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On the Exactness of Semidefinite Relaxation for Nonlinear Optimization over Graphs: Part II

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Abstract— **This work is concerned with finding a global optimization technique for a broad class of nonlinear optimization problems, including quadratic and polynomial optimizations. The main objective of this two-part paper is to investigate how the (hidden) structure of a given real/complex-valued optimization makes the problem easy to solve. To this end, three conic relaxations are proposed and it is proved that some or all of these relaxations are exact if the optimization is highly structured. More precisely, the structure of the optimization is mapped into a generalized weighted graph, where each edge is associated with a weight set extracted from the coefficients of the optimization. In Part I of the paper, it is shown that the relaxations are all exact for general graphs in the real-valued case and for acyclic graphs in the complex-valued case, provided the weight sets satisfy some sign definiteness conditions. In this part of the paper, the complex-valued case is further studied, and three structural properties are derived for the generalized weighted graph, each of which guarantees the exactness of some of the proposed relaxations. It is also shown that this result holds true if the graph can be decomposed as a union of edge-disjoint subgraphs, where each subgraph has one of the derived structural properties. As an application of this paper, it is finally proved that a broad class of optimization problems for power networks are polynomial-time solvable due to the passivity of transmission lines and transformers.**

I. Introduction

Several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program (QCQP) as a special case, are nonlinear/non-convex and NP-hard in the worst case. Due to the complexity of such problems, various convex relaxations based on linear matrix inequality (LMI), semidefinite programming (SDP), and second-order cone programming (SOCP) have gained popularity [1], [2]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value. The SDP relaxation converts an optimization with a vector variable to a convex optimization with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank solution for the SDP relaxation.

This paper is motivated by the fact that real-world optimization problems are highly structured in many ways and their structures could in principle help reduce the computational complexity. The high-level objective is to understand how the computational complexity of a given nonlinear optimization is related to its (hidden) structure. This work is concerned with a broad class of nonlinear real/complex optimization problems, including QCQP. The main feature of this class is that the argument of each objective and constraint function is quadratic (as opposed to linear) with respect to the optimization variable, and the goal is to use three conic relaxations (SDP, reduced SDP, and SOCP) to convexify the argument of the optimization.

In this work, the structure of the nonlinear optimization is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization (e.g., the coefficients). This generalized weighted graph captures both the sparsity of the optimization and possible patterns in the coefficients. In Part I of the paper, it is shown that the proposed relaxations are exact if the variable of the optimization is real-valued, provided the generalized weighted graph satisfies some weak properties. To study the complex-valued case, the notion of "sign-definite complex weight sets" is introduced and it is then proved that the relaxations are exact for a complex optimization if the graph is acyclic with sign definite weight sets (with respect to complex numbers). In this part of the paper, the complex case is further studied for general graphs. In particular, it is shown that if the graph can be decomposed as the union of some edge-disjoint subgraphs in such a way that each subgraph possesses one of the four proposed structural properties, then the SDP relaxation is tight. As an application of this work in optimization for power systems, it is also shown that a broad class of energy optimization problems can be convexified due to the physics of power networks. The results of this paper extend the recent works on energy optimization [3]–[8] and general quadratic optimization [9], [10].

In the next section, we formally state the optimization problem. The main contributions of the paper are outlined in Section II-C, where the plan for the rest of the paper is also given.

II. Problem Statement and Contributions

For the sake of completeness in introducing the notations and stating the problem under study, most of the materials in the next two subsections have been repeated from Part I of the paper.

A. Notations and Definitions

Notation 1: In this work, scalars, vectors, and matrices will be shown by lowercase, bold lowercase, and uppercase

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letters (e.g., x, $\mathbf{x}$, and X). Furthermore, x_i denotes the i^{th} entry of a vector $\mathbf{x}$, and X_{ij} denotes the $(i,j)^{\text{th}}$ entry of a matrix X.

Notation 2: $\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}^n$, and $\mathcal{H}^n$ denote the sets of real numbers, complex numbers, $n \times n$ symmetric matrices, and $n \times n$ Hermitian matrices, respectively.

Notation 3: $\text{Re}\{M\}$, $\text{Im}\{M\}$, M^H, $\text{Rank}\{M\}$, and $\text{Trace}\{M\}$ denote the real part, imaginary part, conjugate transpose, rank, and trace of a given scalar/matrix M, respectively. The notation $M \succeq 0$ means that M is symmetric/Hermitian and positive semidefinite.

Notation 4: The imaginary unit is denoted as "i", while "i" is used for indexing.

Notation 5: Given an undirected graph $\mathcal{G}$, the notation $i \in \mathcal{G}$ means that i is a vertex of $\mathcal{G}$. Moreover, the notation $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of $\mathcal{G}$ and besides $i < j$.

Notation 6: Given a set $\mathcal{T}$, $|\mathcal{T}|$ denotes its cardinality. Given a graph $\mathcal{G}$, $|\mathcal{G}|$ shows the number of its vertices. Given a number (vector) $\mathbf{x}$, $|\mathbf{x}|$ denotes its absolute value (2-norm).

Definition 1: A finite set $\mathcal{T} \subset \mathcal{R}$ is said to be *sign definite with respect to* $\mathcal{R}$ if its elements are either all negative or all nonnegative. $\mathcal{T}$ is called *negative* if its elements are negative and is called *positive* if its elements are nonnegative.

Definition 2: A finite set $\mathcal{T} \subset \mathcal{C}$ is said to be *sign definite with respect to* $\mathcal{C}$ if when the sets $\mathcal{T}$ and $-\mathcal{T}$ are mapped into two collections of points in $\mathcal{R}^2$, then there exists a line separating the two sets (the elements of the sets are allowed to lie on the line).

Definition 3: Given a graph $\mathcal{G}$, a cycle space is the set of all possible cycles in the graph. An arbitrary basis for this cycle space is called a "cycle basis".

Definition 4: In this work, a graph $\mathcal{G}$ is called weakly cyclic if every edge of the graph belongs to at most one cycle in $\mathcal{G}$ (i.e., the cycles of $\mathcal{G}$ are all edge-disjoint).

Definition 5: Consider a graph $\mathcal{G}$, a subgraph $\mathcal{G}_s$ of this graph, and a matrix $X \in \mathcal{C}^{|\mathcal{G}| \times |\mathcal{G}|}$. Define $X\{\mathcal{G}_s\}$ as a submatrix of X obtained by choosing every row and column of X whose index belongs to the vertex set of $\mathcal{G}_s$. For instance, $X\{(i,j)\}$, where $(i,j) \in \mathcal{G}$, has rows i, j and columns i, j of X.

B. Problem Statement

Consider an undirected graph $\mathcal{G}$ with n vertices (nodes), where each edge $(i,j) \in \mathcal{G}$ has been assigned a nonzero edge weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ with k real/complex numbers (note that the superscripts in the weights are not exponents). This graph is called a *generalized weighted graph* as every edge is associated with a set of weights as opposed to a single weight. Consider an unknown vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}$ belonging to $\mathcal{D}^n$, where $\mathcal{D}$ is either $\mathcal{R}$ or $\mathcal{C}$. For every $i \in \mathcal{G}$, x_i is a variable associated with node i of the graph $\mathcal{G}$. Define:

$$\mathbf{y} = \left\{ |x_i|^2 \ \middle|\ \forall i \in \mathcal{G} \right\},$$
$$\mathbf{z} = \left\{ \text{Re}\left\{ c_{ij}^{(t)} x_i x_j^H \right\} \ \middle|\ \forall (i,j) \in \mathcal{G},\ t \in \{1,...,k\} \right\}$$

Note that according to Notation 5, $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of the graph and that $i < j$. The sets $\mathbf{y}$ and $\mathbf{z}$ can be regarded as two vectors, where

- $\mathbf{y}$ collects the quadratic terms $|x_i|^2$'s (one term for each vertex).
- $\mathbf{z}$ collects the cross terms $\text{Re}\{c_{ij}^{(t)} x_i x_j^H\}$'s ($k$ terms for each edge).

This work is concerned with the following optimization:

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{D}^n} \quad & f_0(\mathbf{y}, \mathbf{z}) \\ \text{subject to} \quad & f_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{1}$$

for given functions $f_0, ..., f_m$. The computational complexity of the above optimization depends in part on the structure of the functions f_j's. Regardless of these functions, Optimization (1) is intrinsically hard to solve (NP-hard in the worst case) because $\mathbf{y}$ and $\mathbf{z}$ are both nonlinear functions of $\mathbf{x}$. The objective is to convexify the second-order nonlinearity embedded in $\mathbf{y}$ and $\mathbf{z}$. To this end, notice that there exist two linear functions $l_1 : \mathcal{C}^{n \times n} \to \mathcal{R}^n$ and $l_2 : \mathcal{C}^{n \times n} \to \mathcal{R}^{k\tau}$ such that

$$\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right), \quad \mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$$

where τ denotes the number of edges in $\mathcal{G}$. As discussed in Part I of the paper, Optimization (1) is equivalent to

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{2b}$$
$$X \succeq 0, \tag{2c}$$
$$\text{Rank}\{X\} = 1 \tag{2d}$$

where there is an implicit constraint that $X \in \mathcal{S}^n$ if $\mathcal{D} = \mathcal{R}$ and $X \in \mathcal{H}^n$ if $\mathcal{D} = \mathcal{C}$. In order to reduce the complexity of Optimization (2), three relaxations will be proposed next.

SDP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{3a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j = 1, ..., m \tag{3b}$$
$$X \succeq 0 \tag{3c}$$

Reduced SDP relaxation: Choose a set of cycles $\mathcal{O}_1, ..., \mathcal{O}_p$ in the graph $\mathcal{G}$ such that they form a cycle basis. Let Ω denote the set of all subgraphs $\mathcal{O}_1, ..., \mathcal{O}_p$ as well as all edges of $\mathcal{G}$ that do not belong to any cycle in the graph (i.e., bridge edges). The reduced SDP relaxation is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{4a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{4b}$$
$$X\{\mathcal{G}_s\} \succeq 0, \qquad \forall \mathcal{G}_s \in \Omega \tag{4c}$$

SOCP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{5a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j = 1, ..., m \tag{5b}$$
$$X\{(i,j)\} \succeq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{5c}$$

The above SDP, reduced SDP, and SOCP relaxations are targeted at the non-convexity caused by the nonlinear relationship between $\mathbf{x}$ and $(\mathbf{y}, \mathbf{z})$. Define $f^*, f^*_{\text{SDP}}, f^*_{\text{r-SDP}}$, and f^*_{SOCP} as the optimal solutions of Optimizations (2), (3), (4), and (5), respectively. By comparing the feasible sets of these optimizations, it can be concluded that

$$f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^* \tag{6}$$

Given a particular optimization of the form (1), if any of the above inequalities turns into an equality, the associated relaxation may be able to find the solution of the original optimization. In this case, it is said that the relaxation is "tight" or "exact". The objective of this paper is to relate the exactness of the proposed relaxations to the topology of the graph $\mathcal{G}$ and some properties of its weight sets $\{c^{(1)}_{ij}, c^{(2)}_{ij}, ..., c^{(k)}_{ij}\}$'s.

C. Contributions

Throughout this paper, we assume that $f_j(\mathbf{y}, \mathbf{z})$ is monotonic in every entry of $\mathbf{z}$ for $j = 0, 1, ..., m$ (but possibly nonconvex in $\mathbf{y}$ and $\mathbf{z}$). With no loss of generality, suppose that $f_j(\mathbf{y}, \mathbf{z})$ is an increasing function with respect to all entries of $\mathbf{z}$. A few of the results to be developed in this work do not need this assumption, in which cases the name of the function f_j will be changed to g_j to avoid any confusion in the assumptions.

A number of results have been proved in Part I of the paper, including:

- Necessary and sufficient conditions are obtained for the exactness of the SDP, reduced SDP, and SOCP relaxations.
- It is shown that the relaxations are all exact in the real case $\mathcal{D} = \mathcal{R}$ if a set of conditions are satisfied: one for each weight set and one for each cycle. It is also shown that if some of these conditions are not satisfied, the SDP relaxation has a rank-2 (rather than rank-1) solution for certain generalized weighted graphs.
- It is proved that the proposed relaxations are exact in the complex case $\mathcal{D} = \mathcal{C}$ if $\mathcal{G}$ is acyclic and its weight sets are sign definite with respect to $\mathcal{C}$.

In Section III of this part of the paper, we continue exploring the complex-valued case $\mathcal{D} = \mathcal{C}$ and prove the following statements:

- The SOCP, reduced SDP, and SDP relaxations are tight if each weight set contains only real or imaginary numbers and it also satisfies a specific condition.
- The reduced SDP and SDP relaxations are exact if $\mathcal{G}$ is bipartite and weakly cyclic with positive or negative real weight sets.
- The reduced SDP and SDP relaxations are exact if $\mathcal{G}$ is a weakly cyclic graph with imaginary homogeneous weight sets.

In summary, four types of generalized weighted graphs have been obtained in the two parts of this paper, which make the SDP relaxation exact. It is also shown that if the graph $\mathcal{G}$ can be decomposed as a union of edge-disjoint subgraphs of these four types in an acyclic way, then the SDP relaxation is exact. In Section IV, a detailed discussion is provided to demonstrate how the results of this two-part paper can be used for optimization over power networks. Finally, some illustrative examples are given in section V.

III. Complex-Valued Optimization

In this section, Optimization (1) will be studied in the complex-valued case $\mathcal{D} = \mathcal{C}$.

A. Weakly Cyclic Graph with Real Edge Weights

In Part I of the paper, it is shown that the SDP relaxation is exact, provided $\mathcal{G}$ is acyclic and each weight set is sign definite with respect to $\mathcal{C}$. This result requires the assumption of monotonicity of $f_j(\mathbf{y}, \mathbf{z})$ with respect to $\mathbf{z}$ for $j = 0, 1, ..., m$. The first objective is to show that this assumption is not needed as long as the weight sets are real. To this end, consider the optimization

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{C}^n} \quad & g_0(\mathbf{y}, \mathbf{z}) \\ \text{s.t.} \quad & g_j(\mathbf{y}, \mathbf{z}) \leq 0, \quad j = 1, 2, ..., m \end{aligned} \tag{7}$$

for arbitrary functions $g_i(\cdot, \cdot)$, $i = 0, 1, ..., m$. The difference between the above optimization and (1) is that the functions $g_j(\cdot, \cdot)$'s may not be increasing in $\mathbf{z}$. One can derive the SDP, reduced SDP, and SOCP relaxations for the above optimization by replacing $f_0, ..., f_m$ with $g_0, ..., g_m$ in (3)-(5). In this subsection, it is aimed to investigate the case where the edge weights are all real numbers, while the unknown parameter $\mathbf{x}$ is complex.

Theorem 1: Consider the complex-valued case $\mathcal{D} = \mathcal{C}$ and assume that the edge weights of $\mathcal{G}$ are real numbers. The SDP, reduced SDP, and SOCP relaxations associated with Optimization (7) are all exact if the graph $\mathcal{G}$ is acyclic.

Proof: It is straightforward to show that every real set is sign definite with respect to $\mathcal{C}$. Therefore, the edge weight sets of $\mathcal{G}$ are all sign definite. Let X denote an arbitrary feasible point of the SOCP relaxation. Define $(\alpha_{ij}, \beta_{ij})$ as $(0, 1)$ for every $(i, j) \in \mathcal{G}$. Then,

$$\text{Re}\{c^{(t)}_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})\} = \text{Re}\{c^{(t)}_{ij}\}\alpha_{ij} - \text{Im}\{c^{(t)}_{ij}\}\beta_{ij} = 0$$

for every $t \in \{1, ..., k\}$ (note that $c^{(t)}_{ij} \in \mathcal{R}$ by assumption). As shown in the proof of Theorem 6 in Part I, two properties hold:

- There exists a positive number γ_{ij} such that
$$\left|X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})\right|^2 = X_{ii}X_{jj}$$
- There exists a set of angles $\{\theta_1, \theta_2, ..., \theta_n\}$ such that $\theta_i - \theta_j = \theta_{ij}$ for every $(i, j) \in \mathcal{G}$, where θ_{ij} denotes the phase of the complex number $X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})$.

Define the vector $\mathbf{x}$ as

$$\begin{bmatrix} \sqrt{X_{11}}e^{-\theta_1 \mathrm{i}} & \sqrt{X_{22}}e^{-\theta_2 \mathrm{i}} & \cdots & \sqrt{X_{nn}}e^{-\theta_n \mathrm{i}} \end{bmatrix}^H$$

Therefore,

$$\begin{aligned} \text{Re}\{c^{(t)}_{ij}x_i x^H_j\} &= \text{Re}\{c^{(t)}_{ij}X_{ij}\} + \gamma_{ij}\text{Re}\{c^{(t)}_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})\} \\ &= \text{Re}\{c^{(t)}_{ij}X_{ij}\} \end{aligned}$$

and hence

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) = l_2(X)$$

Given an arbitrary feasible point X for the SOCP relaxation, the above equalities imply that $\mathbf{x}$ is a feasible point of the original optimization (7), and that X and $\mathbf{x}$ both give rise to the same objective value. As a result, $f^* \leq f^*_{\text{SOCP}}$. The proof follows from this inequality and (6). ■

Consider the general optimization (7) in the case when $\mathcal{G}$ is acyclic with real edge weights. As discussed in Part I of the paper, the associated SDP relaxation may not be tight if its variable $\mathbf{x}$ is restricted to real numbers. However, Theorem 1 shows that the relaxation is exact if $\mathbf{x}$ is a complex-valued variable. In what follows, the results of Theorem 1 will be generalized to cyclic graphs for Optimization (1).

Theorem 2: Assume that $\{c_{ij}^{(1)},...,c_{ij}^{(k)}\}$ is a positive or negative real set for every $(i,j) \in \mathcal{G}$. The relations $f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold for Optimization (1) in the complex-valued case $\mathcal{D} = \mathcal{C}$ if the graph $\mathcal{G}$ is bipartite and weakly cyclic.

Proof: For brevity, the proof has been moved to [11]. ■

Note that the SOCP relaxation may not be exact under the assumptions of Theorem 2. As a direct application of this theorem, the class of quadratic optimization problems proposed later in Example 2 is polynomial-time solvable.

B. Cyclic Graph with Real and Imaginary Edge Weights

In this part, there is no specific assumption on the topology of the graph $\mathcal{G}$, but it is assumed that each edge weight is either real or purely imaginary. The definition of the edge sign σ_{ij} introduced in Part I of the paper for real-valued weight sets can be extended as follows:

$$\sigma_{ij} = \begin{cases} 1 & \text{if} \quad c_{ij}^{(1)},...,c_{ij}^{(k)} \geq 0 \\ -1 & \text{if} \quad c_{ij}^{(1)},...,c_{ij}^{(k)} \leq 0 \\ \mathrm{i} & \text{if} \quad c_{ij}^{(1)} \times \mathrm{i},...,c_{ij}^{(k)} \times \mathrm{i} \geq 0 \\ -\mathrm{i} & \text{if} \quad c_{ij}^{(1)} \times \mathrm{i},...,c_{ij}^{(k)} \times \mathrm{i} \leq 0 \\ 0 & \quad\text{otherwise} \end{cases}, \quad \forall (i,j) \in \mathcal{G}$$

By convention, $\sigma_{ij} = -1$ if $c_{ij}^{(1)} = \cdots = c_{ij}^{(k)} = 0$. Define also σ_{ji} as σ_{ij}^H for every $(i,j) \in \mathcal{G}$. The parameter σ_{ij} being nonzero implies that the elements of each edge weight set $\{c_{ij}^{(1)},...,c_{ij}^{(k)}\}$ are homogeneous in type (real or imaginary) and in sign (positive or negative). For every $r \in \{1,2,...,p\}$, let $\vec{\mathcal{O}}_r$ denote a directed cycle corresponding to $\mathcal{O}_r$, meaning that all edges of the undirected cycle $\mathcal{O}_r$ has been oriented consistently.

Theorem 3: The relations $f^*_{\text{SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold for Optimization (1) in the complex-valued case $\mathcal{D} = \mathcal{C}$ with real and purely imaginary edge weight sets if

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{8a}$$

$$\prod_{(i,j)\in\vec{\mathcal{O}}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad \forall r \in \{1,...,p\} \tag{8b}$$

Proof: Consider an arbitrary feasible point X for the SOCP relaxation. Choose a spanning tree of $\mathcal{G}$ and denote it as $\mathcal{T}$. In light of (8a), n numbers $\sigma_1, \sigma_2, ..., \sigma_n$ belonging to the set $\{\pm 1, \pm \mathrm{i}\}$ can be iteratively designed with the property that:

$$\sigma_i \sigma_j^H = -\sigma_{ij}, \qquad \forall (i,j) \in \mathcal{T}$$

This relation together with (8b) yields that

$$\sigma_i \sigma_j^H = -\sigma_{ij}, \qquad \forall (i,j) \in \mathcal{G}$$

Now, define $\mathbf{x}$ as

$$\begin{bmatrix} \sigma_1^H \sqrt{X_{11}} & \sigma_2^H \sqrt{X_{22}} & \cdots & \sigma_n^H \sqrt{X_{nn}} \end{bmatrix}^H$$

In line with the proofs of Theorems 2 and 6 in Part I of the paper, it can be shown that

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) \leq l_2(X)$$

and therefore

$$f_j(\mathbf{y},\mathbf{z}) \leq f_j(l_1(X), l_2(X))$$

for $j = 0,1,...,m$, where $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$. This means that corresponding to every feasible point X of the SOCP relaxation, the original optimization has a feasible point $\mathbf{x}$ with a lower or equal objective value. Therefore, $f^* \leq f^*_{\text{SOCP}}$. The proof is completes by combining this inequality with $f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$. ■

C. Weakly Cyclic Graph with Imaginary Edge Weights

If $\mathcal{G}$ has at least one odd cycle whose edge weight sets contain only imaginary numbers, then the conditions given in Theorem 3 are violated. The reason is that the product of an odd number of imaginary numbers (edge signs) can never become a real number. The high-level goal of this subsection is to show that the SDP relaxation can still be tight in presence of such cycles, while the SOCP relaxation is not guaranteed to be exact. Assume for now that $\mathcal{G}$ is weakly cyclic.

To proceed with the paper, a new SOCP relaxation needs to be introduced. This optimization assigns one real scalar variable q_i to every vertex $i \in \mathcal{G}$ and one 2×2 block matrix variable

$$\begin{bmatrix} U(\mathcal{G}_s) & V(\mathcal{G}_s) \\ V(\mathcal{G}_s)^H & W(\mathcal{G}_s) \end{bmatrix}$$

to every subgraph $\mathcal{G}_s \in \Omega$, where $U(\mathcal{G}_s), W(\mathcal{G}_s) \in \mathcal{S}^{|\mathcal{G}_s|}$, and $V(\mathcal{G}_s) \in \mathcal{R}^{|\mathcal{G}_s|\times|\mathcal{G}_s|}$. Let U, V, and W denote the parameter sets $\{U(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$, $\{V(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$, and $\{W(\mathcal{G}_s) \mid \forall \mathcal{G}_s \in \Omega\}$, respectively.

Notation 7: For every $\mathcal{G}_s \in \Omega$, we arrange the elements in the vertex set of $\mathcal{G}_s$ in an increasing order. Then, we index the rows and columns of each of the matrices $U(\mathcal{G}_s), V(\mathcal{G}_s), V(\mathcal{G}_s)$ according to the ordered vertex set of $\mathcal{G}_s$. For example, if $\mathcal{G}_s$ has three vertices $5,7,1$, the ordered set becomes $\{1,5,7\}$, and therefore the three rows of $U(\mathcal{G}_s)$ are called row 1, row 5, and row 7. As an example, $U_{17}(\mathcal{G}_s)$ refers to the last entry on the first row of $U(\mathcal{G}_s)$.

For every $r \in \{1,2,....,p\}$, let μ_r denote the largest index in the vertex set of $\mathcal{O}_r$. Define $\mathbf{q}$ as $\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$. Recall that $l_2(\mathbf{x}\mathbf{x}^H)$ is a vector corresponding to the set

$$\left\{ \mathrm{Re}\{c_{ij}^{(t)} x_i x_j^H\} \mid \forall (i,j) \in \mathcal{G},\ t \in \{1,...,k\} \right\}$$

Define $\bar{l}(V)$ as a vector obtained from $l_2(\mathbf{x}\mathbf{x}^H)$ by replacing each entry $\mathrm{Re}\{c_{ij}^{(t)} x_i x_j^H\}$ with a new term $\mathrm{Im}\{c_{ij}^t\} \times (V_{ij}(\mathcal{G}_s) - V_{ji}(\mathcal{G}_s))$, where $\mathcal{G}_s$ denotes the unique subgraph in

Ω containing the edge (i,j) (the uniqueness of such subgraph is guaranteed by the weakly cyclic property of $\mathcal{G}$).

Expanded SOCP: This optimization is defined as

$$\min_{\mathbf{q},U,V,W} \quad f_0(\mathbf{q},\bar{l}(V)) \tag{9a}$$

subject to:

$$f_j(\mathbf{q},\bar{l}(V)) \leq 0, \qquad j=1,2,...,m \tag{9b}$$

$$U_{ii}(\mathcal{G}_s)+W_{ii}(\mathcal{G}_s)=q_i, \quad \forall \mathcal{G}_s \in \Omega,\ i\in\mathcal{G}_s \tag{9c}$$

$$\begin{bmatrix} U_{ii}(\mathcal{G}_s) & V_{ij}(\mathcal{G}_s) \\ V_{ij}(\mathcal{G}_s) & W_{jj}(\mathcal{G}_s) \end{bmatrix} \succeq 0,\ \forall \mathcal{G}_s\in\Omega,\ (i,j)\in\mathcal{G}_s \tag{9d}$$

$$\begin{bmatrix} U_{jj}(\mathcal{G}_s) & V_{ji}(\mathcal{G}_s) \\ V_{ji}(\mathcal{G}_s) & W_{ii}(\mathcal{G}_s) \end{bmatrix} \succeq 0,\ \forall \mathcal{G}_s\in\Omega,\ (i,j)\in\mathcal{G}_s \tag{9e}$$

$$W_{\mu_r\mu_r}(\mathcal{O}_r)=0, \qquad r=1,2,...,p \tag{9f}$$

Similar to the argument made for the SOCP relaxation (5), the above optimization is in the form of an SOCP program because its constraints (9d) and (9e) can be replaced by linear and norm constraints. Moreover, this optimization can be regarded as an expanded version of the SOCP relaxation (5). Denote the optimal objective value of this optimization as $f^*_{\text{e-SOCP}}$.

Theorem 4: Consider Optimization (1) in the complex-valued case $\mathcal{D}=\mathcal{C}$, and assume that the graph $\mathcal{G}$ is weakly cyclic with only purely imaginary edge weights. The following statements hold:

i) The expanded SOCP is a relaxation for Optimization (1), meaning that $f^*_{\text{e-SOCP}} \leq f^*$.
ii) The expanded SOCP relaxation is exact if and only if it has a solution $(\mathbf{q}^*,U^*,V^*,W^*)$ for which all 2×2 matrices given in (9d) and (9e) have rank 1.
iii) $f^*_{\text{SOCP}} \leq f^*_{\text{e-SOCP}}$.
iv) $f^*_{\text{e-SOCP}} \leq f^*_{\text{r-SDP}}$.
v) The relations $f^*_{\text{e-SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold if $\sigma_{ij}\neq 0$ for every $(i,j)\in\mathcal{G}$.

Proof: Since the proof is long and involved, it has been moved to [11]. ■

Assume that the graph $\mathcal{G}$ is weakly cyclic and its edge weights are all imaginary numbers. Theorem 4 shows that

$$f^*_{\text{SOCP}} \leq f^*_{\text{e-SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$$

and that the relations

$$f^*_{\text{e-SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^* \tag{10}$$

hold if each edge weight set has homogeneous elements ($\sigma_{ij}=i$ or $-i$). Note that the SOCP relaxation may not be exact, and one needs to use the expanded SOCP relaxation in this case. Interestingly, this result makes no assumption on the signs of the edges belonging to the same cycle in the cycle basis (unlike (8b)).

Although Theorem 4 deals with imaginary coefficients, some of the results derived in this two-part paper for complex/real optimizations with real coefficients are based on this powerful theorem. This is due to the fact that real numbers may be converted to imaginary numbers through a simple multiplication.

D. General Graph with Complex Edge Weight Sets

Given an arbitrary subgraph $\tilde{\mathcal{G}}_s$ of the graph $\mathcal{G}$, four important types will be defined for this subgraph in the following:

- **Type I:** $\tilde{\mathcal{G}}_s$ is acyclic with complex weight sets with the property that $\{c_{ij}^{(1)},...,c_{ij}^{(k)}\}$ is sign definite with respect to $\mathcal{C}$ for every $(i,j)\in\tilde{\mathcal{G}}_s$.
- **Type II:** $\tilde{\mathcal{G}}_s$ is weakly cyclic with imaginary weight sets and nonzero signs σ_{ij} for all $(i,j)\in\tilde{\mathcal{G}}_s$ (i.e., $\sigma_{ij}=\pm\mathrm{i}$).
- **Type III:** $\tilde{\mathcal{G}}_s$ is bipartite and weakly cyclic with the property that $\{c_{ij}^{(1)},...,c_{ij}^{(k)}\}$ is a real weight set with a nonzero sign σ_{ij} (i.e., $\sigma_{ij}=\pm 1$) for every $(i,j)\in\tilde{\mathcal{G}}_s$.
- **Type IV:** $\tilde{\mathcal{G}}_s$ has only real and imaginary weights with the property that

$$\sigma_{ij}\neq 0, \qquad \forall (i,j)\in\tilde{\mathcal{G}}_s \tag{11a}$$

$$\prod_{(i,j)\in\vec{\mathcal{O}}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \quad \forall \mathcal{O}_r\in\{\mathcal{O}_1,...,\mathcal{O}_p\}\cap\tilde{\mathcal{G}}_s \tag{11b}$$

By assuming $\tilde{\mathcal{G}}_s=\mathcal{G}_s$, it follows from the theorems developed in this paper that the SDP relaxation is exact for Optimization (1) if $\mathcal{G}$ is Type I, II, III, or IV. In this subsection, the objective is to show that the relaxation is still tight if $\mathcal{G}$ can be decomposed into a number of Type I-IV subgraphs in an acyclic way.

Theorem 5: Assume that $\mathcal{G}$ can be decomposed as the union of a number of edge-disjoint subgraphs $\tilde{\mathcal{G}}_1,...,\tilde{\mathcal{G}}_\omega$ in such a way that

i) $\tilde{\mathcal{G}}_s$ is Type I, II, III, or IV for every $s\in\{1,...,\omega\}$.
ii) The cycle $\mathcal{O}_r$ is entirely inside one of the subgraphs $\tilde{\mathcal{G}}_1,...,\tilde{\mathcal{G}}_\omega$ for every $r\in\{1,...,p\}$.

Then, the relations $f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold for Optimization (1) in the complex-valued case $\mathcal{D}=\mathcal{C}$.

Proof: The proof has been moved to [11] due to space restrictions. ■

IV. Application in Power Systems

A majority of real-world optimizations are naturally 'optimization over graph", meaning that they are defined over the graphs characterizing some physical systems. For example, optimization problems in circuits, antenna systems, and communication networks can easily be regarded as "optimization over graph". Then, the question of interest is: how is the computational complexity of an optimization related to the structure of the system over which the optimization is performed? This question will be explored here in the context of electrical power grids. Assume that the graph $\mathcal{G}$ corresponds to an AC power network, where:

- The power network has $|\mathcal{G}|$ nodes.
- For every $(i,j)\in\mathcal{G}$, nodes i and j are connected to each other in the power network via a transmission line with the impedance $g_{ij}+b_{ij}\mathrm{i}$.
- Each node $i\in\mathcal{G}$ of the network is connected to an external device, which exchanges electrical power with the power network.

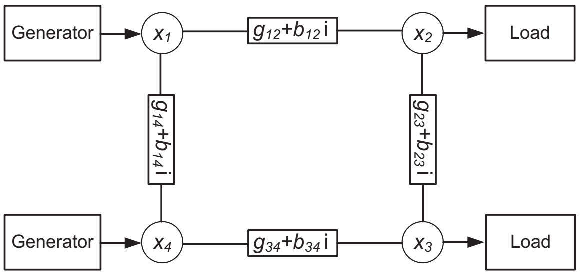

Fig. 1. An example of the power circuit studied in Section IV.

Fig. 2. This figure illustrates that each transmission line has four flows.

Figure 1 exemplifies a sample power network with two external devices generating power and two devices consuming power. As shown in Figure 2, each line $(i,j) \in \mathcal{G}$ is associated with four power flows:

- p_{ij}: Active power entering the line from node i
- p_{ji}: Active power entering the line from node j
- q_{ij}: Reactive power entering the line from node i
- q_{ji}: Reactive power entering the line from node j

Note that $p_{ij}+p_{ji}$ and $q_{ij}+q_{ji}$ represent the active and reactive losses incurred in the line. Let x_i denote the complex voltage (phasor) for node $i \in \mathcal{G}$. One can write:

$$\begin{aligned}
p_{ij}(\mathbf{x}) &= \text{Re}\left\{x_i(x_i-x_j)^H\frac{1}{g_{ij}-b_{ij}\text{i}}\right\}\\
p_{ji}(\mathbf{x}) &= \text{Re}\left\{x_j(x_j-x_i)^H\frac{1}{g_{ij}-b_{ij}\text{i}}\right\}\\
q_{ij}(\mathbf{x}) &= \text{Im}\left\{x_i(x_i-x_j)^H\frac{1}{g_{ij}-b_{ij}\text{i}}\right\}\\
q_{ji}(\mathbf{x}) &= \text{Im}\left\{x_j(x_j-x_i)^H\frac{1}{g_{ij}-b_{ij}\text{i}}\right\}
\end{aligned}$$

Note that since the flows all depend on $\mathbf{x}$, the argument $\mathbf{x}$ has been added to the above equations (e.g., $p_{ij}(\mathbf{x})$ instead of p_{ij}). The flows $p_{ij}(\mathbf{x})$, $p_{ji}(\mathbf{x})$, $q_{ij}(\mathbf{x})$, and $q_{ji}(\mathbf{x})$ can all be expressed in terms of $|x_i|^2$, $|x_j|^2$, and Re$\left\{c_{ij}^{(t)}x_ix_j^H\right\}$ for $t=1,2,3,4$, where

$$\begin{aligned}
c_{ij}^{(1)} &= \frac{-1}{g_{ij}-b_{ij}\text{i}}, \quad & c_{ij}^{(2)} &= \frac{-1}{g_{ij}+b_{ij}\text{i}}\\
c_{ij}^{(3)} &= \frac{\text{i}}{g_{ij}-b_{ij}\text{i}}, \quad & c_{ij}^{(4)} &= \frac{-\text{i}}{g_{ij}+b_{ij}\text{i}}
\end{aligned}$$

(note that Re$\{\alpha x_jx_i^H\}$ = Re$\{\alpha^H x_ix_j^H\}$ and Im$\{\alpha x_jx_i^H\}$ = Re$\{(-\alpha\text{i})x_ix_j^H\}$ for every value of α). Define

$$\begin{aligned}
\mathbf{p}(\mathbf{x}) &= \left\{p_{ij}(\mathbf{x}),p_{ji}(\mathbf{x})\;\middle|\;\forall(i,j)\in\mathcal{G}\right\}\\
\mathbf{q}(\mathbf{x}) &= \left\{q_{ij}(\mathbf{x}),q_{ji}(\mathbf{x})\;\middle|\;\forall(i,j)\in\mathcal{G}\right\}
\end{aligned}$$

Consider the optimization

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{C}^n}\quad & h_0(\mathbf{p}(\mathbf{x}),\mathbf{q}(\mathbf{x}),\mathbf{y}(\mathbf{x}))\\
\text{s.t.}\quad & h_j(\mathbf{p}(\mathbf{x}),\mathbf{q}(\mathbf{x}),\mathbf{y}(\mathbf{x}))\leq 0,\quad j=1,2,...,m
\end{aligned} \tag{12}$$

for given functions $h_0,...,h_m$, where $y(\mathbf{x})$ is the vector of $|x_i|^2$'s. This optimization aims to optimize the flows in a power grid. The constraints of this optimization are meant to limit line flows, voltage magnitudes, power delivered to each load, and power supplied by each generator. Observe that $\mathbf{p}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are both quadratic in $\mathbf{x}$. Assume that $h_j(\cdot,\cdot,\cdot)$ is increasing (or decreasing) in its first and second vector arguments. Since the above optimization can be cast as (1), the SDP, reduced SDP, and SOCP relaxations introduced before can be used to eliminate the effect of quadratic terms. To study under what conditions the relaxations are exact, note that each edge (i,j) of $\mathcal{G}$ has the weight set $\{c_{ij}^{(1)},c_{ij}^{(2)},c_{ij}^{(3)},c_{ij}^{(4)}\}$. Due to the physics of a transmission line, g_{ij} and b_{ij} are both nonnegative real numbers. As a result of this property, the set $\{c_{ij}^{(1)},c_{ij}^{(2)},c_{ij}^{(3)},c_{ij}^{(4)}\}$ turns out to be sign definite (see Definition 2). Now, in light of Theorem 5, the relaxations are all exact as long as $\mathcal{G}$ is acyclic. This result also holds for cyclic power networks with a sufficient number of phase shifters (the graph for a mesh power network with phase shifters can be converted to an acyclic one [5]).

Another interesting case is the optimization of active power flows for lossless networks. In this case, g_{ij} is equal to zero for every $(i,j)\in\mathcal{G}$. Hence, $p_{ji}(\mathbf{x})$ can be simply replaced by $-p_{ij}(\mathbf{x})$. Motivated by this observation, define the reduced vector of active powers as

$$\mathbf{p}_r(\mathbf{x}) = \left\{p_{ij}(\mathbf{x})\;\middle|\;\forall(i,j)\in\mathcal{G}\right\}$$

and consider the optimization

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{C}^n}\quad & \bar{h}_0(\mathbf{p}_r(\mathbf{x}),\mathbf{y}(\mathbf{x}))\\
\text{s.t.}\quad & \bar{h}_j(\mathbf{p}_r(\mathbf{x}),\mathbf{y}(\mathbf{x}))\leq 0,\quad j=1,2,...,m
\end{aligned}$$

for some functions $\bar{h}_0(\cdot,\cdot),...,\bar{h}_m(\cdot,\cdot)$, which are assumed to be increasing in their first vector argument. Now, each edge (i,j) of the graph $\mathcal{G}$ is accompanied by the singleton weight set $\left\{\frac{-\text{i}}{b_{ij}}\right\}$. Due to Theorem 4, the SDP and reduced SDP relaxations are exact if $\mathcal{G}$ is weakly cyclic. This is the generalization of the result obtained in [8] for optimization over lossless networks.

V. EXAMPLES

Example 1: Consider the optimization

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{C}^7}\quad & \mathbf{x}^HM\mathbf{x}\\
\text{s.t.}\quad & |x_i|=1,\quad i=1,2,...,7
\end{aligned} \tag{13}$$

where M is a given Hermitian matrix. Assume that the weighted graph $\mathcal{G}$ depicted in Figure 3 captures the structure of this optimization, meaning that: (1) $M_{ij}=0$ for every pair $(i,j)\in\{1,2,...7\}$ such that $(i,j)\notin\mathcal{G}$, $(j,i)\notin\mathcal{G}$, and $i\neq j$, (2) M_{ij} is equal to the edge weight c_{ij} for every

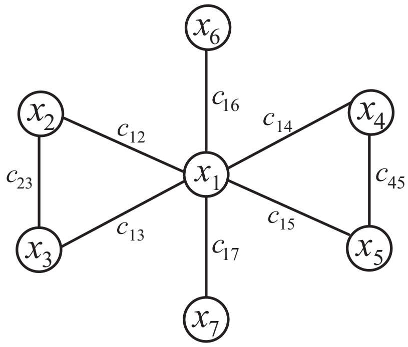

Fig. 3. The weighted Graph $\mathcal{G}$ studied in Example 1.

$(i, j) \in \mathcal{G}$. The SDP relaxation of this optimization is as follows:

$$\begin{aligned} \min_{X \in \mathcal{C}^{7\times 7}} \quad & \text{Trace}\{MX\} \\ \text{s.t.} \quad & X_{jj} = 1, \quad j = 1, 2, ..., 7 \\ & X = X^H \succeq 0 \end{aligned}$$

Define $\mathcal{O}_1$ and $\mathcal{O}_2$ as the cycles induced by the vertex sets $\{1, 2, 3\}$ and $\{1, 4, 5\}$, respectively. Now, the reduced SDP and SOCP relaxations can be obtained by replacing the constraint $X = X^H \succeq 0$ in the above optimization with certain small-sized constraints based on $\mathcal{O}_1$ and $\mathcal{O}_2$. In light of Theorem 5, the following statements hold:

- The SDP, reduced SDP, and SOCP relaxations are all exact in the case when $c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{45}$ are real numbers satisfying the inequalities $c_{12}c_{13}c_{23} \leq 0$ and $c_{14}c_{15}c_{45} \leq 0$.
- The SDP and reduced SDP are exact in the case when $c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{45}$ are imaginary numbers (note that the SOCP relaxation may not be tight).
- The SDP, reduced SDP, and SOCP relaxations are all exact in the case when each of the sets $\{c_{12}, c_{13}, c_{23}\}$ and $\{c_{14}, c_{15}, c_{45}\}$ has at least one zero element.

The above results demonstrate how the combined effect of the graph topology and the edge weights makes various relaxations exact for the quadratic optimization (13).

Example 2: Consider the optimization

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{C}^n} \quad & \mathbf{x}^H M \mathbf{x} \\ \text{s.t.} \quad & |x_j| = 1, \quad j = 1, 2, ..., m \end{aligned} \tag{14}$$

where M is a symmetric real-valued matrix. It has been proven in [12] that this problem is NP-hard even in the case when M is restricted to be positive semidefinite. Consider the graph $\mathcal{G}$ associated with the matrix M. As an application of Theorem 2, the SDP and reduced SDP relaxations are exact for this optimization and therefore this problem is polynomial-time solvable, provided that $\mathcal{G}$ is bipartite and weakly cyclic.

VI. CONCLUSIONS

This work deals with three conic relaxations for a broad class of nonlinear real/complex optimization problems, where the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable. Several types of optimizations, including polynomial optimization, can be cast as the problem under study. To explore the exactness of the proposed relaxations, the structure of the optimization is mapped into a generalized weighted graph with a weight set assigned to each edge. In the case of real-valued optimization, it is shown in Part I that the relaxations are exact if a set of conditions is satisfied, which depends on some weak properties of the underlying generalized weighted graph. A similar result is derived in Part I in the complex-valued case after introducing the notion of "sign-definite complex weight sets", under the assumption that the graph is acyclic. In this part, the complex case is further studied for general graphs, and it is shown that if the graph can be decomposed as the union of edge-disjoint subgraphs, each satisfying one of the four derived structural properties, then two of the relaxations are exact. As an application, it is finally shown that the weight sets are sign definite for power networks due to the passivity of transmission lines, and this makes a broad class of energy optimization problems easy to solve.

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