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Optimal Power Flow in Direct Current Networks

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Abstract— **The optimal power flow (OPF) problem seeks to control power generation/demand to optimize certain objectives such as minimizing the generation cost or power loss. Direct current (DC) networks (e.g., DC-microgrids) are promising to incorporate distributed generation. This paper focuses on the OPF problem in DC networks. The OPF problem is nonconvex, and we study solving it via a second-order cone programming (SOCP) relaxation. In particular, we prove that the SOCP relaxation is exact if there are no voltage upper bounds, and that the SOCP relaxation has at most one solution if it is exact.**

I. Introduction

The optimal power flow (OPF) problem seeks to control power generation/demand to optimize certain objectives such as minimizing the generation cost or power loss [1]. It is one of the fundamental problems in power system operation.

There is increasing interest in direct current (DC) networks, especially DC-microgrids, due to their advantage in accommodating distributed energy resources such as distributed generation, storage devices, and plug-in electric vehicles [2]. DC networks are particularly attractive when loads are DC in nature, e.g., in data centers. This paper focuses on the OPF problem in DC networks.

The OPF problem is difficult to solve due to the nonconvex power flow physical lows, and there are in general three ways to deal with this challenge: (i) linearize the power flow laws; (ii) look for local optima; and (iii) convexify the power flow laws, which are described in turn.

Power flow laws can be approximated by some linear equations in transmission networks, and then the OPF problem reduces to a linear program [3]–[5]. This method is widely used in practice and often quite effective, but there is no guarantee that a feasible solution be found.

Various algorithms have been proposed to find a local optimum of the OPF problem, e.g., successive linear/quadratic programming [6], trust-region based methods [7], [8], Lagrangian Newton method [9], and interior-point methods [10]–[12]. Some of these algorithms, especially Newton-Ralphson based, are quite successful empirically. But in general, there is no guarantee that these algorithms converge, nor converge to nearly optimal solutions.

Convexification methods are the focus of this paper. It is proposed in [13]–[15] to transform the power flow constraints into linear constraints on a positive semidefinite rank-one matrix, and then remove the rank-one constraint to obtain a semidefinite programming (SDP) relaxation. If every solution of the SDP relaxation is of rank one, then a global optimum of the OPF problem can be recovered by solving the SDP relaxation. In this case, the SDP relaxation is called *exact*. Strikingly, the SDP relaxation is exact for the IEEE 14-, 30-, 57-, and 118-bus networks [15], though not exact in general [16]. A through study on the computational speed and exactness of the SDP relaxation, for some publicly available networks, can be found in [17].

Sufficient conditions for the exactness of the SDP relaxation have been found for special networks, e.g., (mesh) DC networks [15], [18], and tree (AC) networks [19]–[23].

Summary

The goal of this paper is to propose a convex relaxation for the OPF problem in (mesh) DC networks, with a sufficient condition for its exactness. In particular, contributions of this paper are threefold.

First, we *propose a second-order cone programming (SOCP) relaxation for the OPF problem in (mesh) DC networks* rather than the SDP relaxation. The SOCP relaxation has a much lower computational complexity than the SDP relaxation, and is exact under existing conditions that guarantee the exactness of the SDP relaxation. Note that the SOCP relaxation is coarser than the SDP relaxation [24].

Second, we prove that *the SOCP relaxation is exact if there are no voltage upper bounds*. Hence, if voltage upper bounds are not binding, then the SOCP relaxation is exact.

Third, we prove that *the SOCP relaxation has at most one solution if it is exact*.

II. The optimal power flow problem

This paper studies the optimal power flow (OPF) problem in direct current (DC) networks. In the following we present a model of this scenario that incorporates nonlinear power flow physical laws, and considers generators/loads as control elements, voltage regulation as control constraints, and generation cost as control objective.

A. Power flow model

A DC network is composed of buses and lines connecting these buses. Its topology can be either a tree or a mesh.

There is a swing bus in the network with a fixed voltage. Index the swing bus by 0 and the other buses by $1, \ldots, n$. Let $\mathcal{N} := \{0, \ldots, n\}$ denote the collection of all buses and define $\mathcal{N}^+ := \mathcal{N} \backslash \{0\}$. Each line connects an unordered pair $\{i, j\}$ of buses. Let $\mathcal{E}$ denote the collection of all lines and abbreviate $\{i, j\} \in \mathcal{E}$ by $i \sim j$.

For each bus $i \in \mathcal{N}$, let V_i denote its voltage, I_i denote its current injection, and p_i denote its power injection. For each line $i \sim j$, let y_{ij} denote its admittance and I_{ij} denote the current flow from bus i to bus j. In a DC network, V_i, I_i, p_i, y_{ij}, and I_{ij} are all real numbers.[1]

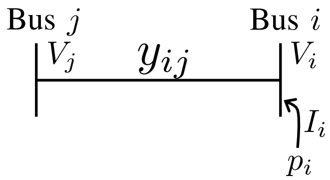

Fig. 1. Summary of the notations.

The notations are summarized in Fig. 1. Further, we use a letter without subscript to denote a vector of the corresponding quantities, e.g., $V = (V_1, \dots, V_n)$, $y = (y_{ij}, i \sim j)$. Note that the subscript 0 is not included in nodal variables to highlight the speciality of the swing bus.

Given the network graph $(\mathcal{N}, \mathcal{E})$, the admittance y, and the swing bus voltage V_0, then the other variables (p, V, I, p_0) are described by the following physical laws.

1) Ohm's Law:
$$I_{ij} = y_{ij}(V_i - V_j), \qquad i \sim j;$$

2) Current balance:
$$I_i = \sum_{j:\, j \sim i} I_{ij}, \qquad i \in \mathcal{N};$$

3) Power balance:
$$p_i = V_i I_i, \qquad i \in \mathcal{N}.$$

If we are only interested in voltages and power, then the three sets of equations can be combined into
$$p_i = V_i \sum_{j:\, j \sim i} (V_i - V_j) y_{ij}, \qquad i \in \mathcal{N}, \tag{1}$$
which is used to model the power flow in this paper.

B. Control elements, constraints, and objective

Power generation/demand p of generators/loads can be controlled, after specifying which the other variables (V, p_0) are determined by the power flow constraint (1).

The power injection p_i of a bus $i \in \mathcal{N}^+$ can be controlled to vary within some externally specified range $[\underline{p}_i, \overline{p}_i]$, i.e.,
$$\underline{p}_i \le p_i \le \overline{p}_i, \qquad i \in \mathcal{N}^+. \tag{2}$$
For example, if bus i has an inelastic load with power consumption d_i, and a generator that can generate any amount of power between 0 and its capacity C_i, then $\underline{p}_i = -d_i$ and $\overline{p}_i = C_i - d_i$. Note that $\underline{p}_i$ and $\overline{p}_i$ can be positive or negative.

There are many operational constraints in a practical OPF problem, including voltage constraints, line (thermal) constraints, security constraints, etc. In this work, we only consider voltage constraints for the following reasons. In distribution networks (DC microgrids are distribution networks), voltages deviate significantly from their nominal values, and therefore voltage regulation is an important operational constraint. In distribution networks, lines are usually over-provisioned to support the demand, and therefore line constraints are not likely to bind. Even if some line constraints are binding, the utility company can upgrade the line (distribution lines are much easier and cheaper to construct than transmission lines). We do not consider security constraints in distribution networks for simplicity.

[1]Voltages, currents, admittances, and power are considered real numbers for DC networks in this paper. While the lines and loads may be inductive or capacitive during transients at the sub-seconds timescale, they can be considered as purely resistive at the minutes timescale of the OPF problem.

Voltage regulation imposes the voltage V_i at bus i to be maintained within externally specified range $[\underline{V}_i, \overline{V}_i]$, i.e.,
$$\underline{V}_i \le V_i \le \overline{V}_i, \qquad i \in \mathcal{N}^+. \tag{3}$$
For example, if 5% voltage deviation from the nominal value is allowed, then $0.95 \le V_i \le 1.05$ per unit [25].

The control objective is minimizing the generation cost. For $i \in \mathcal{N}$, let $f_i(p_i)$ denote the generation cost of bus i where f_i is a real-valued function defined on $\mathbb{R}$. Then, the generation cost is
$$C(p, p_0) = \sum_{i \in \mathcal{N}} f_i(p_i). \tag{4}$$

C. The OPF problem

The OPF problem seeks to minimize the generation cost (4), subject to power flow constraints (1), power injection constraints (2), and voltage constraints (3).

$$\begin{aligned}
\textbf{OPF:}\ \min \quad & \sum_{i \in \mathcal{N}} f_i(p_i) \\
\text{over} \quad & p, V, p_0 \\
\text{s.t.} \quad & p_i = V_i \sum_{j:\, j \sim i} (V_i - V_j) y_{ij}, \quad i \in \mathcal{N}; \\
& \underline{p}_i \le p_i \le \overline{p}_i, \quad i \in \mathcal{N}^+; \\
& \underline{V}_i \le V_i \le \overline{V}_i, \quad i \in \mathcal{N}^+.
\end{aligned}$$

The following assumptions are made throughout this work.

A1 The network graph $(\mathcal{N}, \mathcal{E})$ is connected.
A2 The swing bus voltage V_0 is given, fixed, and non-zero.
A3 The admittances are positive, i.e., $y_{ij} > 0$ for $i \sim j$.
A4 The bounds $\underline{V}_i$ are positive, i.e., $\underline{V}_i > 0$ for $i \in \mathcal{N}^+$.

Assumptions A1, A3, and A4 hold in practice. Regarding Assumption A2, the swing bus voltage V_0 can be changed several times a day, and therefore can be considered as a constant at the minutes timescale of the OPF problem.

OPF is nonconvex due to the quadratic equality constraints (1). To overcome this challenge, one can relax OPF to a convex problem as in [26]. To state the relaxation, define
$$W_{ij} := V_i V_j, \qquad i \sim j \text{ or } i = j, \tag{5}$$
and let $W := (W_{ij})_{i \sim j \text{ or } i=j}$ denote the collection of all such W_{ij}. Define a series of 2×2 symmetric matrices
$$W\{i,j\} := \begin{pmatrix} W_{ii} & W_{ij} \\ W_{ji} & W_{jj} \end{pmatrix}, \qquad i \sim j \;\&\; i < j,$$

then OPF can be equivalently formulated as

$$\begin{aligned}
\textbf{OPF':}\ \min\quad & \sum_{i\in\mathcal{N}} f_i(p_i) \\
\text{over}\quad & p, W, p_0 \\
\text{s.t.}\quad & p_i = \sum_{j:\, j\sim i} (W_{ii} - W_{ij}) y_{ij}, \quad i \in \mathcal{N}; && \text{(6a)}\\
& \underline{p}_i \le p_i \le \overline{p}_i, \quad i \in \mathcal{N}^+; && \text{(6b)}\\
& \underline{V}_i^2 \le W_{ii} \le \overline{V}_i^2, \quad i \in \mathcal{N}^+; && \text{(6c)}\\
& W_{ij} = W_{ji} \ge 0, \quad i \sim j \ \&\ i < j; && \text{(6d)}\\
& \operatorname{rank}(W\{i,j\}) = 1, \quad i \sim j \ \&\ i < j && \text{(6e)}
\end{aligned}$$

according to Theorem 1, which establishes a bijective map between the feasible sets of OPF and OPF', that preserves the objective value. To state the theorem, let $\mathcal{F}_{\text{OPF}}$ / $\mathcal{F}_{\text{OPF'}}$ denote the feasible sets, and let C / C' denote the objective functions, of OPF / OPF'. Besides, for any feasible point $x = (p, V, p_0)$ of OPF, define a map $\phi(x) := (p, W, p_0)$ where W is defined according to (5).

Theorem 1 *For any $x \in \mathcal{F}_{OPF}$, the point $\phi(x) \in \mathcal{F}_{OPF'}$ and satisfies $C(x) = C'(\phi(x))$. Furthermore, the map $\phi : \mathcal{F}_{OPF} \to \mathcal{F}_{OPF'}$ is bijective.*

The theorem is proved in Appendix A.

After transforming OPF to OPF', a second-order cone programming (SOCP) relaxation can be proposed.

$$\begin{aligned}
\textbf{SOCP:}\ \min\quad & \sum_{i\in N} f_i(p_i) \\
\text{over}\quad & p, W, p_0 \\
\text{s.t.}\quad & \text{(6a), (6b), (6c), (6d);} \\
& W\{i,j\} \succeq 0, \quad i \sim j \ \&\ i < j. && \text{(7)}
\end{aligned}$$

The SOCP relaxation is proposed in [26] to convexify OPF for tree networks. In this paper, we use it to convexify OPF for DC networks, which can be either mesh or tree.

If a solution $w = (p, W, p_0)$ of SOCP is feasible for OPF', i.e., w satisfies (6e), then w is a global optimum of OPF'. This motivates a definition of "exactness" for SOCP.

Definition 1 *SOCP is exact if every of its solutions satisfies* (6e).

If SOCP is exact, then a global optimum of OPF can be easily recovered. More specifically, if SOCP is exact and (p, W, p_0) is an optimal solution of SOCP, let $V_i = \sqrt{W_{ii}}$ for $i \in \mathcal{N}$, then (p, V, p_0) is a global optimum of OPF.

D. Related work

A semidefinite programming (SDP) relaxation has also been proposed in literature to convexify OPF [13]–[15]. The SDP relaxation enlarges the feasible set of OPF to a smaller convex set than that of SOCP, and is therefore more likely to be exact [24]. We propose SOCP rather than the SDP relaxation for the following reasons:

R1 SOCP has a significantly lower computational complexity than the SDP relaxation;

R2 SOCP is exact under existing conditions that guarantee the exactness of the SDP relaxation.

To demonstrate R2, we review existing conditions that guarantee the exactness of both relaxations in Propositions 1 and 2, which follow directly from a more general result in [27, Theorem 3.1].

Proposition 1 ([15]) *If $\underline{p}_i = -\infty$ for $i \in \mathcal{N}^+$ and f_i is strictly increasing for $i \in \mathcal{N}$, then the SDP relaxation is exact.*

Proposition 2 ([18]) *If $\underline{p}_i = -\infty$ for $i \in \mathcal{N}^+$ and f_i is strictly increasing for $i \in \mathcal{N}$, then SOCP is exact.*

The conditions in Propositions 1 and 2 are the same, which demonstrates R2.

III. A SUFFICIENT CONDITION

We study the exactness of SOCP in this section. In particular, a sufficient condition is provided in the following theorem, which is proved in Section III-A.

Theorem 2 *If $\overline{V}_i = \infty$ for $i \in \mathcal{N}^+$ and f_0 is strictly increasing, then SOCP is exact.*

The theorem implies that if voltage upper bounds are not binding, then SOCP is exact.

Theorem 2 still holds if power injection constraints (2) are generalized to $p \in \mathcal{P}$ where $\mathcal{P}$ is an arbitrary set, since the proof of Theorem 2 does not require any structure on $\mathcal{P}$.

Theorem 3 *If SOCP is convex and exact, then it has at most one solution.*

The theorem is proved in Appendix B.

Theorem 3 still holds if power injection constraints (2) are generalized to $p \in \mathcal{P}$ where $\mathcal{P}$ is an arbitrary convex set.

A. Proof of Theorem 2

The idea of the proof is as follows. Under the conditions in Theorem 2, for any feasible point w of SOCP that violates (6e), there exists another feasible point w' of SOCP that has a smaller objective value than w. Hence, every solution of SOCP must satisfy (6e), i.e., SOCP is exact.

More specifically, Theorem 2 follows from the following two lemmas, which are proved in Appendix C and D.

Lemma 1 *Let $w = (p, W, p_0)$ be an arbitrary feasible point of SOCP. If $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$, and w violates* (6e) *on some $i \sim j$ where $i, j \in \mathcal{N}^+$, then there exists another feasible point $w' = (p, W', p_0)$ of SOCP that violates* (6e) *on $k \sim l$ whenever $\{k, l\} \cap \{i, j\} \neq \emptyset$.*

Lemma 1 implies that *violation of* (6e) *propagates to neighboring lines*: for any feasible point w of SOCP that violates (6e) on some line $i \sim j$, there exists another feasible point

w' with the same power injection, that violates (6e) on all neighboring lines of $i \sim j$ (including $i \sim j$).

Lemma 2 *If $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$ and f_0 is strictly increasing, then every solution of SOCP must satisfy* (6e) *on all neighboring lines $i \sim 0$ of the swing bus.*

Lemma 2 implies that *every solution w of SOCP satisfies* (6e) *on all neighboring lines of bus 0.*

Combining Lemma 1 and 2 gives the proof of Theorem 2. If there exists a solution w of SOCP that violates (6e) on some line, then since the network is connected, by repeating the propagation procedure described in Lemma 1, one can find a feasible point w' of SOCP with the same power injection as w, that violates (6e) on some neighboring line $k \sim 0$ of the swing bus 0.

The feasible point w' has the same objective value as w and is therefore optimal. This contradicts Lemma 2. Hence, every solution of SOCP must satisfy (6e), i.e., SOCP is exact. This completes the proof of Theorem 2.

IV. Conclusion

We have proposed an SOCP relaxation for the OPF problem in DC networks, with a sufficient condition under which the SOCP relaxation is exact. Specifically, we have proved that the SOCP relaxation is exact if voltage upper bounds are not binding. We have also proved that the SOCP relaxation has at most one solution if it is exact.

There remains many interesting open questions. For example, what happens if we consider line constraints? If the SOCP relaxation is not exact, can a nearly optimal solution of the OPF problem be recovered?

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Appendix

A. Proof of Theorem 1

It is straightforward to show that, for any feasible point $x = (p, V, p_0)$ of OPF, if one defines W according to (5), then the point $\phi(x) = (p, W, p_0)$ is feasible for OPF' and has the same objective value as x. It remains to show that the map ϕ is bijective, i.e., both injective and surjective.

First show that the map ϕ is injective, i.e., for $x, x' \in \mathcal{F}_{\text{OPF}}$, if $\phi(x) = \phi(x')$, then $x = x'$. Denote $x = (p, V, p_0)$ and $x' = (p', V', p_0')$, then $\phi(x) = (p, W, p_0)$ where W satisfies $W_{ij} = V_i V_j$ for $i \sim j$ or $i = j$, and $\phi(x') = (p', W', p_0')$ where W' satisfies $W'_{ij} = V'_i V'_j$ for $i \sim j$ or $i = j$. Therefore it follows from $\phi(x) = \phi(x')$ immediately that $p = p'$ and $p_0 = p_0'$. To show that $x = x'$, it remains to prove that $V = V'$.

It follows from $\phi(x) = \phi(x')$ that $W = W'$, which implies $W_{ii} = W'_{ii}$ for $i \in \mathcal{N}$, i.e.,

$$V_i^2 = V_i'^2, \qquad i \in \mathcal{N}.$$

Since $V_i \geq \underline{V}_i > 0$ and $V_i' \geq \underline{V}_i > 0$, one has $V_i = V_i'$ for $i \in \mathcal{N}$. This completes the proof that the map ϕ is injective.

Finally show that the map ϕ is surjective, i.e., for any $(p, W, p_0) \in \mathcal{F}_{\text{OPF}}$, there exists $x \in \mathcal{F}_{\text{OPF}}$ such that $\phi(x) = (p, W, p_0)$. Set $V_i = \sqrt{W_{ii}}$ for $i \in \mathcal{N}$ and define $x := (p, V, p_0)$. It remains to show that $\phi(x) = (p, W, p_0)$ and x is feasible for OPF.

To show that $\phi(x) = (p, W, p_0)$, it suffices to show that $W_{ij} = V_i V_j$ for $i \sim j$ or $i = j$. It follows from $V_i = \sqrt{W_{ii}}$ immediately that $W_{ij} = V_i V_j$ for $i = j$. Noting that $\operatorname{rank}(W\{i,j\}) = 1$, one has $W_{ij} = \pm\sqrt{W_{ii}W_{jj}} = \pm V_i V_j$ for $i \sim j$. Since $W_{ij} \geq 0$, it must be that $W_{ij} = V_i V_j$ for $i \sim j$. Hence, $\phi(x) = (p, W, p_0)$.

Having established the relationship that $W_{ij} = V_i V_j$ for $i \sim j$ or $i = j$, it is straightforward to show that x is feasible for OPF.

To this end, it has been proved that the map ϕ is bijective. This completes the proof of Theorem 1.

B. Proof of Theorem 3

Assume that SOCP is convex, exact, and has at least one solution. Let $\tilde{w} = (\tilde{p}, \tilde{W}, \tilde{p}_0)$ and $\hat{w} = (\hat{p}, \hat{W}, \hat{p}_0)$ be two arbitrary solutions of SOCP, then it suffices to prove that $\tilde{w} = \hat{w}$.

Define $w := (\tilde{w} + \hat{w})/2$, then w is feasible and optimal for SOCP since SOCP is convex. Denote $w = (p, W, p_0)$. It follows from SOCP being exact that $\tilde{w}$, $\hat{w}$, and w all satisfy (6e). Hence,

$$\tilde{W}_{ii}\tilde{W}_{jj} = \tilde{W}_{ij}^2, \tag{8a}$$
$$\hat{W}_{ii}\hat{W}_{jj} = \hat{W}_{ij}^2, \tag{8b}$$
$$W_{ii}W_{jj} = W_{ij}^2 \tag{8c}$$

for $i \sim j$. Substitute $W = (\tilde{W} + \hat{W})/2$ in (8c), and simplify using (8a) and (8b) to obtain

$$\tilde{W}_{ii}\hat{W}_{jj} + \hat{W}_{ii}\tilde{W}_{jj} = 2\tilde{W}_{ij}\hat{W}_{ij}, \qquad i \sim j.$$

It follows that

$$\begin{aligned} \tilde{W}_{ii}\hat{W}_{jj} + \hat{W}_{ii}\tilde{W}_{jj} &\leq 2\sqrt{\tilde{W}_{ii}\tilde{W}_{jj}\hat{W}_{ii}\hat{W}_{jj}} \\ &\leq \tilde{W}_{ii}\hat{W}_{jj} + \hat{W}_{ii}\tilde{W}_{jj} \end{aligned}$$

for $i \sim j$. The inequality attains equality, which implies $\tilde{W}_{ii}\hat{W}_{jj} = \hat{W}_{ii}\tilde{W}_{jj}$ for $i \sim j$. Since $\hat{W}_{kk}, \tilde{W}_{kk} \geq \underline{V}_k^2 > 0$ for $k \in \mathcal{N}^+$ and $\hat{W}_{00} = \tilde{W}_{00} = V_0^2 \neq 0$, one has

$$\frac{\hat{W}_{ii}}{\tilde{W}_{ii}} = \frac{\hat{W}_{jj}}{\tilde{W}_{jj}}, \qquad i \sim j.$$

Define $\eta_i := \hat{W}_{ii}/\tilde{W}_{ii}$ for $i \in \mathcal{N}$, then $\eta_0 = 1$ and $\eta_i = \eta_j$ if $i \sim j$. Since the network $(\mathcal{N}, \mathcal{E})$ is connected, it follows that $\eta_i = 1$ for $i \in \mathcal{N}$, i.e., $\hat{W}_{ii} = \tilde{W}_{ii}$ for $i \in \mathcal{N}$. Then, it follows from (6e) that

$$\hat{W}_{ij} = \sqrt{\hat{W}_{ii}\hat{W}_{jj}} = \sqrt{\tilde{W}_{ii}\tilde{W}_{jj}} = \tilde{W}_{ij}$$

for $i \sim j$. To this point, we have shown that $\hat{W} = \tilde{W}$. It follows immediately that $\hat{w} = \tilde{w}$, which completes the proof of Theorem 3.

C. Proof of Lemma 1

It suffices to prove that, given any feasible point $w = (p, W, p_0)$ of SOCP that violates (6e) on some line $i \sim j$ where $i, j \in \mathcal{N}^+$, there exists a feasible point $w' = (p', W', p_0')$ of SOCP with $(p', p_0') = (p, p_0)$, that violates (6e) on $k \sim l$ whenever $\{k, l\} \cap \{i, j\} \neq \emptyset$.

Note that (6e) is equivalent to

$$W_{kl} = \sqrt{W_{kk}W_{ll}}, \qquad k \sim l,$$

and (7) is equivalent to

$$0 \leq W_{kl} \leq \sqrt{W_{kk}W_{ll}}, \qquad k \sim l.$$

If w violates (6e) on $i \sim j$ but satisfies (7), then

$$W_{ij} < \sqrt{W_{ii}W_{jj}}.$$

Pick an ϵ such that $0 < \epsilon < \sqrt{W_{ii}W_{jj}} - W_{ij}$, and construct W' as

$$W'_{kl} = \begin{cases} W_{kl} + \epsilon & \text{if } \{k,l\} = \{i,j\} \\ W_{kl} + \frac{y_{ij}}{\sum_{h:\, h\sim k} y_{kh}}\epsilon & \text{if } \{k,l\} = \{i\} \text{ or } \{j\} \\ W_{kl} & \text{otherwise.} \end{cases} \tag{9}$$

After constructing W', construct (p', p_0') as

$$p_k' = \sum_{l:\, l\sim k} (W'_{kk} - W'_{kl}) y_{kl}, \qquad k \in \mathcal{N}. \tag{10}$$

It remains to prove that the point $w' = (p', W', p_0')$ is feasible for SOCP, $(p', p_0') = (p, p_0)$, and w' violates (6e) (i.e., $W'_{kl} \neq \sqrt{W'_{kk}W'_{ll}}$) on $k \sim l$ whenever $\{k, l\} \cap \{i, j\} \neq \emptyset$.

First prove that $(p', p_0') = (p, p_0)$. When $k \notin \{i, j\}$,

$$\begin{aligned} p_k' &= \sum_{l:\, l\sim k} (W'_{kk} - W'_{kl}) y_{kl} \\ &= \sum_{l:\, l\sim k} (W_{kk} - W_{kl}) y_{kl} \\ &= p_k. \end{aligned}$$

When $k \in \{i, j\}$, one has

$$\begin{aligned} p_k' &= \sum_{l:\, l\sim k} (W'_{kk} - W'_{kl}) y_{kl} \\ &= W'_{kk} \sum_{l:\, l\sim k} y_{kl} - \sum_{l:\, l\sim k} W'_{kl} y_{kl} \\ &= W_{kk} \sum_{l:\, l\sim k} y_{kl} + \frac{y_{ij}}{\sum_{l:\, l\sim k} y_{kl}}\epsilon \sum_{l:\, l\sim k} y_{kl} \\ &\quad - \sum_{l:\, l\sim k} W_{kl} y_{kl} - y_{ij}\epsilon \\ &= W_{kk} \sum_{l:\, l\sim k} y_{kl} - \sum_{l:\, l\sim k} W_{kl} y_{kl} \\ &= \sum_{l:\, l\sim k} (W_{kk} - W_{kl}) y_{kl} \\ &= p_k. \end{aligned}$$

Hence, $(p', p_0') = (p, p_0)$.

Next prove that w' satisfies $W'_{kl} \neq \sqrt{W'_{kk}W'_{ll}}$ on $k \sim l$ whenever $\{k, l\} \cap \{i, j\} \neq \emptyset$. Let $k \sim l$ be an arbitrary line that satisfies $\{k, l\} \cap \{i, j\} \neq \emptyset$. If $\{k, l\} = \{i, j\}$, then

$$W'_{kl} = W_{kl} + \epsilon = W_{ij} + \epsilon < \sqrt{W_{ii}W_{jj}} \leq \sqrt{W'_{kk}W'_{ll}}.$$

If $\{k,l\} \neq \{i,j\}$, then one of k,l is in $\{i,j\}$, and the other is not. Without loss of generality, assume that $k=i$ and $l \notin \{i,j\}$. Then,

$$|W'_{kl}| = |W_{kl}| \leq \sqrt{W_{kk}W_{ll}} < \sqrt{W'_{kk}W_{ll}} = \sqrt{W'_{kk}W'_{ll}}.$$

Hence, $W'_{kl} \neq \sqrt{W'_{kk}W_{ll'}}$ on $k \sim l$ whenever $\{k,l\} \cap \{i,j\} \neq \emptyset$.

Finally prove that w' is feasible for SOCP if $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$. The point w' satisfies (6a) according to (10). Since $w = (p, W, p_0)$ is feasible for SOCP, one has $\underline{p}_k \leq p_k \leq \overline{p}_k$ for $k \in \mathcal{N}^+$. Then it follows from $p' = p$ that $\underline{p}_k \leq p'_k \leq \overline{p}_k$ for $k \in \mathcal{N}^+$, i.e., w' satisfies (6b). It follows from (9) that $W'_{kk} \geq W_{kk} \geq \underline{V}_k^2$ for $k \in \mathcal{N}^+$, i.e., w' satisfies (6c) if $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$. To check (7), note that for $k \sim l$, if $\{k,l\} \neq \{i,j\}$, then

$$W'_{kl} = W_{kl} \in \left[0, \sqrt{W_{kk}W_{ll}}\right] \subseteq \left[0, \sqrt{W'_{kk}W'_{ll}}\right];$$

if $\{k,l\} = \{i,j\}$, then

$$W'_{kl} = W_{kl} + \epsilon \in \left[\epsilon, \sqrt{W_{kk}W_{ll}}\right) \subseteq \left[0, \sqrt{W'_{kk}W'_{ll}}\right].$$

This completes the proof of Lemma 1.

D. Proof of Lemma 2

It suffices to prove that, given any feasible point $w = (p, W, p_0)$ of SOCP that violates (6e) on some line $i \sim 0$, there exists a feasible point $w' = (p', W', p'_0)$ of SOCP with $p' = p$ and $p'_0 < p_0$. Since then,

$$\sum_{i\in\mathcal{N}} f_i(p'_i) - \sum_{i\in\mathcal{N}} f_i(p_i) = f_0(p'_0) - f_0(p_0) < 0,$$

i.e., the feasible point w' has a smaller objective value than w. It follows that every solution of SOCP must satisfy (6e) on every neighboring line $i \sim 0$ of the swing bus 0, which completes the proof of Lemma 2.

Note that (6e) is equivalent to $W_{kl} = \sqrt{W_{kk}W_{ll}}$ for $k \sim l$, and (7) is equivalent to $0 \leq W_{kl} \leq \sqrt{W_{kk}W_{ll}}$ for $k \sim l$. If w violates (6e) on $i \sim 0$ but satisfies (7), then

$$W_{0i} < \sqrt{W_{00}W_{ii}}.$$

Pick an ϵ such that $0 < \epsilon < \sqrt{W_{00}W_{ii}} - W_{0i}$, and construct W' as

$$W'_{kl} = \begin{cases} W_{kl} + \epsilon & \text{if } \{k,l\} = \{0,i\} \\ W_{kl} + \frac{y_{0i}}{\sum_{h:\,h\sim k} y_{kh}}\epsilon & \text{if } \{k,l\} = \{i\} \\ W_{kl} & \text{otherwise.} \end{cases} \tag{11}$$

After constructing W', construct (p', p'_0) as

$$p'_k = \sum_{l:\,l\sim k} (W'_{kk} - W'_{kl}) y_{kl}, \qquad k \in \mathcal{N}. \tag{12}$$

It remains to prove that w' is feasible for SOCP, $p' = p$, and $p'_0 < p_0$.

First prove that $p' = p$ and $p'_0 < p_0$. When $k \notin \{0,i\}$,

$$\begin{aligned} p'_k &= \sum_{l:\,l\sim k} (W'_{kk} - W'_{kl}) y_{kl} \\ &= \sum_{l:\,l\sim k} (W_{kk} - W_{kl}) y_{kl} \\ &= p_k. \end{aligned}$$

When $k = i$, one has

$$\begin{aligned} p'_k &= \sum_{l:\,l\sim k} (W'_{kk} - W'_{kl}) y_{kl} \\ &= W'_{kk} \sum_{l:\,l\sim k} y_{kl} - \sum_{l:\,l\sim k} W'_{kl} y_{kl} \\ &= W_{kk} \sum_{l:\,l\sim k} y_{kl} + \frac{y_{0i}}{\sum_{l:\,l\sim k} y_{kl}}\epsilon \sum_{l:\,l\sim k} y_{kl} \\ &\quad - \sum_{l:\,l\sim k} W_{kl} y_{kl} - y_{0i}\epsilon \\ &= W_{kk} \sum_{l:\,l\sim k} y_{kl} - \sum_{l:\,l\sim k} W_{kl} y_{kl} \\ &= \sum_{l:\,l\sim k} (W_{kk} - W_{kl}) y_{kl} \\ &= p_k. \end{aligned}$$

When $k = 0$, one has

$$\begin{aligned} p'_k &= \sum_{l:\,l\sim k} (W'_{kk} - W'_{kl}) y_{kl} \\ &= W'_{kk} \sum_{l:\,l\sim k} y_{kl} - \sum_{l:\,l\sim k} W'_{kl} y_{kl} \\ &= W_{kk} \sum_{l:\,l\sim k} y_{kl} - \sum_{l:\,l\sim k} W_{kl} y_{kl} - y_{0i}\epsilon \\ &= \sum_{l:\,l\sim k} (W_{kk} - W_{kl}) y_{kl} - y_{0i}\epsilon \\ &= p_k - y_{0i}\epsilon < p_k. \end{aligned}$$

Hence, $p' = p$ and $p'_0 < p_0$.

Next prove that w' is feasible for SOCP if $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$. The point w' satisfies (6a) according to (12). Since $w = (p, W, p_0)$ is feasible for SOCP, one has $\underline{p}_k \leq p_k \leq \overline{p}_k$ for $k \in \mathcal{N}^+$. Then it follows from $p' = p$ that $\underline{p}_k \leq p'_k \leq \overline{p}_k$ for $k \in \mathcal{N}^+$, i.e., w' satisfies (6b). It follows from (11) that $W'_{kk} \geq W_{kk} \geq \underline{V}_k^2$ for $k \in \mathcal{N}^+$, i.e., w' satisfies (6c) if $\overline{V}_k = \infty$ for $k \in \mathcal{N}^+$. To check (7), note that for $k \sim l$, if $\{k,l\} \neq \{0,i\}$, then

$$W'_{kl} = W_{kl} \in \left[0, \sqrt{W_{kk}W_{ll}}\right] \subseteq \left[0, \sqrt{W'_{kk}W'_{ll}}\right];$$

if $\{k,l\} = \{0,i\}$, then

$$W'_{kl} = W_{kl} + \epsilon \in \left[\epsilon, \sqrt{W_{kk}W_{ll}}\right) \subseteq \left[0, \sqrt{W'_{kk}W'_{ll}}\right].$$

This completes the proof of Lemma 2.