52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Global Output-Feedback Extremum Seeking Control via Monitoring Functions

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Abstract— **This paper addresses the design of an extremum-seeking controller based on monitoring function for a class of single-input-single-output (SISO) uncertain nonlinear systems. We demonstrate that it is possible to achieve an arbitrarily small neighborhood of the desired optimal point using only output-feedback. The key idea is the combination of a monitoring function with a norm state observer. We show that, as an important advantage, the proposed scheme achieves the extremum of a unknown nonlinear mapping for all domain of initial conditions, i.e., the real-time optimization algorithm has global convergence/stability properties. Moreover, some tuning rules are given to achieve convergence to global maximum in the presence of local extrema. A numerical example illustrates the viability of the proposed approach.**

I. Introduction

The first research about extremum seeking control (ESC) seems to have been introduced by Leblanc in 1922 [1], [2]. It can be defined as a control system used to determine in real-time the extremum (maximum or minimum) of a unknown nonlinear mapping [3]. ESC has become a key area in control theory due to increasing need to optimize plant operation in order to reduce operating costs and meet product specifications [4]. In [5, Section 13.3], the ESC was considered as one of promising area in adaptive control.

In [3] and references therein, we can find a large number of applications, to name a few, the design of antilock braking systems, autonomous vehicles and mobile robots, internal combustion engines, process control, and particle accelerators and plasma control.

The most common algorithms for unconstrained optimization use the derivative or the gradient of the objective function. However, in many applications of control problems such as those mentioned above, the plant model, the gradient information and the cost function to be optimized are not available online. Moreover, "gradient sensors" tend to amplify noise and suffer from instability problems at high frequencies [6].

The most popular ESC method uses a high pass filter in the system output and a small sinusoidal perturbation (dither signal) technique to estimate the gradient of the cost function. This method is characterized by its simplicity and fast adaptation [2], [7]. However, only local stability properties could be guaranteed by assuming full-state measurement and the possibility of dealing with local extrema has been only recently investigated [7]. In [3], [8], [9], under the same assumptions, semi-global practical convergence was obtained, but ESC convergence rate slows down within the domain of attraction.

In [10], [11], [12], the extremum search could be seen as a nonlinear control problem with a state dependent high-frequency gain (HFG) which changes sign (also named control direction) around the optimum point of interest. A scheme utilizing sliding mode control for tracking of uncertain plants with unknown control direction was introduced in [13], [14], [15] using an algorithm of switching based on a monitoring function to the output error. In [16], it was conjectured that the lack of robustness of such monitoring scheme with respect to recurrent changes of HFG sign would preclude the monitoring function to be directly applied to extremum-seeking control.

In this paper, a novel monitoring function is proposed in order to show that the output-feedback tracking controller proposed in [13], [14], [15], indeed can also be applied to ESC of a class of uncertain nonlinear systems, while global convergence properties of the search algorithm are also guaranteed without affecting its rate of convergence. Moreover, the proposed algorithm can achieve global extremum point in the presence of local extrema as shown in the simulation example.

II. Preliminaries and problem formulation

Throughout the paper, the Euclidean norm of a vector x and the corresponding induced norm of a matrix A are denoted by $\|x\|$ and $\|A\|$, respectively. The term $\pi_i(t)$ stands for any exponential decaying function, such that $|\pi_i(t)| \leq Re^{-\beta t}, \forall t$, and some positive scalars R and β. Class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions are defined as in [17]. From a technical standpoint, the theoretical results obtained in this paper are based on Filippov's definition for solution of differential equations with discontinuous right-hand sides [18].

Consider the following nonlinear uncertain system:

$$\dot{x} = f(x,t) + g(x,t)u \quad (1)$$
$$z = h(x,t) \quad (2)$$

in cascade with a static subsystem

$$y = \Phi(z)\,, \quad (3)$$

where $u \in \mathbb{R}$ is the control input, $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}$ and $y \in \mathbb{R}$ are measured outputs of the first subsystem and of the static subsystem, respectively. In order

This work was supported by CNPq, CAPES and FAPERJ.

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to assure existence and forward uniqueness of solutions, the nonlinear uncertain functions f, g and h are locally Lipschitz continuous in x, piecewise continuous in t and sufficiently smooth. Without loss of generality, we assume that the initial time is $t = 0$. For each solution of (1) there exists a maximal time interval of definition given by $[0, t_M)$, where t_M may be finite or infinite.

The control objective of ESC is not "stabilization" or "tracking", but is "real-time optimization" [9]. However, the ESC problem can be reformulated as a tracking problem in which the control direction is unknown [10]. We wish to find an output-feedback control law u so that, from any initial condition, the system is steered to reach the extremum point y^* and remain on such point thereafter, as close as possible. Without loss of generality, we only address the maximum seeking problem.

The system (1)–(3) can be rewritten in the normal form as follows:

$$\begin{aligned} \dot{\eta} &= \phi_0(\eta, z, t)\,, && (4)\\ \dot{z} &= \phi_1(\eta, z, t) + \phi_2(\eta, z, t)u\,, && (5)\\ y &= \Phi(z)\,, && (6) \end{aligned}$$

with state $x := \begin{bmatrix}\eta^T & z\end{bmatrix}^T$, $\eta \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$, and uncertain nonlinear functions $\phi_0 : \mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^{n-1}$ and $\phi_1, \phi_2 : \mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$.

The state η of the η-subsystem, referred to as an "inverse system", is not available for feedback.

With respect to the controlled plant, we assume the following assumptions:

$(\mathbf{A1})$ (*On the uncertainties*): All the uncertain plant parameters belong to a compact set Ω.

This assumption is necessary to obtain the uncertainty bounds for control design.

$(\mathbf{A2})$ (*Relative degree one*): The uncertain function $\phi_2(\eta, z, t)$ is bounded away from zero, *i.e.*,

$$0 < \underline{\phi}_2 \leq |\phi_2|\,, \quad \forall t \in [0, t_M)\,,$$

where the constant lower bound $\underline{\phi}_2$ is known.

According to $(\mathbf{A2})$, the subsystem (4)–(5) has relative degree one *w.r.t.* z since $\phi_2 \neq 0$. It restrict us to the case of relative degree one which is the simplest case amenable by pure Lyapunov design.

By using the notation $\Phi'(z) := \frac{d\Phi(z)}{dz}$ and $\Phi''(z) := \frac{d^2\Phi(z)}{dz^2}$, we consider that (in Ω):

$(\mathbf{A3})$ (*Cost Function*): There exists a unique point $z^* \in \mathbb{R}$ such that $y^* = \Phi(z^*)$ is the extremum (maximum) of $\Phi(z)$: $\mathbb{R} \to \mathbb{R}$, satisfying

$$\Phi'(z^*) = 0, \;\; \Phi''(z^*) < 0$$
$$\Phi(z^*) > \Phi(z), \; \forall z \in \mathbb{R}, \; z \neq z^*$$

and for any given $\Delta > 0$, there exists a constant $L_\Phi(\Delta) > 0$ such that

$$L_\Phi(\Delta) \leq |\Phi'(z)|, \forall z \notin \mathcal{D}_\Delta := \{z : |z - z^*| < \Delta/2\},$$

where $\mathcal{D}_\Delta$ is called Δ-vicinity of z^* and Δ can be made arbitrarily small by allowing a smaller L_Φ.

From (5) and (6), the first time derivative of the output y is given by

$$\dot{y} = \Phi'\phi_1 + k_p u\,, \tag{7}$$

where the plant high frequency gain (HFG), denoted by k_p, is the coefficient of u which appears in the first time derivative of the output y and it is given by

$$k_p = \Phi'\phi_2\,. \tag{8}$$

As in [10], the $\operatorname{sgn}(k_p)$ is also called *control direction*. Assumption $(\mathbf{A3})$ leads us to consider a nonlinear control system with a state dependent HFG which changes sign around the optimum point of interest in a continuous way.

From (8), $(\mathbf{A2})$ and $(\mathbf{A3})$, k_p satisfies $(\forall z \notin \mathcal{D}_\Delta)$

$$0 < \underline{k}_p \leq |k_p| \tag{9}$$

where the lower bound $\underline{k}_p \leq \underline{\phi}_2 L_\Phi$ is a constant.

$(\mathbf{A4})$ (*Norm observability*): The inverse system (4) admits a known first order norm observer of the form:

$$\dot{\bar{\eta}} = -\lambda_0\bar{\eta} + \varphi_0(z, t)\,, \tag{10}$$

with z in (5), input $\varphi_0(z, t)$ and output $\bar{\eta}$ such that: (i) $\lambda_0 > 0$ is a constant; (ii) $\varphi_0(z, t)$ is a non-negative function continuous in z, piecewise continuous and upper-bounded in t; and (iii) for each initial states $\eta(0)$ and $\bar{\eta}(0)$

$$\|\eta\| \leq |\bar{\eta}| + \pi_0\,, \quad \forall t \in [0, t_M)\,, \tag{11}$$

where $\pi_0 := \Psi_0(|\bar{\eta}(0)| + \|\eta(0)\|)e^{-\lambda_0 t}$ and $\Psi_0 \in \mathcal{K}$.

It is well known that, if the inverse system (4) is input-to-state-stable (ISS) w.r.t. z, then it admits such norm observer and the plant is minimum-phase [19]. More examples of nonlinear systems which satisfies such assumption are given in [10].

In order to obtain a norm bound for the term $\Phi'\phi_1$ in (7), we additionally assume that:

$(\mathbf{A5})$ (*Domination Functions*): There exist *known* functions $\bar{\Phi}, \alpha_1 \in \mathcal{K}_\infty$, with α_1 locally Lipschitz, a *known* non-negative function $\varphi_1(z, t)$ continuous in z, piecewise continuous and upperbounded in t such that $|\phi_1(\eta, z, t)| \leq \alpha_1(\|\eta\|) + \varphi_1(z, t)$ and $|\Phi'| \leq \bar{\Phi}(|z|)$.

Note that Assumption $(\mathbf{A5})$ is not restrictive since Φ' is assumed to be smooth and ϕ_1 is locally Lipschitz continuous in η and in z. Furthermore, the domination functions α_1 and φ_1 impose stringent growth condition only *w.r.t.* the time-dependence. Thus, polynomial nonlinearities in η and z can be coped with.

III. Output-feedback Extremum Controller via Monitoring Function

The proposed output-feedback ESC based on monitoring function is represented in Fig. 1. The control law to plants with unknown HFG is defined as in [13], [14]:

$$u = \left\{ \begin{array}{lcll} u^+ &=& -\rho\operatorname{sgn}(e)\,, & t \in T^+\,, \\ u^- &=& \rho\operatorname{sgn}(e)\,, & t \in T^-\,, \end{array} \right. \tag{12}$$

where the monitoring function is used to decide when u should be switched from u^+ to u^- and vice versa. In (12), ρ

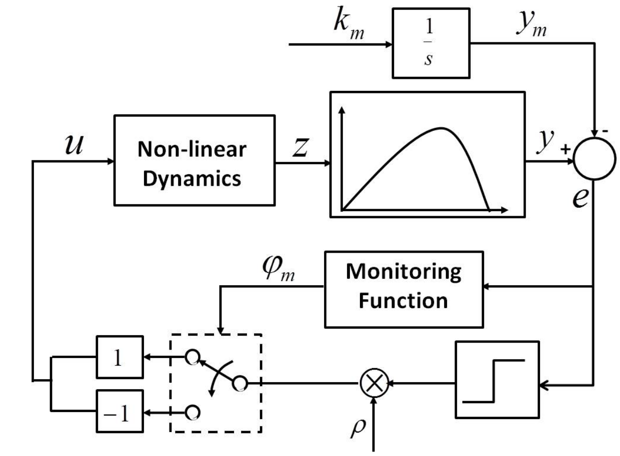

Fig. 1: Extremum seeking controller using a monitoring function.

is the modulation function to be designed later on and the sets T^+ and T^- satisfy $T^+ \cap T^- = \emptyset$ and $T^+ \cup T^- = [0, t_M)$.

The tracking error e is given by the following

$$e(t) = y(t) - y_m(t)\,, \tag{13}$$

where $y_m \in \mathbb{R}$ is a simple ramp time function generated by the reference model

$$\dot{y}_m = k_m, \quad y_m(0) = y_{m0}\,, \tag{14}$$

where k_m and y_{m0} are design constants. In order to avoid an unbounded reference signal $y_m(t)$ in the controller, one can saturate the model output at a rough known norm upper bound of y^* without affecting the control performance.

The modulation function ρ will be designed so that $y(t)$ tracks $y_m(t)$ as long as possible. In this way, y is forced to achieve the vicinity of the maximum $y^* = \Phi(z^*)$ and remains close to the optimum value y^*. To this end, we have to propose ρ such that the output tracking error e tends to zero in finite time at least outside the Δ-vicinity, that is, in the neighborhood of the maximizer z^*.

Thus, it is straightforward to conclude that $y = \Phi(z)$ tries to track y_m (and consequently y must approach the maximum at y^*) as long as y remains away from a small vicinity of y^* where the HFG is away from zero. In contrast, once y reaches the vicinity of y^*, the HFG will approach zero and thus controllability is lost. Consequently, tracking of y_m will cease. But then the neighborhood of the optimum point is already achieved as desired. Our control strategy will guarantee that y will remain close to y^* thereafter. It is apparent that the convergence rate of z to the Δ-vicinity $\mathcal{D}_\Delta$ defined in $(\mathbf{A3})$ is a function of ρ. Although $\mathcal{D}_\Delta$ is not positively invariant, after reaching $\mathcal{D}_\Delta$, it will be shown that z will remain close to z^* where the maximum takes place. It does not imply that $z(t)$ remains in $\mathcal{D}_\Delta$, $\forall t$. However, as shown later on in Theorem 1, one can guarantee that y remains close to the optimum value y^*.

A. Error Dynamics

From (7), (13) and (14), by adding and subtracting λe to the time derivative of the error e one has

$$\dot{e} = \Phi'\phi_1 + k_p u - k_m + \lambda e - \lambda e\,, \tag{15}$$

$$\dot{e} = -\lambda e + k_p(u + d_e)\,, \tag{16}$$

where $\lambda > 0$ is an appropriate constant and

$$d_e := \frac{1}{k_p}[\Phi'\phi_1 - k_m + \lambda e]\,. \tag{17}$$

In [15], it is shown that if the control law

$$u = -\operatorname{sgn}(k_p)\rho\operatorname{sgn}(e) \tag{18}$$

were used with a non negative modulation function ρ satisfying

$$\rho \geq |d_e| + \delta, \tag{19}$$

modulo exponential decaying terms denoted by π (i.e., $\rho \geq |d_e| + \delta - |\pi|$), and $\delta > 0$ is an arbitrarily small constant, then by using the comparison lemma [18], one has $\forall t \in [t_i, t_M)$:

$$|e(t)| \leq \zeta(t)\,, \quad \zeta(t) := |e(t_i)|e^{-\lambda(t-t_i)} + \pi_1\,, \tag{20}$$

where $\pi_1 := \Psi_1(|\bar{\eta}(0)| + \|\eta(0)\|)e^{-\lambda_1 t}$, $0 < \lambda_1 < \min\{\lambda_0, \lambda\}$, $\Psi_1 \in \mathcal{K}$ and λ_0 is defined in 10.

On the other hand, if inequality (19) were verified taking into account the exponential decaying terms, (i.e., $\rho \geq |d_e| + \delta$), the upperbound (20) would be modified to

$$|e(t)| \leq \zeta(t)\,, \quad \zeta(t) := |e(t_i)|e^{-\lambda(t-t_i)}, \tag{21}$$

where $t_i \in [0, t_M)$.

The major problem is that $\operatorname{sgn}(k_p)$ is unknown in both cases, thus we cannot implement it. In what follows, a switching scheme based on monitoring function is developed to cope with the lack of knowledge of the control direction outside the Δ-vicinity.

B. Monitoring Function Design

The detailed description of monitoring function can be found in [15]. Here, only a brief description of how it works is given. Reminding that inequality (21) holds when the control direction is correct, it seems natural to use ζ in (21) as a benchmark to decide whether a switching of u in (12) from u^+ to u^- (or u^- to u^+) is needed, *i.e.*, the switching occurs only when (21) is violated.

Therefore, consider the following function

$$\varphi_k(t) = |e(t_k)|e^{-\lambda(t-t_k)} + r, \tag{22}$$

where the t_k is the switching time and $r \geq 0$ is any arbitrarily small constant. The monitoring function φ_m can be defined as

$$\varphi_m(t) := \varphi_k(t), \quad \forall t \in [t_k, t_{k+1}) \subset [0, t_M). \tag{23}$$

Note that from (22) and (23), one has $|e(t)| < |\varphi_k(t)|$ at $t = t_k$. Hence, t_k is defined as the time instant when the monitoring function $\varphi_m(t)$ meets $|e(t)|$, that is,

$$t_{k+1} := \begin{cases} \min\{t > t_k : |e(t)| = \varphi_k(t)\}, & \text{if it exists}\,, \\ t_M, & \text{otherwise}\,, \end{cases} \tag{24}$$

where $k \in \{1, 2, \ldots\}$ and $t_0 := 0$ (see Fig. 2).

The following inequality is directly obtained from (23)

$$|e(t)| \leq \varphi_m(t), \quad \forall t \in [0, t_M)\,. \tag{25}$$

Fig. 2 illustrates the tracking error norm $|e|$ as well as the monitoring function φ_m.

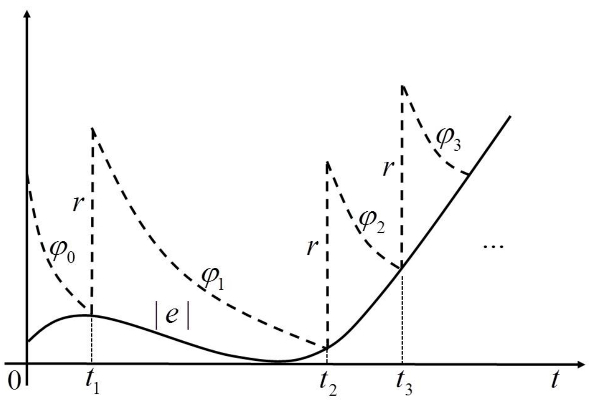

Fig. 2: The trajectories of $\varphi_m(t)$ and $|e(t)|$.

Remark 1 (Main Changes in the Monitoring Function): The main difference with respect to [13], [14], [15] is that the monitoring function earlier proposed was based on the upperbound (20) but such that φ_k in (22) was replaced by

$$\varphi_k(t) = |e(t_k)|e^{-\lambda(t-t_k)} + a(k)e^{-t/a(k)}\,, \tag{26}$$

where $a(k)$ is any positive monotonically increasing unbounded sequence. The purpose of the second term in (26) is to dominate the term π_1 which is not available for measurement. In our ESC application it might be a problem since, as will be shown, the ultimate residual set of the proposed algorithm around the maximum y^* is dependent of the values for which the monitoring function converges. Since (26) can assume arbitrarily large values which eventually result in large transients as a result of a change of the control direction and repetitive switching ($k \to +\infty$). In contrast, this does not occur with the new definition given in (22) whereby the ultimate residual set can be fixed in $\mathcal{O}(r)$.

C. Modulation Function Design

The following auxiliary available upper bounds provide one possible design for the modulation function so that (19) holds and they are obtained, taking into account exponentially decaying terms, by using the norm observer given in **(A4)** and the bounding functions given in **(A5)**.

From **(A5)** and (11), one has $|\phi_1| \leq \alpha_1(|\bar{\eta}|+\pi_0)+\varphi_1(z,t)$. Now, note that $\psi(a+b) \leq \psi(2a)+\psi(2b)$, $\forall a, b \geq 0$ and $\forall \psi \in \mathcal{K}_\infty$. Thus, since $\alpha_1 \in \mathcal{K}_\infty$ one can write $\alpha_1(|\bar{\eta}|+\pi_0) \leq \alpha_1(2|\bar{\eta}|) + \alpha_1(2\pi_0)$ and

$$|\phi_1| \leq \alpha_1(2|\bar{\eta}|) + \alpha_1(2\pi_0) + \varphi_1(z,t)\,.$$

From (11), $\pi_0 := \Psi_0(|\bar{\eta}(0)|+\|\eta(0)\|)e^{-\lambda_0 t}$ is uniformly bounded. Thus, since α_1 is assumed locally Lipschitz in **(A5)**, one can obtain a valid linear upper bound for α_1 such that

$$\alpha_1(2\pi_0) \leq 2k_1\pi_0 = 2k_1\Psi_0(|\bar{\eta}(0)|+\|\eta(0)\|)e^{-\lambda_0 t}\,,$$

where k_1 is a positive constant depending on the Lipschitz constant of α_1. Then, defining

$$\bar{\phi}_1 := \alpha_1(2|\bar{\eta}|) + \varphi_1(z,t) \tag{27}$$

and $\bar{\pi}_1 := 2k_1\Psi_0(|\bar{\eta}(0)|+\|\eta(0)\|)e^{-\lambda_0 t}$, one can also write

$$|\phi_1| \leq \bar{\phi}_1 + \bar{\pi}_1\,, \tag{28}$$

where $\bar{\pi}_1$ decays exponentially.

Furthermore, from (28), the first term $\phi_1\Phi'$ of the y-dynamics in (7) satisfies $|\phi_1\Phi'| \leq |\Phi'||\bar{\phi}_1| + |\Phi'|\bar{\pi}_1 \leq |\Phi'||\bar{\phi}_1| + |\Phi^{'}|^2 + \bar{\pi}_1^2$, where we have used the relationship $|\Phi'|\bar{\pi}_1 \leq |\Phi^{'}|^2 + \bar{\pi}_1^2$. Now, from **(A5)** the following upper bound holds

$$|\phi_1\Phi^{'}| \leq \bar{\phi}_1\bar{\Phi} + \bar{\Phi}^2 + \bar{\pi}_1^2\,. \tag{29}$$

Remind that, outside the Δ-vicinity, the derivative of the cost function does not vanish $\forall z$. Thus, the lower norm bound $\underline{k}_p$ for $k_p = \Phi'\phi_2$ given in (9) holds.

Therefore, one can obtain the following norm bound for d_e defined in (17):

$$|d_e(t)| \leq \bar{d}_e + \pi_2/\underline{k}_p\,, \quad \bar{d}_e := (\bar{\phi}_1\bar{\Phi} + \bar{\Phi}^2 + k_m + \lambda|e|)/\underline{k}_p\,, \tag{30}$$

with the exponential decaying function $\pi_2 = \bar{\pi}_1^2$.

In the proposed scheme, the following proposition provides one possible modulation function implementation so that (19) holds.

Proposition 1: Consider the system (4)–(6), reference model (14) and control law (12). Outside of the Δ-vicinity $\mathcal{D}_\Delta$, if ρ in (12) is designed as

$$\rho := \frac{1}{\underline{k}_p}\left[\bar{\phi}_1\bar{\Phi} + \bar{\Phi}^2 + k_m + \lambda|e|\right] + \Pi(k) + \delta\,, \tag{31}$$

then, while $z \notin \mathcal{D}_\Delta$, one has: **(a)** the monitoring function switching stops, **(b)** no finite-time escape occurs in the system signals ($t_M \to +\infty$), and **(c)** the error e tends to zero in finite time. The term $\Pi(k) = a(k)e^{-t/a(k)}$ with $a(k)$ being any positive monotonically increasing unbounded sequence and δ is any arbitrarily small positive constant.

Proof. See Appendix.

Remark 2 (Modulation Function Reset): The term $\Pi(k)$ in (31) plays a key role in the domination of the exponential decaying term $\pi_2/\underline{k}_p$ in (30). It allows that inequality (19) is satisfied before that $\pi_2/\underline{k}_p$ ultimately vanishes. However, since $\Pi(k) \to +\infty$ as $k \to +\infty$, the modulation function needs a reset mechanism to reinitialize k, from time to time, in order to avoid that the controller amplitudes increase to very high values. In particular, if a first order nonlinear system is considered (i.e., the η-dynamics in (4)–(6) is dropped), the term $\Pi(k)$ can be removed.

D. Global Convergence Result

In the next theorem, we show that the proposed output-feedback controller based on monitoring function drives z to the Δ-vicinity of the unknown maximizer z^* defined in **(A3)**. It does not imply that $z(t)$ remains in $\mathcal{D}_\Delta$, $\forall t$. However, the oscillations around y^* can be made of order $\mathcal{O}(r)$.

Theorem 1: Consider the system (4)–(5), with output or cost function in (6), control law (12), modulation function (31) with $\Pi(k)$ reinitialized as in Remark 2., monitoring function (22)–(23) and reference trajectory (14). Assume that **(A1)–(A5)** hold, then: **(i)** the Δ-vicinity $\mathcal{D}_\Delta$ in **(A3)** is globally attractive being reached in finite time and **(ii)** for L_Φ in **(A3)** sufficiently small, the oscillations around the maximum value y^* of y can be made of order $\mathcal{O}(r)$, with r defined in (22). Since the signal y_m can be saturated in (14), all signals in closed-loop system remain uniformly bounded.

Proof. See Appendix.

E. Multiple Extrema Envisage

ESC applied to the achieve global maximum in the presence of local extrema is an challenging area. Sometimes, the exhaustive search of the solution set may be the only choice as discussed in [9]. The authors have presented a scalar extremum seeking feedback controller that achieves semi-global practical global extremum seeking despite local extrema.

Inspired by the ideas introduced there, it was observed that by tuning a new design parameter in the monitoring function properly it is possible to pass through a local extremum and converge to the global one as well as is done in [9], when the amplitude of the excitation (dither) signal is adaptively adjusted.

In this case, the monitoring function (22)–(23) should be replaced by

$$\varphi_k(t) = |e(t_k)|e^{-\lambda(t-t_k)} + r + c(k)\,, \qquad (32)$$

where $c(k)$ is any positive monotonically decreasing sequence, such that $c(k) \to 0$ as $k \to +\infty$, and $c(0) > \mathcal{O}(r)$.

IV. Illustrative Example

Consider in this example a plant with an unknown output performance characteristic and relative degree one dynamics described by

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (33)$$

$$y = \Phi(z) = e^{-\frac{(z-3)^2}{0.5}} + 1.5e^{-\frac{(z-5)^2}{1.5}}\,, \qquad (34)$$

where $x = [\eta^T \; z]^T$.

This type of problem structure is similar to many real world objectives including a 1-dimensional engine calibration whereby the optimal valve timing to maximize efficiency is sought [3]. Note that the performance map shown in Fig. 3 has multiple maxima, and, in contrast to [3], global convergence (for all domain of initial conditions) can be guaranteed.

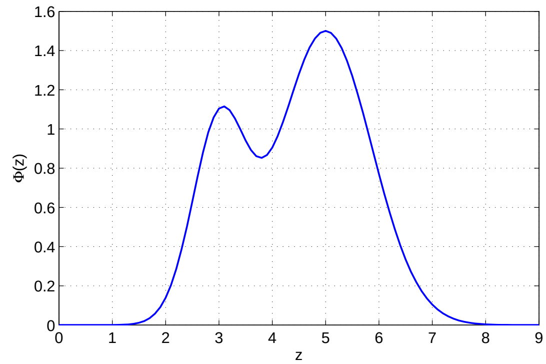

Fig. 3: Performance map $\Phi(z)$.

The modulation function was designed to satisfy (31) with $\Pi(k) = (k+1)e^{-t/(k+1)}$. The norm observer applied to determine $\bar{\eta}$ used in $\bar{\phi}_1(\bar{\eta}, z, t)$ has $\lambda_0 = 0.8$ and $\varphi_0(z,t) = 2z$. The monitoring function (32) used to face the problem of local extrema has $c(k) = 2/(k+1)$. The remaining parameters were: the lower bound $L_\Phi = \frac{2}{3}r$, $\lambda = 2$, $k_m = 1$, $\delta = 0.1$ and $r = 0.1$.

Fig. 4 illustrates the convergence of the scheme for different initial conditions of x, corresponding to $z(0) = 2$, 4 and 7. Note that, differently from [3] (see Fig. 6), where the convergence rate and global maximum are directly dependent on the initial conditions, here the example illustrates that it is possible to reach the global maximum in presence of local maximum without affecting the rate of convergence and independently of the initial conditions. As shown in Fig. 5, y tracks y_m until z reaches the vicinity of the global maximizer $z^* = 5$. In Fig. 3, the small oscillation just after $t = 1$s shows the capacity of the monitoring function to pass through a local maximum at $y = 1.1$ and converge to the global one at $y = 1.5$. Fig. 7 illustrates the monitoring function behavior. It can be seen that after reaching the global maximum, the error starts increasing because the reference trajectory is a ramp. Afterwards, the exact tracking is not obtained but y will be locked at some r-neighborhood of $y^* = 1.5$ and y_m will increase until the saturation value 10.

V. Conclusions

A new extremum-seeking controller based on monitoring function and norm state estimation was developed for a class of uncertain nonlinear plants. The resulting approach guarantees global convergence of the system output to a small neighborhood of the extremum point using only output-feedback. Simulation results were carried out to illustrate the remarkable controller performance even when local extrema were considered. To the best of our knowledge, real-time solutions with global convergence properties and based only on output-feedback did not exist. This paper proposes a solution to this problem.

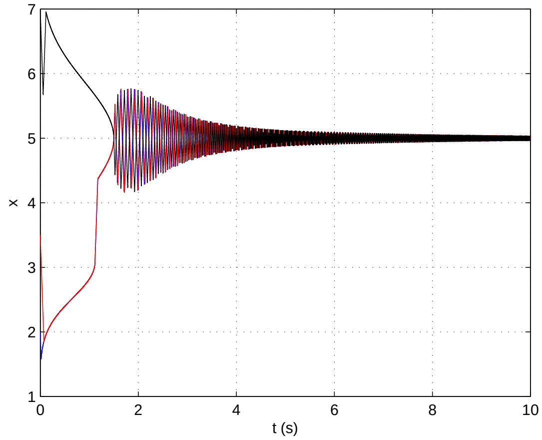

Fig. 4: Parameter z converges to $z^* = 5$ that maximizes y using different initial conditions $z(0)$.

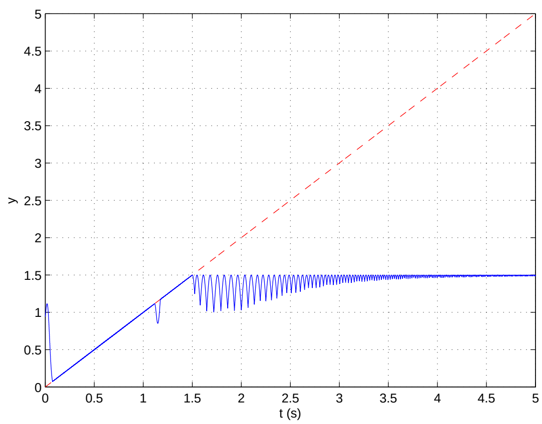

Fig. 5: Time history of the output plant y (solid line) and the output model y_m (dashed line). The output plant tends to the maximum value $y^* = 1.5$.

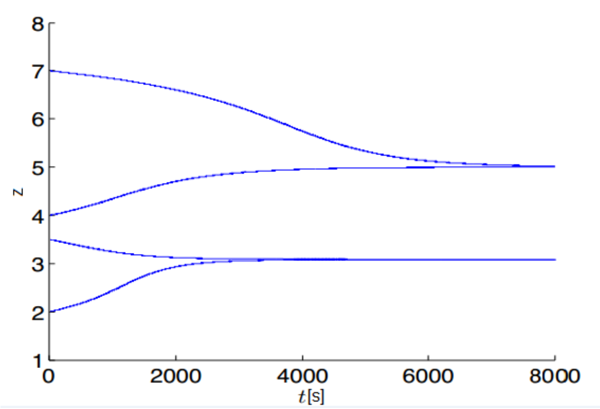

Fig. 6: Figure extracted from [3]. Convergence of z to local maxima for different values of $z(0)$.

APPENDIX

The proofs can be found in the following link: http://www.coep.ufrj.br/~tiagoroux/CDC2013/proofs.pdf

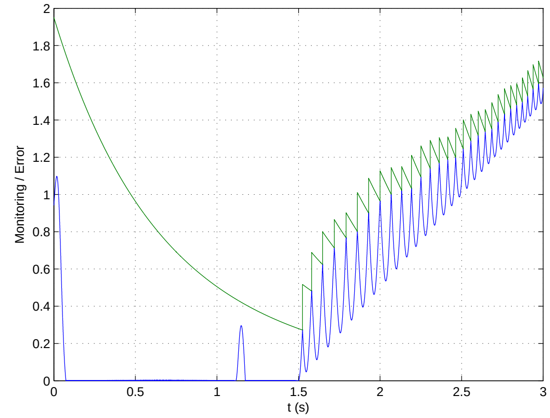

Fig. 7: Monitoring function φ_m and error norm $|e|$.

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