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Extremum Seeking under Input Constraint for Systems with A Time-varying Extremum

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Abstract— **This paper studies extremum seeking under input constraint for static plants with a time-varying extremum. We design an online optimization scheme based on extremum seeking control to solve this problem. No explicit information on the model is needed except that the output is measurable to implement this method. A saturation function is used to constrain the input of the plant while an anti-windup mechanism is employed to eliminate the side effect of the saturation function. We show that the control input converges to a neighborhood of the time-varying extremum input which means that the proposed method is capable to track the time-varying extremum input and hence, the output is kept at the extremum value. Simulation is conducted to demonstrate the effectiveness of the proposed method.**

I. Introduction

Extremum seeking is an online optimization method which produces a control input that is capable to drive the plant to its extremum value without explicit plant information. Extremum seeking control (ESC) has been an active topic [1] since the first rigorous stability proof was provided [2]. Various application cases of ESC have been reported in the literature including pressure rise maximization in an axial--flow compressor [3], optimization of variable cam timing engine operation [4], vehicle target tracking [5], design of the anti-lock braking systems [6], network nodes mobility control [7], stabilization of roll parametric resonance in ships [8], and source tracing [9].

Due to its wide range of applications, significant research efforts have been spent to analyze and improve ESC [10]-[14]. Different from traditional assumption on the time scales of the controller and the plant, the authors in [10] provided a novel idea of approximating the performance function so that linear system analysis tools can be used in analyzing ESC. The restriction that the dynamic plant must be stable in ESC was relieved by the authors in [5]. By utilizing a sliding-mode control based ESC algorithm, nonlinear unstable systems with time-delay performance function and disturbance was studied in [11]. To get rid of predictability produced by periodic, deterministic perturbation, stochastic perturbation was employed in the ESC scheme [12], [13]. Shubert algorithm based ESC [14] opened the door to adapt sampling algorithms for ESC. Despite the attention that ESC has attracted during the past decades, most of the work emphasizes on time-invariant extremum and constant extremum variable, thus leading to the lack of study on the time-varying case which exists extensively in engineering systems. Though traditional ESC can be used for slowly time-varying trajectory tracking [5], if the variation of the extremum input and extremum output is not that slow, traditional ESC is not suitable anymore [18].

To shed light on the implementation of systems with time-varying trajectories, a class of nonlinear systems with periodic orbit are considered in [15] which is further analyzed in [16]. The authors claimed that the proposed ESC is capable to drive the plant to an optimal periodic orbit. However, since the proposed method is based on a Lyapunov optimization algorithm, some explicit information on the model is required which constrains the application of the proposed method. This issue was further addressed by the authors in [17] who considered a more general nonlinear system with a periodic steady-state output. The authors showed the improvement of the performance by utilizing the proposed ESC method. Recently, in [18], the authors provided an approach to address systems with time-varying extremum input by designing a delay-based gradient estimation subsystem and a gradient search subsystem. However, they assumed that the actuation capability is unlimited. For some engineering systems, the input should be within certain range. Motivated by this fact and by [18], in this paper, we consider systems with general time-varying extremum input and time-varying extremum output under input constraint. A saturation function is used to handle the input constraint and an anti-windup mechanism is used to eliminate the side effect of the saturation function. There exist some literatures in which anti-windup based extremum seeking methods are discussed [23]–[25]. However, the analysis on the effectiveness of the anti-windup scheme is absent in most of the work that has been done and in [23], the authors provide a proof by using the relationship between anti-windup mechanism and penalty function. In contrast to [18], [23] and the other previous works, the main contributions of this paper are twofold: 1) we designed an ESC update law that can regulate the plant to its time-varying extremum while keeping the control input within a certain required range; 2) Convergence analysis is provided. We conduct the stability analysis for the proposed scheme directly. Different from [23], we do not prove stability by linking it with penalty function.

The rest of the paper is organized as below. We formulate the problem in Section II, followed by the main results

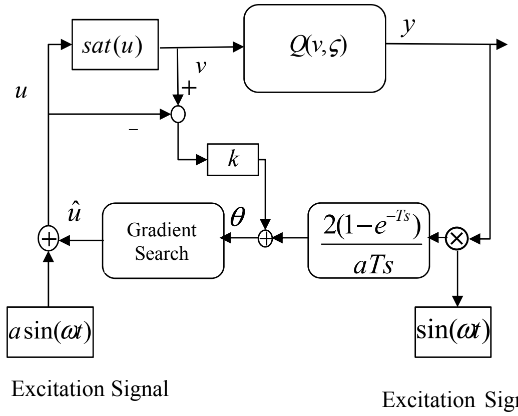

Fig. 1. Schematic of time-varying extremum seeking control loop.

in Section III in which the update law is proposed and stability analysis is shown. Simulation results are provided in Section IV to demonstrate the effectiveness of the proposed algorithm. Finally, a brief conclusion is provided in Section V.

II. Problem formulation

In this paper, we consider time-varying extremum seeking for static plants under input constraint. The problem can be formulated as below.

Problem 1: (Time-varying Extremum Seeking under Input Constraint) Consider a static plant with an output function $y = Q(u(t), \varsigma(t)) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $\varsigma(t)$ is unknown and time-varying vector, the time-varying extremum input under input constraint is defined such that

$$\begin{aligned} Q(u^*(t), \varsigma(t)) &> Q(u(t), \varsigma(t)),\ \forall\ u(t) \neq u^*(t) \\ \text{with } \varpi_1 &\leq u(t) \leq \varpi_2. \end{aligned} \quad (1)$$

The output function $Q(u(t), \varsigma(t))$ is unknown but measurable. Design a control strategy with the input within $[\varpi_1, \varpi_2]$ such that the time-varying output function converges to the unknown extremum value with the input variable at the unknown extremum input $u^*(t)$. We consider the case in which $u^*(t)$ is within $[\varpi_1 + 2\epsilon, \varpi_2 - 2\epsilon]$, where ϵ is a small positive parameter.

Remark 1: An ESC strategy as demonstrated in Fig. 1 will be introduced to solve Problem 1. The sinusoidal perturbations are used to modulate and demodulate the output function so that the gradient of the output function is extractable. Since the extremum input and extremum value of the output function are both time-varying, we employ a Robust Integral of the Sign of the Error (RISE) method [19], [20] in the update law, and meanwhile, we use a saturation function to ensure that the constraint for the control input is satisfied. The detailed control strategy will be discussed in the next section.

The main idea for the proposed extremum seeking method is to extract the gradient firstly and the approximated gradient is fed into gradient search part. The analysis is based on the following assumptions.

Assumption 1: The amplitude of the dither signal a is small.

Assumption 2: The variation of the variable $\varsigma(t)$ and the output function Q as well as the partial derivatives of the output function with respect to u are much slower and negligible when compared with the variation of the dither signal $\sin(\omega t)$.

Assumption 3: The first partial derivative $Q_u(u^*(t), \varsigma(t))$ exists and is equal to zero at the extremum input $u^*(t)$. The second partial derivative $Q_{uu}(u^*(t), \varsigma(t))$ exists at the extremum input $u^*(t)$ and is negative and,

$$\begin{aligned} Q_u(u^*(t) + \xi, \varsigma(t))\xi &< 0, \\ \text{for every } \xi &\neq 0, \end{aligned} \quad (2)$$

where $\xi \in \mathbb{R}$ is defined as

$$\xi \triangleq \hat{u}(t) - u^*(t). \quad (3)$$

Furthermore, if $|u - u^*| \geq \epsilon$, then, $|Q_u(u, \varsigma(t))| \geq \varepsilon$, for a positive ε.

Remark 2: Having the second partial derivative of the output function with respect to u is negative at the extremum input is a sufficient but not necessary condition to ensure the existence of maximum. we will comment on how this assumption can be further relaxed in the following analysis.

[21] A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

III. Main Results

Motivated by the ESC strategy in [18], we design the update law as

$$u = \hat{u} + a \sin(\omega t), \quad (4)$$

where the auxiliary signal $\hat{u}$ is generated by

$$\dot{\hat{u}} = k_1\theta + \phi \quad (5)$$

$$\dot{\phi} = k_1\alpha_1\theta + k_2 sgn(\theta). \quad (6)$$

The signal $\theta = \vartheta + k(sat_1(u) - \hat{u})$ in which ϑ is generated by

$$\vartheta = \frac{2}{a}\mu(u, \varsigma(t)) \quad (7)$$

$$\mu(u, \varsigma(t)) = \frac{1 - e^{-Ts}}{Ts}\left[Q(sat(u), \varsigma(t)) \sin(\omega t)\right]. \quad (8)$$

In the update law, $sat(u)$ is defined as

$$sat(u) = \max\{\min\{\varpi_2, u\}, \varpi_1\},$$

and
$$sat_1(u) = \begin{cases} \varpi_2 - \epsilon, \text{if } u \geq \varpi_2 - \epsilon + \frac{\delta}{2} \\ \kappa_1(u, \varpi_2, \epsilon, \delta), \text{if } \varpi_2 - \epsilon - \frac{\delta}{2} \leq u \\ \quad < \varpi_2 - \epsilon + \frac{\delta}{2} \\ u,\ \text{if } \varpi_1 + \epsilon + \frac{\delta}{2} \leq u < \varpi_2 - \epsilon - \frac{\delta}{2} \\ \kappa_2(u, \varpi_1, \epsilon, \delta), \text{if } \varpi_1 + \epsilon - \frac{\delta}{2} \leq u \\ \quad < \varpi_1 + \epsilon + \frac{\delta}{2} \\ \varpi_1 + \epsilon, \text{if } u \leq \varpi_1 + \epsilon - \frac{\delta}{2} \end{cases},$$

in which $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ are such that the gradient of $sat_1(u)$

is well defined for $u \in \mathbb{R}$, δ is an arbitrarily small positive parameter. Furthermore, the difference between $sat_1(u)$ and $\max\{\min\{\varpi_2 - \epsilon, u\}, \varpi_1 + \epsilon\}$ is much smaller than ϵ and is negligible. The parameter $T = \ell\frac{2\pi}{\omega}$, $\ell \in Z^+$ and k, k_1, k_2, α_1 are positive control gains to be determined.

Remark 3: The control laws shown in (7)-(8) are used to extract the gradient of the output function based on the modulation and demodulation of the output function by using sinusoidal dither signals. The law shown in (5)-(6) is the subsystem denoted as gradient search part which falls into the framework of RISE method. The RISE concept is in accordance with the idea of regarding the bounded terms as uncertainties. The term $k(sat_1(u) - \hat{u})$ in θ is introduced to eliminate the side effect of the saturation function on the performance of the extremum seeking controller.

Since the gradient of $Q(sat(u), \varsigma(t))$ is not well defined at the boundary of the saturation function, we can use an approximation of $Q(sat(u), \varsigma(t))$ denoted as $\hat{Q}(sat(u), \varsigma(t))$ for the subsequent analysis though the integration would smooth out the non-smooth points. The construction of $\hat{Q}(sat(u), \varsigma(t))$ is such that $Q(sat(u), \varsigma(t))$ and $\hat{Q}(sat(u), \varsigma(t))$ are only different at the $o(a^2)$ neighborhood of the boundary and $Q(sat(u), \varsigma(t)) - \hat{Q}(sat(u), \varsigma(t)) = o(a^2)$. Furthermore, Assumption 3 holds for $\hat{Q}(sat(u), \varsigma(t))$ for all $u = sat(u)$. For details of the construction of $\hat{Q}(sat(u), \varsigma(t))$, readers are referred to [23].

Lemma 1: Given that Assumptions 1 and 2 are satisfied, the signal ϑ defined in (7) can be related to the gradient of the output mapping as

$$\vartheta = \hat{Q}_u(sat(\hat{u}), \varsigma(t)) + o(a),$$

where $\hat{Q}_u(sat(\hat{u}), \varsigma(t))$ is the gradient of the approximated output function $\hat{Q}(sat(u), \varsigma(t))$ at $u = \hat{u}$.

Proof: Substituting $u = \hat{u} + a\sin(\omega t)$ into the output function $Q(sat(u), \varsigma(t))$ and employing Taylor expansion to analyze $\mu(u, \varsigma(t))$ defined in (8), we have

$$\begin{aligned} & \mu(u, \varsigma(t)) \qquad (9) \\ = \; & \frac{1 - e^{-Ts}}{Ts}\left[Q(sat(u), \varsigma(t))\sin(\omega t)\right] \\ = \; & \frac{1}{T}\int_{t-T}^{t} Q(sat(u(\tau)), \varsigma(\tau))\sin(\omega\tau)d\tau \\ = \; & \frac{1}{T}\int_{t-T}^{t} [\hat{Q}(sat(\hat{u} + a\sin(\omega\tau)), \varsigma(\tau)) \\ & + o(a^2)]\sin(\omega\tau)d\tau \\ = \; & \frac{1}{T}\int_{t-T}^{t} [\hat{Q}(sat(\hat{u}(\tau)), \varsigma(\tau)) + o(a^2) + \\ & \frac{\partial \hat{Q}}{\partial u}(sat(\hat{u}(\tau)), \varsigma(\tau))a\sin(\omega\tau)]\sin(\omega\tau)d\tau \\ = \; & \frac{a}{2}\hat{Q}_u(sat(\hat{u}), \varsigma(t)) + o(a^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \vartheta &= \frac{2}{a}\mu(u, \varsigma(t)) \qquad (10) \\ &= \hat{Q}_u(sat(\hat{u}), \varsigma(t)) + o(a). \end{aligned}$$

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Remark 4: Since

$$\vartheta = \hat{Q}_u(\hat{u}, \varsigma(t)) + o(a).$$

The update law in (5)-(6) can be written as

$$\begin{aligned} \dot{\hat{u}} &= k_1(\hat{Q}_u + o(a) + k(sat_1(u) - \hat{u})) + \phi \qquad (11) \\ \dot{\phi} &= k_1\alpha_1(\hat{Q}_u + o(a) + k(sat_1(u) - \hat{u})) \\ &\quad + k_2 sgn(\hat{Q}_u + o(a) + k(sat_1(u) - \hat{u})), \end{aligned}$$

in which $o(a)$ is introduced by gradient estimation part and it can be made arbitrarily small by choosing a small a.

Since the approximation of Q_u is such that Assumption 3 holds for all $u = sat(u)$, and for positive k, $k(sat_1(u) - \hat{u})\xi < 0$ for every $\xi \neq 0$ and $sat_1(u) \neq \hat{u}$. Hence, Assumption 3 holds for $\hat{Q}_u + k(sat_1(u) - \hat{u})$. Define,

$$e = \hat{Q}_u + k(sat_1(u) - \hat{u}) + o(a),$$

and make the following assumption to facilitate the subsequent analysis.

Assumption 4: The partial derivative of e with respect to $\hat{u}$ and ς (denoted as $\frac{\partial e}{\partial \hat{u}}, \frac{\partial e}{\partial \varsigma}$, respectively) are both independent of $\hat{u}$. The term $\frac{\partial e}{\partial \hat{u}}$ is negative and bounded, and $\frac{\partial^2 e}{\partial \varsigma^2}, \frac{\partial e}{\partial \varsigma}, \dot{\varsigma}, \ddot{\varsigma}, \dddot{\varsigma}$ as well as the time derivative of $\frac{\partial e}{\partial \hat{u}}, \frac{\partial^2 e}{\partial \varsigma^2}, \frac{\partial e}{\partial \varsigma}$ are bounded.

Theorem 1: Suppose Assumptions 1-4 hold and a is such that $\varepsilon >> o(a)$ and $\epsilon >> o(a)$, in which $o(a)$ is the gradient estimation error. Then, there exists a class $\mathcal{KL}$ function β and parameter $\tilde{v}$ such that

$$\begin{aligned} \|\hat{u}(t) - u^*(t)\| &\leq \beta(\|\hat{u}(t_0) - u^*(t)\|, t - t_0) \qquad (12) \\ &\quad + \tilde{v} \\ \forall t &\geq t_0 \geq 0, \end{aligned}$$

for any $|\hat{u}(t_0) - u^*(t_0)| \in \mathbb{R}$ by appropriately choosing the control gains.

Proof: Define

$$\eta = \dot{e} + \alpha_1 e.$$

Take the time derivative of η, we get that

$$\begin{aligned} \dot{\eta} &= \ddot{e} + \alpha_1\dot{e} \\ &= \frac{\partial^2 e}{\partial \hat{u}^2}(\dot{\hat{u}})^2 + 2\frac{\partial^2 e}{\partial \hat{u}\partial\varsigma}\dot{\varsigma}\dot{\hat{u}} + \frac{\partial^2 e}{\partial \varsigma^2}(\dot{\varsigma})^2 \\ &\quad + \frac{\partial e}{\partial \hat{u}}\ddot{\hat{u}} + \frac{\partial e}{\partial \varsigma}\ddot{\varsigma} + \alpha_1\dot{e} \\ &= \frac{\partial e}{\partial \hat{u}}(k_1(\dot{e} + \alpha_1 e) + k_2 sgn(e)) \\ &\quad + \frac{\partial e}{\partial \varsigma}\ddot{\varsigma} + \alpha_1\dot{e} + \frac{\partial^2 e}{\partial \hat{u}^2}(\dot{\hat{u}})^2 \\ &\quad + 2\frac{\partial^2 e}{\partial \hat{u}\partial\varsigma}\dot{\varsigma}\dot{\hat{u}} + \frac{\partial^2 e}{\partial \varsigma^2}(\dot{\varsigma})^2 \\ &= \frac{\partial e}{\partial \hat{u}}(k_1\eta + k_2 sgn(e) - N_c(t)) + \tilde{N}(t), \end{aligned}$$

where

$$N_c(t) = \frac{1}{\frac{\partial e}{\partial \hat{u}}}\left(-\frac{\partial e}{\partial \varsigma}\ddot{\varsigma} - \frac{\partial^2 e}{\partial \varsigma^2}(\dot{\varsigma})^2\right). \qquad (13)$$

and

$$\tilde{N}(t) = \alpha_1 \dot{e} = \alpha_1(\eta - \alpha_1 e). \quad (14)$$

By the definition of $N_c(t)$ in (13), we get that there exist constants b and c such that

$$|N_c(t)| \le b,$$

and

$$\left|\frac{dN_c(t)}{dt}\right| \le c.$$

By (14), we can get that there exists a positive d such that

$$\left|\tilde{N}(t)\right| \le d \left\|\chi\right\|, \quad (15)$$

in which $\chi = [e\ \eta]^T$.

Noticing that $\frac{\partial e}{\partial \hat{u}}$ is negative, we define $V(\mathcal{Y}, t) \in R$ as

$$V(\mathcal{Y}, t) \triangleq \frac{1}{2}e^2 - \frac{1}{2\frac{\partial e}{\partial \hat{u}}}\eta^2 + W, \quad (16)$$

where $\mathcal{Y} = [\chi^T, \sqrt{W}]^T$, and

$$W = k_2 |e(0)| - e(0)N_c(0) - \int_0^t L(\tau)d\tau \quad (17)$$

$$L(t) \triangleq \eta\left[N_c(t) - k_2 sgn(e)\right]. \quad (18)$$

Since,

$$\begin{aligned}
& \int_0^t \eta\left[N_c(\tau) - k_2 sgn(e)\right]d\tau \\
= & \int_0^t (\frac{de}{d\tau} + \alpha_1 e)(N_c(\tau) - k_2 sgn(e))d\tau \\
= & \int_0^t \alpha_1 e(N_c(\tau) - k_2 sgn(e))d\tau \\
& + \int_0^t \frac{de}{d\tau}N_c(\tau)d\tau - k_2\int_0^t \frac{de}{d\tau}sgn(e)d\tau \\
= & \int_0^t \alpha_1 e(N_c(\tau) - k_2 sgn(e))d\tau \\
& + eN_c(\tau)\left.\right|_0^t - \int_0^t e\frac{dN_c(\tau)}{d\tau}d\tau - k_2|e|\left.\right|_0^t \\
\le & \int_0^t \alpha_1|e|(|N_c| + \frac{1}{\alpha_1}\left|\frac{dN_c}{d\tau}\right| - k_2)d\tau \\
& + |e|(|N_c| - k_2) + k_2|e(0)| - e(0)N_c(0) \\
\le & \ k_2|e(0)| - e(0)N_c(0),
\end{aligned}$$

if we choose $k_2 \ge |N_c| + \frac{1}{\alpha_1}\left|\frac{dN_c}{d\tau}\right|$, we have, $W \ge 0$.

Therefore, under this condition, the Lyapunov candidate function satisfies

$$\frac{1}{2}\min\{1, \left|\frac{1}{\frac{\partial e}{\partial \hat{u}}}\right|\}\left\|\mathcal{Y}\right\|^2 \le V(\mathcal{Y}, t) \le \max\{\left|\frac{1}{2\frac{\partial e}{\partial \hat{u}}}\right|, 1\}\left\|\mathcal{Y}\right\|^2. \quad (19)$$

The time derivative of the Lyapunov candidate function

$$\begin{aligned}
\dot{V} \ = \ & e\dot{e} - \frac{1}{\frac{\partial e}{\partial \hat{u}}}\eta\dot{\eta} - \eta\left[N_c(t) - k_2 sgn(e)\right] \quad (20)\\
& + \frac{d\frac{\partial e}{\partial \hat{u}}}{dt}\frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\eta^2 \\
= \ & e(\eta - \alpha_1 e) + \frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\frac{d\frac{\partial e}{\partial \hat{u}}}{dt}\eta^2 \\
& + \eta(-k_1\eta - k_2 sgn(e) + N_c(t) - \frac{1}{\frac{\partial e}{\partial \hat{u}}}\tilde{N}(t)) \\
& - \eta\left[N_c(t) - k_2 sgn(e)\right] \\
= \ & e(\eta - \alpha_1 e) + \frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\frac{\partial\frac{\partial e}{\partial \hat{u}}}{\partial t}\eta^2 \\
& - k_1\eta^2 - \frac{\alpha_1}{\frac{\partial e}{\partial \hat{u}}}\eta^2 + \frac{\alpha_1^2}{\frac{\partial e}{\partial \hat{u}}}\eta e \\
\le \ & \frac{1}{2}(1 - \frac{\alpha_1^2}{\frac{\partial e}{\partial \hat{u}}})(e^2 + \eta^2) - \alpha_1 e^2 \\
& - (k_1 + \frac{\alpha_1}{\frac{\partial e}{\partial \hat{u}}} - \frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\frac{\partial\frac{\partial e}{\partial \hat{u}}}{\partial t})\eta^2 \\
\le \ & -(\alpha_1 - \frac{1}{2} + \frac{\alpha_1^2}{2\frac{\partial e}{\partial \hat{u}}})e^2 - (\alpha_1 - \frac{1}{2} + \frac{\alpha_1^2}{2\frac{\partial e}{\partial \hat{u}}})\eta^2 \\
& - (k_1 + \frac{\alpha_1}{\frac{\partial e}{\partial \hat{u}}} - \frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\frac{\partial\frac{\partial e}{\partial \hat{u}}}{\partial t} - \alpha_1)\eta^2 \\
\le \ & -k_x\left\|\chi\right\|^2 - k_y\eta^2 \\
\le \ & -k_x\left\|\chi\right\|^2,
\end{aligned}$$

in which $k_x = \alpha_1 - \frac{1}{2} + \frac{\alpha_1^2}{2\frac{\partial e}{\partial \hat{u}}}$, $k_y = k_1 + \frac{\alpha_1}{\frac{\partial e}{\partial \hat{u}}} - \frac{1}{2(\frac{\partial e}{\partial \hat{u}})^2}\frac{\partial\frac{\partial e}{\partial \hat{u}}}{\partial t} - \alpha_1$, choose α_1, k_1 such that $k_x, k_y > 0$.

Hence, $\|\chi\| \to 0$ as $t \to \infty$, that is to say, $\left|\hat{Q}_u + k(sat_1(u) - \hat{u})\right| \to |o(a)|$ as $t \to \infty$. Choosing a arbitrarily small means that $\left|\hat{Q}_u + k(sat_1(u) - \hat{u})\right|$ is within a small neighborhood of zero. Actually, since $\hat{Q}_u(sat_1(u) - \hat{u}) \ge 0$, it is equivalent to $\left|\hat{Q}_u\right| + |k(sat_1(u) - \hat{u})| \to |o(a)|$ with k to be positive. We discuss this result in three cases:

1). The input $u \ge \varpi_2$ or $u \le \varpi_1$. Under this condition $\hat{Q}_u = 0$ since the input u is saturated. However, in this case, $|sat_1(u) - \hat{u}| > \epsilon$. Hence, in this case, $\left|\hat{Q}_u + k(sat_1(u) - \hat{u})\right| \to |o(a)|$ is not satisfied.

2). The input $\varpi_2 - \epsilon < u < \varpi_2$ or $\varpi_1 < u < \varpi_1 + \epsilon$. In this case, $\left|\hat{Q}_u + k(sat_1(u) - \hat{u})\right|$ can not tend to be in the $o(a)$ neighborhood of zero since $|u - u^*| \ge \epsilon$ and hence, $\left|\hat{Q}_u\right| >> o(a)$.

3). The input $\varpi_1 + \epsilon \le u \le \varpi_2 - \epsilon$. For this case, $sat_1(u) - \hat{u} = 0$ and $\hat{Q}_u$ is at a small neighborhood of zero if and only if $|u - u^*|$ converges to a small neighborhood of zero.

Remark 5: Choosing the control gains in the proposed controller is a remarkable issue. We should firstly choose a k_2 to satisfy $k_2 \ge |N_c| + \frac{1}{\alpha_1}\left|\frac{dN_c}{d\tau}\right|$, and then choose large enough α_1, k_1 such that k_x, k_y are both positive. In our method, we only need to ensure that the basic requirement on the the positiveness of k_x, k_y are satisfied to achieve a global result. Actually, we can't choose arbitrarily large control gains in the proposed method since it violates Assumption 2. A good news is that the frequency of the dither signal is tunable which relaxes the restriction of Assumption 2 to some extend. For the amplitude of the dither signal, normally, we want a small a to make the ultimate convergence bound $\tilde{v}$ in (12) to be small.

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Remark 6: For Assumption 3, we assume that the first partial derivative of Q with respect to u exists and is equal to zero at the extremum input $u^*(t)$ while second partial derivative of Q with respect to u exists at the extremum input $u^*(t)$ and is negative. This condition can be relaxed by assuming that the first N partial derivative of the output function Q with respect to u exists and is equal to zero and the $(N+1)$th partial derivative of Q with respect to u exists and is negative with N an odd number. In this case, the Nth partial derivative of Q has similar property with the first partial derivative in Assumption 3. Therefore, we can feed the gradient search part with the Nth partial of Q. The method to get an approximation of Nth is to extend the gradient estimation method in (8) by defining

$$\begin{aligned} &\mu_i(u(t), \varsigma(t)) \\ = &\frac{1-e^{-Ts}}{Ts}\left[Q(sat(u(t)), \varsigma(t))\sin^i(\omega t)\right], \end{aligned}$$

where $i \in \{0, 1, 2 \cdots N\}$. By solving the N+1 equations of $\mu_i(u(t), \varsigma(t))$, $i \in \{0, 1, 2 \cdots N\}$, we can get the approximated first N partial derivatives of Q at $u = \hat{u}$ as

$$\boldsymbol{\Xi} = \Delta^{-1}A^{-1}\boldsymbol{\mu} = \mathbf{D_N}Q(sat(\hat{u}(t)), \varsigma(t)) + \mathbf{o}(\mathbf{a}),$$

where Δ, A are both $(N+1)\times(N+1)$ matrices and

$$\Delta = \begin{bmatrix} \Delta_0 & \frac{\Delta_1}{1!} & \cdots & \frac{\Delta_{N-1}}{(N-1)!} & \frac{\Delta_N}{N!} \\ \Delta_1 & \frac{\Delta_2}{1!} & & \frac{\Delta_N}{(N-1)!} & \frac{\Delta_{N+1}}{N!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_N & \frac{\Delta_{N+1}}{1!} & \cdots & \frac{\Delta_{2N-1}}{(N-1)!} & \frac{\Delta_{2N}}{N!} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^N \end{bmatrix}$$

$$\Delta_i = \frac{1}{T}\int_{t-T}^{t} \sin^i(\omega\tau)d\tau,$$

and

$$\begin{aligned} &\mathbf{D_N}Q(sat(\hat{u}(t)), \varsigma(t)) \\ = &[Q(sat(\hat{u}(t)), \varsigma(t)), \cdots, \frac{\partial^N Q}{\partial u^N}(sat(\hat{u}(t)), \varsigma(t))]^T, \end{aligned}$$

$$\boldsymbol{\mu} = [\mu_1, \mu_2, \cdots, \mu_N]^T.$$

Noticing that the idea of the gradient estimation part is equivalent with averaging method, readers are referred to *Proposition 1* of [22] for proof.

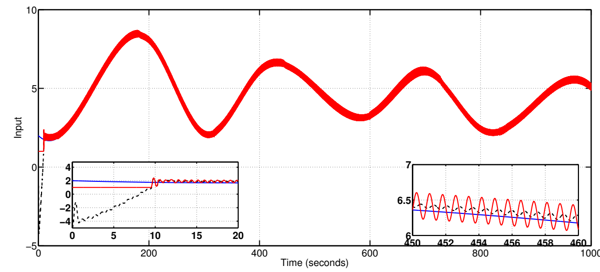

Fig. 2. The blue line denotes the actual optimal trajectory $u^*(t)$, control input $\hat{u}(t)$ produced by the proposed ESC method is shown in black dashed line and the saturated input of the output function $sat(u)$ is in red line.

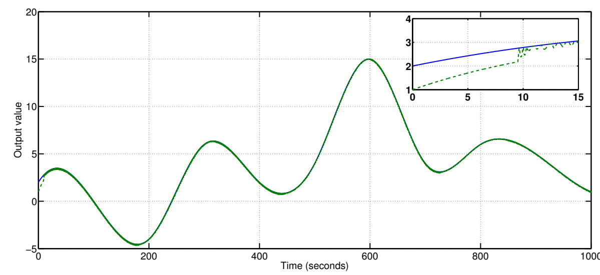

Fig. 3. The blue line denotes the actual optimal value of the output function and green line is the output value that is derived by using the proposed ESC method.

IV. Numerical Example

Consider a quadratic function

$$y = -(u - \varsigma_1(t))^2 + \varsigma_2(t), u \in [1, 9], \tag{21}$$

where $\varsigma_1(t), \varsigma_2(t)$ are time-varying functions and u is the control input. The extremum trajectory for this plant is $u = \varsigma_1(t)$ at which the output achieves the extremum $\varsigma_2(t)$. Both the extremum trajectory and extremum output of the plant are time-varying. Simulation results are shown in Fig. 2 and Fig. 3. The parameters in the simulation are selected as $a = 0.2, \omega = 8, T = \frac{2\pi}{\omega}, k_1 = 1.2, k_2 = 1, \alpha_1 = 0.6, k = 0.2$ and the initial condition is $\hat{u}(0) = -5$.

To verify the effectiveness of the anti-windup mechanism in the extremum seeking controller, we conduct an another simulation in which we get rid of the anti-windup mechanism in the control loop. The function to be maximized is the same as in (21). The parameters in this case are set as $a = 0.2, \omega = 5, T = \frac{2\pi}{\omega}, k_1 = 1.2, k_2 = 0.4, \alpha_1 = 0.6, \hat{u}(0) = -5$. The results are shown in Figs. 4-5.

From the simulation results, we see that it takes about 10 seconds for the control input produced by the proposed method to converge to the time-varying extremum input, while it takes about 35 seconds if we get rid of the anti--windup module. Hence, we conclude that the anti-windup module included in the control loop would make the performance of the proposed controller better by shortening the convergence time, i.e., it takes longer time for the estimated

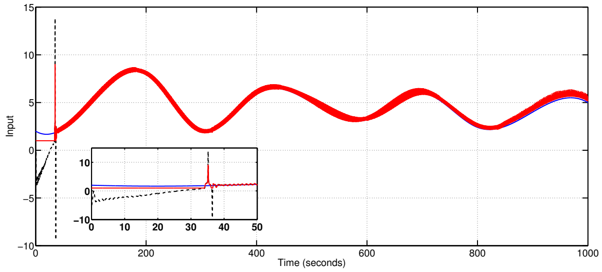

Fig. 4. The blue line denotes the actual optimal trajectory $u^*(t)$, control input $\hat{u}(t)$ produced by the proposed ESC method without the anti-windup mechanism in the control loop is shown in black dashed line and the saturated input of the output function $sat(u)$ is in red line.

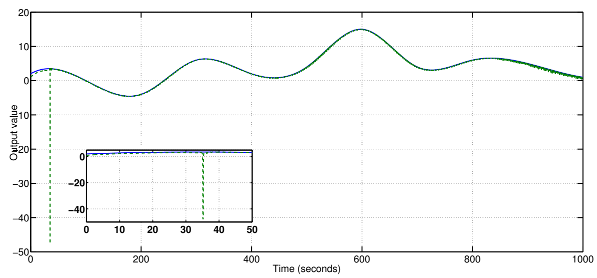

Fig. 5. The blue line denotes the actual optimal value of the output function and green line is the output value that is derived by using the proposed ESC method without the anti-windup mechanism in the control loop.

control input to converge to the extremum input if the anti-windup mechanism is deleted from the proposed method. Actually, a direct saturation function to constrain the input may even lead to the instability of the controller.

V. Conclusions

This paper studies the problem of online non-model-based optimization with input constraint. An ESC method is proposed. We prove that the proposed method enables the control input to converge to a neighborhood of the extremum. Simulation is conducted to verify the effectiveness of the proposed extremum seeking method. The input can be constrained within a required range which makes the proposed method more practical for some engineering systems where the input is not allowed to be too large or too small.

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