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Power Line Control under Uncertainty of Ambient Temperature

Jose Blanchet, Daniel Bienstock[1] and Juan Li[2]

Abstract— **This paper discusses a control scheme for maintaining low tripping probability of a transmission system power line under thermal stress. We construct a stochastic differential equation to describe the temperature evolution in a line subject to randomness of the ambient temperature. When the distribution of the ambient temperature changes, so does the dependence of the tripping probability as a function of line current. The theory of extremes of Gaussian random fields is used to guide the size of the underlying frequency inspection so as to insure that current is effective for controlling the risk of overheating. In particular, we show that only when the change of temperature is suitably light-tailed (according to a precise definition discussed in the paper), the current provides a powerful enough mechanism to control tripping probabilities due to overheating. We then provide bounds that can be used to control the tripping probability in our stochastic model.**

I. INTRODUCTION

Power grids play a significant role in economic and social development. As people rely more and more on the robust operation of power grids, grid trippings, in the form of power outages, could have disastrous consequences. It is thus desirable to develop control schemes that lower operational costs while simultaneously maintaining low "risk". The development of appropriate models is a technical challenge, due to the complexity of the underlying physics. In addition, power grids are exposed to external factors that are difficult to model in a mechanistic fashion, in particular requiring stochastic models.

Power outages are typically triggered by an exogenously caused tripping or set of trippings, for example involving a line or set of lines. Following the initial tripping the system may "cascade" due to overloading of a line or lines as a result of the topology change. In this context, grid operators attempt to control a line's internal temperature, so as to prevent excessive heating, followed by line sagging which could damage the line itself or cause it to contact a foreign object, see [1], pages 57-59; periodic inspection of the status of a line helps insure safe operation. Roughly speaking, when a line reaches an excessive temperature point it will "trip", thereby taking it out of operations, a highly undesirable feature.

This paper is motivated by the fact that very frequent inspections might be inconvenient and costly. On the other hand, a policy that results in long intervals between inspections (and thus, control actions) might result in a limited ability to mitigate risk due to variations of random factors.

We suggest a set of guidelines for control of a power line subject to risk constraints. More specifically, a guideline for the choice of inspection frequency and ways to control current (such control may result in intermittent load shedding) is provided. In order to produce these guidelines we study a model for the temperature of a line which is based on a diffusion equation for heat conduction with an additional stochastic component used to model variations in ambient temperature. This model is flexible enough to allow an ambient temperature distribution which is long-tailed or light-tailed (these notions are quantitative and will be discussed in the body of this paper). We explain how tail behavior of the maximum ambient temperature variations are linked to the capability to keep the safety of the line through controlling current. If the stochastic ambient temperature distribution is long-tailed in a precise way to be described, then we conclude that potential high variations can be so wide that the ability to maintain low risk through a static current policy is limited. If, however, the distribution is light-tailed, then we study a class of models that illustrate how to manipulate the current to achieve a certain risk level.

We believe that our results can provide guidance in two aspects. Firstly, one can use our results to choose an appropriate time horizon that is long enough so that it does not require impractical frequent monitoring but short enough so that risk can be effectively managed (by selecting a time horizon under which empirical statistics of ambient temperature data support the light-tailed assumption). Secondly, in the case of light-tailed temperature distributions, we explain how the theory of Gaussian random fields allows us to easily and practically estimate the risk of tripping, thereby providing a computable upper bound that can be used in optimization routines as a side constraint.

This paper is organized as follows. In Section 2, we discuss the mathematical model used to describe the evolution of line temperature incorporating randomness of the ambient temperature. The features of our model, the definition of tripping and a general concept of current control are then discussed. In Section 3, we discuss the dependence of tripping probability on current under general distributions of the ambient temperature, and an asymptotical result of the logarithm of tripping probability is given. In Section 4, two specific forms of distributions are discussed, one long-tailed, and the other light-tailed. In the latter case an upper bound of tripping probability is given which can be used as a guidance for current control. In Section 5, one numerical light-tailed example is given based on simulated data, and we further

[1]J. Blanchet and D. Bienstock are faculty members of the IEOR Department at Columbia University, 116th Street and Broadway, New York, NY 10027 jose.blanchet@columbia.edu, dano@ieor.columbia.edu

[2]J. Li is a PhD student in the IEOR Department at Columbia University jl3035@columbia.edu

discuss how to limit tripping probability at a certain level by controlling current. A more detailed version of this report is available at [9].

II. MODEL

A. Stochastic Differential Equation

We focus on grid trippings due to line overloading. [2] introduced several interesting ideas to describe line outages:

1) A random line tripping model that incorporates line-specific information,
2) A physical overloaded-line tripping model in which outages happen due to thermal causes,
3) A model of the dynamics of line-tripping incorporating operator control.

We construct our model based on the physical overloaded-line tripping model. As described in [2], For a given line, its internal temperature $T(x,t)$ at point x, at time t, $x \in [0,L]$, $t \in [0,\tau]$ for some $L > 0$, $\tau > 0$, follows the heat equation:

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2 - \nu(T(x,t) - T^{ext}), \quad (1)$$

where κ, α and ν are constant that depend on the line material, I is the current and T^{ext} is the ambient temperature.

If we assume that fluctuations in power flows along the line propagate fast enough, and assume that the heat source is uniformly distributed along the line, or if κ is small enough, we could let $\kappa = 0$ and get a simple equation:

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - T^{ext}). \quad (2)$$

This is the approach used in [2]; it is furthermore consistent with IEEE Standard 738, used to compute temperature of a line as a function of current, see [8] (also see [7]).

One drawback of this model is that it assumes T^{ext} is constant, which may not hold. Note that T^{ext} is actually location-dependent. Moreover, T^{ext} is not deterministic since it is greatly affected by external conditions such as wind speed which is stochastic and location-dependent. Therefore, variations in external conditions have a large impact on line temperature and for very long lines, the variations can be very large. In our model, we set $T^{ext} = G(h(x))$, where G is a monotone strictly increasing and continuous function which we will discuss later, $h(x) \sim N(\mu(x), \sigma^2(x))$ follows Gaussian distribution with mean and variance depending on x. For $x \neq y$ we have that $h(x)$ and $h(y)$ are jointly distributed Gaussian random variables with some correlation structure which satisfies mild assumptions required to make the $h(\cdot)$ a Hölder continuous function with probability one (i.e. every realization of the function $h(\cdot)$ satisfies $|h(x) - h(y)| \le K|x-y|^{\rho}$ for a deterministic constant $\rho \in (0,1)$ and some random $K > 0$). Then T^{ext} is both location-dependent and stochastic.

Now (1) becomes:

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2 - \nu(T(x,t) - G(h(x))), \quad (3)$$

and when $\kappa = 0$, we have

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(h(x))). \quad (4)$$

If we know the boundary condition $T(x,0) = l(x)$, where $l(x)$ is some continuous function, then making use of Feynman-Kac formula gives us

$$T(x,t) = e^{-\nu t} \int_{-\infty}^{\infty} l(x)\phi(y;x,2\kappa t)\,dx + \frac{\alpha I^2}{v}\left(1 - e^{-vt}\right) + \int_0^t v e^{-vs} \int_{-\infty}^{\infty} G(h(x))\,\phi(y;x,2\kappa s)\,dxds, \quad (5)$$

where

$$\phi(y;x,2\kappa t) = \frac{\exp(-(y-x)^2/(4\kappa t))}{(4\pi\kappa t)^{1/2}}.$$

For the simplified version when $\kappa = 0$, we have that

$$T(x,t) = (1 - e^{-\nu t})(G(h(x)) + \frac{\alpha I^2}{\nu}) + e^{-\nu t} l(x). \quad (6)$$

Note that in both solutions (5) and (6) we have that x is not bounded. However, we might only interested in x inside some compact interval $[0,L]$, so we look only at $x \in [0,L]$, for some $L > 0$. While in our technical development in this paper we will only consider the simplified case in which $\kappa = 0$, some of our insights can be easily seen to also apply to the case $\kappa > 0$. In particular, as we shall briefly discuss the fact that the tail of the temperature distribution is relatively unsensitive to the current in the case of long-tailed temperature distributions is a feature that remains valid in the case of $\kappa > 0$.

B. Model Flexibility

We want to point out here that the choice of $T^{ext} = G(h(x))$ makes our model very flexible. Actually it might be used to describe virtually all marginal distributions. Indeed, let $\Phi(\cdot)$ denote the distribution function of $h(x)$, then it is a simple statistical fact that $\Phi(h(x))$ is uniformly distributed over $[0,1]$. Suppose one considers any continuous distribution $F(\cdot)$ of ambient temperature, then $F^{-1}(\Phi(h(x)))$ has precisely distribution function $F(\cdot)$. In this case, we could let $G(\cdot) = F^{-1}(\Phi(\cdot))$, which is indeed an increasing function as we have postulated. Now, the underlying correlation structure between $h(x)$ and $h(y)$ for every pair x, y allows to capture the dependence in the temperature distribution across different locations.

C. Tripping Probability

If the internal temperature of a line is too high, it (or connectors) might suffer physical damage. In a more likely scenario, the line may sag and contact a foreign object such as a tree. In either case the line will trip and be taken out of operation. Such an event (or set of events) could trigger a cascading failure of the underlying grid. Therefore, we want to limit the tripping probability which, in order to capture the effects described previously, we define as follows:

$$P(\max_{x \in [0,L], t \in [0,\tau]} T(x,t) > k), \quad (7)$$

where k is some critical temperature. In order to obtain robust insights in the level of generality that we are aiming at, we shall perform an asymptotic analysis in the context of k large. We believe that this asymptotic environment is meaningful in the applications considered because, presumably, failure events happen mostly when temperatures reach a threshold.

D. Current Control

Notice that $T(x,t)$ is a function of the current, I. To limit the tripping probability (7) to a certain level, I should satisfy some constraint, which gives us a guidance how we should control the current in the line.

We set the length of each time window equals to τ. At the beginning of each time window, operators inspect the line and input updated κ, α, ν and $l(x)$ at that time, and modify I such that the tripping probability in this window is limited to a certain level to guarantee the line is safe before the next inspection.

III. STUDY OF TRIPPING PROBABILITY

Now we want to learn when and where the line temperature reaches its maximum. Combining (4) and (6) we get

$$\frac{\partial T(x,t)}{\partial t} = e^{-\nu t}[\nu G(h(x)) + \alpha I^2 - \nu l(x)]. \tag{8}$$

Therefore,

$$\max_{x\in[0,L],t\in[0,\tau]} T(x,t) = \begin{cases} \max\limits_{x\in[0,L]} T(x,\tau), & \text{if} \quad q(x,I) > 0 \\ \max\limits_{x\in[0,L]} T(x,0), & \text{if} \quad q(x,I) \le 0 \end{cases}$$

where $q(x,I) = \nu G(h(x)) + \alpha I^2 - \nu l(x)$.

If $q(x,I) \le 0$, then $\max_{x\in[0,L],t\in[0,\tau]} T(x,t) = \max_{x\in[0,L]} l(x)$. The line trips if $\max_{x\in[0,L]} l(x) > k$, which means that this line is already tripped when operators inspect it. We believe that this case is not of practical interest for our purposes since we aim to find controls that prevent potential future tripping. We also might be able to modify I such that $q(x,I) > 0$ with large probability. Thus when the critical temperature k is large, tripping is most likely to happen at time τ. Therefore, we are more interested in the probability:

$$P(\max_{x\in[0,L]} T(x,\tau) > k). \tag{9}$$

For the rest of paper, when we say tripping probability, we refer to probability (9) by default unless otherwise stated.

We conclude that the tripping probability behaves very differently in the case when $\max_{x\in[0,L]} G(h(x))$ is long-tailed from the case when it is light-tailed. Before we explain the difference, we would first introduce some definitions.

Definition 1: The distribution of a random variable X is called *long-tailed* if for all $c > 0$, $\lim_{x\to\infty} P(X > x + c|X > x) = 1$. It is called *heavy-tailed* if for all $\lambda > 0$, $\lim_{x\to\infty} e^{\lambda x} P(X > x) = \infty$. It is called *light-tailed* if there exists some $\lambda > 0$, $\lim_{x\to\infty} e^{\lambda x} P(X > x) < \infty$.

Theorem 1: When $k \to \infty$, if $\max_{x\in[0,L]} G(h(x))$ is long-tailed, then the tripping probability is independent of I; the tripping probability depends on I otherwise.

Proof:

$$\begin{aligned} &P(\max_{x\in[0,L]} T(x,\tau) > k) \\ &= P(\max_{x\in[0,L]} (1-e^{-\nu\tau})(G(h(x)) + \frac{\alpha I^2}{\nu}) + e^{-\nu\tau} l(x) > k) \qquad (10) \\ &\sim P(\max_{x\in[0,L]} (1-e^{-\nu\tau}) G(h(x)) > k), \quad \text{as} \quad k \to \infty. \qquad (11) \end{aligned}$$

The last asymptotic equivalence is obtained by the definition of long-tailed distribution and the boundedness of $l(x)$. Notice that (11) doesn't involve I, so the tripping probability is (asymptotically) independent of current. On the other hand, (10) does involve I. ■

We believe that this simple result provides useful and important guidance. If the variations of the maximum ambient temperature are long-tailed, the capability to control tripping probability through controlling current is limited. To make sure our control is effective, we need to make sure the variations of maximum ambient temperature are light-tailed, which required that the length of time window τ should not be too large. Given historical data of ambient temperature, we could choose a proper τ. Although we have used the simplified model with $\kappa = 0$, note that (as in equation (6)), in equation (5), I^2 only appears as in a term added to the only term that involves randomness. Thus the same insights indicated in the theorem prevail in the case when $\kappa > 0$ although the mathematical proof will be somewhat more complicated.

Define

$$\begin{aligned} &g(x) = h(x) - \mu(x), \quad \bar{\mu} = \max_{x\in[0,L]} \mu(x), \\ &\bar{\sigma}^2 = \max_{x\in[0,L]} E[g^2(x)], \quad \bar{\lambda} = E[\max_{x\in[0,L]} g(x)], \\ &\bar{l} = \max_{x\in[0,L]} l(x), \quad \tilde{l} = \min_{x\in[0,L]} l(x). \end{aligned}$$

We assume that $\mu(x)$, $\sigma(x)$ and $l(x)$ are continuous, then $\bar{\mu} < \infty$, $\bar{\sigma}^2 < \infty$ and $\bar{l} < \infty$. If we further assume that the Gaussian random fields $h(x)$ is almost surely continuous on $[0,L]$, then $\bar{\lambda} < \infty$. The next result allows us to obtain asymptotic approximations (at least in logarithmic scale). These approximations, although somewhat coarse, allow to quantify the tripping probability in great generality (including both light-tailed and long-tailed distributions).

Theorem 2: If $G^{-1}(\cdot)$ satisfies $G^{-1}(x) \to \infty$ as $x \to \infty$ and that for any $c > 0$,

$$\lim_{k\to\infty} \frac{G^{-1}(k-c)}{G^{-1}(k)} = 1, \tag{12}$$

then as $k \to \infty$,

$$\log P(\max_{x\in[0,L]} T(x,\tau) > k) \sim -\frac{(G^{-1}(\frac{k}{1-e^{-\nu\tau}}))^2}{2\sigma^2(x^*)}, \tag{13}$$

where $x^* = \arg\max\limits_{x\in[0,L]} \sigma(x) \in [0,L]$.

Proof: We start with deriving an upper bound.

$$
\begin{aligned}
&P(\max_{x\in[0,L]} T(x,\tau) > k) \\
&\le P(\max_{x\in[0,L]} (1-e^{-\nu\tau})(G(g(x)+\bar{\mu}) + \frac{\alpha I^2}{\nu}) + e^{-\nu\tau}\bar{l} > k) \\
&= P\Big(\max_{x\in[0,L]} g(x) > G^{-1}(\frac{k-e^{-\nu\tau}\bar{l}}{1-e^{-\nu\tau}} - \frac{\alpha I^2}{\nu}) - \bar{\mu}\Big) \\
&\le e^{-\frac{(y_u-\bar{\mu}-\bar{\lambda})^2}{2\sigma^2(x^*)}},
\end{aligned} \tag{14}
$$

where $y_u = G^{-1}(\frac{k-e^{-\nu\tau}\bar{l}}{1-e^{-\nu\tau}} - \frac{\alpha I^2}{\nu})$ and the last inequality is obtained by Borell-TIS inequality, see [3], page 50.

Now we consider the lower bound.

$$
\begin{aligned}
&P(\max_{x\in[0,L]} T(x,\tau) > k) \\
&\ge P\Big((1-e^{-\nu\tau})(G(g(x^*)+\mu(x^*)) + \frac{\alpha I^2}{\nu}) + e^{-\nu\tau}\tilde{l} > k\Big) \\
&= P\Big(g(x^*) > G^{-1}(\frac{k-e^{-\nu\tau}\tilde{l}}{1-e^{-\nu\tau}} - \frac{\alpha I^2}{\nu}) - \mu(x^*)\Big) \\
&\ge (\frac{\sigma(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))} - \frac{\sigma^3(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))^3})e^{-\frac{(y_l-\mu(x^*))^2}{2\sigma^2(x^*)}},
\end{aligned} \tag{15}
$$

where $y_l = G^{-1}(\frac{k-e^{-\nu\tau}\tilde{l}}{1-e^{-\nu\tau}} - \frac{\alpha I^2}{\nu})$.

Combining (14) and (15), and taking logarithm, we have

$$
\begin{aligned}
&-\frac{(y_u-\bar{\mu}-\bar{\lambda})^2}{2\sigma^2(x^*)} \\
&\ge \log P(\max_{x\in[0,L]} T(x,\tau) > k) \\
&\ge \log(\frac{\sigma(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))} - \frac{\sigma^3(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))^3}) \\
&-\frac{(y_l-\mu(x^*))^2}{2\sigma^2(x^*)}.
\end{aligned}
$$

Since $G^{-1}(\cdot)$ satisfies (12), then

$$
\begin{aligned}
&\lim_{k\to\infty} \frac{-\frac{(y_u-\bar{\mu}-\bar{\lambda})^2}{2\sigma^2(x^*)}}{-(G^{-1}(\frac{k}{1-e^{-\nu\tau}}))^2} \\
&= \lim_{k\to\infty} \frac{\log(\frac{\sigma(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))} - \frac{\sigma^3(x^*)}{\sqrt{2\pi}(y_l-\mu(x^*))^3}) - \frac{(y_l-\mu(x^*))^2}{2\sigma^2(x^*)}}{-(G^{-1}(\frac{k}{1-e^{-\nu\tau}}))^2} \\
&= \frac{1}{2\sigma^2(x^*)}.
\end{aligned}
$$

Therefore, as $k\to\infty$,

$$
\log P(\max_{x\in[0,L]} T(x,\tau) > k) \sim -\frac{(G^{-1}(\frac{k}{1-e^{-\nu\tau}}))^2}{2\sigma^2(x^*)}.
$$

■

After the discussion regarding to the general function $G(\cdot)$, next we will discuss two specific functions: $G(x) = e^x$ and $G(x) = x$. In the former case, $\max_{x\in[0,L]} G(h(x))$ is long-tailed, and in the latter case, it is light-tailed.

IV. EXAMPLES

We first need to introduce some definitions and a theorem on the maximum of Gaussian non-centered filed, see [6], pages 190-193.

Definition 2: Let the collection $\alpha_1,\ldots,\alpha_k$ of positive number is be given, as well as the collection $l_1,\ldots,l_k$ of positive integers such that $\sum_{i=1}^k l_i = n$. We set $l_0 = 0$. These two collections will be called a *structure*. For any vector $\mathbf{x} = (x_1,\ldots,x_n)^T$ its *structural module* is defined by

$$
|\mathbf{x}|_\alpha = \sum_{i=1}^k \Big(\sum_{j=E(i-1)+1}^{E(i)} x_j^2\Big)^{\frac{\alpha_i}{2}},
$$

where $E(i) = \sum_{j=0}^i l_j, j = 1,\ldots,k$.

Definition 3: Let an $\alpha-$structure be given on R^n. We say that $h(\mathbf{x})$, $\mathbf{x}\in A\subset R^n$, has a local $(\alpha, D_\mathbf{x})$-stationary structure, or $h(\mathbf{x})$ is locally $(\alpha, D_\mathbf{x})$-stationary, if for any $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ such that for any $\mathbf{s}\in A$ one can find a non-degenerate matrix $D_\mathbf{s}$ such that the covariance function $r(\mathbf{x}_1,\mathbf{x}_2)$ of $h(\mathbf{x})$ satisfies

$$
1-(1+\epsilon)|D_\mathbf{s}(\mathbf{x}_1,\mathbf{x}_2)|_\alpha \le r(\mathbf{x}_1,\mathbf{x}_2) \le 1-(1-\epsilon)|D_\mathbf{s}(\mathbf{x}_1,\mathbf{x}_2)|_\alpha,
$$

provided $\|\mathbf{x}_1-\mathbf{s}\| < \delta(\epsilon)$ and $\|\mathbf{x}_2-\mathbf{s}\| < \delta(\epsilon)$.

Lemma 1: Let $h(\mathbf{x})$, $\mathbf{x}\in R^n$, be a Gaussian locally $(\alpha, D_\mathbf{x})$-stationary field, with some $\alpha > 0$ and continuous matrix function $D_\mathbf{x}$). Let $\mathcal{M}\subset R^n$ be a smooth p-dimensional compact, $0 < p \le n$. Let the expectation $m(\mathbf{x}) = \mathbf{E}h(\mathbf{x})$ be continuous on $\mathcal{M}$ and attains its maximum on $\mathcal{M}$ at the only point $\mathbf{x}_0$, with

$$
m(\mathbf{x}) = m(\mathbf{x}_0) - (\mathbf{x}-\mathbf{x}_0)B(\mathbf{x}-\mathbf{x}_0)^T + O(\|\mathbf{x}-\mathbf{x}_0\|^{2+\beta}),
$$

as $\mathbf{x}\to\mathbf{x}_0$, for some $\beta > 0$ and positive matrix B. Then,

$$
P(\sup_{\mathbf{x}\in\mathcal{M}} h(\mathbf{x}) > u) = bu^\theta\Psi(u-m(\mathbf{x}_0))(1+o(1)), \text{ as } u\to\infty, \tag{16}
$$

where b and θ are constants, and $\Psi(u) = \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-x^2/2}dx$.

For the rest of this section, we assume that $h(x)$ satisfies the conditions required in the above theorem.

A. *Long-tailed Example:* $G(h(x)) = e^{h(x)}$

1) $\max_{x\in[0,L]} e^{h(x)}$ is long-tailed

Proof: Making use the above theorem, we have

$$
\begin{aligned}
&P(\max_{x\in[0,L]} e^{h(x)} > k+c) \\
&= P(\max_{x\in[0,L]} h(x) > \log(k+c)) \\
&\sim b[\log(k+c)]^\theta\Psi(\log(k+c) - m(x_0)),
\end{aligned}
$$

where $m(x_0)$ is defined as the same as in the theorem.

Since

$$
\begin{aligned}
&\Psi(\log(k+c) - m(x_0)) \\
&\sim \frac{1}{\log(k+c) - m(x_0)}\frac{1}{\sqrt{2\pi}}e^{-(\log(k+c)-m(x_0))^2/2},
\end{aligned}
$$

as $k \to \infty$, it is easy to see that

$$P(\max_{x\in[0,L]} e^{h(x)} > k+c) \sim b(\log k)^{\theta}\Psi(\log k - m(x_0)) \sim P(\max_{x\in[0,L]} e^{h(x)} > k).$$

■

2) Asymptotic behavior of tripping probability
In this case, $G^{-1}(x) = \log x$. Therefore,

$$\log P(\max_{x\in[0,L]} T(x,\tau) > k) \sim -\frac{(\log(\frac{k}{1-e^{-\nu\tau}}))^2}{2\sigma^2(x^*)}, \text{ as } k \to \infty.$$

B. Light-tailed Example: $G(h(x)) = h(x)$

1) $\max_{x\in[0,L]} h(x)$ is light-tailed

Proof: Since

$$P(\max_{x\in[0,L]} h(x) > k) \sim ck^{\theta-1}\frac{1}{\sqrt{2\pi}}e^{-k^2/2},$$

for any $\lambda > 0$,

$$e^{\lambda k}P(\max_{x\in[0,L]} h(x) > k) < \infty, \quad \text{as} \quad k \to \infty.$$

■

2) Asymptotic behavior of tripping probability In this case, $G^{-1}(x) = x$. Therefore,

$$\log P(\max_{x\in[0,L]} T(x,\tau) > k) \sim -\frac{(\frac{k}{1-e^{-\nu\tau}})^2}{2\sigma^2(x^*)}, \quad \text{as} \quad k \to \infty.$$

3) A useful upper bound for current control

As we said earlier, in this case, we could control the risk effectively through current control. The reason is we can find an upper bound of the tripping probability.

Define

$$\hat{\mu} = \max_{x\in[0,L]} E[h(x)] < \infty, \quad \hat{\sigma}^2 = \max_{x\in[0,L]} Var[h(x)] < \infty.$$

According to the following Borell-Sudakov-Tsirelson inequality in [6], page 192, we can find an upper bound of the tripping probability.

Lemma 2: If there exists some a such that

$$P(\max_{x\in[0,L]} h(x) - E[h(x)] \geq a) \leq \frac{1}{2}$$

Then, for all b,

$$P(\max_{x\in[0,L]} h(x) > b) \leq 2\Psi(\frac{b-\hat{\mu}-a}{\hat{\sigma}}). \tag{17}$$

In fact, it is not difficult to prove that such a exists using Borell-TIS inequality because we are assuming that the underlying Gaussian process is continuous. In practice, such a could be estimated through Monte Carlo simulation.

Theorem 3: There exists a, such that

$$P(\max_{x\in[0,L]} T(x,\tau) > k) \leq 2\Psi(\frac{\hat{k}-\hat{\mu}-a}{\hat{\sigma}}), \tag{18}$$

where $\hat{k} = \frac{k-(1-e^{-\nu\tau})\frac{\alpha I^2}{\nu}-e^{-\nu\tau}\bar{l}}{1-e^{-\nu\tau}}$.

Proof: $P(\max_{x\in[0,L]} T(x,\tau) > k)$

$$\leq P(\max_{x\in[0,L]}(1-e^{-\nu\tau})(h(x)+\frac{\alpha I^2}{\nu}) + e^{-\nu\tau}\bar{l} > k)$$
$$= P\Big(\max_{x\in[0,L]} h(x) > \frac{k-(1-e^{-\nu\tau})\frac{\alpha I^2}{\nu}-e^{-\nu\tau}\bar{l}}{1-e^{-\nu\tau}}\Big)$$
$$\leq 2\Psi(\frac{\hat{k}-\hat{\mu}-a}{\hat{\sigma}}).$$

■

Note that the upper bound (18) is increasing in I. Suppose we need to keep the risk of tripping at a certain level η, or in other words, we require that $P(\max_{x\in[0,L]} T(x,\tau) > k) \leq \eta$. We only need to let $2\Psi(\frac{\hat{k}-\hat{\mu}-a}{\hat{\sigma}}) \leq \eta$, which gives us an upper bound of current. This provides a mechanism for periodic control of the line. Note that current magnitude is directly proportional to (active, or real) power flow magnitude; in other words the risk exposure that we compute as inputed by a given current level can be used as a guide for load shedding in an emergency situation.

Suppose in the presence of bound (18), we are interested in maximizing an increasing function of I, subject to a constraint that guarantees the tripping probability is less than η. We can easily compute an optimal value of I_* satisfying

$$I_*^2 = \frac{\nu}{\alpha(1-e^{-\nu\tau})}\Big(k - e^{-\nu\tau}\bar{l} - (1-e^{\nu\tau})(\hat{\sigma}\Psi^{-1}(\frac{\eta}{2})+\hat{\mu}+a)\Big). \tag{19}$$

Next, we would look into a numerical example based on this current control methodology.

V. NUMERICAL LIGHT-TAILED EXAMPLE: $G(h(x)) = h(x)$

Basically, we need to do three things.

1) Define $h(x), x \in [0,L]$ such that it is both Hölder continuous and locally stationary.
2) Estimate a and calculate I_*^2.
3) Set the current equal to I_*, simulate $h(x)$ up to time τ, estimate the tripping probability.

A. Definition of $h(x), x \in [0,L]$

We assume that $h(x)$ follows a Ornstein-Uhlenbeck process, i.e.

$$dh(x) = (c_0 - h(x))dx + \sigma B(x), \tag{20}$$

where c_0 and $\sigma > 0$, and $B(x)$ denotes the standard Brownian motion. We further assume that $h(0) \sim N(c_0, \sigma^2/2)$, which is the stationary distribution. Therefore, $h(x) \sim N(c_0, \sigma^2/2), \forall x \in [0,L]$. Then $E[h(x)] = c_0, \forall x \in [0,L]$, and $cov(h(x), h(y)) = e^{-|x-y|}/2, \forall x, y \in [0,L]$.

It is known that the Ornstein-Uhlenbeck process has almost surely Hölder continuous sample path, and it is obviously that $h(x)$ is also locally stationary. We have

$$\hat{\mu} = c_0, \quad \hat{\sigma}^2 = \sigma^2/2.$$

Note that $E[h(x)] = c_0$ does not satisfy the conditions in Lemma 1. However, $h(x)$ can be considered as a centered field. Therefore, we can make use of Theorem 7.1 in [4], page 108, to find a similar equation as (16).

B. Estimation of a and I_*^2

We want to find a, such that $P(\max_{x\in[0,L]} h(x) - c_0 \geq a) \leq \frac{1}{2}$. Since the solution to the stochastic differential equation (20) is

$$h(x) = \frac{e^{-x}}{\sqrt{2}} B(e^{2x} - 1) + e^{-x} h(0) + (1 - e^{-x}) c_0,$$

we could estimate a using Monte Carlo simulation. Then I_* could be calculated using formula (19).

C. Estimation of the Tripping Probability by Simulating $h(x)$ up to Time τ

For simplicity, we assume $l(x) = \bar{l}, \forall x \in [0, L]$, then the tripping probability

$$P(\max_{x\in[0,L]} T(x,\tau) > k) = P\Big(\max_{x\in[0,L]} h(x) > \hat{k}\Big),$$

where $\hat{k} = \frac{k-(1-e^{-\nu\tau})\frac{\alpha I_*^2}{\nu} - e^{-\nu\tau}\bar{l}}{1-e^{-\nu\tau}}$.

We want to efficiently compute the tail probability for the suprema of Gaussian random fields. We apply algorithm 7.3 in [5]. Before we explain this algorithm, let us first introduce a definition.

Definition 4: We call $\tilde{X} = \{x_1, x_2, \ldots, x_M\} \subset [0, L]$ a *θ-regular discretization* of $[0, L]$ if, and only if,

$$\min_{i\neq j} |x_i - x_j| \geq \theta, \qquad \sup_{x\in[0,L]} \min_i |x_i - x| \leq 2\theta.$$

To decide the number of replications n, we could choose $n = O(\varepsilon^{-2}\delta^{-1})$ to achieve ε relative error with probability at least $1 - \delta$.

Let $H = (h(x_1), h(x_2), \ldots, h(x_M))$, we define distribution Q as

$$Q(H \in B) = \sum_{i=1}^{M} \frac{1}{M} P(H \in B | h(x_i) > \hat{k} - \frac{1}{\hat{k}}). \quad (21)$$

Given a number of replications n and an $\varepsilon/\hat{k}$-regular discretization, the simulation algorithm 7.3 is as follows:

Step 1: Sample $H^{(1)}, H^{(2)}, \ldots, H^{(n)}$ i.i.d copies of H with distribution Q given by (21).

Step 2: Compute and output $\hat{L}_n = \frac{1}{n}\sum_{i=1}^{n} \tilde{L}_{\hat{k}}^{(i)}$, where

$$\tilde{L}_{\hat{k}}^{(i)} = \frac{M \times P(h(x_1) > \hat{k} - 1/\hat{k})}{\sum_{j=1}^{M} \mathbf{1}(h(x_j)^{(i)} > \hat{k} - 1/\hat{k})} \mathbf{1}(\max_{1\leq i\leq M} h(x_j)^{(i)} > \hat{k}).$$

D. Results

We set parameters as follows:

$$\alpha = 1, \nu = 1, L = 1, \tau = 1/4, \bar{l} = 70, c_0 = 70, \sigma = 10,$$
$$k = 100, \eta = 0.05, \varepsilon = 0.05, \delta = 0.05, n = 8000.$$

In other words, We want the tripping probability

$$P(\max_{x\in[0,1]} T(x, 1/4) > 100) \leq 0.05.$$

The simulation result is:

$$I_*^2 = 99.97, P(\max_{x\in[0,1]} T(x, 1/4) > 100) = 7.4\times10^{-6} < 0.05.$$

Therefore, if we set the risk level $\eta = 0.05$, and let the current $I_*^2 = 99.97$, we could make sure that the tripping probability is below this level. The simulation results confirm the validity of the current control methodology we develop.

VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

We are interested in the probability that the maximum temperature over a fixed amount of time along a single line exceeds a critical threshold. Based on a stochastic heat equation used to describe the temperature evolution, we conclude that the tripping is most likely to happen at the end of time window when the threshold is large. When the maximum ambient temperature follows long-tailed distribution, limiting the risk of tripping through current control is not effective. Therefore, to better control risk, operators need to appropriately choose the frequency of inspection. This paper provides a guidance for the choice of frequency and the way to modify current so that the tripping probability is kept at a safe level.

B. Future Work

We are now pursuing several extensions.

1) The parameters in the heat equation, α and ν are also position-dependent and stochastic,
2) When $\kappa > 0$, the tripping probability problem is more challenging, specially if one is interested in computing useful upper bounds that can later be used inside optimization routines.
3) There are some other ways to define tripping, not only through maximum temperature thresholds as we have done here. We are investigating approaches on stochastic hazard rate models.
4) The derivation of bounds for functions $G(\cdot)$ (beyond $G(x) = x$ treated here) that can be used to calibrate a wide class of statistical behavior observed in practice. In particular, we believe that $G(x) = |x|^p$, inducing exponential tails for $p = 2$ and Weibullian tails for $p > 2$ are of special interest.

REFERENCES

[1] U.S.-Canada Power System Outage Task Force. Report on the august 14, 2003 blackout in the united states and canada: Causes and recommendations. https://reports.energy.gov, (2004).

[2] M. Anghel, K.A. Werley, and A.E. Motter, Stochastic Model for Power Grid Dynamics. Proceedings of the 40th Hawaii Int. Conf. on System Sciences, 2007.

[3] R. J. Adler, and J. E. Taylor, Random Fields and Geometry. New York: Springer Monographs in Mathematics, 2010.

[4] V. I. Piterbarg, Asymptotic Methods in the Theory of Gaussian Processes and Fields. Amer Mathematical Society, 1996.

[5] R. J. Adler, J. H. Blanchet and J. Liu, Efficient Monte Carlo for High Excursions of Gaussian Random Fields. Annals of Aplied Probability.

[6] N. Balakrishnan, I.A. Ibragimov and V.B. Nevzorov, Asymptotic Methods in Probability and Statistics with Applications. Birkhauser Boston, 2001.

[7] Mathematical Model for Evaluation of Conductor Temperature in the Steady (or Quasi-Steady) State (Normal Operation, CIGRE, ELECTRA 44, 1992.

[8] IEEE Standard for Calculating the Current-Temperature of Bare Overhead Conductors, IEEE Std. 738-2006, 2006.

[9] http://www.columbia.edu/ jb2814/.