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Output Consensus Control for Heterogeneous Multi-Agent Systems

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***Abstract*— We study distributed output feedback control of a heterogeneous multi-agent system (MAS), consisting of N different continuous-time dynamic systems. Conditions on the existence of distributed output feedback control protocols are derived and a solution is proposed to synthesize the stabilizing and consensus control protocols over directed graphs. The results in this paper complement the existing ones, and are illustrated by a numerical example.**

I. Introduction

In the last decade, research on cooperative MASs has intensified mainly due to the wide variety of applications that make use of the MAS framework, cf. [7], [16] and references therein. Early work, among others, includes [5] where the agents' dynamics are represented by linear switched systems, and [15], [17] where agents consist of a scalar integrator. More recently the majority of existing results have been concerned with homogeneous agents represented by a state-space description [9], [13], [20]. Such MASs are more general and include integrator dynamics as a special case. The results in these papers and others solve the problems of designing distributed and local control protocols for state feedback and state estimation. More importantly the separation principle holds, and thus the solutions to cooperative consensus control based on output feedback are also available.

Motivated by the recent developments, we study control of heterogeneous MASs which is in general more difficult than that of homogeneous MASs. Although there are some new results reported in [6], [12], [18], how to design distributed and local control protocols to achieve not only feedback stability but also output consensus remains a significant challenge. This paper is aimed at developing a general method for consensus control and deriving the consensusability condition for heterogeneous MASs. It will be shown that parallel results for homogeneous MASs in [9], [13], [20] are also true for heterogeneous MASs. More importantly it will be shown that the existing design methods, such as linear quadratic control (LQG) [1] and $\mathcal{H}_\infty$ loop shaping [14], developed for multi-input/multi-output (MIMO) feedback control systems can be employed to synthesize consensus controllers for heterogeneous MASs.

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The notation in this paper is more or less standard. Denote $\mathbb{R}^N$ as the N-dimensional real space. The vector $1_N \in \mathbb{R}^N$ has all its entries equal to one. Let $A = [\ a_{ij}\ ]$ be a matrix with a_{ij} the (i,j)th entry. Its i^{th} singular value is denoted by $\sigma_i(A)$ arranged in descending order with $\overline{\sigma}(A) = \sigma_1(A)$. For square A, its i^{th} eigenvalue is denoted by $\lambda_i(A)$. The real square A is called *row dominant* if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$, *column dominant* if $|a_{jj}| \geq \sum_{i\neq j} |a_{ij}|$, and doubly dominant if it is both row and column dominant. If the inequalities are strict then one calls such matrices strictly row or column or doubly dominant. The rest of the notation will be made clear as we proceed.

II. Preliminaries

This section prepares the results in later sections by reviewing some graph theory and formulating the consensus problem.

A. Graph and Its Associated Matrices

We focus on directed graphs (digraphs), although our results also hold for MASs over the undirected feedback graph. Consider a weighted digraph specified by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_i\}_{i=1}^N$ is the set of nodes and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges, where an edge starting at node i and ending at node j is denoted by $(v_i, v_j) \in \mathcal{E}$. The node index set is denoted by $\mathcal{N} = \{1, \cdots, N\}$. The neighborhood of node i is denoted by the set $\mathcal{N}_i = \{j \mid (v_j, v_i) \in \mathcal{E}\}$. A path on the digraph is an ordered set of distinct nodes $\{v_{i_1}, \cdots, v_{i_K}\}$ such that $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}$. If there is a path in $\mathcal{G}$ from node v_i to node v_j, then v_j is said to be reachable from v_i, denoted as $v_i \to v_j$. The digraph is called strongly connected if $v_i \to v_j$ and $v_j \to v_i\ \forall\ i,j \in \mathcal{N}$. The set of descendants of node v_k is denoted as $\mathcal{S}_k = \{v_j \in \mathcal{V} :\ \exists$ a path $v_k \to v_j\}$. The digraph is called connected if there exists a node v_k such that $v_j \in \mathcal{S}_k$ for $j = 1, \ldots, N,\ j \neq k$.

Let $\mathcal{A} = [\ a_{ij}\ ] \in \mathbb{R}^{N\times N}$ be weighted adjacency matrix. The value of $a_{ij} \geq 0$ represents the coupling strength of edge (v_i, v_j). Self edges are not allowed, i.e., $a_{ii} = 0\ \forall\ i \in \mathcal{N}$. Denote the degree matrix for $\mathcal{A}$ by $\mathcal{D} = \operatorname{diag}\{\deg_1, \cdots, \deg_N\}$ with $\deg_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ and the Laplacian matrix as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. It is clear that $\mathcal{L}1_N = 0$ and thus it has at least one zero eigenvalue. It is also known that $\operatorname{Re}\{\lambda_i(\mathcal{L})\} \geq 0\ \forall\ i$. In fact the only eigenvalues of the Laplacian matrix on the imaginary axis are zero in light of the Gershgorin circle theorem. In addition zero is a simple eigenvalue of $\mathcal{L}$, if $\mathcal{G}$ is a connected digraph [11].

B. Problem Formulation

We consider N heterogeneous agents with the dynamics of the ith agent described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^m$ is the input, and $y_i(t) \in \mathbb{R}^p$ is the output. Thus the ith agent admits transfer matrix $P_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i$ with I_n the $n \times n$ identity matrix. Note that the state dimension n_i can be different from each other. However, all agents have the same number of inputs and outputs.

For heterogenous MASs, the consensus problem is concerned with the agents' outputs and requires that

$$\lim_{t\to\infty} [y_i(t) - y_j(t)] = 0 \;\; \forall\, i, j \;\in \mathcal{N}. \tag{2}$$

In studying output consensus, tracking performance is often taken into account [20]. In particular, the N outputs of the MAS are required to track the output of some exosystem or reference model described by

$$\dot{x}_0(t) = A_0 x_0(t), \quad y_0(t) = C_0 x_0(t) \tag{3}$$

with zero steady-state error. We restrict the eigenvalues of A_0 to lie on the imaginary axis. To reduce the communication overhead, the reference signal $r(t) = y_0(t)$ is often transmitted to only one or a few of the N agents, introducing the concept of pinning control [10]. Whether or not pinning control is feasible for achieving output consensus is clearly hinged on the underlying feedback graph.

Assume that realizations of N agents are all stabilizable and detectable. In this paper we will study under what condition for the feedback graph, there exist distributed stabilizing controllers and consensus control protocols. Moreover we will study how to synthesize the required distributed and local controllers in order to achieve output consensus, taking performance into account.

We conclude this section with a useful fact.

Fact 1: Let $X \geq 0$ be the stabilizing solution to the following algebraic Riccati equation (ARE)

$$A'X + XA' - XBB'X + Q = 0, \quad Q \geq 0. \tag{4}$$

Then with $F = B'X$, the transfer matrix

$$T_F(s) = F(sI - A + BF)^{-1}B$$

is positive real (PR) [1] (page 106), i.e., $T_F(s)^* + T_F(s) \geq 0$ for all Re$[s] \geq 0$ with superscript * conjugate transpose.

III. Distributed Stabilization

Let $e_i \in \mathbb{R}^N$ be a vector with 1 in the ith entry and zeros elsewhere. We first state the following lemma that is instrumental to the main results of this paper.

Lemma 1 *Let $\mathcal{L}$ be the Laplacian associated with the directed graph $\mathcal{G}$. There exist diagonal matrices $D > 0$ and $G \geq 0$, with rank of G equal to 1, such that*

$$(D\mathcal{L} + G) + (D\mathcal{L} + G)' > 0, \tag{5}$$

if and only if $\mathcal{G}$ is connected, and

$$\text{rank}\left\{\begin{bmatrix} \mathcal{L} & e_i \\ -e_i' & 0 \end{bmatrix}\right\} = N + 1 \tag{6}$$

for at least one index $i \in \mathcal{N} := \{1, 2, \cdots, N\}$.

Proof: We omit the proof due to space limitations. □

Although Lemma 1 considers only the directed graph, the result holds for the undirected graph with a similar and much simpler proof. The connectivity of the feedback graph has been reported in several papers to ensure consensusability [2], [5], whereas the condition (6) is new. In practice N is large, and hence condition (6) holds generically for some $i \in \mathcal{N}$.

Remark 1 (i) If the conditions in Lemma 1 hold, then there exist a diagonal matrix $D > 0$ and a rank 1 diagonal matrix $G \geq 0$ such that

$$(D\mathcal{L} + G) + (D\mathcal{L} + G)' > 2\kappa I \tag{7}$$

for some $\kappa > 0$. Efficient algorithms for linear matrix inequality (LMI) can be used to search for D and G. In fact $G = g_i e_i e_i'$ with $g_i > 0$ for those i satisfying (6) can be taken. Hence computation of the required D and G in Lemma 1 is not an issue. (ii) For MIMO agents with m-input/p-output, a commonly adopted graph has the weighted adjacency matrix in form of $\mathcal{A} = \{a_{ij} I_q\}$ with $q = m$ or $q = p$. Thus

$$D = \text{diag}(d_1 I_q, \cdots, d_N I_q),\; G = \text{diag}(g_1 I_q, \cdots, g_N I_q). \tag{8}$$

In this case Lemma 1 holds true, and (6) is extended to

$$\text{rank}\left\{\begin{bmatrix} \mathcal{L} & I_q \otimes e_i \\ -I_q \otimes e_i' & 0 \end{bmatrix}\right\} = (N + 1)q. \tag{9}$$

□

A. State Feedback

We now focus on the distributed control protocol over the connected graph $\mathcal{G}$, represented by its Laplacian matrix $\mathcal{L}$. Distributed stabilization will be studied for the case of state feedback, and the stabilizability condition will be derived. Consider the control protocol for the ith agent given by (recall $r(t) = C_0 x_0(t)$)

$$u_i(t) = g_i(r - F_i x_i) - d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) \quad (10)$$

with $d_i > 0$, $g_i \geq 0$, and F_i the state feedback gain for the state vector of agent i where $1 \leq i \leq N$. Basically the control signal for the ith agent consists of relative information, i.e., its error signals with respect to the neighboring agents, plus its tracking error with respect to the reference signal. In order to minimize the communication overhead, only one of $\{g_i\}_{i=1}^{n}$ is nonzero that is why condition (6) in Lemma 1 becomes useful. It is argued that even in the case when more than one $g_i \neq 0$ is used, the condition (6) should also be enforced in order to provide stability and consensus robustness. Substituting (10) into (1) yields

$$\dot{x}_i = A_i x_i - B_i d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) - B_i g_i (F_i x_i - r).$$

Let D and G be in (8) with $q = m$, and $\mathcal{L}$ be the corresponding Laplacian matrix as in (ii) of Remark 1. By denoting $x(t)$ as the collective state, i.e., the stacked vector of $\{x_i(t)\}_{i=1}^{N}$, the collective closed loop dynamics are now described by

$$\dot{x} = [\underline{A} - \underline{B}(D\mathcal{L} + G)\underline{F}]\, x + \underline{B}G\, [1_N \otimes r] \quad (11)$$

where $\underline{A} = \text{diag}(A_1, \cdots, A_N)$, $\underline{B} = \text{diag}(B_1, \cdots, B_N)$, and $\underline{F} = \text{diag}(F_1, \cdots, F_N)$. Since $\mathcal{L}(1_N \otimes v) = 0 \;\forall\, v \in \mathbb{R}^m$, (11) is equivalent to

$$\dot{x} = [\underline{A} - \underline{B}\mathcal{M}\underline{F}]\, x + \underline{B}\mathcal{M}\, [1_N \otimes r] \quad (12)$$

where $\mathcal{M} := D\mathcal{L} + G$. The following result is concerned with stabilization for the underlying MAS under state feedback.

Theorem 1 *Suppose that (A_i, B_i) is stabilizable $\forall\, i \in \mathcal{N}$. There exists a stabilizing state feedback control protocol* (10) *for the underlying* MAS *over the directed graph $\mathcal{G}$, if $\mathcal{G}$ is connected, and the condition* (9) *holds for $q = m$.*

Proof: If $\mathcal{G}$ is connected and (9) holds for $q = m$, then Lemma 1 and Remark 1 imply the existence of a diagonal $G \geq 0$ with only one of $\{g_i\}_{i=1}^{N}$ nonzero and a diagonal $D > 0$ such that the inequality (5) holds. In fact (7) holds for some $\kappa > 0$. Thus $Z + Z' > 0$ by taking $Z = (D\mathcal{L} + G)/\kappa - I$, i.e., $(D\mathcal{L} + G) = \kappa(Z + I)$. Feedback stability of the underlying MAS requires that

$$\det\,[sI - \underline{A} + \underline{B}(D\mathcal{L} + G)\underline{F}] \neq 0 \quad \forall\; \text{Re}\{s\} \geq 0. \quad (13)$$

Substituting $(D\mathcal{L} + G) = \kappa(Z + I)$ into the above inequality yields

$$\det\,(sI - \underline{A} + \underline{B}_\kappa \underline{F} + \underline{B}_\kappa Z \underline{F}) \neq 0 \quad \forall\; \text{Re}\{s\} \geq 0$$

where $\underline{B}_\kappa = \kappa \underline{B}$. Simple manipulation shows the equivalence of the above inequality to

$$\det\,[I + T(s)Z] \neq 0 \;\forall\; \text{Re}\{s\} \geq 0 \quad (14)$$

where $T(s) = \underline{F}(sI - \underline{A} + \underline{B}_\kappa \underline{F})^{-1}\underline{B}_\kappa$. Stabilizability of (A_i, B_i) or equivalently $(A_i, B_{i\kappa})$ with $B_{i\kappa} = \kappa B_i$ assures the existence of a stabilizing state feedback control gain F_i such that

$$T_i(s) = F_i(sI - A_i + B_{i\kappa}F_i)^{-1}B_{i\kappa}$$

is PR for all $i \in \mathcal{N}$. Recall Fact 1 in Section 2. It follows that

$$T(s) = \text{diag}\,\{T_1(s), \ldots, T_N(s)\}$$

is PR. That is, $T(s) + T(s)^* \geq 0 \;\forall\; \text{Re}\{s\} \geq 0$. Thus the inequality (14) holds by $(Z + Z') > 0$ □

Remark 2 Theorem 1 provides a sufficient condition for stabilizabiity under the distributed state feedback control. This sufficient condition becomes necessary if we require the existence of the distributed stabilizing control law for each MAS with stabilizable $(A_i, B_i) \;\forall\; i \in \mathcal{N}$. Indeed heterogeneous MASs include homogeneous MASs as a special case, implying $\underline{F}(sI - \underline{A})^{-1}\underline{B} = I_N \otimes F(sI - A)^{-1}B$ where $(A_i, B_i, F_i) = (A, B, F) \;\forall\; i \in \mathcal{N}$. Hence if either the graph is not connected or (9) does not hold, then there does not exist diagonal $D > 0$ and $G \geq 0$ with only one of $\{g_i\}_{i=1}^{N}$ nonzero such that inequality (5) is true. That is, $\det(D\mathcal{L} + G) = 0$. Using the same procedure as in [9], [17], the feedback stability condition in (13) can be shown to be the same as

$$\det[I + F(sI - A)^{-1}B\lambda_i(D\mathcal{L} + G)] \neq 0 \;\forall\; \text{Re}[s] \geq 0.$$

The above indicates that the internal stability of the underlying feedback MAS cannot be true if A has unstable eigenvalues. The reason lies in the fact that $\lambda_i(D\mathcal{L} + G) = 0$ for some i, implying that some subsystem of the feedback MAS is not stabilizable. □

B. Output Feedback

When the states of the MAS are not available for feedback, a distributed observer can be designed to estimate the state of each agent, which can then be used for feedback control. See [9], [20] for homogeneous MASs. We will modify some of the distributed observers in [20] for design of distributed output feedback controllers in the case of heterogeneous MASs.

Our first distributed observer has the local form

$$\begin{aligned}\dot{\hat{x}}_i &= A_i \hat{x}_i + B_i u_i - L_i(\hat{y}_i - y_i) \\ &= (A_i - L_i C_i)\hat{x}_i + L_i C_i x_i + B_i u_i \end{aligned} \quad (15)$$

for each i. Since $\dot{x}_i = A_i x_i + B_i u_i$, taking the difference of the two leads to

$$\dot{e}_{x_i} = (A_i - L_i C_i) e_{x_i}, \quad e_{x_i} = x_i - \hat{x}_i.$$

By packing $\{e_{x_i}\}_{i=1}^N$ into a single vector e_x, we obtain $\dot{e}_x = (\underline{A} - \underline{LC}) e_x$ with $\underline{L} = \mathrm{diag}(L_1, \cdots, L_N)$. Using the estimated states $\{\hat{x}_i\}$ for the control input in (10) leads to

$$u_i(t) = g_i[\hat{r} - F_i \hat{x}_i] - d_i \sum_{j=1}^N a_{ij}(F_i \hat{x}_i - F_j \hat{x}_j) \quad (16)$$

with $\hat{r}(t) = C_0 \hat{x}_0(t)$. Substituted into $\dot{x} = \underline{A}x + \underline{B}u$ yields

$$\dot{x} = [\underline{A} - \underline{B}\mathcal{M}\underline{F}]\, x + \underline{B}\mathcal{M}\underline{F} e_x + \underline{B}\mathcal{M}\,(1_N \otimes C_0 \hat{x}_0) \quad (17)$$

by $\mathcal{M} = D\mathcal{L} + G$. Hence the overall MAS satisfies the state space equation

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} \underline{A} - \underline{B}\mathcal{M}\underline{F} & \underline{B}\mathcal{M}\underline{F} \\ 0 & \underline{A} - \underline{LC} \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} \underline{B}\mathcal{M} \\ 0 \end{bmatrix} (1_N \otimes r) \quad (18)$$

It follows that the internal stability holds, if and only if $[\underline{A} - \underline{B}\mathcal{M}\underline{F}]$ and $(\underline{A} - \underline{LC})$ are both Hurwitz.

A drawback for the local form in (15) and also for the results reported in [18] lies in the use of $\{y_i(t)\}$. For applications to vehicle formation, $y_i(t)$ often represents the absolute position of the ith agent and requires a GPS signal which may not be available to all agents for feedback except the one with $g_i \neq 0$. Instead the relative positions are available from and to the neighboring vehicles, which motivates the second observer, termed neighborhood observer [20]. We propose the following observer for heterogeneous MASs as follows:

$$\begin{aligned} \dot{\hat{x}}_i &= A_i \hat{x}_i + B u_i + g_i[L_i C_i(\hat{x}_i - x_i) - L_0 C_0(\hat{x}_0 - x_0)] \\ &\quad + d_i L_i \sum_{j=1}^N a_{ij}[C_i(\hat{x}_i - x_i) - C_j(\hat{x}_j - x_j)]\ \forall\ i \in \mathcal{N}. \end{aligned} \quad (19)$$

The above observer employs the error signals with respect to neighbors of the ith agent plus the reference signal, and thus can be more preferred in some applications. Taking the difference from $\dot{x}_i = A_i x_i + B_i u_i$ leads to

$$\begin{aligned} \dot{e}_{x_i} &= A_i e_{x_i} - d_i L_i \sum_{j=1}^N a_{ij}(C_i e_{x_i} - C_j e_{x_j}) \\ &\quad - g_i L_i C_i e_{x_i} + g_i L_0 C_0 e_{x_0} \end{aligned}$$

which results in the collective error dynamics

$$\begin{aligned} \dot{e}_x &= [\underline{A} - \underline{L}\,(D\mathcal{L} + G)\,\underline{C}]\, e_x \\ &\quad + (I_N \otimes L_0)\mathcal{M}\,(1_N \otimes C_0 e_{x_0}) \end{aligned}$$

with $\mathcal{M} = (D\mathcal{L} + G)$. Denote $\underline{L}_0 = I_N \otimes L_0$. In connection with (17), the overall MAS has the state space equation

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} \underline{A} - \underline{B}\mathcal{M}\underline{F} & \underline{B}\mathcal{M}\underline{F} \\ 0 & \underline{A} - \underline{L}\mathcal{M}\underline{C} \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} \underline{B}\mathcal{M}\,(1_N \otimes C_0 \hat{x}_0) \\ \underline{L}_0 \mathcal{M}\,(1_N \otimes C_0 e_0) \end{bmatrix}. \quad (20)$$

For both local and neighborhood observers, the separation principle holds true as manifested in collective dynamics (18) and (20). Hence we have the following result for which we omit the proof.

Theorem 2 *Suppose that (A_i, B_i, C_i) is both stabilizable and detectable for all $i \in \mathcal{N}$. Then there exist distributed output feedback stabilizing controllers for the underlying heterogeneous MAS, if the feedback graph is connected and (9) holds for $q = p$ and $q = m$.*

We point out that the observer design requires estimation of $x_0(t)$ as well. In addition the state estimation gain L_i is required not only to be stabilizing but also satisfy the PR property for the resulting $C_i(sI - A_i + L_i C_i)^{-1} L_i$ which is dual to the state feedback case. Consequently $e_x(t) \to 0$ as $t \to \infty$ or all $i \in \mathcal{N}$. Hence the observer-based control results in the same closed-loop transfer matrix in steady-state. Many known output feedback controllers are observer based and satisfy the required PR property, these include those designed using $\mathcal{H}_\infty$ loop shaping and LQG/LTR methods.

We conclude this section with the existence of static output stabilizing control law for heterogeneous MASs.

Corollary 1 *Suppose that (A_i, B_i, C_i) is both stabilizable and detectable satisfying $\det(C_i B_i) \neq 0$ and (A_i, B_i, C_i) are strictly minimum phase for all $i \in \mathcal{N}$. Then there exist distributed static output stabilizing controllers for the underlying heterogeneous MAS, if the feedback graph is connected and (9) holds for $q = m$.*

Proof: We omit the proof due to space limitations. □.

IV. OUTPUT CONSENSUS

Several results exist regarding the condition for heterogeneous MASs to achieve output consensus, including [6], [18]. Our results differ from the existing work due to the absence of a local reference model at each agent and in the explicit conditions for the consensusability in terms of the connected graph and the rank condition in Lemma 1. Hence synchronization of the local reference models can be avoided and consensus control can be achieved directly using local and distributed feedback control protocols. More importantly the existing well-developed design methods such as $\mathcal{H}_\infty$ loop shaping [14] and LQG/LTR [1] can be used to synthesize the output consensus control law.

Prior to study of output consensus, we introduce a known result from [4].

Lemma 2 *Let the plant model be described by*

$$\dot{x}_{\rm a}(t) = A_{\rm a}x(t) + B_{\rm a}u_{\rm a}(t), \quad y_{\rm a}(t) = C_{\rm a}x_{\rm a}(t),$$

where $A_{\rm a} \in \mathbb{R}^{n_{\rm a}\times n_{\rm a}}$, $B_{\rm a} \in \mathbb{R}^{n_{\rm a}\times m_{\rm a}}$, and $C_{\rm a} \in \mathbb{R}^{p_{\rm a}\times n_{\rm a}}$, and the reference model be described in (3) *with $A_0 \in \mathbb{R}^{n_0\times n_0}$, $C_0 = I_p$, and $p = p_{\rm a}$. Assume that $(A_{\rm a}, B_{\rm a})$ is stabilizable and consider the control law $u_{\rm a}(t) = -F_{\rm a}x_{\rm a}(t) + F_{0\rm a}r(t)$. Then for each stabilizing state feedback gain $F_{\rm a} \in \mathbb{R}^{m_{\rm a}\times n_0}$, there exists a reference feed-forward gain $F_{0\rm a} \in \mathbb{R}^{m_{\rm a}\times p}$ such that*

$$\lim_{t\to\infty}[y_{\rm a}(t) - r(t)] = 0, \tag{21}$$

i.e., the output of the plant model tracks the reference input with zero steady-state error, if and only if

$$\text{rank}\left\{\begin{bmatrix} \lambda I - A_{\rm a} & B_{\rm a} \\ C_{\rm a} & 0 \end{bmatrix}\right\} = n + p \tag{22}$$

at $\lambda = \lambda_\ell(A_0)$ for $\ell = 1, \cdots, n_0$.

The above lemma implicitly assumes $p \leq m$. That is, the plant model $P_{\rm a}(s) = C_{\rm a}(sI - A_{\rm a})^{-1}B_{\rm a}$ is a wide or square transfer matrix. Thus the feed-forward gain $F_{0\rm a}$ is a tall or square matrix. It follows that the closed-loop transfer matrix from the reference input $r(t)$ to output $y_{\rm a}(t)$ is square and given by

$$T_{\rm a}(s) = C_{\rm a}(sI - A_{\rm a} + B_{\rm a}F_{\rm a})^{-1}B_{\rm a}F_{0\rm a}. \tag{23}$$

Computation of $F_{0\rm a}$, given a stabilizing $F_{\rm a}$, requires first computing the solution $(W_{\rm a}, U_{\rm a})$ to the equation

$$R_1\begin{bmatrix} W_{\rm a} \\ U_{\rm a} \end{bmatrix} A_0 - R_2\begin{bmatrix} W_{\rm a} \\ U_{\rm a} \end{bmatrix} = \begin{bmatrix} 0 \\ C_0 \end{bmatrix} \tag{24}$$

where $R_1 = \begin{bmatrix} I_{n_{\rm a}} & 0 \\ 0 & 0 \end{bmatrix}$ and $R_2 = \begin{bmatrix} A_{\rm a} & B_{\rm a} \\ C_{\rm a} & 0 \end{bmatrix}$, and then setting $F_{0\rm a} = U_{\rm a} - F_{\rm a}W_{\rm a}$ [4] (page 7-9). Generally $F_{0\rm a}$ has full rank, and in fact, the full rank condition can be assured if the synthesis of the stabilizing state feedback $F_{\rm a}$ admits design degrees of freedom.

It is important to observe that the tracking condition in (21) does not require that the plant model include the modes $\{\lambda_\ell(A_0)\}$ due to the existence of the feed-forward gain $F_{0\rm a}$. In practice, though, the inclusion of the modes $\{\lambda_\ell(A_0)\}$ in the plant dynamics help to improve performance of both tracking and disturbance rejection. For this reason we assume the following:

Assumption 1 Each distinct eigenvalue of $\{\lambda_\ell(A_0)\}$ is a pole of $P_i(s)$ and satisfies

$$\text{rank}\left\{\lim_{s\to\lambda_\ell(A_0)}[s - \lambda_\ell(A_0)]P_i(s)\right\} = \text{full} \ \forall\ i.$$

If Assumption 1 does not hold, then dynamic weighting functions $\{W_i(s)\}$ (having poles at the missing modes of $\{\lambda_\ell(A_i)\}$) can be employed so that the weighted plant $P_{W_i}(s) = P_i(s)W_i(s)$ satisfies Assumption 1 $\forall\ i$. In fact adding weighted dynamics such as integrators and lead/lag compensators to obtain a desired frequency shape has been a standard procedure in LQG/LTR and $\mathcal{H}_\infty$ loop shaping design methods [1], [14]. Controller design can then proceed for $P_{W_i}(s)$ and implementation of the controller needs to take $W_i(s)$ as part of the controller. Assumption 1 then results in no loss of generality, since it can always be made true. The next result provides the output consensusability condition for heterogeneous MASs in the case $p = m$.

Theorem 3 *Consider the heterogeneous* MAS *with equal number of inputs and outputs, and agent model $P_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i$ having stabilizable and detectable realization for all i. Let the reference model be described in* (3) *with $C_0 = I_p$. Under* Assumption 1*, the given* MAS *over the feedback graph $\mathcal{G}$ is output consensusable, if $\mathcal{G}$ is connected, the condition* (9) *holds for $q = p = m$, and* (22) *is true for all $\rm a = i \in \mathcal{N}$.*

Proof: We omit the proof due to space limitations. □

It needs to be reminded that the tracking performance is also influenced by the eigenvalues of the Laplacian matrix $\mathcal{L}$. How to take $\mathcal{L}$ into consideration for design of high performance feedback controllers remains a challenging issue.

V. EXAMPLE

Following [8], consider a system of N point masses moving in one spatial dimension. Dynamics are governed by

$$\begin{aligned} \dot{x}_i &= \frac{1}{m_i}u_i \\ y_i &= x_i \end{aligned}$$

for $i = 1, 2, \ldots, \ell$ and

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= C_ix_i \end{aligned}$$

for $i = \ell + 1, \ldots, N$, where

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & -f_{d_i} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix},$$

and $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The first set of dynamics represents agents whose velocity is directly controlled, while the second set represents agents that experience drag forces and whose acceleration is directly controlled. The output signal corresponds to the position of the point mass. Figure 1 shows the interconnection graph for a network of 4 agents.

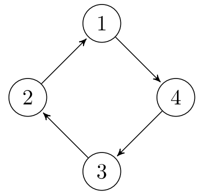

Fig. 1. Graph for N=4 point masses.

For this set of agents with parameters $\{m_i\} = \{0.5, 2, 2.5, 3\}$ and $\{f_{d_i}\} = \{0, 0, 0.5, 0.9\}$, we want to asymptotically regulate the position such that the final positions are 2, 4, 6, and 8 for agent 1, 2, 3, and 4, respectively. Figure 2 shows each agent's position in closed loop using the state feedback control signal (10) and the local observer based control signal (16). It is important to note that for both examples, the network graph is connected and that $g_i \neq 0$ only for $i = 1$, i.e., only agent 1 has direct access to $r(t)$. In addition, the rank condition (6) is satisfied with $i = 1$.

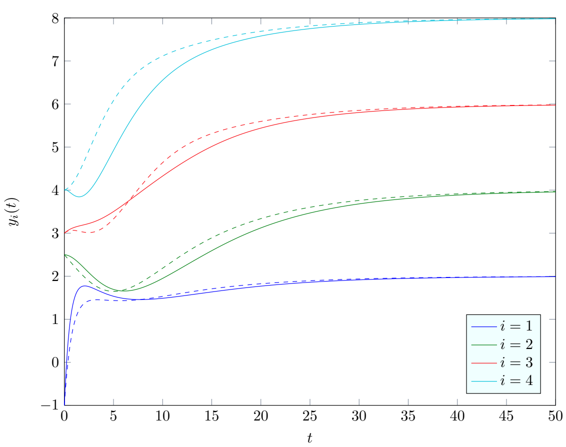

Fig. 2. Evolution of the output signals under state feedback (dashed) and local observer-based feedback (solid).

VI. Conclusion

We have shown that under some mild and reasonable conditions involving the connected feedback graph, there exist distributed stabilizing controllers and consensus control protocols for heterogeneous MASs. We have also shown how to synthesize the required distributed and local controllers in order to achieve distributed feedback stability and output consensus. The controller synthesis is based on loop shaping and loop transfer recovery methods, and therefore can accommodate performance requirements. Robustness and performance issues of the observer based-controller design, such as robustness in the presence of unmodeled plant dynamics, is currently under study.

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