52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Achieving String Stability in Irrigation Channels Under Distributed Distant-Downstream Control

Laven Soltanian[1] and Michael Cantoni[2]

Abstract— **A new distributed distant-downstream controller for irrigation channels is studied. The controller ensures upstream propagation of transients, like those considered previously, but now in a spatially stable fashion. This is achieved by foregoing zero steady-state water-level errors in response to step changes in flow off-takes, via a mechanism that adjusts water-level references on the basis of downstream flow load. While structurally different, the approach is inspired by time headway schemes in vehicle platoons. A trade-off between steady-state water-level errors and the spatial attenuation of the water-flow peaks is identified.**

I. INTRODUCTION

Irrigation channels are structures used to supply water at requested flow rates to farms or secondary channels. A channel is decomposed into pools, which are smaller stretches of water bounded between neighboring gates or flow actuators. The local control objectives in each pool are the following: 1) to match the flow over the upstream gate to the local flow off-takes to farms and the downstream flow load; and 2) to keep the downstream water-levels close to reference set-points which reflect the capacity to supply flow. This can be achieved via distributed distant-downstream control schemes, [1], [2], whereby the upstream gate is manipulated on the basis of downstream water-level measurements. Flow transients are confined to spatially propagate in the upstream direction, which has merit from a distribution efficiency perspective. Although the decentralized PI (with and without feedforward of downstream flow) and H_∞ loop-shaping based distributed controllers considered in [1] are able to accomplish the local control goals just mentioned, in these cases a component of the transient water-level errors and gate flows are amplified as they propagate upstream. This paper aims to overcome this "string instability" [3], [4], [5]. There are different types of string stability considered in the literature, involving different measures of input and output size, such as L_2 or L_∞ norm of the signals in the time-domain. From an actuator saturation perspective, L_∞ string stability is of particular interest here; i.e. it is desirable for the L_∞ norms of the gate flows, in response to a bounded step off-take flow in the bottom pool, to be uniformly bounded with respect to pool index. Considering step off-takes in all pools at once does not yield a useful measure of performance since, in this case, the flow over a gate grows linearly with its distance from the bottom whenever the flow matching requirement is satisfied in steady state.

To start, this paper models an irrigation channel under fully decentralized distant-downstream PI control in a way that draws focus to the interaction between neighboring pools. While water-levels are still to be controlled, as proxies for the capacity to locally supply flow under the power of gravity alone, the focus is on flow load matching and on the identification of a control structure for achieving an attenuation of transients as they propagate spatially. Ultimately, the latter is accomplished via a feedforward-like adjustment of the local water-level reference based on the downstream flow load as set by the downstream controller. This gain comes at the cost of water-level off-sets, which equate to the exploitation of storage in the pools. The modeling style is inspired by the vehicle platoons setup and ideas of time headway, as discussed in [6], [3], [7], for example. In that context, string stability can be achieved by adjustment of the spacing reference, via a modification of the forward path of each local control loop, without changing the closed-loop characteristics. In the irrigation channel context of this paper, it is not possible to modify the local control loops in this way as it would require physical modification to the channel infrastructure. In the new set-up introduced, a feedforward-information path provides the extra degree of freedom needed to modify the interaction between neighboring pools under closed-loop control. In the recent work [8], MPC based water-level reference planning is being considered from the perspective of ensuring the satisfaction of operational constraints, given an uncertain load schedule. As such,

*Work supported by the Australian Research Council (LP0989497).
L. Soltanian and M. Cantoni are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville 3010, Australia. Emails: lavens@pgrad.unimelb.edu.au; cantoni@unimelb.edu.au

978-1-4673-5716-6/13/$31.00 ©2013 IEEE

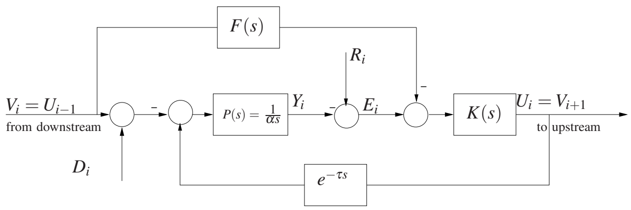

Fig. 1. Block diagram of a local control loop with a feedforward of the downstream flow

although the offsets resulting from the feedforward path gives rise to a new trade-off to be considered, the offsets present for constant references is of minor concern.

An approach to the modeling of irrigation channels under distant-downstream control is presented in section II. The impact of non-constant water-level reference on achieving string stability of the distributed control structure is analyzed in III and illustrated by some simulation results. A trade-off between steady-state water-level errors and flow transients attenuation is discussed in section IV. We conclude by a discussion of some directions for future research.

II. Modeling of Irrigation Channels Under Distant-Downstream Control

For the purpose of controller design, the dynamics of a pool are described well by the following frequency-domain model which reflects mass balance and transport delay:

$$Y_i = \frac{1}{\alpha_i s}[U_i e^{-\tau_i s} - (V_i + D_i)], \tag{1}$$

where τ_i denotes the delay, α_i is a measure of pool surface area, and Y_i, U_i, D_i, V_i denote Laplace transforms of the downstream water-level $y_i(t)$, flow over the upstream gate $u_i(t)$, the off-take flow onto farms and secondary channels $d_i(t)$, and the flow over the downstream gate $v_i(t)$ of pool i, respectively [9], [10]. Note that $U_{i-1} = V_i$ and that the pools along the channel (i.e. the path-graph interconnection of pools) are indexed from downstream towards upstream, with the bottom and top pools indexed by 0, and N, respectively. In this paper, a channel of homogeneous structure is considered, in that all the pools are of the same physical specifications (i.e. $\alpha_i = \alpha$ and $\tau_i = \tau$), and the local controllers are taken to be the same.

Given that the interconnection of the pools is via the input and output flows, Fig. 1 with $F(s) = 0$ is a natural representation of a pool model under a well studied decentralized distant-downstream controller considered in [1]. The following relationship holds in the frequency domain:

$$U_i = \frac{K(s)}{1 + K(s)\frac{e^{-\tau s}}{\alpha s}}[R_i + \frac{1}{\alpha s}(V_i + D_i)], \tag{2}$$

where the local controller

$$K(s) = \frac{\kappa(1+\phi s)}{s(1+\rho s)}$$

is PI with additional roll-off. The integrator of $K(s)$ in addition to the integrator in the pool dynamics guarantee the local control objectives are achieved, i.e. $\lim_{t\to\infty} e_i(t) := r_i - y_i(t) = 0$ and $\lim_{t\to\infty} u_i(t) = \lim_{t\to\infty}(v_i(t) + d_i(t))$ in response to step changes in flow load $d_0(t) = d_0$, where r_i is the constant local water-level reference.

The completely decentralized control architecture (i.e. $F(s) = 0$) results in amplification of components of the flow transients towards upstream end of the channel [11], [1]. This spatial instability property is due to the two integrators in the local open-loop and the fact that

$$|T_{V_i \to U_i}(j\omega)| = |\frac{L(j\omega)}{1 + L(j\omega)e^{-j\tau\omega}}| = |T(j\omega)|,$$

where $L(s) = K(s)P(s)$ is the open-loop transfer function without the delay and

$$T(s) = \frac{L(s)e^{-\tau s}}{1 + L(s)e^{-\tau s}}$$

is the complementary sensitivity function of the loop. As such, there exists a frequency ω_0 s.t. $|T(j\omega_0)| > 1$ resulting in string instability. In order to prevent amplification of water-flow transients under a distant-downstream scheme, the transfer function $T_{V_i \to U_i}(s)$ of the interaction between the pools should be modified such that $|T_{V_i \to U_i}(j\omega)| \leq 1 \ \forall\ \omega$, while still satisfying the flow matching requirement. For simplicity, by change of variable the references all considered to be zero; i.e. $r_i = r = 0$. In light of the flow load matching requirement, the consideration of off-takes in each pool does not lead to a useful definition of string stability or performance in this case [12]. As such, in the rest of this paper, the off-take that generates flow load is applied at the bottom pool.

Consider now a non-zero feedforward path, $F(s)$, as depicted in Fig. 1. This provides an extra degree of freedom for modifying the $T_{V_i \to U_i}$ characteristic of the purely decentralized scheme. The inclusion of the feedforward path yields a distributed distant-downstream control structure, which also ensures transient propagate in the upstream direction. The feedforward scheme can be viewed in terms of non-constant water-level references $R_{\text{new}} = R - F(s)V = -F(s)V$. The integrator of the local decentralized PI controller is retained to sustain capacity to supply under the power of gravity without steady-state offsets due to the local feedback action alone. The following now holds in the frequency domain:

$$U_i = \frac{L(s)}{1+L(s)e^{-\tau s}}[(1-\frac{F(s)}{P(s)})V_i + D_i]. \quad (3)$$

With

$$G(s) := \frac{L(s)}{1+L(s)e^{-\tau s}}(1-\frac{F(s)}{P(s)}), \quad (4)$$

it follows that the modified $T_{V_i \to U_i}(s) = G(s)$. Since, the local steady-state flow matching is still required under the new structure, application of the Final Value Theorem implies

$$\lim_{s\to 0} G(s) = 1 \quad (5)$$

must be satisfied. In the following, to achieve string stability the requirement of zero steady-state water-level error is relaxed. This introduces a trade-off to be more discussed in section IV. The focus in next section is synthesis of the feedforward path, $F(s)$, to achieve string stability; i.e. attenuation of transients as they propagate upstream

Before continuing, it is instructive to recall the following. A transfer function $Q(s)$ and the input-output relation $Y = Q(s)U$, with impulse response $q = \mathscr{L}^{-1}(Q)$ in the time domain, where $\mathscr{L}$ denotes the Laplace transform, is called stable if Q is analytic on $\Re(s) > -\varepsilon$ for some $\varepsilon > 0$; this is equivalent to bounded-input-bounded-output stability in the time domain [13]. For a stable transfer function Q, the following inequalities prove to be useful:

$$\|y\|_\infty \le \|q(\cdot)\|_1 \|u\|_\infty, \quad (6)$$

$$\|y\|_\infty \le \|Q(j\cdot)\|_2 \|u\|_2, \quad (7)$$

$$\|y\|_2 \le \|Q(j\cdot)\|_\infty \|u\|_2, \quad (8)$$

where $\|q(\cdot)\|_1 = \int_0^\infty |q(t)|dt$, $\|Q(j\cdot)\|_2 = (\frac{1}{2\pi}\int_{-\infty}^{\infty}|Q(j\omega)|^2 d\omega)^{\frac{1}{2}}$ is the H_2 norm of Q, $\|Q(j\cdot)\|_\infty = \sup_\omega |Q(j\omega)|$ is the H_∞ norm of Q, $\|u\|_\infty = \sup_t |u(t)|$, and $\|u\|_2 = (\int_{-\infty}^{\infty} u(t)^2 dt)^{\frac{1}{2}}$.

III. Achieving String Stability via $F(s)$

Synthesis of the feedforward path to attenuate flow transients as they propagate upstream is discussed in this section. To this end, $T_{V_i \to U_i}(s)$ is modified. Analysis corresponding to the following definition of L_∞ string stability is carried out first.

*Definition 1 (L_∞ **string stability**):* A string of pools under the distributed distant-downstream control scheme shown in Figure 1 is L_∞ string stable if there exists an $0 < M < \infty$ such that, with $d_i = 0$ for $i = 1, 2, \dots$ and d_0 bounded, $\|u_i\|_\infty \le M\|d_0\|_\infty$ for all $i = 0, 1, 2, \dots$.

In the following $F(s)$ is chosen to achieve L_∞ string stability in the sense defined above. To this end, using (3) and the spatial boundary condition $V_0 = 0$, note that the following holds:

$$U_0 = \frac{L(s)}{1+L(s)e^{-\tau s}}D_0,$$
$$U_i = G(s)V_i \text{ for all } i = 1, 2, \dots, \quad (9)$$

whereby

$$U_i = G^i(s)\frac{L(s)}{1+L(s)e^{-\tau s}}D_0 \text{ for } i = 0, 1, 2, \dots\ . \quad (10)$$

Recall that (5) must also hold. For each $i = 0, 1, 2, \dots$, u_i is bounded for a bounded input d_0 if, and only if, $T_{D_0 \to U_i}(s)$ is a stable transfer function. For this to be the case, it is clearly sufficient for $F(s)$ to be chosen such that $G(s)$ is stable. Indeed, this is also necessary as $L(s)/(1+L(s)e^{-\tau s})$ has no unstable zeros. In view of this, repeated application of (6) gives

$$\|u_i\|_\infty \le \|g\|_1^i \cdot \|\mathscr{L}^{-1}(\frac{L(s)}{1+L(s)e^{-\tau s}})\|_1 \cdot \|d_0\|_\infty. \quad (11)$$

As such, we have the following result.

Theorem 1: The distributed distant-downstream control scheme shown in Figure 1 is L_∞ string stable in the sense of Definition 1 if $G(s)$ in (4) is stable and $\|g\|_1 \le 1$.

In view of Theorem 1, ensuring satisfaction of local control objectives and L_∞ string stability reduces to deriving an $F(s)$ that gives a stable $G(s)$ which satisfies $\lim_{s\to 0} G(s) = 1$ and $\|g\|_1 \le 1$. There are many possible choices for such a $G(s)$, one of which is a constant transfer function of 1; i.e.

$$G(s) = (\frac{L(s)}{1+L(s)e^{-\tau s}})(1-\frac{F(s)}{P(s)}) = 1.$$

However, inverting over all frequencies in this way will be sensitive to uncertainties, particularly in the high frequency dynamics which have not been modeled here. An alternative is a first-order low pass filter with DC gain 1; i.e.

$$G(s) = (\frac{L(s)}{1+L(s)e^{-\tau s}})(1-\frac{F(s)}{P(s)}) = \frac{1}{1+T_c s}, \quad (12)$$

for some $T_c > 0$. The required feedforward transfer function $F(s)$ is obtained as

$$F(s) = \frac{-1}{K(s)(1+T_c s)} + P(s)(1-\frac{e^{-\tau s}}{1+T_c s}). \quad (13)$$

TABLE I

POOL MODEL AND CONTROLLER PARAMETERS

Pool number	Pool Model Parameters		Controller Parameters		
	time-delay τ (mins)	α (m^2)	κ	ρ	ϕ
0,1,2	16	43806	7.72	15.2	128

Remark 1: If $F(s)$ is chosen such that the condition of Theorem 1 is satisfied, then it follows that $\|G\|_\infty \leq 1$, since $|G(j\omega)| \leq \|g\|_1 \leq 1$ for all ω. As such, in this case, it follows by (10) and repeated application of (8) that controlled channel is also L_2 string stable in the sense that there exists a constant $0 < M < \infty$ such that $\|u_i\|_2 \leq M\|d_0\|_2$.

Simulation results presented in Fig. 2 are carried out to illustrate the impact of adding the feed-forward path $F(s)$. The last 3 pools from [1] with the specifications as in table I are used in this paper. It is assumed that an off-take of $17\mathrm{m}^3/\mathrm{min}$ is taken from the most downstream pool of the channel from 100mins to 1000mins. Flow transients attenuation is reached while having non-zero steady-state water-level errors which can be noticed comparing the top and bottom rows. Indeed, water storage of the pools are utilized to overcome amplification of transients towards upstream. The roll-off frequency of the low-pass filter selected as the interactions transfer function is an extra freedom to deal with the existing trade-off to be discussed in section IV.

$F(s)$ consists of an infinite dimensional delay component which can be realized with a Pade approximation whereas the rest are proper rational transfer functions. Fig. 4 shows the simulation result carried out for performance comparison when a first order Pade approximation of the delay is used. Comparing Fig. 2 and 4 reveals that such approximation of the delay does not deteriorate performance.

IV. TRADE-OFF BETWEEN STEADY-STATE WATER-LEVEL ERROR AND ATTENUATION RATE OF THE FLOW TRANSIENTS

As illustrated with simulations, by adjusting the water level reference to $R_{\text{new}} = R - F(s)V$ via the addition of the feedforward path to the purely decentralized PI control scheme, string stability is achieved at the cost of not necessarily zero steady-state water-level errors for step changes in flow load. In this section, we will see how the roll-off frequency of the low-pass filter as the interconnection dynamics (12), identifies a trade-off between steady-state water-level errors and the peaks in flow transients.

As shown in Fig. 2, the response of u_0 to a step change of the off-take d_0 has a peak in the transient component. Note that $u_i,\ i = 0, 1, \ldots$ is passed through the low-pass filter, $G(s)$, to produce u_{i+1}. Therefore, the lower the bandwidth of $G(s)$, larger T_c, the more attenuation of the flow peaks towards upstream of the channel. However, this trades-off with the water-level offsets of the pools as presented in the next Theorem.

Theorem 2: The automated channel shown in Fig. 1 with $F(s)$ as in (13) in response to a unit step $d_0(t)$ performs such that

$$\lim_{t\to\infty} e_0(t) = 0,$$
$$\lim_{t\to\infty} e_i(t) = \frac{T_c+\tau}{\alpha} \text{ for } i = 1, 2, \ldots \ . \tag{14}$$

Proof: The closed-loop equation of each pool, according to Fig. 1, is

$$E_i = \frac{P(s)}{1+L(s)e^{-\tau s}}[(1+K(s)F(s)e^{-\tau s})V_i + D_i].$$

Note that $v_i(t) = u_{i-1}(t) = d_0(t) + \tilde{v}_i(t)$ due to flow matching property, where $\tilde{v}_i(t)$ is the transinet component tending to zero over time. Moreover, considering $D_i = 0$ for $i = 1, 2, \ldots$ and the integrator of $K(s)$ and two integrators of $L(s)$, the steady-state water-level error of pool $i = 1, 2, \ldots$ is computed as follows

$$\begin{aligned}
\lim_{t\to\infty} e_i(t) &= \lim_{s\to 0} sE_i(s) \\
&= \lim_{s\to 0} s\frac{P(s)}{1+L(s)e^{-\tau s}}[(1+K(s)F(s)e^{-\tau s})V_i(s)] \\
&= 0 + \lim_{s\to 0} s\frac{L(s)e^{-\tau s}}{1+L(s)e^{-\tau s}}F(s)V_i(s) \\
&= \lim_{s\to 0} s\frac{L(s)e^{-\tau s}}{1+L(s)e^{-\tau s}}F(s)(\frac{d_0}{s} + \tilde{V}_i(s)) \\
&= \lim_{s\to 0} \frac{L(s)e^{-\tau s}}{1+L(s)e^{-\tau s}}F(s)d_0 \\
&= 1 \times \lim_{s\to 0} F(s)d_0 \\
&= 0 + \lim_{s\to 0} P(s)(1 - \frac{e^{-\tau s}}{1+T_c s})d_0 \\
&= \lim_{s\to 0} \frac{1}{\alpha s}\frac{1+T_c s - e^{-\tau s}}{1+T_c s}d_0 \\
&= \lim_{s\to 0} \frac{1}{\alpha}\frac{T_c + \tau e^{-\tau s}}{1+2T_c s}d_0 \\
&= \frac{T_c+\tau}{\alpha}d_0 \text{ for } i = 1, 2, \ldots,
\end{aligned} \tag{15}$$

where $\tilde{V}_i(s)$ is the Laplace transform of $\tilde{v}_i(t)$ with stable singularities and the second last inequality is derived by applying the L'Hopital's rule when $s \to 0$ along the real axis. Steady-state water-level error of the bottom pool, $e_0(t)$, is zero as the feed-forward path is not applied, $F(s) = 0$. ■

Remark 2: The steady-state water-level error off-sets are proportional to T_c and decrease with a low-pass filter of a larger bandwidth.

In other words, increasing T_c results in larger water-level errors and smaller peaks of the flow over gates.

This is illustrated via simulations in two cases $T_c = 10,\ 100$ in Fig. 3. As plotted, higher attenuation rate of the flow peaks is achieved with a low-pass filter of a smaller cut-off frequency, $\frac{1}{T_c}$, while steady-state water-level errors are increased.

Although flow transients attenuation due to flow off-take disturbances are desired, water-levels are also important as they are considered as a proxy to provide flow at supply points. Therefore, even though transients are attenuated with the non-constant level reference scheme introduced, water-level errors should be taken into consideration at steady-state since a non-zero error means water storage of pool is utilized.

V. CONCLUSIONS

Attenuation of the effect of flow disturbances propagating along an irrigation channel is gained with the non-constant water-level references considered. This shows, if the knowledge of the interaction is used in reference adjustment, performance will improve. Future research directions will be analysis of the robustness of the feed-forward scheme to uncertainties in the plant model, particularly to the water transportation delay. Also, extension of the results for a more realistic heterogenous channel is of interest. In order to accommodate for the case that off-takes are drawn at several points, linear growth of the transients along the string needs to be allowed. This is inevitable due to the steady-state flow matching requirement along the channel. This will be pursued further in future.

REFERENCES

[1] M. Cantoni, E. Weyer, Y. Li, S. Ooi, I. Mareels, and M. Ryan, "Control of large-scale irrigation networks," *in Proceedings of the IEEE*, vol. 95, pp. 75–91, 2007.

[2] X. Litrico and V. Fromion, *Modeling and control of hydrosystems*. Springer, 2009.

[3] S. Klinge and R. H. Middleton, "Time Headway Requirements for String Stability of Homogeneous Linear Unidirectionally Connected Systems," *in Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pp. 1992–1997, December 2009.

[4] L. E. Peppard, "String Stability of Relative-Motion PID Vehicle Control Systems," *IEEE Transactions on Automatic Control*, vol. 19, no. 5, pp. 579–581, 1974.

[5] S. Darbha and J. K. Hedrick, "String Stability of Interconnected Systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 349–357, 1996.

[6] S. Darbha, J. K. Hedrick, and P. A. Ioannou, "A Comparison of Spacing and Headway Control Laws for Automatically Controlled Vehicles," *Vehicle Systems Dynamic*, vol. 23, no. 1, pp. 597–625, 1994.

[7] M. Sebek and Z. Hurak, "2-D polynomial approach to control of leader following vehicular platoons," *The 18th IFAC World Congress*, pp. 6017–6022, August 28 - September 2 2011.

[8] M. Kearney and M. Cantoni, "MPC-based reference management for automated irrigation channels," *in Proceedings of the Australian Control Conference*, Nov 2012.

[9] J. Schuurmans, O. Bosgra, and R. Brouwer, "Open-channel flow model approximation for controller design," *Applied Mathematical Modelling*, vol. 19, no. 9, pp. 525–530, 1995.

[10] E. Weyer, "System identification of an open water channel," *Control engineering practice*, vol. 9, no. 12, pp. 1289–1299, 2001.

[11] Y. Li, M. Cantoni, and E. Weyer, "On Water-Level Error Propagation in Controlled Irrigation Channels," *in the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pp. 2101–2106, Dec 2005.

[12] M. Jovanovic and B. Bamieh, "On the ill-posedness of certain vehicular platoon control problems," *IEEE Transactions on Automatic Control*, vol. 50, no. 9, pp. 1307–1321, 2005.

[13] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. Dover Publications, 1992.

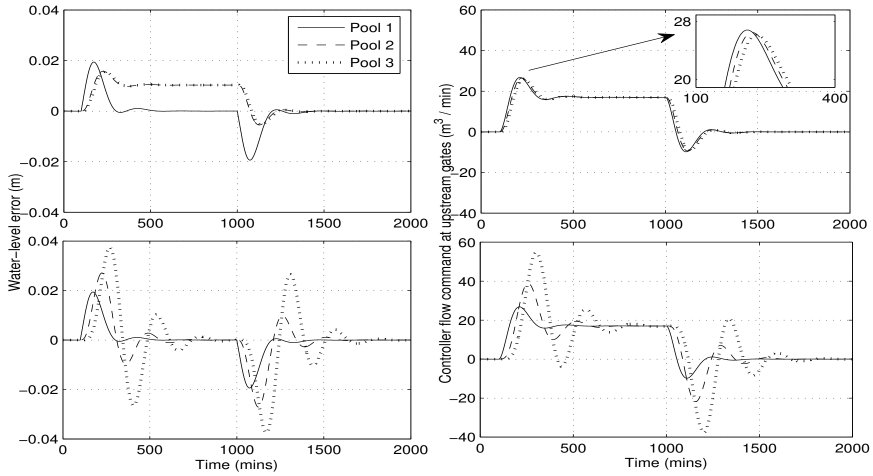

Fig. 2. Simulations with top plot: $T_{V_i \to V_{i+1}}(s) = \frac{1}{1+T_c s}$, $T_c = 10$, bottom plot: purely decentralized scheme.

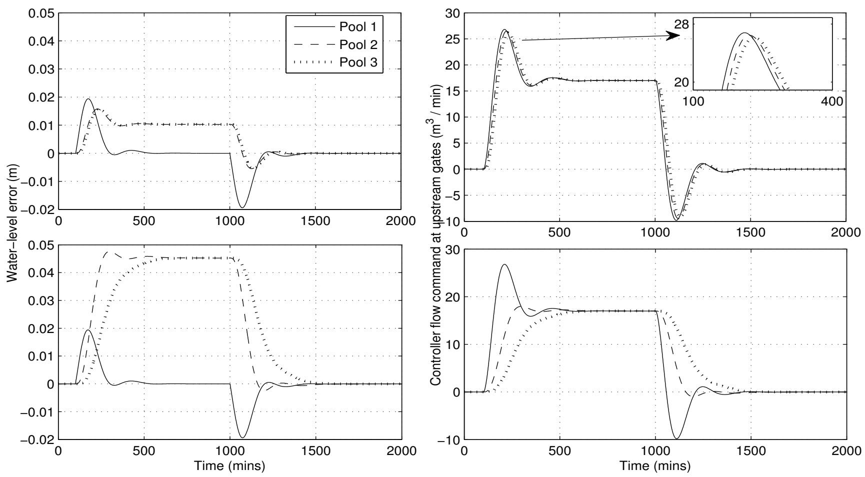

Fig. 3. Simulations with $T_{V_i \to V_{i+1}}(s) = \frac{1}{1+T_c s}$, top plot: $T_c = 10$, bottom plot: $T_c = 100$.

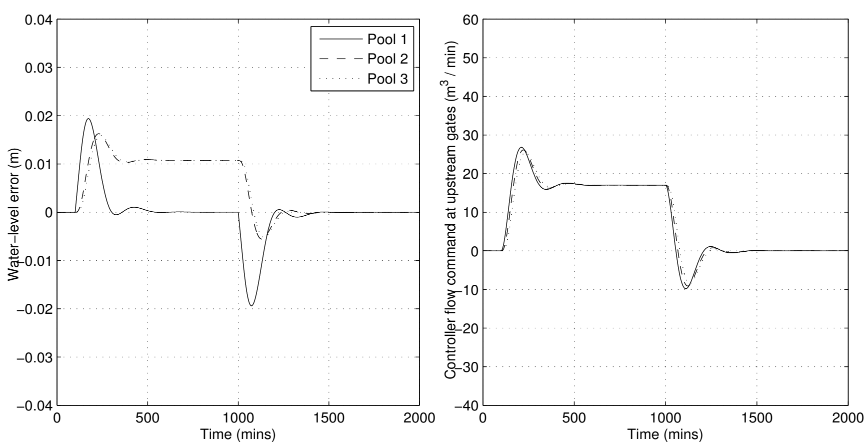

Fig. 4. Simulations with $T_{V_i \to V_{i+1}}(s) = \frac{1}{1+10s}$ and first order approximation of the delay in $F(s)$.