CHAPTER 4

PARALLEL SHEAR FLOWS

The transition of laminar flow, with its clean layers of flow tubes, to strongly mixed, irregular turbulent flow is one of the principal problems of modern hydrodynamics. It is certain that this fundamental change in type of motion of the fluid is traceable to an instability in the laminar flow, for laminar flows of themselves would always be possible solutions of the hydrodynamic equations. – W. Tollmien (1935)

20 Introduction

In this chapter we wish to consider the stability of steady two-dimensional or axisymmetric flows with parallel streamlines. Flows of this type were first studied experimentally by Reynolds (1883), who observed that instability could occur in quite different ways depending on the form of the basic velocity distribution. Thus, when the velocity profile is of the form shown in Fig. 4.1(a) he observed that 'eddies showed themselves reluctantly and irregularly' whereas when the profile is as shown in Fig. 4.1(b) the 'eddies appeared in the middle regularly and readily'. From these observations he was led to consider the role of viscosity in flows of this type. By comparing the flow of a viscous fluid with that of an inviscid fluid, both flows being assumed to have the same basic velocity distribution, he was led to formulate two fundamental hypotheses which can be stated as follows:

First Hypothesis. The inviscid fluid may be unstable and the viscous fluid stable. The effect of viscosity is then purely stabilizing.

Second Hypothesis. The inviscid fluid may be stable and the viscous fluid unstable. In this case viscosity would be the cause of the instability.

Although Reynolds was unable to suggest a physical mechanism by which viscosity could cause instability he refused to exclude such a possibility. In this chapter, therefore, we will examine the circumstances under which these hypotheses are valid and give, so far as possible, a physical description of the mechanism of instability.

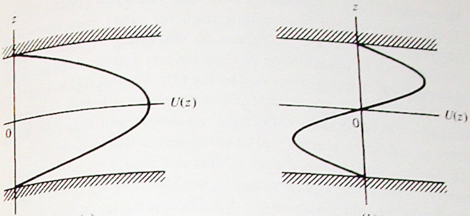

Fig. 4.1. The velocity profiles considered by Reynolds (1883).

The analytical study of inviscid flows of this type had been initiated somewhat earlier by Helmholtz (1868), Kelvin (1871) and Rayleigh (1880), who considered the purely inertial instability of an incompressible fluid of constant density. Perhaps the most important result of this period, of which Reynolds was apparently unaware, was Rayleigh's famous theorem on the role of inflexion points in the velocity profile. According to this theorem an inviscid flow of the form shown in Fig. 4.1(a) would be stable but the one shown in Fig. 4.1(b) would be unstable. This part of the theory would now appear to be reasonably complete, both physically and mathematically, and provides quite general criteria under which the First Hypothesis is valid. A general review of the inviscid theory has been given by Drazin & Howard (1966) who also discussed the effects of rotation and stratification.

Progress has been much slower, however, in our understanding of the stability of viscous flows. One of the main steps in this direction was not taken until much later by Orr (1907) and Sommerfeld (1908), who derived the celebrated equation which now bears their names. The importance of this equation can hardly be exaggerated, and much of the subsequent work on the stability of viscous flows has concentrated on it. At the time the Orr–Sommerfeld equation was derived the existing methods of asymptotic analysis were not sufficiently well developed to deal with it effectively, and various heuristic methods of approximation were later suggested by

Heisenberg (1924), Tollmien (1929, 1947), and Lin (1945, 1955). These methods have been widely used for many computational purposes and they will therefore be discussed in this chapter together with an examination of their defects and limitations. The mathematical challenge provided by the Orr–Sommerfeld equation has also stimulated substantial further developments in the asymptotic solution of ordinary differential equations and some of these more recent developments will be discussed in the following chapter. The existing viscous theory is thus not nearly as complete or general as the inviscid theory, and it provides only a partial understanding of the role of viscosity in those circumstances when it is the cause of instability.

THE INVISCID THEORY

21 The governing equations

For a parallel two-dimensional flow, the basic steady flow is of the form

$$\mathbf{U}_* = U_*(z_*)\mathbf{i} \quad (z_{1*} \leq z_* \leq z_{2*}), \tag{21.1}$$

where $\mathbf{i}$ denotes a unit vector in the x_*-direction and the asterisks denote dimensional quantities. For an inviscid fluid $U_*(z_*)$ can be an arbitrary function of z_*. The flow is assumed to be bounded by the two planes $z_* = z_{1*}$ and z_{2*} which may be either rigid or free. On a rigid boundary the normal component of the velocity must vanish and on a free boundary the pressure must be constant. More generally, one of the boundaries may be at infinity as in the case of boundary layers or they may both be at infinity as in the case of shear layers, jets and wakes.

It is convenient, as usual, to write the governing equations in terms of dimensionless quantities, and for this purpose we introduce a characteristic length L and a characteristic velocity V associated with the basic flow. The choice of L and V is, of course, not unique; considerable variation in their definition exists in the literature and some care is required therefore in comparing the results of different writers. In the present discussion, however, we will usually take

$$V = \max_{z_{1*} \leq z \leq z_{2*}} |U_*(z_*)|$$

and, for flows in a channel, $L = \frac{1}{2}(z_{2*} - z_{1*})$. If we now let

$$\left.\begin{aligned} t = t_* V/L, \quad \mathbf{x} = \mathbf{x}_*/L, \quad \mathbf{u} = \mathbf{u}_*/V, \quad p = p_*/\rho V^2, \\ \text{and} \qquad \mathbf{U} = \mathbf{U}_*/V \equiv U(z)\mathbf{i}, \end{aligned}\right\} \tag{21.2}$$

where ρ is the (constant) density of the fluid, then the Euler equations of motion and the equation of continuity become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0. \tag{21.3}$$

The basic flow $\mathbf{U} = U(z)\mathbf{i}$ with $U(z)$ an arbitrary function of z automatically satisfies both of the boundary conditions and equations (21.3) are also satisfied provided $\nabla p = 0$, i.e. the pressure is constant. To study the stability of this flow we let

$$\left.\begin{aligned} \mathbf{u}(\mathbf{x}, t) = U(z)\mathbf{i} + \mathbf{u}'(\mathbf{x}, t) \\ \text{and} \qquad p(\mathbf{x}, t) = \text{constant} + p'(\mathbf{x}, t), \end{aligned}\right\} \tag{21.4}$$

where $\mathbf{u}'$ is the disturbance velocity and p' is the disturbance pressure. On substituting these expressions into equations (21.3) and neglecting the term $\mathbf{u}'\cdot\nabla\mathbf{u}'$, which is quadratic in the disturbance velocity, we then obtain the *linearized equations of motion*

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\mathbf{u}' + w'\frac{\mathrm{d}U}{\mathrm{d}z}\mathbf{i} = -\nabla p' \quad \text{and} \quad \nabla\cdot\mathbf{u}' = 0. \tag{21.5}$$

In studying the linearized stability problem for an inviscid fluid, one way to proceed would be to consider a suitably posed *initial-value problem*. Although the formulation of such an initial-value problem is not difficult, the subsequent analysis, even for basic flows of simple form, rapidly becomes complicated. Since one of the major aims of the inviscid theory is to provide general criteria by which one can decide whether a given basic flow is stable or not, it would be desirable if we could avoid having to solve the initial-value problem in detail. This can be partially achieved by considering a *normal-mode analysis* of equations (21.5). These two approaches are equivalent if it can be shown that the normal modes are complete and that an arbitrary initial disturbance can be expanded in terms of them. As we will see, however, the normal modes of the linearized inviscid stability problem are, in general, not complete, but they do play an important role in the initial-value approach, further discussion of which will be given in § 24. There it will be shown that instability, if it exists, is always associated with the discrete part of the spectrum; thus, in seeking general criteria for instability, it is sufficient to consider only the normal modes.

Since the coefficients in equations (21.5) depend only on z, the equations admit solutions which depend on x, y, and t exponentially. We consider therefore solutions of the form

$$\left.\begin{aligned} \mathbf{u}'(\mathbf{x},t) &= \hat{\mathbf{u}}(z)\exp[\mathrm{i}(\alpha x+\beta y-\alpha ct)] \\ \text{and}\qquad p'(\mathbf{x},t) &= \hat{p}(z)\exp[\mathrm{i}(\alpha x+\beta y-\alpha ct)], \end{aligned}\right\} \tag{21.6}$$

in which it is understood that the real parts of these expressions must be taken to obtain physical quantities. The requirement that the solutions remain bounded as $x,\ y\to\pm\infty$ implies that the wavenumbers α and β must be real. The wave speed c may be complex, i.e. $c=c_r+\mathrm{i}c_i$, and the expressions (21.6) thus represent waves which travel in the direction $(\alpha,\beta,0)$ with phase speed $\alpha c_r/(\alpha^2+\beta^2)^{1/2}$ and which grow or decay in time like $\exp(\alpha c_i t)$. Such a wave is said to be stable if $\alpha c_i\leqslant 0$, unstable if $\alpha c_i>0$, and neutrally stable if $\alpha c_i=0$.

If we now let $\mathrm{D}=\mathrm{d}/\mathrm{d}z$, then on substituting the expressions (21.6) into equations (21.5) we obtain the system of ordinary differential equations:

$$\left.\begin{aligned} \mathrm{i}\alpha(U-c)\hat{u}+U'\hat{w} &= -\mathrm{i}\alpha\hat{p}, \\ \mathrm{i}\alpha(U-c)\hat{v} &= -\mathrm{i}\beta\hat{p}, \\ \mathrm{i}\alpha(U-c)\hat{w} &= -\mathrm{D}\hat{p}, \\ \text{and}\qquad \mathrm{i}(\alpha\hat{u}+\beta\hat{v})+\mathrm{D}\hat{w} &= 0. \end{aligned}\right\} \tag{21.7}$$

For rigid boundaries, this being the usual case, we impose the boundary conditions

$$\hat{w}=0 \quad \text{at } z=z_1 \text{ and } z_2. \tag{21.8}$$

Thus it is clear from equations (21.7) and the boundary conditions (21.8) that we have an eigenvalue problem which will lead to an eigenvalue relation of the form

$$\mathcal{F}(\alpha,\beta,c)=0. \tag{21.9}$$

For a given basic flow $U(z)$ and a given vector wavenumber $(\alpha,\beta,0)$ this eigenvalue relation determines the allowable values of c. Since the allowed values of c are, in general, finite in number the normal modes are obviously not complete and, in addition to this discrete part of the spectrum, there is also a continuous part which arises from the singularity in the equations where $U-c=0$. The continuous part of the spectrum must be included in order to represent an arbitrary initial disturbance and will be discussed further in § 24 in connexion with the initial-value problem.

In deriving equations (21.7) we have considered general three-dimensional disturbances and we now wish to show how the three-dimensional problem defined by equations (21.7) and (21.8) can be reduced to an *equivalent two-dimensional problem.* For this purpose we use the transformation first introduced by Squire (1933) for the more general viscous problem. Thus if we let

$$\left.\begin{aligned} \tilde{\alpha} &= (\alpha^2+\beta^2)^{1/2}, \quad \tilde{\alpha}\tilde{u}=\alpha\hat{u}+\beta\hat{v}, \\ \tilde{p}/\tilde{\alpha} &= \hat{p}/\alpha, \quad \tilde{w}=\hat{w}, \text{ and } \tilde{c}=c, \end{aligned}\right. \tag{21.10}$$

then equations (21.7) can be combined to give

$$\left.\begin{aligned} \mathrm{i}\tilde{\alpha}(U-\tilde{c})\tilde{u}+U'\tilde{w} &= -\mathrm{i}\tilde{\alpha}\tilde{p}, \\ \mathrm{i}\tilde{\alpha}(U-\tilde{c})\tilde{w} &= -\mathrm{D}\tilde{p}, \\ \text{and}\qquad \mathrm{i}\tilde{\alpha}\tilde{u}+\mathrm{D}\tilde{w} &= 0, \end{aligned}\right\} \tag{21.11}$$

and the boundary conditions are

$$\tilde{w}=0 \quad \text{at } z=z_1 \text{ and } z_2. \tag{21.12}$$

These equations have exactly the same mathematical form as the original equations with $\beta=\hat{v}=0$ and they thus define the equivalent two-dimensional problem. It is sufficient, therefore, to consider only two-dimensional disturbances; for, once the solution of equations (21.7) and (21.8) with $\beta=\hat{v}=0$ has been obtained, we can immediately obtain the corresponding solution of the equivalent two-dimensional problem by a trivial change in notation and from this, by means of Squire's transformation, we can then obtain the solution of the original three-dimensional problem. From these results we can now easily prove

Squire's theorem for an inviscid fluid. To each unstable three-dimensional disturbance there corresponds a more unstable two-dimensional one.

To prove the theorem observe first that if $c = f(\alpha)$ is the solution of the two-dimensional problem, i.e. the solution of equations (21.7) and (21.8) with $\beta = \hat{v} = 0$, then $\tilde{c} = f(\tilde{\alpha})$ is the solution of the equivalent two-dimensional problem and, by Squire's transformation, $c = f((\alpha^2+\beta^2)^{1/2})$ is the solution of the three-dimensional problem. Thus, to each unstable three-dimensional disturbance with growth rate αc_i there corresponds a two-dimensional disturbance with growth rate $\tilde{\alpha} c_i$ which is more unstable since $\tilde{\alpha} > \alpha$ if $\beta \neq 0$.

An important consequence of this theorem is that in seeking sufficient criteria for instability we need consider only two-dimensional disturbances, and it is then convenient to introduce a stream function $\psi'(x, z, t)$ such that the two components of the disturbance velocity are given by

$$u' = \partial\psi'/\partial z \quad \text{and} \quad w' = -\partial\psi'/\partial x. \tag{21.13}$$

If we next let

$$\psi'(x, z, t) = \phi(z)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{21.14}$$

then

$$\hat{u} = \phi' \quad \text{and} \quad \hat{w} = -\mathrm{i}\alpha\phi, \tag{21.15}$$

and the first of equations (21.7) gives

$$\hat{p} = U'\phi - (U-c)\phi'. \tag{21.16}$$

On substituting this result for $\hat{p}$ into the third of equations (21.7) we obtain *Rayleigh's stability equation*

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \tag{21.17}$$

which, together with the boundary conditions

$$\alpha\phi = 0 \quad \text{at } z = z_1 \quad \text{and} \quad z_2, \tag{21.18}$$

defines the basic eigenvalue problem for inviscid parallel shear flows. In fact Rayleigh's stability equation is the vorticity equation of the disturbance (see Problem 4.1).

Note that Rayleigh's equation and the boundary conditions are unchanged when α is replaced by $-\alpha$. Thus, without loss of generality, we can take $\alpha \geqslant 0$, and the criterion for instability then becomes that there exists a solution with $c_i > 0$ for some $\alpha > 0$. Furthermore, if ϕ is an eigenfunction with eigenvalue c for some α, then so too is ϕ^* with eigenvalue c^* for the same α. Thus, to each unstable mode there is a corresponding stable mode, and it is convenient therefore to adopt the convention of taking $c_i > 0$ as the criterion of instability and to ignore the complex conjugate eigenvalue with $c_i < 0$.

Rayleigh's equation is not self-adjoint but its adjoint is easily found to be

$$(\mathrm{D}^2 - \alpha^2)(U-c)\phi^\dagger - U''\phi^\dagger = 0, \tag{21.19}$$

where $\phi^\dagger$ must satisfy the same boundary conditions. On comparing this equation with equation (21.17) it immediately follows that $\phi^\dagger = \text{constant} \times \phi/(U-c)$. Equation (21.19) can also be written in the self-adjoint form

$$\mathrm{D}[(U-c)^2\mathrm{D}\phi^\dagger] - \alpha^2(U-c)^2\phi^\dagger = 0, \tag{21.20}$$

and this equation will be used later in the proof of Howard's semicircle theorem.

22 General criteria for instability

A distinctive feature of the velocity profiles shown in Fig. 4.1 is that one has an inflexion point but the other does not. The importance of this fact and its bearing on the stability or instability of the flow was first recognized by Rayleigh (1880), who proved

> *Rayleigh's inflexion-point theorem.* A *necessary* condition for instability is that the basic velocity profile should have an inflexion point.

To prove this theorem first rewrite Rayleigh's equation in the form

$$\phi'' - \alpha^2\phi - \frac{U''}{U-c}\phi = 0, \tag{22.1}$$

and suppose that $c_i > 0$ so that the equation is non-singular. On multiplying this equation by ϕ^*, integrating from z_1 to z_2, and then integrating the first term by parts, we obtain

$$\int_{z_1}^{z_2} (|\mathrm{D}\phi|^2 + \alpha^2|\phi|^2)\,\mathrm{d}z + \int_{z_1}^{z_2} \frac{U''}{U-c}|\phi|^2\,\mathrm{d}z = 0. \tag{22.2}$$

The imaginary part of this equation is

$$c_i \int_{z_1}^{z_2} \frac{U''}{|U-c|^2} |\phi|^2 \, dz = 0, \tag{22.3}$$

from which it follows that U'' must change sign at least once in the open interval (z_1, z_2).

A stronger form of this condition was obtained later by Fjørtoft (1950) who proved

Fjørtoft's theorem. A *necessary* condition for instability is that $U''(U-U_s)<0$ somewhere in the field of flow, where z_s is a point at which $U''=0$ and $U_s = U(z_s)$.

To prove this theorem consider the real part of equation (22.2):

$$\int_{z_1}^{z_2} \frac{U''(U-c_r)}{|U-c|^2} |\phi|^2 \, dz = -\int_{z_1}^{z_2} (|D\phi|^2 + \alpha^2 |\phi|^2) \, dz. \tag{22.4}$$

If we now add

$$(c_r - U_s) \int_{z_1}^{z_2} \frac{U''}{|U-c|^2} |\phi|^2 \, dz = 0$$

to the left-hand side of equation (22.4) we obtain

$$\int_{z_1}^{z_2} \frac{U''(U-U_s)}{|U-c|^2} |\phi|^2 \, dz = -\int_{z_1}^{z_2} (|D\phi|^2 + \alpha^2 |\phi|^2) \, dz < 0, \tag{22.5}$$

from which the result follows. Thus, if $U(z)$ is a monotone function with only one inflexion point then a necessary condition for instability is that $U''(U-U_s) \leqslant 0$ for $z_1 \leqslant z \leqslant z_2$ with equality only at $z = z_s$, and this condition is illustrated in Fig. 4.2.

Unfortunately, neither of these conditions for instability is sufficient. This is perhaps most clearly seen from Tollmien's (1935) simple counter-example with $U = \sin z$ which will be discussed later in this section. Tollmien also gave a heuristic argument, however, which suggests that the conditions are sufficient for symmetric profiles in a channel and for monotone profiles of the boundary-layer type. His argument was based on first showing the existence of a neutrally stable eigensolution

$$\phi = \phi_s, \quad \alpha = \alpha_s > 0, \quad c = c_s \tag{22.6}$$

and then, by perturbing this solution, to construct neighbouring unstable modes for α close to α_s with $\alpha < \alpha_s$.

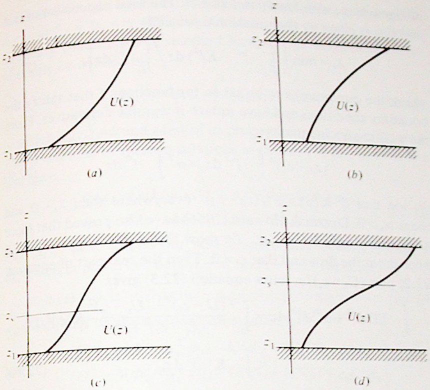

Fig. 4.2. (*a*) Stable: $U''<0$; (*b*) stable: $U''>0$; (*c*) stable: $U''=0$ at z_s but $U''(U-U_s) \geqslant 0$; (*d*) possibly unstable: $U''=0$ at z_s but $U''(U-U_s) \leqslant 0$. (From Drazin & Howard 1966.)

It is worth noting that if $\alpha = 0$ then $\phi = U - c$ is a neutral mode provided c can be chosen so as to satisfy the boundary conditions. This apparently trivial solution is merely a perturbed form of the basic flow but it does play a minor role in the viscous theory in determining the asymptotic behaviour of the curve of marginal stability. To demonstrate the existence of a neutrally stable eigensolution with $\alpha_s > 0$ we suppose, following Friedrichs (Mises & Friedrichs 1942), that $K(z) = -U''/(U-U_s)$ is regular at z_s, i.e. $U_s'' = 0$, and let $\lambda = -\alpha^2$. If $c = c_s = U_s$ then Rayleigh's stability equation can be rewritten in the form

$$\phi'' + \{\lambda + K(z)\}\phi = 0, \tag{22.7}$$

which, together with the boundary conditions (21.18), is a standard Sturm–Liouville problem for which there exists an infinite sequence of eigenvalues with limit point at $+\infty$. The least eigenvalue of this problem is given by the variational principle

$$\lambda_s = \min\left\{ \int_{z_1}^{z_2} (f'^2 - Kf^2)\,dz \Big/ \int_{z_1}^{z_2} f^2\,dz \right\}, \tag{22.8}$$

where the minimum is to be taken for functions f that satisfy the boundary conditions and have square-integrable derivatives. From the well-known inequality

$$(z_2 - z_1)^2 \int_{z_1}^{z_2} f'^2\,dz \geqslant \pi^2 \int_{z_1}^{z_2} f^2\,dz \tag{22.9}$$

we see that if $K(z) > \pi^2/(z_2 - z_1)^2$ everywhere then $\lambda_s < 0$ and hence $\alpha_s > 0$. Drazin & Howard (1966) have also proved that there is instability only for $\alpha < \alpha_s$. To show this suppose that $K(z) > 0$ throughout the flow and that $c_i \neq 0$. Then the real part of equation (22.2) plus $(U_s - c_r)/c_i$ times equation (22.3) gives

$$\int_{z_1}^{z_2} (|D\phi|^2 + \alpha^2|\phi|^2)\,dz = \int_{z_1}^{z_2} \frac{(U-c_r)^2 - (U_s - c_r)^2}{(U-c_r)^2 + c_i^2} K|\phi|^2\,dz$$
$$< \int_{z_1}^{z_2} K|\phi|^2\,dz, \tag{22.10}$$

from which it immediately follows that

$$\alpha^2 < -\lambda_s = \alpha_s^2. \tag{22.11}$$

Hence instability is possible only when $0 < \alpha < \alpha_s$ and we have stability ($c_i = 0$) for $\alpha \geqslant \alpha_s$. But this argument does *not* show that if $0 < \alpha < \alpha_s$ then $c_i \neq 0$, for we have not excluded the possibility of the eigensolution defined by equation (22.6) being an isolated neutral mode.

If, however, we *assume* the existence of unstable modes for α close to α_s with $\alpha < \alpha_s$, whose limit as $c_i \downarrow 0$ is the neutrally stable eigensolution defined by equation (22.6), then, as Tollmien (1935) and Lin (1945, 1955) have shown, they can be found by a simple perturbation procedure. The requirement that c_i tend to zero through positive values is a consequence of considering either the inviscid initial-value problem or the inviscid limit of the viscous problem. We follow here a method suggested by Hughes & Reid (1965b) which, with minor modifications, lends itself to certain applications in the viscous theory. For this purpose it is necessary to

consider a second solution, ψ_s (say), of Rayleigh's equation with $\alpha = \alpha_s$ and $c = U_s = c_s$. (This solution does not, of course, satisfy the boundary conditions.) A standard form of this solution can conveniently be defined by

$$\psi_s(z) = \phi_s(z) \int_{z_s}^{z} \{\phi_s(z)\}^{-2}\,dz \tag{22.12}$$

provided $\phi_s(z_s) \neq 0$ and a few of its properties may be briefly noted. The Wronskian of the two solutions is given by $\mathscr{W}(\phi_s, \psi_s) = 1$. We also have

$$\left.\begin{aligned} &\psi_s(z_1) = -1/\phi_s'(z_1), \quad \psi_s(z_2) = -1/\phi_s'(z_2), \\ &\psi_s(z_s) = 0 \quad \text{and} \quad \psi_s'(z_s) = 1/\phi_s(z_s). \end{aligned}\right. \tag{22.13}$$

For (α, c) near (α_s, c_s) we now assume that $\phi(z; \alpha, c)$ can be expanded in powers of both $\alpha - \alpha_s$ and $c - c_s$ in the form

$$\phi(z) = \phi_s(z) + \Phi_1(z)(\alpha - \alpha_s) + \Phi_2(z)(c - c_s) + \cdots, \tag{22.14}$$

where Φ_1 and Φ_2 must satisfy the equations

$$\left.\begin{aligned} &(U - c_s)(\Phi_1'' - \alpha_s^2 \Phi_1) - U''\Phi_1 = 2\alpha_s(U - c_s)\phi_s \\ \text{and} \quad &(U - c_s)(\Phi_2'' - \alpha_s^2 \Phi_2) - U''\Phi_2 = U''(U - c_s)^{-1}\phi_s. \end{aligned}\right\} \tag{22.15}$$

The solutions of these equations that vanish at $z = z_1$ (say) are

$$\Phi_1 = 2\alpha_s\left(\psi_s \int_{z_1}^{z} \phi_s^2\,dz - \phi_s \int_{z_1}^{z} \phi_s\psi_s\,dz\right) \tag{22.16}$$

and

$$\Phi_2 = \psi_s \int_{z_1}^{z} \frac{U''}{(U - c_s)^2} \phi_s^2\,dz - \phi_s \int_{z_1}^{z} \frac{U''}{(U - c_s)^2} \phi_s\psi_s\,dz. \tag{22.17}$$

In accordance with the requirement that c_i tend to zero through positive values, the path of integration for the first integral in equation (22.17) must lie below z_s if $U_s' > 0$ and above z_s if $U_s' < 0$; the integrand of the second integral, however, is regular at z_s. At $z = z_2$, Φ_1 and Φ_2 have the values

$$\left.\begin{aligned} &\Phi_1(z_2) = -\frac{2\alpha_s}{\phi_s'(z_2)} \int_{z_1}^{z_2} \phi_s^2\,dz \\ \text{and} \quad &\Phi_2(z_2) = -\frac{1}{\phi_s'(z_2)} \int_{z_1}^{z_2} \frac{U''}{(U - c_s)^2} \phi_s^2\,dz, \end{aligned}\right\} \tag{22.18}$$

and are thus independent of ψ_s. It may be noticed that $\Phi_1(z_2)$ is real but that $\Phi_2(z_2)$ is complex with real and imaginary parts given by

$$\Phi_{2r}(z_2) = -\frac{1}{\phi_s'(z_2)}\mathscr{P}\int_{z_1}^{z_2}\frac{U''}{(U-c_s)^2}\phi_s^2\,dz$$

and

$$\Phi_{2i}(z_2) = -\pi\frac{U_s'''}{U_s'^2}\frac{\phi_s^2(z_s)}{\phi_s'(z_2)}\,\mathrm{sgn}\,U_s', \tag{22.19}$$

where $\mathscr{P}$ denotes the Cauchy principal value of the integral. With Φ_1 and Φ_2 determined in this manner, ϕ in equation (22.14) automatically vanishes at $z = z_1$; the requirement that it also vanish at $z = z_2$ shows that

$$c - c_s \sim \frac{\Phi_1(z_2)\Phi_2^*(z_2)}{|\Phi_2(z_2)|^2}(\alpha_s - \alpha) \text{ as } \alpha \uparrow \alpha_s, \tag{22.20}$$

and this result is equivalent to Lin's formula (1955), p.123 for $(\partial c/\partial\alpha^2)_{\alpha=\alpha_s}$. In particular, the imaginary part of equation (22.20) is

$$c_i \sim -\frac{\Phi_1(z_2)\Phi_{2i}(z_2)}{|\Phi_2(z_2)|^2}(\alpha_s - \alpha) \text{ as } \alpha \uparrow \alpha_s. \tag{22.21}$$

The sign of the coefficient in this expression is determined by the sign of $U_s'''\mathrm{sgn}\,U_s'$; alternatively if $K(z_s) = -U_s'''/U_s' > 0$ then c_i is positive for α just less than α_s and we have instability. Perturbation formulae of this type can also be derived without difficulty for semi-bounded flows of the boundary-layer type (Hughes & Reid 1965b) and unbounded flows of the jet and shear-layer type.

To illustrate some of the consequences of these results consider a *sinusoidal basic flow* with $U = \sin z$ $(z_1 \leq z \leq z_2)$. The points of inflexion for this flow are where $z = z_s = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. If there are no values of z_s in the interval (z_1, z_2) then the flow is stable by Rayleigh's theorem. Next suppose that there is at least one value of z_s in the interval which, without loss of generality, we can take to be $z_s = 0$ so that $z_1 < 0 < z_2$. Thus, with $c = c_s = 0$ Rayleigh's equation becomes

$$\sin z\{\phi'' + (1-\alpha^2)\phi\} = 0 \quad \text{with } \phi = 0 \quad \text{at } z = z_1 \quad \text{and} \quad z_2. \tag{22.22}$$

If we now simply drop the factor $\sin z$, thereby ignoring the continuous spectrum, then we have

$$\left.\begin{aligned}\phi_s &= \sin\{n\pi(z-z_1)/(z_2-z_1)\}\\ \text{and}\qquad \alpha_s &= \{1 - n^2\pi^2/(z_2-z_1)^2\}^{1/2}\end{aligned}\right\} \tag{22.23}$$

for each integer $n < (z_2 - z_1)/\pi$. Thus, if $z_2 - z_1 < \pi$ the flow is stable even though it has an inflexion point and this is Tollmien's counter-example to the sufficiency of Rayleigh's theorem. If $z_2 - z_1 > \pi$, however, then the flow is unstable; this condition also follows from the inequality (22.9) on noting that $K(z) \equiv 1$ for this velocity profile. Suppose now that $z_1 = -\pi$ and $z_2 = \pi$ so that the flow is like the one shown in Fig. 4.1(*b*). We then have the neutral mode

$$\phi_s = \cos\tfrac{1}{2}z, \quad \alpha_s = \tfrac{1}{2}\sqrt{3}, \quad c_s = 0, \tag{22.24}$$

and from equation (22.21) we have $c_i \sim \sqrt{3}(\alpha_s - \alpha)$ as $\alpha \uparrow \alpha_s$. In addition there is also the trivial neutral mode $\phi_s = \sin z$, $\alpha_s = 0$, $c_s = 0$ and for this mode $\phi_s(z_s) = 0$.

Tollmien's inviscid solutions. Consider now the solutions of Rayleigh's equation when c is not necessarily equal to c_s. A point $z = z_c$ where $U - c = 0$ and $U_c' \neq 0$ is a regular singular point of equation (21.17) with exponents 0 and 1. Thus, in a neighbourhood of z_c there exists one solution which is analytic at $z = z_c$. It is convenient to write this solution in the form

$$\phi_1(z) = (z - z_c)P_1(z), \tag{22.25}$$

where $P_1(z)$ is analytic at z_c and $P_1(z_c) \neq 0$. For convenience, we shall choose $P_1(z_c) = 1$. The second linearly independent solution of equation (21.17), however, has a logarithmic branch point at $z = z_c$ and is of the form

$$\phi_2(z) = P_2(z) + (U_c''/U_c')\phi_1(z)\ln(z - z_c), \tag{22.26}$$

where $P_2(z)$ is also analytic at z_c with $P_2(z_c) = 1$. To make this second solution definite, it is convenient to suppose that $\phi_2(z)$ contains no multiple of $\phi_1(z)$, i.e. that the coefficient of $z - z_c$ in the power series expansion of $P_2(z)$ is zero. The solutions of Rayleigh's equation were first given in this form by Tollmien (1929) in connexion with his discussion of the Orr–Sommerfeld equation and they are often referred to as Tollmien's inviscid solutions. A more descriptive terminology will be helpful in our later discussion of the viscous problem and we will therefore call $\phi_1(z)$ the 'regular inviscid solution', $\phi_2(z)$ the 'singular inviscid solution', and $P_2(z)$ the 'regular part' of the singular inviscid solution.

In the case of neutral stability c, and hence z_c, is real and it is then necessary to specify the correct branch of the multivalued solution given by equation (22.26). By again letting c_i tend to zero through positive values we see that if $U_c' > 0$ and we let $\ln(z-z_c) = \ln|z-z_c|$, for $z > z_c$ then we have $\ln(z-z_c) = \ln|z-z_c| - \pi i$ for $z < z_c$. Later, in our discussion of the viscous problem, we will be concerned with the circumstances under which these solutions of Rayleigh's equation provide approximations to the solutions of the Orr–Sommerfeld equation.

The first few terms in the power series expansion of P_1 and P_2 are

$$\left.\begin{aligned} P_1(z) &= 1 + \frac{U_c''}{2U_c'}(z-z_c) + \frac{1}{6}\left(\frac{U_c'''}{U_c'} + \alpha^2\right)(z-z_c)^2 + \cdots \\ &\text{and} \\ P_2(z) &= 1 + \left(\frac{U_c'''}{2U_c'} - \frac{U_c''^2}{U_c'^2} + \frac{1}{2}\alpha^2\right)(z-z_c)^2 + \cdots. \end{aligned}\right\} \quad (22.27)$$

For velocity profiles with a sufficiently simple analytical form, the summation of these series is often feasible; more generally, however, ϕ_1 and the regular part of ϕ_2 can be obtained by direct numerical integration. When this latter method is used, ϕ_1 can conveniently be defined as the solution of Rayleigh's equation that satisfies the initial conditions $\phi_1(z_c) = 0$ and $\phi_1'(z_c) = 1$. Similarly, as suggested by Conte & Miles (1959), P_2 can be obtained as the solution of the inhomogeneous equation

$$(U-c)(P_2'' - \alpha^2 P_2) - U''P_2 = -\frac{U_c''}{U_c'} \cdot \frac{U-c}{z-z_c}\{2(z-z_c)P_1' + P_1\} \tag{22.28}$$

that satisfies the initial conditions $P_2(z_c) = 1$ and $P_2'(z_c) = 0$. The Wronskian of the solutions ϕ_1 and ϕ_2 is a constant with the value $\mathscr{W}(\phi_1, \phi_2) = -1$, and this relation often provides a useful check on numerical work.

The corresponding solutions of the adjoint equation can conveniently be taken in the form

$$\phi_j^\dagger = \frac{U_c'}{U-c}\phi_j \quad (j = 1, 2) \tag{22.29}$$

from which we have immediately

$$\left.\begin{aligned} \phi_1^\dagger(z) &= P_1^\dagger(z) \\ &\text{and} \\ \phi_2^\dagger(z) &= (z-z_c)^{-1}P_2^\dagger(z) + (U_c''/U_c')\phi_1^\dagger(z)\ln(z-z_c), \end{aligned}\right\} \quad (22.30)$$

where $P_1^\dagger$ and $P_2^\dagger$ are again analytic at z_c and $P_1^\dagger(z_c) = P_2^\dagger(z_c) = 1$. Since the power series expansion of $P_1^\dagger$ has no linear term, $\phi_2^\dagger$ contains no multiple of $\phi_1^\dagger$. The solution $\phi_1^\dagger$, like ϕ_1, is regular at z_c, but $\phi_2^\dagger$ is more singular than ϕ_2 and this has important consequences in the asymptotic theory of the adjoint Orr–Sommerfeld equation.

The modified Heisenberg expansions. The solutions of Rayleigh's equation had been obtained even earlier by Heisenberg (1924) as power series in α^2 of the form

$$\phi_j(z; \alpha^2, c) = \frac{U-c}{U_c'}\sum_{n=0}^{\infty}\alpha^{2n}q_{jn}(z; c) \quad (j = 1, 2), \tag{22.31}$$

where

$$\left.\begin{aligned} &q_{10} = 1, \quad q_{20} = \int (U-c)^{-2}\,dz \\ &\text{and} \\ &q_{j,n+1} = \int (U-c)^{-2}\,dz \int (U-c)^2 q_{jn}\,dz \quad (n \geqslant 0). \end{aligned}\right\} \quad (22.32)$$

The lower limits of integration in these expressions are, of course, arbitrary but they are usually taken as z_1. When the limits of integration are fixed in this manner the relationship between the expansions (22.31) and the Tollmien solutions is neither simple nor direct. This difficulty can be avoided, however, by a simple re-definition of the coefficients q_{jn} as discussed by Nield (1972). Thus we let

$$\left.\begin{aligned} q_{10} = 1,\ q_{20} &= \frac{1}{z-z_c} + \frac{U_c''}{U_c'}\ln(z-z_c) - \frac{U_c''}{2U_c'} \\ &\quad - \int_{z_c}^{z}\left\{\left(\frac{U_c'}{U-c}\right)^2 - \frac{1}{(z-z_c)^2} + \frac{U_c''}{U_c'(z-z_c)}\right\}dz \\ &\text{and} \\ q_{j,n+1} &= \int_{z_c}^{z}(U-c)^{-2}\,dz\int_{z_c}^{z}(U-c)^2 q_{jn}(z)\,dz. \end{aligned}\right\} \quad (22.33)$$

It is then an easy matter to show that these modified Heisenberg expansions are simply different representations of the Tollmien solutions. The series (22.31) are uniformly convergent for bounded values of α and fixed values of $z \neq z_c$. They are particularly useful in the viscous theory in connexion with the determination of the asymptotic behaviour of the curves of marginal stability. For unbounded or semi-bounded flows, however, different forms of approximations are needed and these will be discussed later in this section.

Suppose now that $U(z)$ is monotone with a single inflexion point at z_s $(z_1 < z_s < z_2)$. Then both ϕ_1 and ϕ_2 are regular at z_s, and ϕ_s must be a linear combination of ϕ_1 and P_2. Since $\phi_s(z_s) \neq 0$ (except when $\alpha = 0$), it is convenient to let

$$\phi_s(z) = A\phi_1(z) + P_2(z), \tag{22.34}$$

so that $\phi_s(z_s) = 1$ and the two boundary conditions then determine A and α_s. This method of determining ϕ_s remains applicable to monotone profiles with more than one inflexion point and to non-monotone profiles for which $U'' = 0$ whenever $U - U_s = 0$ (in particular, this includes symmetric flows). For non-monotone profiles which do not satisfy this condition† the determination of the neutrally stable eigensolution is more difficult for then ϕ_s must necessarily be singular at points where $U - c_s = 0$, and the following argument, due to Foote & Lin (1950), is helpful in such cases.

The average of the (dimensionless) Reynolds stress over one wavelength is

$$\begin{aligned}\tau = -\langle u'w'\rangle &= -\frac{\alpha}{2\pi}\int_0^{2\pi/\alpha} u'w'\,\mathrm{d}x \\ &= \tfrac{1}{4}\mathrm{i}\alpha(\phi\mathrm{D}\phi^* - \phi^*\mathrm{D}\phi)\exp(2\alpha c_i t)\end{aligned} \tag{22.35}$$

and on using Rayleigh's equation we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = \frac{1}{2}\alpha c_i \frac{U''}{|U-c|^2}|\phi|^2 \exp(2\alpha c_i t). \tag{22.36}$$

The integral of $\mathrm{d}\tau/\mathrm{d}z$ over (z_1, z_2) must vanish since $\tau = 0$ at the boundaries. Suppose now, however, that we have an unstable mode for some value of α, and that, as $\alpha \uparrow \alpha_s$ (say), the corresponding value of c becomes real, i.e. $c_i(\alpha) \downarrow 0$. For this neutral mode $\mathrm{d}\tau/\mathrm{d}z = 0$ everywhere, i.e. τ = constant, except possibly at $z = z_c$ where

† A simple example is the free-convection boundary layer with suction for which $U = ze^{-z}$ when $Pr = 1$. This profile has an inflexion point at $z_s = 2$ but $c_s = U_s \neq 2e^{-2}$.

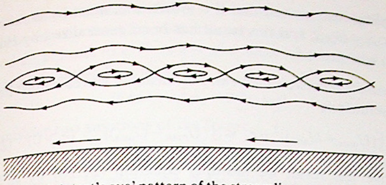

Fig. 4.3. Kelvin's 'cat's eye' pattern of the streamlines near the critical level as viewed by an observer moving with the wave.

$U - c = 0$. On integrating equation (22.36) across the critical layer $z = z_c$ and then letting $c_i \downarrow 0$ we find that the 'jump' in τ is given by

$$\Delta\tau = \tfrac{1}{2}\alpha\pi(U_c''/U_c')|\phi_c|^2, \tag{22.37}$$

where $\Delta\tau = \tau(z_c + 0) - \tau(z_c - 0)$. Thus, if the profile is monotone, then there can be only one jump and this implies that $U_c'' = 0$ since $\phi_c \neq 0$ except possibly when $\alpha = 0$. More generally, the algebraic sum of all such jumps must be zero. In the case of non-monotone profiles for which $c_s \neq U_s$ this condition must be used in the determination of α_s and c_s and the corresponding eigenfunction ϕ_s will be (mildly) singular whenever $U - c_s = 0$.

The form of the streamlines in the neighbourhood of the critical level where $U - c = 0$ has been given by Kelvin (1880b). If we impose a velocity equal to the phase velocity on the whole system then the motion becomes steady and the streamlines are identical with the particle paths. The physical stream function for this steady flow is

$$\Psi(z) + A\,\mathrm{Re}\{\phi(z)\,\mathrm{e}^{\mathrm{i}\alpha x}\},$$

where

$$\Psi(z) = \int_{z_c}^{z} (U - c)\,\mathrm{d}z$$

and A is proportional to the amplitude of the wave. Near the critical level $z = z_c$ the equation for the streamlines is

$$\tfrac{1}{2}U_c'(z - z_c)^2 + A\phi(z_c)\cos\alpha x = \text{constant}, \tag{22.38}$$

where we have taken $\phi(z_c)$ to be real. The streamlines therefore have the famous 'cat's eye' pattern shown in Fig. 4.3.

Rayleigh had proved that if $c_i \neq 0$ then c_r must lie in the range $U_{\min} < c_r < U_{\max}$, and this result has been generalized by Howard (1961) who proved

Howard's semicircle theorem. For unstable waves c must lie in the semicircle

$$\{c_r - \tfrac{1}{2}(U_{\max} + U_{\min})\}^2 + c_i^2 \leq \{\tfrac{1}{2}(U_{\max} - U_{\min})\}^2 \quad (c_i > 0). \tag{22.39}$$

To prove this result we multiply equation (21.20) by $\phi^{\dagger *}$ and integrate from z_1 to z_2 to obtain

$$\int_{z_1}^{z_2} (U-c)^2 Q \, dz = 0 \quad \text{where } Q \equiv |D\phi^{\dagger}|^2 + \alpha^2 |\phi^{\dagger}|^2 > 0, \tag{22.40}$$

and we have assumed that $c_i \neq 0$ so that $\phi^{\dagger}$ is non-singular. The real and imaginary parts of this integral are

$$\int_{z_1}^{z_2} \{(U-c_r)^2 - c_i^2\} Q \, dz = 0 \quad \text{and} \quad 2c_i \int_{z_1}^{z_2} (U-c_r) Q \, dz = 0. \tag{22.41}$$

The second integral here gives Rayleigh's result that c_r must lie in the range of U. Observe further that

$$0 \geq \int_{z_1}^{z_2} (U - U_{\min})(U - U_{\max}) Q \, dz$$

$$= \int_{z_1}^{z_2} \{(c_r^2 + c_i^2) - (U_{\max} + U_{\min}) c_r + U_{\max} U_{\min}\} Q \, dz$$

and hence that

$$c_r^2 + c_i^2 - (U_{\max} + U_{\min}) c_r + U_{\max} U_{\min} \leq 0,$$

from which the theorem follows.

Unbounded flows. For unbounded flows the Heisenberg expansions of the solutions of Rayleigh's equation are not uniformly convergent as $z \to \pm\infty$ and, following Drazin & Howard (1962), we now wish to consider the long-wave approximation for such flows. We assume, firstly, that U approaches constant values $U(\pm\infty)$ as $z \to \pm\infty$. If $U(+\infty) = U(-\infty)$ then, by a simple re-definition of c, we can normalize U so that $U(\pm\infty) = 0$ and this will be referred to as the *jet case*. Similarly, if $U(+\infty) \neq U(-\infty)$, then we can normalize U so that $U(\pm\infty) = \pm 1$ and this will be referred to as the *shear-layer case*. Secondly, we assume that $U \to$ constant as $z \to \pm\infty$ sufficiently rapidly so that (at least for $c \neq 0$) the solutions of Rayleigh's equation are asymptotic to $e^{\mp\alpha z}$ as $z \to \pm\infty$. Now let $W = U - c$ so that Rayleigh's equation becomes

$$W(\phi'' - \alpha^2 \phi) - W'' \phi = 0 \tag{22.42}$$

and consider two solutions of the form

$$\phi_{\pm}(z) = e^{\mp\alpha z} \chi_{\pm}(z). \tag{22.43}$$

It is convenient to normalize these solutions so that $\chi_{\pm}(\pm\infty) = W(\pm\infty) \equiv W_{\pm\infty}$. This normalization does not require that $\phi_+(0) = \phi_-(0)$. But if $\phi = \phi_+$ for $z > 0$ then its continuation to $z < 0$ must be a multiple of ϕ_-, i.e.

$$\phi_+(0) = K\phi_-(0) \quad \text{and} \quad \phi'_+(0) = K\phi'_-(0),$$

where $K = K(\alpha, c)$. Eliminating K between these equations then gives the eigenvalue relation

$$\phi_+(0)\phi'_-(0) - \phi'_+(0)\phi_-(0) = 0, \tag{22.44}$$

which is simply the Wronskian of $\phi_{\pm}(z)$. For small values of α we now let

$$\chi_{\pm}(z) = \sum_{n=0}^{\infty} (\pm\alpha)^n \chi_{\pm n}(z). \tag{22.45}$$

The coefficients in this expansion can be expressed explicitly in terms of W and repeated integrals of W, and on expanding the eigenvalue relation (22.44) for small values of α we obtain

$$\alpha(W_{\infty}^2 + W_{-\infty}^2) + \alpha^2 \int_{-\infty}^{\infty} (W^2 - W_{\infty}^2)(W^2 - W_{-\infty}^2) W^{-2} \, dz$$

$$+ O(\alpha^3) = 0. \tag{22.46}$$

In a first approximation we have $(U_{\infty} - c)^2 + (U_{-\infty} - c)^2 = 0$ and, on taking the root with $c_i > 0$, this gives

$$c \to \tfrac{1}{2}(U_{\infty} + U_{-\infty}) + \tfrac{1}{2}\mathrm{i}(U_{\infty} - U_{-\infty}) \text{ as } \alpha \downarrow 0. \tag{22.47}$$

Thus, in the shear-layer case we get $c = \mathrm{i}$. For the jet case it is necessary to go to a second approximation to get

$$c \sim \mathrm{i}\alpha^{1/2} \left(\frac{1}{2} \int_{-\infty}^{\infty} U^2 \, dz \right)^{1/2} \text{ as } \alpha \downarrow 0. \tag{22.48}$$

23 Flows with piecewise-linear velocity profiles

The eigenvalue problem defined by equations (21.17) and (21.18) is difficult to solve explicitly when $U(z)$ is a smoothly varying function. When $U(z)$ is piecewise linear, however, the solutions of Rayleigh's equation are simple exponential or hyperbolic functions which must satisfy certain matching conditions at a discontinuity of $U(z)$ or $U'(z)$. The use of piecewise-linear profiles thus provides a simple method of modelling some features of smoothly varying profiles.

Suppose then that U or U' are discontinuous at $z = z_0$ (say) and let $\Delta f = f(z_0+0) - f(z_0-0)$ denote the 'jump' in $f(z)$ at z_0. To derive the first matching condition rewrite Rayleigh's equation in the form

$$[(U-c)\phi' - U'\phi]' - \alpha^2(U-c)\phi = 0. \qquad (23.1)$$

On integrating this equation across the discontinuity from $z_0-\varepsilon$ to $z_0+\varepsilon$ and then letting $\varepsilon \downarrow 0$ we obtain

$$\Delta[(U-c)\phi' - U'\phi] = 0 \quad \text{at } z = z_0. \qquad (23.2)$$

This condition also follows immediately from equation (21.16) by requiring that the pressure be continuous across the material interface, i.e. $\Delta\hat{p} = 0$ at $z = z_0 + \zeta(x,t)$, where $\zeta(x,t) = \zeta_0 \exp\{i\alpha(x-ct)\}$. To first order in the small perturbation ζ_0 this gives equation (23.2).

To derive the second matching condition divide the pressure equation (21.16) by $(U-c)^2$ to get

$$\left(\frac{\phi}{U-c}\right)' = -\frac{\hat{p}}{(U-c)^2}. \qquad (23.3)$$

On integrating across the discontinuity, we have

$$\Delta\left(\frac{\phi}{U-c}\right) = -\lim_{\varepsilon\downarrow 0}\int_{z_0-\varepsilon}^{z_0+\varepsilon} \frac{\hat{p}}{(U-c)^2}\,dz. \qquad (23.4)$$

The right-hand side of this equation vanishes provided either $c_i \neq 0$ as $\varepsilon \downarrow 0$ or, if $c_i = 0$, then $U - c \neq 0$ for z in the interval $[z_0-\varepsilon, z_0+\varepsilon]$. Thus the second matching condition becomes

$$\Delta[\phi/(U-c)] = 0 \quad \text{at } z = z_0. \qquad (23.5)$$

This condition can also be derived from the requirement of streamline tangency, i.e. by requiring that the normal velocity of the fluid be continuous at the material interface.

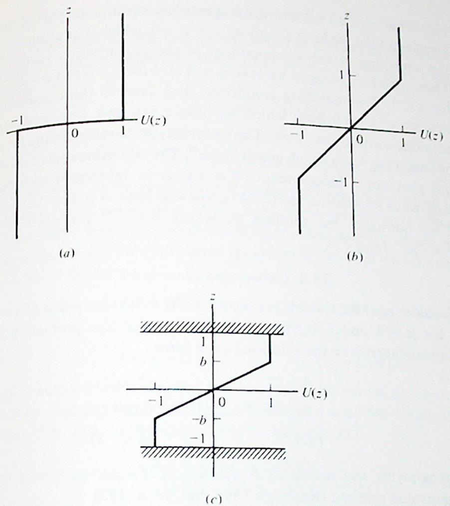

Fig. 4.4. (*a*) Unbounded vortex sheet; (*b*) unbounded shear layer; (*c*) bounded shear layer.

Since $\phi^\dagger = \phi/(U-c)$, the corresponding matching conditions for the adjoint problem are

$$\Delta\phi^\dagger = \Delta[(U-c)^2 D\phi^\dagger] = 0 \quad \text{at } z = z_0. \qquad (23.6)$$

To illustrate some of the general stability characteristics discussed in § 22, consider the three flows shown in Fig. 4.4.

23.1 *Unbounded vortex sheet*

The Kelvin–Helmholtz instability of a vortex sheet has already been discussed in some detail in § 4 on the assumption that the disturbed flow was irrotational. When $U'' \equiv 0$ Rayleigh's equation reduces to

$(U-c)(\phi''-\alpha^2\phi)=0$ and if we ignore the continuous part of the spectrum then we have simply $\phi''-\alpha^2\phi=0$, which is equivalent to the vanishing of the y-component of the perturbation vorticity. Thus, with $U(z)=\operatorname{sgn} z$ we take $\phi=Ae^{-\alpha z}$ for $z>0$ and $\phi=Be^{\alpha z}$ for $z<0$. The matching conditions then lead to the eigenvalue relation $c^2+1=0$ from which we have $c_r=0$ and, in accordance with our convention, $c_i=1$. The instability is therefore in the form of a standing wave which grows like $e^{\alpha t}$. The eigenvalue relation for this problem is independent of α and is in agreement with the long-wave approximation (22.47). We also have $A-iB=0$ so that if we normalize the solution by letting $\phi=e^{-\alpha z}$ for $z>0$ then $\phi=-ie^{\alpha z}$ for $z<0$.

23.2 *Unbounded shear layer*

Consider next the shear layer shown in Fig 4.4(b) for which $U(z)=z$ for $|z|<1$ and $U(z)=z/|z|$ for $|z|>1$. For this problem it is convenient to take the solution in the form

$$\phi=\begin{cases} Ae^{-\alpha(z-1)} & (z>1) \\ Be^{-\alpha(z-1)}+Ce^{\alpha(z+1)} & (|z|<1) \\ De^{\alpha(z+1)} & (z<-1). \end{cases}$$

On applying the matching conditions at $z=\pm1$ we obtain the eigenvalue relation (Rayleigh 1894, vol. II, p. 393)

$$c^2=(4\alpha^2)^{-1}\{(1-2\alpha)^2-e^{-4\alpha}\}. \tag{23.7}$$

If we let $\alpha_s\cong0.64$ be the root of $1-2\alpha+e^{-2\alpha}=0$ then c is purely imaginary for $0<\alpha<\alpha_s$ and we have instability. The growth rate αc_i is greatest when $\alpha\cong0.40$, and in the long-wave limit we again recover equation (22.47). The eigenfunction ϕ_s associated with the neutral mode $\alpha=\alpha_s$ and $c=0$ is also of some interest. If we fix the normalization of ϕ_s by letting $A=1$ then we have

$$\phi=\begin{cases} \exp\{-\alpha_s(z-1)\} & (z>1) \\ (\cosh\alpha_s)^{-1}\cosh\alpha_s z & (|z|<1) \\ \exp\{\alpha_s(z+1)\} & (z<-1) \end{cases}$$

so that ϕ_s is an even function of z. In the long-wave limit $\phi\to$ constant $\times\, U(z)$ as $\alpha\downarrow0$.

23.3 *Bounded shear layer*

To further illustrate the lack of sufficiency in Rayleigh's inflexion-point theorem consider the shear layer shown in Fig. 4.4(c) for which

$$U(z)=\begin{cases} 1 & (b<z\leqslant1) \\ z/b & (|z|<b) \\ -1 & (-1\leqslant z<-b) \end{cases}$$

with $0<b\leqslant1$. In the limit as $b\to0$ we have a vortex sheet which is unstable for all values of α. When $b=1$, however, we have plane Couette flow which has no discrete eigenvalues and for which, as discussed in § 24, the continuous spectrum is stable. If we take the solution in the form

$$\phi=\begin{cases} A\sinh\alpha(1-z) & (b<z\leqslant1) \\ B\sinh\alpha z+C\cosh\alpha z & (|z|<b) \\ D\sinh\alpha(1+z) & (-1\leqslant z<-b) \end{cases}$$

which automatically satisfies the boundary conditions at $z=\pm1$, then on applying the matching conditions at $z=\pm b$ we obtain the eigenvalue relation (Rayleigh 1894, vol. II, p. 388)

$$c^2=1-\frac{\alpha b(1+X^2)Y^2+2\alpha bXY-XY^2}{\alpha^2b^2\{(1+X^2)Y+X(1+Y^2)\}}, \tag{23.8}$$

where $X=\tanh\alpha b$ and $Y=\tanh\alpha(1-b)$. A little analysis of this result shows that $c^2\to2b-1$ as $\alpha\to0$ and that $c^2\to1$ as $\alpha\to\infty$. Since c^2 is a monotone increasing function of α it then follows that the flow is unstable provided $b<\frac{1}{2}$.

24 The initial-value problem

In this section we shall digress a little from the topic of normal modes, which forms the basis of the development of this chapter. In Chapter 1 we introduced the idea of seeking the development in time of an arbitrary small disturbance of some basic flow to test whether the flow was stable. This led to the linearization of the equations of motion and to the resolution of the arbitrary disturbance into independent wave components, each a normal mode.

Even if one accepts the approximation of linearization without reservation, however, one would expect to see in practice not only one normal mode but some superposition of many normal modes which is determined by the nature of the initial disturbance. Here we shall elaborate these ideas in the context of the linear stability of steady parallel flows.

We model this situation first by supposing that at some instant, say $t=0$, there are given arbitrary smooth distributions of $\mathbf{u}'$ and p' in space subject to the constraint of incompressibility. We then seek to trace the subsequent development of the solutions $\mathbf{u}'(\mathbf{x}, t)$ and $p'(\mathbf{x}, t)$ of the linearized equations for $t>0$. This is the classic *initial-value problem*. If $\mathbf{u}'$ and p' remain bounded for all time we deem the disturbance stable. Of course the initial-value problem is closely related to the spectrum of normal modes, because if the spectrum is complete then any solution of the initial-value problem may be represented as sums and integrals of the normal modes.

We noted in § 21 that for a given flow there is in fact only a finite number of non-singular normal modes at each value of the wavenumber. Therefore it is impossible in general to represent any smooth initial disturbance as a superposition of those normal modes, and we may say that they are incomplete. We found in § 22 the finiteness of the number of non-singular normal modes at marginal stability, and showed how to find their eigenfunctions ϕ_s. For many flows there is only one non-singular normal mode, and for any stable flow there is none. However, the spectrum of normal modes is in fact made complete by the inclusion of a continuous spectrum of singular normal modes associated with the singularity of the Rayleigh stability equation where $U(z)=c$. Because this singularity arises only when c is real (and lies within the range of U over the domain of flow), this continuous spectrum is composed of only stable modes and may be ignored when seeking only a criterion for stability.

These ideas are well exemplified by the simple case of plane Couette flow, even though plane Couette flow is stable and so has no discrete spectrum of non-singular normal modes for any value of the wavenumber. Thus we take

$$U(z)=z \quad \text{for } -1\leqslant z\leqslant 1. \tag{24.1}$$

Orr (1907), pp. 26–29 was the first to treat this problem. He noted that the equation governing two-dimensional disturbances was simply of the form

$$\left(\frac{\partial}{\partial t}+z\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial z^2}\right)=0. \tag{24.2}$$

(This is the linearized vorticity equation, which takes this simple form because the basic flow has uniform vorticity. On taking the normal mode described by equation (21.14), it reduces to the Rayleigh stability equation.) This equation gives

$$\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial z^2}=F(x-zt, z), \tag{24.3}$$

where F is an arbitrary function of integration which is differentiable with respect to x. Any initial disturbance specified by $\psi'(x, z, 0)$ may be used to determine F. Orr went on to solve the Poisson equation (24.3) for $\psi'(x, z, t)$ by the use of Fourier series in z.

An alternative, and more generally applicable, approach to the problem was developed by Eliassen, Høiland & Riis (1953) and Case (1960a,b). They took the Laplace transform of equation (24.2) with respect to t, and the Fourier transform with respect to x. This leads to an inhomogeneous linear ordinary differential equation in z for the transform of ψ', the inhomogeneity being due to the initial disturbance. The solution of this inhomogeneous equation and its boundary conditions can be found in terms of a Green's function which has a discontinuous derivative. Finally the transforms may be inverted in the usual way to give $\psi'(x, z, t)$. In fact it is found that in general $\psi'=O(t^{-2})$ and $u'=\partial\psi'/\partial z=O(t^{-1})$ as $t\to\infty$ for fixed x and z, and that the flow is stable.

This method of Fourier–Laplace transforms has been applied to a general basic flow by Case (1960a) and Dikii (1960a). Their argument gives a solution of the form

$$\psi'(x, z, t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{1}{2\pi \mathrm{i}}\int_{\varepsilon-\mathrm{i}\infty}^{\varepsilon+\mathrm{i}\infty}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}\alpha x+st}\phi_{s\alpha}(z) \tag{24.4}$$

for $t>0$, $-\infty<x<\infty$, $z_1<z<z_2$, and ε is chosen so that the Bromwich contour of integration in the complex s-plane is to the

right of all singularities of the integrand. The Fourier–Laplace transform is then determined by

$$\phi_{s\alpha}(z)=\int_{z_1}^{z_2} G(z, z_0; s)\,\frac{\Psi(z_0; 0)}{s+\mathrm{i}\alpha U(z_0)}\,\mathrm{d}z_0, \tag{24.5}$$

where G is the appropriate Green's function, and Ψ is determined from the initial distribution of the stream function $\psi'(x, z, 0)$. In the inversion of the Laplace transform the discrete spectrum emerges from the residues at the poles of G, if any, the poles occurring wherever the normal-mode problem gives an eigenvalue for a non-singular eigenfunction. Also the continuous spectrum is associated with the zeros of $s+\mathrm{i}\alpha U(z_0)$.

Yet another method is to use generalized functions, directly or indirectly. Eliassen, Høiland & Riis (1953) solved the initial-value problem for plane Couette flow in this way, attributing their solution to unpublished work by Fjørtoft and Høiland. We shall follow the later treatment of Case (1960a), who invoked the method of Fourier–Laplace transforms to justify his use of generalized functions. For plane Couette flow defined by equation (24.1) the normal-mode eigenvalue problem becomes

$$(z-c)(\mathrm{D}^2-\alpha^2)\phi=0, \tag{24.6}$$

and

$$\phi=0 \quad \text{at } z=\pm 1. \tag{24.7}$$

Now if c does not lie in the range $(-1, 1)$ of the basic velocity given by equation (24.1) then we may divide both sides of equation (24.6) by $z-c$ to get

$$(\mathrm{D}^2-\alpha^2)\phi=0. \tag{24.8}$$

It follows at once that there is no non-trivial solution of the system described by equations (24.8) and (24.7); thus, there is no non-singular normal mode and the discrete spectrum is empty. If $-1<c<1$, however, then division of equation (24.6) by $z-c$ gives

$$(\mathrm{D}^2-\alpha^2)\phi=\delta(z-c), \tag{24.9}$$

where an arbitrary normalization has been chosen and δ is the Dirac delta function. The solution of the system described by equations

(24.9) and (24.7) is readily seen to be

$$\phi=G(z,c)\equiv\begin{cases}\dfrac{\sinh\alpha(c-1)\sinh\alpha(z+1)}{\alpha\sinh 2\alpha} & \text{for } -1\leq z\leq c\\[2ex] \dfrac{\sinh\alpha(c+1)\sinh\alpha(z-1)}{\alpha\sinh 2\alpha} & \text{for } c\leq z\leq 1.\end{cases} \tag{24.10}$$

This function is related to the Green's function in the Fourier–Laplace transform method for this problem, and can be seen to have a discontinuous derivative at $z=c$. It gives the continuous spectrum of singular eigenfunctions corresponding to values of c in the interval $(-1, 1)$ for each value of α. The physical quantity $\psi'(x, z, t)$, however, which is composed of a double integral of the eigenfunctions given by equations (24.10) is non-singular and may be used to represent the development in time of any smooth initial disturbance.

Although the formal treatment of the inviscid initial-value problem is straightforward in principle, detailed analysis of the solution, even for simple basic flows and specially chosen initial conditions, is technically complicated (Case 1960a). For the viscous initial-value problem, Gaster (1975) has made a theoretical study of the development of a model wavepacket in a laminar boundary layer and good agreement was obtained with the experimental results of Gaster & Grant (1975) during the initial stages of growth. In physical terms, the linear instability may be described qualitatively as follows. If the initial disturbance is localized in space, then the disturbance propagates much like a wavepacket of the most unstable modes, travelling with their group velocity and growing exponentially like them as it travels. (One may note that the group velocity may be zero, but is typically in the direction of the basic flow, and that interference of the most unstable modes leads to algebraic moderation of the exponential growth of a wavepacket.) The disturbance decays rapidly up-stream and down-stream of its centre as it travels. Thus, relative to an observer moving with the group velocity, the disturbance grows exponentially in time. But the disturbance measured at any fixed station x eventually decays rapidly with time because the localized disturbance travels away; indeed, at a distant station the disturbance may appear first to grow and then decay rapidly as its centre passes by. This instability is called *convected instability* by plasma physicists; if, however, the group velocity is zero, the instability remains stationary as it grows exponentially and then is called *absolute* or *non-convected*.

So far we have treated the classic initial-value problem. But this is a poor model of many experiments for which some device, such as a small loudspeaker or an oscillating ribbon, is introduced into a basic flow to produce a controlled wave disturbance. At some instant the device is turned on, and thereafter it is maintained as a wave source of fixed amplitude form and fixed angular frequency, ω (say). If the group velocity for that frequency has the same direction as the basic flow, the resultant forced oscillation of the flow is observed to have the real frequency $\omega = -\alpha c$ downstream of the source after the transients have propagated away. This situation and its mathematical model will be discussed more fully for the viscous problem in § 32, but it suffices to state now that what little work has been done on the model suggests that the forced disturbance of frequency ω develops at any fixed station x as $t \to \infty$ and is composed of modes each of which varies downstream like $e^{i\alpha x}$, where $\alpha = \alpha(\omega)$ is real if the mode is stable and has negative imaginary part if it is unstable. At each instant the disturbance is bounded as $x \to \pm\infty$ because the transients travel with their group velocities, but the disturbance may seem unbounded if one examines it by letting $t \to \infty$ before one lets $x \to +\infty$.

Closely related to this kind of initial-value problem is the use of *spatially growing modes*, sometimes called simply *spatial modes*. They were first applied to hydrodynamic stability by Watson (1962) and Gaster (1962), but seem to have been first used by Landau (1946) for a problem of plasma physics, a subject for which they have since been used extensively (see, for example, Clemmow & Dougherty (1969), Chap. 6).

The essential theoretical ideas behind the use of spatial modes are as follows. The dispersion relation can be written in the form

$$\mathscr{F}(\alpha, \omega) = 0, \tag{24.11}$$

where $\omega = -\alpha c$, and may be regarded as determining either the complex value or values of ω for any real value of α or of α for any real value of ω, or, indeed, as determining the complete functional relationship between the complex variables ω and α. Hitherto we have treated only *temporally growing modes*, or *temporal modes*, for

which we take a real value of the wavenunber α and seek complex c or ω to determine the temporal growth of the mode through the factor $e^{-i\alpha ct} = e^{i\omega t}$. For spatial modes we take a real value of the frequency ω and seek the complex eigenvalues $\alpha = \alpha_r + i\alpha_i$. Consequently the spatial modes behave like $\exp\{i(\alpha x + \omega t)\} = \exp\{-\alpha_i x + i(\alpha_r x + \omega t)\}$; and so they grow or decay exponentially with x unless $\alpha_i = 0$ and also oscillate in x with wavelength $2\pi/\alpha_r$. These spatial modes resemble the forced oscillation due to a source of frequency ω, so one might think that the criterion of instability of the basic flow is simply that $\alpha_i \neq 0$ for some real value of ω, but unfortunately the criterion is not always as simple as that.

It must be remembered that spatial modes (and, indeed, temporal modes) are significant because of their role in describing solutions of initial-value problems. A spatial mode is itself inadmissible as a solution when it is unbounded at infinity, but may nevertheless describe the solution a long time after a localized source of given frequency ω has been turned on. In this way the physical properties of spatial modes are closer than those of temporal modes to the instability observed in most experiments on parallel flows, and for that reason spatial modes are often used when making comparisons with experimental results as discussed in § 32.

To interpret spatial modes without ambiguity, however, one must go to the initial-value problem, and this is not easy. One can see that to evaluate Fourier–Laplace integrals such as in equation (24.4) one may seek to integrate first with respect to either α or s $(= i\omega)$. Also one can change the contours of integration in the complex planes of α and ω if one knows the nature of the singularities, as is often desirable in evaluating the integral asymptotically as $t \to \infty$. It is for these reasons that the relationship (24.11) between the complex variables α and ω is important.

These ideas are discussed further by Clemmow & Dougherty (1969), Chap. 6 and applied to viscous fluid by Gaster (1965); see also § 47.

THE VISCOUS THEORY

25 The governing equations

In discussing the governing equations for a viscous fluid we shall again suppose that the basic steady flow is of the form given by