52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Distributed H_∞ Tracking Control for Discrete-Time Multi-Agent Systems with a High-Dimensional Leader

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Abstract— **This paper considers the distributed H_∞ leader-following tracking problem for a class of discrete-time multi-agent systems with a high-dimensional dynamic leader. It is assumed that output information about the leader is only available to designated followers, and the dynamics of the followers are subject to perturbations. To achieve distributed H_∞ leader-following tracking, a new class of control protocols is proposed which is based on the feedback from the nearest neighbors as well as a distributed state estimator. Under the assumptions that dynamics of the leader are detectable and the communication topology contains a directed spanning tree, sufficient conditions are obtained that enable all followers to track the leader while achieving a desired H_∞ leader-following tracking performance. Numerical simulations illustrate the effectiveness of the theoretical analysis.**

I. Introduction

Distributed control of multi-agent systems has been receiving a great deal of attention in the recent literature due to its broad applications in a number of areas; e.g., see [1]-[3]. The leader-follower tracking problem represents a particular class of distributed control problems which is concerned with the design of control protocols for each agent based only on the information from the nearest neighbours, with the aim to guarantee that states of all followers converge to that of a dynamic leader; e.g, see [4].

This paper considers the leader-following tracking problem for a class of discrete-time linear multi-agent systems with a high-dimensional leader and undirected communications between followers. Closely related work includes [5]–[9]. These references exemplify a common trend in the existing literature, which has a significant focus on consensus of discrete-time multi-agent systems with first or second-order integrator dynamics. In contrast, consensus tracking for discrete-time multi-agent systems with identical linear higher-order node dynamics was studied in [8], [9]. The work in [8], [9] covers the results on consensus tracking for multi-agent systems consisting of first- and second-order integrator dynamics as special cases, respectively. Also, in the majority of the existing papers on this topic, including some of the previously mentioned references, the leader and all the followers are assumed to have identical dynamics models. This assumption allows one to directly analyze dynamics of the tracking error arising in the corresponding multi-agent networks consisting of closed-loop agent systems. On the contrary, the case where dynamics of the leader and those of the followers have different models (e.g., are described by state-space equations of different order) has not received as much attention.

In this paper, we focus on the case where dynamics of the leader are more complex than those of the followers. Therefore, the existing theoretical approaches for analyzing leader-following tracking problems which have been developed for networks of identical agents cannot be directly applied in this case. Furthermore, the state of the leader which evolves independently of the followers is not directly measurable by all of the followers. It is assumed that only a partial information about the state of the leader can be sensed by a small group of followers which are subject to uncertainty. Thus, the multi-agent system under consideration is more general and contains some other commonly studied classes of leader-following multi-agent systems such as, e.g., the multi-agent systems in [9], as special cases. The control goal here is to design a tracking protocol for each agent such that the closed-loop system of agents achieves a desired level of H_∞ leader-following tracking performance. Note that the proposed problem of distributed H_∞ leader-following tracking for multi-agent systems with a high-dimensional leader is meaningful in a number of practical applications such as the design of distributed sensor networks [10] where dynamics of the leader are more complex than those of the followers.

The fact that the followers can only sense partial information about the state of the leader prompts us to introduce a dynamic protocol where a local controller together with a neighbor-based state observer is designed for each follower. In the present framework, the state observer embedded in the followers performs the task of estimating the unmeasurable states of the leader in a distributed way. Under the assumptions that the leader is detectable and the communication topology contains a directed spanning tree, we propose a procedure for the design of a tracking protocol which involves a solution to a modified algebraic Riccati equation. The analysis of distributed H_∞ leader-following tracking performance of this protocol when applied to a multi-agent system with a high-dimensional leader is then presented.

The protocol design to achieve a pre-specified level of H_∞ leader-following tracking performance for a system of agents whose dimension are different form that of the leader is the main contribution of this paper. We note that dynamic protocols have been considered in a number of recent papers.

This research was supported under Australian Research Council's Discovery Projects funding scheme (Project number DP120102152). Part of this work was carried out during the second author's visit to the Australian National University.

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For example, a dynamic protocol for synchronization of multiagent systems has been proposed recently in [16]. Similar to this paper, the analysis in [16] is based on the reduction of the problem to the analysis of N decoupled systems. However, in contrast to our paper, robust performance issues are not considered in [16]. It is also worth noting that there is another possible way to solve tacking problems under partial information about the leader, by using the distributed output regulation theory and internal model principle [11].

The remainder of this paper is organized as follows. In Section II, some preliminaries from the graph theory and the problem formulation are given. In Section III, the main results are presented. A numerical example and simulations to illustrate our theoretical analysis are provided in Section IV. Section V concludes the paper.

Notation: Let $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ be the sets of $n\times n$ real matrices and complex matrices, respectively. Let $\mathbb{N}$ and ℓ_2^n be, respectively, the sets of natural numbers and the n-dimensional real square summable functions. If not explicitly stated, all matrices are assumed to have compatible dimensions. The superscripts T and H denote the transpose and the Hermitian adjoint of a matrix, respectively. A matrix $U\in\mathbb{C}^{n\times n}$ is a unitary matrix if $U^HU=UU^H=I_n$. $\mathrm{diag}(a_1,a_2,\cdots,a_n)$ represents a diagonal matrix with a_i, $i=1,2,\cdots,n$ on its diagonal. The notation $\mathbf{1}_n\in\mathbb{R}^n$ denotes the vector whose elements are equal to 1. Let O_n and I_n be the $n\times n$ zero and identity matrices, respectively. A square matrix is said to be Schur stable if the magnitude of all of its eigenvalues is less than 1. The symbols $\otimes$ and $\|\cdot\|$ denote, respectively, the Kronecker product and the Euclidian norm.

II. Preliminaries and the problem formulation

A. Preliminaries

Let $\mathscr{G}(\mathscr{V},\mathscr{E},\mathscr{A})$ be a directed graph with a set of nodes $\mathscr{V}=\{\upsilon_1,\upsilon_2,\cdots,\upsilon_N\}$, a set of directed edges $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\mathscr{A}=[a_{ij}]_{N\times N}$. We will use the simplified notation $\mathscr{G}$ for the graph when this causes no confusion. A directed edge e_{ij} is associated with an ordered pair of nodes (υ_j,υ_i), where υ_j and υ_i are called the parent and child nodes, respectively, and $e_{ij}\in\mathscr{E}$ if and only if $a_{ij}>0$. Furthermore, self-loops are not allowed, i.e., $a_{ii}=0$ for all $i=1,2,\cdots,N$. A directed path from node υ_i to υ_j is an ordered sequence of edges, $\{(\upsilon_i,\upsilon_{k_1}),(\upsilon_{k_1},\upsilon_{k_2}),\ldots,(\upsilon_{k_l},\upsilon_j)\}\subseteq\mathscr{E}$, with distinct nodes $\upsilon_{k_m}, m=1,2,\cdots,l$. A directed tree is a directed graph such that (a) its every node υ_k $(k\neq r)$, except for the root node υ_r, has exactly one parent, and (b) there exists a unique directed path from υ_r to each node υ_k $(k\neq r)$. A directed spanning tree of $\mathscr{G}$ is a directed tree that has the same node set $\mathscr{V}$ and whose edge set is a subset of $\mathscr{E}$. $\mathscr{G}$ will reduce to an undirected graph if and only $a_{ij}=a_{ji}$, for all $i,j=1,2,\cdots,N$. A matrix $D=[d_{ij}]_{N\times N}$ is called a row-stochastic matrix associated with the graph $\mathscr{G}$, if the following properties hold: $d_{ii}>0$; $d_{ij}>0$ if $(j,i)\in\mathscr{E}$, and $d_{ij}=0$ otherwise; and $\sum_{j=1}^N d_{ij}=1$, for all $i=1,2,\cdots,N$.

Lemma 1 ([12]): For any row-stochastic matrix D associated with the graph $\mathscr{G}$, 1 is an eigenvalue of D, and all other eigenvalues of D lie in the open unit disk. Furthermore, 1 is a simple eigenvalue of D if and only if the graph $\mathscr{G}$ contains a directed spanning tree.

B. The multi-agent system

Consider a group of N agents indexed by $1,2,\cdots,N$. Without loss of generality, it is assumed that the agent labeled 1 is the leader, whose dynamics are governed by the following equations

$$\begin{aligned}\Theta(k+1)&=\hat{A}\Theta(k),\\ y(k+1)&=\hat{C}\Theta(k+1),\quad k\in\mathbb{N}.\end{aligned}\tag{1}$$

The vector $\Theta(k)\in\mathbb{R}^{nm_0}$ represents the state of the leader at time instant k, m_0 and n are two positive integers. This vector is assumed to be partitioned as $\Theta(k)=(\theta_1(k)^T,\theta_2(k)^T,\cdots,\theta_n(k)^T)^T$, where $\theta_i(k)\in\mathbb{R}^{m_0}$, $i=1,2,\cdots,n$. Accordingly, the state matrix $\hat{A}=[\hat{a}_{ij}]$ is in $\mathbb{R}^{nm_0\times nm_0}$, and admits a compatible partition of the following form:

$$\hat{A}=\begin{pmatrix}\hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1n}\\ \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \hat{A}_{n1} & \hat{A}_{n2} & \cdots & \hat{A}_{nn}\end{pmatrix},\tag{2}$$

where $\hat{A}_{ij}\in\mathbb{R}^{m_0\times m_0}$, $i,j=1,2,\cdots,n$. The vector $y(k+1)$ represents the output information about the leader that is made available to the followers for sensing at time instant $k+1$. However, as will be seen later, it will be assumed that only a partial information about this vector is sensed by some of the followers. Therefore, without loss of generality it is assumed that $\hat{C}=(O_{m_0},\cdots,O_{m_0},I_{m_0})\in\mathbb{R}^{m_0\times nm_0}$, i.e., $y(k)=\theta_n(k)$, for $k\in\mathbb{N}$. The above model for the leader reflects the common situation where the leader of the group plays the role of a command generator providing reference states to be tracked by the followers. Therefore, it is natural to assume that the state of the leader evolves in accordance with its intrinsic nominal model and is not influenced by the followers.

The dynamics of the followers, labeled as i, $i=2,3,\cdots,N$, are described by the state equations

$$x_i(k+1)=Ax_i(k)+u_i(k)+B_\omega\omega_i(k),\quad k\in\mathbb{N},\tag{3}$$

where $A\in\mathbb{R}^{m_0\times m_0}$ and $B_\omega\in\mathbb{R}^{m_0\times m_\omega}$ are constant matrices, $u_i(k)\in\mathbb{R}^{m_0}$ is the control input to be designed, and $\omega_i(k)\in\ell_2^{m_\omega}$ is a disturbance input. As mentioned, we assume that some of the followers are able to sense the output of the leader. In a general form, the output $\hat{y}_i(k)$ of the leader sensed by the follower i, $2\leq i\leq N$, is expressed as

$$\hat{y}_i(k)=c_iEy(k),\tag{4}$$

where $c_i\geq 0$, and $c_i>0$ if and only if the leader is a neighbor of follower i. The matrix $E\in\mathbb{R}^{m_y\times m_0}$ characterizes the protocol for information exchange in the network, and $m_y\leq m_0$. This reflects the situation where only a partial information about the leader is made available to selected followers. Clearly, if the agent i is not connected to the leader, then $\hat{y}_i(k)=0$.

The present framework has several features that distinguish it from similar problems considered in the literature. Firstly, while we assume that the state matrix $\hat{A}$ is known to the followers, though the initial condition $\Theta(0)$ is unknown, the leader and the followers have significantly different models in that the states of the leader and the followers have different dimensions, and are also governed by different state matrices. Secondly, the followers employ different output matrices c_iE; the scaling parameters c_i may reflect differences in the strength of the leader's signal received by the followers positioned at a different distance from the leader. Finally, dynamics of the followers are subject to uncertain perturbations. As is well known, within the H_∞ framework, such perturbations are often associated with unmodeled uncertain dynamics, and can be used to account for imperfections in the followers' models.

In regard to communications between the followers, define the information output of follower i to be

$$y_i(k) = Ex_i(k), \quad 2 \le i \le N,$$

where $E \in \mathbb{R}^{m_y \times m_0}$ is the matrix from equation (4). We will assume that each follower i can only use for control the information outputs of its neighbours relative its own output, $y_i(k) - y_j(k)$. As mentioned previously, the matrix E characterizes the protocol for information exchange in the network. The matrix E is in general a rectangular matrix; this allows for the vectors $y_i(k) - y_j(k)$ exchanged between the neighboring followers i and j to have a lower dimension than their states. Note that all agents are assumed to use the same matrix E. As discussed in [17], in certain applications, using a common communication matrix by all followers is not a significant limitation.

For the notational convenience, let[1] $a_{i1} = c_i$ and introduce the relative information available to follower i, $2 \le i \le N$, given by

$$\begin{aligned} \varepsilon_i(k) &= \frac{1}{\kappa}\left[\sum_{j=2}^{N} a_{ij}(y_i(k) - y_j(k)) + (c_i y_i(k) - \hat{y}_i(k))\right], \\ &= \frac{1}{\kappa}\left[\sum_{j=2}^{N} a_{ij}E(x_i(k) - x_j(k)) + a_{i1}E\,(x_i(k) - \theta_n(k))\right], \end{aligned} \tag{5}$$

where $\kappa = \kappa_0 + h$, $\kappa_0 = \max_{i=2,3,\cdots,N}\{\sum_{s=1}^{N} a_{is}\}$, and h is a given positive constant. The factor $1/\kappa$ on the right hand side of (5) is a weighting factor which scales the actual communication weights between follower i and its neighbors into positive scalars within the interval $(0,1)$, for each $i = 2,3,\cdots,N$. Note that such scaling technique is commonly used in consensus problems for discrete-time multi-agent systems [9], [12], [14].

Before closing this section, we state the standing assumptions about the structure of the system communication topology. The first assumption is concerned with interactions between the followers, while the second assumption describes the communication topology between the leader and the rest of the network.

Assumption 1: The adjacency matrix of graph $\mathscr{G}$, $\mathscr{A} = [a_{ij}]_{N\times N}$ has the property that for all $i, j = 2,3,\cdots,N$, $a_{ij} = a_{ji}$.

Remark 1: Note that Assumption 1 indicates that the subgraph describing the communication topology between the followers is undirected. However, this subgraph is not required to be connected in the present framework.

Assumption 2: The communication topology graph $\mathscr{G}$ contains a directed spanning tree with the leader node being its root.

Remark 2: Note that Assumption 2 is not restrictive. For example, it holds when the subgraph describing the communication topology between the followers is connected, and also at least one follower senses the output of the leader. More generally, when the communication topology between the followers consists of p separate connected components, Assumption 2 will be satisfied if each component of the graph includes a node which directly senses the output of the leader.

C. The leader-following H_∞ tracking problem

The control problem in this paper is to design a distributed protocol $u_i(k)$, $i = 2,3,\cdots,N$, to enable the closed-loop multi-agent system (3), equipped with this protocol, to achieve a prescribed level of H_∞ leader-following tracking performance. The mathematical definition of the H_∞ leader-following tracking performance index will be given later. To guarantee the H_∞ leader-following consensus tracking performance, the following observer-based dynamic distributed tracking protocol is proposed for each follower i, $i = 2,3,\cdots,N$. The protocol consists of two parts:

i) The neighbor-based local controller:

$$u_i(k) = \check{A}x_i(k) + \sum_{j=1}^{n-1}\hat{A}_{nj}z_i^j(k) - F_n\varepsilon_i(k), \tag{6}$$

where $\check{A} = \hat{A}_{nn} - A$, $\hat{A}_{ij}$, $i, j = 1,2,\cdots,n$, are defined in (2) and $\varepsilon_i(k)$ is given in (5).

ii) Distributed state estimator:

$$z_i^s(k+1) = \hat{A}_{sn}x_i(k) + \sum_{j=1}^{n-1}\hat{A}_{sj}z_i^j(k) - F_s\varepsilon_i(k), \tag{7}$$
$$s = 1,2,\cdots,n-1.$$

The gain matrix $F = (F_1^T, F_2^T, \cdots, F_n^T)^T \in \mathbb{R}^{nm_0\times m_y}$ is the design parameter of the protocol which will be defined later.

Combining equations (3), (6) and (7) yields the closed-loop system describing dynamics of each follower governed by the proposed protocol:

$$\zeta_i(k+1) = \hat{A}\zeta_i(k) - F\varepsilon_i(k) + \hat{B}_\omega\omega_i(k), \quad k \in \mathbb{N}, \tag{8}$$

where

$$\zeta_i(k) = \left(z_i^1(k)^T, z_i^2(k)^T, \cdots, z_i^{n-1}(k)^T, x_i(k)^T\right)^T \in \mathbb{R}^{nm_0} \tag{9}$$

is the state of the closed-loop system, $i = 2,3,\cdots,N$, and $\hat{B}_\omega = \left(O^T_{m_0\times m_\omega}, \cdots, O^T_{m_0\times m_\omega}, B_\omega^T\right)^T \in \mathbb{R}^{nm_0\times m_\omega}$. Then, it is

[1]The assumption is not restrictive and is made for simplicity. We can dispense with this assumption by introducing a modified adjacency matrix in which a_{i1} is replaced with $a_{i1}c_i$, and then using this modified matrix instead of $\mathscr{A}$ in the subsequent derivations.

easy to see that the difference between the state of the leader and the state of the i-th closed-loop system, $\rho_i(k) = \zeta_i(k) - \Theta(k)$, satisfies the following equation

$$\rho_i(k+1) = \hat{A}\rho_i(k) - F\varepsilon_i(k) + \hat{B}_\omega \omega_i(k), \tag{10}$$

$i = 2,3\cdots,N$.

To characterize performance of the proposed tracking protocol, define the performance variable $e(k) = \left(e_2(k)^T, e_3(k)^T, \cdots, e_N(k)^T\right)^T$ where $e_i(k) = C\rho_i(k)$ where $C \in \mathbb{R}^{nm_1 \times nm_0}$ is a given performance output matrix, $i = 2,3,\cdots,N$.

Using the expression for $\varepsilon_i(k)$ given in (5), we have

$$\begin{cases} \rho(k+1) = \left\{\left(I_{N-1} \otimes \hat{A}\right) - \left[\left(I_{N-1} - \check{D}\right) \otimes (F\tilde{C})\right]\right\} \rho(k) \\ \qquad\qquad + (I_{N-1} \otimes \hat{B}_\omega)\omega(k), \\ e(k+1) = (I_{N-1} \otimes C)\,\rho(k+1), \end{cases} \tag{11}$$

where $\check{D}$ denotes the $(N-1) \times (N-1)$ matrix defined as $\check{D} = [\check{d}_{ij}]_{(N-1)\times(N-1)}$ with $\check{d}_{ii} = (h+\delta_i)/\kappa$, $\check{d}_{ij} = a_{(i+1)(j+1)}/\kappa$, $\omega(k) = (\omega_2(k)^T, \omega_3(k)^T, \cdots, \omega_N(k)^T)^T$ and $\delta_i = \kappa_0 - \sum_{j=1}^{N} a_{(i+1)j}$, κ and κ_0 are the constants defined in (5), $\tilde{C} = (O_{m_y}, \cdots, O_{m_y}, E) \in \mathbb{R}^{m_y \times nm_0}$ where E is the matrix from equation (4). Denote by $T_{\omega e}(\mathrm{z})$ the transfer function matrix of the system (11) from disturbance input $\omega(k)$ to the performance output $e(k)$.

We are now in a position to formulate the leader-following H_∞ tracking problem under consideration in this paper.

Definition 1: The multi-agent system consisting of the leader (1) and the followers (3) and equipped with the protocol (6) is said to solve the distributed H_∞ leader-following tracking problem with performance index $\gamma > 0$, if the following two conditions hold:

i) The multi-agent system described by (1) and (3) with $\omega_i(k) \equiv 0$, $i = 2,3,\cdots,N$, achieves consensus in the sense of $\lim_{k\to\infty} \|\zeta_i(k) - \Theta(k)\| = 0$, where ζ_i is the state of the closed loop system defined in (9), $i = 2,3,\cdots,N$.
ii) The H_∞ norm of $T_{\omega e}(\mathrm{z})$ satisfies the following condition: $\|T_{\omega e}(\mathrm{z})\|_\infty < \gamma$.

III. Main results

In this section, the main theoretical results are presented.

Since the leader has no neighbors, the matrix D associated with communication topology $\mathscr{G}$, has the following structure

$$D = \begin{pmatrix} 1 & 0 \\ \check{d} & \check{D} \end{pmatrix}, \tag{12}$$
$$\check{d} = (a_{21}/\kappa, a_{31}/\kappa, \cdots, a_{N1}/\kappa)^T \in \mathbb{R}^{N-1},$$

and is a row-stochastic matrix; κ is the constant defined in (5). By Assumption 1, the block $\check{D}$ in (12) is symmetric. Let λ_i, $i = 1,2,\cdots,N-1$, be the eigenvalues of $\check{D}$. It then follows from Lemma 1 and Assumption 2 that $0 < \lambda_i < 1$, for all $i = 1,2,\cdots,N-1$.

Theorem 1: Suppose the communication graph $\mathscr{G}$ satisfies Assumptions 1 and 2. Then, for a given $\gamma > 0$, the distributed H_∞ leader-following tracking problem stated in Definition 1 admits a solution if and only if the following $N-1$ systems are simultaneously internally stable and have the H_∞ norm less than γ:

$$\begin{cases} \tilde{\rho}_i(k+1) = (\hat{A} - (1-\lambda_i)F\tilde{C})\tilde{\rho}_i(k) + \hat{B}_\omega \tilde{\omega}_i(k), \\ \tilde{e}_i(k+1) = C\tilde{\rho}_i(k+1), \end{cases} \tag{13}$$

where $i = 1,2,\cdots,N-1$.

Remark 3: Theorem 1 shows that the distributed H_∞ tracking problem for the networked agent system (11) can been converted into a collection of H_∞ control problems for a group of uncoupled systems (13), each having the same dimension. Thus, the complexity of the design reduces significantly. Note also that the effect of topology on the distributed H_∞ leader-following tracking performance is characterized by the eigenvalues of the $\check{D}$.

Although Theorem 1 gives necessary and sufficient conditions for the distributed H_∞ leader-following tracking problem to admit a solution, it does not explain how the feedback gain matrix F should be selected in order to obtain such a solution. The following theorem shows that this issue can be addressed using tools from the H_∞ control theory, based on the result of Theorem 1.

Theorem 2: Suppose that Assumptions 1 and 2 hold, and let $\lambda_0 = \max_{i=1,2,\cdots,N-1} |\lambda_i|$. Given a constant $\gamma > 0$, suppose there exist real matrices $P = P^T > 0$, V and a positive scalar $\varepsilon > 0$ such that

$$\begin{pmatrix} -P & P\hat{A} - V\tilde{C} & P\hat{B}_\omega & V \\ \hat{A}^T P - \tilde{C}^T V^T & -P + \frac{1}{\gamma^2}C^T C + \varepsilon\lambda_0^2 \tilde{C}^T \tilde{C} & O & O \\ \hat{B}_\omega^T P & O & -I & O \\ V^T & O & O & -\varepsilon I \end{pmatrix} < 0, \tag{14}$$

and

$$\begin{pmatrix} I & C \\ C^T & \gamma^2 P \end{pmatrix} > 0. \tag{15}$$

Then the protocol (6) augmented with the distributed state estimator (7), with the feedback gain matrix F, defined as $F = P^{-1}V$, solves the H_∞ leader-following tracking problem for the multi-agent system described by (1) and (3), with a disturbance attenuation level γ.

Remark 4: Theorem 2 provides sufficient conditions for solvability of the distributed H_∞ leader-following tracking problem for the multi-agent system described by (1) and (3). It is not hard to see that a necessary condition for this tracking problem to have a solution is that the matrix pair $(\tilde{C}, \hat{A})$ must be detectable.

Remark 5: Note that performance of the proposed H_∞ tracking protocol is determined by the matrix $C \in \mathbb{R}^{nm_1 \times nm_0}$ in (11) which defines the performance variable $e(k)$. In the special case where the performance variable of interest is the tracking error $\rho(k)$, the conditions of Theorem 2 are simplified by letting $C = I_{nm_0}$.

IV. Example

Consider a leader whose dynamics are described by equation (1), with

$$\Theta(k) = \begin{pmatrix} \theta_1(k) \\ \theta_2(k) \\ \theta_3(k) \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0.5 \end{pmatrix}, \quad \hat{C} = (0,0,1).$$

Furthermore, let $m_0 = 1$, $E = 1$, and $B_\omega = 1.5$. Then the corresponding matrices $\tilde{C}$ and $\hat{B}_\omega$ are $\tilde{C} = (0,0,1)$, $\hat{B}_\omega = (0,0,1.5)^T$. Clearly, the pair $(\tilde{C}, \hat{A})$ is detectable. To illustrate Theorem 2, let us consider a network of agents of the form (3) connected over the communication graph $\mathscr{G}$ shown in Fig. 1. The adjacency matrix of this graph is

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1.2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1.5 & 0 & 0 & 0 & 1.9 \\ 0 & 0 & 0 & 1.9 & 0 \end{pmatrix}.$$

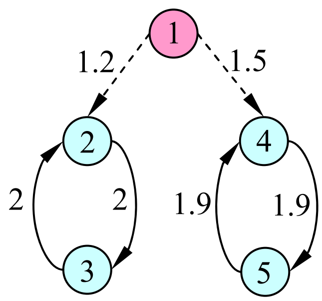

Fig. 1. The communication topology $\mathscr{G}$.

It is easy to check that Assumption 2 holds. Thus, the neighbor-based protocol consisting of the local controller of the form (6) and the distributed state estimator of the form (7) can be designed by solving the conditions in Theorem 2. To design the protocol in this example, the parameter h in (5) is set to be equal 0.20. Calculations show that the eigenvalues of $\check{D}$ defined in (12) are $\lambda_1 = -0.2857$, $\lambda_2 = -0.2844$, $\lambda_3 = 0.8336$, $\lambda_4 = 0.8597$. Then, $\lambda_0 = \max_{i=1,\cdots,4} |\lambda_i| = 0.8579$. Also, the output matrix $C = 0.15 I_3$ and performance level $\gamma = 1$ were chosen in this example.

Solving the linear matrix inequality (14) with $\varepsilon = 0.25$ gives that $V = (-0.8698, 0.2105, 0.1083)$ and $F = (0.0003, 0.0551, 0.4660)^T$. To illustrate properties of this protocol, we simulated the closed loop system without disturbances and also with disturbance inputs of the form $\omega_i(k) = 25\sin(i(k-1))\overline{\omega}(k)$, where $i = 2, \cdots, 5$, $k \in \mathbb{N}$, and

$$\overline{\omega}(k) = \begin{cases} 1 & 0 \le k \le 200, \\ 0 & \text{otherwise}. \end{cases} \tag{16}$$

To illustrate asymptotic convergence of the tracking agents in the absence of disturbances, the corresponding state trajectories of the closed-loop multi-agent system (3) with a high-dimensional leader (1), are shown in Figs. 2–4. Let $E(k) = \sum_{j=2}^{5} \|\zeta_j(k) - \theta(k)\|^2$ be the square of the norm of the consensus tracking error vector for the multi-agent system, where $\zeta_j(k) = (z_j^1(k), z_j^2(k), x_j(k))^T$, $j = 2, \cdots, 5$. Fig. 5 indicates that the proposed distributed dynamic tracking protocol indeed ensures consensus tracking in the absence of disturbances, i.e., when $\omega_i(k) \equiv 0$, $i = 2, \cdots, 5$. Next, the H_∞ consensus tracking under disturbances is considered. Under zero initial conditions, the 'energy trajectories' $\|\sum_{k=1}^{T_0} e^T(k)e(k)\|$ and $\gamma\|\sum_{k=1}^{T_0} \omega^T(k)\omega(k)\|$ were computed as functions of the evolution time T_0 and were plotted in Fig. 6. It can be seen from Fig. 6 that the proposed distributed dynamic tracking protocol indeed ensures the set level of disturbance attenuation.

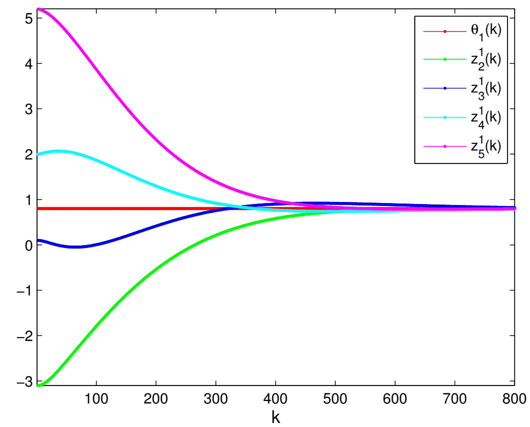

Fig. 2. Trajectories of the leader's first state variable $\theta_1(k)$ and the corresponding estimate of this variable, $z_i^1(k)$, produced by the estimator embedded in the ith follower, $i = 2, 3, 4, 5$.

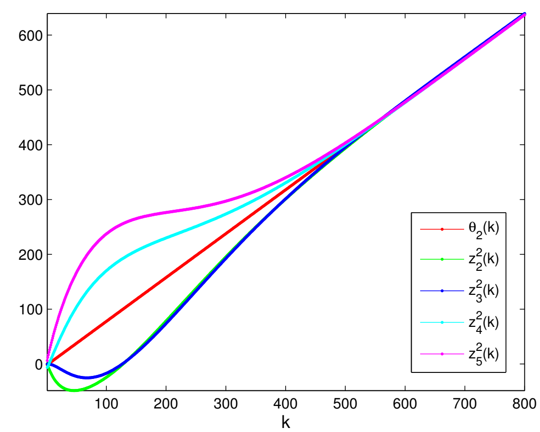

Fig. 3. Trajectories of the leader's second state variable $\theta_2(k)$ and the corresponding estimate of this variable, $z_i^2(k)$, produced by the estimator embedded in the ith follower, $i = 2, 3, 4, 5$.

V. Conclusions

The distributed H_∞ leader-following tracking problem for a class of discrete-time multi-agent systems with a high-dimensional active leader has been investigated in this paper. In the presented framework, the outputs of the leader are only sensed by some informed followers. A new kind of dynamic tracking protocol consisting of a local controller and a distributed state estimator has been constructed and employed to solve such a coordination problem. Using tools from the H_∞ control theory, it has been proved that distributed H_∞ leader-following tracking can be ensured if the underlying topology graph contains a directed spanning tree with the leader being its root while the communication

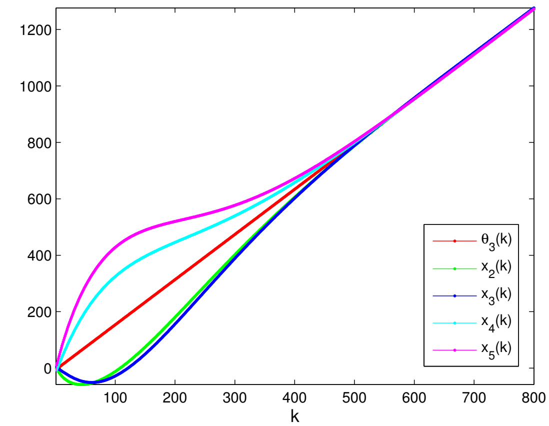

Fig. 4. Trajectories of the leader's third state variable $\theta_3(k)$ and the followers' state variables $x_i(k)$, $i = 2,3,4,5$.

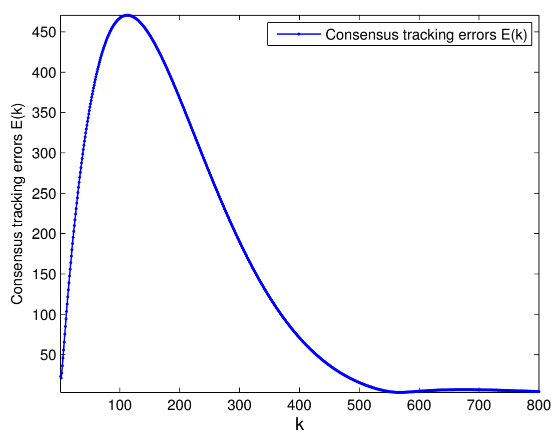

Fig. 5. The squared norm of the consensus tracking error $E(k)$, in the absence of disturbances.

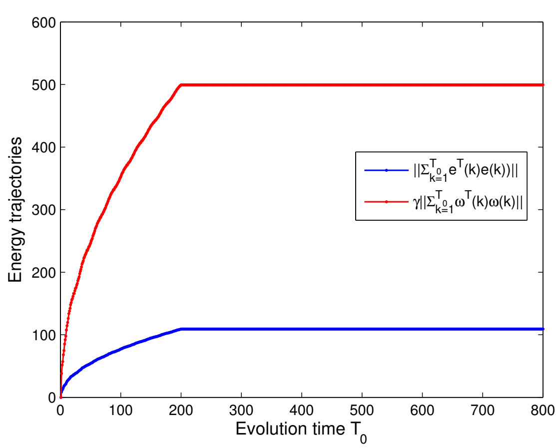

Fig. 6. Energy trajectories of the performance output $e(k)$ (the bottom curve) and the disturbance $\omega(k)$.

topology among the followers is undirected. Future work will focus on solving the distributed tracking problem for multi-agent systems with a high-dimensional leader and time-varying topologies as well as leader-following tracking for multi-agent systems with nonlinear dynamics.

ACKNOWLEDGMENT

G. Wen would like to thank Prof. Yiguang Hong for the inspiring discussions and helpful suggestions.

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