]^2 + c_i^2 - r^2 \right\} \int_{z_1}^{z_2} \left(|F'|^2 + k^2 |F|^2 \right) dz$$
$$\leq - \int_{z_1}^{z_2} g\beta |F|^2 \, dz \leq [-g\beta]_{\max} \int_{z_1}^{z_2} |F|^2 \, dz .$$

The proposition now follows from these and (3.2).

Remark 3.21 *It is clear from propositions [3.16] and [3.20] together with Howard's semicircle theorem that for an arbitrary flow (u, β), the set of complex wave velocities of its normal modes is bounded.*

Theorem 3.22 *Let (k, c) be an unstable mode for the flow (u, β). Let u' and Δ be nonnegative functions and let $\beta(z)$ be negative somewhere. Let $\nu_c = \nu(z_c)$ where $u(z_c) = c_r$ and let $a = \max_z \{ \nu'(z)/u'(z) \}$ then one of the conditions (a) or (b) holds.*

(a) *Parts (1) and (2) of theorem [2.7] hold.*

$$(b) \qquad c_i \leq \frac{(\nu_m - 1)\pi r_o}{\left[\tan^{-1}(r_o/2r) + (\nu_m - 1)\tan^{-1}(2r_o/r) \right]} \qquad \text{and}$$

$$\nu_c \geq 1 - \frac{2(\nu_m - 1) a r_o}{\left[\tan^{-1}(r_o/2r) + (\nu_m - 1)\tan^{-1}(2r_o/r) \right]},$$

so that $c_i \to 0$ as $a r_o (\nu_m - 1) \to 0$.

Proof: Let $\phi(z)$ be as in (2.7). Suppose the condition: $-\pi < \phi(z) < 0$ for every z_o, z in $[z_1, z_2]$, holds. In this case the same argument that is used to prove parts (1) and (2) of theorem [2.7], proves part (a).

On the other hand suppose for some z_o and z in $[z_1, z_2]$, the condition: $-\pi < \phi(z) < 0$ does not hold

$$\text{then} \qquad c_i \int_{z_1}^{z_2} \frac{u'(1-\nu)\,dz}{|u-c|^2} \leq -\min(A, B) \tag{3.6}$$

so that $\quad (\nu_m - 1)(\pi - A - B) \geq \min(A, B)$.

The definition of the angles A and B given above, gives in this case

$$\min\{A, B\} \geq \tan^{-1}\left(\frac{c_i}{2r}\right) \geq \frac{c_i}{r_o}\tan^{-1}(r_o/2r)$$

$$\text{and} \qquad A + B \geq \tan^{-1}(2c_i/r) \geq \frac{c_i}{r_o}\tan^{-1}(2r_o/r),$$

so that $(\nu_m - 1)\pi \geq \dfrac{c_i}{r_o}[\tan^{-1}(r_o/2r) + (\nu_m - 1)\tan^{-1}(2r_o/r)]$.

The first inequality in part (b) follows now from this equation. To obtain the second inequality, sharper estimation of the integral in (3.6) is needed. It may be assumed that $\nu_c < 1$. Let $s = (1-\nu_c)/a$ then

$$\nu(z) - 1 < \nu_c - 1 + as = 0 \qquad \text{if} \quad |u(z) - c| < s$$

Let $E = \dfrac{1}{2}\pi - \tan^{-1}(\dfrac{c_i}{s})$. Equation (3.6) then gives

$$(\nu_m - 1)(\pi - A - B - 2E) \geq \min(A, B)$$

so that $\quad (\nu_m - 1)(\pi - 2E) \geq \min(A, B) + (\nu_m - 1)(A + B)$.

Further $\quad \pi - 2E = 2\tan^{-1}(c_i/s) < 2c_i/s$. It follows that

$$(\nu_m - 1)a \geq \frac{(1-\nu_c)}{2r_o}[\tan^{-1}(r_o/2r) + (\nu_m - 1)\tan^{-1}(2r_o/r)].$$

The second inequality in part (b) of the theorem follows from this.

***Remark* 3.23** *It is clear from the proofs that in both the theorems above, results similar to those of § 3 will hold if the condition in part (b) does not hold.*

4 Fundamental solutions of TGE

In this section TGE will be solved for a class of flows. It is seen here in a partially discreet form as a quadratic recursion relationship on a sequence of smooth functions. A pair of linearly independent solutions is obtained in some neighbourhood of the critical layer $z = z_c$ where $u(z_c) = c_r$, in theorem [4.1] when $c_i > 0$ is sufficiently small. To prove the smoothness of the solutions it is assumed that $[\, g\,\beta(z) / u'^2(z)\,]$ is a constant. It is very likely that the results hold without this. Corollary [4.3] gives a smooth solution of Rayleigh's equation which vanishes at $z = z_c$ when $c_i = 0$.

Let U be an open subinterval of (z_1, z_2). Let z_c be a point in U. Let $u(z)$, $\beta(z)$, and $v(z)$ be $C^\infty-$functions on U. For z in U let $w(z) = \exp\left[\ \int_{z_c}^{z} v(z)\,dz\right]$ satisfy (1.1), then

$$v'(z) + v^2(z) - k^2 - \frac{u''(z)}{\{u(z) - c\}} + \frac{g\,\beta(z)}{\{u(z) - c\}^2} = 0\,.$$

For $n \geq 0$ let $a_n(z)$ be $C^\infty-$functions on U and let

$$v(z) = \frac{u'_{\pm}(z)}{\{u(z) - c\}} + \sum_{n=0}^{\infty} a_n(z)\,Y^n(z)\,, \quad \text{where} \quad Y(z) = \frac{u(z) - c}{u'(z)}\,.$$

Suppose these series converge uniformly on some open subset J of U. Substituting these values of $v(z)$ in turn, in the equation above, one gets

$$\frac{-\,u''_{\mp}(z) + 2\,u'_{\pm}(z)\,a_0}{\{u(z) - c\}} + \left[\left\{1 + \frac{2\,u'_{\pm}(z)}{u'(z)}\right\} a_1 + a'_o + a_0^2 - k^2\right] +$$
$$\sum_{n=1}^{\infty}\left[\left\{n + 1 + \frac{2\,u'_{\pm}(z)}{u'(z)}\right\} a_{n+1} + a'_n - n\,a_n\,\frac{u''(z)}{u'(z)} + \sum_{j=0}^{n} a_j\ a_{n-j}\right] Y^n = o\,.$$

This proves part (a) of the following :

Theorem 4.1 *Let* (u, β) *be as above. Let* $\beta(z)$, $\Delta(z) \geq 0$ *and* $u'(z) > 0$ *for every* z *in* U. *Let* (k, c) *be given and* $u(z_c) = c_r$.

$$\textit{Let} \qquad w_1(z) = \exp\left[\ \int_{z_c}^{z}\left\{\frac{u'_{+}(z)}{\{u(z) - c\}} + \sum_{n=0}^{\infty} a_n(z)\,Y^n(z)\right\} dz\right]$$

$$\textit{and} \qquad w_2(z) = \exp\left[\ \int_{z_c}^{z}\left\{\frac{u'_{-}(z)}{\{u(z) - c\}} + \sum_{n=0}^{\infty} b_n(z)\,Y^n(z)\right\} dz\right]$$

$$\textit{where} \qquad a_0(z) = \frac{u''_{-}(z)}{2u'_{+}(z)}\,, \quad a_1(z) = \frac{\{k^2 - a'_0(z) - a_0^2(z)\}}{1 + 2\{u'_{+}(z) / u'(z)\}} \qquad \textit{and}$$

$$a_{n+1}(z) = \frac{1}{(n+1) + 2\{u'_{+} / u'\}}\left[-a'_n + n\,a_n\left\{\frac{u''}{u'}\right\} - \sum_{j=0}^{n} a_j\ a_{n-j}\right]$$

for $n \geq 1$ *and* $b_n(z)$ *is obtained from* $a_n(z)$ *on interchanging* $u_+(z)$ *and* $u_-(z)$.

(a) *Suppose the series* $\sum a_n(z)\, Y^n(z)$ [*resp.* $\sum b_n(z)\, Y^n(z)$] *converges uniformly in some neighbourhood* J *of* z_c, *then* $w_1(z)$ [*resp.* $w_2(z)$] *is a smooth solution of TGE in* J.

(b) *Let Richardson number* $\{ g\,\beta(z) / u'^2(z) \}$ *be a constant in* U. *Let* $\mathrm{s} \geq 1 + k^2$ *be a constant. Suppose for every* z *in* U,

(1) $\left| \dfrac{d^j}{dz^j} \left\{ \dfrac{u''(z)}{u'(z)} \right\} \right| \leq \mathrm{s}^{\,j+1}$ *for every* $j \geq 0$ *and*

(2) $|\, a_0(z) \,|$ (*resp.* $|\, b_0(z) \,|$) $\leq \mathrm{s}$, *then*

for sufficiently small $c_i > 0$, $\sum a_n(z)\, Y^n(z)$ [*resp.* $\sum b_n(z)\, Y^n(z)$] *converges absolutely and uniformly in some neighbourhood* J *of* z_c.

Proof: Under the hypothesis of part (b), $u'_\pm(z)$ are constant multiples of $u'(z)$. It is then clear that a_n and b_n are polynomials in $[\, u''(z) / u'(z) \,]$ and its derivatives with constant coefficients.

Let $t_j(z) = \dfrac{d^j}{dz^j} \left\{ \dfrac{u''(z)}{u'(z)} \right\}$ for $j \geq 0$. Let $R = \mathbb{R}[\, t_0,\, t_1,\, \ldots]$ be the ring of polynomial functions of $t_0,\, t_1,\, \ldots$ with real coefficients. For a multi–index $m = (\, m_0,\, m_1,\, \ldots)$ with only finite number of nonzero entries, let $deg\,(m) = \sum_{j=0}^{\infty} (j+1)\, m_j$ and $t^m = \prod_{j=0}^{\infty} t_j^{m_j}$.

For $q(z) = \sum_m q_m\, t^m$ in R, let $deg\,(q) = \max\limits_m \{\, deg\,(m) \mid q_m \neq 0 \,\}$ and $\|\, q \,\|_{\mathrm{s}} = \sum_m |\, q_m \,|\, \mathrm{s}^{\,deg\,(m)}$. It is easily checked that if $q,\, q_1,\, q_2 \in R$ and $\lambda \in \mathbb{R}$, then

(1) $\sup\limits_z |\, q(z) \,| \leq \|\, q \,\|_{\mathrm{s}}$; (2) $\|\, q_1 + q_2 \,\|_{\mathrm{s}} \leq \|\, q_1 \,\|_{\mathrm{s}} + \|\, q_2 \,\|_{\mathrm{s}}$;

(3) $\|\, q_1\, q_2 \,\|_{\mathrm{s}} \leq \|\, q_1 \,\|_{\mathrm{s}} \,\|\, q_2 \,\|_{\mathrm{s}}$; (4) $\|\, \lambda\, q \,\|_{\mathrm{s}} = |\, \lambda \,|\, \|\, q \,\|_{\mathrm{s}}$ and

(5) $\|\, q' \,\|_{\mathrm{s}} \leq \mathrm{s}\; deg\,(q)\; \|\, q \,\|_{\mathrm{s}}$.

The last part follows because on differentiating a monomial of degree n one obtains a sum of at most n monomials of degree $(\, n+1 \,)$.

It is clear from the hypothesis that $a_n, b_n \in R$ for every $n \geq 0$ and $deg\,(a_n) = deg\,(b_n) = (\, n+1 \,)$. Further $\|\, a_0 \,\|_{\mathrm{s}} \leq \mathrm{s}$ and $\|\, a_1 \,\|_{\mathrm{s}} \leq 3\,\mathrm{s}^2$. We now prove that $\|\, a_j \,\|_{\mathrm{s}} \leq 3^j\, \mathrm{s}^{\,j+1}$ for every $j \geq 0$. Inductively we assume that this result holds for $j < n$, then

$$\| a_n \|_{\mathrm{s}} \;\leq\; \frac{1}{n} \, \| a'_{n-1} \|_{\mathrm{s}} \;+\; \mathrm{s} \, \| a_{n-1} \|_{\mathrm{s}} \;+\; \frac{1}{n} \, \sum_{j=0}^{n-1} 3^{\,n-1} \, \mathrm{s}^{\,n+1} .$$

$$\leq\; 3^{\,n} \, \mathrm{s}^{\,n+1} \;,$$

so that
$$\limsup_{n \to \infty} \left[\; \sup_{z} \; | \, a_n(z) \, |^{\frac{1}{n}} \right] \;\leq\; 3 \, \mathrm{s} \, .$$

It follows that $\sum_{n=0}^{\infty} a_n(z) \, Y^n$ converges absolutely and uniformly for z in U, and $| \, Y \, | \leq r$, for any $r < 1 \, / \, [\, 3 \, \mathrm{s} \,]$.

A similar argument with a_n replaced by b_n gives the same result about the convergence of $\sum_{n=0}^{\infty} b_n(z) \, Y^n(z)$. Part (b) now follows from the fact that for sufficiently small $c_i > 0$, the set $\{ \, | \, Y(z) \, | \leq r \, \}$ is a closed neighbourhood of z_c in $[\, z_1 \, , \, z_2 \,]$.

***Remark* 4 .2** (1) *The function $u(z)$ together with its two derivatives can be uniformly approximated by a function $q(z) = \int_{z_0}^{z} [\exp \{ \int_{z_0}^{z} p \, (z) \, dz \, \} \, dz \,]$ where $p \, (z)$ is a polynomial in z. Clearly then $[\, q''(z) \, / \, q'(z) \,]$ together with all its derivatives is uniformly bounded.*

(2) *Suppose $c_i = 0 = \beta(z_c) = u''_{+}(z_c)$ and $\mathrm{T}(z)$ has a smooth extension to $[\, z_1 \, , \, z_2 \,]$, then $w_2(z)$ is a smooth solution of TGE.*

When $\beta \equiv 0$, we have $u_{+}(z) = u(z)$, and $u_{-}(z) = 0$. This leads to the regular solution of Rayleigh's equation.

Corollary 4 .3 *Suppose $\beta(z) = 0$ and $u'(z) > 0$ for every z in U. Let $a_1(z) = k^2 \, / \, 3$, and for $n \geq 1$*

$$a_{n+1}(z) \;=\; \frac{1}{n+3} \; \left[- \, a'_n \;+\; n \, a_n \left(\frac{u''}{u'} \right) \;-\; \sum_{j=1}^{n-1} a_j \; a_{\,n-j} \; \right] .$$

Let
$$w(z) \;=\; \{ \, u(z) \;-\; c \, \} \; \exp \left[\; \int_{z_0}^{z} \sum_{n=1}^{\infty} a_n(z) \; Y^n(z) \; dz \right] .$$

Suppose for some $\mathrm{s} \;\geq\; 1 + k^2 \, / \, 3$, $\left| \, \frac{d^j}{dz^j} \left\{ \frac{u''(z)}{u'(z)} \right\} \right| \;\leq\; \mathrm{s}^{\,j+1}$ for every $j \geq 0$ and every z in U. Let $c_i \geq 0$ be sufficiently small then the series $\sum_{n=0}^{\infty} a_n(z) \, Y^n(z)$ converges absolutely and uniformly in some neighbourhood J of z_c where $u(z_c) = c_r$ and $w(z)$ is a smooth solution of Rayleigh's equation in J.

§ 4.1 Existence of unstable flows.

So many parallel flows are believed to be unstable because they are not seen to persist, yet very few can be proved to be so. The only known example of an unstable parallel flow with rigid and finite boundaries is a piecewise linear flow called a shear- layer (see Drazin & Reid 1981). The difficulty is that no general result for the existence of solutions of Taylor-Goldstein boundary value problem is known. A good beginning in this direction has been made (see Friedlander & Howard 1998 ; Friedlander 2001) but much remains to be done here. An existence theorem will provide deep insight into the subject. We expect :

(1) Let (u, β) be a flow satisfying condition (A) and let $\mathrm{T}(z)$ be bounded. Suppose for every $k^2 \leq |\mathrm{T}|_{\max}$, every c inside Howard's semicircle and every piecewise$-C^2$ function $f(z)$ satisfying $Im\,[\,f^{\,2}\,] < 0$ everywhere, the condition : $Im\left[\,ff'' + Af^2\,\right] < 0$ for some z, holds then the flow is unstable.

(2) Suppose for some $t_o < t_1 < t_2$ in $[\,z_1\,,\,z_2\,]$, $\Delta(t_o) < 0$, $\Delta(t_2) < 0$ and $\Delta(t_1) > 0$ then the flow is unstable.

(3) If $\beta(z) < 0$ for some z then the flow is unstable.

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