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Near-Optimal Execution Policies for Demand-Response Contracts in Electricity Markets

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Abstract— **Demand side participation is essential for achieving real-time energy balance in today's electricity grid. Demand-response contracts, where an electric utility company buys options from consumers to reduce their load in future, are one of the important tools to increase the demand-side participation.**

In this paper, we consider the operational problem of optimally exercising the available contracts over the planning horizon such that the total cost to satisfy the demand is minimized. In particular, we consider the objective of minimizing the sum of the expected ℓ_β-norm of the load deviations from given thresholds and the contract execution costs over the planning horizon. We present a data driven near-optimal algorithm for the contract execution problem. Our algorithm is a sample average approximation (SAA) based and we provide a sample complexity bound on the number of demand samples required to compute a $(1+\epsilon)$-approximate policy for any $\epsilon > 0$. Our SAA algorithm is quite general and can be adapted to quite general demand models and objective function. For the special case where the demand in each period is i.i.d., we show that a static solution is optimal for the dynamic problem. We also conduct a numerical study to compare the performance of our SAA based DP algorithm. Our numerical experiments show that we can achieve a $(1+\epsilon)$-approximation in significantly smaller number of samples than what is implied by the theoretical bounds. Moreover, the structure of the approximate policy also shows that it can be well approximated by a simple piecewise linear function of the state.

I. INTRODUCTION

Due to an increasing integration of renewable sources such as wind and solar power on the grid, the supply uncertainty in the electricity market has increased significantly. Demand-side participation has become extremely important to maintain a real-time energy balance in the grid. There are several ways to increase the demand-side participation including time of use pricing, real-time pricing for smart appliances and interruptible demand-response contracts. In this paper, we focus on the interruptible demand-response contracts as a tool for increased demand-side participation. A demand-response contract is a contract or an option that an electric utility company can buy from the customers to interrupt or reduce their load by a specified amount, a specified number of times until the expiration of the contract.

Typically an electric utility forecasts the day-ahead load and buys the forecast load in the day-ahead market. If the actual load turns out to be higher than the forecast, the utility can buy the difference in the real-time market by paying the real-time spot price that can be significantly higher than the day-ahead price, especially when the supply is scarce. Alternatively, the utility can exercise the available demand-response contracts (if any) to offset the imbalance instead of paying the real-time spot price. Therefore, these contracts help to achieve the real-time supply-demand balance.

In this paper, we consider the operational problem of optimally exercising the demand-response contracts over the planning horizon from the perspective of the utility. At each time period, the goal is to decide on the number of contracts to exercise such that the total cost of satisfying demand is minimized over the planning horizon. As is the case with all option exercising problems, there is a tradeoff between exercising the options now or saving them for future periods. In order to minimize the total cost one needs to model the dynamics for the real-time electricity price in addition to the uncertainty in demand. The real-time price of electricity at any location, also referred to as the *locational marginal price*, is computed from the optimal dual prices of the energy balance constraint corresponding to that location in a linearized power flow problem. This makes modeling the real-time price dynamics quite challenging. To avoid this, we use a different objective function that provides a good proxy to the total cost. The real-time electricity price is high typically when the demand is high. Therefore, we consider the objective of minimizing the expected ℓ_β norm of demand deviations above a threshold. When $\beta = \infty$, the objective reduces to minimizing the expected peak load. This provides a good approximation to minimizing the total cost. Moreover, this allows a data-driven approach with minimal model assumptions where historical demand data can be used to model the uncertain future demand.

Interruptible load contracts have been considered in the literature for improving the reliability of power systems. The literature is divided in two broad problems: i) designing and pricing of interruptible load contracts, and ii) optimal dispatch or exercising of these contracts to minimize costs or improve reliability. In the regulated markets in the past, these contracts were used mainly to improve system reliability in situations of supply-demand imbalances (see Oren and Smith [8] and Caves et al. [2]). With deregulation, these contracts are used as an ancillary service or as an equivalent price-based generating resource (see Tuan and Bhattacharya [11], Yu et al. [12], [13]).

The problem of designing and pricing such contracts has also been studied extensively (see Fahrioglu and Alvarado [3], Kamat and Oren [6], and Oren [9]). Strauss and Oren [10] propose a methodology for designing priority pricing of interruptible load programs with an early notification option. Oren [9] propose a double-call option which captures

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the effects of early notification of interruptions. Kamat and Oren [6] discuss the valuation of interruptible contracts with multiple notification times using forward contracts and option derivatives, for the supply and procurement of interruptible load programs. Baldick, Kolos and Tompaidis [1] discuss both the design and execution problem and provide a good overview of the literature. Most of this work makes assumptions about the demand model and the real-time price dynamics. In this paper, we consider a data-driven approach where we make minimal assumptions about the model of demand uncertainty.

A. Our contribution

Our main contributions in this paper are the following.

1) We consider the demand-response contract execution problem to minimizing the . We present a data driven near-optimal algorithm for the demand-response contract execution problem where the goal is to minimize the sum of the expected ℓ_β-norm of the observed load deviations from given thresholds and the contract execution costs over the planning horizon. Our algorithm is based on a sample average approximation (SAA) dynamic program and we provide a sample complexity bound of $O(T^2/\epsilon^2)$ on the number of demand samples required to compute a $(1+\epsilon)$-approximate policy for a planning horizon of T periods (where $O(\cdot)$ is the standard Big-O notation [4]). The main challenge is to show that the sampling and discretization error remains bounded in the multi-period problem. Our SAA algorithm is quite general and can be adapted to quite general demand models and objective function.
2) We consider the special case where the demand is i.i.d. or exchangeable, and the contract execution costs are zero. For this case, we show that a static solution is optimal for the dynamic problem of minimizing the ℓ_β norm objective of the load deviations from given thresholds. In particular, we show that executing an equal number of contracts evenly across the planning horizon is optimal irrespective of the realized demand. When either the demands are not exchangeable or there is a non-zero contract execution costs, a static solution is no longer optimal.
3) We also conduct a computational study to compare the performance of our SAA based DP algorithm. In particular, we compare the performance of our algorithm with respect to the number of samples. In our numerical experiments, we observe that we can obtain a $(1+\epsilon)$-approximation for significantly smaller number of samples that the bound of $O(T^2/\epsilon^2)$ given by the theoretical analysis. This is indicated by a fast convergence of the value function and the cost of policy as the sample size increases. We also analyze the near-optimal policy computed by our algorithm and observe that the policy can be well approximated by a piecewise linear function of the states with appropriate rounding to obtain a integer decision on the number of contracts.

B. Model, Notation and Assumptions

We formulate the contract execution problem in terms of the following quantities.

$$\begin{aligned} S_t &= \text{number of contracts available in period } t. \\ X_t &= \text{random load in period } t. \\ \Gamma_t &= \text{threshold base load in period } t. \\ g_t &= \text{execution cost in period } t. \\ n_t(X_{[t-1]}) &= \text{number of contracts to exercise at time } t. \\ \alpha_t &= \text{shortfall penalty in period } t \end{aligned}$$

Here, $X_{[t]} = (X_1, X_2, \cdots, X_t)$ denotes the historical demands up to period t. We assume that the demand is realized at the end of each period and the contract execution decision is made at the beginning of the period. Therefore, the number of contracts n_t to exercise at time t must only be a function of $X_{[t-1]}$. Each contract reduces load by δ. The threshold base load Γ_t denotes the power the utility has procured in the day-ahead market for period t. To satisfy demand, X_t in period t, the utility can either buy $(X_t - \Gamma_t)_+$ from the real-time market or exercise a demand-response contract to reduce the load (for any $x \in \mathbb{R}$, x_+ denotes the positive part of x). Let S_1 denote the total number of demand-response contracts available in period 1. Therefore, we can formulate the optimal execution problem as follows.

$$\begin{aligned} \min \mathbb{E}_{X_{[T]}} \Bigg[\left(\sum_{t=1}^{T} \alpha_t \left(X_t - \Gamma_t - n_t(X_{[t-1]})\delta \right)_+^\beta \right)^{1/\beta} \\ + \sum_{t=1}^{T} g_t \left(n_t(X_{[t-1]}) \right) \Bigg] \qquad (1) \\ \text{s.t.} \sum_{t=1}^{T} n_t(X_{[t-1]}) \le S_1 \quad \text{with probability 1.} \end{aligned}$$

The ℓ_β-norm objective captures the fact that the price grows faster as the power demand increases. We also include the function $g_t\left(n_t(X_{[t-1]})\right)$ that models the cost for each execution of the contract. Note that when $g_t = 0$, we can consider the objective as sum of the load deviations raised to the power of β and ignore the $1/\beta$ exponent on the sum. For $\beta = \infty$, the ℓ_β-norm objective reduces to the peak load and we obtain the following special case of minimizing the sum of expected peak load and the execution cost.

$$\begin{aligned} \min \mathbb{E}_{X_{[T]}} \Bigg[\max_{t=1,\cdots,T} \alpha_t \left(X_t - \Gamma_t - n_t(X_{[t-1]})\delta \right)_+ \\ + \sum_{t=1}^{T} g_t \left(n_t(X_{[t-1]}) \right) \Bigg] \qquad (2) \\ \text{s.t.} \sum_{t=1}^{T} n_t(X_{[t-1]}) \le S_1 \quad \text{with probability 1.} \end{aligned}$$

II. Structure of Optimal Execution Policies

In this section, we study the structural property of optimal execution policies for a special setting when the demand sequence $\{X_t : t \ge 1\}$ is *exchangeable* and the cost of executing the contracts $g_t \equiv 0$ for all $t = 1, \ldots, T$. A sequence of random variables is *exchangeable* if the joint probability distribution of every permuted sequence is the same as the original sequence. We prove that a static solution is optimal in this case.

Without loss of generality, we assume that $\Gamma_t = 0$ and $\alpha_t = 1$ for all $t = 1, \ldots, T$ (otherwise, we can appropriately change the demand distribution). Therefore, at each period, we can formulate the optimization problem as follows.

$$\begin{aligned} V_t(S_t) = \min \mathbb{E}_{X_u: t \le u \le T} & \left[\sum_{u=t}^{T} \left(X_u - n_u(X_{[u-1]})\delta \right)_+^\beta \right] \\ s.t. & \sum_{u=t}^{T} n_u(X_{[u-1]}) \le S_t, \end{aligned} \tag{3}$$

where S_t is the number of contracts available at time t. We can reformulate (3) as follows.

$$\begin{aligned} V_t(S_t) = \min_{n_t} \ & \mathbb{E}_{X_t} \left[(X_t - n_t\delta)_+^\beta + V_{t+1}(S_t - n_t) \right] \\ & s.t. \ n_t \le S_t \end{aligned} \tag{4}$$

Theorem 1: Suppose demand X_t is exchangeable and the execution cost $g_t = 0$ for all $t = 1, \ldots, T$. Then a static solution is optimal for the dynamic problem (3). Furthermore, for each period, the optimal solution is

$$n_t^*(X_{[t-1]}) = \frac{S_t}{T - t + 1}, \ \forall X_{[t-1]}.$$

Proof: We denote $n_t(X_{[t-1]})$ as n_t for simplicity. We prove the claim by a backward induction.

Base Case. For period $j = T$, since there is only one period left, we execute all the options available. For time $j = T - 1$, it is easy to verify that the optimal solution, $n_{T-1} = S_{T-1}/2$.

Induction Step. Suppose the induction hypothesis holds for $j \ge t + 1$. For period t,

$$\begin{aligned} V_t(S_t) &= \min_{0 \le n_t \le S_t} \mathbb{E}_{X_t} \left[(X_t - n_{T-1}\delta)_+^\beta + V_{t+1}(S_t - n_t) \right] \\ &= \min_{0 \le n_t \le S_t} \mathbb{E}_{X_t} \Bigg[(X_t - n_{T-1}\delta)_+^\beta \\ &\qquad + \mathbb{E}_{X_{t+1}, \cdots, X_T} \Bigg[\sum_{j=t+1}^{T} \left(X_j - \frac{S_t - n_t}{T - t}\delta \right)_+^\beta \Bigg] \Bigg] \\ &= \min_{0 \le n_t \le S_t} \mathbb{E}_{X_t, \cdots, X_T} \Bigg[(X_t - n_t\delta)_+^\beta \\ &\qquad + \sum_{j=t+1}^{T} \left(X_j - \frac{S_t - n_t}{T - t}\delta \right)_+^\beta \Bigg], \end{aligned}$$

where the second equality follows from the induction hypothesis. We can show that the above function is convex in n_t. Therefore, $n_t = \frac{S_t}{T-t+1}$ is an optimal solution. ∎

The peak load objective is a special case of ℓ_β norm objective with $\beta = \infty$. The above arguments can be used to show static policy is optimal if we want to minimize the expected peak realized load. Theorem 1 indicates that if the demands are identically distributed and there is no cost of execution, we shall exercise the available contract uniformly over the remain time periods. The optimal decision is static and independent of the history of demand. A static solution is not necessarily optimal if either of the two assumptions in Theorem 1 are violated. We provide a counter-example in the full version of the paper.

III. Approximate Dynamic Program for Peak Load Minimization

In this section, we consider the general demand-response contract execution problem. We present an efficient data-driven near-optimal algorithm for the execution policy problem under mild assumptions. For exposition purposes, we consider the objective of minimizing the sum of expected peak load and execution cost, for the case of i.i.d. demand. However, our algorithm can be easily adapted for more general demand models (including Markovian demand) and objective functions. We defer the details to the full version of the paper.

Our algorithm is based on the sample average approximation. We consider an appropriate discretization of the state space, similar in spirit to the knapsack problem [4] and approximate the expectation by the empirical mean in a data-driven approach. Therefore, we have two sources of error in discretization and sampling, and the main challenge is to show that the error propagation remains bounded in the multi-period dynamic program. We give a bound on the number of samples such that the approximation error remains bounded over T periods.

To formulate the problem, let us introduce a few notations. Let Y_t denote the peak realized load up to time t. i.e.,

$$Y_t = \max_{j=1,\ldots,t-1} \left(X_j - n_j(X_{[j-1]})\delta \right)_+.$$

Let $n_t(Y_t, S_t)$ denote the number of contracts to exercise in period t when the current peak load is Y_t and the number of available contracts is S_t. We can assume without loss of generality that the optimal decision in period t is a function of Y_t and S_t. The optimization problem in period t can be formulated as

$$\begin{aligned} \min \mathbb{E}_{X_u: t \le u \le T} & \left[\max_{u=t,\cdots,T} (X_u - n_u(S_u, Y_u)\delta)_+ \right] \\ s.t. & \sum_{u=t}^{T} n_u(S_u, Y_u) \le S_t \end{aligned} \tag{5}$$

We define the value function $V_t(S_t, Y_t)$ as the minimum sum of increase in peak realized demand above Y_t and the execution cost. For simplicity, let n_t denote $n_t(X_{[t-1]})$. Therefore, we can reformulate (5) as follows.

$$\begin{aligned} V_t(S_t, Y_t) = \min_{0 \le n_t \le S_t} \mathbb{E}_{X_t} & \Bigg[(X_t - n_t\delta - Y_t)_+ + g_t(n_t) \\ & + V_{t+1}\left(S_t - n_t, Y_t + (X_t - n_t\delta - Y_t)_+ \right) \Bigg]. \end{aligned} \tag{6}$$

for $t = 1, 2, \cdots, T$. We make the following two mild assumptions.

Assumption 1. Let $X_{max} = \max_{1 \le t \le T} X_t$ denote the peak demand. There is a known constant $M \in \mathbb{R}$ such that

i) $\mathbb{P}(X_{max} > M) < \eta$, and

ii) $\mathbb{E}(X_{max}) \ge c_1 M$ for some constant $0 < c_1 < 1$.

Our algorithm requires an estimate of the peak demand for an appropriate discretization for the approximate dynamic program. Since this is a random quantity, the above two conditions suffice for the analysis of our algorithm. This assumption is not restrictive and holds easily for large T. For instance, consider the case when $P(X_t > M/2) \ge 1/T$ for all $t = 1, \ldots, T$. For large T, this implies a small lower bound on the tail probability for demand in each period. In this case, $\mathbb{E}(X_{max}) \ge (1 - 1/e)M/2$.

Assumption 2. $S_1 \le c_2 M$, for some constant $c_2 < 1$.

This assumption is reasonable because demand-response contracts are used to manage only peak loads and random variability which is only a small fraction of the total demand.

A. Sample Average Approximation (SAA)

We first discretize the state space. Let $K = \epsilon M/T$. We consider discrete values of Y_t and X_t on $[0, M]$ in multiples of K. For all $t = 1, \ldots, T$, let

$$\tilde{Y}_t \triangleq \left\lceil \frac{Y_t}{K} \right\rceil,\ \tilde{X}_t \triangleq \left\lfloor \frac{X_t}{K} \right\rfloor,\ \tilde{\delta} \triangleq \left\lceil \frac{\delta}{K} \right\rceil.$$

Note that there are only $O(T/\epsilon)$ possible values of Y_t and X_t. We define our approximate value function, $\bar{V}_t(S_t,\ \tilde{Y}_t)$, on the discrete state space where we approximate the expectation with the sample average. The detailed description appears in Algorithm 1.

B. Analysis of algorithm

We make two approximations for an efficient computation of the dynamic solution. First, we discretize the state space such that the total number of states become polynomial. Secondly, we approximate the expectation in each period by a sample average. These two sources of error can propagate in the multi-period computation and possibly lead to a highly suboptimal decision. We show that if the number of samples is sufficiently large, then the solution computed in Algorithm 1 gives a $(1+\epsilon)$-approximation of the optimal execution policy with high probability. In order to analyze the performance of Algorithm 1, we introduce two more value functions. For all $t = 1, \ldots, T$, let $\hat{V}_t$ denote the optimal dynamic solution on the discrete states. We can compute $\hat{V}_t$ as follows: $\hat{V}_{T+1}(S_{T+1}, \tilde{Y}_{T+1}) = V_{T+1}(S_{T+1}, \tilde{Y}_{T+1})$ and for all $t \le T$,

$$\begin{aligned} &\hat{V}_t\left(S_t, \tilde{Y}_t\right) \\ &= \min_{0 \le n_t \le S_t} \Big\{ \mathbb{E}_{X_t}\Big[\left(X_t - n_t\delta - \tilde{Y}_t\right)_+ + g_t(n_t) \\ &\qquad + \hat{V}_{t+1}\Big(S_t - n_t, \tilde{Y}_t + \big(\tilde{X}_t - n_t\tilde{\delta} - \tilde{Y}_t\big)_+\Big)\Big]\Big\}, \quad (8) \end{aligned}$$

Algorithm 1: FPTAS for a ϵ - optimal solution

Given $\epsilon > 0$, let $K = \epsilon(1+a)M/T$. Define the terminal value

$$\bar{V}_{T+1}(S_{T+1}, \tilde{Y}_{T+1}) = V_{T+1}(S_{T+1}, \tilde{Y}_{T+1}) = 0.$$

for $t = T$ **to** 1 **do**

solve the following DP for the optimal strategy $\bar{n}_t^*$ (as a function of (S_t, Y_t)) by SAA:

$$\begin{aligned} &\bar{V}_t\left(S_t, \tilde{Y}_t\right) \\ &= \min_{0 \le n_t \le S_t} \left\{ \frac{1}{N_t} \sum_{i=1}^{N_t} \left[\left(X_t^{(i)} - n_t\delta - \tilde{Y}_t\right)_+ + g_t(n_t) \right.\right. \\ &\quad + \bar{V}_{t+1}\Bigg(S_t - n_t, \\ &\qquad\qquad \left.\left. \tilde{Y}_t + \left(\tilde{X}_t^{(i)} - n_t\tilde{\delta} - \tilde{Y}_t\right)_+ \Bigg)\right]\right\}, \quad (7) \end{aligned}$$

where

N_t : number of samples of X_t.

$X_t^{(i)}$: i^{th} sample of X_t.

end

return $\{\bar{n}_t^* : t = 1, \cdots, T\}$

Also, let $\bar{U}_t$ denote the true cost of the approximate solution computed by Algorithm 1. Therefore, $\bar{U}_{T+1}(S_{T+1}, Y_{T+1}) = V_{T+1}(S_{T+1}, Y_{T+1})$ and for all $t \le T$

$$\begin{aligned} &\bar{U}_t(S_t, Y_t) \\ &= \mathbb{E}_{X_t}\Big[(X_t - \bar{n}_t^*\delta - Y_t)_+ + g_t(n_t) \\ &\qquad + \bar{U}_{t+1}\left(S_t - \bar{n}_t^*,\ Y_t + (X_t - \bar{n}_t^*\delta - Y_t)_+\right)\Big]. \quad (9) \end{aligned}$$

where $\bar{n}_t^*$ is the optimal solution computed by Algorithm 1. We prove the following sample complexity bound for Algorithm 1.

Theorem 2: Suppose the number of samples $N = O(T^2 \log(TS_1)/\epsilon^2)$. Then the cost of the execution policy computed by Algorithm 1 is a $(1 + O(\epsilon))$-approximation of the dynamic optimal solution with probability at least $1 - O(1/TS_1) - \eta$.

Using the fact that $K = \epsilon M/T$, it is straightforward to establish that the discretization error is small and we have the following lemma.

Lemma 1: For all S_t and Y_t, $t = 1, 2, \cdots, T$,

$$V_t(S_t, Y_t) - K(T - t + 1) \le \hat{V}_t(S_t,\ \tilde{Y}_t) \le V_t(S_t, Y_t)$$

Next, we show that for N sufficiently large, the sampling error is small.

Lemma 2: Suppose the number of sample $N = O(T^2 \log(TS_1)/\epsilon^2)$. Then for any S_t and Y_t, $t =$

$1, 2, \cdots, T$,

$$\hat{V}_t(S_t, \tilde{Y}_t) - 2M\epsilon \le \bar{V}_t(S_t, \tilde{Y}_t) \\ \le \hat{V}_t(S_t, \tilde{Y}_t) + 2M\epsilon$$

with probability at least $1 - 1/T^3 S_1^3$.

From the union bound over all states, we have that the value functions $\bar{V}$ and $\hat{V}$ are close point-wise with probability at least $(1 - 1/(TS_1))$. In the following lemma, we establish that approximated true cost function $\bar{U}_t$ is a good approximation of $\bar{V}_t$, i.e., the error $|\bar{U}_t - \bar{V}_t|$ is small.

Lemma 3: Suppose the number of sample $N = O(\frac{T^2 \log(TS_1)}{\epsilon^2})$. Then for any S_t and Y_t, $t = 1, 2, \cdots, T$,

$$\bar{V}_t(S_t,\ \tilde{Y}_t) - M\epsilon \le \bar{U}_t(S_t,\ Y_t) \\ \le \bar{V}_t(S_t,\ \tilde{Y}_t) + 2M\epsilon$$

with probability $\ge 1 - 1/T^3 S_1^3$.

The proofs of these results are deferred to the full version. We are now ready to prove Theorem 2.

Proof of Theorem 2 At time $t = 1$, $Y_1 = 0$. We know from Lemma 2 that for all S_1,

$$\bar{V}_1(S_1, 0) \le \hat{V}_1(S_1, 0) + 2M\epsilon$$

with probability at least $1 - 1/(TS_1)$. Similarly, from Lemma 3 we have that for all S_1,

$$\bar{U}_1(S_1, 0) \le \bar{V}_1(S_1, 0) + 2M\epsilon$$

From Lemmas 1, 2 and 3, we have

$$\bar{U}_1(S_1, 0) \le \bar{V}_1(S_1, 0) + 2M\epsilon \\ \le \hat{V}_1(S_1, 0) + 4KT \le V_1(S_1, 0) + 4M\epsilon$$

with probability at least $1 - 2/(TS_1)$ for all S_1.

From Assumptions 1 and 2, the expected peak load is at least $c_1 M$ and the number of contracts S_1 is at most $c_2 M$ where $c_1 > c_2$ are constants.

$$V_1(S_1, 0) \ge \mathbb{E}\left[X_{max}\right] - S_1 \ge cM,$$

for some constant c. Therefore, with probability at least $1 - 2/(TS_1) - \eta$,

$$\bar{U}_1(S_1, 0) \le (1 + O(\epsilon))\, V_t(S_1, 0)$$

■

IV. Computational Study

In this section, we present the results of a computational study to test the performance of our SAA based dynamic programming algorithm and contrast it with the theoretical bounds. We also analyze the structure of the near-optimal policy computed by our algorithm.

TABLE I

Relative Error of Value Function and True Cost for $\epsilon = 0.1(10\%)$. Relative error = $\frac{\hat{V}_1^{\epsilon,N} - \hat{V}_1^{\epsilon,12800}}{\hat{V}_1^{\epsilon,12800}}$, $\frac{\bar{U}_1^{\epsilon,N} - \bar{U}_1^{\epsilon,12800}}{\bar{U}_1^{\epsilon,12800}}$.

N	$\hat{V}_1^{\epsilon}(S_1, 0)$	Relative Error	$\bar{U}_1^{\epsilon}(S_1, 0)$	Relative Error
100	42.9945	0.20%	43.4140	0.67%
200	43.0937	0.02%	43.2392	0.26%
400	43.1430	0.14%	43.1936	0.16%
800	43.0485	0.07%	43.1696	0.10%
1600	43.1067	0.05%	43.1068	0.05%
3200	43.0910	0.02%	43.1348	0.02%
6400	43.1060	0.05%	43.1388	0.03%
12800	43.0819	-	43.1264	-

TABLE II

Relative Error of Value Function and True Cost for $\epsilon = 0.05(5\%)$. Relative error = $\frac{\hat{V}_1^{\epsilon,N} - \hat{V}_1^{\epsilon,12800}}{\hat{V}_1^{\epsilon,12800}}$, $\frac{\bar{U}_1^{\epsilon,N} - \bar{U}_1^{\epsilon,12800}}{\bar{U}_1^{\epsilon,12800}}$.

N	$\hat{V}_1^{\epsilon}(S_1, 0)$	Relative Error	$\bar{U}_1^{\epsilon}(S_1, 0)$	Relative Error
100	43.0974	0.21%	43.0647	0.16%
200	43.1046	0.20%	43.2835	0.34%
400	43.1378	0.12%	43.2595	0.28%
800	43.1873	0.01%	43.2147	0.19%
1600	43.1823	0.02%	43.1258	0.02%
3200	43.2013	0.02%	43.1812	0.10%
6400	43.2019	0.02%	43.1539	0.04%
12800	43.1911	-	43.1349	-

A. Dependence on the Sample Size

We show that with $O(T^2 \log(TS_0)/\epsilon^2)$ samples, Algorithm 1 guarantees a $(1+\epsilon)$-approximation of optimal policy with high probability. In practice, we may be able to obtain a $(1+\epsilon)$-approximation with significantly smaller numbers of samples. We conduct computational experiments to study the effect of number of samples on the performance of our algorithm. In particular, we compare the value function of SAA approximated DP, $\hat{V}_1(S_1, 0)$, and the cost of the policy computed by the DP, $\bar{U}_1(S_1, 0)$, with respect to the sample size. For our numerical experiment, we use the following parameters: $S_1 = 30, T = 30, \delta = 1$ and there is a linear execution cost of 0.01 per contract. The load $\overset{i.i.d.}{\sim} Unif(0, 45)$.

Tables I and II, show the value function and cost function as the sample size increases for different level of accuracy ϵ. We consider the value corresponding to the highest number of samples $N = 12800$ as the best approximation of the true value and compute the relative error with respect to the best approximation for different sample sizes.

Tables I and II shows that both the DP value function and the true cost function of the DP policy converge rapidly as the sample size increases. The relative error is less than 1% which is small compared with ϵ. The theoretical bounds imply that one requires around 10^6 samples to compute a $(1 + \epsilon)$-approximation for $\epsilon = 0.01$. The computational results indicate that one can compute a very high quality solution with a relatively modest number of samples.

B. True Cost Distribution

In Table III, we compare the true cost under the approximate policy with the approximate DP value function for different values of ϵ. We also give the 99% confidence interval of true cost function for each value of ϵ.

TABLE III

TRUE COST DISTRIBUTION. RELATIVE ERROR = $\frac{|\bar{U}_1^\epsilon - \hat{V}_1^\epsilon|}{\hat{V}_1^\epsilon}$.

ϵ	$\hat{V}_1^\epsilon(S_1, 0)$	$\bar{U}_1^\epsilon(S_1, 0)$	Confidence Interval	Relative Error
0.15	42.8996	43.1109	(39.3878, 46.8340)	0.493%
0.10	43.0461	43.1084	(39.3716, 46.8451)	0.145%
0.05	43.1834	43.1219	(39.3167, 46.9272)	0.142%
0.02	43.2150	43.1148	(39.3455, 46.8841)	0.023%
0.01	43.2067	43.1115	(39.4824, 46.7405)	0.022%

Table III shows that the relative error between the value function and the true cost is at most 0.493% and is much smaller than accuracy level ϵ. We also observe that the relative error decreases as ϵ decreases. This result agrees with Lemmas 2 and 3 for our algorithm which indicate that the error $|\bar{V}_t - \hat{V}_t|$ and $|\bar{U}_t - \bar{V}_t|$ are both small with respect to the level of accuracy. The discretized value function $\hat{V}_1(S_1, 0)$ is inside the confidence interval in all the cases and is a good approximation of the true cost of the DP policy.

C. Structure of Policy

We also study the structure of the near-optimal policy computed by our approximate SAA based DP algorithm. We plot the approximated decision n_t with respect to the state variable S_t and Y_t for $t = T - 1$ and $T - 2$ in Figures 1 and 2.

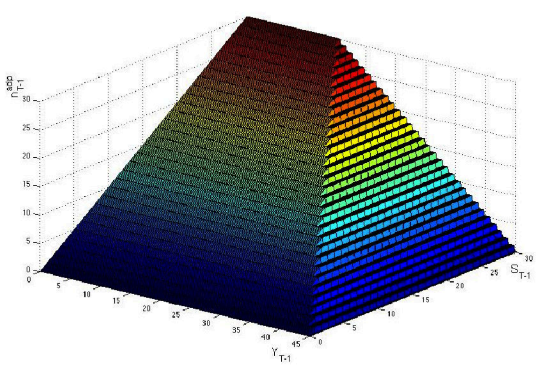

Fig. 1. Near-optimal Policy at Period T-1

We observe from Figure 1 and 2 that the optimal number of contracts to execute is almost linear in S_t and Y_t and remains constant after it reaches a threshold. We can therefore, approximate the policy by a piecewise linear function of the

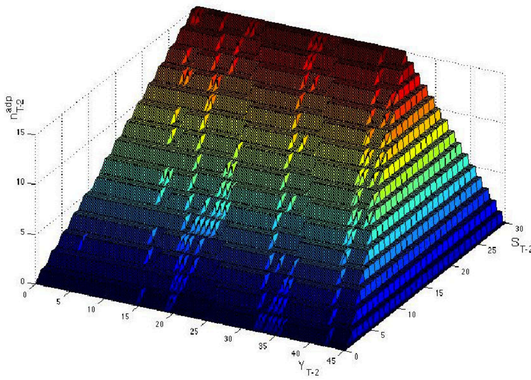

Fig. 2. Near-optimal Policy at Period T-2

states with some rounding to obtain a integer decision on the number of contracts. Such a functional approximation is quite useful as it provides a compact representation of the approximate execution policy. Moreover, it provides important insights into valuation, pricing and designing of the demand-response contracts. We defer the details to the full version of the paper.

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