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Integral input-to-state stable saddle-point dynamics for distributed linear programming

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Abstract— **This paper studies the robustness properties of a class of saddle-point dynamics for linear programming. This dynamics is distributed over a network in which every node controls one component of the optimization variable. In this multi-agent setting, communication noise, computation errors, and mismatches in the agents' knowledge about the problem data all enter into the dynamics as unmodeled disturbances. We show that the saddle-point dynamics is integral input-to-state stable and hence robust to disturbances of finite energy. This result also allows us to establish the robustness of the dynamics when the communication graph is recurrently connected because of link failures. Several simulations illustrate our results.**

I. Introduction

In this paper we establish some robustness properties for a certain saddle-point dynamics (i.e., gradient descent in one set of variables, gradient ascent in another). The trajectories of this dynamics converge to solutions of linear programs. In principle, the saddle-point dynamics in this paper can be applied to general linear programs with compact solution space. With regards to distributed implementation over a network of processors we consider the specific case where (i) each processor's state corresponds to a component of the decision vector and (ii) processors can communicate with each other if their states appear in a common constraint. Note that this setup differs from consensus-type optimization problems where agents agree on the entire solution vector.

Literature review: This work is related to the literature on linear programming, distributed optimization, and robustness of nonlinear systems. Linear programs arise in many real-world decision making scenarios, such as portfolio optimization [1], operator placement [2], network flow [3], perimeter patrolling [4], among others. Efficient algorithms to solve linear programs, such as the simplex algorithm [5] or interior point methods [6], are well-established in the optimization literature. Lately, there has been interest in the distributed implementation of such methods. Recent advances in distributed optimization include [7], [8], [9], [10] (and references therein), with linear programs specifically considered in [11]. The saddle-point dynamics we consider appear in [12] and bear some semblance to the dynamics found in [13] (a major difference being that the dual variable dynamics is smooth in the former reference). All of the aforementioned methods can be applied to so-called robust optimization problems [14] (incidentally, robust linear programs are a special case of the setup considered in this paper). However, robust optimization problems only account for uncertainty in the problem data and do not study the effect of disturbances and communication failures in the algorithms themselves. To this end, let us recall various notions of robustness for nonlinear system dynamics. In particular, under mild regularity conditions, asymptotic stability of the dynamics imply a certain degree of qualitative robustness to sufficiently small perturbations [15]. Towards a more quantitative description of robustness, input-to-state stability (ISS) [16] guarantees that small disturbances give rise to small state deviations. Finally, integral input-to-state stability (iISS) [17] is a weaker notion of robustness than ISS, but stronger than asymptotic stability. In iISS, the state deviations depend on the energy of the disturbance. The iISS property can also be used to establish stability in supervisory control setups [18] and cascade interconnections [19].

Statement of contributions: Our starting point is a class of provably correct saddle-point dynamics for distributed linear programming. We examine the robustness of this dynamics against disturbances that may be due to communication noise, computation errors, unmodeled dynamics, or mismatches in the agents' knowledge about the problem data. The main contributions of the paper are as follows. Our first contribution is a general result showing that no dynamics for linear programming is input-to-state stable. Our second contribution is the characterization of the integral input-to-state stability properties of the saddle-point dynamics. In particular, this property implies the asymptotic convergence to the solution set of the linear program when disturbance have finite energy. Our analysis allows us to extend this convergence result to the solution set of a perturbed linear program when disturbances have finite variation. Our third and final contribution is the analysis of the robustness properties of the saddle-point dynamics when communication among the agents is subject to link failures. Specifically, we establish its asymptotic correctness properties under scenarios modeled via the notion of recurrently connected graphs. Several simulations illustrate our results. For reasons of space, all proofs will appear elsewhere.

II. Preliminaries

This section introduces preliminaries on notation, nonsmooth analysis, and set-valued dynamical systems.

A. Notation

The set of real numbers is $\mathbb{R}$. For $x \in \mathbb{R}^n$, $x \geq 0$ means that all components of x are nonnegative. For $x \in \mathbb{R}^n$, we

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define $\max\{0,x\} = (\max\{0,x_1\},\dots,\max\{0,x_n\}) \in \mathbb{R}^n_{\geq 0}$. We use $\|\cdot\|$ and $\|\cdot\|_\infty$ to denote the 2- and ∞-norms in $\mathbb{R}^n$. The Euclidean distance from a point $x \in \mathbb{R}^n$ to a set $X \subset \mathbb{R}^n$ is denoted by $\|\cdot\|_X$. The open ball around x with radius $\delta > 0$ is $\mathbb{B}(x,\delta)$. The set $X \subset \mathbb{R}^n$ is convex if it fully contains the segment connecting any two points in X. Given a matrix $A \in \mathbb{R}^{m\times n}$, $\mathrm{row}_\ell(A) \in \mathbb{R}^n$ denotes its ℓ^{th} row. The function $L : X \times Y \to \mathbb{R}$ defined on the convex set $X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$ is convex-concave if it is convex on its first argument and concave on its second. A point $(\bar{x},\bar{y}) \in X \times Y$ is a saddle point of L if $L(x,\bar{y}) \geq L(\bar{x},\bar{y}) \geq L(\bar{x},y)$ for all $(x,y) \in X \times Y$. $\mathcal{K}$ is the class of functions $[0,\infty) \to [0,\infty)$ that are continuous, zero at zero, and strictly increasing. The subset of class $\mathcal{K}$ functions that are unbounded are called class $\mathcal{K}_\infty$. $\mathcal{L}$ is the set of functions $[0,\infty) \to [0,\infty)$ that are continuous, decreasing, and converging to zero as its argument tends to ∞. A class $\mathcal{KL}$ function $[0,\infty)\times[0,\infty) \to [0,\infty)$ is class $\mathcal{K}$ in its first argument and class $\mathcal{L}$ in its second. A function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite with respect to X if $V(x) = 0$ for all $x \in X$ and $V(x) > 0$ for all $x \notin X$. If $X = \{0\}$, we simply say that V is positive definite. Finally, a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ maps elements in $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$.

B. Nonsmooth analysis

Here we review some basic notions from nonsmooth analysis following [20]. A function $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz if for every $x \in \mathbb{R}^n$ there exists a $\delta > 0$ and $L \geq 0$ such that $|V(y)-V(z)| \leq L\|y-z\|$ for $y,z \in \mathbb{B}(x,\delta)$. The generalized gradient of a locally Lipschitz function V at $x \in \mathbb{R}^n$ is

$$\partial V(x) = \mathrm{co}\{\lim_{i\to\infty} \nabla V(x_i) : x_i \to x, x_i \notin S \cup \Omega_V\},$$

where $\mathrm{co}\{\cdot\}$ is the convex hull, $S \subset \mathbb{R}^n$ is any set with zero Lebesgue measure, and $\Omega_V \subset \mathbb{R}^n$ is the set of points where V is not differentiable. If $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, then we use $\partial_x V(x,y)$ and $\partial_y V(x,y)$ to denote the generalized gradients of the maps $x \mapsto V(x,y)$ and $y \mapsto V(x,y)$, respectively. A set-valued map $F : X \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is upper semi-continuous if for every $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $F(y) \subseteq F(x) + \mathbb{B}(0,\epsilon)$ for all $y \in \mathbb{B}(x,\delta)$. F is locally bounded if for every $x \in X$ there exists an $\epsilon > 0$ and $M > 0$ such that $\|z\| \leq M$ for all $z \in F(y)$ and all $y \in \mathbb{B}(x,\epsilon)$.

C. Set-valued dynamical systems

Our exposition on set-valued dynamical systems follows [20]. A time-invariant set-valued dynamical system is given by the differential inclusion

$$\dot{x} \in F(x), \tag{1}$$

where $F : X \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set valued map. If F is locally bounded, upper semi-continuous and takes nonempty, convex, and compact values, then from any initial condition in X there exists an absolutely continuous curve $x : \mathbb{R}_{\geq 0} \to X$, called a solution, satisfying (1) almost everywhere. The set-valued Lie derivative of a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ along the trajectories of (1) is defined as

$$\mathcal{L}_F V(x) = \{\nabla V(x)^T v : v \in F(x)\}.$$

Differential inclusions are especially useful to handle differential equations with discontinuities. Specifically, let $f : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise continuous vector field and consider the differential equation

$$\dot{x} = f(x). \tag{2}$$

The classical notion of solution is not applicable to (2) because of the discontinuities. Instead, consider the Filippov set-valued map associated to f, defined by

$$\mathcal{F}[f](x) := \overline{\mathrm{co}}\{\lim_{i\to\infty} f(x_i) : x_i \to x, x_i \notin \Omega_f\}, \tag{3}$$

where $\overline{\mathrm{co}}\{\cdot\}$ denotes the closed convex hull and Ω_f are the points where f is discontinuous. The set-valued map $\mathcal{F}[f]$ is locally bounded, upper semi-continuous and takes nonempty, convex, and compact values, and hence solutions exist to

$$\dot{x} \in \mathcal{F}[f](x), \tag{4}$$

starting from any initial condition. The solutions of (4) are, by definition, the solutions of (2) in the sense of Filippov.

III. Problem Statement

This section recalls the dynamics developed in our previous work [12] whose trajectories converge to solutions of a linear program. In particular, we (i) introduce the set-valued saddle-point dynamics and state a convergence property of it, (ii) state a discontinuous version of the set-valued dynamics and associated convergence result, and (iii) discuss the distributed implementation of the discontinuous dynamics. In subsequent sections, we study the robustness properties of this dynamics. The linear program we consider is,

$$\min \quad c^T x \tag{5a}$$

$$\text{s.t.} \quad Ax = b, \quad x \geq 0, \tag{5b}$$

where $x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. The set of all solutions to (5) is $\mathcal{X}$. The dual formulation of (5) is

$$\max \quad -b^T z \tag{6a}$$

$$\text{s.t.} \quad A^T z - \lambda + c = 0, \quad \lambda \geq 0. \tag{6b}$$

We denote by $\mathcal{Z}$ the set of solutions of (6). In this paper, we only consider feasible linear programs with compact primal-dual solution space (i.e., $\mathcal{X} \times \mathcal{Z}$ is compact).

A. Saddle-point and discontinuous saddle-point dynamics

The following result relates the solutions of (5) to the saddle points of a modified Lagrangian function.

Proposition III.1 (Solutions of a linear program as saddle points [12]). *Consider $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ given by*

$$L(x,z) = c^T x + \frac{1}{2}(Ax-b)^T(Ax-b) + z^T(Ax-b) + \kappa \mathbf{1}_n^T \max\{0,-x\}, \tag{7}$$

where $\kappa \in \mathbb{R}_{>0}$ and $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of ones. Then, L is convex in x and concave (in fact, linear) in z. Let $\kappa > ||A^T z_* + c||_\infty$ for some $z_* \in \mathcal{Z}$. Then,

(i) *if $x_* \in \mathbb{R}^n$ is a solution of* (5)*, then there exists $z_* \in \mathcal{Z}$ such that (x_*, z_*) is a saddle point of L,*
(ii) *if $(x_*, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a saddle point of L, then $x_* \in \mathbb{R}^n$ is a solution of* (5).

According to the above result, finding saddle points of L is an equivalent problem to finding a solution to (5). This fact motivates the use of saddle-point dynamics (gradient descent in x, gradient ascent in z) associated with L to solve (5),

$$\dot{x} + c + A^T(Ax - b + z) \in -\kappa\partial\max\{0, -x\}, \tag{8a}$$
$$\dot{z} = Ax - b. \tag{8b}$$

For notational convenience, let $F^{\kappa}_{\text{sdl}} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ denote the dynamics (8). The following result characterizes the asymptotic convergence of (8) to the solutions of (5)-(6).

Theorem III.2 (Asymptotic convergence of saddle-point dynamics [12]). *Let κ satisfy $\max_{z_* \in \mathcal{Z}} \|A^T z_* + c\|_\infty < \kappa < \infty$. Then, the projection onto the first (resp. second) component of any trajectory of* (8) *asymptotically converges to a solution of* (5) *(resp.* (6)*).*

Since the above convergence property requires κ to be sufficiently large, and the lower bound on κ is unknown a priori (since it depends on solutions to the dual), [12] proposes a discontinuous dynamics which we formulate next. First, define the *nominal flow function* $f^{\text{nom}} : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$f^{\text{nom}}(x, z) := -c - A^T(Ax - b + z).$$

Then the *discontinuous saddle-point dynamics* is,

$$\dot{x}_i = \begin{cases} f^{\text{nom}}_i(x, z), & \text{if } x_i > 0, \\ \max\{0, f^{\text{nom}}_i(x, z)\}, & \text{if } x_i = 0, \end{cases} \tag{9a}$$
$$\dot{z} = Ax - b, \tag{9b}$$

for all $i \in \{1, \dots, n\}$. We use the discontinuous vector field $f_{\text{dis}} : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ to denote the dynamics (9). The solutions of (9) are understood in the Filippov sense and the following result relates the Filippov set-valued map of f_{dis} to the saddle-point dynamics F^{κ}_{sdl}.

Proposition III.3 (Asymptotic convergence of the discontinuous saddle-point dynamics [12]). *For every compact set $X \times Z \subset \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m$, there exists a finite $\kappa \in \mathbb{R}$ such that the inclusion $\mathcal{F}[f_{dis}](x, z) \subseteq F^{\kappa}_{sdl}(x, z)$ holds for every $(x, z) \in X \times Z$. As a consequence, for any initial condition in $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^m$, the projection onto the first (resp. second) component of any trajectory of* (9) *asymptotically converges to a solution of* (5) *(resp.* (6)*).*

Note that (9) and its convergence properties do not depend on κ.

B. Distributed implementation

The motivation for implementing dynamics (9) rather than other linear programming methods is that it is well-suited for distributed implementation. We consider scenarios in which each component of $x \in \mathbb{R}^n$ corresponds to an independent decision maker or agent. The interconnection between them is modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ are vertices (which represent the agents) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are edges. Agents i and j are called neighbors if $(i, j) \in \mathcal{E}$.

Let us express the dynamics (9) component-wise to see under what conditions it can be implemented by an agent using local information. First, the nominal flow function for agent i is,

$$f^{\text{nom}}_i(x, z) = -c_i - \sum_{\{\ell \,:\, a_{\ell,i} \neq 0\}} a_{\ell,i}\Big[z_\ell + \sum_{\{k \,:\, a_{\ell,k} \neq 0\}} a_{\ell,k} x_k - b_\ell\Big],$$

and the z-dynamics for each $\ell \in \{1, \dots, m\}$ is

$$\dot{z}_\ell = \sum_{\{i \,:\, a_{\ell,i} \neq 0\}} a_{\ell,i} x_i - b_\ell. \tag{10}$$

Then i can compute $f^{\text{nom}}_i(x, z)$ and any z_ℓ used therein if

(D1) for each $i \in \mathcal{V}$, agent i knows x_i as well as
  a) $c_i \in \mathbb{R}$,
  b) every $b_\ell \in \mathbb{R}$ for which $a_{\ell,i} \neq 0$,
  c) the non-zero elements of every $\text{row}_\ell(A) \in \mathbb{R}^n$ for which $a_{\ell,i} \neq 0$,

(D2) agent $i \in \mathcal{V}$ implements $\dot{x}_i$,

(D3) for each $\ell \in \{1, \dots, m\}$ such that $a_{\ell,i} \neq 0 \neq a_{\ell,j}$, it holds that $(i, j) \in \mathcal{E}$, and

(D4) agent i can measure every x_j for which j is a neighbor of i.

When we say that i can compute z_ℓ, we mean that i can implement $\dot{z}_\ell$ if it so requires z_ℓ in computing $f^{\text{nom}}_i(x, z)$. Dynamics (9) is *distributed over* $\mathcal{G}$ when (D1)-(D4) hold.

IV. Robustness against disturbances and noise

In this section we explore the robustness properties of the distributed set-valued dynamics (8). Because the solutions of the discontinuous saddle-point dynamics are solutions to (8) for sufficiently large κ (cf. Proposition III.3), the robustness properties of (8) translate equivalently to the discontinuous saddle-point dynamics (9). Thus, throughout this section and without loss of generality, we consider the dynamics (8).

We consider external disturbances to the saddle-point dynamics (8). Specifically,

$$\dot{x} + c + A^T(Ax - b + z) + u_x \in -\kappa\partial\max\{0, -x\}, \tag{11a}$$
$$\dot{z} = Ax - b - u_z, \tag{11b}$$

where $u = (u_x, u_z)$ takes values in $\mathbb{R}^n \times \mathbb{R}^m$ and is locally essentially bounded. For notational purposes, we use $F^{\kappa,u}_{\text{sdl}} : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ to denote the dynamics (11). The asymptotic stability of (8), cf. Theorem III.2, ensures robustness in the sense of [15]. This definition of robustness is a qualitative statement of the form: there exists a perturbation small enough such that the equilibria is still asymptotically stable. Our objective here is to obtain a more precise quantitative description of the robustness properties of (11).

Remark IV.1 (Noisy dynamics model). The disturbance u in (11) captures unmodeled dynamics, and both measurement and computation noise. In addition, any error in an agent's knowledge of the problem data (c, A, and b) can be interpreted as a specific manifestation of u. For example, if agent i uses an estimate $\hat{c}_i$ of c_i when computing its dynamics, this can be modeled in (11) by considering $u_{x,i}(t) = c_i - \hat{c}_i$. •

A. No linear programming dynamics is input-to-state stable

The first notion of robustness that we consider is input-to-state stability (ISS). Essentially, ISS corresponds to the idea that small disturbances give rise to small deviations from the equilibrium set. We state this formally below.

Definition IV.2 (Input-to-state stability [16]). *The system* (11) *is ISS with respect to the set $\mathcal{X} \times \mathcal{Z}$ if there exist functions $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for any trajectory $t \mapsto (x(t), z(t))$ of* (11)*, one has*

$$\|(x(t), z(t))\|_{\mathcal{X}\times\mathcal{Z}} \leq \beta(\|(x(0), z(0)\|_{\mathcal{X}\times\mathcal{Z}}, t) + \gamma(\|u\|_\infty),$$

for all $t \geq 0$. Here, $\|u\|_\infty := \operatorname{ess\,sup}_{s\geq 0} \|u(s)\|$ is the essential supremum of $u(t)$.

In the next result, we establish that any dynamics that solve any feasible linear program and uncertainties in the problem data (A, b, and c) enter as disturbances is not ISS. In particular, this implies that the noisy saddle-point dynamics (11) is not ISS either.

Theorem IV.3 (No dynamics for linear programming is ISS). *Consider the generic dynamics*

$$(\dot{x}, \dot{z}) = \Phi(x, z, v) \tag{12}$$

with disturbance $t \mapsto v(t)$. Assume uncertainties in the problem data are modeled by v. That is, there exists a surjective function $g = (g_1, g_2) : \mathbb{R}^{n+m} \to \mathbb{R}^n \times \mathbb{R}^m$ with $g(0) = (0, 0)$ such that, for $\bar{v} \in \mathbb{R}^{n+m}$, the primal-dual solution set $\mathcal{X}(\bar{v}) \times \mathcal{Z}(\bar{v})$ of the linear program

$$\min \quad (c + g_1(\bar{v}))^T x \tag{13a}$$
$$\text{s.t.} \quad Ax = b + g_2(\bar{v}), \quad x \geq 0. \tag{13b}$$

is the stable equilibrium set of $(\dot{x}, \dot{z}) = \Phi(x, z, \bar{v})$ whenever $\mathcal{X}(\bar{v}) \times \mathcal{Z}(\bar{v}) \neq \emptyset$. Then, the dynamics (12) *is not ISS with respect to $\mathcal{X} \times \mathcal{Z}$.*

The above result holds because, for some finite uncertainty in the problem data, the associated perturbed optimization problem has an unbounded solution set. Since any point in that unbounded set is an equilibrium, it is clear that the dynamics is not ISS.

Remark IV.4 (Small-signal ISS). In [21], we show that the discontinuous saddle-point dynamics satisfy the ISS inequality when the disturbance $u(t)$ is sufficiently small. •

In the subsequent section, we show that (11) satisfies a weaker notion of robustness.

B. Saddle-point dynamics is integral input-to-state stable

Here we establish that the dynamics (11) is integral input-to-state stable (iISS). Informally, iISS guarantees that disturbances with small energy give rise to small deviations from the equilibria. The definition below states this formally.

Definition IV.5 (Integral input-to-state stability [17]). *The system* (11) *is iISS with respect to the set $\mathcal{X} \times \mathcal{Z}$ if there exist functions $\alpha \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for any trajectory $t \mapsto (x(t), z(t))$ of* (11) *and all $t \geq 0$, one has*

$$\alpha(\|(x(t), z(t))\|_{\mathcal{X}\times\mathcal{Z}}) \leq \beta(\|(x(0), z(0)\|_{\mathcal{X}\times\mathcal{Z}}, t) + \int_0^t \gamma(\|u(s)\|)ds. \tag{14}$$

Our ensuing discussion is based on a suitable adaptation of the exposition in [17] to the setup of asymptotically stable sets for differential inclusions. A useful tool for establishing iISS is the notion of iISS Lyapunov function, whose definition we review next

Definition IV.6 (iISS Lyapunov function). *A differentiable function $V : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ is an iISS Lyapunov function with respect to the set $\mathcal{X} \times \mathcal{Z}$ for system* (11) *if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$, and a continuous positive definite function α_3 such that*

$$\alpha_1(\|(x, z)\|_{\mathcal{X}\times\mathcal{Z}}) \leq V(x, z) \leq \alpha_2(\|(x, z)\|_{\mathcal{X}\times\mathcal{Z}}), \tag{15a}$$
$$a \leq -\alpha_3(\|(x, z)\|_{\mathcal{X}\times\mathcal{Z}}) + \sigma(\|u\|), \tag{15b}$$

for all $a \in \mathcal{L}_{F^{\kappa,u}_{sdl}} V(x, z)$ and $x \in \mathbb{R}^n, z \in \mathbb{R}^m, u \in \mathbb{R}^{n+m}$.

The existence of a iISS Lyapunov function is critical in establishing iISS, as the following result states.

Theorem IV.7 (iISS Lyapunov function implies iISS). *If there exists an iISS Lyapunov function with respect to $\mathcal{X} \times \mathcal{Z}$ for* (11)*, then the system is iISS with respect to $\mathcal{X} \times \mathcal{Z}$.*

This result is stated in [17, Theorem 1] for differential equations with an asymptotically stable origin and can be trivially extended to include differential inclusions and asymptotically stable sets, as considered here. We use Theorem IV.7 next to establish that (11) is iISS.

Theorem IV.8 (iISS of saddle-point dynamics). *Assume $\kappa > \lambda_{\max}$. Then the system* (11) *is iISS with respect to $\mathcal{X} \times \mathcal{Z}$ with $\gamma(s) = 4s$.*

Based on the discussion in Section IV-A, we believe that the iISS property of (11) is an accurate representation of the overall robustness of the dynamics when considering general linear programs, and not a weakness of our analysis. A consequence of iISS is that the asymptotic convergence of the system is preserved under finite energy disturbances [22, Proposition 6]. In the case of (11), a stronger convergence property is true under finite variation disturbances ($u(t)$ has *finite variation* if there exists a $\bar{u} \in \mathbb{R}$ s.t. $\int_0^\infty \|u(s) - \bar{u}\| < \infty$). The following formalizes this idea.

Corollary IV.9 (Finite variation disturbances). *Suppose $t \mapsto u(t) \in \mathbb{R}^{n+m}$ is such that $\int_0^\infty \|u(s)-(\bar{u}_x, \bar{u}_z)\|ds < \infty$ for some $(\bar{u}_x, \bar{u}_z) \in \mathbb{R}^n \times \mathbb{R}^m$ and that*

$$\min \quad (c + \bar{u}_x + A^T \bar{u}_z)^T x \tag{16a}$$

$$\text{s.t.} \quad Ax = b + \bar{u}_z, \quad x \geq 0, \tag{16b}$$

is feasible. Denote the set of solutions to (16) *(resp. the dual of* (16)*) as $\mathcal{X}^{\bar{u}}$ (resp. $\mathcal{Z}^{\bar{u}}$). Assume $\mathcal{X}^{\bar{u}} \times \mathcal{Z}^{\bar{u}}$ is compact, and let $\kappa > \max_{z_* \in \mathcal{Z}^{\bar{u}}} \|A^T z_* + c + \bar{u}_x - A^T \bar{u}_z\|_\infty$. Then* (11) *is iISS with respect to $\mathcal{X}^{\bar{u}} \times \mathcal{Z}^{\bar{u}}$. In this case, solutions to* (11) *converge asymptotically to a point in $\mathcal{X}^{\bar{u}} \times \mathcal{Z}^{\bar{u}}$.*

We simulate (11) for the following linear program

$$\begin{aligned}
\min_{x \geq 0} \quad & x_1 + x_3 + x_5 + x_7 \\
\text{s.t.} \quad & x_1 - x_2 + x_3 = 4, \\
& x_3 - x_4 + x_5 = 3, \\
& x_5 - x_6 + x_7 = 2.
\end{aligned} \tag{17}$$

The primal-dual solution set to (17) is $\mathcal{X}^{\text{sim}} \times \mathcal{Z}^{\text{sim}}$ where,

$$\begin{aligned}
\mathcal{X}^{\text{sim}} = \{x \geq 0 : & \; x_2 = x_6 = 0, x_1 + x_3 = 4, \\
& x_3 - x_4 + x_5 = 3, \text{ and } x_5 + x_7 = 2\}, \\
\mathcal{Z}^{\text{sim}} = \{(-1, & 0, -1)\}.
\end{aligned}$$

In Figure 1(a) asymptotic convergence is achieved under a finite energy signal (i.e., $\int_0^\infty \|u(s)\| < \infty$), which supports Corollary IV.9. Even under a finite power signal (i.e., $\lim_{T\to\infty} \frac{1}{T} \int_0^T \|u(s)\| < \infty$) the trajectories are well-behaved (see Figure 1(b)), verifying the small-signal ISS property (see Remark IV.4). This simulation may also suggest a noise-to-state stability property for the dynamics.

V. Robustness in recurrently connected graphs

Here we study the convergence properties of the saddle-point dynamics (8) when agents do not receive updated state information from their neighbors at all times because of communication link failures. As such, agents use the last known value of neighbor states to compute their dynamics. The type of link failures we consider are characterized by recurrently connected graphs (RCG), which we define next.

Definition V.1 (Recurrently connected graphs). *Given a strictly increasing sequence $\{t_k\}_{k=0}^\infty \subset \mathbb{R}_{\geq 0}$ and a base graph $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$, we call $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$* recurrently connected with respect to *$\mathcal{G}_b$* and *$\{t_k\}_{k=0}^\infty$ if $\mathcal{E}(t) = \mathcal{E}_b$ for all $t \in [t_{2k+1}, t_{2k+2}), k \in \mathbb{Z}_{\geq 0}$.*

In a RCG, no assumption is made on edges during the intervals $[t_{2k}, t_{2k+1})$. In what follows and for simplicity of presentation, we only consider the worst-case scenario where edges in the base graph fail over the entire intervals $[t_{2k}, t_{2k+1})$. The results stated here also apply to the more general scenario where edges may fail and reconnect multiple times during the intervals $[t_{2k}, t_{2k+1})$. We start by noting that, under link failures, the implementation of (10) across different agents would yield in general different outcomes (given that different agents have access to different information at different times). To avoid this problem, we assume that, for each $\ell \in \{1, \ldots, m\}$, the agent

$$j = \mathbb{S}(\ell) := \min\{i \in \{1, \ldots, n\} : a_{\ell,i} \neq 0\},$$

implements the z_ℓ-dynamics and communicates this value to its neighbors (incidentally, only neighbors of $j = \mathbb{S}(\ell)$ need to know z_ℓ). Next, we are ready to describe the network dynamics under link failures. With the notation of Definition V.1, let $\mathbb{F}(k)$ be the set of failing communication edges for $t \in [t_k, t_{k+1})$. In other words, if $(i,j) \in \mathbb{F}(k)$ then agents i and j do not receive updated state information from each other during the whole interval $[t_k, t_{k+1})$. The nominal flow function of i on a RCG for $t \in [t_k, t_{k+1})$ is

$$\begin{aligned}
f_i^{\text{nom,RCG}}(x,z) = & -c_i - \sum_{\substack{\ell=1 \\ (i,\mathbb{S}(\ell)) \notin \mathbb{F}(k)}}^m a_{\ell,i} z_\ell - \sum_{\substack{\ell=1 \\ (i,\mathbb{S}(\ell)) \in \mathbb{F}(k)}}^m a_{\ell,i} z_\ell(t_k) \\
& - \sum_{\ell=1}^m a_{\ell,i} \Big[\sum_{\substack{j=1 \\ (i,j) \notin \mathbb{F}(k)}}^n a_{\ell,j} x_j + \sum_{\substack{j=1 \\ (i,j) \in \mathbb{F}(k)}}^n a_{\ell,j} x_j(t_k) - b_\ell \Big].
\end{aligned}$$

Thus the x-dynamics during $[t_k, t_{k+1})$ for $i \in \{1, \ldots, n\}$ is

$$\dot{x}_i = \begin{cases} f_i^{\text{nom,RCG}}(x,z), & \text{if } x_i > 0, \\ \max\{0, f_i^{\text{nom,RCG}}(x,z)\}, & \text{if } x_i = 0. \end{cases} \tag{18a}$$

Likewise, the z-dynamics for $\ell \in \{1, \ldots, m\}$ is

$$\dot{z}_\ell = \sum_{\substack{i=1 \\ (i,\mathbb{S}(\ell)) \notin \mathbb{F}(k)}}^n a_{\ell,i} x_i + \sum_{\substack{i=1 \\ (i,\mathbb{S}(\ell)) \in \mathbb{F}(k)}}^n a_{\ell,i} x_i(t_k) - b_\ell. \tag{18b}$$

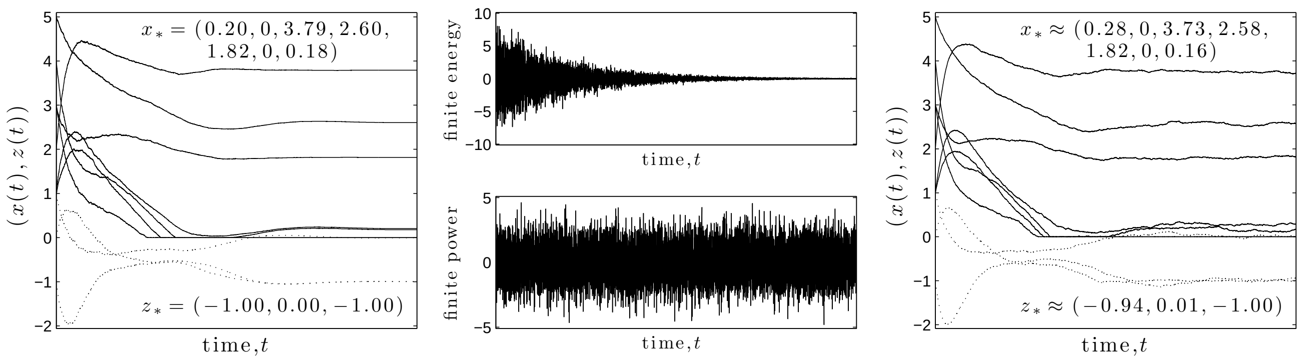
It is worth noting that (18) and (9) coincide when $\mathbb{F}(k) = \emptyset$.

Proposition V.2 (Convergence of saddle-point dynamics under recurrently connected graphs). *Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ be recurrently connected with respect to $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ and $\{t_k\}_{k=0}^\infty$. Suppose that* (18) *is distributed over $\mathcal{G}_b$ and $T_{\text{disconnected}}^{\max} := \sup_{k \in \mathbb{Z}_{\geq 0}} (t_{2k+1} - t_{2k}) < \infty$. Let $(x(t), z(t))$ be a trajectory of* (18)*. Then there exists a $T_{\text{connected}}^{\min} > 0$ (depending on $T_{\text{disconnected}}^{\max}$, $x(t_0)$, and $z(t_0)$) such that $\inf_{k \in \mathbb{Z}_{\geq 0}} (t_{2k+2} - t_{2k+1}) > T_{\text{connected}}^{\min}$ implies that $\|(x(t_{2k}), z(t_{2k}))\|_{\mathcal{X} \times \mathcal{Z}} \to 0$ as $k \to \infty$.*

Figure 2 illustrates the result of Proposition V.2.

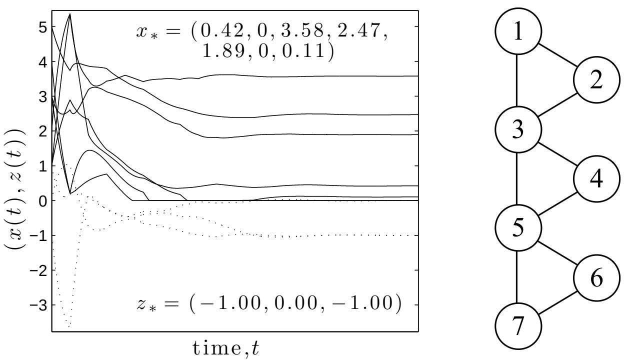
VI. Conclusions

We have studied the robustness properties of a saddle-point dynamics for distributed linear programming against disturbances induced by communication noise, computation errors, and mismatches in the agents' knowledge about the problem data. We stated that no dynamics for linear programming is input-to-state stable when uncertainty in the problem data is modeled as a disturbance. We show instead that the dynamics is integral input-to-state stable, and hence robust against disturbances of finite energy. In addition, we also showed that asymptotic convergence is achieved under disturbances of finite variation. Finally, we have also



(a) Trajectories subject to finite energy noise. (b) Noise used in simulations. (c) Trajectories subject to finite power noise.

Fig. 1. Trajectories of the saddle-point dynamics under disturbances (11) for the linear program (17) with two types of noise: finite energy and finite power. In (a) and (c), the solid lines are x-trajectories and the dotted lines are z-trajectories. The final values of the state are indicated on each plot. In (b) we provide a sample of the noise affecting the x_1-dynamics in both cases. Asymptotic convergence to the primal-dual solution set $\mathcal{X}^{\text{sim}} \times \mathcal{Z}^{\text{sim}}$ is achieved when the noise has finite energy, as expected (cf. Corollary IV.9). Even when the noise has finite power (white Gaussian noise at 2dBW in this simulation), trajectories remain close to the primal-dual solution set (see Remark IV.4).



(a) (b)

Fig. 2. Trajectories of (18) for linear program (17) under a recurrently connected graph. In (a), the solid lines are x-trajectories and the dotted lines are z-trajectories. (b) shows the base graph. A random number of random links fail in recurring intervals of time. The time ratio of link failures to no-link failures is 100 : 1. Convergence to $\mathcal{X}^{\text{sim}} \times \mathcal{Z}^{\text{sim}}$ is achieved, suggesting that in this example $T^{\min}_{\text{connected}} \leq T^{\max}_{\text{disconnected}}/100$.

established the robustness of the saddle-point dynamics under link failures modeled by a recurrently connected graph. Future work will include a constructive characterization of the bound $T^{\min}_{\text{connected}}$, the study of more general link failure scenarios, the analysis of noise-to-state stability properties of the saddle-point dynamics, and the design of event-triggered implementations of the saddle-point dynamics.

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