52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Risk Limiting Dispatch in Congested Networks

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***Abstract*—Increased penetration of renewables requires improved power dispatch processes to limit integration cost. Existing Monte Carlo sample based approaches are difficult to implement in practice and result in difficult to interpret solutions. Recently, a two-stage dispatch procedure called Risk Limiting Dispatch (RLD) was proposed, and analytical solution was derived for uncongested networks. In this paper we extend RLD to a network with multiple congested links. We show that the network size can be reduced tremendously by carefully considering the possible congestions in the network. The key insight is to use first stage forecast values of renewables to predict the likely real-time congestions. Once the real-time congestions are predicted, we derive an equivalent second stage problem with dimension one greater than the number of congested lines in the original network. The detailed dispatch algorithm is given and illustrated with example networks.**

I. Introduction

Existing practice in electric grid operations utilizes worst case dispatch: generation is dispatched to meet predicted peak demand and an additional reserve capacity is allocated to account for uncertainties due to forecast errors and outages in generation. The reserve capacity is scheduled following a '3-σ' rule [1], where σ is the standard deviation of the forecast error. Typical values today are around 1% to 2% of total load. Renewable generation is included as a negative load under worst case dispatch. Consequently, increased penetration of renewables increases the required reserve capacities [2], [3] as the typical forecast errors for renewable generation are significantly higher. These higher reserve capacities result in significantly increased operation costs. For example, for California ISO (CAISO) each additional 1% in reserves costs \$50MM in 2009 dollars.

Various alternative stochastic dispatch procedures have been studied to reduce these costs [4], [5]. These procedures rely on exploiting the sequential nature of the dispatch process to incorporate up-to-date forecast information. In particular, Monte-Carlo based formulations including DC power flow balance and power network constraints have been investigated [4], [6]–[11]. They result in complex optimization problems that can only be evaluated with limited Monte-Carlo samples, without offering significant insights in the resulting dispatch. They also require significant changes in the existing dispatch systems. In some cases, the forecast error distributions are not utilized appropriately or at all [12]. MPC approaches have also been considered [13], [14] but are not very accurate as the number of optimization stages is small in grid operations. Simplified single stage dispatch models are more tractable [15]–[18] but do not capture recourse or congestion.

Risk Limiting Dispatch (RLD) [19], [20] was proposed as an alternative dispatch framework that provides a simple analytic operation rule by simplifying various constraints, in particular network constraints. The approach significantly reduces the cost of integrating renewables in uncongested and lossless networks. Reliable estimates of metrics such as integration cost, emissions and costs due to forecasting performance are easy to obtain [21]. The present paper develops an RLD dispatch procedure for network with multiple congested links. Under congestion novel structural properties of the dispatch process need to be considered for deriving a simple analytic solution. In particular, the paper significantly extends the formulation in [22] that addressed a single congestion link. The main structural result is that a general congested network can always be reduced to an equivalent network that only contains congested transmission lines and a few additional lines. Congested links still allow flow- *back flow* - in the direction opposing congestion. The reduction is valid under small-σ, i.e, when uncertainty is moderate, so that transmission lines predicted to be congested are actually congested when the dispatch is realized. This reduction forms the basis of computing analytic dispatch solutions. The resulting solution is very easy to compute and can be directly incorporated in existing software for grid dispatch.

The remainder of the paper is organized as follows. Section II sets up the dispatch and uncertainty model, including the small-σ assumption, in detail. Section III derives the dispatch procedure for multiple congested links. Section IV provides the proof of the main results. Section V concludes with future works.

II. Model and Problem Formulation

A. Model Setup

A typical Independent System Operator(ISO) (see, e.g. [23] for New England ISO) in the US schedules generators and decide their output in a three step process as shown in Figure 1. The first step occurs a day (24 hours) ahead of the actual time of delivery of electricity, and is a purely financial procedure. At this step, generators and loads bid cost curves and offers respectively, and the ISO clears the market through a double auction [24]. A majority of generator are committed through this process[1]. The second step is called the reliability process, and is ran by the ISO with actual load forecasts to ensure that there would be enough generation to meet demand at the time of delivery. A few additional generator may be committed at this stage. The last step is balancing the real-time generation and demand of the power system.

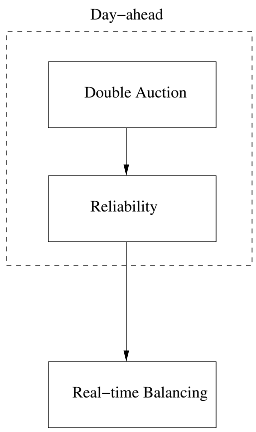


Fig. 1. The market structure of a US ISO. The first two steps occurs day-ahead of the real-time balancing of the system.

In this paper, we primarily deal with the reliability process and the real-time balancing of the network. We do not consider the double auction step for two reasons: i) the double auction deal with purely financial bids, so the ISO cannot directly model the randomness of renewables at this step; ii) most of generators are committed by the double auction, therefore the number of binary variables are small in the reliability process. In fact, we make the assumption that all commitment decisions are made by the double auction, and the reliability process and the real-time balancing only involve the generation levels of the generators.

The joint optimization of the reliability and the real-time balances steps is modelled as the as a two-stage stochastic optimization problem. Throughout the paper, let n denote the number of buses in the network, and m denote the number of transmission lines. We assume the network is connected. Let l_i denote the load at bus i. Let w_i denote the amount of renewable energy available at bus i (w_i could be 0). We assume at the renewable resources and loads are not dispatchable, and the net-demand at each bus is defined as

$$d_i = l_i - w_i.$$

In the first stage (day-ahead), d_i is a random variable. The ISO has some information about d_i at this stage, and we model this knowledge by decomposing d_i as

$$d_i = \hat{d}_i + e_i; \tag{1}$$

where $\hat{d}_i$ can be thought as the predicted net-demand, and e_i is the prediction error. The average forecast error of load is less than 2% of the total load [25]. The average forecast error of wind is significantly higher, with the state of art being about 20% of total wind power [21]. Let σ_e, σ_l, and σ_w denote the standard deviation of prediction error of net-demand, load and renewable respectively.

At the first stage, the vector of forecast net-demands, $\hat{\mathbf{d}} = [\hat{d}_1 \dots \hat{d}_n]^T$ is known. The *joint distribution* of the errors $\mathbf{e} = [e_1 \dots e_n]^T$ are also known at the first stage. The ISO decides generation levels, g_i, at each of the buses. Once g_i's are decided, they are fixed in the second stage. We assume that generation must be positive in the first stage, that is, $g_i \geq 0$. At the second stage (real-time), the actual realization of $\mathbf{d} = [d_1 \dots d_n]^T$ is known. To balance the network, the ISO decides real-time generation levels g_i^R at each bus to balance the network. The g_i^R's can be thought as outputs of fast ramping generators, lost load or excessive energy being disposed. Therefore g_i^R need not be positive.

The cost of generation at bus i is $c_i(g_i)$ for the first stage and $q_i(g_i^R)$ for the second stage. In general, both are convex and increasing functions. In this paper, we assume c_i's are linear and q_i's are piecewise linear[2]. Specifically, let

$$c_i(g_i) = \alpha_i g_i, \; q_i(g_i^R) = \beta (g_i^R)^+,$$

where α_i and β are in the unit of dollars per MW and $(x)^+ = \max(0, x)$. The piecewise structure of q_i means that in real time, excess energy can be disposed for free. Note all buses have the same stage two cost. This is reasonable since the cost in stage two is often thought as value of loss load, and is set to be a uniform number for all buses by the ISOs (see, e.g. [23]). A typical day-ahead price is 50 \$/MW and a typical real-time price (penalty) is 1000 \$/MW.

B. Problem Formulation

Before giving the main problem definition, it is useful to define a generic DC optimal power flow in (2) [26] where

[1] A generator is committed means it will be turned on at the time of delivery of electricity

[2] Note the (piecewise) linear assumption on the costs are not essential to the results in this paper.

the price is $\mathbf{h}$ and the demand is $\mathbf{x}$.

$$J(\mathbf{h}, \mathbf{x}) = \min_{\mathbf{y}} \mathbf{h}^T(\mathbf{y})^+ \quad (2a)$$

$$\text{s.t. } \mathbf{y} - \boldsymbol{\nabla}^T\mathbf{f} - \mathbf{x} = \mathbf{0} \quad (2b)$$

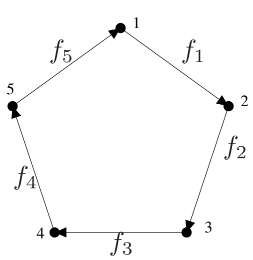
$$\mathbf{K}\mathbf{f} = \mathbf{0} \quad (2c)$$

$$|\mathbf{f}| \leq \mathbf{c}, \quad (2d)$$

where (2b) is the power balance constraint, $\nabla^T \in \mathbb{R}^{n\times m}$ is the mapping from branch flows to bus injections [27], $\mathbf{f}$ is the $m \times 1$ vector of branch flows, (2c) is Kirchoff's voltage law that states a weighted sum of flows in a cycle must be 0, $\mathbf{c} = [c_1 \dots c_m]^T$ is the vector of line capacities, and (2d) is capacity constraint of the flows. The constraint in (2c) states that not all flows in the network can be decided independently. From the theory of fundamental flows in graphs [28], [29], the $n-1$ flows in any spanning tree of a graph can be thought as a set of fundamental flows where all other flows can be derived by considering the Kirchoff constraints in (2c). Then (2b) and (2c) can be combined into one constraint as

$$\mathbf{y} - \mathbf{A}\mathbf{f} - \mathbf{x} = \mathbf{0}, \quad (3)$$

where $\mathbf{A} \in \mathbb{R}^{n\times n-1}$ captures the mapping from flows to bus injections together with (2c), and $\mathbf{f} \in \mathbb{R}^{n-1}$ is now a sect of fundamental flows. $\mathbf{A}$ has full rank and the null space of $\mathbf{A}^T$ is spanned by the all ones vector [29]. An example for a 5-cycle network with equal inductance on each line is shown in Fig. 2. The this network, constraint (2c) becomes all flows must sum to 0 where the orientation of the flows are shown in Fig. 2.



$$\boldsymbol{\nabla}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -2 \end{bmatrix}$$

Fig. 2. A 5-cycle network with equal inductance on each branch. The orientation of each flow is along the arrows.

Given a network, since not all flows in cycle can be independently chosen, for all values of capacity except a set of measure 0, not all flows in a cycle can be at their capacity. Therefore, we say that for a *generic* set of capacities, the congested flow belong to some set of fundamental flows. Similarly, when we say a condition is true for a generic set of demands, we mean for all demands except for a set of measure 0.

Main Problem: The two-stage optimization problem is the stochastic problem in (4):

$$V^*(\hat{d}) = \min_{\mathbf{g}\geq 0}\{\alpha^T\mathbf{g} + \mathbb{E}[J(\beta\mathbf{1}, \mathbf{d}-\mathbf{g})|\hat{\mathbf{d}}]\}, \quad (4)$$

where the dependence on $\hat{\mathbf{d}}$ represents the fact $\mathbf{g}$ is decided based on the first stage forecast and the optimal generation $\mathbf{g}^*$ is called the *Risk Limiting Dispatch* [19].

C. Analytical Solutions

It is possible to write down a set of deterministic equations whose solution give the risk limiting dispatch, $\mathbf{g}^*$. Since the second stage of (4) is a linear problem, the dual gives the Lagrange multipliers that can be interpreted as the sensitivity of the power balance equations. Then set of equations can be found by taking the derivative of (4) [22]. However, the size of the equations grow exponentially with respect to the size of the network. Therefore most existing literature take an simulation approach to solving (4). In Section III-A we show that how large networks can be reduced to small networks with a few buses, thus allowing us to analytically solve the problem.

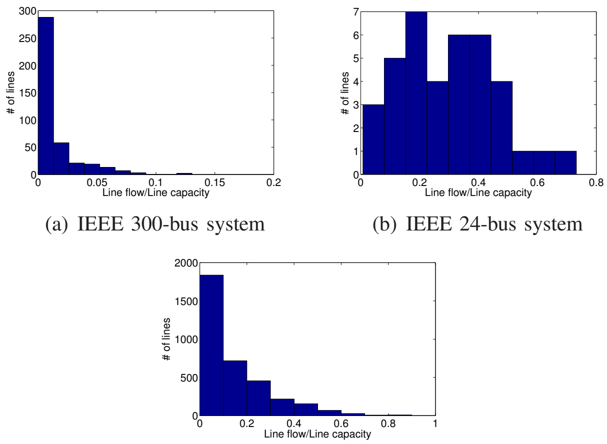
D. Small-σ Assumption

The size of practical electrical networks ranges from tens of buses to tens of thousands of buses. The high dimensionality of the second stage optimization makes problem in (4) is difficult to solve. However, under closer inspection, the dimension of (4) can be reduced significantly. The key observation is that even though the uncertainties in a system with moderate renewable penetration is significant in financial terms, from an operation point of view, the uncertainties do not change the qualitative behaviour of the power network. This is called the *small-σ* assumption.

In a power system with 20% penetration of wind, the total uncertainty in the net-demand is about

$$\sigma_e = \sigma_l + \sigma_w \approx 7\% \quad (5)$$

assuming a load prediction error of 2%, 25% prediction error of wind. Figure 3 shows the histograms of the ratio of the power flow on a line to the capacity of that line for three networks: the IEEE 300-bus system, the IEEE 24-bus system and a Polish 2736-bus system. The power flows on the lines are calculated by an OPF from nominal data for these networks. The histograms provide a measure of how congested the networks are. In Fig. 3(a), all lines are loaded below 0.2 of the capacity, therefore even when 7% of uncertainties are introduced into the system, the flows would be much lower than their limits. In Fig. 3(b), lines are more evenly loaded, and when 7% of uncertainties are introduced,



(a) IEEE 300-bus system (b) IEEE 24-bus system

(c) Polish 2736-bus system

Fig. 3. Histogram of the ratio of the power flow on a line to the capacity. The power flows are calculated by an DC-OPF using data from MatPower [30].

some lines may hit their capacities. Similarly this is the case for Fig. 3(c).

Suppose that the ISO calculates the flows in the system based on the predicted net-demand (nominal demand). The main observation from Fig. 3 is that the qualitative feature of the power system does not change if the actual demand is used to calculate the flows. More specifically, the congestion pattern in a network would not change. For example, if a line is far away from congestion under the predicted net-demand, then it will not become congested in at real-time. This observation allows us to reduce the dimension of the optimization problem in (4) significantly.

III. Solution Method and Main Result

A. Solution Strategy

Based on Section II-D, we propose the following two-step solution strategy [3]:

1) **Nominal Problem** Solve the OPF problem J in (2) with price $\boldsymbol{\alpha}$ and demand $\hat{\mathbf{d}}$. Let $\overline{g}$ denote the optimal solution of $J(\boldsymbol{\alpha}, \hat{\mathbf{d}})$, and let $\overline{f}$ denote the corresponding flows. Without loss of generality, we pick the directions of the flow such that if a flow on line k is at capacity, then $\overline{f}_k = c_k$. If $\overline{g}_i < 0$, it means that the bus is dumping energy in the nominal problem. Label the each bus of the network as follows: bus i is labelled $+$ if $\overline{g}_i > 0$, $-$ if $\overline{g}_i < 0$ and 0 if $\overline{g}_i = 0$.
2) **Perturbed Two-Stage Problem** To find the solution to the original problem in (4), solve the following problem:

$$\min_{\boldsymbol{\Delta}} \alpha'^T \Delta + \mathbb{E}[\tilde{J}(\boldsymbol{\beta}', \mathbf{e})] \tag{6a}$$
$$\text{s.t. } \Delta_i \geq 0 \text{ if } i \text{ is labelled or } 0 \tag{6b}$$

where

$$\tilde{J}(\boldsymbol{\beta}', \mathbf{e}) = \min \boldsymbol{\beta}'^T (\mathbf{y})^+ \tag{7a}$$
$$\text{s.t. } \mathbf{y} - \boldsymbol{\nabla}^T \mathbf{f} - (\mathbf{e} - \boldsymbol{\Delta}) = 0 \tag{7b}$$
$$\mathbf{K}\mathbf{f} = 0 \tag{7c}$$
$$f_k < 0 \text{ if } \overline{f}_k = c_k, \tag{7d}$$

$\alpha'_i = \alpha_i$ ($\beta'_i = \beta$) if $\overline{g}_i \geq 0$ and $\alpha'_i = 0$ ($\beta'_i = 0$) if $\overline{g}_i < 0$. The solution to the original problem in (4) is given by

$$\mathbf{g}^* = (\overline{\mathbf{g}} + \boldsymbol{\Delta})^+.$$

It may seem that (6) is no simpler than the original problem in (4). The key observation is that (6) is a perturbation of (4) only flow constraints at are binding in the nominal problem are included in the second stage of the perturbed problem. The dimension of the perturbed problem can be orders of magnitudes smaller than the dimension of the original problem: e.g., a few lines compared to thousands of lines. Note that the transformation from the original problem is *linear*, and therefore can be computed efficiently. The modified prices $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ captures the intuition that if a bus is dumping energy in the nominal case, then that bus need not to buy reserve. That is, if the bus is labelled $-$, that means it is dumping energy in the nominal case, then in real time it still have excess energy for free. This strategy is given graphically in Fig. 4.

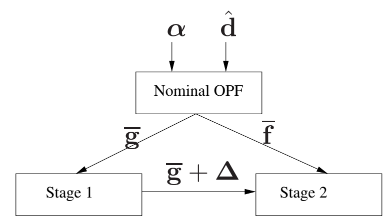


Fig. 4. The solution strategy. The nominal problem is the OPF problem $J(\boldsymbol{\alpha}, \hat{\mathbf{d}})$. Its optimal solution is $\overline{\mathbf{g}}$ and $\overline{\mathbf{f}}$ is the associated flows. The optimal solution the original problem in (4) is $\overline{\mathbf{g}}+\boldsymbol{\Delta}$, and the congestion information in $\overline{\mathbf{f}}$ is used to reduce the dimension of the second stage problem.

B. Main Result

The arguments in Section II-D is formalized as:

Small-σ assumption: Given a power system and the forecast net-demand $\hat{\mathbf{d}}$. We say a power system and the forecast error satisfies the small sigma assumption if for all error realizations $\mathbf{e}$ the following hold

1) If a line is not congested in the solution of the nominal problem, then it will not be congested in real-time.

[3]This strategy was proposed in [22], we repeat it here for completeness

2) If a line is congested in the solution of the nominal problem, then the congestion direction will not reverse in real-time.
3) If a bus is dumping energy in the solution of the nominal problem ($\overline{g}_i < 0$), then it still dumps energy at real-time.

If the errors have unbounded support, then the small-σ assumption should be modified to hold with high probability. However, the probability of violation would be very small and can be ignored in practice [31]. Therefore even when the errors have unbounded support(e.g. Gaussian), we still make the small-σ assumption.

The main result of the paper is stated in Theorem 1.

Theorem 1. *Suppose the small-σ assumption hold. Then the two step procedure in Section III-A is optimal. Furthermore, let K be the number of congested links from the nominal problem. The the perturbed problem in* (6) *reduces to an equivalent problem on at most $K+1$ buses with at most K congested lines. The perturbed network is constructed by Algorithm 1.*

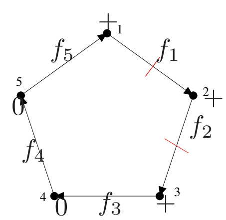
Algorithm 1: Construction of the perturbed network

1) Solve the nominal problem $J(\boldsymbol{\alpha}, \hat{\mathbf{d}})$ to get $\overline{\mathbf{g}}$ and $\overline{\mathbf{f}}$.
2) Renumber the flows such that the K congested flows are numbered 1 to K. Let the matrix $\mathbf{A}$ be defined with respect to this numbering of the flows (3).
3) Let $\mathbf{A}_1$ be the matrix formed from $\mathbf{A}^T$ be keeping the first K rows and $\mathbf{A}_2$ be the matrix formed from $\mathbf{A}^T$ by removing the first K rows. Therefore $\mathbf{A}_1$ has size $K \times n$ and $\mathbf{A}_2$ has size $(n-1-K) \times n$.
4) Consider the equation
$$\mathbf{A}_2 \boldsymbol{\lambda} = \mathbf{0}. \tag{8}$$
From [28], all rows of $\mathbf{A}$ are linear independent, $\mathbf{A}_2$ has rank $n-K-1$ and null space of dimension of $K+1$. Therefore each λ as a linear combination of $K+1$ independent λ's.
5) The $K+1$ independent λ's include all buses with a $+$ labelling. If the number of $+$ labelled buses is less than $K+1$, the pick other buses to complete this set. Renumber the buses such that these independent buses at numbered from 1 to $K+1$. Rearrange the columns $\mathbf{A}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ accordingly.
6) Find $\mathbf{B}$ such that $[\lambda_{K+2}, \ldots, \lambda_n]^T = \mathbf{B}[\lambda_1 \ldots \lambda_{K+1}]^T$. Define $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{A}}$ as
$$\tilde{\mathbf{e}} = \begin{bmatrix}\mathbf{I} & \mathbf{B}^T\end{bmatrix} \mathbf{e} \qquad \tilde{\mathbf{A}} = \mathbf{A}_1 \begin{bmatrix}\mathbf{I} \\ \mathbf{B}\end{bmatrix} \tag{9}$$
where $\mathbf{I}$ is the identity matrix of size $K+1$, $\tilde{\mathbf{e}}$ is a vector of length $K+1$, and $\tilde{\mathbf{A}}$ is of size $K \times K+1$. Since the all one's vector spans the null space of $\mathbf{A}^T$, the rows of $\mathbf{B}$ sum up to 1.
7) The perturbed network has $K+1$ buses, with the flow to bus mapping determined by $\tilde{\mathbf{A}}^T$ and the prediction error at bus i is $\tilde{e}_i$. The topology of the reduced network can be determined by the method in Chapter 12 of [29], but is not needed in solving the perturbed problem.
8) The first and second stage prices are given by
$$\tilde{\boldsymbol{\alpha}} = \begin{bmatrix}\mathbf{I} & \mathbf{B}^T\end{bmatrix} \boldsymbol{\alpha}' \qquad \tilde{\boldsymbol{\beta}} = \begin{bmatrix}\beta_1' & \beta_2' & \ldots & \beta_{K+1}'\end{bmatrix}^T. \tag{10}$$
The perturbed two stage problem becomes
$$\min_{\tilde{\boldsymbol{\Delta}}} \tilde{\boldsymbol{\alpha}}^T \tilde{\boldsymbol{\Delta}} + \mathbb{E}[\tilde{J}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}})].$$
This problem has dimension $K+1$. Let $\tilde{\boldsymbol{\Delta}}$ the risk limiting dispatch for the perturbed problem. The original $\boldsymbol{\Delta}$ is obtained by solving
$$\min_{\boldsymbol{\Delta}} \boldsymbol{\alpha}'^T \boldsymbol{\Delta} \tag{11a}$$
$$\text{s.t. } \begin{bmatrix}\mathbf{I} & \mathbf{B}^T\end{bmatrix} \boldsymbol{\Delta} = \tilde{\boldsymbol{\Delta}} \tag{11b}$$
$$\Delta_i \geq 0 \text{ if } \overline{g}_i = 0. \tag{11c}$$

The proof of Theorem 1 mainly involves showing each step in Algorithm 1 is possible and is somewhat technical. We delay the proof until Sec IV. For the rest of this section, we illustrate the usefulness of Theorem 1 with examples.

C. Example 1

Consider the 5-cycle network in Fig. 2. For some $\hat{\mathbf{d}}$ and α, the sign pattern of the buses after solving the nominal problem is given in Fig. 5. In this case renumbering of buses



$$\boldsymbol{B} = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 \end{bmatrix}$$
$$\tilde{\mathbf{A}} = \begin{bmatrix} 4/3 & -1 & -1/3 \\ 1/3 & 1 & -4/3 \end{bmatrix}$$

Fig. 5. The resulting flows and signs of buses of a 5-cycle network after solving a nominal problem. The congested edges are label by red crosses and the directions are given by arrows (f_1 and f_2 are congested). The matrices are computed according to Algorithm 1.

or the flows are not needed. Following Algorithm 1, we compute $\tilde{\mathbf{A}}$ and $\mathbf{B}$ and they are given in Fig. 5. The last step of Algorithm 1 gives the perturbed network is a triangle, and the direction of flows consistent with $\tilde{\mathbf{A}}$ is given in Fig. 6.

For this perturbed network, the second stage optimization problem becomes
$$\tilde{J}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}}) = \min_{\mathbf{y}} \tilde{\boldsymbol{\beta}}^T \mathbf{y} \tag{12a}$$
$$\text{s.t. } \mathbf{y} - \tilde{\mathbf{A}}^T \mathbf{f} - (\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \geq \mathbf{0} \tag{12b}$$
$$\mathbf{f} \leq \mathbf{0} \tag{12c}$$
$$\mathbf{y} \geq 0, \tag{12d}$$

$$\tilde{e}_1 = e_1 + 1/3e_4 + 2/3e_5$$
$$\tilde{e}_2 = e_2$$
$$\tilde{e}_3 = e_3 + 2/3e_4 + 1/3e_5$$
$$\tilde{\mathbf{A}} = \begin{bmatrix} 4/3 & -1 & -1/3 \\ 1/3 & 1 & -4/3 \end{bmatrix}$$

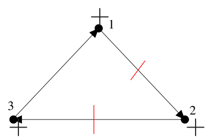


Fig. 6. The perturbed network. The error at each bus is a linear combination of the original errors. The fundamental flows in the network is the flow from bus 1 to bus 2 and bus 2 to bus 3. The congestion is denoted by red lines.

where $\mathbf{f}$ is the two fundamental flows in Fig. 6. By standard duality theory, the dual is

$$\tilde{J}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}}) = \max_{\boldsymbol{\lambda}} \boldsymbol{\lambda}^T(\mathbf{e} - \tilde{\boldsymbol{\Delta}}) \tag{13a}$$
$$\text{s.t. } \mathbf{0} \le \boldsymbol{\lambda} \le \tilde{\boldsymbol{\beta}} \tag{13b}$$
$$\tilde{\mathbf{A}}\boldsymbol{\lambda} \le \mathbf{0}, \tag{13c}$$

where $\boldsymbol{\lambda}$ are the Lagrange multipliers of the energy balance constraint (12b). The overall problem is

$$\min_{\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3} \sum_{i=1}^{3} \tilde{\alpha}_i \tilde{\Delta}_i + \mathbb{E}[J(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}})]. \tag{14}$$

To solve for the first stage control $\tilde{\boldsymbol{\Delta}}$, we can set up a system of equilibrium equations. The equations are found by considering trade-off between procuring more one unit of energy in the first stage and the potential benefit of that unit of energy in the second stage. The cost of procuring one more unit of energy at bus i is α_i and the expected benefit of that unit of energy is given by $\mathbb{E}[\lambda_i^*(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}})]$ [32]. To obtain the optimal $\tilde{\boldsymbol{\Delta}}$, we solve the system of equilibrium equations:

$$\tilde{\alpha}_i = \mathbb{E}[\lambda_i^*(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}})], \ \forall i. \tag{15}$$

Since $\tilde{J}$ is a linear program, the optimal occurs at a vertex of the polyhedron defined by (13b) and (13c). The vertices can be listed [33], and for each $(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}})$, the corresponding optimal vertex can be found be simple comparisons. For the network in Fig. 6, there are 6 possible vertices, and due to space constraints, we do not list them here. By comparison of the vertices,

$$\lambda_1^* = \begin{cases} \beta & \text{if } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1/4 \\ 3/4 & 1 & 0 \end{bmatrix} (\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0 \\ \beta/4 & \text{if } \begin{bmatrix} 1/4 & 0 & 1 \\ 1/4 & 0 & 0 \\ 1/4 & -1 & 0 \\ -3/4 & -1 & 0 \end{bmatrix} (\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \beta & \text{if } \mathbf{R}_1(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0 \\ \beta/4 & \text{if } \mathbf{R}_2(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore the first equilibrium equation is given by

$$\tilde{\alpha}_1 = \beta \Pr(\mathbf{R}_1(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0) + \beta/4 \Pr(\mathbf{R}_2(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \ge 0)$$

and equations for $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ can be generated in the same way. Solving this set of equations gives $\tilde{\boldsymbol{\Delta}}^*$. And the optimal solution to the original problem is $(\overline{\mathbf{g}}+\boldsymbol{\Delta})^+$ where Δ is found by solving (11).

D. Example 2

Consider the IEEE 9-bus benchmark network in Fig. 7 [34]. For some predicted demand and cost $\boldsymbol{\alpha}$, the congestion

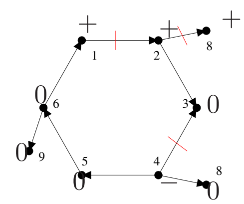


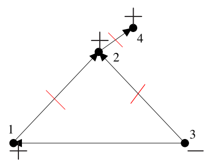
Fig. 7. The IEEE 9-bus benchmark network with a particular sign and congestion pattern. The direction of the flows are given by the arrows, and the congested lines are denoted by red marks.

pattern and the labelling of the buses are given in Fig. 7 as well. The perturbed network is given in Fig. 8.

The equilibrium conditions can be written down using the same procedure in Section III-C.

IV. Proof

The proof of Theorem 1 mainly follows duality of the DC-OPF problem. The statement that the two step strategy is optimal is not surprising, since the small-σ assumption guarantees that the generation pattern (the sign of g_i at bus i) and the congestion pattern (which lines are congested) are the same at day-ahead and real-time. The interesting part is to show that the size of the perturbed problem can be reduced



$$\begin{aligned}\tilde{e}_1 &= e_1 + 1/3(e_3+e_5) + 2/3e_6 \\ &\quad + e_9 \\ \tilde{e}_2 &= e_2 + e_3 \\ \tilde{e}_4 &= e_7 \\ \tilde{\mathbf{A}} &= \begin{bmatrix} 4/3 & -1 & -1/3 & 0 \\ 2/3 & 1 & -5/3 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\end{aligned}$$

Fig. 8. The perturbed network. The error at each bus is a linear combination of the original errors. The error at bus 3 does not matter since it is labelled $(-)$ ($\tilde{\beta}=0$). The fundamental flows in the network is the flow from bus 1 to bus 2, bus 3 to bus 2 and bus 2 to bus 4. The congestion is denoted by red lines.

to a problem of size $K+1$. The rest of the proof shows that each step of Algorithm 1 is valid.

Let $\overline{\mathbf{g}}$ be the solution of $J(\boldsymbol{\alpha}, \hat{\mathbf{d}})$. Let the lines be numbered as in step 2 of Algorithm 1. Lemma 2 states that the total number of buses with $+$ label can be no greater than $K+1$.

Lemma 2. *Let N_+ be the number of buses labelled with $+$. Then $N_+ \le K+1$. Furthermore, solution of $\mathbf{A}_2\boldsymbol{\lambda} = \mathbf{0}$ can be expressed as a linear combination of λ_i where i is labelled with $+$, together $K+1-|N_+|$ other λ's.*

Suppose Lemma 2 is true and renumber the buses as in step 5 of Algorithm 1. For a generic set of capacities, all congested flows are fundamental. Since only the first K lines are congested, $J(\boldsymbol{\alpha}, \hat{\mathbf{d}})$ can be equivalently written as

$$J(\boldsymbol{\alpha}, \hat{\mathbf{d}}) = \min_{\overline{\mathbf{g}}} \alpha^T (\overline{\mathbf{g}})^+ \tag{16a}$$
$$\text{s.t. } \overline{\mathbf{g}} - \mathbf{A}\overline{\mathbf{f}} - \hat{\mathbf{d}} = \mathbf{0} \tag{16b}$$
$$\overline{f}_k = c_k,\ k = 1, \ldots, K. \tag{16c}$$

The dual is

$$\max \overline{\boldsymbol{\lambda}}^T \hat{\mathbf{d}} - \sum_{k=1}^{K} \overline{\mu}_k c_k \tag{17a}$$
$$\text{s.t. } \mathbf{0} \le \overline{\boldsymbol{\lambda}} \le \boldsymbol{\alpha} \tag{17b}$$
$$\mathbf{A}^T\overline{\boldsymbol{\lambda}} + \begin{bmatrix}\overline{\mu}_1 & \ldots & \overline{\mu}_K & 0 & \ldots & 0\end{bmatrix}^T = \mathbf{0} \tag{17c}$$

where $\overline{\boldsymbol{\lambda}}$ are the Lagrange multipliers to the energy balance constraint (16b) and $\overline{\boldsymbol{\mu}}$ are the Lagrange multipliers to the flow constraints (16c). Part of the vector constraint in (17c) can be written as $\mathbf{A}_2\overline{\boldsymbol{\lambda}} = \mathbf{0}$ where $\mathbf{A}_2$ is defined in Algorithm 1. By the choice of $\lambda_1, \ldots, \lambda_{K+1}$, $\overline{\lambda}_{K+1}, \ldots, \overline{\lambda}_n$ can be written as a linear combination of $\overline{\lambda}_1$ to $\overline{\lambda}_{K+1}$. This fact is used later for the dual of the second stage problem as well.

Now write the solution of the original problem (4) as $\overline{\mathbf{g}} + \boldsymbol{\Delta}$. The objective function becomes

$$\alpha^T(\overline{\mathbf{g}} + \boldsymbol{\Delta})^+.$$

And denote the flows by $\overline{\mathbf{f}} + \mathbf{f}$. Using the small-$\sigma$ assumption, if $\overline{g}_i < 0$, then Δ_i is small enough such that $\overline{g}_i + \Delta_i < 0$ always and the price of Δ_i can be thought as 0; if $\overline{g}_i > 0$, then $\overline{g}_i + \Delta_i$ always and there are no constraint on Δ_i. The power balance constraints in the second stage becomes

$$\mathbf{y} + \overline{\mathbf{g}} + \boldsymbol{\Delta} - \mathbf{A}(\overline{\mathbf{f}} + \mathbf{f}) - (\hat{\mathbf{d}} + \mathbf{e}) = \mathbf{0},$$

where $\mathbf{y}$ is the energy purchased at the second stage. By definition of $\overline{\mathbf{g}}$ and $\overline{\mathbf{f}}$, this constraint becomes

$$\mathbf{y} + \boldsymbol{\Delta} - \mathbf{A}\mathbf{f} - \mathbf{e} = \mathbf{0}.$$

The flow limits are

$$\overline{f}_k + f_k \le c_k,\ k = 1, \ldots, K$$

and by the small-σ assumption and $\overline{f}_k = c_k$, they become

$$f_k \le 0,\ k = 1, \ldots, K.$$

Therefore under the small-σ assumption, the two stage optimization problem becomes

$$\min_{\boldsymbol{\Delta}} \boldsymbol{\alpha'}^T \Delta + \mathbb{E}[\min_{\mathbf{y}} \boldsymbol{\beta'}(\mathbf{y})^+\}] \tag{18}$$
$$\text{s.t. } \mathbf{y} + \boldsymbol{\Delta} - \mathbf{A}\mathbf{f} - \mathbf{e} = \mathbf{0}$$
$$f_k \le 0\ k = 1, \ldots, K,$$

where the constraint $\Delta_i = 0$ if $\overline{g}_i = 0$. Recall the $\alpha'_i = 0$ ($\beta'_i = 0$) if bus i is labelled $-$ and α_i (β) otherwise. This means that under the small-σ assumption, we can assume that a bus dumping energy in the nominal problem as a source of energy with cost 0.

The dual of the second stage problem in (18) is

$$\max \boldsymbol{\lambda}^T(\mathbf{e} - \boldsymbol{\Delta}) \tag{19a}$$
$$\text{s.t. } \mathbf{0} \le \boldsymbol{\lambda} \le \boldsymbol{\beta}' \tag{19b}$$
$$\mathbf{A}^T\boldsymbol{\lambda} + \begin{bmatrix}\mu_1 & \ldots & \mu_K & 0 & \ldots & 0\end{bmatrix}^T = \mathbf{0} \tag{19c}$$
$$\mu_k \ge 0,\ k = 1, \ldots, K. \tag{19d}$$

This is exactly the form of (17). Therefore we can express λ_{K+2}, λ_n as a function of $\lambda_1, \ldots, \lambda_{K+1}$ through the $\mathbf{B}$ matrix defined in step 4 of Algorithm 1. Let $\tilde{\boldsymbol{\lambda}} = [\lambda_1 \ \ldots \ \lambda_{K+1}]^T$ and write (19) in terms of $\tilde{\boldsymbol{\lambda}}$ gives

$$\max \tilde{\boldsymbol{\lambda}}^T(\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) \tag{20a}$$
$$\text{s.t. } \mathbf{0} \le \tilde{\boldsymbol{\lambda}} \le \tilde{\boldsymbol{\beta}} \tag{20b}$$
$$\tilde{\mathbf{A}}\tilde{\boldsymbol{\lambda}} \le \mathbf{0}, \tag{20c}$$

where the definitions for $\tilde{\mathbf{A}}$, $\tilde{\mathbf{e}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\Delta}}$ are given in Algorithm 1. Since each column of B sum up to 1, and β'_i is either 0 or β, it suffices to consider the first $K+1$ as in (20b). Taking the dual of (20), we get back the primal problem

$$\tilde{J}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}}) = \min_{\mathbf{y}} \tilde{\boldsymbol{\alpha}}^T(\mathbf{y})^+ \tag{21a}$$
$$\text{s.t. } \mathbf{y} - \tilde{A}\tilde{\mathbf{f}} - (\tilde{\mathbf{e}} - \tilde{\boldsymbol{\Delta}}) = \mathbf{0} \tag{21b}$$
$$\tilde{\mathbf{f}} \le 0, \tag{21c}$$

but this is exactly the primal of a second stage problem with a network of size $K+1$, with flows governed by the matrix $\tilde{\mathbf{A}}$. Therefore the two stage problem is

$$\min_{\tilde{\boldsymbol{\Delta}}} \tilde{\boldsymbol{\alpha}}^T \tilde{\boldsymbol{\Delta}} + \mathbb{E}[\tilde{J}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\Delta}}], \tag{22}$$

and this is precisely the risk limiting dispatch problem for a network of size $K+1$.

A. Proof of Lemma 2

Since $\mathbf{A}_2$ has full rank, by the rank-nullity theorem, the null space of $\mathbf{A}_2$ has rank $K+1$. Therefore all of λ_i can be expressed as linear combination of some $K+1$ λ's. By complementary slackness [35], the buses with label $+$ ($\overline{g}_i > 0$) has $\lambda_i = \alpha_i$. Intuitively, this means buses that are generating positive power are generating at their marginal costs. Since these are the marginal generators, they cannot dependent on other λ_i's [32]. Therefore any set of independent λ_i's must include the buses labelled $+$.

V. Conclusions

The paper derived a procedure for Risk Limiting Dispatch in arbitrarily congested networks. The procedure is very efficient to compute, and can be easily adapted to existing dispatch software. There are several extensions for future work: extending the dispatch procedure in the presence of storage, generation ramping constraints and deferrable loads; considering multiple dispatch stages and considering a full AC power flow model for distribution networks.

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