PHY478F Undergraduate Research Project on Hydrodynamic Stability

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Abstract

This undergraduate research project focused upon the problem of hydrodynamic stability, specifically on the aspect of this general class of problems that involves the transition to turbulence in density stratified shear flows. The conservation laws of mass and momentum were studied. The equation of motion, Navier-Stokes equation, was derived from the conservation laws. Reyleigh's stability equation and Taylor-Goldstein equation were used to investigate the mechanisms of transition of the Kelvin-Helmholtz and Holmboe modes of instability.

Contents

1 Background

1.1 Material Derivative

$$
\begin{aligned}
\frac{Df}{Dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \\
&= \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f \qquad (1)
\end{aligned}
$$

where $\mathbf{u}$ is the velocity of the packet.

1.2 Conservation Laws

1.2.1 Conservation of Mass

Consider a small test volume V with surface A. The rate of change of the mass in the volume is

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV \tag{2}$$

The derivative can be moved inside the integral because the test volume is fixed in position.

The mass flow out of the surface is given by

$$\int_A \rho \mathbf{u} \cdot \mathbf{n}\, dA = \int_V \nabla \cdot (\rho \mathbf{u})\, dV \tag{3}$$

where divergence theorem was used to change the equation into a volume integral.

From the law of conservation of mass, the rate of change of the mass in the volume must be equal to the mass flow *into* the surface, thus

$$\begin{aligned}\int_V \frac{\partial \rho}{\partial t}\, dV &= -\int_V \nabla \cdot (\rho \mathbf{u})\, dV \\ \int_V \left[(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] dV &= 0 \end{aligned} \tag{4}$$

Since the test volume is arbitrary, the function inside the bracket [ ] must vanish everywhere. Hence

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \tag{5}$$

It can be written in material derivative:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{6}$$

For an incompressible fluid, $D\rho/Dt = 0$, the equation of continuity is reduced to

$$\nabla \cdot \mathbf{u} = 0 \tag{7}$$

1.2.2 Conservation of Momentum

From the law of conservation of momentum, consider a same fixed volume V:

$$\frac{d}{dt}(\text{momentum in } V) = -\text{momentum leakage} + \sum(\text{applied force}) \tag{8}$$

The rate of change of mometum in the volume is

$$\frac{d}{dt}\int_V \rho u_i \, dV = \int_V \frac{\partial \rho u_i}{\partial t} \, dV \tag{9}$$

where u_i is the ith component of the velocity $\mathbf{u}$. Similar to (2), the derivative can be moved inside the integral.

The momentum leakage (in component form) is given by

$$\int_A (\rho u_i) u_j (n_j \, dAj) = \int_V \frac{\partial}{\partial x_j} (\rho u_i u_j) \, dV \tag{10}$$

where divergence theorem was applied again.

The applied forces consist of body force and surface force.

The body force is simply

$$\text{body force} = \int_V \rho f_i \, dV \tag{11}$$

where f_i is the component of the body force per unit mass.

Considered a tetrahedral volume of fluid.

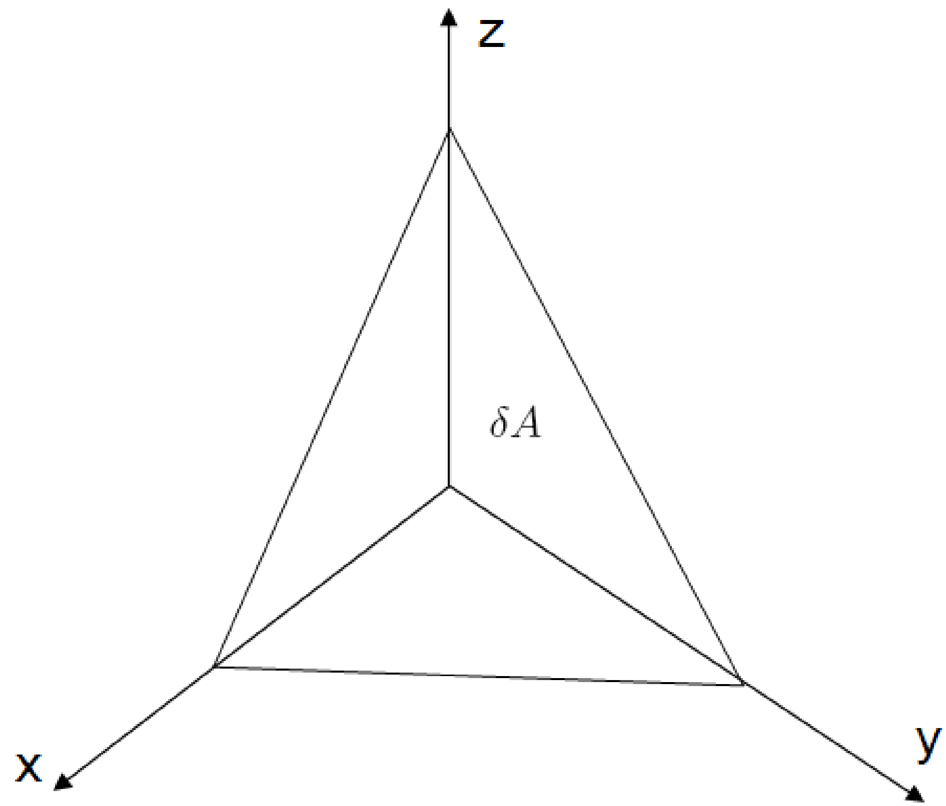

Figure 1: Stress on a tetrahedral volume of fluid

The sum of the surface force must be zero by Newton's second law[a]. The area of the "big triangle" is given by δA, while the area of other three surfaces are given by

$$\delta A_1 = \mathbf{a} \cdot \mathbf{n} \delta A \tag{12a}$$

$$\delta A_2 = \mathbf{b} \cdot \mathbf{n} \delta A \tag{12b}$$

$$\delta A_3 = \mathbf{c} \cdot \mathbf{n} \delta A \tag{12c}$$

[a] Otherwise it will be accelerating.

The sum of surface force is

$$\Sigma_i(\mathbf{n})\delta A + \Sigma_i(-\mathbf{a})\delta A_1 + \Sigma_i(-\mathbf{b})\delta A_2 + \Sigma_i(-\mathbf{c})\delta A_3 = 0$$

$$\Sigma_i(\mathbf{n})\delta A - \Sigma_i(\mathbf{a})\mathbf{a}\cdot\mathbf{n}\delta A - \Sigma_i(\mathbf{b})\mathbf{b}\cdot\mathbf{n}\delta A - \Sigma_i(\mathbf{c})\mathbf{c}\cdot\mathbf{n}\delta A = 0 \quad (13)$$

Since Σ is an odd function by Newton's third law. Then

$$\Sigma_i(\mathbf{n}) = [a_j\Sigma_i(\mathbf{a}) + b_j\Sigma_i(\mathbf{b}) + c_j\Sigma_i(\mathbf{c})]n_j \quad (14)$$

Define the stress vector as

$$\sigma_{ij} = a_j\Sigma_i(\mathbf{a}) + b_j\Sigma_i(\mathbf{b}) + c_j\Sigma_i(\mathbf{c}) \quad (15)$$

The total force on a surface is

$$\text{surface force} = \int_A \sigma_{ij}n_j\,dA = \int_V \frac{\partial}{\partial x_j}\sigma_{ij}\,dV \quad (16)$$

Combining (9), (10), (11) and (16), (8) can be written as:

$$\int_V \frac{\partial \rho u_i}{\partial t}\, dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV + \int_V \rho f_i\, dV + \int_V \frac{\partial}{\partial x_j}\sigma_{ij}\, dV \tag{17}$$

Since the test volume is arbitrary, the integrands can be taken out from the integrals:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho f_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{18}$$

Applying the equation of continuity (5) in component form,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

The left side of (18) can be further simplified:

$$
\begin{aligned}
\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) &= \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial}{\partial x_j}(u_i) \\
&= \rho \frac{\partial u_i}{\partial t} + u_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right] + \rho u_j \frac{\partial}{\partial x_j}(u_i) \\
&= \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j}(u_i) \\
&= \rho \frac{D u_i}{D t}
\end{aligned}
$$

The momentum equation is

$$\boxed{\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial}{\partial x_j}\sigma_{ij}} \tag{19}$$

Navier-Stokes equation will be derived from the conservation of momentum equation.

1.3 Navier-Stokes Equation

The fundamental equation in fluid mechanics, Navier-Stokes equation, is derived from conservation of momentum.

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial}{\partial x_j} \sigma_{ij}$$

The stress tensor is symmetric and isotropic in a static fluid. Define the static stress as

$$\sigma_{ij} = -p\delta_{ij} \tag{20}$$

In general, for a moving fluid we can expand σ_{ij} into

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \tag{21}$$

where d_{ij} is due to the fluid motion alone, called the deviatoric stress tensor. d_{ij} cannot be only related to the coordinate x_j since it is independent of reference frame[a]. Use the Newtonian hypothesis that it is a linear function of $\partial u_i/\partial x_j$, we can express the relationship with a 4-th rank tensor:

$$d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l} \tag{22}$$

[a] If d_{ij} only depends on x_j, upon changing to a co-moving frame with the fluid, d_{ij} will be zero. But stress should be invariant from coordinate transformation.

$\partial u_i/\partial x_j$ can be decomposed into a symmetric and an antisymmetic part:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) + \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big)$$
$$= \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) - \frac{1}{2}\epsilon_{ijk}\,\omega_k$$

where $\omega = \nabla \times \mathbf{u}$ is the fluid vorticity. Define $e_{ij} = \dfrac{1}{2}\Big(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\Big)$ and $\zeta_{ij} = -\frac{1}{2}\epsilon_{ijk}\,\omega_k$,

$$d_{ij} = A_{ijkl}\frac{\partial u_i}{\partial x_j} = A_{ijkl}(e_{kl} + \zeta_{kl}) \tag{23}$$

Assume the fluid is isotropic in motion, that d_{ij} is independent of the orientation of the fluid. Then A_{ijkl} must also be isotropic. From Jeffreys [6, p.70], the general 4th order isotropic tensor has the form

$$A_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \mu''\delta_{ij}\delta_{kl} \tag{24}$$

Since the stress is symmetric,

$$\begin{aligned} \sigma_{ij} &= \sigma_{ji} \\ \mu &= \mu' \\ A_{ijkl} &= 2\mu\delta_{ik}\delta_{jl} + \mu''\delta_{ij}\delta_{kl} \end{aligned} \tag{25}$$

We can see that A_{ijkl} is also symmetric in l and k. Then

$$\begin{aligned}
A_{ijkl}\zeta_{ij} &= -\frac{1}{2}A_{ijkl}\,\epsilon_{ijk}\,\omega_k = 0 \\
d_{ij} &= A_{ijkl}e_{kl} \\
&= (2\mu\delta_{ik}\delta_{jl} + \mu''\delta_{ij}\delta_{kl})e_{kl} \\
&= 2\mu e_{ij} + \mu'' e_{ll}\delta_{ij} \\
&= 2\mu e_{ij} + \mu''\Delta\delta_{ij} \qquad (26)
\end{aligned}$$

where $\Delta = \nabla \cdot \mathbf{u}$.

By definition of d_{ij},

$$\mathrm{tr}(d_{ij}) \equiv 0 = d_{ii}$$

$$d_{ii} = 2\mu e_{ii} + 3\mu''\Delta = 0$$

$$\text{if } e_{ii} \neq 0, \qquad \mu'' = -\frac{2}{3}\mu \tag{27}$$

Therefore

$$d_{ij} = 2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij}) \tag{28}$$

μ is called the molecular viscosity.

Finally we get the Navier-Stokes equation by substituting (28) into the conservation of momentum equation:

$$\boxed{\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\Big(2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\Big)} \tag{29}$$

If $\Delta = \nabla \cdot \mathbf{u} = 0$, with μ is constant,

$$\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \tag{30}$$

Or in vector form:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{u} \tag{31}$$

1.4 Reyleigh's Stability Equation

To deal with the simple problem in hydrodynamic stability, adopt the small amplitude linear approximation to define the velocity, pressure, and assume the fluid is incompressible:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(z) + \mathbf{u}'(\mathbf{x}, t) \tag{32a}$$

$$p(\mathbf{x}, t) = P(z) + p'(\mathbf{x}, t) \tag{32b}$$

$$\rho = \rho_0 \quad \text{everywhere} \tag{32c}$$

where $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ is the velocity and pressure at different location and time, $\mathbf{U}(z)$ is the background velocity , $P(z)$ is the background pressure. The background fields only depend on z. $\mathbf{u}'(\mathbf{x}, t)$ and $p'(\mathbf{x}, t)$ are the perturbed velocity and pressure fields respectively.

Then write the Navier-Stokes equation explicitly

$$\frac{\partial}{\partial t}(\mathbf{U}(z) + \mathbf{u}') + (\mathbf{U}(z) + \mathbf{u}') \cdot \nabla(\mathbf{U}(z) + \mathbf{u}') = -\frac{\nabla(P(z) + p')}{\rho_0} - g\mathbf{k} \tag{33}$$

Since $g = -(1/\rho_0)(\partial P/\partial z)$, the background pressure term and the gravitational force term canceled each other.

(33) can be linearized by neglecting products of the small perturbed quantities (denotes by primes).

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\mathbf{u}' + \frac{dU}{dz}w'\mathbf{i} = -\frac{\nabla p'}{\rho_0} \tag{34}$$

where $\mathbf{u}' = (u', v', w')$.

The equation has coefficients indepedent of x, y, t but not z.

Separate the variables by taking independent normal modes of the form

$$\mathbf{u}'(\mathbf{x},t) = \hat{\mathbf{u}}(z)e^{i(\alpha x+\beta y-\alpha ct)}, \qquad p'(\mathbf{x},t) = \hat{p}(z)e^{i(\alpha x+\beta y-\alpha ct)} \quad (35)$$

Together with (7) the equation of continuity $\nabla \cdot \mathbf{u} = 0$, (60) gives

$$i\alpha(U-c)\hat{u} + \frac{dU}{dz}\hat{w} + \frac{i\alpha}{\rho_0}\hat{p} = 0 \tag{36a}$$

$$i\alpha(U-c)\hat{v} + \frac{i\beta}{\rho_0}\hat{p} = 0 \tag{36b}$$

$$i\alpha(U-c)\hat{w} + \frac{1}{\rho_0}\frac{d\hat{p}}{dz} = 0 \tag{36c}$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{d\hat{w}}{dz} = 0 \tag{36d}$$

Apply the Squire's transformation

$$\tilde{\alpha} = (\alpha^2 + \beta^2)^{1/2}, \qquad \tilde{u} = (\alpha\hat{u} + \beta\hat{v})/\tilde{\alpha}, \qquad \tilde{p} = \tilde{\alpha}\hat{p}/\alpha \tag{37}$$

Then multiply (36a) by α and (36b) by β, take the sum and divide by α to get (38a). (36c) and (36d) can be transformed in a similar way.

$$i\tilde{\alpha}(U - c)\tilde{u} + \frac{dU}{dz}\hat{w} + \frac{i\tilde{\alpha}}{\rho_0}\tilde{p} = 0 \tag{38a}$$

$$i\tilde{\alpha}(U - c)\hat{w} + \frac{1}{\rho_0}\frac{d\tilde{p}}{dz} = 0 \tag{38b}$$

$$i\tilde{\alpha}\tilde{u} + \frac{d\hat{w}}{dz} = 0 \tag{38c}$$

The three-dimensional mode in (36) is essentially reduced to a two-dimensional mode of wave number vector $\boldsymbol{\alpha} = \alpha\mathbf{i} + \beta\mathbf{j}$, for $\tilde{\alpha} = |\boldsymbol{\alpha}|$ and $\tilde{u} = \hat{\boldsymbol{\alpha}} \cdot \hat{\mathbf{u}}$, where $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/\tilde{\alpha}$. This represents a wave traveling in the direction of $\boldsymbol{\alpha}$. (36) is the same as (38) when $\beta = \hat{v} = 0$.

Introduce a perturbed stream function ψ' into the linearized equations (38). Define that

$$u' = \frac{\partial \psi'}{\partial z}, \qquad v' = 0, \qquad w' = -\frac{\partial \psi'}{\partial x} \tag{39}$$

and take independent normal modes of the form

$$\psi'(x, z, t) = \phi(z) e^{i\alpha(x - ct)} \tag{40}$$

It follows that

$$\hat{u} = \frac{d\phi}{dz}, \qquad \hat{w} = -i\alpha\phi \tag{41}$$

which will automatically satisfy the equation of continuity (36d).

From (36a),

$$\frac{\hat{p}}{\rho_0} = \frac{dU}{dz}\phi - (U-c)\frac{d\phi}{dz} \tag{42}$$

From (36c),

$$(U-c)\alpha^2\phi + \frac{1}{\rho_0}\frac{d\hat{p}}{dz} = 0 \tag{43}$$

Differentiate (42) with respect to z and substitute into (43), we get the Rayleigh's stability equation:

$$\boxed{(U-c)(\frac{d^2\phi}{dz^2} - \alpha^2\phi) - \frac{d^2U}{dz^2}\phi = 0} \tag{44}$$

For a piecewise linear velocity profile, the term d^2U/dz^2 vanishes, the Rayleigh's stability equation reduces to a second order linear differential equation

$$\frac{d^2\phi}{dz^2} - \alpha^2\phi = 0 \tag{45}$$

The general solution is in the form $\phi = Ae^{-\alpha z} + Be^{\alpha z}$.

1.5 Taylor-Goldstein Equation

To treat the problem with compressible fluid with variable density profile, the small amplitude linear approximations (32) were modified to:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(z) + \mathbf{u}'(\mathbf{x},t) \tag{46a}$$

$$p(\mathbf{x},t) = P(z) + p'(\mathbf{x},t) \tag{46b}$$

$$\rho(\mathbf{x},t) = \rho(z) + \rho'(\mathbf{x},t) \tag{46c}$$

where $\rho(z)$ is the background density which only depends on z. $\rho'(\mathbf{x},t)$ is the perturbed density from the background.

Apply the Boussinesq approximation that the variation in density only appears in the buoyancy term, the Navier-Stokes equation (33) was modified to

$$\frac{\partial}{\partial t}(\mathbf{U}(z)+\mathbf{u}')+(\mathbf{U}(z)+\mathbf{u}')\cdot\nabla(\mathbf{U}(z)+\mathbf{u}') = -\frac{\nabla(P(z)+p')}{\rho_0}-\frac{g}{\rho_0}(\rho(z)+\rho')\mathbf{k} \tag{47}$$

Here $g\rho/\rho_0 = -(1/\rho_0)(\partial P/\partial z)$, thus the background pressure term and the gravitational force term canceled each other again.

Linearize (47) by neglecting products of the small perturbed quantities.

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\mathbf{u}' + \frac{dU}{dz}w'\mathbf{i} = -\frac{\nabla p'}{\rho_0} - \frac{g\rho'}{\rho_0}\mathbf{k} \tag{48}$$

where $\mathbf{u}' = (u', 0, w')$.

Assume there is no diffusion taking place in the flow, the density must be kept constant during movement $D\rho/Dt = 0$,

$$\frac{\partial}{\partial t}(\rho + \rho') + (\mathbf{U} + \mathbf{u}')\nabla(\rho + \rho') = 0$$

Linearizing and dropping the second order perturbed terms,

$$\frac{\partial}{\partial t}\rho' + U\frac{\partial \rho'}{\partial x} + w'\frac{d\rho}{dz} = 0$$

Define the Brunt-Väisälä frequency $N^2 \equiv -(g/\rho_0)(d\rho/dz)$ as a constant[a], then

$$\frac{\partial}{\partial t}\rho' + U\frac{\partial \rho'}{\partial x} - \frac{\rho_0 N^2}{g}w' = 0 \tag{49}$$

[a]The density profile is $\rho_0 e^{-z/H}$ where H is a scale height.

Since the equation has coefficients independent of x, t but not z, I can separate the variables by taking independent normal modes of the form

$$\mathbf{u}'(\mathbf{x},t) = \hat{\mathbf{u}}(z)e^{i(\alpha x-\alpha ct)}, p'(\mathbf{x},t) = \hat{p}(z)e^{i(\alpha x-\alpha ct)}, \rho'(\mathbf{x},t) = \hat{\rho}(z)e^{i(\alpha x-\alpha ct)} \tag{50}$$

(49) and (48) gives

$$i\alpha(U-c)\hat{u} + \frac{dU}{dz}\hat{w} + \frac{i\alpha}{\rho_0}\hat{p} = 0 \tag{51a}$$

$$i\alpha(U-c)\hat{w} + \frac{1}{\rho_0}\frac{d\hat{p}}{dz} + \frac{g}{\rho_0}\hat{\rho} = 0 \tag{51b}$$

$$i\alpha(U-c)\hat{\rho} - \frac{\rho_0 N^2}{g}\hat{w} = 0 \tag{51c}$$

To satisfy the equation of continuity, use the definition from (39), (40) and (41), it turns out to be

$$(U-c)\frac{d\phi}{dz} - \frac{dU}{dz}\phi + \frac{\hat{p}}{\rho_0} = 0 \tag{52a}$$

$$\alpha^2(U-c)\phi + \frac{1}{\rho_0}\frac{d\hat{p}}{dz} + \frac{g}{\rho_0}\hat{\rho} = 0 \tag{52b}$$

$$(U-c)\hat{\rho} + \frac{\rho_0 N^2}{g}\phi = 0 \tag{52c}$$

From (52c),

$$\frac{g}{\rho_0}\hat{\rho} = -\frac{N^2\phi}{U-c} \tag{53}$$

Differentiate (52a) with respect to z and substitute with (53) into (52b), we get the Taylor-Goldstein equation:

$$\boxed{\left(\frac{d^2}{dz^2}-\alpha^2\right)\phi - \frac{d^2U/dz^2}{U-c}\phi + \frac{N^2\phi}{(U-c)^2} = 0} \tag{54}$$

For a piecewise linear velocity profile, the term d^2U/dz^2 vanishes, the Taylor-Goldstein equation reduces to

$$\frac{d^2\phi}{dz^2} - \left(\alpha^2 - \frac{N^2}{(U-c)^2}\right)\phi = 0 \tag{55}$$

2 Kelvin-Helmholtz Instability

Kelvin-Helmholtz instability is the mode of instability at the interface of two horizontal parallel shear flows. The velocity profile is in general:

$$\mathbf{U}(z) = \begin{cases} U_1\mathbf{i} & \text{if } z > 0, \\ U_2\mathbf{i} & \text{if } z < 0. \end{cases} \tag{56}$$

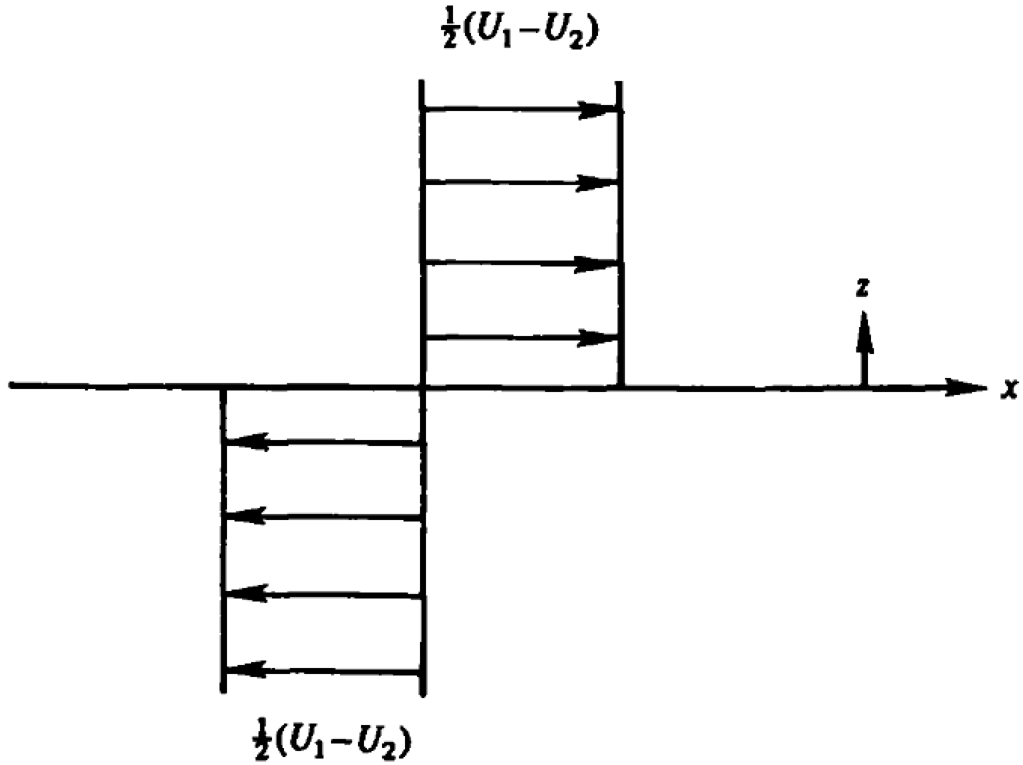

Figure 2: Velocity profile of a Kelvin-Helmholtz mode

Kelvin-Helmholtz modes of instability is well found in the nature. Figure 3 shows a billow cloud with the shape of Kelvin-Helmholtz mode of waves.

Figure 3: Billow cloud near Denver, Colorado (from *Hydrodynamic Stability* by Drazin and Weid, [3])

2.1 Constant Density Flow

For the velocity profile shown as below[a]

$$\mathbf{U}(z) = \begin{cases} U_0\mathbf{i} & \text{if } z > 0, \\ -U_0\mathbf{i} & \text{if } z < 0. \end{cases} \tag{57}$$

with $\rho = \rho_0$ everywhere and the boundary conditions $\phi \to 0$ as $z \to \pm\infty$, the solution of the Reyleigh's Stability equation (45) is in the form

$$\phi = \begin{cases} Ae^{-\alpha z} & \text{if } z > 0, \\ Be^{\alpha z} & \text{if } z < 0. \end{cases}$$

[a] Here I used the transformation of reference frame to make $U_0 = (U_1 - U_2)/2$ in (56) without the loss of generality.

There are two more boundary conditions at $z = 0$:

(i) The pressure at the interface must be continuous. From (42)

$$\frac{\hat{p}}{\rho_0} = \frac{dU}{dz}\phi - (U - c)\frac{d\phi}{dz} \quad \text{is continuous.} \tag{58}$$

(ii) The displacement of the perturbed interface must be continuous. Let $z = z_0 + \xi(x, t)$ be the displacement of the interface. The movement of the interface is given by

$$w' = \frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + (\mathbf{u} \cdot \nabla)\xi, \qquad \text{where } \mathbf{u} = \mathbf{U}(z) + \mathbf{u}'$$

Apply the linearization to drop the product of perturbed terms, let $\xi(x,t) = \hat{\xi} e^{i\alpha(x-ct)}$

$$
\begin{aligned}
w' &= \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} = -i\alpha c\, \xi + i\alpha U \xi \\
\hat{w} &= i\alpha(U-c)\hat{\xi} \\
-\hat{\phi} &= (U-c)\hat{\xi} \\
\frac{\phi}{U-c} &= -\hat{\xi} \quad \text{is continuous at } z=0.
\end{aligned} \tag{59}
$$

Apply the boundary conditions from (58) and (59), I get

$$(U_0 - c)(-\alpha)A = (-U_0 - c)(\alpha)B \tag{60a}$$

$$\frac{A}{U_0 - c} = \frac{B}{-U_0 - c} \tag{60b}$$

Solving (60), The solution is

$$\boxed{c = \pm iU_0} \tag{61}$$

The result is plotted in Figure 4. There is one unstable mode and one decaying mode corresponding to positive and negative value of c_I respectively. In Kelvin-Helmholtz instability, every wave number α has a corresponding unstable mode.

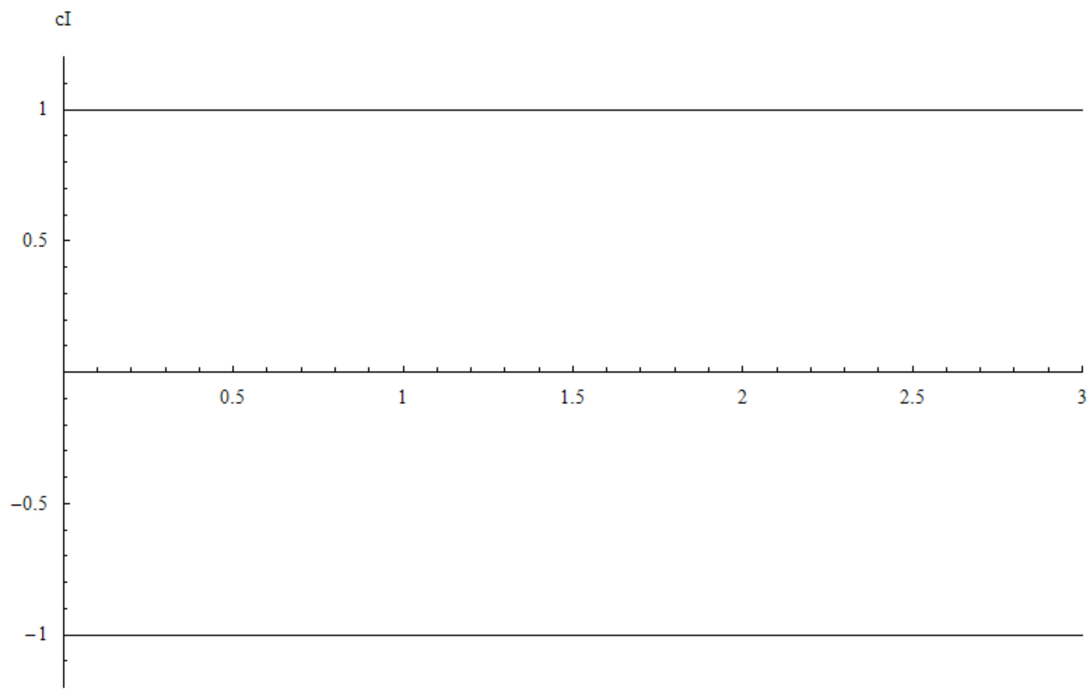

Figure 4: c_I vs. α for $U_0 = 1$ of a KH mode

2.2 Density Stratified Flow

For a velocity profile the same as in (57)

$$\mathbf{U}(z) = \begin{cases} U_0\mathbf{i} & \text{if } z > 0, \\ -U_0\mathbf{i} & \text{if } z < 0. \end{cases}$$

with the boundary conditions $\phi \to 0$ as $z \to \pm\infty$, $c = c_R + ic_I$ is complex in general. But the density is stratified by a constant $N^2 \equiv -(g/\rho_0)(d\rho/dz)$. The Boussinesq approximation is applied so that the variation in density only appears in the buoyancy term.

Taylor-Goldstein equation (55) is required to solve this problem. By matching the boundary conditions at $z = 0$ in general case of (56), I found $c_R = (U_1 + U_2)/2$, which is

$$c_R = \frac{U_0 - U_0}{2} = 0 \tag{62}$$

We have two types of solutions.

(A) Neutral Solutions: $c_I = 0$

For $\alpha^2 > N^2/U^2$, the trial solution is

$$\phi = \begin{cases} Ae^{-nz} & \text{if } z > 0, \\ Be^{nz} & \text{if } z < 0. \end{cases} \qquad \text{where } n = \sqrt{\alpha^2 - \frac{N^2}{U^2}} \tag{63}$$

Matching boundary conditions at $z = 0$:

(i) pressure: $-nU_0A = n(-U_0)B \quad \Rightarrow A = B$

(ii) displacement: $A/U_0 = B/-U_0 \quad \Rightarrow A = -B$

Therefore $A = B = 0$. There is no neutral solution for $\alpha^2 > N^2/U^2$.

For $\alpha^2 < N^2/U^2$, the trial solution is

$$\phi = \begin{cases} Ae^{inz} & \text{if } z > 0, \\ Be^{inz} & \text{if } z < 0. \end{cases} \qquad \text{where } n = \pm\sqrt{\frac{N^2}{U^2} - \alpha^2} \tag{64}$$

Matching boundary conditions at $z = 0$:

(i) pressure: $-inU_0 A = -in(-U_0)B \quad \Rightarrow A = -B$

(ii) displacement: $A/U_0 = B/-U_0 \quad \Rightarrow A = -B$

To satisfy the radiative boundary conditions at $z \longrightarrow \pm\infty$, we have the vertical components of the energy flux as[a]

$$\begin{aligned}
\langle F_z \rangle &= \langle \mathrm{Re}(p')\mathrm{Re}(w') \rangle \\
&= \langle \mathrm{Re}(-inU\phi e^{i(\alpha x - \alpha ct)})\mathrm{Re}(-i\alpha\phi e^{i(\alpha x - \alpha ct)}) \rangle \\
&= \frac{1}{2}\mathrm{Re}(nU\alpha\phi^* e^{-i(\alpha x - \alpha ct)}\phi e^{i(\alpha x - \alpha ct)}) \\
&= \frac{1}{2}\mathrm{Re}(nU\alpha\phi^*\phi) \\
&= \frac{1}{2}nU\alpha|A|^2 \qquad (65)
\end{aligned}$$

[a] $\langle \quad \rangle$ means average value.

For $z > 0$,

$$U = U_0, \quad \langle F_z \rangle > 0 \quad \Rightarrow n > 0$$

For $z < 0$,

$$U = -U_0, \quad \langle F_z \rangle < 0 \quad \Rightarrow n > 0$$

Therefore the natural solution is

$$\phi = \begin{cases} Ae^{inz} & \text{if } z > 0, \\ -Ae^{inz} & \text{if } z < 0. \end{cases} \qquad \text{where } n = \sqrt{\frac{N^2}{{U_0}^2} - \alpha^2}, \quad \alpha^2 < \frac{N^2}{{U_0}^2} \tag{66}$$

$$\boxed{c = 0, \qquad \alpha^2 < \frac{N^2}{{U_0}^2}} \tag{67}$$

(B) Unstable Solutions: $c_I \neq 0$

The trial solution is

$$\phi = \begin{cases} Ae^{-nz} & \text{if } z > 0, \\ Be^{n^* z} & \text{if } z < 0. \end{cases} \qquad \text{where } n^2 = \alpha^2 - \frac{N^2}{(U - ic_I)^2} \quad (68)$$

n is chosen to have positive real part so that the solution vanishes when $z \to \pm\infty$. When $\mathrm{Re}(n) = 0$, $\mathrm{Im}(n) < 0$ which automatically satisfied the radiative boundary conditions as shown in (65).

Matching boundary conditions at $z = 0$:

(i) pressure: $n(U_0 - ic_I)A = n^*(U_0 + ic_I)B$

(ii) displacement: $A/(U_0 - ic_I) = B/(-U_0 - ic_I)$

Combining the two equations to get

$$n(U_0 - ic_I)^2 = -n^*(U_0 + ic_I)^2 \tag{69}$$

Which can be written as $n(U_0 - ic_I)^2 = -(n(U_0 - ic_I)^2)^*$, therefore I have

$$\mathrm{Re}\Big(n(U_0 - ic_I)^2\Big) = 0 \tag{70}$$

Then expand $n(U_0 - ic_I)^2$ as

$$n(U_0 - ic_I)^2 = ({U_0}^2 - 2iU_0c_I - {c_I}^2)(n_R + in_I) \tag{71}$$

By comparing real part of (70) and (71),

$$n_R = -\frac{2c_IU_0}{{U_0}^2 - {c_I}^2}n_I \tag{72}$$

Write n as $n = n_R + in_I = \left(-\frac{2c_I U_0}{{U_0}^2 - {c_I}^2} + 1\right) n_I$, take square and compare the real and imaginary parts with the definition in (68), I get

$$
{c_I}^2 = {U_0}^2 - \frac{N^2}{2\alpha^2}, \qquad \text{where } \alpha^2 > \frac{N^2}{2{U_0}^2} \tag{73}
$$

$$
n^2 = -\alpha^2 \frac{(U + ic_I)^2}{(U - ic_I)^2} \tag{74}
$$

c is found to be:

$$\boxed{c = \pm i\sqrt{{U_0}^2 - \frac{N^2}{2\alpha^2}}, \qquad \alpha^2 > \frac{N^2}{2{U_0}^2}} \tag{75}$$

The unstable solution is plotted in Figure 5. The density stratification tends to stabilize long waves with smaller wave number α.

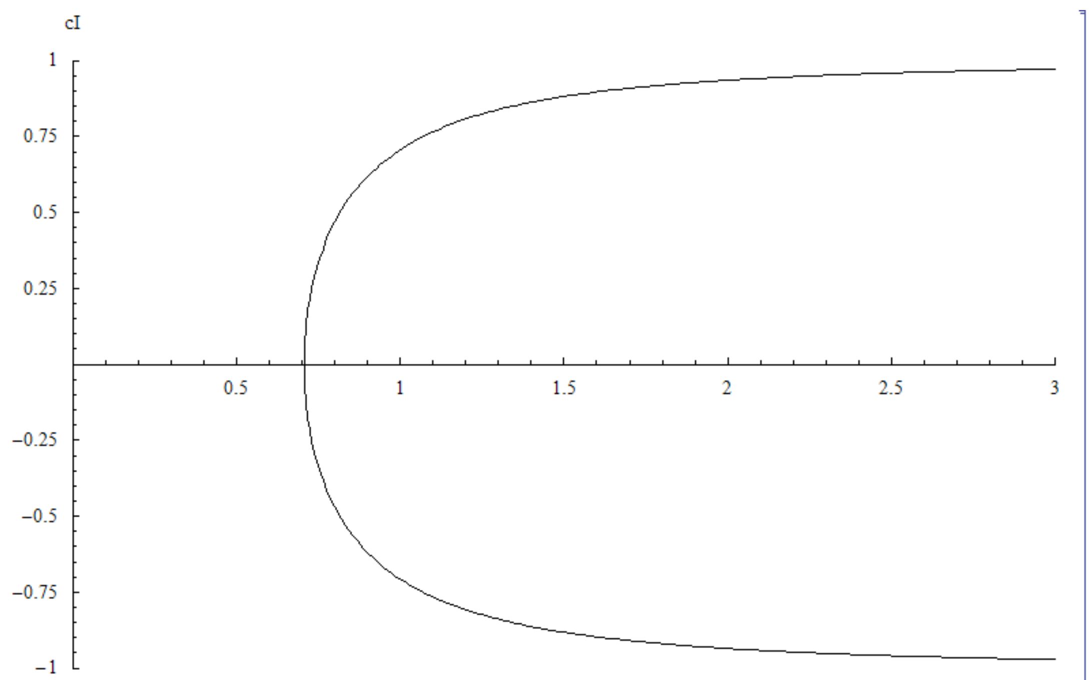

Figure 5: c_I vs. α for $U_0 = 1$ of a stratified KH mode

3 Holmboe Instability

Holmboe instability is defined by the velocity profile:

$$\mathbf{U}(z) = \begin{cases} U_0\mathbf{i} & \text{if } z > d, \\ \dfrac{z}{d}U_0\mathbf{i} & \text{if } -d < z < d, \\ -U_0\mathbf{i} & \text{if } z < -d. \end{cases} \tag{76}$$

Or, in dimensionless form:

$$\mathbf{U}(z) = \begin{cases} 1\mathbf{i} & \text{if } z > 1, \\ \dfrac{z}{d}U_0\mathbf{i} & \text{if } -1 < z < 1, \\ -1\mathbf{i} & \text{if } z < -1. \end{cases} \tag{77}$$

The velocity profile is shown in Figure 6.

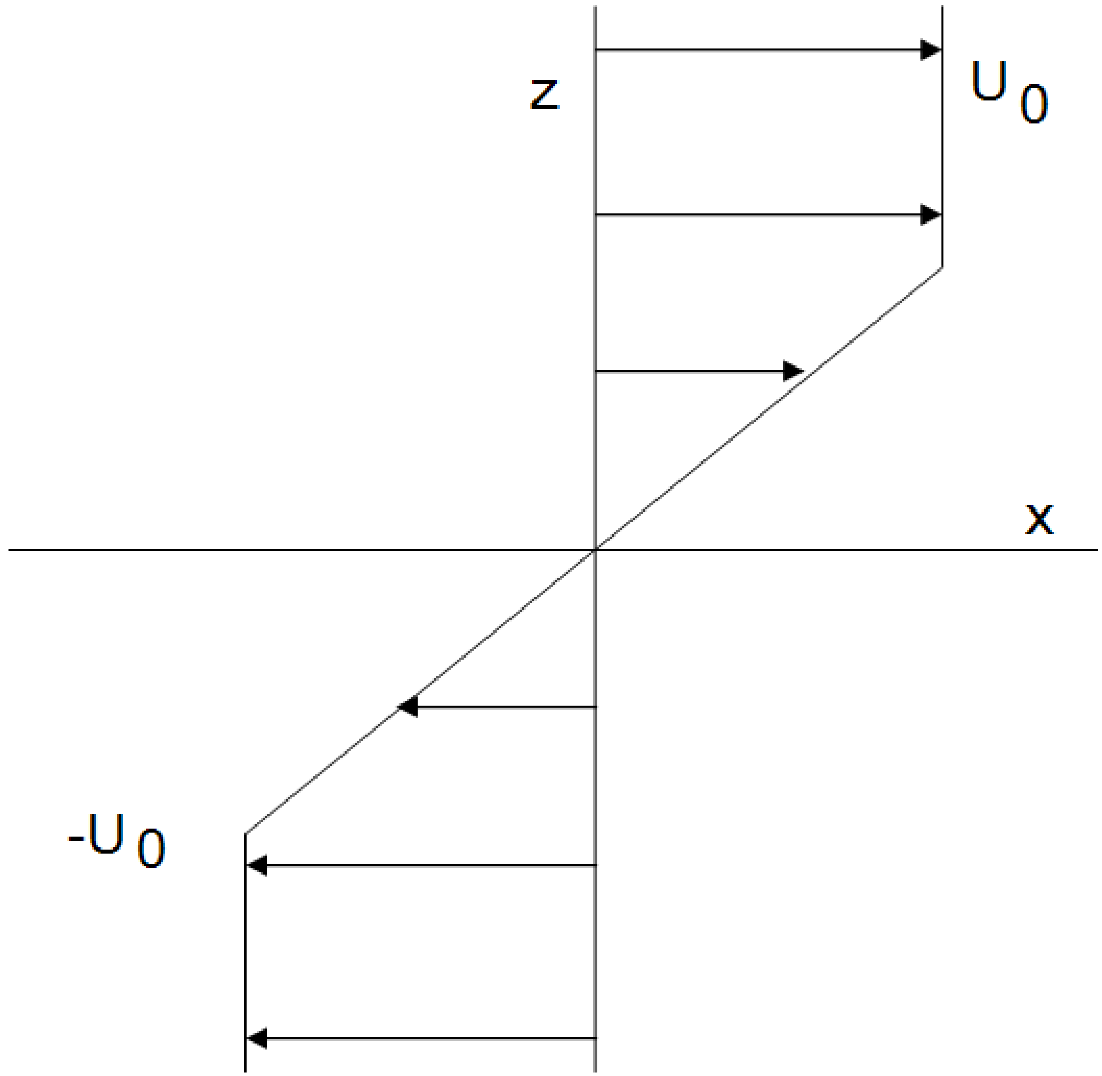

Figure 6: Velocity profile of a Holmboe mode

3.1 Constant Density Flow

For the profile in (77) with constant density $\rho = \rho_0$ everywhere, the trial solution of the Reyleigh's Stability equation (45) is in the form

$$\phi = \begin{cases} Ae^{-\alpha z} & \text{if } z > 1, \\ Be^{-\alpha z} + Ce^{\alpha z} & \text{if } -1 < z < 1, \\ De^{\alpha z} & \text{if } z < -1. \end{cases} \tag{78}$$

At $z = 1$, the boundary conditions are:

(i) pressure:

$$\hat{p} = U'\phi - (U - c)\phi' \quad \text{is continuous}$$

$$-(1 - c)Ae^{-\alpha} = Be^{-\alpha} + Ce^{\alpha} - (1 - c)\alpha(-Be^{-\alpha} + Ce^{\alpha}) \quad (79)$$

(ii) displacement:

$$\phi \quad \text{is continuous}$$

$$Ae^{-\alpha} = Be^{-\alpha} + Ce^{\alpha} \quad (80)$$

Substitute (80) into (79), I get

$$2(1 - c)\alpha C = Be^{-2\alpha} + C \quad (81)$$

At $z = -1$, the boundary conditions are:

(i) pressure:

$$-(-1-c)Ae^{\alpha} = Be^{\alpha} + Ce^{-\alpha} - (-1-c)\alpha(-Be^{\alpha} + Ce^{-\alpha}) \quad (82)$$

(ii) displacement:

$$Be^{\alpha} + Ce^{-\alpha} = De^{\alpha} \quad (83)$$

Substitute (83) into (82), I get

$$2(1+c)\alpha B = B + Ce^{-2\alpha} \quad (84)$$

Combining (81) and (84) to eliminate the constants B and C, I found that the non-dimensional phrase speed satisfy the equation:

$$\boxed{c^2 + \left(\frac{e^{-4\alpha} - (2\alpha - 1)^2}{4\alpha^2}\right) = 0} \tag{85}$$

The solution is

$$c = \pm\sqrt{\left(1 - \frac{1}{2\alpha}\right)^2 - \left(\frac{1}{4\alpha^2}\right)e^{-4\alpha}} \tag{86}$$

The value of α was solved numerically. For $\alpha > 0.639232$, c is real, the solution is neutral. For $0 \leq \alpha \leq 0.639232$, c is imaginary, one of the solutions is unstable. The value of c_R and c_I is plotted in Figure 7 and 8. From the graph we can verify that when $\alpha \to 0$, $c = \pm i$, this recovers the result of Kelvin-Helmholtz instability.

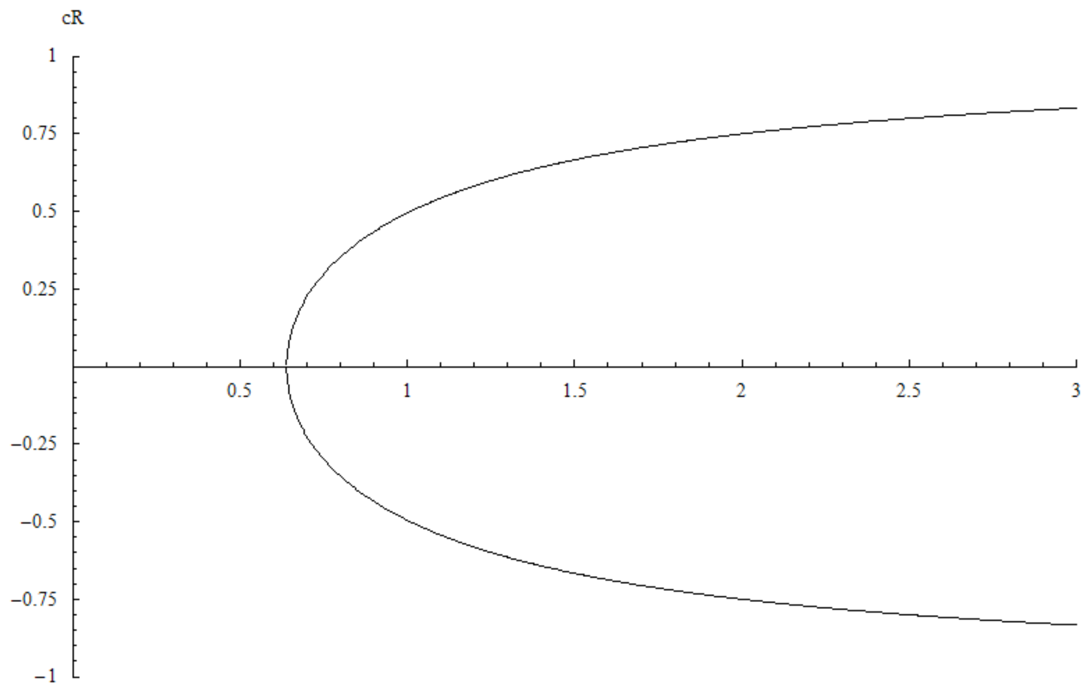

Figure 7: c_R vs. α for $U_0 = 1$ of a Holmboe mode

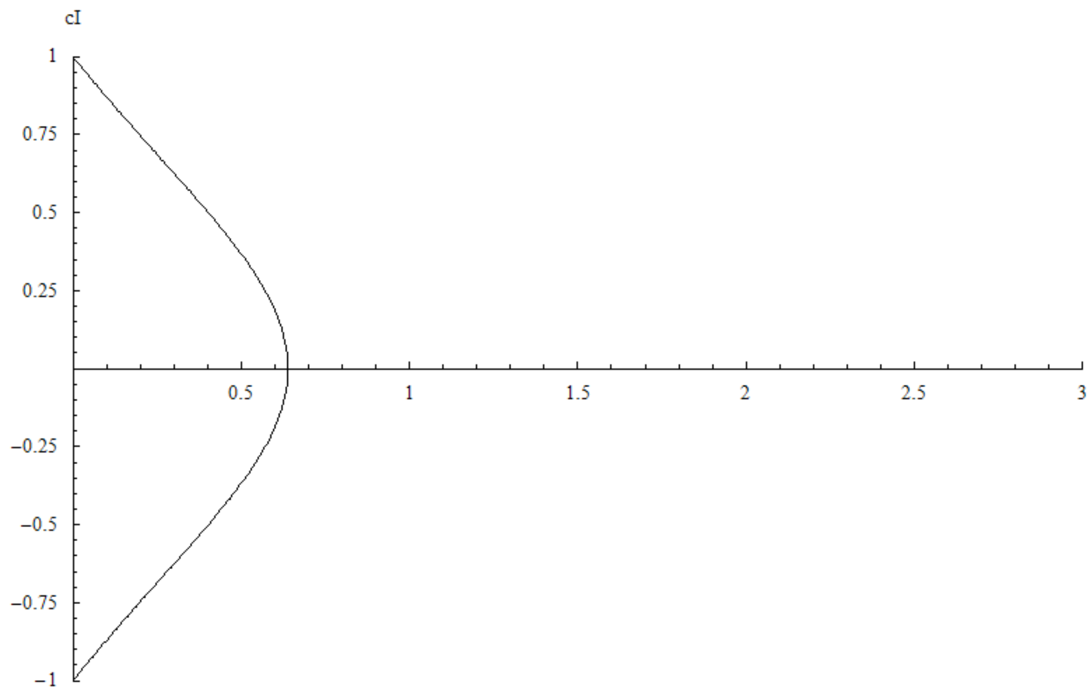

Figure 8: c_I vs. α for $U_0 = 1$ of a Holmboe mode

3.2 Sheared Density Flow

Consider the same velocity profile as (77) but with a sheared density profile:

$$\rho = \begin{cases} \rho_0 & \text{if } z > 0, \\ \rho_0 + \Delta\rho & \text{if } z < 0, \end{cases} \tag{87}$$

The trial solution is

$$\phi = \begin{cases} Ae^{-\alpha z} & \text{if } z > 1, \\ Be^{-\alpha z} + Ce^{\alpha z} & \text{if } 0 < z < 1, \\ De^{-\alpha z} + Ee^{\alpha z} & \text{if } -1 < z < 0, \\ Fe^{\alpha z} & \text{if } z < -1. \end{cases} \tag{88}$$

Using the same technique as last section, we get at $z = 1$

$$2(1 - c)\alpha C = Be^{-2\alpha} + C \tag{89}$$

and at $z = -1$

$$2(1 + c)\alpha D = D + Ee^{-2\alpha} \tag{90}$$

However, the fluid near $z = 0$ is affected by the buoyancy force due to the density shearing, so I have to treat it with a different method. First, by matching ϕ, I get

$$B + C = D + E \tag{91}$$

Using the vorticity equation cited in Caulfield [2][a], the x component can be written as

$$\frac{D}{Dt}\Big(\frac{\partial u}{\partial z}\Big) = \Big(g'\frac{\partial \xi}{\partial x}\Big)\delta(z-\xi) \tag{92}$$

where $u = U(z) + u'$ is the total velocity in x direction, $g' = g\Delta\rho/\rho$ is the reduced gravitational acceleration, ξ is the perturbed displacement of the interface with density jump $\Delta\rho$. Note that $U(0) = 0$ at $z = 0$ and linearizing, I found that $u = u'$ and $D/Dt = \partial/\partial t$ at $z = 0$.

[a]The result is following Hoiland, 1948.

Integrate (92) over an arbitrary small region near the density interface, I obtain

$$\frac{\partial}{\partial t}(u'_+ - u'_-) = g' \frac{\partial \xi}{\partial x} \quad \text{at } z = 0. \tag{93}$$

From (59),

$$\hat{\xi} = -\frac{\phi}{U - c} = \frac{\phi}{c} \quad \text{at } z = 0. \tag{94}$$

Recall that $u' = (d\phi/dz)e^{i\alpha(x-ct)}$ and $\xi = \hat{\xi}e^{i\alpha(x-ct)}$, I can write (93) in

$$\frac{\partial}{\partial t}\Big[\frac{d(\phi_+ - \phi_-)}{dz}e^{i\alpha(x-ct)}\Big] = g'\frac{\partial}{\partial x}(\hat{\xi}e^{i\alpha(x-ct)})$$

$$-i\alpha c[\alpha(-B + C + D - E)\phi e^{i\alpha(x-ct)}] = g'(i\alpha\hat{\xi}e^{i\alpha(x-ct)})$$

$$c\alpha(B - C - D + E)\phi = g'\frac{\phi}{c}$$

Define the bulk Richardson number as

$$Ri_0 = \frac{g \Delta \rho d}{\rho (\Delta u)^2} = \frac{g' 2}{2^2} = \frac{g'}{2} \tag{95}$$

where d is the thickness of the shear layer and Δu is the velocity difference of the upper and lower region. Finally I get

$$B - C = D - E + \frac{2 Ri_0}{\alpha c^2} \tag{96}$$

Put together (89), (90), (91) and (96), we get the stability equation for a sheared density flow:

$$\boxed{c^4 + \Big(\frac{e^{-4\alpha} - (2\alpha - 1)^2}{4\alpha^2} - \frac{Ri_0}{\alpha}\Big)c^2 + \frac{Ri_0}{\alpha}\Big(\frac{e^{-2\alpha} + (2\alpha - 1)}{2\alpha}\Big)^2 = 0} \tag{97}$$

The result is visualized by Lawrence et al. [5] in Figure 9. The region I shows Kelvin-Helmholtz instability while region II shows Holmboe instability.

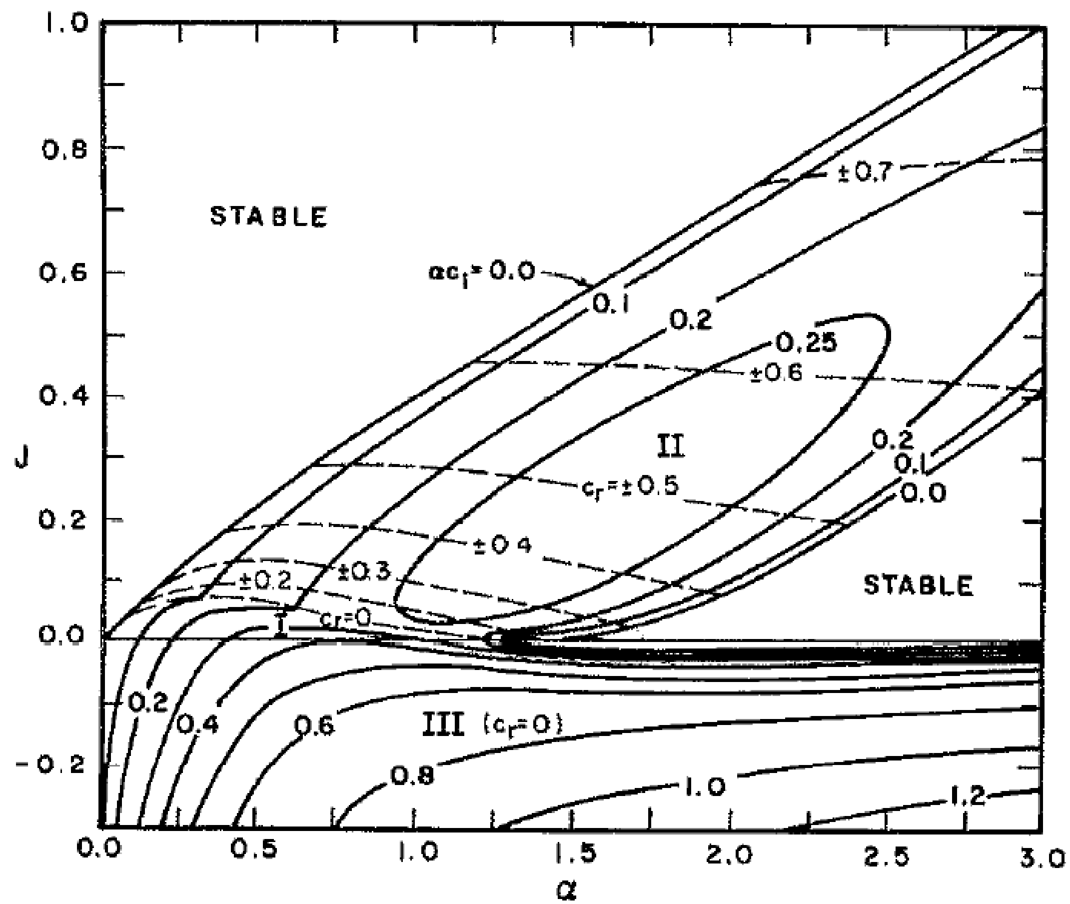

Figure 9: Stability diagram for different Ri_0 and α

4 Conclusion

- Kelvin-Helmholtz and Holmboe modes were investigated
- More complicated problem requires computational technique
- 3D Evolution by Smyth and Winters [9]

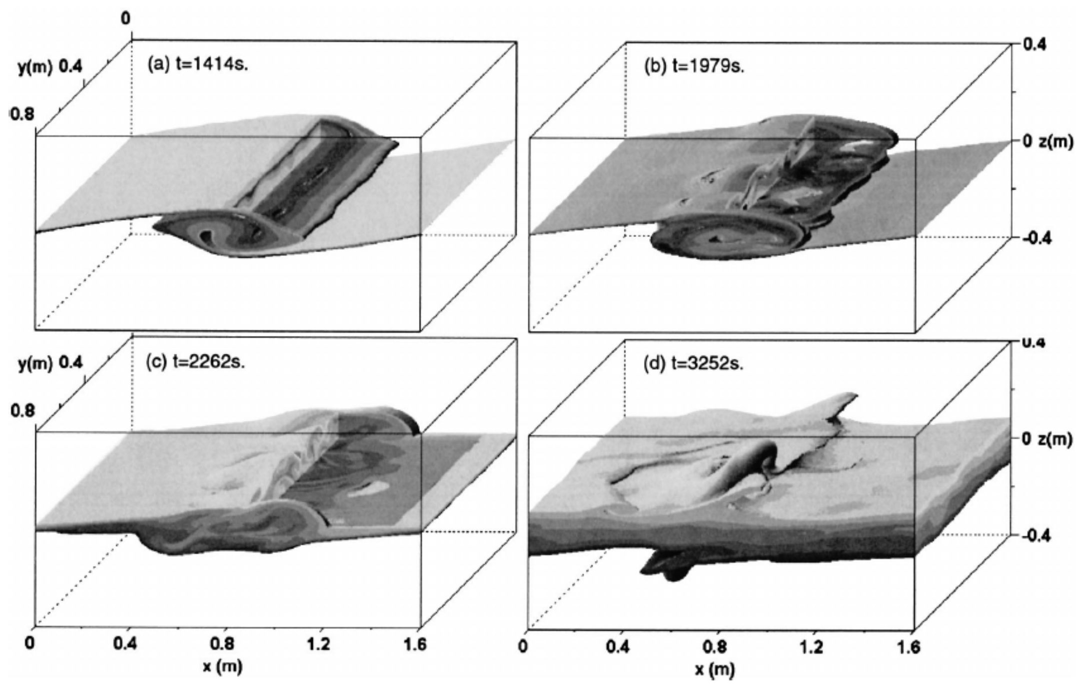

Figure 10: Time evolution for a Kelvin-Helmholtz mode (Smyth and Winters [9])

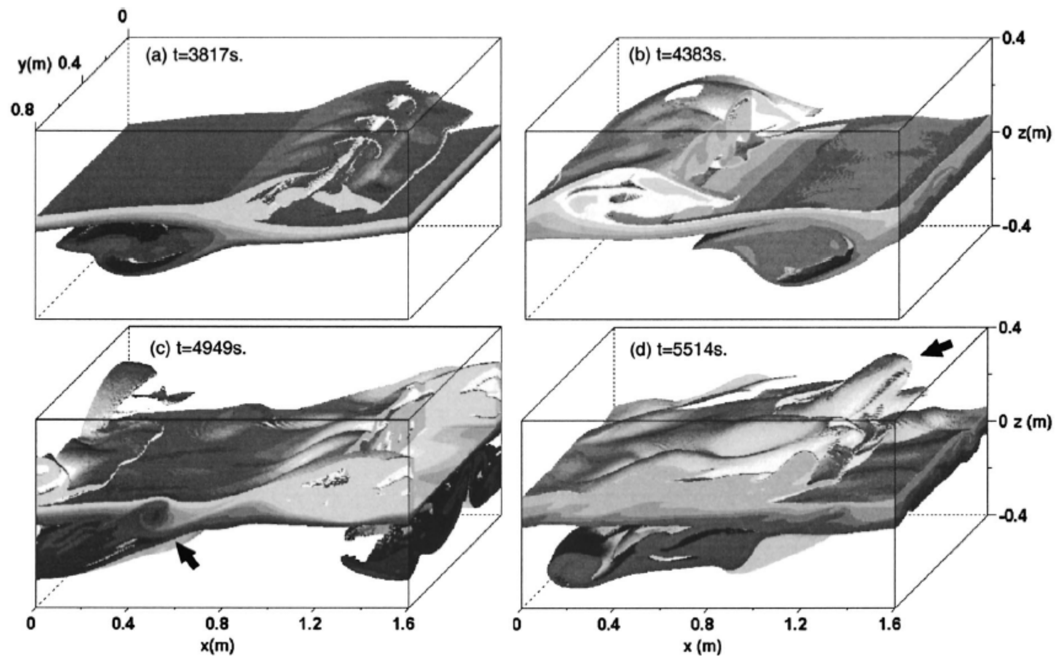

Figure 11: Time evolution for a Holmboe mode (Smyth and Winters [9])

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