

Robust Modeling of Probabilistic Uncertainty in Smart Grids: Data Ambiguous Chance Constrained Optimum Power Flow

Daniel Bienstock, Michael Chertkov, Sean Harnett

Abstract—Future Grids will integrate time-intermittent renewables and demand response whose fluctuating outputs will create perturbations requiring probabilistic measures of resilience. When smart but uncontrollable resources fluctuate, Optimum Power Flow (OPF), routinely used by the electric power industry to dispatch controllable generation over control areas of transmission networks, can result in higher risks. Our Chance Constrained (CC) OPF corrects the problem and mitigates dangerous fluctuations with minimal changes in the current operational procedure. Assuming availability of a reliable forecast parameterizing the distribution function of the uncertain resources, our CC-OPF satisfies all the constraints with high probability while simultaneously minimizing the cost of economic dispatch. For linear (DC) modeling of power flows, and parametrization of the uncertainty through Gaussian distribution functions the CC-OPF turns into convex (conic) optimization, which allows efficient and scalable cutting-plane implementation. When estimates of the Gaussian parameters are imprecise we robustify CC-OPF deriving its data ambiguous and still scalable implementation.

I. CHANCE-CONSTRAINED OPTIMUM POWER FLOW MODELS

A. Transmission Grids: Controls and Limits

Electric power (alternating current, or AC) transmission systems balance consumption/load and generation using a complex strategy that spans three different time scales (see e.g. [2]). At any point in time, generators produce power at a previously computed base level. In real time, changes in loads are registered at generators through (opposite) changes in frequency, for example in the case of a load increase generators will marginally slow down – frequency will start to drop. So-called **primary** and **secondary** frequency control is responsible for arresting this drop and then restoring system frequency, all in a matter of minutes. Base generator output levels are set using an OPF algorithm, typically run as frequently as every fifteen minutes (to one hour), and using estimates for loads during the next time window. This computation takes into account not only load levels but also other parameters of importance, such as line transmission

limits. OPF computation represents the shortest time scale where actual off-line and network wide optimal computations are employed.

A central goal of the controls concerns “stability,” especially maintaining stable frequency and voltage levels. Additionally line power flows should be kept within given bounds. An excessively large power flow across a line will increase line temperature to a physical failure point or so that the line sags, and potentially arcs or trips due to a physical contact. For each line there is a given parameter, the *line rating* (or *limit*) which upper bounds flow level during satisfactory operation.

Of the three “stability” criteria described above, the first two (maintaining synchrony and voltage) are a concern only in a truly nonlinear regime which under normal circumstances occur rarely. Discussion normal or close to normal operations, we focus on the third – observing line limits.

B. OPF – Standard Generation Dispatch

OPF is a key underlying algorithm in power engineering; see the review in [3] and e.g. [4], [2]. The task of OPF, usually executed off-line at periodic intervals, is to reset generator output levels over a control area of the transmission grid. The generic OPF problem can be stated as follows:

- The goal is to determine the vector $p \in \mathcal{R}^{\mathcal{G}}$, where for $i \in \mathcal{G}$, p_i is the output of generator i , so as to minimize an objective function $c(p)$. This function is, usually, a convex, separable quadratic function of p :

$$c(p) = \sum_{i \in \mathcal{G}} c_i(p_i),$$

where each c_i is convex quadratic.

- The problem is endowed by three types of constraints: power flow, line limit and generation bound constraints.

The simplest constraints are the generation bounds, which are box constraints on the individual p_i . The thermal line limits place an upper bound on the power flows in each line; and how power is transported in the given network is described by the power flow constraints. In keeping to common transmission system practice, here we use the so-called “DC-approximation,” a linearized version of the power flow equations. In this approximation the (real) power flow over line $(i, j) \in \mathcal{E}$, where \mathcal{E} is the set of all lines of the grid, with line susceptance β_{ij} ($= \beta_{ji}$) is related linearly to the respective phase difference,

$$f_{ij} = \beta_{ij}(\theta_i - \theta_j). \quad (1)$$

D. Bienstock is with the Departments of Industrial Engineering and Operations Research, and Applied Physics and Applied Mathematics, Columbia University, New York, NY, 10027 USA. email: dano@columbia.edu

M. Chertkov is with Theoretical Division and Center for Non-linear Studies, Los Alamos National Laboratory, NM 87545 USA. email: chertkov@lanl.gov

S. Harnett is with Department of Applied Physics and Applied Mathematics, Columbia University, 500 West 120th St. New York, NY 10027 USA and Center for Nonlinear Studies, Los Alamos National Laboratory, NM 87545 USA.

Extended version of this paper containing additional examples and proofs is available at [1].

Suppose, for convenience of notation, that we extend the vector p to include an entry for every bus $i \in \mathcal{V}$ with the proviso that $p_i = 0$ whenever $i \notin \mathcal{G}$. Likewise, denote by $d \in \mathcal{R}^{\mathcal{V}}$ the vector of (possibly zero) demands and by $\theta \in \mathcal{R}^{\mathcal{V}}$ the vector of phase angles. Then, the power flow constraint (in the DC approximation) states that a vector f of power flows is feasible if and only if

$$\sum_{j:(i,j) \in \mathcal{E}} f_{ij} = p_i - d_i, \quad \text{for each bus } i. \quad (2)$$

Eqs. (1,2) can be rewritten in the following matrix form:

$$B\theta = p - d, \quad (3)$$

where the $n \times n$ matrix B is a weighted-Laplacian defined as follows:

$$\forall i, j \in \mathcal{V}: \quad B_{ij} = \begin{cases} -\beta_{ij}, & (i, j) \in \mathcal{E} \\ \sum_{k:(k,j) \in \mathcal{E}} \beta_{kj}, & i = j \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

It is readily seen that the system (3) is feasible if and only if total generation equals total demand. Thus, the standard OPF in DC-formulation can be stated as the following constrained optimization problem:

$$\text{OPF: } \min_p \quad c(p), \quad \text{s.t.} \quad (5)$$

$$B\theta = p - d, \quad (6)$$

$$\forall i \in \mathcal{G}: \quad p_i^{\min} \leq p_i \leq p_i^{\max}, \quad (7)$$

$$\forall (i, j) \in \mathcal{E}: \quad |f_{ij}| \leq f_{ij}^{\max}, \quad (8)$$

Note that the p_i^{\min}, p_i^{\max} quantities can be used to enforce the convention $p_i = 0$ for each $i \notin \mathcal{G}$; if $i \in \mathcal{G}$ then p_i^{\min}, p_i^{\max} are lower and upper generation bounds which are generator-specific. Here, Constraint (8) is the line limit constraint for (i, j) ; f_{ij}^{\max} represents the line limit (typically a thermal limit), which is assumed to be strictly enforced in constraint (8). This conservative condition will be relaxed in the following.

The above is a convex quadratic program, easily solved using modern optimization tools. The vector d of demands is *fixed* in this problem and is obtained through estimation. In practice, however, demand will fluctuate around d ; generators then respond by adjusting their output (from the OPF-computed quantities) proportionally to the overall fluctuation.

C. Chance constrained OPF: motivation

Suppose OPF yields output p_i for each generator i assuming constant demands d . Let $\hat{d}(t)$ be the vector of real-time demands at time t . Then primary and secondary controls in combination resets output at generator $i \in \mathcal{G}$ to

$$\hat{p}_i(t) = p_i - \rho_i \sum_j (d_j - \hat{d}_j(t)). \quad (9)$$

In this equation, the quantities $\rho_i \geq 0$ are fixed and satisfy $\sum_i \rho_i = 1$. Thus, from (9) we obtain $\sum_i \hat{p}_i(t) = \sum_i p_i - \sum_j (d_j - \hat{d}_j(t)) = \sum_j \hat{d}_j(t)$, in other words, demands are met. The quantities $\rho_i \geq 0$ are generator dependent but

essentially chosen far in advance and without regard to short-term demand forecasts.

This scheme has worked well in the past because of the slow time scales of change in uncontrolled resources (mainly loads); i.e. frequency control and load changes are well-separated. Note that a large error in the forecast or an under-estimation of possible d for the next –e.g., fifteen minute–period may lead to an operational problem in standard OPF (see e.g. the discussions in [5], [6]) because even though the vector $\hat{p}(t)$ is sufficient to meet demands, the phase angles $\hat{\theta}(t)$ computed from $B\hat{\theta}(t) = \hat{p}(t) - \hat{d}(t)$ can give rise to real-time power flows $\hat{f}_{ij}(t) \doteq \beta_{ij}[\hat{\theta}_i(t) - \hat{\theta}_j(t)]$ that violate constraints (8). This has not been considered a handicap, however, primarily because the deviations $\hat{d}_i(t) - d_i$ are small in the time scale of interest.

Now assume that a subset \mathcal{W} of the buses holds uncertain power sources (wind farms); for each $j \in \mathcal{W}$, write the amount of power generated by source j at time t as $\mu_j + \omega_j(t)$, where μ_j is the forecast output of farm j in the time period of interest. For ease of exposition, we will assume in what follows that \mathcal{G} refers to the set of buses holding *controllable* generators, i.e. $\mathcal{G} \cap \mathcal{W} = \emptyset$. Renewable generation can be incorporated into the OPF formulation (5)–(8) by simply setting $p_i = \mu_i$ for each $i \in \mathcal{W}$. Assuming constant demands but fluctuating renewable generation, the application of the frequency control yields the following analogue to (9):

$$\hat{p}_i(t) = p_i - \rho_i \sum_{j \in \mathcal{W}} \omega_j(t) \quad \text{for each } i \in \mathcal{G}, \quad (10)$$

e.g. if $\sum_{j \in \mathcal{W}} \omega_j(t) > 0$, that is to say, there is a net increase in wind output, then (controllable) generator output will proportionally decrease.

Eq. (10) describes how the controllable generation adjusts to wind changes, under current power engineering practice. The hazard here is that the quantities $\omega_j(t)$ can be large resulting in sudden and large changes in power flows, large enough to substantially overload power lines and thereby cause their tripping, a highly undesirable feature that compromises grid stability.

D. Using chance constraints

Power lines do not fail (trip) instantly when their flow limits are exceeded; an overloaded line will heat up and the longer this condition is maintained, the higher the probability that it will trip. Even though an exact representation of line tripping seems difficult, we can however state a practicable alternative. To ensure safety, we will impose a *chance constraint* [7], [8], [9]: “the probability that any given line is overloaded is small”.

To formalize this notion, we assume:

- W.1.** For each $i \in \mathcal{W}$, the (stochastic) amount of power generated by source i is of the form $\mu_i + \omega_i$, where
- W.2.** μ_i is constant, assumed known from the forecast, and ω_i is a zero mean independent random variable with known standard deviation σ_i .

Here and in what follows, we use bold face to indicate uncertain quantities. Let \mathbf{f}_{ij} be the flow on a given line (i, j) , and let $0 < \epsilon_{ij}$ be small. The chance constraint for line (i, j) , is:

$$P(\mathbf{f}_{ij} > f_{ij}^{max}) < \epsilon_{ij} \text{ and } P(\mathbf{f}_{ij} < -f_{ij}^{max}) < \epsilon_{ij}. \quad (11)$$

Likewise, for a generator g we will require that

$$P(\mathbf{p}_g > p_g^{max}) < \epsilon_g \text{ and } P(\mathbf{p}_g < p_g^{min}) < \epsilon_g.$$

The parameter ϵ_g will be chosen *extremely* small, so that for all practical purposes the generator's will be guaranteed to stay within its bounds.

E. Related Bibliography

[10] considers the standard OPF problem under stochastic demands, and describes a method that computes *fixed* generator output levels to be used throughout the period of interest, independent of demand levels, using a simulation-based local optimization system consisting of an outer loop used to assess the validity of a control (and estimate its gradient) together with an inner loop that seeks to improve the control. Experiments are presented using a 5-bus and a 30-bus example. This approach, though general, requires a number of technical assumptions and elaborations to guarantee convergence and feasibility; and it appears to entail a very high computational cost. [11] describes a security constrained optimization for reserve scheduling with fluctuating wind generation. Security constraints are represented via a set of outage scenarios. In [11] wind generation is assumed located at a single bus in a 30 bus network. Assuming that the probability distribution function of the aggregated wind resources is known, the cost of generation over the time horizon is optimized under the chance constraints for line overloads and generation limits. The stochastic optimization is tackled in [11] via transformation to a convex optimization at the expense of a number of approximations; however convergence to global optimum is not guaranteed. For other uses of chance constrained optimization in a power engineering context, see [12], [13], [14], [15].

F. Uncertain power sources

We assume that wind power fluctuations are (stochastically) independent at different farms. (This assumption is well justified for wind farms located larger than $L \approx T * v$ apart, where T is time period between two consecutive CC-OPFs, and v is the typical wind velocity. For $T = 15\text{min}$ and $v = 15\text{m/s}$, one gets $L \approx 15\text{km}$.) In our basic construction we assume Gaussianity of the ω_i . Then, we also consider a data-robust version of our chance-constrained problem where the parameters for the Gaussian distributions are assumed unknown, but lying in a window. This allows both for parameter mis-estimation and for model error, that is to say the implicit approximation of non-Gaussian distributions with Gaussians; our approach, detailed in [1], remains computationally sound in this robust setting. We also perform in [1] out-of-sample analyses of our computed control, in particular testing its behavior against non-normal distributions, in particular other

fitting distributions considered in the wind-modeling literature, e.g. Weibull distributions and logistic distributions [16], [17].

G. Affine Control

To handle variability of the ω we propose the following control, which can be viewed as a refinement Eq. (10):

$$\forall \text{ bus } i \in \mathcal{G} : \quad \mathbf{p}_i = \bar{p}_i - \alpha_i \sum_{j \in \mathcal{W}} \omega_j. \quad \alpha_i \geq 0. \quad (12)$$

Here the quantities $\bar{p}_i \geq 0$ and $\alpha_i \geq 0$ are design variables satisfying (among other constraints) $\sum_{i \in \mathcal{G}} \alpha_i = 1$. Notice that we do not set any α_i to a standard (fixed) value, but instead leave the optimization to decide the optimal value. Observe that $\sum_i p_i = \sum_i \bar{p}_i - \sum_i \omega_j$, that is, the total power generated equals the average production of the generators minus any additional wind power above the average case.

This affine control scheme creates the possibility of requiring a generator to produce power beyond its limits. With unbounded wind, this possibility is inevitable, though we can restrict it to occur with arbitrarily small probability in our chance constraint for generators.

II. SOLVING THE MODELS

A. Chance-constrained optimal power flow: formal expression

In what follows we outline our solution procedure. Proofs are omitted for the sake of brevity. Following the W.1 and W.2 notations, Eqs. (12) explain the *affine* control, given that the α_i are decision variables in our CC-OPF, additional to the standard \bar{p}_i decision variables already used in the standard OPF (5). For $i \notin \mathcal{W}$ write $\mu_i = 0$, thereby obtaining a vector $\mu \in \mathcal{R}^n$. Likewise, extend \bar{p} and α to vectors in \mathcal{R}^n by writing $\bar{p}_i = \alpha_i = 0$ whenever $i \notin \mathcal{G}$.

Definition. We say that the pair \bar{p}, α is *viable* if the generator outputs under control law (12), together with the uncertain outputs, always exactly match total demand.

The following simple result characterizes this condition as well as other basic properties of the affine control. Here and below, $e \in \mathcal{R}^n$ is the vector of all 1's.

Lemma 2.1: Under the control law (12) the net output of bus i equals $\bar{p}_i + \mu_i - d_i + \omega - \alpha_i(e^T \omega)$ and thus the (stochastic) power flow equations can be written as $B\theta = \bar{p} + \mu - d + \omega - (e^T \omega)\alpha$. Consequently, the pair \bar{p}, α is viable if and only if $\sum_{i \in \mathcal{V}} (\bar{p}_i + \mu_i - d_i) = 0$.

For convenience of notation we will assume that bus n is neither a generator nor a wind farm bus, that is to say, $n \notin \mathcal{G} \cup \mathcal{W}$, and we denote by \hat{B} the submatrix obtained from B by removing row and column n , and write

$$\check{B} = \begin{pmatrix} \hat{B}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (13)$$

Note that we can assume without loss of generality that $\theta_n = 0$.

Lemma 2.2: Suppose the pair \bar{p}, α is viable. Then under the control law (12) a vector of (stochastic) phase angles is

$$\theta = \bar{\theta} + \check{B}(\omega - (e^T \omega) \alpha), \quad \text{where} \quad (14)$$

$$\bar{\theta} = \check{B}(\bar{p} + \mu - d). \quad (15)$$

As a consequence, $E_\omega \theta = \bar{\theta}$, and given any line (i, j) , $E_\omega f_{ij} = \beta_{ij}(\bar{\theta}_i - \bar{\theta}_j)$. Furthermore, each quantity θ_i or f_{ij} is an affine function of the random variables ω_i .

In our chance-constrained problem the objective function is the expected cost incurred by the stochastic generation vector $p = \bar{p} - (e^T \omega) \alpha$ over the varying wind power output w . In standard power engineering practice generation cost is convex, quadratic and separable, i.e. for any vector p , $c(p) = \sum_i c_i(p_i)$ where each c_i is convex quadratic. Note that for any $i \in \mathcal{G}$ we have

$$p_i^2 = \bar{p}_i^2 + (e^T \omega)^2 \alpha_i^2 - 2e^T \omega \bar{p}_i \alpha_i,$$

from which we obtain, since the ω_i have zero mean,

$$E_w(p_i^2) = \bar{p}_i^2 + \text{var}(\Omega) \alpha_i^2,$$

where “var” denotes variance and $\Omega \doteq \sum_j \omega_j$. Consequently the objective function is convex quadratic, as a function of \bar{p} and α . Denote:

- For $j \in \mathcal{W}$, the variance of ω_j is denoted by σ_j^2 .
- For $1 \leq i, j \leq n$ let π_{ij} denote the i, j entry of the matrix \check{B} given above, that is to say,

$$\pi_{ij} = \begin{cases} [\hat{B}^{-1}]_{ij}, & i < n, \\ 0, & \text{otherwise.} \end{cases}, \quad (16)$$

- Given α , for $1 \leq i \leq n$ write

$$\delta_i \doteq [\check{B}\alpha]_i = \begin{cases} [\hat{B}^{-1}\alpha]_i, & i < n, \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

Lemma 2.3: Assume that the ω_i are independent random variables. Given α , for any line (i, j) ,

$$\text{var}(f_{ij}) = \beta_{ij}^2 \sum_{k \in \mathcal{W}} \sigma_k^2 (\pi_{ik} - \pi_{jk} - \delta_i + \delta_j)^2. \quad (18)$$

Remark. Lemma 2.3 holds for any distribution of the ω_i so long as independence is assumed. Similar results are easily obtained for higher-order moments of the f_{ij} .

Above we assumed that the ω_i random variables have zero mean. To obtain an efficient solution procedure we will additionally assume that they are (a) pairwise independent and (b) normally distributed. Since the f_{ij} are affine functions of the ω_i (because the θ are, by eq. (14)), it turns out that there is a simple statement of the chance-constraints in a computationally practicable form. See [18] for a general treatment of linear inequalities with stochastic coefficients. For any real $0 < r < 1$ we write $\eta(r) = \phi^{-1}(1-r)$, where ϕ is the cdf of a standard normally distributed random variable.

Lemma 2.4: Let \bar{p}, α be viable. Assume that the ω_i are normally distributed and independent. Then:

For any line (i, j) , $P(f_{ij} > f_{ij}^{max}) < \epsilon_{ij}$ and $P(-f_{ij} > f_{ij}^{max}) < \epsilon_{ij}$ if and only if

$$|\bar{\theta}_i - \bar{\theta}_j| \leq \frac{f_{ij}^{max}}{\beta_{ij}} - \eta(\epsilon_{ij}) \left[\sum_{k \in \mathcal{W}} \sigma_k^2 (\pi_{ik} - \pi_{jk} - \delta_i + \delta_j)^2 \right]^{1/2}, \quad (19)$$

where as before $\bar{\theta} = \check{B}(\bar{p} + \mu - d)$ and $\delta = \check{B}\alpha$.

Note that a similar statement applies to the generator bound constraints.

We can now present a formulation of our chance-constrained optimization as a convex optimization problem, on variables $\bar{p}, \alpha, \theta, \delta$ and s :

$$\min \sum_{i \in \mathcal{G}} \left\{ c_{i1} \bar{p}_i^2 + \left(\sum_k \sigma_k^2 \right) c_{i1} \alpha_i^2 + c_{i2} \bar{p}_i + c_{i3} \right\}, \quad (20)$$

$$\text{for } 1 \leq i \leq n-1: \quad \sum_{j=1}^{n-1} \hat{B}_{ij} \delta_j = \alpha_i, \quad (21)$$

$$\text{for } 1 \leq i \leq n-1: \quad \sum_{j=1}^{n-1} \hat{B}_{ij} \bar{\theta}_j - \bar{p}_i = \mu_i - d_i, \quad (22)$$

$$\sum_i \alpha_i = 1, \quad \alpha \geq 0, \quad \bar{p} \geq 0, \quad (23)$$

$$\bar{p}_n = \alpha_n = \delta_n = \bar{\theta}_n = 0, \quad (24)$$

$$\forall (i, j), \quad \beta_{ij} |\bar{\theta}_i - \bar{\theta}_j| + \beta_{ij} \eta(\epsilon_{ij}) s_{ij} \leq f_{ij}^{max}, \quad (25)$$

$$\left[\sum_{k \in \mathcal{W}} \sigma_k^2 (\pi_{ik} - \pi_{jk} - \delta_i + \delta_j)^2 \right]^{1/2} - s_{ij} \leq 0, \quad (26)$$

$$\forall g \in \mathcal{G}, \quad -\bar{p}_g + \eta(\epsilon_g) \left(\sum_{k \in \mathcal{W}} \sigma_k^2 \right)^{1/2} \leq -p_g^{min}, \quad (27)$$

$$\forall g \in \mathcal{G}, \quad \bar{p}_g + \eta(\epsilon_g) \left(\sum_{k \in \mathcal{W}} \sigma_k^2 \right)^{1/2} \leq p_g^{max}. \quad (28)$$

Constraints (26), (27) and (28) are *second-order cone* inequalities [19]. A problem of the above form is solvable in polynomial time using well-known methods of convex optimization; several commercial software tools such as Cplex [20], Gurobi [21], Mosek [22] and others are available.

B. Solving the conic program

Even though optimization theory guarantees that the above problem is efficiently solvable, experimental testing shows that in the case of large grids (thousands of lines) the problem proves challenging. For example, in the Polish 2003-2004 winter peak case¹, we have 2746 buses, 3514 lines and 8 wind farms, and Cplex [20] reports (after pre-solving) 36625 variables and 38507 constraints, of which 6242 are conic. On this problem, a recent version of Cplex on a (current) 8-core workstation ran for 3392 seconds (on 16 parallel threads, making use of “hyperthreading”) and was unable to produce a feasible solution. On the same problem Gurobi reported “numerical trouble” after 31.1 cpu seconds, and stopped. All of the commercial solvers [20], [21], [22] we experimented

¹ Available with MATPOWER [23]

with reported numerical difficulties with problems of this size.

To address this issue we implemented an effective cutting-plane algorithm for solving problem (20)-(28). For brevity we will focus on constraints (26) ((27) and (28) are similarly handled). For a line (i, j) define

$$C_{ij}(\delta) \doteq \left(\sum_{k \in \mathcal{W}} \sigma_k^2 (\pi_{ik} - \pi_{jk} - \delta_i + \delta_j)^2 \right)^{1/2}.$$

Constraint (26) can thus be written as $C_{ij}(\delta) \leq s_{ij}$. For completeness, we state the following result:

Lemma 2.5: Constraint (26) is equivalent to the infinite set of linear inequalities: $\forall \hat{\delta} \in \mathcal{R}^n$

$$C_{ij}(\hat{\delta}) + \frac{\partial C_{ij}(\hat{\delta})}{\partial \delta_i} (\delta_i - \hat{\delta}_i) + \frac{\partial C_{ij}(\hat{\delta})}{\partial \delta_j} (\delta_j - \hat{\delta}_j) \leq s_{ij}. \quad (29)$$

Constraints (29) express the *outer envelope* of the set described by (26) [19]. This observation directly leads to an effective *cutting-plane* algorithm that iterates by solving a linearly constrained relaxation of our optimization problem, and dynamically adding constraints (29).

Termination is declared when the maximum constraint violation is less than 10^{-6} . Table I displays typical performance of the cutting-plane algorithm on (comparatively more difficult) large problem instances. In the Table, 'Polish1'- 'Polish3' are the three Polish cases included in MATPOWER [23] (in Polish1 we increased loads by 30%). All Polish cases have uniform random costs on $[0.5, 2.0]$ for each generator and ten arbitrarily chosen wind sources. The average wind power penetration for the four cases is 8.8%, 3.0%, 1.9%, and 1.5%. 'Iterations' is the number of linearly-constrained subproblems solved before the algorithm converges, and 'Time' is the total (wallclock) time required by the algorithm. For each case, line tolerances are set to two standard deviations and generator tolerances three standard deviations. These instances all prove unsolvable if directly tackled by CPLEX or Gurobi.

TABLE I

PERFORMANCE OF CUTTING-PLANE METHOD ON A NUMBER OF LARGE CASES.

| Case | Buses | Generators | Lines | Time (s) | Iterations |
|---------|-------|------------|-------|----------|------------|
| BPA | 2209 | 176 | 2866 | 5.51 | 2 |
| Polish1 | 2383 | 327 | 2896 | 13.64 | 13 |
| Polish2 | 2746 | 388 | 3514 | 30.16 | 25 |
| Polish3 | 3120 | 349 | 3693 | 25.45 | 23 |

Table II provides additional, typical numerical performance for the cutting-plane algorithm on an instance of the Polish grid model. Each row of Table II shows that maximum relative error and objective value at the end of several iterations. The total run-time was 25 seconds. Note the "flatness" of the objective. This makes the problem nontrivial – the challenge is to find a *feasible* solution (with respect to the chance constraints); at the onset of the algorithm the computed solution is quite infeasible and it is

this attribute that is quickly improved by the cutting-plane algorithm.

TABLE II

TYPICAL CONVERGENCE BEHAVIOR OF CUTTING-PLANE ALGORITHM ON A LARGE INSTANCE.

| Iteration | Max rel. error | Objective |
|-----------|----------------|-----------|
| 1 | 1.2e-1 | 7.0933e6 |
| 4 | 1.3e-3 | 7.0934e6 |
| 7 | 1.9e-3 | 7.0934e6 |
| 10 | 1.0e-4 | 7.0964e6 |
| 12 | 8.9e-7 | 7.0965e6 |

Note the small (typical) number of iterations needed to attain numerical convergence. Thus at termination only a very small number of conic constraints (26) have been incorporated into the master system. This validates the expectation that only a small fraction of the conic constraints in CC-OPF are active at optimality. The cutting-plane algorithm can be viewed as a procedure that opportunistically discovers these constraints.

III. EXPERIMENTS/RESULTS

A. Failure of standard OPF

We first consider the IEEE 118-bus model with a quadratic cost function, and four sources of wind power added at arbitrary buses to meet 5% of demand in the case of average wind. The standard OPF solution is safely within the thermal capacity limits for all lines in the system. However, when accounting fluctuations in wind assuming independent Gaussian fluctuations with standard deviations set to 30% of their respective means, five lines exceed their limits 8% or more of the time. This situation translates into an unacceptably high risk of failure. Further, after scaling up all loads by 10% to simulate a more highly stressed system, we added wind power to ten buses for a total of 2% penetration. The standard solution results in six lines exceeding their limits over 45% of the time, and in one line over 10% of the time.

Similarly we studied the Polish national grid (from MATPOWER) under simulated 20 % renewable penetration spread over 18 wind farms, co-located with the 18 largest generators; the risk associated with renewable fluctuation should be partially "absorbed" by the co-located generators. Under standard OPF, the results are unacceptable: one line is overloaded at least 50% of the time, another at least 30% and two other lines at least 5% of the time. In contrast, using CC-OPF no line is overloaded more than 0.24% of the time. This is attained with a minor increase in cost ($< 1\%$) while the computational time is on the order of 10 seconds.

B. Cost of reliability under high wind penetration

Transmission line congestion has forced temporary shutdown of wind farms even during times of high wind (see [28]). Curtailment, however, results in a lost opportunity for savings. We experiment by keeping the mean power outputs of the wind sources in a fixed proportion to one another and proportionally scaled so as to vary total amount of penetration, and likewise with the standard deviations. First,

we run our CC-OPF under a high penetration level. Second, we add a 10% buffer to the line limits and reduce wind penetration (i.e., curtail) until under *standard* OPF solution line overloads are reduced to an acceptable level. Assuming zero cost for wind power, the difference in cost for the high-penetration CC-OPF solution and the low-penetration standard solution are the savings produced by our model (generously, given the buffers). On the 39-bus New England system (from [23]) our CC-OPF solution is feasible under 30% of wind penetration, but the standard solution has 5 lines with excessive overloads, even when solved with the 10% buffer. Reducing the penetration to 5% relieves the lines, but more than quadruples(!) the cost over the CC-OPF solution.

C. Increasing penetration

To investigate the impact of increased wind penetration we consider the 39-bus case with four farms, and line flow limits scaled by .7 to simulate a heavily loaded system. The quadratic cost terms are set to $\text{rand}(0,1) + .5$. We fix the four wind generator average outputs in a ratio of 5/6/7/8 and standard deviations at 30% of the mean. We first solve our model using $\epsilon = .02$ for each line and assuming zero wind power, and then increase total wind output until the optimization problem becomes infeasible. The results illustrate that at least for the model considered, the 30% of wind penetration with rather strict probabilistic guarantees enforced by our CC-OPF may be feasible, but in fact lies rather close to the dangerous threshold. Further investment would thus be needed to attain higher penetration.

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REFERENCES

- [1] D. Bienstock, C. M., and S. Harnett, "Chance constrained optimum power flow: Risk-aware network control under uncertainty," <http://arxiv.org/abs/1209.5779>, 2012.
- [2] A. Bergen and V. Vittal, *Power System Analysis*. Prentice Hall, Inc, 2000.
- [3] M. Huneault, "A survey of the optimal power flow literature," *Power Systems, IEEE Transactions*, no. 2, 1991. [Online]. Available: http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=76723
- [4] P. Kundur, *Power System Stability and Control*. New York, NY, USA: McGraw-Hill, 1994.
- [5] CAISO, "Integration of renewable resources: Transmission and operating issues and recommendations for integrating renewable resources on the california iso-controlled grid," <http://www.caiso.com/1ca5/1ca5a7a026270.pdf>, 2007.
- [6] Y. Makarov, C. Loutan, J. Ma, and P. de Mello, "Operational impacts of wind generation on california power systems," *Power Systems, IEEE Transactions on*, vol. 24, no. 2, pp. 1039–1050, may 2009.
- [7] A. Prékopa, "On probabilistic constrained programming," *Proc. Princeton Symposium on Math. Programming*, pp. 113–138, 1970.
- [8] A. Charnes, W. Cooper, and G. Symonds, "Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil," *Management Science*, vol. 4, pp. 235–263, 1958.
- [9] L. Miller and H. Wagner, "Chance-constrained programming with joint constraints," *Operations Research*, vol. 13, pp. 930–945, 1965.
- [10] H. Zhang, "Chance Constrained Programming for Optimal Power Flow Under Uncertainty," *Power Systems, IEEE Transactions on*, vol. 26, no. 4, pp. 2417–2424, 2011. [Online]. Available: http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=5871715
- [11] M. Vrakopoulou, K. Margellos, J. Lygeros, and G. Andersson, "A probabilistic framework for security constrained reserve scheduling of networks with wind power generation," in *Energy Conference and Exhibition (ENERGYCON), 2012 IEEE International*, sept. 2012, pp. 452–457.
- [12] U. Ozturk and M. Mazumdar, "A solution to the stochastic unit commitment problem using chance constrained programming," *Power Systems, IEEE*, vol. 19, no. 3, pp. 1589–1598, 2004. [Online]. Available: http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1318698
- [13] J. Wang, M. Shahidepour, and Z. Li, "Security-constrained unit commitment with volatile wind power generation," *Power Systems, IEEE Transactions on*, vol. 23, no. 3, pp. 1319–1327, aug. 2008.
- [14] S. Zhang, Y. Song, Z. Hu, and L. Yao, "Robust optimization method based on scenario analysis for unit commitment considering wind uncertainties," in *Power and Energy Society General Meeting, 2011 IEEE*, july 2011, pp. 1–7.
- [15] M. Hajian, M. Glavic, W. Rosehart, and Z. H., "A chance-constrained optimization approach for control of transmission voltages," *IEEE Transaction on Power Systems*, vol. 27, no. 3, pp. 1568–1576, 2012.
- [16] H. Bludszweit, J. Dominguez-Navarro, and A. Llombart, "Statistical analysis of wind power forecast error," *Power Systems, IEEE Transactions on*, vol. 23, no. 3, pp. 983–991, aug. 2008.
- [17] B.-M. Hodge and M. Milligan, "Wind power forecasting error distribution over multiple timescales," *Detroit Power Engineering Society Meeting*, pp. 1–8, 2011.
- [18] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," *SIAM Journal on Optimization*, vol. 17, no. 4, pp. 969–996, 2006. [Online]. Available: <http://link.aip.org/link/?SJE/17/969/1>
- [19] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.
- [20] CPLEX, "ILOG CPLEX Optimizer," <http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/>, 2012.
- [21] GUROBI, "Optimizer," <http://www.gurobi.com/>, 2012.
- [22] MOSEK, "Optimizer," <http://www.mosek.com/>, 2012.
- [23] R. Zimmerman, C. Murillo-Sanchez, and R. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *Power Systems, IEEE Transactions on*, vol. 26, no. 1, pp. 12–19, feb. 2011.
- [24] E. Erdogan and G. Iyengar, "Ambiguous chance constrained problems and robust optimization," *Mathematical Programming*, vol. 107, pp. 37–61, 2007.
- [25] D. Bertsimas and M. Sim, "Price of robustness," *Operations Research*, vol. 52, pp. 35–53, 2004.
- [26] R. Ben-Tal and A. Nemirovski, "Robust solutions of linear programming problems contaminated with uncertain data," *Mathematical Programming*, vol. 88, pp. 411–424, 2000.
- [27] D. Goldfarb and G. Iyengar, "Robust portfolio selection problems," *Mathematical Programming*, vol. 28, pp. 1–38, 2001.
- [28] M. Wyld, "Wind energy bumps into power grid limits," *New York Times*, 2008/08/27. [Online]. Available: <http://www.nytimes.com/2008/08/27/business/27grid.html>
- [29] A. Sjodin, D. Gayme, and U. Topcu, "Risk-mitigated optimal power flow with high wind penetration," <http://smart.caltech.edu/papers/riskoptimal.pdf>, 2012.
- [30] L. Roald, F. Oldewurtel, T. Krause, and G. Andersson, "Analytical reformulation of security constrained optimal power flow with probabilistic constraints," *Proceedings of the Grenoble PowerTech, Grenoble, France*, 2013.