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On the Distribution of Energy Storage in Electricity Grids

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***Abstract*— Distributed energy storage is a promising emerging technology for smart grid. In this paper, we address the question of optimally placing and sizing distributed storage resources in a network to minimize the cost of generation given a budget of available storage. For a non-decreasing convex generation cost, we prove that it is always optimal to place zero storage at generator buses that connect to rest of the grid via single links, regardless of demand profiles and network parameters. Hence, this defines a robust investment strategy for network planners. Besides, for a star network where the center is a generator bus and the other nodes are demand buses, we show that it is optimal not to place storage resources at the generator bus for small enough and large enough storage budget.**

I. INTRODUCTION

Over the last decade or so, grid-level energy storage technologies have shown significant technical improvements and cost drops [1], [2] and show potential to have a major impact on the emerging smart grid. At faster time scales (seconds to mintues), storage can be used to reduce variability of renewable sources of energy like wind or solar [3]–[5]. At slower time scales (over hours), it can be used for load shifting [1], [6], i.e., generate when it is cheaper and use storage dynamics to follow the demand. It is expected to play a more important role in the wake of future smart grid programs like demand response and distribution system markets [7]–[9]; for a detailed survey of the applications of storage, see [10], [11]

Several authors have investigated the following questions with storage: (a) What is the optimal control policy for a storage device? (b) How to size the storage devices in a power transmission network? A common performance metric in these studies is the reduction in generation cost due to the effect of storage. In this paper, we formulate both problems in a common framework and present results on sizing such storage units in a network and a charging/ discharging policy for the installed units, i.e., we solve the investment decision problem and the control policy for the storage devices.

Next, we present a brief overview of the literature in this area. Authors in [12], [13] examine the control of a single storage device without a network, while [14]–[16] explicitly model the role of the networks in the operation of distributed storage resources. Optimal sizing of a single storage device has been studied, e.g., in [17], [18] using purely economic arguments and in [13], [19] with constraints to model the engineering of the power network. Also, Kanoria et al. [14] compute the effect of sizing of distributed storage resources on generation cost for specific networks and Gayme et. al [15] study a similar problem in IEEE benchmark systems [20] through simulations. Recently, a more general framework to study the optimal storage placement problem in generic networks has been formulated and studied through simulations in [21], [22]. While Sjödin et al. in [21] model the underlying network using a linearized DC power flow equations [23], [24], Bose et al. in [22] use a conic relaxation of the AC power flow model [25]–[27].

Our main contribution in this paper is the following result: *when minimizing a convex and nondecreasing generation cost with any fixed available storage budget, there always exists an optimal storage allocation that assigns zero storage at generator nodes that connect via single transmission lines to the network, for arbitrary demand profiles and other network parameters*. It suggests that in most distribution networks and isolated transmission networks, it is always optimal to allocate the entire available storage budget among demand buses. This result generalizes our work in [28]. Authors in [15], [21], [22] empirically observe and conjecture that optimal storage allocation depends mainly on the network structure and not on the total available storage budget. This work provides an analytic justification of this observation. Furthermore, we study the same optimal storage placement problem for a star network where a single generator bus is connected to several demand nodes. When the storage budget is the minimum required to support the loads or when large enough, it is optimal not to place any storage units at the generator bus.

The paper is organized as follows. We formulate the optimal storage placement problem in section II. The main result is proven in section III. In section IV, we present our result on a star network and conclude in Section V.

II. PROBLEM FORMULATION

Consider a power network defined by an undirected connected graph $\mathcal{G}$ on n nodes $\mathcal{N} = \{1, 2, \dots, n\}$. We use the term buses and nodes interchangeably. For two nodes k and l in $\mathcal{N}$, let $k \sim l$ denote that k is connected to l in $\mathcal{G}$ by a transmission line.

For the power flow model, we use the DC approximation; see [23] for a detailed survey. In this model, the network is

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lossless, the voltage magnitudes are assumed to be 1 per unit (p.u.) at all buses and voltage angle differences are small.

Time is discrete and indexed by t. Now we introduce the following notation.

- $d_k(t)$ is the real power demand at bus $k \in \mathcal{N}$ at time t, which is assumed to be known.
- $g_k(t)$ is the real power generation at bus $k \in \mathcal{N}$ at time t and it satisfies
$$0 \leq g_k(t) \leq \overline{g}_k, \tag{1}$$
where, $\overline{g}_k$ is the generation capacity of the generator at bus k.
- $c_k(g_k)$ denotes the cost of generating power g_k at bus $k \in \mathcal{N}$. Note that the cost of generation is independent of time t and depends only on the generation technology at bus k. For the purpose of our analysis, assume that the function $c_k : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, differentiable and convex. A cost function commonly found in the literature [15], [26], [27] that satisfies the above assumption is:
$$c_k(g_k) = \gamma_{k,2} g_k^2 + \gamma_{k,1} g_k + \gamma_{k,0},$$
where $\gamma_{k,2}, \gamma_{k,1}, \gamma_{k,0}$ are known nonnegative coefficients.
- $\theta_k(t)$ is the voltage angle at bus $k \in \mathcal{N}$ at time t.
- For two nodes $k \sim l$, let $p_{kl}(t)$ be the power flow from bus k to bus l at time t. It satisfies
$$p_{kl}(t) = y_{kl}\left[\theta_k(t) - \theta_l(t)\right], \tag{2}$$
where y_{kl} is the admittance of the line joining buses k and l. Also, the power delivered over this line is limited by thermal effects and stability constraints and hence
$$|p_{kl}(t)| \leq f_{kl}, \tag{3}$$
where f_{kl} is the known flow limit of the line.
- $\gamma_k(t)$ and $\delta_k(t)$ are the average charging and discharging powers of the storage unit at bus $k \in \mathcal{N}$ at time t, respectively. The energy transacted over a time-step is converted to power units by dividing it by the length of the time-step. This transformation conveniently allows us to formulate the problem in units of power [22].
Let $0 < \alpha_\gamma, \alpha_\delta < 1$ denote the charging and discharging efficiencies, respectively of the storage technology used. Then the roundtrip efficiency is $\alpha = \alpha_\gamma \alpha_\delta < 1$. The power flowing in and out of the storage device at node $k \in \mathcal{N}$ at time t are $\alpha_\gamma \gamma_k(t)$ and $\frac{1}{\alpha_\delta}\delta_k(t)$, respectively [13], [29].
- $s_k(t)$ denotes the storage level at node $k \in \mathcal{N}$ at time t and s_k^0 is the storage level at node k at time $t = 0$. From the definitions above, we have that
$$s_k(t) = s_k^0 + \sum_{\tau=1}^{t}\left(\alpha_\gamma \gamma_k(\tau) - \frac{1}{\alpha_\delta}\delta_k(\tau)\right). \tag{4}$$
For each $k \in \mathcal{N}$, assume $s_k^0 = 0$, so that the storage units are empty at installation time.
- $b_k \geq 0$ is the storage capacity at bus k. Thus, $s_k(t)$ for all t satisfies
$$0 \leq s_k(t) \leq b_k. \tag{5}$$
- h is the available storage budget and denotes the total amount of storage capacity that can be installed in the network. Our optimization algorithm decides the allocation of storage capacity b_k at each node $k \in \mathcal{N}$ and thus we have
$$\sum_{k \in \mathcal{N}} b_k \leq h. \tag{6}$$
- We assume that the charging and discharging ramp rates of each storage device are proportional to the capacity of the corresponding device. In particular, for all $k \in \mathcal{N}$,
$$0 \leq \gamma_k(t) \leq \epsilon_\gamma b_k, \tag{7a}$$
$$0 \leq \delta_k(t) \leq \epsilon_\delta b_k, \tag{7b}$$
where $\epsilon_\gamma \in (0, \frac{1}{\alpha_\gamma}]$ and $\epsilon_\delta \in (0, \alpha_\delta]$ are fixed constants.

To maintain appropriate power balance at each bus $k \in \mathcal{N}$, we have for all t,
$$g_k(t) - d_k(t) - \gamma_k(t) + \delta_k(t) = \sum_{l \sim k} p_{kl}(t). \tag{8}$$
Demand profiles often show diurnal variations [30], i.e., they exhibit cyclic behavior. Let T time-steps denote the common cycle length of the variation. In particular, for all $k \in \mathcal{N}$ and $t \geq 0$, assume
$$d_k(t+T) = d_k(t).$$
Optimally placing storage over an infinite horizon is then equivalent to solving this problem over a singe cycle, provided the state of the storage levels at the end of a cycle is the same as its initial condition [22]. Thus for each $k \in \mathcal{N}$, we have
$$\sum_{t=1}^{T}\left(\alpha_\gamma \gamma_k(t) - \frac{1}{\alpha_\delta}\delta_k(t)\right) = 0. \tag{9}$$

For convenience, denote $[T] := \{1, 2, \ldots, T\}$. Using the above notation, we define the following optimization problems.

Storage placement problem P:
$$\begin{aligned}
\text{minimize} \quad & \sum_{k \in \mathcal{N}} \sum_{t=1}^{T} c_k(g_k(t)) \\
\text{over} \quad & (g_k(t), \gamma_k(t), \delta_k(t), \theta_k(t), p_{kl}(t), b_k), \\
& k \in \mathcal{N},\ k \sim l,\ t \in [T], \\
\text{subject to} \quad & (1), (2), (3), (4), (5), (6), (7), (8), (9).
\end{aligned}$$
Restrict attention to network topologies where each bus either has generation or load but not both. Partition the set of buses $\mathcal{N}$ into two groups $\mathcal{N}_G$ and $\mathcal{N}_D$ where they represent the generation-only and load-only buses respectively and assume $\mathcal{N}_G$ and $\mathcal{N}_D$ are non-empty. For any subset $\mathcal{K}$ of $\mathcal{N}_G$, define the following optimization problem.

Restricted storage placement problem $\Pi^{\mathcal{K}}$:

$$\begin{aligned}
\text{minimize} \quad & \sum_{k\in\mathcal{N}} \sum_{t=1}^{T} c_k\left(g_k(t)\right) \\
\text{over} \quad & (g_k(t), \gamma_k(t), \delta_k(t), \theta_k(t), p_{kl}(t), b_k), \\
& k \in \mathcal{N},\ k \sim l,\ t \in [T], \\
\text{subject to} \quad & (1), (2), (3), (4), (5), (6), (7), (8), (9), \\
& b_i = 0, \quad i \in \mathcal{K}.
\end{aligned}$$

Problem $\Pi^{\mathcal{K}}$ corresponds to placing no storage at the (generation) buses of the network that belong to the subset $\mathcal{K}$. The rest of the paper deals with the connection between problems P and $\Pi^{\mathcal{K}}$.

We say bus $k \in \mathcal{N}$ has *a single connection* if it has exactly one neighboring node $l \sim k$. Similarly, a bus $k \in \mathcal{N}$ has *multiple connections* if it has more than one neighboring nodes in $\mathcal{G}$. To illustrate the notation, consider the network shown in Figure 1. $\mathcal{N}_G = \{1, 2, 7\}$ and $\mathcal{N}_D = \{3, 4, 5, 6\}$. Buses 1 and 2 have single connections and all other buses in the network have multiple connections.

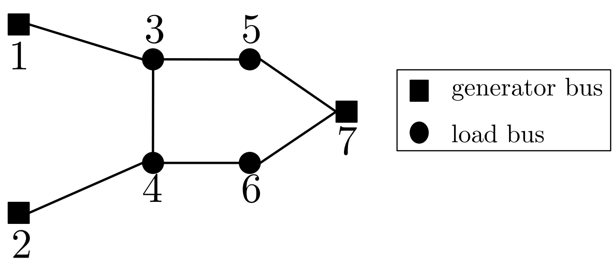

Fig. 1: A sample network.

III. On Generator Buses with Single Connections

Let p_* and ${\pi_*}^{\mathcal{K}}$ be the optimal values for the problems P and $\Pi^{\mathcal{K}}$ respectively for some subset $\mathcal{K} \subseteq \mathcal{N}_G$. Next, we relate p_* and ${\pi_*}^{\mathcal{K}}$ when all generators in $\mathcal{K}$ have single connections in our main result:

Theorem 1: Suppose $\mathcal{K} \subseteq \mathcal{N}_G$ and each node $i \in \mathcal{K}$ has a single connection. If P is feasible, then $\Pi^{\mathcal{K}}$ is feasible and $p_* = \pi_*^{\mathcal{K}}$.

The theorem states that, for any available storage budget and any subset of generation buses with single connections, there *always* exists an optimal storage allocation that assigns *zero* storage at those buses, regardless of the demand patterns and other network parameters, such as line flow constraints. In our model, we have considered deterministic demand profiles only. However, Theorem 1 holds for arbitrary demand profiles and hence the result generalizes to the case with stochastic demand profiles. The theorem restricts attention to generator buses with single connections only. This is applicable in many practical scenarios, as discussed in section III-B. For generator buses with multiple connections, however, we show in section IV that the optimal storage placement may not, in general, place zero storage capacity at such buses.

A. Proof of Theorem 1

Assume P is feasible throughout. For any variable z in problem P, let z^* be the value of the corresponding variable at the optimum. Consider node $i \in \mathcal{K}$ and $j \sim i$. Node j is uniquely defined as i has a single connection. Note that problem P, in general, has multiple optima. In the following result, we characterize only a subset of these optima.

Lemma 1: There exists an optimal point of P such that for all $t \in [T]$ and all $i \in \mathcal{K}, j \sim i$,

(a) $g_i^*(t)\gamma_i^*(t)\delta_i^*(t) = 0$,
(b) $g_i^*(t) \leq f_{ij}$.

The first part of lemma 1 essentially says that at some optimum point of P, the storage units should not charge and discharge at the same time step if there is positive generation at the same bus at that time step. This is expected since the round-trip efficiency of the storage devices $\alpha = \alpha_\gamma \alpha_\delta$ is less than one. The second part can be interpreted as follows. The power that flows from bus i to bus j at each $t \in [T]$ is $p_{ij}(t) = g_i(t) - \gamma_i(t) + \delta_i(t)$ and we have $p_{ij}(t) \leq f_{ij}$. But lemma 1 states that there exists an optimum for which, $g_i^*(t), t \in [T]$ itself defines a feasible flow over this line.

Proof:

The feasible set of problem P is a bounded polytope and the objective function is a continuous convex function. Hence the set of optimum of P is a convex and compact set [31]. Now, with every point in the feasible set, consider the function $\sum_{i\in\mathcal{K},t\in[T]} \left(\gamma_i(t) + \delta_i(t)\right)$. This is a linear and hence continuous function on the compact set of optima of P and hence attains a minimum. Consider the optimum of P where this minimum is reached. We show that for this optimum, $g_i^*(t)\gamma_i^*(t)\delta_i^*(t) = 0$ and $g_i^*(t) \leq f_{ij}$ for all $t \in [T]$ and $i \in \mathcal{K}, j \sim i$.

(a) Suppose, on the contrary, we have $g_i^*(t_0) > 0$, $\gamma_i^*(t_0) > 0$ and $\delta_i^*(t_0) > 0$ for some $t_0 \in [T]$. Define

$$\Delta g' := \min\left\{(1-\alpha)\gamma_i^*(t_0)\ ,\ \frac{1-\alpha}{\alpha}\delta_i^*(t_0)\ ,\ g_i^*(t_0)\right\}.$$

Note that $\Delta g' > 0$. Now, for bus i, construct modified generation, charging and discharging profiles $\tilde{g}_i(t), \tilde{\delta}_i(t), \tilde{\gamma}_i(t), t \in [T]$ that differs from $g_i^*(t), \delta_i^*(t), \gamma_i^*(t)$ only at t_0 as follows:

$$\begin{aligned}
\tilde{g}_i(t_0) &:= g_i^*(t_0) - \Delta g', \\
\tilde{\gamma}_i(t_0) &:= \gamma_i^*(t_0) - \frac{1}{1-\alpha}\Delta g', \\
\tilde{\delta}_i(t_0) &:= \delta_i^*(t_0) - \frac{\alpha}{1-\alpha}\Delta g'.
\end{aligned}$$

It can be checked that the storage level and the power flowing from bus i to bus j remain unchanged throughout. Also, the modified profiles define a feasible point of P. Since $c_i(\cdot)$ is non-decreasing, we have $c_i\left(\tilde{g}_i(t_0)\right) \leq c_i\left(g_i^*(t_0)\right)$ and hence this feasible point has an objective function value of at most p_*. It follows that this feasible point defines an optimal point of P. However, we have $\tilde{\gamma}_i(t_0) + \tilde{\delta}_i(t_0) < \gamma_i^*(t_0) + \delta_i^*(t_0)$ and thus, this optimum of P has a strictly lower $\sum_{i\in\mathcal{K},t\in[T]} \left(\gamma_i(t) + \delta_i(t)\right)$, contradicting our hypothesis. This completes the proof of $g_i^*(t_0)\gamma_i^*(t_0)\delta_i^*(t_0) = 0$.

(b) If $g_i^*(t) = 0$ for all $t \in [T]$, then $g_i^*(t) \le f_{ij}$ clearly holds. Henceforth, assume $\max_{t\in[T]} g_i^*(t) > 0$, and consider any $t_0 \in [T]$, such that $g_i^*(t_0) = \max_{t\in[T]} g^*(t)$. If $\gamma_i^*(t_0) = 0$, then,

$$\begin{aligned}\max_{t\in[T]} g_i^*(t) &= g_i^*(t_0) \\ &= \underbrace{p_{ij}^*(t_0)}_{\le f_{ij}} + \underbrace{\gamma_i^*(t_0)}_{=0} - \underbrace{\delta_i^*(t_0)}_{\ge 0} \\ &\le f_{ij}. \end{aligned} \tag{10}$$

and lemma 1(b) holds.

Suppose now that $\gamma_i^*(t_0) > 0$ and hence $\delta_i^*(t_0) = 0$ from lemma 1(a). First, we show that the storage device discharges at some point after t_0.

$$s_i^*(t_0) = \underbrace{s_i^*(t_0-1)}_{\ge 0} + \underbrace{\alpha_\gamma \gamma_i^*(t_0)}_{>0} > 0.$$

We also have $s_i^*(T) = s_i^*(0) = 0$ by hypothesis. Thus the storage device at node i needs to discharge in $[t_0+1, T]$ and hence $\alpha_\gamma \gamma_i^*(t) - \frac{1}{\alpha_\delta}\delta_i^*(t) < 0$ for some $t \in [t_0+1, T]$. Let $t_1 \in [t_0+1, T]$ be the first time instant after t_0 when the storage device at bus i is discharged, i.e.

$$t_1 := \min\left\{ t \in [t_0+1, T] \mid \alpha_\gamma \gamma_i^*(t) - \frac{1}{\alpha_\delta}\delta_i^*(t) < 0 \right\}. \tag{11}$$

Thus, $\delta_i^*(t_1) > 0$. Define

$$\Delta g := \min\left\{ \gamma_i^*(t_0)\ ,\ \frac{1}{\alpha}\delta_i^*(t_1)\ ,\ g_i^*(t_0) \right\}. \tag{12}$$

Then, $\Delta g > 0$. Now, consider the case where:

$$g_i^*(t_1) > 0, \quad \text{and} \quad g_i^*(t_0) \le g_i^*(t_1) + \alpha\Delta g. \tag{13}$$

Since $g_i^*(t_1) > 0$, then $\gamma_i^*(t_1) = 0$, by lemma 1(a). In that case, $g_i^*(t_1) + \delta_i^*(t_1) = p_{ij}^*(t_1)$ is the power that flows from bus i to bus j at time t_1. Combining (12) and (13), we have

$$\begin{aligned}\max_{t\in[T]} g_i^*(t) &= g_i^*(t_0) \\ &\le g_i^*(t_1) + \alpha\Delta g \\ &\le g_i^*(t_1) + \delta_i^*(t_1) \\ &= p_{ij}^*(t_1) \le f_{ij}. \end{aligned}$$

Hence, 1(b) holds when (13) is satisfied. Next, we show that if (13) does not hold, then we can construct an optimum of P with a lower $\sum_{i\in\mathcal{K}, t\in[T]} (\gamma_i(t) + \delta_i(t))$ and this contradicts our hypothesis.

Suppose (13) does not hold. If $g_i^*(t_1) = 0$, then we have

$$g_i^*(t_0) \ge \Delta g > \alpha\Delta g = g_i^*(t_1) + \alpha\Delta g.$$

Thus, it suffices to only consider the following case:

$$g_i^*(t_0) > g_i^*(t_1) + \alpha\Delta g. \tag{14}$$

Construct the modified generation, charging and discharging profiles at node i, $\tilde{g}_i(t), \tilde{\delta}_i(t), \tilde{\gamma}_i(t), t \in [T]$ using (12), that differ from $g_i^*(t), \delta_i^*(t), \gamma_i^*(t)$ only at t_0 and t_1 as follows:

$$\begin{aligned} \tilde{g}_i(t_0) = g_i^*(t_0) - \Delta g, &\quad \tilde{g}_i(t_1) = g_i^*(t_1) + \alpha\Delta g, \\ \tilde{\gamma}_i(t_0) = \gamma_i^*(t_0) - \Delta g, &\quad \tilde{\gamma}_i(t_1) = \gamma_i^*(t_1), \\ \tilde{\delta}_i(t_0) = \delta_i^*(t_0) = 0, &\quad \tilde{\delta}_i(t_1) = \delta_i^*(t_1) - \alpha\Delta g. \end{aligned}$$

Also, define the modified storage level $\tilde{s}_i(t)$ using $\tilde{\gamma}_i(t)$ and $\tilde{\delta}_i(t)$. To provide intuition to the above modification, we essentially generate and store less at time t_0 by an amount Δg. This means at a future time t_1, we can discharge $\alpha\Delta g$ less from the storage device and hence have to generate $\alpha\Delta g$ more to compensate. To check feasibility, it follows from (12), that for $t = t_0, t_1$, we have

$$\begin{aligned} 0 \le \tilde{g}_i(t) &\le \overline{g}_i, \\ 0 \le \tilde{\gamma}_i(t) &\le \epsilon_\gamma b_i^*, \\ 0 \le \tilde{\delta}_i(t) &\le \epsilon_\delta b_i^*. \end{aligned}$$

The line flows $p_{ij}(t)$ remain unchanged. For the storage levels, it can be checked that the following holds:

$$\begin{aligned} 0 \le s_i^*(t_0-1) \le \tilde{s}_i(t) \le s_i^*(t) \le b_i^*, &\ \text{for } t \in [t_0, t_1-1], \\ \tilde{s}_i(t) = s_i^*(t), &\ \text{otherwise}. \end{aligned}$$

This proves that the modified profiles define a feasible point for P. The cost satisfies

$$\begin{aligned} & c_i(\tilde{g}_i(t_0)) + c_i(\tilde{g}_i(t_1)) \\ &\quad \le c_i(g_i^*(t_0) - \alpha\Delta g) + c_i(g_i^*(t_1) + \alpha\Delta g) \quad \text{(15a)} \\ &\quad \le c_i(g_i^*(t_0)) + c_i(g_i^*(t_1)). \quad \text{(15b)} \end{aligned}$$

Equation (15a) follows from the non-decreasing nature of $c_i(\cdot)$ and equation (15b) follows from using (14) and applying the mean-value theorem [31] to the convex differentiable function $c_i(\cdot)$. Thus the modified profiles $\tilde{g}_i(t), \tilde{\delta}_i(t), \tilde{\gamma}_i(t)$ define a feasible point of P with a cost at most p_* and hence optimal for P. However, we also have

$$\begin{aligned} & \tilde{\gamma}_i(t_0) + \tilde{\gamma}_i(t_1) + \tilde{\delta}_i(t_0) + \tilde{\delta}_i(t_1) \\ &\quad = \gamma_i^*(t_0) + \gamma_i^*(t_1) + \delta_i^*(t_0) + \delta_i^*(t_1) - \underbrace{(1+\alpha)\Delta g}_{>0}. \end{aligned}$$

Thus the modified profiles define an optimum of P with a lower $\sum_{i\in\mathcal{K}, t\in[T]} (\gamma_i(t) + \delta_i(t))$. This is a contradiction and completes the proof of the lemma.

■

To prove theorem 1, consider the optimal point of P that satisfies 1(b). For all $i \in \mathcal{K}$, $g_i^*(t)$ itself defines a feasible flow over the line joining buses i and j, where j is the unique neighboring node of i. Now the proof idea is as follows. For $i \in \mathcal{K}$, transfer all storage capacities b_i^* and the associated charging/ discharging profiles $(\gamma_i^*(t), \delta_i^*(t))$, to the neighboring node j. In particular, consider the point

$\Big(g_k^*(t), \hat{\gamma}_k(t), \hat{\delta}_k(t), \hat{\theta}_k(t), \hat{p}_{kl}(t), \hat{b}_k,\ k \in \mathcal{N}, k \sim l, t \in [T]\Big)$ defined as follows.

$$\begin{aligned}
\hat{\gamma}_i(t) &= 0, \quad \hat{\gamma}_j(t) = \gamma_i^*(t) + \gamma_j^*(t), \\
\hat{\gamma}_k(t) &= \gamma_k^*(t), \quad k \in \mathcal{N} \setminus \{i, j\}, \\
\hat{\delta}_i(t) &= 0, \quad \hat{\delta}_j(t) = \delta_i^*(t) + \delta_j^*(t), \\
\hat{\delta}_k(t) &= \delta_k^*(t), \quad k \in \mathcal{N} \setminus \{i, j\}, \\
\hat{\theta}_i(t) &= \theta_i^*(t) + \frac{1}{y_{ij}}(\gamma_i^*(t) - \delta_i^*(t)), \\
\hat{\theta}_k(t) &= \theta_k^*(t), k \in \mathcal{N} \setminus \{i\}, \\
\hat{b}_i &= 0, \quad \hat{b}_j = b_i^* + b_j^*, \\
\hat{b}_k &= b_k^*, \quad k \in \mathcal{N} \setminus \{i, j\}, \\
\hat{p}_{ij}(t) &= p_{ij}^*(t) + \gamma_i^*(t) - \delta_i^*(t), \\
\hat{p}_{kl}(t) &= p_{kl}^*(t), \quad k \sim l, (k, l) \neq (i, j).
\end{aligned}$$

We do this successively for each $i \in \mathcal{K}$ to obtain a feasible point of $\Pi^{\mathcal{K}}$. Since the generation profiles remained invariant, the resulting point is optimal for $\Pi^{\mathcal{K}}$. This completes the proof of theorem 1.

B. Discussion

Theorem 1 defines a structural property of the investment decision strategy for storage devices in a network. The optimization framework in the storage placement problem P, however, applies more generally and solves for each node $k \in \mathcal{N}$: (a) the optimal size of the storage resource b_k^*, and (b) the optimal control policy $\gamma_k^*(t), \delta_k^*(t)$ for operating the storage devices.

Next, we explore applications of theorem 1 and its potential usefulness for a network planner. In particular, consider the following networks:

- A network with a single generator and a single load connected by a transmission line as shown in figure 2. This models topologies where generators and loads are geographically separated and connected by a long transmission line [32],
- A radial network as in figure 2 that models most distribution networks [33] and isolated transmission networks, e.g., power network in Catalina island [4],
- A generic mesh networks with generators with single connections, as in figure 1.

In these examples, set $\mathcal{K}$ includes generator buses with single connections. The rest of the buses $\mathcal{N} \setminus \mathcal{K}$ include all load buses and generator buses with multiple connections. Theorem 1 states that, for any available storage budget h, it is always optimal to place *no storage* on generator buses that have single connections. Note that problem P, in general, has multiple optima but there always exists an optimum that assigns zero storage to nodes in the set $\mathcal{K}$, i.e. $b_i = 0$ for all i in $\mathcal{K}$.

An important property of our result is that it is robust to changes in demand profiles, generation capacities, line flow capacities and admittances in the entire network, i.e., it remains optimal not to place any storage at buses in set $\mathcal{K}$ under any changes to the above-mentioned parameters.

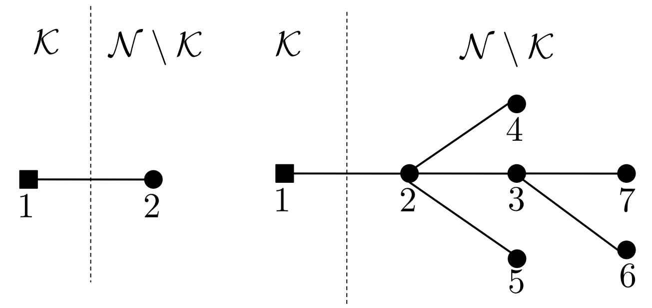

Fig. 2: Examples of power networks (a) Single generator single load system, (b) Radial network.

The optimal sizing and the operation of the storage devices, however, might vary with changes in these parameters. The result is also robust to extensions of the network. To illustrate this, suppose another generator is built to supply the load in figure 2. In that case, our result suggests that the optimal placement still has no storage at bus 1 and hence defines a sound investment strategy for the network planner. As a final remark, our analysis also holds in the presence of renewable generation with negligible marginal cost of production; see [34] for more details.

IV. On Generator Buses with Multiple Connections

Theorem 1 only provides insights into the storage placement for generator buses with single connections. The result does not generalize to generation buses with multiple connections, as illustrated through a simple counterexample.

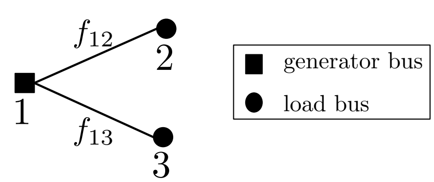

Fig. 3: A generator with multiple connections. For specific values of demand profiles ($d_2(t)$, $d_3(t)$), line flow limits (f_{12}, f_{13}) and storage budget h, it can be verified that $p_* < \pi_*^{\{1\}}$.

Consider a 3-node network as shown in figure 3. All quantities are in per units. Let the cost of generation at node 1 be $c_1(g_1) = g_1^2$. Let $T = 4$ and the demand profiles at nodes 2 and 3 be

$$d_2 = (9, 10, 0, 10) \qquad \text{and} \qquad d_3 = (0, 10, 10, 10).$$

Also, suppose that the line capacities are $f_{12} = f_{13} = 9.5$ and the available storage budget is $h = 5$. Finally, assume no losses and ignore the ramp constraints in the charging and discharging processes, i.e. $\alpha = 1$ and $\epsilon_\gamma = \epsilon_\delta = 1$. Then, it can be checked that $p_* = 877 < \pi_*^{\{1\}} = 900.75$.

A. Star Network

Consider a star network on $n \geq 2$ nodes as shown in figure 4. $\mathcal{N}_G = \{1\}$, $\mathcal{N}_D = \{2, 3, \ldots, n\}$ and $\mathcal{K} = \{1\}$. For fixed demand profiles $d_k(t), t \in [T], k \in \mathcal{N}_D$, line flow

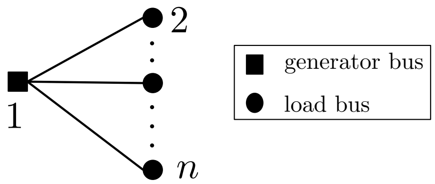

Fig. 4: A star network on n buses.

capacities f_{1k}, $k \in \mathcal{N}_D$ and capacity of the generator $\overline{g}_1$, let $P(h)$ and $\Pi^{\{1\}}(h)$ denote the storage placement problem and its restricted version as functions of the available storage budget h. Also, let $p_*(h)$ and $\pi_*^{\{1\}}(h)$ be their optimal costs respectively.

For the purposes of this section assume that the cost of generation $c_1(\cdot)$ is strictly convex. Further assume no losses and ignore any ramp constraints in the charging and discharging process of the storage devices, i.e. $\alpha = 1$ and $\epsilon_\gamma = \epsilon_\delta = 1$.

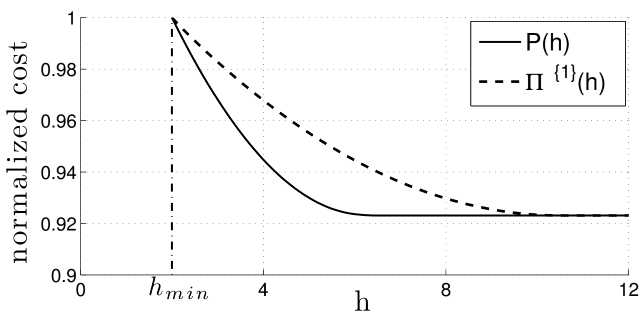

Fig. 5: $P(h)$ and $\Pi^{\{1\}}(h)$ for the simple 3-node star network in figure 3 and for $c_1(g_1) = g_1^2$.

We showed that placing zero storage at the generator bus of a star network is not optimal, i.e., $p_*(h) \neq \pi_*^{\{1\}}(h)$ in general. In figure 5, we plot $p_*(h)$ and $\pi_*^{\{1\}}(h)$ over a range of values of the total storage budget h , for the 3-node star network shown in figure 3 and for a quadratic cost of generation $c_1(g_1) = g_1^2$. Observe that $p_*(h) < \pi_*^{\{1\}}(h)$ for some value of h but they coincide at:

- Minimum value of h for which $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible.
- Large enough values of h.

We now formally state these observations for a general n-node star network. Suppose $\overline{g}_1 = \infty$ for simplicity.

Theorem 2: Suppose $f_{1k} \geq \max_{t\in[T]} \left(\frac{\sum_{\tau=1}^{t} d_k(\tau)}{t} \right)$ for all $k \in \mathcal{N}_D$. Then, $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible iff $h \geq h_{min}$, where

$$h_{min} = \sum_{k\in\mathcal{N}_D} \max \left\{ \max_{0\leq t_1 < t_2 \leq T} \left\{ \sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right\}, 0 \right\}. \tag{16}$$

Moreover:

(a) $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$,

(b) There exists $h_o \geq h_{min}$ such that $p_*(h) = \pi_*^{\{1\}}(h)$ for all $h \geq h_o$.

The details of the proof are omitted for brevity, but we outline the main steps of the proof in the rest of this section; see [34] for details.

Suppose $P(h)$ is feasible. First, we find a lower bound to b_k for the load at bus k to be satisfied. We derive this using the power flow constraint on the line connecting buses 1 and k and the constraint on storage level at node k. Thus, we have

$$b_k \geq \max \left\{ \max_{0\leq t_1 < t_2 \leq T} \left\{ \sum_{\tau=t_1+1}^{t_2} (d_k(\tau) - f_{1k}) \right\} \ , \ 0 \right\}. \tag{17}$$

This holds for all $k \in \mathcal{N}_D$. Summing the above expression for all $k \in \mathcal{N}_D$, we get that $h \geq h_{min}$ is necessary for $P(h)$ to be feasible. Next, we show that this is sufficient. Note that it suffices to show that $P(h)$ and $\Pi^{\{1\}}(h)$ are feasible for $h = h_{min}$ and we can explicitly construct a feasible solution for both problems in this case.

To prove that the two problems agree on their objective values, i.e., $p_*(h_{min}) = \pi_*^{\{1\}}(h_{min})$, let $b_k^*, k \in \mathcal{N}$ be the optimal storage capacities for problem $P(h_{min})$. Note that these optimal storage capacities satisfy the following pair of equations:

$$\sum_{k\in\mathcal{N}_D} b_k^* \geq h_{min},$$

$$b_1^* + \sum_{k\in\mathcal{N}_D} b_k^* \leq h_{min}.$$

where the first one follows from (17) and the second one follows from the constraint on the total available storage capacities. Manipulating the above equations, it can be shown that $b_1^* = 0$. This completes the proof sketch of theorem 2(a).

Next, we present the proof idea of theorem 2(b). The cost function $c_1(\cdot)$ is convex and hence flatter the generation profile, lower the cost. Using this fact, it can be argued that $p_*(h)$ and $\pi_*^{\{1\}}(h)$ remain constant for h above a certain threshold and analyzing $P(h)$ or $\Pi^{\{1\}}(h)$ at large finite h is equivalent to analyzing the corresponding problems at $h = \infty$.

Consider such a large h. Similar to the proof of lemma 1, we consider an optimum of $P(h)$ where the function $\sum_{t\in[T]} \gamma_1(t) + \delta_1(t)$ is minimized. We show that for this optima $\sum_{t\in[T]} \gamma_1^*(t) + \delta_1^*(t) = 0$, i.e. $\gamma_1^*(t) = \delta_1^*(t) = 0$ for all $t \in [T]$ and hence it defines a feasible and optimal point of $\Pi^{\{1\}}(h)$.

The proof is by contradiction, i.e. assume there exists a time t_0 where the storage device at the generator bus is charged. Then it must also be discharged at a subsequent time; let t_1 be the first time instant after t_0 when the storage device at the generator bus discharges. Define $\Delta = \min\{\gamma_1^*(t_0) \ , \ \delta_1^*(t_1)\}$ and consider two cases based on how $g_1^*(t_0)$ compares to $g_1^*(t_1) + \Delta$.

If $g_1^*(t_0) > g_1^*(t_1) + \Delta$, then we construct modified generation, charging and discharging profiles for bus 1 that define an optimum of P with a strictly lower $\sum_{t\in[T]} \gamma_1(t) + \delta_1(t)$, and hence contradicting our hypothesis.

If $g_1^*(t_0) \leq g_1^*(t_1) + \Delta$, then we use Farkas' Lemma [31] to show that the storage at node 1 can be distributed to add to the storage capacities at nodes $k \in \mathcal{N}_D$ and still define a feasible solution to $P(h)$. Note that the generation profile remains invariant and hence this again defines an optimal solution to $P(h)$ with a strictly lower $\sum_{t\in[T]} \gamma_1(t) + \delta_1(t)$, and hence contradicts our hypothesis.

Thus, we have $\sum_{t\in[T]} \gamma_1^*(t) + \delta_1^*(t) = 0$ at this optimum. This completes the proof sketch.

V. Conclusions and Future work

We have derived analytic results on the problem of optimal placement of large-scale storage in the grid. A natural direction for future work is to study the same problem with performance metrics other than generation cost. We also intend to investigate the optimal storage allocation with stochastic demand and generation models.

References

[1] D. Rastler, *Electricity Energy Storage Technology Options: A White Paper Primer on Applications, Costs and Benefits*. Electric Power Research Institute, 2010.

[2] "2020 strategic analysis of energy storage in california," *Califonia Energy Commision*, November 2011.

[3] Y. M. Atwa and E. F. El-Saadany, "Optimal allocation of ESS in distribution systems with a high penetration of wind energy," *IEEE Trans. on Power Sys.*, vol. 25, no. 4, pp. 1815–1822, Nov. 2010.

[4] H. Xu, U. Topcu, S. Low, C. Clarke, and K. Chandy, "Load-shedding probabilities with hybrid renewable power generation and energy storage," in *Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on*. IEEE, 2010, pp. 233–239.

[5] C. Budischak, D. Sewell, H. Thomson, L. Mach, D. Veron, and W. Kempton, "Cost-minimized combinations of wind power, solar power and electrochemical storage, powering the grid up to 99.9% of the time," *Journal of Power Sources*, 2012.

[6] J. Eyer and G. Corey, "Energy storage for the electricity grid: Benefits and market potential assessment guide," *Sandia Nat. Lab.*, 2010.

[7] I. C. Paschalidis, B. Li, and M. C. Caramanis, "A market-based mechanism for providing demand-side regulation service reserves," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 21–26.

[8] M. Caramanis, E. Goldis, P. Ruiz, and A. Rudkevich, "Power market reform in the presence of flexible schedulable distributed loads. new bid rules, equilibrium and tractability issues," in *Communication, Control, and Computing (Allerton), 2012 50th Annual Allerton Conference on*, 2012, pp. 1089–1096.

[9] L. Chen, N. Li, L. Jiang, and S. H. Low, "Optimal demand response: Problem formulation and deterministic case," in *Control and Optimization Methods for Electric Smart Grids*. Springer, 2012, pp. 63–85.

[10] S. Chu and A. Majumdar, "Opportunities and challenges for a sustainable energy future," *Nature*, vol. 488, no. 7411, pp. 294–303, 2012.

[11] "Electrical energy storage," *IEC WHITE PAPER EES ed1.0*, 2011.

[12] I. Koutsopoulos, V. Hatzi, and L. Tassiulas, "Optimal energy storage control policies for the smart power grid," in *Smart Grid Communications (SmartGridComm), 2011 IEEE International Conference on*. IEEE, 2011, pp. 475–480.

[13] H. Su and A. Gamal, "Modeling and analysis of the role of fast-response energy storage in the smart grid," *arXiv preprint arXiv:1109.3841*, 2011.

[14] Y. Kanoria, A. Montanari, D. Tse, and B. Zhang, "Distributed storage for intermittent energy sources: Control design and performance limits," in *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*. IEEE, 2011, pp. 1310–1317.

[15] D. Gayme and U. Topcu, "Optimal power flow with large-scale energy storage integration," *IEEE Trans. on Power Sys., to appear*, 2012.

[16] M. Chandy, S. Low, U. Topcu, and H. Xu, "A simple optimal power flow model with energy storage," in *Proc. of Conf. on Decision and Ctrl.*, 2010.

[17] M. Kraning, Y. Wang, E. Akuiyibo, and S. Boyd, "Operation and configuration of a storage portfolio via convex optimization," in *Proc. of the IFAC World Congress*, 2010, pp. 10 487–10 492.

[18] P. Denholm and R. Sioshansi, "The value of compressed air energy storage with wind in transmission-constrained electric power systems," *Energy Policy*, vol. 37, pp. 3149–3158, 2009.

[19] P. Harsha and M. Dahleh, "Optimal management and sizing of energy storage under dynamic pricing for the efficient integration of renewable energy," in preparation.

[20] Univ. of Washington, "Power systems test case archive." [Online]. Available: http://www.ee.washington.edu/research/pstca/

[21] A. E. Sjödin, D. Gayme, and U. Topcu, "Risk-mitigated optimal power flow for wind powered grids," in *Proc. of the American Ctrl. Conf.*, 2012.

[22] S. Bose, F. Gayme, U. Topcu, and K. Chandy, "Optimal placement of energy storage in the grid," in *Decision and Control (CDC), 2012 51st IEEE Conference on*.

[23] K. Purchala, L. Meeus, D. Van Dommelen, and R. Belmans, "Usefulness of dc power flow for active power flow analysis," in *Power Engineering Society General Meeting*. IEEE, 2005, pp. 454–459.

[24] K. S. Pandya and S. K. Joshi, "A survey of optimal power flow methods," *J. of Theoretical and App. Info. Tech.*, vol. 4, no. 5, pp. 450–458, 2008.

[25] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite programming for optimal power flow problems," *Int'l J. of Electrical Power & Energy Sys.*, vol. 30, no. 6-7, pp. 383–392, 2008.

[26] J. Lavaei and S. Low, "Zero duality gap in optimal power flow problem," *IEEE Trans. on Power Sys.*, vol. 27, Feb. 2012.

[27] S. Bose, D. Gayme, S. Low, and K. Chandy, "Quadratically constrained quadratic programs on acyclic graphs with application to power flow," *To appear in IEEE Transactions on Automatic Control*, 2012.

[28] C. Thampoulidis, S. Bose, and B. Hassibi, "Optimal large-scale storage placement in single generator single load networks," in *Power Engineering Society General Meeting*. IEEE, 2013.

[29] M. Korpaas, A. T. Holen, and R. Hildrum, "Operation and sizing of energy storage for wind power plants in a market system," *International Journal of Electrical Power & Energy Systems*, vol. 25, no. 8, pp. 599–606, 2003.

[30] Hourly load data. [Online]. Available: http://www.pjm.com/markets-and-operations/energy/real-time/loadhryr.aspx

[31] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge Univ. Press, 2004.

[32] L. Gaillac, J. Castaneda, A. Edris, D. Elizondo, C. Wilkins, C. Vartanian, and D. Mendelsohn, "Tehachapi wind energy storage project: Description of operational uses, system components, and testing plans," in *Transmission and Distribution Conference and Exposition (T D), 2012 IEEE PES*, may 2012, pp. 1 –6.

[33] IEEE Distribution Test Feeders. [Online]. Available: http://www.ewh.ieee.org/soc/pes/dsacom/testfeeders/index.html

[34] C. Thampoulidis, S. Bose, and B. Hassibi, "Optimal placement of distributed energy storage in power networks," *arXiv preprint arXiv:1303.5805*, 2013.