Convergence Time for Unbiased Quantized Consensus*

Seyed Rasoul Etesami, Tamer Başar

Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801
Email: (etesami1, basar1)@illinois.edu

Abstract— We revisit the quantized consensus problem on undirected connected graphs, and obtain some strong results on expected time to convergence. This is *unbiased* consensus, because the edges emanating from a node have equal probability of being selected. The paper first develops an approach that bounds the expected convergence time of the underlying discrete-time dynamics. The bounds are tight for some simple networks when there exists some symmetry in the network. Following this, the paper provides a tight expression for the expected convergence time of unbiased quantized consensus over general networks. Finally, the paper shows that the expected convergence time can be expressed in terms of the effective resistances of the associated Cartesian product graph. The approach adopted in the paper uses the theory of harmonic functions for reversible Markov chains.

Index Terms— Quantized consensus, convergence time, social networks, Markov chains, random walk.

I. INTRODUCTION

With the appearance of myriads of online social networks and availability of huge data sets, modeling of the opinion dynamics in a social network has gained a lot of attention in recent years. In a distributed averaging algorithm, agents will exchange their information and update their values based on others' opinions so that eventually they reach the same outcome. Among many problems that arise in such an application is the one of computing the average of agents' initial values. However, in many applications, due to limited memory and energy, working with continuous variables as agent opinions is difficult and inefficient. This justifies finite quantization of opinions. One of the models that involves such dynamics is the gossip quantized model, where agents' opinions are constrained to be integer valued as it was introduced in [1]. The same problem without integer constraints has been studied in many forms; see, for example, [2], [3], [4].

It is known that, for any initial profile, the quantized dynamics introduced in [1] will converge to the consensus set with probability 1. However, depending on the initial profile and the distribution of choosing the edges at each time instant, the convergence time in expectation may vary. Studies on the behavior of the quantized consensus algorithm based on natural random walks and biased random walks can be found in [5] and [6].

In this paper, we consider the quantized consensus algorithm when each edge has equal probability of being chosen at each time step (unbiased). The convergence of such dynamics to the set of quantized consensus points has been shown earlier in [1]. However, an exact expression for the expected convergence time for these dynamics based on the topology of the network has not yet been given. In this paper we address this issue and provide some tight bounds on the expected convergence time based on some parameters of the underlying graph.

The paper is organized as follows: In Section II, we review the unbiased quantized consensus dynamics and some of its basic properties. In Section III, we present some preliminary results for later use. In Section IV, we obtain our main results, providing expected convergence time for these dynamics. We include some simulation results for simple graphs in Section V. Finally, we conclude the paper in Section VI.

Notations: For a positive integer n, we let $[n] := \{1, \ldots, n\}$. For a vector $v \in \mathbb{R}^n$, we let v_i to be the ith entry of v, and v^T be the transpose of v. We say that v is stochastic if $v_i \geq 0$ for all $i \in [n]$ and $\sum_{i=1}^{n} v_i = 1$. Similarly, we say that a matrix A is stochastic if each row of A is stochastic. If both A and A^T are stochastic, we say that A is doubly stochastic. For a random walk $\mathcal{Z}$ with transition probability matrix P, we let the random variable τ_z be the first time that the random walk hits the state z. Also, we let $H_{\mathcal{Z}}(a, z)$ denote the expected time that the random walk $\mathcal{Z}$ initiated at a hits z for the first time. We take $G_{\tau_z}(a, x)$ to be the expected number of visits to x before τ_z when the random walk has been initiated from a. For an undirected graph $\mathcal{G} = (V, \mathcal{E})$, we let $N(x)$ be the set of neighbors of x. We also let $\mathcal{G} \times \mathcal{G} = (V \times V, \mathcal{E}')$ be the Cartesian product of $\mathcal{G}$, i.e. $\big((x, y), (r, s)\big) \in \mathcal{E}'$ if and only if $x = r, s \in N(y)$ or $y = s, r \in N(x)$. We let $\mathcal{R}(x \leftrightarrow y)$ be the effective resistance between two nodes x and y in an electric network when every edge has resistance equal to 1. We denote by $\mathbb{V}(\cdot)$ the voltage function over an electric circuit and let $\mathbb{V}_{xy} = \mathbb{V}(x) - \mathbb{V}(y)$. Finally, we define the distribution of a vector v as the list $\{(v_1, n_1), (v_2, n_2), \ldots\}$ in which n_i is the number of entries of v which have value v_i.

II. UNBIASED QUANTIZED CONSENSUS

In this section, we describe the discrete-time quantized consensus model as introduced in [1].

- There is a set of n agents $V = \{1, 2, \ldots, n\}$ which are connected on some undirected graph $\mathcal{G}(V, \mathcal{E})$.
- Each agent has an initial value $x_i(0)$, which is a positive integer.
- At each time instant $t = 1, 2, \ldots$, one edge is chosen uniformly at random among the set of all the edges $\mathcal{E}$, and the incident nodes on the sides of this edge (let us call them i and j) update their values according to:

$$x_i(t+1) = \begin{cases} x_i(t) - 1, & \text{if } x_i(t) > x_j(t) \\ x_i(t) + 1, & \text{if } x_i(t) < x_j(t) \\ x_i(t), & \text{if } x_i(t) = x_j(t), \end{cases} \quad (1)$$

and the same holds for agent j. We refer to $x_i(t)$ as the *opinion of agent i* at time t and $x(t)$ as the *opinion profile* at time t.

It has been shown in [1] that at each time instant t, the Lyapunov function defined by:

$$V(x(t)) = \sum_{\ell=1}^{n} \left(x_\ell(t) - \frac{\sum_{k=1}^{n} x_k(0)}{n} \right)^2 \quad (2)$$

will decrease by at least 2 if a nontrivial update occurs at t. By nontrivial update we mean that the values of the incident nodes of the chosen edge (i, j) at time instant t differ by at least 2, i.e. $|x_i(t) - x_j(t)| \geq 2$. Therefore, to bound the expected convergence time for the above algorithm, one can think of the maximum expected time it takes for a nontrivial update to occur. Also, using the Lyapunov function given in (2), one can see that the number of nontrivial updates is of the order of $O(n)$. Therefore, the problem reduces to that of finding the maximum expected time it takes for a nontrivial update to take place, which we denote by $\bar{T}(\mathcal{G})$. In the rest of this paper we are interested in computing $\bar{T}(\mathcal{G})$. Let $T_1(x(0))$ be a random variable denoting the first nontrivial averaging when the initial profile is $x(0)$. Through some manipulation, it is not hard to see that $\bar{T}(\mathcal{G}) = \max_{x(0) \in \mathcal{T}(\mathcal{G})} T_1(x(0))$ where $\mathcal{T}(\mathcal{G}) = \{x | \text{distribution of x is } \{(0,1), (1, n-2), (2, 1)\}\}$. Therefore, the main issue is to find an expression for $\bar{T}(\mathcal{G}) = \max_{x(0) \in \mathcal{T}(\mathcal{G})} T_1(x)$.

In the above setting, we now assume that all the agents on the graph $\mathcal{G}$ have value 1 except two of them which are 0 and 2. At each time instant t, one edge will be selected with equal probability $\frac{1}{m}$ where m is the number of edges, and the incident nodes update their values based on (1). Therefore, we can interpret this problem in an alternative way. Consider two random walkers, let us call them 0 and 2, who start a random walk on the vertices of the graph $\mathcal{G}$. Then, $\bar{T}(\mathcal{G})$ is equal to the maximum expected time it takes for these two walkers to meet. One important fact is that both of these walkers have the same source of randomness, which selects an edge at each time instant. Therefore, these random walks are jointly correlated. The next lemma provides us with an upper bound for the meeting time based on the parameters of the network $\mathcal{G}$. We will see later that these upper bounds are indeed asymptotically exact for some simple graphs such as paths or cycles. We begin with the following definition.

Definition 1: A *birth-and-death chain* of length $n+1$ has state space $\Omega = \{0, 1, \ldots, n\}$ such that in one step the state can increase or decrease by at most 1.

Lemma 1: Assume that $\mathcal{G}$ is a connected graph with diameter D. Then, $\bar{T}(\mathcal{G})$ is bounded from above by the maximum hitting time of a birth-and-death chain of length $D+1$ and positive transition probabilities greater than $\frac{1}{m}$.

Proof: We partition all the different states of the above coupled random walks into different classes. For each state x we define $d(x^{(0)}, x^{(2)})$ to be the length of the shortest path between walker 0 and walker 2 in state x. Let

$$S_\ell = \{x \in \mathcal{T}(\mathcal{G}) | \; d(x^{(0)}, x^{(2)}) = \ell\}, \; \ell = 0, 1, \ldots, D.$$

It is clear that $\{S_\ell\}_{\ell=1}^{D}$ is a partitioning of all the states. Furthermore, S_0 contains just one state. In other words, when we reach class S_0, it means that the walkers have met. Now, we construct a new Markov chain, in a way that we let each class to be one state by itself, and we denote it by S_ℓ. Finally, we assign the following transition probabilities to the new Markov chain. For each $\ell = 1, 2, \ldots, D$, let

1. $\mathbb{P}\{S_\ell \to S_{\ell-1}\} = \min\limits_{x \in S_\ell, y \in S_{\ell-1}} \mathbb{P}\{x \to y\}$,

2. $\mathbb{P}\{S_\ell \to S_\ell\} = \min\limits_{x \in S_\ell} \mathbb{P}\{x \to x\}$,

3. $\mathbb{P}\{S_\ell \to S_{\ell+1}\} = 1 - \min\limits_{x \in S_\ell} \mathbb{P}\{x \to x\} - \min\limits_{x \in S_\ell, y \in S_{\ell-1}} \mathbb{P}\{x \to y\}$.

Also, note that $\mathbb{P}\{S_\ell \to S_{\ell+1}\} \geq 0$, and hence the above transition probabilities are well defined. Based on the above construction, it is not hard to see that the probability of moving toward the state S_0 in a new birth-and-death chain is always non-increasing, and hence the expected time to hit the state S_0 is always an upper bound for meeting time in the original coupled Markov chain. In fact, if we look at every sample path of states until the process terminates, then its equivalent process in the new birth-and-death chain has smaller probability of hitting the class S_0. Finally, we note that since in the original process each edge is chosen with probability $\frac{1}{m}$, the above assigned probabilities cannot be smaller than $\frac{1}{m}$. ∎

Corollary 1: Assume that $\mathcal{G}$ is a cycle of n nodes. Then,

$$\bar{T}(\mathcal{G}) \leq \frac{n(n-1)(n-3)}{16} + \frac{2n+1}{2}.$$

Proof: Analyzing the birth-and-death chain described in Lemma 1 for a cycle with n nodes, we can bound $\bar{T}(\mathcal{G})$ from above. For such a graph, the new Markov chain has the structure shown in Figure 1.
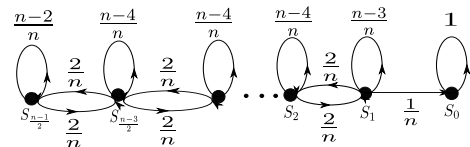

Fig. 1. Birth-and-death chain for a cycle with n nodes.

Therefore, $\bar{T}(\mathcal{G})$ is bounded from above by $H(S_{\lceil \frac{n-1}{2} \rceil}, S_0)$ in the above birth-and-death diagram. A simple calculation

shows that $H(S_{[\frac{n-1}{2}]}, S_0) = \frac{n(n-1)(n-3)}{16} + \frac{2n+1}{2}$ and the result follows. ∎

Corollary 2: If $\mathcal{G}$ is a line graph with n nodes, then

$$\bar{T}(\mathcal{G}) \leq \frac{(n-1)^2(n+1)}{4}.$$

Proof: This follows from a similar argument as in the proof of Corollary 1, and the bound coincides with the result given in [1]. ∎

Although Lemma 1 gives us a process which works well when we benefit from inherent symmetry in the underlying graph $\mathcal{G}$, in general it does not give us an explicit tight bound based on the parameters of the network. In the Section IV we take it a step further, and find an explicit form for $\bar{T}(\mathcal{G})$ for a general network, but first we provide some preliminary results.

III. PRELIMINARY RESULTS

In this section, we discuss some preliminary results which will be used to prove our main results. We start our discussion by defining some notation that will be used throughout this work. A simple random walk on a graph $\mathcal{G}$ is a Markov chain with transition probabilities

$$P(x,y) = \begin{cases} \frac{1}{d(x)}, & \text{if } y \in N(x) \\ 0, & \text{otherwise,} \end{cases}$$

where $d(x)$ denotes the degree of a node x in the graph $\mathcal{G}$. Note that a simple random walk is a special case of a weighted random walk when the weights of all edges in $\mathcal{G}$ are equal to 1. It is well known that every reversible Markov chain is a weighted random walk on a network. Suppose P is a transition matrix on a finite set S which is reversible with respect to the probability distribution $\pi(\cdot)$. Define conductance on edges by $c(x,y) = \pi(x)P(x,y)$ and $c(x) := \sum_{y: y \in N(x)} c(x,y)$. Also, the resistance of each edge e is defined to be the inverse of conductance, i.e. $r(e) = \frac{1}{c(e)}$.

Lemma 2: $\frac{G_{\tau_z}(a,x)}{d(x)}$ is equal to the induced voltage between x and z, i. e. V_{xz} when we define the terminal voltages to be $V_{zz} = 0, V_{az} = \frac{G_{\tau_z}(a,a)}{d(a)}$. Moreover, for all x we have

$$\frac{1}{2}\big[\mathcal{R}(a \leftrightarrow z) + \mathcal{R}(z \leftrightarrow x) - \mathcal{R}(a \leftrightarrow x)\big] = \frac{G_{\tau_z}(a,x)}{d(x)} = V_{xz}.$$

Proof: This is the result of Corollary 3 in [7]. ∎

By taking summation over the above equality and noting that $\sum_x G_{\tau_z}(a,x)$ is equal to the expected hitting time of a simple random walk when it starts from a and hits z, we get:

$$H(a,z) = \frac{1}{2}\sum_x d(x)\big[\mathcal{R}(a \leftrightarrow z) + \mathcal{R}(z \leftrightarrow x) - \mathcal{R}(a \leftrightarrow x)\big]$$
(3)

Definition 2: A function $h : \Omega \to R$ is harmonic for a transition probability matrix P at vertex x if

$$h(x) = \sum_{y \in \Omega} P(x,y)h(y).$$

Lemma 3: Let $\{X_t\}$ be a Markov chain with an irreducible transition matrix P, let $B \subset \Omega$, and $h_B : B \to R$ be a function defined on B. The function $h : \Omega \to R$ defined by $h(x) := \mathbb{E}_x h_B(X_{\tau_B})$ is the unique extension $h(\cdot)$ of h_B such that $h(x) = h_B(x)$ for all $x \in B$ and h is harmonic for P at all $x \in \Omega \setminus B$.

Proof: The proof can be found in [8]. ∎

Given a graph $\mathcal{G}$, we choose an edge at random and uniformly among all the set of edges at each time instance. Let $\mathcal{Z}$ be the lazy random walk which is generated based on marginal distribution of this setting. In other words, the walker will move to one of his neighbors with equal probability if he is located at one of the incident vertices. It is not hard to see that $\mathcal{Z}$ has the following transition probabilities:

$$P_{\mathcal{Z}}(x,y) = \begin{cases} 1 - \frac{d(x)}{m}, & \text{if } y = x \\ \frac{1}{m}, & \text{if } y \in N(x) \\ 0, & \text{else.} \end{cases}$$
(4)

Since the above transition matrix is doubly stochastic, $\pi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ is its stationary distribution. This results in $\pi_i P(i,j) = \pi_j P(j,i) \; \forall i,j \in 1, \ldots, n$, and hence $\mathcal{Z}$ is a reversible Markov chain.

We already know [9] that every reversible Markov chain has a hidden vertex w such that the hitting time from w to every state is less than or equal to the hitting time from that particular node to state w, i.e w is a hidden vertex for $\mathcal{Z}$ if $H_{\mathcal{Z}}(w,x) \leq H_{\mathcal{Z}}(x,w) \; \forall x$.

Definition 3: Assume that w is a hidden vertex for the reversible Markov chain $\mathcal{Z}$. As in [9] and [6], we define the potential function $\Phi(\cdot, \cdot) : V \times V \to R$ to be

$$\Phi(x,y) = H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y).$$

Now consider the original joint process for two walkers on a finite graph. However, this time when the walkers are each other's neighbors, we count the connecting edge twice in our edge probability distribution. We refer to this new process a *virtual* process and denote the meeting time function of the virtual process by $M(x,y)$ for every two initial states x and y. In fact we will show in the Section IV that the meeting time function of the virtual process $M(x,y)$ follows almost the same rule as $\Phi(x,y)$.

IV. MAIN RESULTS

We open this section with the following lemma.

Lemma 4: A function $f : V \times V \to \mathbb{R}$ defined by $f(x,y) = \frac{1}{2}\Phi(x,y) - M(x,y)$ is harmonic for the simple random walk on $\mathcal{G} \times \mathcal{G}$, i.e.

$$f(x,y) = \sum_{(r,s) \in V \times V} \mathcal{Q}\big((x,y),(r,s)\big)f(r,s),$$

where $\mathcal{Q}$ is the transition matrix of the simple random walk on $\mathcal{G} \times \mathcal{G}$, i.e.

$$\mathcal{Q}\big((x,y),(r,s)\big) = \begin{cases} \frac{1}{d(x)+d(y)}, & \text{if } (r,s) \in N_{\mathcal{G} \times \mathcal{G}}(x,y) \\ 0, & \text{else.} \end{cases}$$

Proof: By the transitivity property of reversible Markov chains, we note that $\Phi(x,y)$ is symmetric, i.e. for any hidden vertex w,

$$\Phi(x,y) = H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)$$
$$= H_{\mathcal{Z}}(y,x) + H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x) = \Phi(y,x).$$

Therefore, we can write:

$$\Phi(x,y) = \frac{d(x)}{d(x)+d(y)}\big[H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big]$$
$$+ \frac{d(y)}{d(x)+d(y)}\big[H_{\mathcal{Z}}(y,x) + H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x)\big]$$
$$= \frac{d(x)}{d(x)+d(y)}\big(H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big)$$
$$+ \frac{d(y)}{d(x)+d(y)}\big(H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x)\big)$$
$$+ \Big[\frac{d(x)}{d(x)+d(y)}H_{\mathcal{Z}}(x,y) + \frac{d(y)}{d(x)+d(y)}H_{\mathcal{Z}}(y,x)\Big]$$
$$\tag{5}$$

Also, by expanding $H_{\mathcal{Z}}(x,y)$ one step we get:

$$H_{\mathcal{Z}}(x,y) = \frac{m}{d(x)} + \frac{1}{d(x)}\sum_{j\in N(x)}H_{\mathcal{Z}}(x_j,y), \tag{6}$$

and similarly by switching x and y we have:

$$H_{\mathcal{Z}}(y,x) = \frac{m}{d(y)} + \frac{1}{d(y)}\sum_{j\in N(y)}H_{\mathcal{Z}}(y_j,x). \tag{7}$$

Using (6) and (7) in (5), we get

$$\Phi(x,y) = \frac{2m}{d(x)+d(y)} + \frac{d(x)}{d(x)+d(y)}\big(H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big)$$
$$+ \frac{d(y)}{d(x)+d(y)}\big(H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x)\big)$$
$$+ \frac{1}{d(x)+d(y)}\Big(\sum_{j\in N(x)}H_{\mathcal{Z}}(x_j,y) + \sum_{j\in N(y)}H_{\mathcal{Z}}(y_j,x)\Big).$$
$$\tag{8}$$

Also, from the definition of $\Phi(\cdot,\cdot)$, we have $\Phi(x_j,y) = H_{\mathcal{Z}}(x_j,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)$, $\forall j\in N(x)$. By taking summation over all $j\in N(x)$ and multiplying by the factor $\frac{1}{d(x)+d(y)}$, we arrive at

$$\frac{1}{d(x)+d(y)}\sum_{j\in N(x)}\Phi(x_j,y) = \frac{1}{d(x)+d(y)}\sum_{j\in N(x)}H_{\mathcal{Z}}(x_j,y)$$
$$+ \frac{d(x)}{d(x)+d(y)}\big[H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big].$$
$$\tag{9}$$

By the same argument, and since $\Phi(x,y_j) = H_{\mathcal{Z}}(y_j,x) + H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x)$, we have

$$\frac{1}{d(x)+d(y)}\sum_{j\in N(x)}\Phi(x_j,y) = \frac{1}{d(x)+d(y)}\sum_{j\in N(y)}H_{\mathcal{Z}}(y_j,x)$$
$$+ \frac{d(y)}{d(x)+d(y)}\big[H_{\mathcal{Z}}(x,w) - H_{\mathcal{Z}}(w,x)\big],$$
$$\tag{10}$$

Substituting (10) and (9) in (8) gives us

$$\Phi(x,y) = \frac{2m}{d(x)+d(y)}$$
$$+ \frac{1}{d(x)+d(y)}\Big(\sum_{j\in N(x)}\Phi(x_j,y) + \sum_{j\in N(y)}\Phi(y_j,x)\Big).\tag{11}$$

On the other side, we consider that regardless of whether $y\notin N(x)$ or $y\in N(x)$, the meeting time of the virtual process is equal to

$$M(x,y) = (1 - \frac{d(x)+d(y)}{m})\big(1 + M(x,y)\big)$$
$$+ \sum_{j\in N(x)}\frac{1}{m}\big(1 + M(x_j,y)\big) + \sum_{j\in N(y)}\frac{1}{m}\big(1 + M(y_j,x)\big)$$

from which by simplifying and rearranging the terms we get

$$M(x,y) = \frac{m}{d(x)+d(y)}$$
$$+ \frac{1}{d(x)+d(y)}\Big(\sum_{j\in N(x)}M(x_j,y) + \sum_{j\in N(y)}M(y_j,x)\Big).\tag{12}$$

Let $S(x,y) = \frac{\Phi(x,y)}{2}$. From (11) it is not hard to see that

$$S(x,y) = \frac{m}{d(x)+d(y)}$$
$$+ \frac{1}{d(x)+d(y)}\Big(\sum_{j\in N(x)}S(x_j,y) + \sum_{j\in N(y)}S(y_j,x)\Big).\tag{13}$$

We consider the simple random walk $\mathcal{Q}$ on the Cartesian product graph $\mathcal{G}\times\mathcal{G}$. The cover time and hitting time of such graphs have been extensively studied in [10], [11] and [12]. We show that the function $f(x,y) = S(x,y) - M(x,y)$ is harmonic on $\mathcal{G}\times\mathcal{G}$ for the transition matrix $\mathcal{Q}$. In fact,

$$f(x,y) = \frac{1}{d(x)+d(y)}\sum_{j\in N(x)}\big(S(x_j,y) - M(x_j,y)\big)$$
$$+ \frac{1}{d(x)+d(y)}\sum_{j\in N(y)}\big(S(y_j,x) - M(y_j,x)\big)$$
$$= \frac{1}{d(x)+d(y)}\Big(\sum_{j\in N(x)}f(x_j,y) + \sum_{j\in N(y)}f(x,y_j)\Big)$$
$$= \sum_{(r,s)\in V\times V}\mathcal{Q}\big((x,y),(r,s)\big)f(r,s).$$

This completes the proof. ∎

Now we are ready to state and prove our main results.

Theorem 1: The meeting time of the virtual process is equal to

$$M(x,y) =$$
$$\frac{1}{2}\Big[H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y) - \big(H_{\mathcal{Z}}(a,w) - H_{\mathcal{Z}}(w,a)\big)$$
$$\times \Big(\frac{1}{2} + \frac{\mathcal{R}\big((x,y)\leftrightarrow(w,w)\big) - \mathcal{R}\big((a,a)\leftrightarrow(x,y)\big)}{2\mathcal{R}\big((w,w)\leftrightarrow(a,a)\big)}\Big)\Big].$$

Proof: Define a function $g: V\times V\to R$ by

$$g(x,y) = \frac{1}{2}\big[H_{\mathcal{Z}}(a,w) - H_{\mathcal{Z}}(w,a)\big]\frac{\mathbb{V}(x,y) - \mathbb{V}(w,w)}{\mathbb{V}(a,a) - \mathbb{V}(w,w)},$$

where, w is the hidden vertex and a is an arbitrary but fixed node in $\mathcal{G}$. Also, $\mathbb{V}(\cdot)$ is the voltage function defined on the network $\mathcal{G} \times \mathcal{G}$ while each edge has resistance equal to 1. Now we have

$$f(w,w) = S(w,w) - M(w,w) = S(w,w)$$
$$= \frac{1}{2}\Big(H(w,w)+H(w,w)-H(w,w)\Big) = 0 = g(w,w),$$
$$f(a,a) = S(a,a) = \frac{1}{2}\Big(H(a,a)+H(a,w)-H(w,a)\Big)$$
$$= \frac{1}{2}\Big(H(a,w)-H(w,a)\Big) = g(a,a).$$

Since f and g are both harmonic functions for the transition matrix $\mathcal{Q}$ and also they have the same values at the nodes (w,w) and (a,a), using Lemma 3 they must be equal. Hence,

$$\frac{1}{2}\big[H_{\mathcal{Z}}(a,w)-H_{\mathcal{Z}}(w,a)\big]\frac{\mathbb{V}(x,y)-\mathbb{V}(w,w)}{\mathbb{V}(a,a)-\mathbb{V}(w,w)}$$
$$= \frac{1}{2}\big[H_{\mathcal{Z}}(x,y)+H_{\mathcal{Z}}(y,w)-H_{\mathcal{Z}}(w,y)\big]-M(x,y). \quad (14)$$

Furthermore, Lemma 2 gives us that

$$\frac{\mathbb{V}(x,y)-\mathbb{V}(w,w)}{\mathbb{V}(a,a)-\mathbb{V}(w,w)} = \frac{\mathcal{R}\big((x,y)\leftrightarrow(w,w)\big)+\mathcal{R}\big((w,w)\leftrightarrow(a,a)\big)-\mathcal{R}\big((a,a)\leftrightarrow(x,y)\big)}{2\mathcal{R}\big((w,w)\leftrightarrow(a,a)\big)}.$$

Substituting this relation in (14) and rearranging the terms we get the desired result. ∎

Corollary 3: If there is more than one hidden vertex in a graph G, then we have:

$$M(x,y) = \frac{1}{2}\big(H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big).$$

Proof: Indeed if we have at least two hidden vertices w and w', then we can choose a in the proof of Theorem 1 to be w'. From the definition of hidden vertex, we must have, $H_{\mathcal{Z}}(w',w) \geq H_{\mathcal{Z}}(w,w')$ (since w is a hidden vertex) and also $H_{\mathcal{Z}}(w,w') \geq H_{\mathcal{Z}}(w',w)$ (since w' is a hidden vertex). Putting these two together, we get $H_{\mathcal{Z}}(w,w') = H_{\mathcal{Z}}(w',w)$. Therefore, the function g in the above theorem would be zero everywhere and the result follows. ∎

Theorem 2: Consider a network $\mathcal{G} = (V,\mathcal{E})$. Then,

$$\max_{x,y} M(x,y) \leq \bar{T}(\mathcal{G}) \leq 2\max_{x,y} M(x,y).$$

Proof: Initiating from arbitrary nodes x and y, we note that both the virtual process and the original process follow the same joint distribution until they are each other's neighbor. However, when the walkers are each other's neighbor, then, with higher probability they are going to meet in the virtual process than in the original process. Therefore, $M(x,y) \leq \bar{T}(\mathcal{G})$ for all x,y. Since by definition $\bar{T}(\mathcal{G})$ is independent of the initial states of walkers, $\max_{x,y} M(x,y) \leq \bar{T}(\mathcal{G})$. For the upper bound, we use the same argument used in [6]. Indeed half of the time when two walkers meet in the virtual process, they do not meet in the original process and they stay in the same position. In fact, at each time that the walkers in the virtual process meet, the original random process terminates (two walkers meet each other) with a probability of 1/2, independently. Therefore,

$$\bar{T}(\mathcal{G}) \leq \sum_{k=1}^{\infty}(\frac{1}{2})^k k \max_{x,y} M(x,y) = 2\max_{x,y} M(x,y).$$

Next, we will proceed by computing $H_{\mathcal{Z}}(x,y)$. Note that if at the time instant t the walker is at the node $\mathcal{Z}(t)$, then the probability of staying in that state is $1-\frac{d(\mathcal{Z}(t))}{m}$. Because of Markov property of the random walk, the probability of moving out from each state follows the geometric distribution. Therefore, the expected time that the walker waits in state $\mathcal{Z}(t)$ is $\frac{m}{d(\mathcal{Z}(t))}$. However, when the walker is moving to the next state, he will see all of his neighbors with the same probability. Therefore, we can split the expected hitting time of the random walk $\mathcal{Z}(t)$ namely $H_{\mathcal{Z}}(x,y)$ to summation of two parts:

- the expected time that the walker spends in each state $\mathcal{Z}(t)$, which is $\frac{m}{d(\mathcal{Z}(t))}$,
- the expected hitting time of a simple random walk $\mathcal{Z}'(t)$, namely $H(x,y)$.

Therefore, we have:

$$H_{\mathcal{Z}}(x,y)=\sum_{t=1}^{H(x,y)}\big(1+\frac{m}{d(\mathcal{Z}'(t))}\big)=H(x,y)+m\sum_{t=1}^{H(x,y)}\frac{1}{d(\mathcal{Z}'(t))}$$
$$= H(x,y) + m\sum_i \frac{G_{\tau_y}(x,i)}{d(i)}$$
$$= H(x,y)+\frac{m}{2}\sum_i \big[\mathcal{R}\big(x\leftrightarrow y\big)+\mathcal{R}\big(y\leftrightarrow i\big)-\mathcal{R}\big(x\leftrightarrow i\big)\big]$$

$$(15)$$

where the last equality is due to Lemma 2.

Corollary 4: For a line graph and cycle with n nodes, we have $C_1 n^3 \leq \bar{T}(\mathcal{G}) \leq C_2 n^3$, where $0 < C_1 < C_2$ are two constants.

Proof: We prove the result for the line graph; for cycle graph the proof is similar. From (3), since the degree of each node is at most 2, we have

$$2\sum_i \big[\mathcal{R}\big(x\leftrightarrow y\big) + \mathcal{R}\big(y\leftrightarrow i\big)-\mathcal{R}\big(x\leftrightarrow i\big)\big] \geq H(x,y).$$

Replacing this inequality in (15), and since $m = n-1$, we get

$$H_{\mathcal{Z}}(x,y) \geq \frac{n+3}{4}H(x,y).$$

Also, since a path has two hidden vertices (vertices 1 and n), using Corollary 3 and Theorem 2 we can write

$$\bar{T}(\mathcal{G})\geq \max_{x,y} M(x,y)$$
$$= \frac{1}{2}\max_{x,y}\big(H_{\mathcal{Z}}(x,y) + H_{\mathcal{Z}}(y,w) - H_{\mathcal{Z}}(w,y)\big)$$
$$\geq \frac{1}{2}\max_{x,y} H_{\mathcal{Z}}(x,y)\geq \frac{n+3}{8}\max_{x,y} H(x,y)=\frac{n+3}{8}(n-1)^2$$

This relation in view of Corollary 2 completes the proof. ∎

We are now ready to state the main result of this paper.

Theorem 3: Consider a connected network $\mathcal{G}$ with n nodes and m edges and diameter D. Then, for unbiased quantized consensus, we have

$$\frac{1}{2}H_{\mathcal{Z}} \leq \bar{T}(\mathcal{G}) \leq 2H_{\mathcal{Z}} \leq 2n(n^2 + mD),$$

where, $H_{\mathcal{Z}} = \max_{x,y} H_{\mathcal{Z}}(x, y)$.

Proof: Using the triangle inequality for the effective resistance in a circuit, we know that

$$\frac{\mathcal{R}\big((x,y) \leftrightarrow (w,w)\big) - \mathcal{R}\big((a,a) \leftrightarrow (x,y)\big)}{2\mathcal{R}\big((w,w) \leftrightarrow (a,a)\big)} \leq \frac{1}{2}. \quad (16)$$

Using (16) in Theorem 1 and by transitivity of reversible Markov chains, i.e. $H_{\mathcal{Z}}(y, w) + H_{\mathcal{Z}}(w, a) + H_{\mathcal{Z}}(a, y) = H_{\mathcal{Z}}(y, a) + H_{\mathcal{Z}}(a, w) + H_{\mathcal{Z}}(w, y)$, we get

$$M(x, y) \geq \frac{1}{2}\Big[H_{\mathcal{Z}}(x, y) + H_{\mathcal{Z}}(y, a) - H_{\mathcal{Z}}(a, y)\Big].$$

Since the above relation must hold for every particular choice of a (Theorem 1), for $a = y$ we get $M(x, y) \geq \frac{1}{2}H_{\mathcal{Z}}(x, y)$. On the other hand, since w is a hidden vertex, $H_{\mathcal{Z}}(a, w) - H_{\mathcal{Z}}(w, a) \geq 0$. Applying this inequality in the expression for $M(x, y)$ in Theorem 1 gives us

$$\frac{1}{2}H_{\mathcal{Z}}(x, y) \leq M(x, y) \leq \frac{1}{2}\big(H_{\mathcal{Z}}(x, y) + H_{\mathcal{Z}}(y, w) - H_{\mathcal{Z}}(w, y)\big). \quad (17)$$

Using (17) and Theorem 2, and the definition of $H_{\mathcal{Z}}$, we can see that $\frac{1}{2}H_{\mathcal{Z}} \leq \bar{T}(\mathcal{G}) \leq 2H_{\mathcal{Z}}$. Furthermore, by relation (15), we get

$$H_{\mathcal{Z}} = \max_{x,y} \Big\{ H(x, y)$$
$$+ \frac{m}{2} \sum_{i \in V(\mathcal{G})} \big[\mathcal{R}\big(x \leftrightarrow y\big) + \mathcal{R}\big(y \leftrightarrow i\big) - \mathcal{R}\big(x \leftrightarrow i\big)\big] \Big\}$$
$$\leq \max_{x,y} H(x, y) + \frac{m}{2}n(2D) \leq n^3 + mnD,$$

where, in the first inequality we have used the fact that the effective resistance between any two nodes cannot exceed the diameter of $\mathcal{G}$ and in the second inequality we have used the well-known result [8] that the hitting time of a simple random walk on a general network is bounded from above by n^3. ∎

V. SIMULATION RESULTS

In this section we present some simulation results to provide a comparison between the maximum expected meeting time $\bar{T}(\mathcal{G})$ and the proposed upper and lower bounds given above. We consider four different types of graphs with n nodes: line graph, star graph, lollipop graph, and semi-regular graph. In lollipop graph each of its side clusters has $\left[\frac{n}{4}\right]$ nodes and they are connected with a single path. Also, for the semi-regular graph we consider a graph with n nodes arranged around a circle, such that each node is linked to its next four nodes when we move clockwise around the circle. In Figure 2, the ratio $\frac{\bar{T}(\mathcal{G})}{mnD}$ is depicted for each graph. As it can be seen, this ratio for the line graph converges asymptotically to a constant as n goes to infinity. This is the same as what we would have expected from Corollary 4. Moreover, using the transition probabilities given in (4) for the star graph, a simple calculation shows that $H_{\mathcal{Z}} = \frac{n(n-1)}{2}$, and hence from Theorem 3 we get $\frac{n(n-1)}{4n^2} \leq \frac{\bar{T}(\mathcal{G})}{mnD} \leq \frac{n(n-1)}{n^2}$, which is consistent with the ratio given in Figure 2. Finally, for lollipop and semi-regular graphs, although the ratio $\frac{\bar{T}(\mathcal{G})}{mnD}$ is

oscillating, it is clearly bounded from above by 1, which confirms the upper bound provided in Theorem 3.
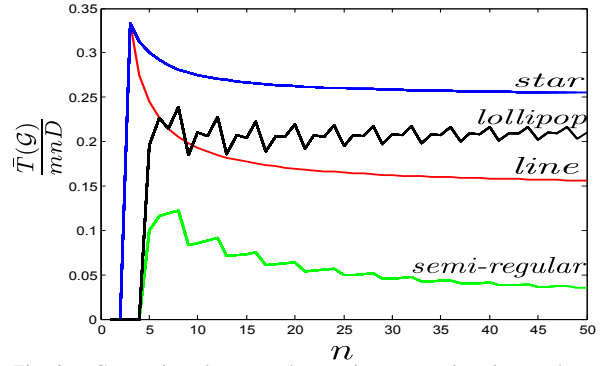

Fig. 2. Comparison between the maximum meeting time and mnD four different types of graphs.

VI. CONCLUSION

In this paper, we have studied the unbiased quantized consensus problem under the assumption that the underlying network $\mathcal{G}$ is connected. We provided tight upper and lower bounds for the maximum expected convergence time of the model. Moreover we gave an exact asymptotic value for the convergence time when the network is a line graph or a cycle.

As a future direction of research, an interesting problem is to consider the model when the choice of edges at each time instant is based on some specific, not necessarily uniform, distribution. Also, given a network $\mathcal{G}$, one can think of adding an edge (or removing an edge) so as to minimize (or maximize) the expected convergence time.

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