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Mean Square Formation and Containment Control of Multi-Agent Systems under Noisy Measurements

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Abstract— **A multi-agent system with multiple leaders under noisy measurements is considered. A two-level control protocol is proposed to make the leaders achieve formation and meanwhile to make the followers be contained in the formation asymptotically in the mean square sense. Decaying gain functions are introduced to attenuate the noises. The communication graph is dynamically switching. Some connectivity conditions of the switching communication graphs are derived to guarantee the mean square formation and containment.**

I. INTRODUCTION

In the last decade, distributed cooperative control received a lot of attentions from the researchers, e.g., [13], [12], [7], [8]. There are many applications of distributed control, for example distributed formation control [4], [11], [1], distributed optimization [17], [20], distributed coverage control [3].

Distributed formation control is to control the multiple intelligent agents such as UAVs to achieve a certain formation in a distributed way. In [5], it is proved that under the proposed protocol, the formation control problem is equivalent to a robust stabilization problem. Moreover, a static feedback gain is given by using Nyquist plot. [22] investigates the finite time formation problem which means to achieve the formation in a finite time. The distance-based formation control is considered in [4] and [1]. Due to the implementation problem of the common used consensus and formation control protocol, a delayed neighbors' input information based formation control protocol is developed in [13]. The problem for discrete-time systems is also studied in [11].

The containment tracking problem where a collection of agents are to be driven to a certain compact set [9] has recently attracted much interest. The containment tracking can be considered as a leader-following problem with multiple leaders [15]. All the follower agents are to be controlled to track the convex hull of the leaders [6]. In [2], the followers can converge to the convex hull formed by the leaders for both stationary and dynamic leaders cases. For dynamically switching topologies case, the containment control protocol is studied in [18] when the communication subgraphs induced by the leader set and the follower set are both undirected. For switching directed graph case, necessary and sufficient conditions on the graph are provided in [19] to guarantee set input-to-state stability and set integral input-to-state stability based on a nonlinear protocol. Multi-agent systems with general linear dynamics are considered in [10]. In many practical systems, measurement noises need to be considered in protocol design since it can affect the stability of the system. In [21], a containment control problem is considered in the presence of measurement noises under a fixed topology. In [16], the same problem under the dynamically switching and randomly switching topologies is studied and a weaker condition to the control gain function is provided.

In this paper, we will consider both distributed formation control for the leaders and containment control for the followers with noisy information. In order to attenuate the noises, we will show that the control gain function for the followers needs to be non-summable and convergent to zero. The same condition is also required for the control gain function of the leaders when there is a reference signal to the leaders. The control gain function for the leaders needs to satisfy a further condition, i.e. square summable, when there is no reference information to the leaders. This is because the centroid of the formation the leaders are achieving can drift to infinity if the square summation of the control gain for the leaders is unbounded. The communication topology to be considered in this paper is dynamically switching. A uniformly joint connectivity condition is assumed to guarantee the mean square formation and containment.

Some notations are introduced as follows. $\mathbb{R}$ (or $\mathbb{R}^+$) denotes the set of real numbers (or positive real numbers). $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote n dimensional real vector and $n \times m$ real matrix, respectively. $\mathbb{Z}^+$ denotes the set of all positive integers and $\bar{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{0\}$. Given a matrix T, we use $T^{i,j}$ to denote its (i,j)-th element. $\lambda_i[T]$ is the i-th eigenvalue of matrix T. For matrix T, T' means its transpose. $\mathbf{1}_n$ stands for a column vector with dimension n and every element of 1. $\mathbf{0}$ is a zero vector or matrix with appropriate dimension. $\|\cdot\|_2$ denotes the 2-norm. $\delta_{i,j}$ is the Kronecker delta function, i.e. $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. $\Pi^M_{i,j}$ is the transition matrix generated by $M(k)$ which is equal to $M(i)M(i-1)\cdots M(j)$, if $i \geq j$ and I otherwise. We denote $E(X)$ the mathematical expectation. A vector is called stochastic vector if it is nonnegative and the sum is equal to 1. A matrix is called row stochastic matrix if each row of it is a stochastic vector.

Let $Z = \{z_1, \cdots, z_k\}$ be a finite set in $\mathbb{R}$. The convex hull of Z, denoted by $co(Z)$, is defined as $co(Z) = \{\sum_{i=1}^k \lambda_i z_i \mid \sum_{i=1}^k \lambda_i = 1,\ \lambda_i \geq 0\}$. $dist(x, \mathcal{S})$ means the distance from

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$x \in \mathbb{R}^n$ to the set $\mathcal{S} \subseteq \mathbb{R}^n$ in the sense of 2-norm, i.e.

$$dist(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} \|x - y\|_2.$$

II. PRELIMINARIES IN GRAPH THEORY

We shall use graph to model the communication protocol of the multi-agent systems. In the following we shall introduce some basics of graph theory.

A directed graph, is denoted by $\mathcal{G}$, with node set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$. For simplicity, we define $\mathcal{V} = \{1, 2, \ldots, n\}$, where n is the number of nodes. For any pair $(i, j) \in \mathcal{E}(\mathcal{G})$, i is called parent node whose information is sent to agent j. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \mid j \in \mathcal{V},\ (j, i) \in \mathcal{E}(\mathcal{G})\}$. We denote by $a_{i,j} \geq 0$ the weighting on the edge (j, i). When $j \notin \mathcal{N}_i$, which means there is no information flow from node j to node i, it has $a_{i,j} = 0$. $a_{i,j} \neq 0$ indicates that there exists an edge from node j to node i, i.e. $j \in \mathcal{N}_i$. There is a path from node i to node j if there exists a sequence $l_1, \ldots, l_p \in \mathcal{V}$ satisfying $(l_1, l_2), \ldots, (l_{p-1}, l_p) \in \mathcal{E}(\mathcal{G})$ where $l_1 = i$ and $l_p = j$. Given a graph $\mathcal{G}$, it contains a spanning tree if there exists one node i such that for any other node j, there is a path from i to j. Moreover, if $\mathcal{N}_i$ is empty, node i is called leader and the corresponding graph is a leader-following graph. A graph is balanced if $\sum_{j \in \mathcal{V} \setminus \{i\}} a_{i,j} = \sum_{j \in \mathcal{V} \setminus \{i\}} a_{j,i}$ for all $i \in \mathcal{V}$. Given a graph $\mathcal{G}$ with $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ and a node subset $\mathcal{V}_s \subset \mathcal{V}(\mathcal{G})$, a subgraph induced by $\mathcal{V}_s$ is a graph with node set $\mathcal{V}_s$ and edge set $\mathcal{E}(\mathcal{G}) \cap \mathcal{V}_s \times \mathcal{V}_s$.

When we consider time-varying graphs, the notion of union graph needs to be introduced. Given a sequence of graphs, the union graph is a graph whose node and edge sets are respectively the union of the node and edge sets of these graphs. For example, we denote $\mathcal{G}(k, h)$ the union graph of $\{\mathcal{G}(i), k \leq i < k + h\}$, then we have

$$\mathcal{V}(\mathcal{G}(k, h)) = \bigcup_{i=k}^{k+h-1} \mathcal{V}(\mathcal{G}(i)),\ \mathcal{E}(\mathcal{G}(k, h)) = \bigcup_{i=k}^{k+h-1} \mathcal{E}(\mathcal{G}(i)).$$

We divide the node set $\mathcal{V}(\mathcal{G})$ of a graph $\mathcal{G}$ into two groups $\mathcal{V}_l$, $\mathcal{V}_f$ which satisfy $\mathcal{V}_l \cup \mathcal{V}_f = \mathcal{V}$, $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$. If the divided node sets further satisfy $\forall i \in \mathcal{V}_l$, $\mathcal{N}_i \cap \mathcal{V}_f = \emptyset$ and $\forall j \in \mathcal{V}_f$, $\exists i \in \mathcal{V}_l$ such that there is a path from i to j, then we say the graph contains a united spanning tree, where $\mathcal{V}_l$ and $\mathcal{V}_f$ are called leader set and follower set, respectively. $\{\mathcal{G}(i), k \leq i < k+h\}$ jointly contains a united spanning tree if the union graph $\mathcal{G}(k, h)$ contains a united spanning tree. $\{\mathcal{G}(i), i \in \bar{\mathbb{Z}}^+\}$ uniformly jointly contains a united spanning tree if there exists $T \in \mathbb{Z}^+$ such that $\forall k \in \bar{\mathbb{Z}}^+$, $\{\mathcal{G}(i), k \leq i < k + T\}$ jointly contains a united spanning tree.

III. PROBLEM STATEMENT

In this section, we shall introduce the two-level control problem. We want to control the multiple leaders to form a formation in a distributed way and the followers are controlled to be driven to the convex hull generated by the leaders asymptotically. The communication graph of all the agents is described by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$. The system to be considered consists of n autonomous agents and $\mathcal{V}(\mathcal{G}) = \{1, \ldots, n\}$. Each agent is assumed to have the following dynamics

$$x_i(k + 1) = x_i(k) + u_i(k),\ i \in \mathcal{V}, \tag{1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and input of agent i, respectively. Without loss of generality, we assume that all the followers are labeled by $1, \cdots, n_f$ and $\mathcal{V}_f = \{1, \cdots, n_f\}$. In this paper we consider the measurements with additive noises. Hence, we denote the received state value of j by i as

$$y_{i,j}(k) = x_j(k) + w_{i,j}(k),\ j \in \mathcal{N}_i,\ i \in \mathcal{V}. \tag{2}$$

We have the following assumptions:

A1. The noises $\{w_{i,j}(k), k \in \bar{\mathbb{Z}}^+, i \in \mathcal{V}, j \in \mathcal{N}_i\}$ are white noises with zero means and bounded covariances.

A2. The subgraphs of $\mathcal{G}$ induced by $\mathcal{V}_l$ and $\mathcal{V}_f$ are balanced at each time instant.

The objective is to design the input of the leaders $u_i(k)$, $i \in \mathcal{V}_l$ such that the leaders achieve a given formation in a stochastic sense. Meanwhile, it is needed to control the followers such that the followers asymptotically converge into the convex hull formed by the leaders in a stochastic sense. The definitions of distributed formation and containment control in mean square sense are given below:

Definition 1. (Mean square formation): Given formation information h_i, $i \in \mathcal{V}_l$, The agents achieve mean square formation if $\forall i, j \in \mathcal{V}_l$,

$$\lim_{k \to \infty} E\left((x_i(k) - h_i) - (x_j(k) - h_j)\right)^2 = 0. \tag{3}$$

Definition 2. (Mean square containment): The agents are said to achieve mean square containment if $\forall i \in \mathcal{V}_f$,

$$\lim_{k \to \infty} E dist^2\left(x_i(k), co\{\mathcal{V}_l\}\right) = 0.$$

In Definition 2 we slightly abuse the notation by denoting $co\{x_i,\ i \in \mathcal{V}_l\}$ as $co\{\mathcal{V}_l\}$.

IV. TWO-LEVEL DISTRIBUTED CONTROL PROTOCOL DESIGN AND CONVERGENCE ANALYSIS

The protocol is called two-level distributed control protocol because we want to control the agents in two different levels, i.e. leader level and follower level. The control protocols for the leaders and followers are given as follows:

$$u_i(k) = c_1(k) \sum_{j \in \mathcal{V}} a_{i,j}(k)(y_{i,j}(k) - x_i(k)),\ i \in \mathcal{V}_f, \tag{4}$$

$$u_i(k) = c_2(k) \sum_{j \in \mathcal{V}_l} b_{i,j}(k)(y_{i,j}(k) - h_j - x_i(k) + h_i),\quad i \in \mathcal{V}_l, \tag{5}$$

where $c_1 > 0$ and $c_2 > 0$ are control gains to be designed; $a_{i,j}, b_{i,j} \in \mathcal{C} \cup \{0\}$ and $\mathcal{C}$ is a compact set on $\mathbb{R}^+$. Before providing the result, the following assumption is introduced:

A3. $$0 < \inf_{k\in\bar{\mathbb{Z}}^+} \frac{c_i(k)}{c_i(k+1)} \le \sup_{k\in\bar{\mathbb{Z}}^+} \frac{c_i(k)}{c_i(k+1)} < +\infty,\ i=1,2. \quad (6)$$

Now we are going to show under what conditions the mean square formation and containment control are achievable.

Theorem 4.1: Consider the multi-agent system (1)-(2) with interaction protocol (4) and (5). Under Assumption A1-A3, mean square formation and containment are achieved by the leaders and followers respectively if the communication graph contains a uniformly joint united spanning tree, the subgraph induced by $\mathcal{V}_l$ contains a uniform joint spanning tree and the control gain functions satisfy the persistence condition:

$$\sum_{k=0}^{\infty} c_1(k) = +\infty,\ \lim_{k\to\infty} c_1(k) = 0, \quad (7)$$

$$\sum_{k=0}^{\infty} c_2(k) = +\infty,\ \sum_{k=0}^{\infty} c_2^2(k) < +\infty. \quad (8)$$

Proof. Define $X_f = [x_1\ x_2\ \cdots\ x_{n_f}]'$ and $X_l = [x_{n_f+1}\ \cdots\ x_n]'$. By substituting (4)-(5) into (1)-(2), we rewrite the closed-loop system as

$$\begin{aligned} X_f(k+1) &= \big(I - c_1(k)A(k)\big)X_f(k) - c_1(k)B(k)X_l(k) \\ &\quad + c_1(k)W_f(k), \quad (9) \\ \bar{X}_l(k+1) &= \big(I - c_2(k)L_l(k)\big)\bar{X}_l(k) + c_2(k)W_l(k), \quad (10) \end{aligned}$$

where $\bar{X}_l(k) = X_l(k) - H$ and

$$\begin{aligned} W_f(k) &= [w_{f,1}\ \cdots\ w_{f,n_f}]',\ w_{f,i} = \textstyle\sum_{j=1}^n a_{i,j}(k)w_{i,j}(k), \\ W_l(k) &= [w_{l,1}\ \cdots\ w_{l,n_l}]',\ w_{l,i} = \textstyle\sum_{j=1}^n b_{i,j}(k)w_{i,j}(k), \\ H &= [h_1\ h_2\ \cdots\ h_{n_l}]'. \end{aligned}$$

According to A1, $\{W_f(k), W_l(k), k \in \bar{\mathbb{Z}}^+\}$ are white noises with uniformly bounded covariances.

$A(k) \in \mathbb{R}^{n_f\times n_f}$, $B(k) \in \mathbb{R}^{n_f\times n_l}$ and $L_l(k) \in \mathbb{R}^{n_l\times n_l}$ are defined as follows

$$A^{i,j}(k) = \begin{cases} \sum_{s=1}^{n} a_{i,s}(k), i = j \\ -a_{i,j}(k), \quad i \ne j \end{cases},\ B^{i,j}(k) = -a_{i,n_f+j}(k),$$

$$L_l^{i,j}(k) = \begin{cases} \sum_{s=1}^{n_l} b_{i,s}(k), i = j, \\ -b_{i,j}(k), \quad i \ne j. \end{cases}$$

It is clear that

$$A(k)\mathbf{1}_{n_f} + B(k)\mathbf{1}_{n_l} = \mathbf{0}. \quad (11)$$

Next, we will prove that $X_l(k)$ converges to H in mean square sense. Since the subgraph induced by $\mathcal{V}_l$ is balanced, there exists an orthogonal matrix $P = [\frac{1}{n_l}\mathbf{1}_{n_l}\ \psi]$ with $\psi \in \mathbb{R}^{n_l\times(n_l-1)}$ orthogonal to $\mathbf{1}_{n_l}$ such that $\hat{L}_l(k) = Pdiag\{0, \hat{L}_{l,1}(k)\}P'$ where $\hat{L}_l(k) = L_l(k) + L_l'(k)$. Moreover, since the subgraph contains a uniformly joint spanning tree, there exists $h > 0$ such that $\forall k \ge 0$ the union of the subgraphs over $[k, k+h)$ contains a spanning tree and therefore $\sum_{i=k}^{k+h-1} \hat{L}_{l,1}(i) > 0$. So we can define

$$\lambda_l \triangleq \inf_{k\ge 0} \min_j \mathrm{Re}\left\{\lambda_j\Big[\sum_{i=kh}^{(k+1)h-1} \hat{L}_{l,1}(i)\Big]\right\} > 0.$$

Define $\hat{X}_l(k) = P'\bar{X}_l(k) \triangleq [\hat{X}_{l,1}(k)\ \ \hat{X}_{l,2}'(k)]'$ with $\hat{X}_{l,1}(k) \in \mathbb{R}$ and $\hat{X}_{l,2}(k) \in \mathbb{R}^{n_l-1}$. Then we have

$$\hat{X}_l(k+1) = (I - c_2(k)P^{-1}L_l(k)P)\hat{X}_l(k) + c_2(k)P'W_l(k), \quad (12)$$

and

$$\begin{aligned} \hat{X}_{l,2}(k+1) &= (I - c_2(k)L_{l,1}(k))\hat{X}_{l,2}(k) \\ &\quad + c_2(k)\psi' W_l(k), \quad (13) \end{aligned}$$

where $L_{l,1}$ is derived from $P'L_lP$ by deleting the first row and first column. It is clear that $\hat{L}_{l,1} = L_{l,1} + L_{l,1}'$.

Since $c_2(k)$ satisfies (6), there exist constants $C_{l,h}, C_{u,h} \in \mathbb{R}^+$ such that $C_{l,h}c_2(kh) \le c_2(kh+i) \le C_{u,h}c_2(kh)$, $i = 0, \ldots, h-1$. Denote $\bar{C} = \sup_{k\ge0} c_2(k)$. In view of (8) it has that $0 < \bar{C} < +\infty$. Then we have

$$\begin{aligned} \left\|\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\right\|_2^2 &= \left\|\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\Big(\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\Big)'\right\|_2 \\ &\le \left\|I - \sum_{i=kh}^{(k+1)h-1} c(i)\hat{L}_{l,1}(i)\right\|_2 + \|M(k,h)\|_2, \quad (14) \end{aligned}$$

where

$$\begin{aligned} M(k,h) &\triangleq \Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\Big(\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\Big)' - I \\ &\quad + \sum_{i=kh}^{(k+1)h-1} c(i)\hat{L}_{l,1}(i). \end{aligned}$$

By tedious calculation, it can be found that

$$\|M(k,h)\|_2 \le m(h)c^2(kh), \quad (15)$$

where

$$\begin{aligned} m(h) = \frac{C_{u,h}^2}{\bar{C}^2}\Big[&(1 - \bar{C}\sup_{k\ge0}\|L_{l,1}(k)\|_2)^{2h} - 1 \\ &+ 2h\bar{C}\sup_{k\ge0}\|L_{l,1}(k)\|_2\Big] < +\infty. \quad (16) \end{aligned}$$

Define

$$\lambda_u \triangleq \sup_{k\ge0} \max_j \mathrm{Re}\left\{\lambda_j\Big[\sum_{i=kh}^{(k+1)h-1} \hat{L}_{l,1}(i)\Big]\right\} < +\infty.$$

There must exist a time instant k_1 such that $\forall k \ge k_1$, $C_{l,h}c(kh)\lambda_u < 1$. Then we have $\forall k \ge k_1$

$$\left\|I - \sum_{i=kh}^{(k+1)h-1} c(i)\hat{L}_{l,1}(i)\right\|_2 \le 1 - C_{l,h}c(kh)\lambda_l. \quad (17)$$

There exists a time instant k_2 such that $\forall k \geq k_2$, $\sup_{k\geq k_2} c(kh) < C_{l,h}\lambda_1/m(h)$. Then by considering (14)-(17), for any $k \geq k_3 \triangleq \max\{k_1, k_2\}$, we have

$$\left\|\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\right\|_2^2 \leq 1-\varepsilon_1 c_2(kh) < 1, \tag{18}$$

where $\varepsilon_1 = C_{l,h}\lambda_1 - m(T)\sup_{k\geq k_2} c_2(kh) > 0$.

By defining $\phi(k,h) = \Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}$, it can be proved that $\forall k \geq k_3$,

$$\|\Pi^{\phi(\cdot,h)}_{k,k_3}\|_2^2 \leq e^{-\varepsilon_1\sum_{i=k_3}^{k} c_2(kh)}.$$

By considering (7) and (6), we have

$$\sum_{k=0}^{\infty} c(kh) \geq \frac{1}{C_{u,h}h}\sum_{k=0}^{\infty} c(k) = +\infty,$$

which implies that

$$\lim_{k\to\infty}\|\Pi^{\phi(\cdot,h)}_{k,0}\|_2^2 = \lim_{k\to\infty}\|\Pi^{I-c_2L_{l,1}}_{k,0}\|_2^2 = 0. \tag{19}$$

According to (13), we have

$$\begin{aligned}\hat{X}_{l,2}((k+1)h) &= \Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\hat{X}_{l,2}(kh)\\ &\quad + \sum_{j=kh}^{(k+1)h-1}\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,j+1}c_2(j)\psi' W_l(j).\end{aligned}$$

Since h is finite, $\Pi^{I-c_2L_{l,1}}_{(i+1)h-1,j+1}$ is uniformly bounded with respect to i and there is a positive matrix Q such that

$$\sup_{i\geq 0}\max_{ih\leq j<(i+1)h} cov\big(\Pi^{I-c_2L_{l,1}}_{(i+1)h-1,j+1}\psi' W_l(j)\big) \leq Q.$$

In view of the independence of $W_l(k)$, we have

$$\begin{aligned}E\|\hat{X}_{l,2}((k+1)h)\|_2^2 &\leq \|\Pi^{I-c_2L_{l,1}}_{(k+1)h-1,kh}\|_2^2 E\|\hat{X}_{l,2}(kh)\|_2^2\\ &\quad + \text{trace}(Q)hC_{u,h}^2 c_2^2(kh)\\ &\leq (1-\varepsilon_1 c_2(kh))E\|\hat{X}_{l,2}(kh)\|_2^2\\ &\quad + \text{trace}(Q)hC_{u,h}^2 c_2^2(kh).\end{aligned} \tag{20}$$

According to Lemma 6 in [14], the above inequality implies that

$$\lim_{k\to\infty} E\|\hat{X}_{l,2}(kh)\|_2^2 \leq \lim_{k\to\infty}\frac{\text{trace}(Q)hC_{u,h}^2 c_2^2(kh)}{\varepsilon_1 c_2(kh)} = 0. \tag{21}$$

Letting $m_1(h) = \sup_{k\geq 0,\ 0\leq i<h}\|I - c_2(k)L_{l,1}(k)\|_2^{2i}$, according to (13) it has

$$\begin{aligned}\lim_{k\to\infty} E\|\hat{X}_{l,2}(kh+i)\|_2^2 &\leq m_1(h)\lim_{k\to\infty} E\|\hat{X}_{l,2}(kh+i)\|_2^2\\ &\quad + \lim_{k\to\infty}\text{trace}(Q)hC_{u,h}^2 c_2^2(kh) = 0,\end{aligned}$$

which together with (21) implies $\lim_{k\to\infty} E\|\hat{X}_{l,2}(k)\|_2^2 = 0$.

On the other hand, since the subgraph induced by $\mathcal{V}_l$ is balanced, it is easy to check that

$$\begin{aligned}\hat{X}_{l,1}(k+1) &= \hat{X}_{l,1}(k) + c_2(k)\frac{1}{n_l}\mathbf{1}'_{n_l}W_l(k)\\ &= \hat{X}_{l,1}(0) + \sum_{i=0}^{k} c_2(i)\frac{1}{n_l}\mathbf{1}'_{n_l}W_l(i).\end{aligned} \tag{22}$$

Thanks to $\sum_{k=0}^{\infty} c_2^2(k) < \infty$, one has

$$\begin{aligned}&\lim_{k\to\infty} E\|\hat{X}_{l,1}(k) - \hat{X}_{l,1}(0)\|_2^2\\ &= \lim_{k\to\infty} E\left\|\sum_{i=0}^{k} c_2(i)\frac{1}{n_l}\mathbf{1}'_{n_l}W_l(i)\right\|_2^2 < +\infty.\end{aligned} \tag{23}$$

By denoting $\pi = e_1P^{-1}(\bar{X}_l(0) - H)$ and $\lim_{k\to\infty}\sum_{i=0}^{k} c_2(i)\frac{1}{n_l}\mathbf{1}'_{n_l}W_l(i) = \bar{\omega}$, we know that

$$\begin{aligned}&\lim_{k\to\infty}\|\bar{X}_l(k) - \mathbf{1}_{n_l}(\pi+\bar{\omega})\|_2^2\\ &\leq \|P\|_2^2\lim_{k\to\infty} E\|\hat{X}_l(k) - [\pi+\bar{\omega}\ \ \mathbf{0}]'\|_2^2 = 0,\end{aligned}$$

which, according to the definition of $\bar{X}_l(k)$, implies (3).

Next, we need to prove the mean square containment. Denote $\hat{A}(k) = A(k) + A'(k)$. Since the subgraph induced by $\mathcal{V}_f$ is balanced and the graph contains a uniformly joint united spanning tree, it can be verified that

$$\lambda_{l,f} \triangleq \inf_{k\geq 0}\min_j \text{Re}\left\{\lambda_j\Big[\sum_{i=kh}^{(k+1)h-1}\hat{A}(i)\Big]\right\} > 0.$$

Similarly, we can prove that there exists a time k_5 such that $\forall k \geq k_5$,

$$\left\|\Pi^{I-c_1A}_{(k+1)h-1,kh}\right\|_2^2 \leq 1-\varepsilon_2 c_1(kh) < 1,$$

where ε_2 is a positive constant. Denote $\phi_A(k,h) = \Pi^{I-c_1A}_{(k+1)h-1,kh}$. Then it can be verified that

$$\lim_{k\to\infty}\|\Pi^{\phi_A(\cdot,h)}_{k,0}\|_2^2 = \lim_{k\to\infty}\|\Pi^{I-c_1A}_{k,0}\|_2^2 = 0. \tag{24}$$

According to (9) and (10), we have

$$\begin{aligned}X_f((k+1)h) &= \Pi^{I-c_1A}_{(k+1)h-1,kh}X_f(kh)\\ &\quad + \sum_{j=kh}^{(k+1)h-1}\Pi^{I-c_1A}_{(k+1)h-1,j+1}[-c_1(j)B(j)X_l + c(j)W_f(j)],\end{aligned}$$

and

$$\begin{aligned}X_f((k+1)h) &= \Pi^{\phi_A(\cdot,h)}_{k,0}X_f(0) + \sum_{i=0}^{k}\Pi^{\phi_A(\cdot,h)}_{k,i+1}\\ &\quad\cdot\sum_{j=ih}^{(i+1)h-1}\Pi^{I-c_1A}_{(i+1)h-1,j+1}[-c_1(j)B(j)X_l + c_1(j)W_f(j)].\end{aligned}$$

Denote

$$F(k,h) = \sum_{i=0}^{k}\Pi^{\phi_A(\cdot,h)}_{k,i+1}\sum_{j=ih}^{(i+1)h-1}\Pi^{I-c_1A}_{(i+1)h-1,j+1}c_1(j)B(j).$$

Note that

$$\begin{aligned}&-F(k,h)\mathbf{1}_{n_f}\\ &= \sum_{i=0}^{k}\Pi^{\phi(\cdot,h)}_{k,i+1}\sum_{j=ih}^{(i+1)h-1}\left[\Pi^{I-c_1A}_{(i+1)h-1,j+1} - \Pi^{I-c_1A}_{(i+1)h-1,j}\right]\mathbf{1}_{n_f}\\ &= \sum_{i=0}^{k}\left[\Pi^{\phi_A(\cdot,h)}_{k,i+1} - \Pi^{\phi(\cdot,h)}_{k,i}\right]\mathbf{1}_{n_f}\\ &= \left(I - \Pi^{\phi_A(\cdot,h)}_{k,0}\right)\mathbf{1}_{n_f},\end{aligned}$$

which together with (24) yields that $-F(k,h)$ converges to a row stochastic matrix as $k\to\infty$. Next, define

$$S(k)=\sum_{i=0}^{k}\Pi_{k,i+1}^{\phi_A(\cdot,h)}\left(\Pi_{k,i+1}^{\phi_A(\cdot,h)}\right)'c_1^2(ih).$$

Similar to the proof of the mean square formation, we have $\lim_{k\to\infty}\|S(k)\|_2=0$. Meanwhile, it can be found that $\Pi_{(i+1)h-1,j+1}^{I-c_1A}$ is uniformly bounded with respect to i and there exists Q_1 such that

$$\sup_{i\geq 0}\max_{ih\leq j<(i+1)h} cov\left(\Pi_{(i+1)h-1,j+1}^{I-c_1A}W_f(j)\right)\leq Q_1.$$

Then we arrive at that

$$\begin{aligned}&\lim_{k\to\infty}E\left\|X_f((k+1)h)+F(k,h)X_l\right\|_2^2\\ &\leq 2\lim_{k\to\infty}\left\|\Pi_{k,i+1}^{\phi_A(\cdot,h)}\right\|_2^2\|X_f(0)\|_2^2+2h\lim_{k\to\infty}\|S(k)\|_2Q_1\\ &=0.\end{aligned}\tag{25}$$

The last inequality is due to that $\{W_f(k),k\geq 0\}$ is a white noise process. From (25), we know that $E\|X_f(kh)\|_2$ is uniformly bounded. Then according to (9) we have

$$\begin{aligned}&E\|X_f(kh+i)-X_f(kh)\|_2^2\\ &\leq 2\left\|(\Pi_{kh+i-1,kh}^{I-cA}-I)X_f(kh)\right\|_2^2\\ &\quad+2\left\|\sum_{j=kh}^{kh+i-1}\Pi_{kh+i-1,j+1}^{I-cA}c(j)B(j)X_l\right\|_2^2\\ &\quad+E\left\|\sum_{j=kh}^{kh+i-1}\Pi_{kh+i-1,j+1}^{I-cA}c(j)W_f(j)\right\|_2^2\\ &\leq 2\varepsilon_2c_1(kh)\sup_{k\geq 0}\|X_f(kh)\|_2^2\\ &\quad+2(i-1)^2C_{u,h}^2c_1^2(kh)\sup_{k\geq 0}\|B(k)\|_2^2\|X(l)\|_2^2\\ &\quad+(i-1)C_{l,h}c_1^2(kh)Q_1,\end{aligned}$$

which implies that $\lim_{k\to\infty}E\|X_f(kh+i)-X_f(kh)\|_2^2=0$. Then $\forall i\in\mathcal{V}_f$, it follows that

$$\begin{aligned}&\lim_{k\to\infty}Edist^2\left(x_i(k),co\{\mathcal{V}_l\}\right)\\ &\leq 3\lim_{k\to\infty}Edist^2\left(x_i(k),x_i\left(\left\lfloor\frac{k}{h}\right\rfloor h\right)\right)\\ &\quad+3\lim_{k\to\infty}Edist^2\left(x_i\left(\left\lfloor\frac{k}{h}\right\rfloor h\right),-F\left(\left\lfloor\frac{k}{h}\right\rfloor,h\right)X_l\right)\\ &\quad+3\lim_{k\to\infty}Edist^2\left(-F\left(\left\lfloor\frac{k}{h}\right\rfloor,h\right)X_l,co\{\mathcal{V}_l\}\right)\\ &=0.\end{aligned}$$

□

In Theorem 4.1, we have proved that under some connectivity condition, the mean square formation of the leaders and mean square containment of the followers can be achieved when the control gain functions c_1 and c_2 satisfy (7) and (8), respectively. Next, we will show that when one of the leaders is fixed to a reference point, persistence condition of c_2 can be further weakened.

Assume that there exists one leader which can access the reference input r. We slightly modify the protocol of the leaders in the following way:

$$\begin{aligned}u_i(k)=c_2(k)\Big[&\sum_{j\in\mathcal{V}_l}b_{i,j}(k)(y_{i,j}(k)-h_j-x_i(k)+h_i)\\ &+\delta_{i,i_r}b(r-x_i)\Big],\ i\in\mathcal{V}_l,\end{aligned}\tag{26}$$

where i_r is the node index corresponding to the leader with access to r, b is a positive constant. Then we have the following result.

Theorem 4.2: Consider the multi-agent system (1)-(2) with interaction protocol (4) and (26). Under Assumption A1-A3, mean square formation and containment are achieved by the leaders and followers respectively if the communication graph contains a uniformly joint united spanning tree, the subgraph induced by $\mathcal{V}_l$ contains a uniformly joint spanning tree and c_1 and c_2 satisfy

$$\sum_{k=0}^{\infty}c_1(k)=+\infty,\quad\lim_{k\to\infty}c_1(k)=0,$$
$$\sum_{k=0}^{\infty}c_2(k)=+\infty,\quad\lim_{k\to\infty}c_2(k)=0.$$

Proof. Define g and G n-dimensional column vector and diagonal matrix with the (i_r-n_f)-th element equal to b, respectively, $\check{X}_l(k)=[r\ \bar{X}_l'(k)]'$. Then we have

$$\check{X}_l(k+1)=(I-c_2(k)\check{L}_l(k))\check{X}_l(k)+c_2(k)\begin{bmatrix}0\\ W_l(k)\end{bmatrix},\tag{27}$$

where

$$\check{L}_l(k)=\begin{bmatrix}0 & 0\\ -g & L_l(k)+G\end{bmatrix}.$$

Since the subgraph induced by $\mathcal{V}_l$ is balanced and contains a joint spanning tree, along the same line of proof of (18), we know that there exist large number k_6 and small positive number ε_3 such that $\forall k\geq k_6$

$$\left\|\Pi_{(k+1)h-1,kh}^{I-c_2(L_l+G)}\right\|_2^2\leq 1-\varepsilon_3c_2(kh)<1.\tag{28}$$

Then it follows that

$$\lim_{k\to\infty}\|\Pi_{k,0}^{I-c_2(L_l+G)}\|_2^2=0.$$

On the other hand, from (27) we can get

$$\begin{aligned}\bar{X}_l(k+1)=&[I-c_2(k)(L_l(k)+G)]\bar{X}_l(k)+c_2(k)gr\\ &+c_2(k)W_l(k),\end{aligned}$$

and

$$\begin{aligned}\bar{X}_l(k+1)=&\Pi_{k,0}^{I-c_2(L_l+G)}\bar{X}_l(0)\\ &+\sum_{i=0}^{k}\Pi_{k,i+1}^{I-c_2(L_l+G)}[c_2(i)gr+c_2(i)W_l(i)].\end{aligned}$$

Note that

$$\begin{aligned}\sum_{i=0}^{k}\Pi_{k,i+1}^{I-c_2(L_l+G)}c_2(i)g &= \sum_{i=0}^{k}\Pi_{k,i+1}^{I-c_2(L_l+G)}c_2(i)(L_l+G)\mathbf{1}_{n_l}\\ &= \sum_{i=0}^{k}\Pi_{k,i+1}^{I-c_2(L_l+G)}\mathbf{1}_{n_l}\\ &\quad -\sum_{i=0}^{k}\Pi_{k,i}^{I-c_2(L_l+G)}\mathbf{1}_{n_l}\\ &= (I-\Pi_{k,0}^{I-c_2(L_l+G)})\mathbf{1}_{n_l}.\end{aligned}$$

Meanwhile, we can prove that

$$\lim_{k\to\infty}\sum_{i=0}^{k}\left(\Pi_{k,i+1}^{I-c_2(L_l+G)}\right)'\Pi_{k,i+1}^{I-c_2(L_l+G)}c_2^2(i)=0.$$

Then we have

$$\begin{aligned}&\lim_{k\to\infty}E\|\bar{X}_l(k)-\mathbf{1}_{n_l}r\|_2^2\\ =&\lim_{k\to\infty}E\Big\|\Pi_{k,0}^{I-c_2(L_l+G)}[\bar{X}_l(0)+\mathbf{1}_{n_l}r]\\ &+\sum_{i=0}^{k}\Pi_{k,i+1}^{I-c_2(L_l+G)}c_2(i)W_l(i)\Big\|_2^2\\ \le&\lim_{k\to\infty}E\Big\|\Pi_{k,0}^{I-c_2(L_l+G)}[\bar{X}_l(0)+\mathbf{1}_{n_l}r]\Big\|_2^2\\ &+\lim_{k\to\infty}\text{trace}\Big(\sum_{i=0}^{k}\left(\Pi_{k,i+1}^{I-c_2(L_l+G)}\right)'\Pi_{k,i+1}^{I-c_2(L_l+G)}c_2^2(i)Q_2\Big)\\ =&\,0,\end{aligned}$$

where $Q_2=\sup_{k\geq 0}cov(W_l(k))$. The mean square formation follows immediately. The proof of mean square containment of the followers is similar to the case in Theorem 4.1 and is omitted. □

V. Conclusion

In this paper, the containment tracking problem for multi-agent systems with noises in transmission channels has been studied. The case of dynamically changing graphs was studied. A decaying gain function has been introduced to attenuate the noises. Sufficient conditions on the gain function and the graph have been provided to guarantee the containment tracking in the mean square sense. The results can be extended to multi-agent systems with continuous dynamics.

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