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Structured–LMI Conditions for Stabilizing Network–Decentralized Control

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***Abstract*— In this paper we consider a set of dynamically decoupled systems interconnected by control agents. We can visualize this architecture as a graph where subsystems are associated to nodes and control agents are associated to arcs. The interconnection between subsystems is determined by their input matrix. The decisions of each control agent can directly affect only the nodes connected by the corresponding arc. We seek a stabilizing control framework in which each control agent has information only about the state components associated with the nodes it influences; we say this control architecture is decentralized in the sense of networks. This problem setup involves block–structured feedback matrices, with structural zero blocks. We provide a constructive, sufficient condition based on an LMI with block–diagonal constraints, which guarantees stabilizability through a network–decentralized state–feedback control law. We show that under some structural conditions, concerning local stabilizability and connection with the external environment, the LMI condition we provide is always feasible. Thus, the desired controller can be found in an efficient way.**

I. Introduction and Motivation

Many systems, consisting of naturally independent units, become interactive once a control action is applied. A typical example is given by water distribution systems [3], [15]: the water level in each reservoir has its own dynamics, yet, for a proper management of the system, we need to regulate incoming or outgoing water fluxes to achieve an efficient distribution service. In this case, the reservoirs are pairwise connected by pipes in which the fluxes are controlled. Another application example is distributed traffic control. In large platoons of vehicles, individual vehicle control including information from other neighboring vehicles can assure optimal speed and safety distance, increasing the overall throughput and avoiding collisions and congestions [9], [11], [12]. In this case, the subsystems become globally interacting if we add a control which acts pairwise (each vehicle must keep a certain distance from the one in front and from the one behind). Formation flight of aircrafts [9] is another significant example: independently piloted planes may be cooperatively controlled, for instance, in order to keep a common height. Additional examples include transportation networks [2], [18], routing in telecommunication and data communication networks [17], [16], [10], [14], [13], inventory management and production–distribution systems [4], [5], [6], [7], [8], [22], [23] and network flows in general [4], [1], [21].

In many cases it is too expensive or actually impossible to implement a centralized controller having information about all the subsystems (for instance, when a very large number of subsystems are geographically sparse). Then, the control has to be computed locally: each control law acting on a certain subset of subsystems has to be decided according to the information about that subset of subsystems only. We call this type of control *network–decentralized*. Pioneering work along these lines can be found in [14], [11], [13]; more recent contributions are [7], [3], although limited to the case in which the subsystems are characterized by first–order integrators. Consensus problems have become increasingly popular for distributed computation (see [20], [19] and the references therein), yet the main focus is on agreement among agents (nodes) rather than stabilizability. We approach the problem using Linear Matrix Inequalities (LMIs), whose advantages are: 1) they can express a variety of classical control constraints (including Lyapunov and Riccati inequalities) for general dynamical systems and 2) they are readily solvable with off–the–shelf software.

In this paper, we consider a set of decoupled subsystems with possibly unstable dynamics. We look for a state–feedback control which must be network–decentralized. As we will see in Section II, this is equivalent to requiring that the feedback matrix has the same structure as the transpose of the overall input matrix. Our main results are:

- under the condition of local stabilizability, we provide in Section III a sufficient condition, based on a constrained LMI, for the solvability of the problem;
- we show that the obtained condition is sufficient only, because there are systems which admit a network–decentralized stabilizing feedback matrix, even though this LMI is not feasible;
- we prove that, for a class of systems structurally characterized by a graph connected with the external environment, our decentralized problem is solvable because the constrained LMI is always feasible (Section IV).

We will see that the structural *external connection condition* is crucial, since it is necessary to have global reachability when all the subsystems have the same dynamics and the same type of interactions with the agents. Finally, in Section V, we provide a numerical example.

II. Decentralized Control of Networks: Problem Formulation

A system consisting of N subsystems, connected by several control agents, can be written as

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ includes the state variables associated with each subsystem, $u(t) \in \mathbb{R}^m$ is the control vector, $d(t) \in \mathbb{R}^n$

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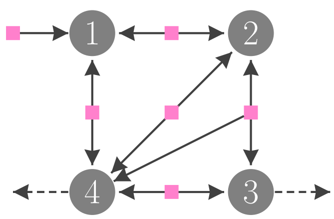

Fig. 1. The graph corresponding to Example 1

is the vector representing an external, non–controllable signal affecting the system, E is a generic matrix, while A and B are block–structured: $A \in \mathbb{R}^{n \times n}$ is a block–diagonal matrix

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_N \end{bmatrix} \tag{2}$$

and $B \in \mathbb{R}^{n \times m}$ is a matrix of a given structure, such as

$$B = \begin{bmatrix} B_{**} & B_{**} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ B_{**} & \mathbf{0} & B_{**} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_{**} & \mathbf{0} & \dots & B_{**} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{**} & \dots & B_{**} \end{bmatrix}$$

where blocks denoted by $\mathbf{0}$ are structural zero blocks, while the other entries B_{**} are arbitrary.

Matrix B can be represented by a hypergraph having N nodes (corresponding to the blocks of matrix A, i.e. to the subsystems): each column of B is associated to a hyperarc connecting all the nodes directly affected by the corresponding control. In the following, for simplicity, hypergraphs and hyperarcs will be referred to as graphs and arcs. Figure 1 shows the graph corresponding to Example 1 below: gray circles are nodes associated to subsystems and pink squares on the arcs indicate network controllers.

All the block dimensions must be coherent with the block structure of A. If each diagonal block of A, A_i, has dimension n_i and each block of B has m_i columns, it must be that: $\sum_{i=1}^{N} n_i = n$ and $\sum_i m_i = m$.

Each of the N connected subsystems has its own dynamics. We say that a state–feedback control law is decentralized when each control component, which affects a certain subset of nodes, has information about the state components associated with those nodes only. This property forces the feedback matrix to have the same structure as B^T.

Definition 1: A control of the form $u = -Kx$ is *decentralized* in the sense of networks if K has the same structural zero blocks as B^T; we write that $K \in \mathcal{S}(B^T)$.

Example 1: Consider the case

$$A = \text{blockdiag}\{A_1, A_2, A_3, A_4\}$$

$$B = \begin{bmatrix} B_{11} & B_{12} & \mathbf{0} & \mathbf{0} & B_{15} & \mathbf{0} \\ \mathbf{0} & B_{22} & B_{23} & B_{24} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{34} & \mathbf{0} & B_{36} \\ \mathbf{0} & \mathbf{0} & B_{43} & B_{44} & B_{45} & B_{46} \end{bmatrix}$$

$$E = \text{blockdiag}\{0, 0, -I, -I\}$$

The graph corresponding to B is shown in Fig. 1. K must have the structure

$$K = \begin{bmatrix} K_{11}^T & K_{12}^T & \mathbf{0} & \mathbf{0} & K_{15}^T & \mathbf{0} \\ \mathbf{0} & K_{22}^T & K_{23}^T & K_{24}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K_{34}^T & \mathbf{0} & K_{36}^T \\ \mathbf{0} & \mathbf{0} & K_{43}^T & K_{44}^T & K_{45}^T & K_{46}^T \end{bmatrix}^T$$

Note that the "restricted information" condition is reflected in this constraint on K. For instance, the second component of the control is the sub–vector

$$u^{(2)}(t) = K_{12}x^{(1)}(t) + K_{22}x^{(2)}(t)$$

so that $u^{(2)}(t)$, which affects $x^{(1)}$ and $x^{(2)}$, has to be decided without any information about $x^{(3)}$ and $x^{(4)}$.

Remark 1: As a special case, when $A = 0$,

$$\dot{x}(t) = Bu(t) + Ed$$

the control $u = -\gamma B^T x$, with $\gamma > 0$, is a decentralized solution. If B has full row rank, then the closed loop system is asymptotically stable. Optimality of this control has been proved even under saturation [3].

III. Sufficient LMI Conditions for Network Decentralized Stabilization

It is possible to find a general condition expressed by a Linear Matrix Inequality (LMI) which, if satisfied, guarantees that the system (1) can be stabilized through a decentralized state–feedback control. The expression of this decentralized control is also provided. Unfortunately, this condition is merely *sufficient* and not necessary.

A. Stable subsystems

If the system is stable (each subsystem A_i is asymptotically stable) we have the following.

Proposition 1: If system (1) is open loop stable, then stability is preserved by the network–decentralized control $K = \gamma B^T P$, $\gamma > 0$, where P is a positive definite matrix with the same block–diagonal structure as matrix A, such that $A^T P + PA < 0$.

Proof: Due to the open loop stability assumption, system (1) satisfies the Lyapunov equation

$$A^T P + PA = -Q$$

where Q and P are positive definite matrices, with the same block–diagonal structure as matrix A.

$$P = \begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_N \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_N \end{bmatrix}$$

Then, if we apply the control $K = \gamma B^T P$, $\gamma > 0$, the closed loop matrix $A - BK$ is stable as well, since

$$\begin{aligned} & (A - BK)^T P + P(A - BK) = \\ = \; & (A^T P + PA) - \gamma PBB^T P - \gamma PBB^T P \\ = \; & -(Q + 2\gamma PBB^T P) < 0 \end{aligned}$$

■

B. Unstable subsystems

We consider now the non–trivial case in which the system is marginally stable or unstable (i.e. at least one of the subsystems A_i is marginally stable or unstable).

Proposition 2: Consider system (1), where matrix A has a block–diagonal structure and B is block–structured. If it is possible to find a matrix S, positive definite, with the same block–diagonal structure as A, which satisfies the LMI condition

$$SA^T + AS - 2\gamma BB^T < 0, \tag{3}$$

then there exists a stabilizing decentralized control law.

Proof: Assume that there exist a positive definite matrix S, block–diagonal as A,

$$S = \begin{bmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_N \end{bmatrix}, \tag{4}$$

and a matrix $R \in \mathcal{S}(B^T)$, which satisfy the LMI

$$AS + SA^T - BR - R^T B^T < 0. \tag{5}$$

Then, the control $u = -Kx$ with $K = RS^{-1}$ is decentralized and stabilizing. In fact, because of the block structure of R and S, $K \in \mathcal{S}(B^T)$, thus the controller is decentralized. In addition, if $P = S^{-1}$ (which is still block–diagonal and positive definite), by replacing $R = KS$ in (5) we obtain $AS + SA^T - BKS - SK^T B^T < 0$ and, by pre- and post-multiplying by P, we have $PA + A^T P - PBK - K^T B^T P < 0$, which corresponds to $(A - BK)^T P + P(A - BK) < 0$, thus ensuring the stability of the closed loop system.

Since the inequality

$$x^T[(A - BK)^T P + P(A - BK)]x < 0$$

is satisfied $\forall\, x \neq 0$, a suitable feedback is always provided by the control $u = -\gamma B^T Px$, with $\gamma > 0$ and large enough, which satisfies the same inequality. By substituting $K = \gamma B^T P$, we obtain

$$\begin{aligned} x^T(A^T P + PA)x - 2\gamma x^T PBB^T Px < 0,\ \forall\, x \neq 0 \\ \Rightarrow\ A^T P + PA - 2\gamma PBB^T P < 0. \end{aligned}$$

Pre- and post-multiplication by matrix S, block–diagonal as in (4), gives the LMI

$$SA^T + AS - 2\gamma BB^T < 0$$

which must be solved with the structural condition $S > 0$. The existence of such a matrix S assures that the control of the form $u = -\gamma B^T Px$ is decentralized, since P has the same block–diagonal structure as A and thus $B^T P \in \mathcal{S}(B^T)$. ■

We have thus shown that, given a system with A block–diagonal and B block–structured, a decentralized control $u = -\gamma B^T S^{-1} x$ exists if (3) holds for a block–diagonal $S > 0$ (or, equivalently, $A^T P + PA - 2\gamma PBB^T P < 0$ for a block–diagonal $P > 0$, thus $u = -\gamma B^T Px$). This condition is sufficient, but not necessary, for the existence of a stabilizing decentralized control. In fact, we can provide the counter-example of a system which *can* be stabilized by means of a proper decentralized control, even though it is *impossible* to find a block–diagonal P satisfying the inequality.

C. Why the Condition is not Necessary: an Example

Example 2: Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$$

with $\lambda > 0$ (therefore the system is unstable).

In order to find a state–feedback control $u = -Kx$, we look for a matrix $K \in \mathcal{S}(B^T)$:

$$K = \begin{bmatrix} a & -b & 0 \\ 0 & -c & d \end{bmatrix}$$

Therefore

$$A - BK = \begin{bmatrix} -a & b & 0 \\ a & \lambda - (b + c) & d \\ 0 & c & -d \end{bmatrix}$$

and we are free to assign its eigenvalues in order to obtain an asymptotically stable system. For instance, if we take $a = b = c = 10\lambda$ and $d = -\lambda$, we obtain a stable matrix.

Thus it is possible to stabilize the system by means of a suitable *decentralized* state–feedback control. However, it is impossible to find a stabilizing decentralized control $u = -\gamma B^T Px$ with a block–diagonal positive definite matrix P which satisfies the Lyapunov condition $(A - BK)^T P + P(A - BK) < 0 \Rightarrow A^T P + PA - 2\gamma PBB^T P < 0$. In fact, if we choose

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \text{ with } p_1, p_2, p_3 > 0,$$

then $-(A^T P + PA) + 2\gamma PBB^T P =$

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2p_2\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\gamma p_1^2 & -2\gamma p_1 p_2 & 0 \\ -2\gamma p_1 p_2 & 4\gamma p_2^2 & -2\gamma p_2 p_3 \\ 0 & -2\gamma p_2 p_3 & 2\gamma p_3^2 \end{bmatrix} \\ &= \begin{bmatrix} 2\gamma p_1^2 & -2\gamma p_1 p_2 & 0 \\ -2\gamma p_1 p_2 & 4\gamma p_2^2 - 2p_2\lambda & -2\gamma p_2 p_3 \\ 0 & -2\gamma p_2 p_3 & 2\gamma p_3^2 \end{bmatrix} \end{aligned}$$

should be positive definite, so all the leading principal minors should be positive.

This can be true for the first two, yet it is *impossible* for the determinant $2\gamma p_1^2(8\gamma^2 p_2^2 p_3^2 - 4\gamma p_2 \lambda p_3^2 - 4\gamma^2 p_2^2 p_3^2) - 8\gamma^3 p_1^2 p_2^2 p_3^2 = 8\gamma^3 p_1^2 p_2^2 p_3^2 - 8\gamma^2 p_1^2 p_2 p_3^2 \lambda - 8\gamma^3 p_1^2 p_2^2 p_3^2 = -8\gamma^2 p_1^2 p_2 p_3^2 \lambda$ to be positive, since $\lambda > 0$.

So, we have proved that the presence of a block–diagonal matrix $P > 0$ with the same structure as A and such that the inequality $A^T P + PA - 2\gamma PBB^T P < 0$ holds is a sufficient, but not necessary, condition in order to guarantee the presence of a network stabilizing decentralized control.

IV. Structural conditions for solvability

We now consider a system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6}$$

with A block–diagonal as in (2) and B block–structured. We now assume that the system has the structure of a connected (undirected) graph (not a hypergraph). Thus, we assume that each undirected arc corresponding to a control agent connects *a pair* of nodes. This means that, in each block column of matrix B, there are at most two non–zero blocks.

Definition 2: The system is *connected* if its graph is connected, namely each of its nodes can be reached starting from any other, by following the existing arcs. We say that the system is *connected with the external environment* if it is connected and there is at least one block column of matrix B with a single non–zero block (i.e. the system is connected with an "external node").

Definition 3: The system is *separately stabilizable* if each control agent can separately stabilize each of the nodes which it affects. Precisely, denoting by B_{ij} the block of B representing the action of control j on subsystem i, then (A_i, B_{ij}) is stabilizable.

Remark 2: It is worth stressing that a control agent may not be able to stabilize simultaneously both subsystems to which it is connected. For instance, the non–stabilizable system

$$\dot{x}_1 = x_1 + u, \quad \dot{x}_2 = x_2 + u,$$

is separately stabilizable.

Then, we can obtain the following general result, which guarantees the existence of a decentralized control.

Theorem 1: If system (6) is connected with the external environment and separately stabilizable, then a stabilizing decentralized control always exists and can be found by solving LMI (3), which is feasible.

Proof: We provide a constructive proof in three steps: 1) tree–graph pre–stabilization; 2) completion with the insertion of the neglected control arcs; 3) general LMI feasibility.

1) First of all, we suitably reorder the subsystems, starting from one of the nodes connected with the external environment, and we create a tree by choosing a subset of the arcs. This is always possible, since by assumption we consider a *connected* graph (and not a hypergraph). We choose and order the N arcs so that B can be written as $B = \begin{bmatrix}\bar{B} & \tilde{B}\end{bmatrix}$ with a block–triangular matrix $\bar{B}$:

$$\begin{bmatrix}\bar{B} & \tilde{B}\end{bmatrix} = \left[\begin{array}{cccc|c} \bar{B}_{11} & * & \dots & * & \tilde{B}_1 \\ \mathbf{0} & \bar{B}_{22} & \dots & * & \tilde{B}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \bar{B}_{NN} & \tilde{B}_N \end{array}\right]$$

where the "*" are suitable elements. We can thus choose a control $u = -\hat{K}x = -\begin{bmatrix}\bar{K}^T & \mathbf{0}\end{bmatrix}^T x$, which stabilizes the system. Precisely, $\bar{K}$ is formed by N diagonal blocks $K_1, K_2, \dots, K_N$, each computed to stabilize $(A_i - \bar{B}_{ii}K_i)$. Thus, the resulting closed–loop matrix is

$$A - B\hat{K} = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_N \end{bmatrix} -$$

$$\left[\begin{array}{cccc|c} \bar{B}_{11} & * & \dots & * & \tilde{B}_1 \\ \mathbf{0} & \bar{B}_{22} & \dots & * & \tilde{B}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \bar{B}_{NN} & \tilde{B}_N \end{array}\right] \left[\begin{array}{cccc} K_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & K_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & K_N \\ \hline \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{array}\right]$$

which is block–triangular and stable. As a consequence, it always admits a Lyapunov function, expressed by a block–diagonal matrix $P > 0$ such that

$$(A - B\hat{K})^T P + P(A - B\hat{K}) = -Q < 0.$$

This in turn implies that there exists a suitable control law $\hat{K} = \begin{bmatrix}(\gamma\bar{B}^T P)^T & \mathbf{0}\end{bmatrix}^T$, where, for γ large enough, the system with closed loop matrix $(A - \bar{\gamma}\bar{B}\bar{B}^T P)$ is stable and satisfies

$$A^T P + PA - 2\bar{\gamma}P\bar{B}\bar{B}^T P = -\bar{Q} < 0$$

with the given block–diagonal P.

2) If we include all the control components, after the transformation the closed loop matrix becomes $A - \bar{\gamma}\bar{B}\bar{B}^T P - \bar{\gamma}\tilde{B}\tilde{B}^T P = A - \bar{\gamma}BB^T$. Then

$$\begin{aligned} & (A - \bar{\gamma}BB^T)^T P + P(A - \bar{\gamma}BB^T) \\ = \; & A^T P + PA - 2\bar{\gamma}P\bar{B}\bar{B}^T P - 2\bar{\gamma}P\tilde{B}\tilde{B}^T P \\ = \; & -\bar{Q} - 2\bar{\gamma}P\tilde{B}\tilde{B}^T P < 0 \end{aligned}$$

3) Since $(A - \bar{\gamma}BB^T)^T P + P(A - \bar{\gamma}BB^T) < 0$, feasibility of the original LMI (3) can be proved as in the proof of Proposition 2. ■

Remark 3: While the proof of Theorem 1 is constructive, it is not effective to follow its steps to compute the control action. As long as it is demonstrated that the LMI condition is always feasible, one can directly solve it using an LMI software toolbox.

A. Comments on the external connection Assumption

Requiring the system to be connected with the external environment in Theorem 1 might seem uselessly restrictive. However, external connections are absent in Example 2 (the counterexample highlighting how our conditions are sufficient but not necessary): this suggests that the external connection assumption might be important.

We now show that the external connection assumption is necessary for reachability in systems where all the diagonal blocks of matrix A are equal and the pairs of non–zero blocks in the block columns of matrix B are equal but have opposite signs. For example:

$$B = \begin{bmatrix} B_{bl} & B_{bl} & -B_{bl} & \dots \\ \mathbf{0} & -B_{bl} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & B_{bl} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \end{bmatrix}$$

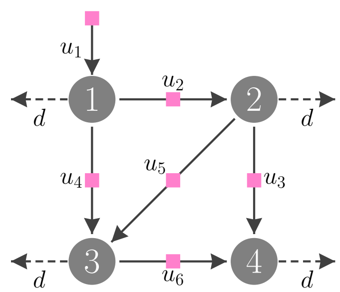

Fig. 2. Network graph corresponding to the example in Section V

For these systems, under the assumption of local reachability, the external connection assumption is equivalent to reachability.

Proposition 3: Assume that in system (6) all the diagonal blocks of matrix A are equal, $A_i = A_{bl}\ \forall i = 1, \dots, N$, and, if there are two non–zero blocks in a block column of matrix B, they differ for the sign only. Assume that we have local reachability, namely that (A_k, B_{bl}) are all reachable pairs. Then, system (A, B) is reachable if and only if it is connected with the external environment.

Proof: The "if" part can be demonstrated along the lines of Theorem 1: local reachability guarantees that, by repeating the same procedure, we can arbitrarily assign all the eigenvalues, thus the overall system (A, B) is reachable.

Then we need to show that, if the system is *not* connected with the external environment, i.e. if all the block columns of B have two non–zero blocks with opposite sign, the system is not reachable. Popov criterion states that (A, B) is reachable if and only if $\begin{bmatrix}\lambda I - A & B\end{bmatrix}$ has full rank $\forall \lambda \in \sigma(A)$: this means that $\nexists z$ such that $z^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$. Yet if we choose $z^T = \begin{bmatrix} v^T & v^T & \dots & v^T\end{bmatrix}$, where v is a left eigenvector of matrix A_{bl}, then $z^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$. ■

Therefore, if the external connection assumption fails, we are forced to rely on the stability of the unreachable modes.

V. Example

Consider the graph in Fig. 2, which represents a flow network: each node is a reservoir and the directed arcs are associated to controlled *flows*. To ensure that at steady state the nodes reach the exact prescribed level (fixed as 0) they are all equipped by a supplementary integrator. This corresponds to the eight state model

$$\dot{x} = Ax + Bu + Ed$$

with

$$A = \begin{bmatrix} A_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_2 & -B_2 & \mathbf{0} & -B_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & -B_2 & \mathbf{0} & -B_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_2 & B_2 & -B_2 \\ \mathbf{0} & \mathbf{0} & B_2 & \mathbf{0} & \mathbf{0} & B_2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = -\begin{bmatrix} B_2 \\ B_2 \\ B_2 \\ B_2 \end{bmatrix}$$

The even states represent the effective levels, while the odd states represent the integral variables, whose derivatives are equal to the even states. Vector d represents a constant uniform demand on each node. The control for this system can be $u = -Kx = -\gamma B^T P x$, where γ is a constant scalar and P is a Lyapunov block diagonal matrix. P and γ can be found by numerically solving an LMI; however, we can find an analytical solution if we note that

$$P = \begin{bmatrix} P_2 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \gamma = 2$$

render the closed loop system stable.

If we start from an initial condition different from the equilibrium one, the integral control guarantees the exact recovery of the equilibrium values of the even states, as expected (see Fig. 3, 5), while the odd states are asymptotically constant (Fig. 4, 6). Note that the integral variables are "communicated" by each node to its controls in a network–decentralized way, in order to achieve the zero–error goal. The numerical and analytical solution differ for the equilibrium values obtained, but lead to the same steady state control vector. In both cases, any deviation from the equilibrium values at the nodes is completely eliminated by this integral control.

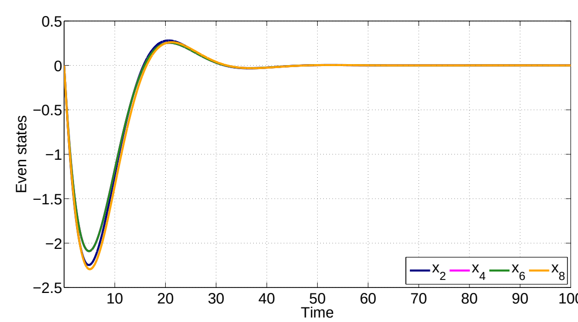

Fig. 3. Double integrator, numerical solution: even states ($x_4 = x_6$)

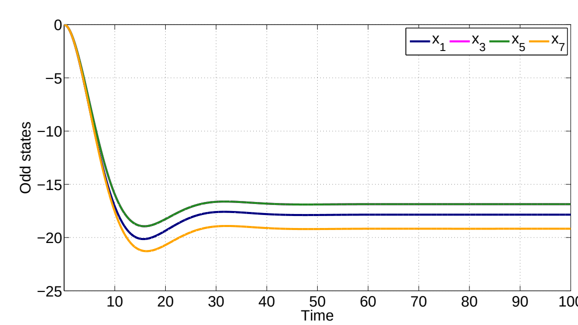

Fig. 4. Double integrator, numerical solution: odd states ($x_3 = x_5$)

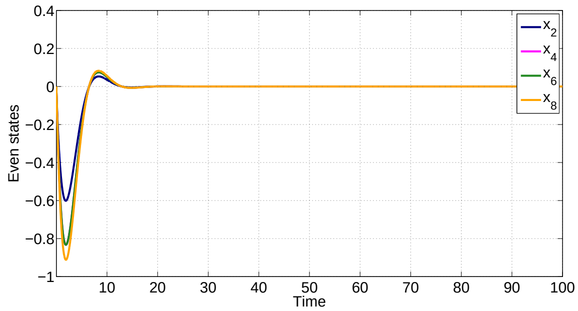

Fig. 5. Double integrator, analytical solution: even states ($x_4 = x_6$)

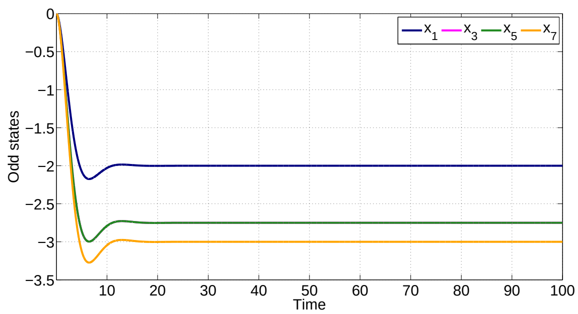

Fig. 6. Double integrator, analytical solution: odd states ($x_3 = x_5$)

VI. Conclusions

We have considered the problem of designing a network–decentralized control strategy for a system formed by a set of independent subsystems (nodes) which become connected when a control is applied. The control matrix is associated with a graph and the constraints on the control agents, associated with the arcs of the graph, impose that each agent makes its decisions based only on the information regarding the nodes it affects. The novel contribution of this paper is that it considers nodes which have their own arbitrary dynamics, rather than being first–order integrators as in previous work [3].

We have provided a sufficient structured LMI condition for decentralized stabilizability. However, we have seen that such an LMI is *always feasible* under the assumptions of local stabilizability and connection of the system with the external environment.

Further developments of this work may extend the results to systems with switching topologies and with uncertain local node dynamics.

References

[1] A. Atamturk and M. Zhang, "Two–stage robust network flow and design under demand uncertainty," *Operations Research*, vol. 55, no. 4, pp. 662–673, 2007.

[2] B. Ataslar and A. Iftar, "A decentralized control approach for transportation networks," *Proc. of the 8^{th} IFAC Symposium on Large Scale Systems*, pp. 348-353, 1998.

[3] D. Bauso, F. Blanchini, L. Giarré, and R. Pesenti, "The linear saturated control strategy for constrained flow control is asymptotically optimal," *Automatica*, vol. 49, no. 7, pp. 2206–2212, 2013.

[4] D. Bauso, F. Blanchini, and R. Pesenti, "Optimization of Long-run Average-Flow Cost in Networks with Time-Varying Unknown Demand," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 20–31, 2010.

[5] D. Bertsimas and A. Thiele, "A Robust Optimization Approach to Inventory Theory," *Operations Research*, vol. 54, no. 1, pp. 150–168, 2006.

[6] F. Blanchini, F. Rinaldi, and W. Ukovich, "Least Inventory Control of Multi-Storage Systems with Non-Stochastic Unknown Input," *IEEE Transactions on Robotics and Automation*, vol. 13, pp. 633–645, 1997.

[7] F. Blanchini, S. Miani, and W. Ukovich, "Control of production-distribution systems with unknown inputs and system failures," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1072–1081, 2000.

[8] E. K. Boukas, H. Yang, and Q. Zhang, "Minimax production planning in failure–prone manufacturing systems," *Journal of Optimization Theory and Applications*, vol. 82, no. 2, pp. 269–286, 1995.

[9] R. D'Andrea, "A Linear Matrix Inequality Approach to Decentralized Control of Distributed Parameter System," *Proc. of the American Control Conference*, Philadelphia, PA, pp. 1350–1354, 1998.

[10] A. Ephremides and S. Verdú, "Control and Optimisation Methods in Communication Networks," *IEEE Transactions on Automatic Control*, vol. 34, pp. 930–942, 1989.

[11] A. Iftar, "A linear programming based decentralized routing controller for congested highways," *Automatica*, vol. 35, no. 2, pp. 279–292, 1999.

[12] A. Iftar, "A decentralized routing control strategy for semi-congested highways," *Proc. of the 13^{th} IFAC World Congress*, Vol. P, pp. 319-324, 1996.

[13] A. Iftar and E. J. Davison, "Decentralized control strategies for dynamic routing," *Optimal Control Applications and Methods*, vol. 23, pp. 329-355, 2002.

[14] A. Iftar and E. J. Davison, "Decentralized Robust Control for Dynamic Routing of Large Scale Networks," *Proc. of the American Control Conference*, San Diego, CA, pp. 441–446, 1990.

[15] R. E. Larson and W. G. Keckler, "Applications of Dynamic Programming to the Control of Water Resource Systems," *Automatica*, vol. 5, pp. 15–26, 1969.

[16] J. C. Moreno and M. Papageorgiou, "A Linear Programming Approach to Large-scale Linear Optimal Control Problems," *IEEE Transactions on Automatic Control*, vol. 40, pp. 971–977, 1995.

[17] F. H. Moss and A. Segall, "An Optimal Control Approach to Dynamic Routing in Data Communication Networks," *IEEE Transactions on Automatic Control*, pp. 1–7, 1978.

[18] S. Mudchanatongsuk, F. Ordóñez, and J. Liu, "Robust Solutions for Network Design under Transportation Cost and Demand Uncertainty," *Journal of the Operational Research Society*, vol. 59, 2008, pp. 652–662.

[19] W. Ren, R. W. Beard, and E. Atkins, "Information Consensus in Multivehicle Cooperative Control: Collective Group Behavior through Local Interaction," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, April 2007.

[20] R. Olfati–Saber, R. M. Murray, "Consensus Problems in Networks with Switching Topology and Time–Delays", *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1523, 2004.

[21] F. Ordóñez and J. Zhao, "Robust capacity expansion of network flows," *Network*, vol. 38, pp. 136–145, 2007.

[22] H. Sarimveisa, P. Patrinos, C. D. Tarantilis, and C. T. Kiranoudis, "Dynamic modeling and control of supply chain systems: A review," *Computers & Operations Research* vol. 35, pp. 3530–3561, 2008.

[23] E. A. Silver and R. Peterson, *Decision System for Inventory Management and Production Planning*, Wiley, New York, NY, 1985.