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Optimal Power Flow in Tree Networks

Lingwen Gan, Na Li, Ufuk Topcu, and Steven H. Low

Abstract— **The optimal power flow (OPF) problem seeks to control power generation/demand to optimize certain objectives such as minimizing the generation cost or power loss. It is becoming increasingly important for tree distribution networks due to the emerging distributed generation and controllable loads. The OPF problem is nonconvex. We prove that after modifying the OPF problem, its global optimum can be recovered via a second-order cone programming (SOCP) relaxation for tree networks, under a condition that can be checked in advance. Empirical studies justify that the modification is "small", and that the condition holds, for the IEEE 13-bus network and two real-world networks.**

I. Introduction

The optimal power flow (OPF) problem seeks to control power generation/demand to optimize certain objectives such as minimizing the generation cost or power loss. It was first proposed by Carpentier in 1962 [1], and has been one of the fundamental problems in power system operation ever since.

The OPF problem is gaining importance for tree distribution networks due to the advent of distributed generation and controllable loads like electric vehicles. Distributed generation is difficult to predict, calling the traditional "generation follows demand" control strategy into question. Meanwhile, controllable loads provide significant potential to compensate for the randomness in distributed generation. To achieve this, solving the OPF problem in real-time is inevitable.

The OPF problem is difficult to solve due to the nonconvex power flow physical laws, and there are in general three ways to deal with this challenge: (i) linearize the power flow laws; (ii) look for local optima; and (iii) convexify power flow laws, which are described in turn.

The power flow laws can be approximated by linear equations in transmission networks, and then the OPF problem reduces to a linear program [2]–[4]. This method is widely used in practice for transmission networks, but does not apply to distribution networks where line resistances are high and voltages deviate significantly from the nominal values.

Various algorithms have been proposed to find local optima of the OPF problem, e.g., successive linear/quadratic programming [5], trust-region based methods [6], [7], Lagrangian Newton method [8], and interior-point methods [9]–[11]. However, these algorithms may not convergence, nor converge to optimal solutions.

Convexification methods are the focus of this paper. It is proposed in [12]–[14] to transform the nonconvex power flow laws into linear constraints on a positive semidefinite rank-one matrix, and then remove the rank-one constraint to obtain a semidefinite programming (SDP) relaxation. If the solution of the SDP relaxation is feasible for the OPF problem, then a global optimum of the OPF problem can be recovered. In this case, the SDP relaxation is called *exact*. Strikingly, the SDP relaxation is exact for the IEEE 14-, 30-, 57-, and 118-bus test transmission networks [14], and a more recent study on the computational speed and exactness of the SDP relaxation can be found in [15].

There is another type of convex relaxations for the OPF problem, i.e., second-order cone programming (SOCP) relaxations [16]–[19]. While computationally much more efficient than the SDP relaxation, the SOCP relaxations are exact if and only if the SDP relaxation is exact, for tree networks [20]. Hence, we focus on the SOCP relaxations, more specifically, the one proposed in [19].

Up to date, existing conditions that guarantee the exactness of the SOCP relaxation are difficult to satisfy. For example, the conditions in [16], [21], [22] require some/all of the buses to be able to draw infinite power; and the condition in [23] requires bus voltages to be fixed constants.

Summary of contributions

The goal of this paper is to provide a priori guarantee that the SOCP relaxation be exact. Specifically, contributions of this paper are threefold.

First, we prove that *if optimal power injections lie in some region, and maximum power injections are sufficiently small, then the SOCP relaxation is exact*. We have checked that maximum power injections are indeed sufficiently small, for the IEEE 13-bus network and two real-world networks.

Second, we *propose a modified OPF problem whose power injections are further restricted*. A modification is necessary for an exact SOCP relaxation since otherwise examples, in which the SOCP relaxation is not exact, exist. Remarkably, with the proposed modification, only feasible points "close" to voltage upper bounds are eliminated, and the SOCP relaxation is guaranteed exact. Empirical studies justify that the modification is "small" for the same test networks.

Third, we prove that *the SOCP relaxation has at most one solution if it is exact*.

II. The optimal power flow problem

This paper studies the optimal power flow (OPF) problem in tree distribution networks, which includes Volt/VAR

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Lingwen Gan, Na Li, and Steven H. Low are with the Engineering and Applied Science Department, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: lgan@caltech.edu).

Ufuk Topcu is with the Electrical and Systems Engineering Department, University of Pennsylvania, Philadelphia, PA 19104 USA.

control and demand response as its special cases. In the following we present a model of this scenario, that incorporates nonlinear power flow physical laws, and considers a variety of controllable devices including distributed generators, inverters, controllable loads, and shunt capacitors.

A. Power flow model

A distribution network is composed of buses and lines connecting them. It has a tree topology in normal operation.

There is a substation in the network with a fixed voltage. Index the substation bus by 0 and the other buses by $1, \ldots, n$. Let $\mathcal{N} := \{0, \ldots, n\}$ denote the collection of all buses and define $\mathcal{N}^+ := \mathcal{N} \backslash \{0\}$. Each line connects an ordered pair (i, j) of buses where bus j is in the middle of bus i and bus 0. Let $\mathcal{E}$ denote the collection of all lines and abbreviate $(i, j) \in \mathcal{E}$ by $i \to j$. If $i \to j$ or $j \to i$, denote $i \sim j$.

For each bus $i \in \mathcal{N}$, let V_i denote its voltage and I_i denote its current injection. Specifically, the substation voltage, V_0, is fixed. Let $s_i = p_i + \mathbf{i} q_i$ denote the power injection of bus i where p_i and q_i denote the real and reactive power injections respectively. Let $\mathcal{P}_i$ denote the path (a collection of buses in $\mathcal{N}$ and lines in $\mathcal{E}$) from bus i to bus 0.

For each line $i \sim j$, let $y_{ij} = g_{ij} - \mathbf{i} b_{ij}$ denote its admittance, and define $z_{ij} := r_{ij} + \mathbf{i} x_{ij} := 1/y_{ij}$.

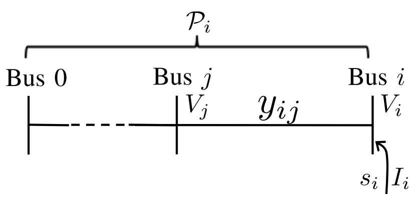

Fig. 1. Some of the notations.

Some of the notations are summarized in Fig. 1. Further, we use a letter without subscript to denote a vector of the corresponding quantity, e.g., $V = (V_i)_{i \in \mathcal{N}^+}$, $y = (y_{ij})_{i \sim j}$. Note that subscript 0 is not included in nodal variables.

Given the network graph $(\mathcal{N}, \mathcal{E})$, the admittance y, and the swing bus voltage V_0, then the other variables (s, V, I, s_0) are described by (the superscript H denotes hermitian)

- Ohm's law: $I_{ij} = y_{ij}(V_i - V_j)$ for $i \sim j$;
- Current balance: $I_i = \sum_{j:\, j \sim i} I_{ij}$ for $i \in \mathcal{N}$;
- Power balance: $s_i = V_i I_i^H$ for $i \in \mathcal{N}$.

If only voltages and power are concerned, then the three sets of equations can be combined into

$$s_i = V_i \sum_{j:\, j \sim i} (V_i^H - V_j^H) y_{ij}^H, \qquad i \in \mathcal{N}, \tag{1}$$

which is used to model the power flow in this paper.

B. Controllable devices and control objective

Controllable devices in a distribution network include distributed generators, inverters that connect distributed generators to the grid, controllable loads like electric vehicles and smart appliances, and shunt capacitors.

Real and reactive power generation/demand of these devices can be controlled to achieve certain objectives. For example, in Volt/VAR control, reactive power injection of the inverters and shunt capacitors are controlled to regulate the voltages; in demand response, real power demand from controllable loads are reduced or shifted in response to power supply conditions. Mathematically, power injection s is the control variable, after specifying which the other variables V and s_0 are determined by (1).

Constraints on the power injection s_i of a bus $i \in \mathcal{N}^+$ is captured by some externally specified set $\mathcal{S}_i$, i.e.,

$$s_i \in \mathcal{S}_i, \qquad i \in \mathcal{N}^+. \tag{2}$$

The set $\mathcal{S}_i$ for typical control devices are summarized below.

- If bus i only has a shunt capacitor with nameplate capacity $\overline{q}_i$, then
$$\mathcal{S}_i = \{s \mid \mathrm{Re}(s) = 0,\ \mathrm{Im}(s) = 0 \text{ or } \overline{q}_i\}.$$
- If bus i has a solar photovoltaic panel with real power generation capacity $\overline{p}_i$, and an inverter with nameplate capacity $\overline{s}_i$, then
$$\mathcal{S}_i = \{s \mid 0 \le \mathrm{Re}(s) \le \overline{p}_i,\ |s| \le \overline{s}_i\}.$$
- If bus i only has a controllable load with constant power factor η, whose real power consumption can vary continuously from $-\overline{p}_i$ to $-\underline{p}_i$, then
$$\mathcal{S}_i = \{s \mid \underline{p}_i \le \mathrm{Re}(s) \le \overline{p}_i,\ \mathrm{Im}(s) = \sqrt{1-\eta^2}\mathrm{Re}(s)/\eta\}.$$

Note that constraint (2) may or may not be convex, depending on the structure of $\mathcal{S}_i$. In this paper, nonconvexity from (2) is not considered.

The control objective in a distribution network is twofold. The first one is regulating the voltages within certain range, i.e., there exists $\underline{V}_i$ and $\overline{V}_i$ such that

$$\underline{V}_i \le |V_i| \le \overline{V}_i, \qquad i \in \mathcal{N}^+. \tag{3}$$

For example, if 5% voltage deviation from the nominal value is allowed, then $0.95 \le |V_i| \le 1.05$ per unit [24].

The second objective is minimizing the power loss

$$L(s, s_0) := \sum_{i \in \mathcal{N}} \mathrm{Re}(s_i). \tag{4}$$

C. The OPF problem

We can now formally state the OPF problem that we seek to solve: minimize the power loss (4), subject to power flow constraints (1), power injection constraints (2), and voltage regulation constraints (3).

$$\begin{aligned}
\textbf{OPF:}\ \min \quad & \sum_{i \in \mathcal{N}} \mathrm{Re}(s_i) \\
\text{over} \quad & s, V, s_0 \\
\text{s.t.} \quad & s_i = V_i \sum_{j:\, j \sim i} (V_i^H - V_j^H) y_{ij}^H, \qquad i \in \mathcal{N}; \\
& s_i \in \mathcal{S}_i, \qquad i \in \mathcal{N}^+; \\
& \underline{V}_i \le |V_i| \le \overline{V}_i, \qquad i \in \mathcal{N}^+.
\end{aligned}$$

The following assumptions are made throughout this work:

A1 The network $(\mathcal{N}, \mathcal{E})$ is a tree. Distribution networks are indeed tree networks in normal operation.

A2 The substation voltage V_0 is given and fixed. In practice, V_0 can be modified several times a day, therefore can be considered as a fixed constant at the minutes timescale of the OPF problem.

A3 Line resistances and reactances are positive, i.e., $r_{ij} > 0$ and $x_{ij} > 0$ for $i \sim j$. In practice, $r_{ij} > 0$ since lines are passive, and $x_{ij} > 0$ since lines are inductive.

A4 Voltage lower bounds are positive, i.e., $\underline{V}_i > 0$ for $i \in \mathcal{N}^+$. In practice, $\underline{V}_i$ is around 0.95.

The challenge in solving OPF comes from the nonconvex constraints (1). To overcome this challenge, one can enlarge the feasible set of OPF to a convex set. Define

$$W_{ij} := V_i V_j^H, \qquad i \sim j \text{ or } i = j; \tag{5}$$
$$W\{i,j\} := \begin{pmatrix} W_{ii} & W_{ij} \\ W_{ji} & W_{jj} \end{pmatrix}, \qquad i \to j;$$

and $W := (W_{ij})_{i=j \text{ or } i \sim j}$, then OPF is equivalent to

$$\begin{aligned}
\textbf{OPF':} \ \min \ & \sum_{i\in\mathcal{N}} \mathrm{Re}(s_i) \\
\text{over} \ & s, W, s_0 \\
\text{s.t.} \ & s_i = \sum_{j:\, j\sim i} (W_{ii} - W_{ij}) y_{ij}^H, \quad i \in \mathcal{N}; && \text{(6a)} \\
& s_i \in \mathcal{S}_i, \quad i \in \mathcal{N}^+; && \text{(6b)} \\
& \underline{V}_i^2 \le W_{ii} \le \overline{V}_i^2, \quad i \in \mathcal{N}^+; && \text{(6c)} \\
& W\{i,j\} = W\{i,j\}^H, \quad i \to j; && \text{(6d)} \\
& W\{i,j\} \succeq 0, \quad i \to j; && \text{(6e)} \\
& \mathrm{rank}(W\{i,j\}) = 1, \quad i \to j && \text{(6f)}
\end{aligned}$$

for tree networks, where for a hermitian matrix A,

$$A \succeq 0 \quad \overset{\text{def}}{\Longleftrightarrow} \quad A \text{ is positive semidefinite.}$$

After transforming OPF to OPF', one can obtain a convex relaxation by removing the rank constraints (6f) as in the following second-order cone programming (SOCP) [19].

$$\begin{aligned}
\textbf{SOCP:} \ \min \ & \sum_{i\in\mathcal{N}} \mathrm{Re}(s_i) \\
\text{over} \ & s, W, s_0 \\
\text{s.t.} \ & \text{(6a)} - \text{(6e)}.
\end{aligned}$$

If a solution $w = (s, W, s_0)$ of SOCP is feasible for OPF', i.e., w satisfies (6f), then w is a global optimum of OPF'. This motivates a definition of "exactness" for SOCP.

Definition 1 *SOCP is exact if every of its solutions satisfies* (6f).

A global optimum of OPF can be recovered if SOCP is exact.

D. Related work

This paper studies the exactness of SOCP. Several sufficient conditions have already been derived for the exactness of the SOCP relaxation [16], [21]–[23], [25].

It is proved in [16] that SOCP is exact if power injection constraints take specific forms. The results in [21], [22] improve over this condition.

Proposition 1 ([16]) *SOCP is exact provided that* $\mathcal{S}_i = \{s \mid \mathrm{Re}(s) \le \overline{p}_i,\ \mathrm{Im}(s) \le \overline{q}_i\}$ *for* $i \in \mathcal{N}^+$.

In practice, $\mathcal{S}_i$ may take more general forms.

Reference [25] considers more general forms of $\mathcal{S}_i$, but ignores upper bounds on the voltages. To state the result, for every line $i \to j$, let

$$\hat{S}_{ij}(p + \mathbf{i}q) := \hat{P}_{ij}(p) + \mathbf{i}\hat{Q}_{ij}(q) := \sum_{k:\, i\in\mathcal{P}_k} p_k + \mathbf{i} \sum_{k:\, i\in\mathcal{P}_k} q_k \tag{7}$$

denote its downstream total power injection.

Proposition 2 ([25]) *SOCP is exact provided that* $\mathcal{S}_i \subseteq \{s \mid \mathrm{Re}(s) \le \overline{p}_i,\ \mathrm{Im}(s) \le \overline{q}_i\}$ *for* $i \in \mathcal{N}^+$, $\overline{V}_i = \infty$ *for* $i \in \mathcal{N}^+$, *and any one of the following conditions hold:*

(i) $\hat{P}_{ij}(\overline{p}) \le 0$ *and* $\hat{Q}_{ij}(\overline{q}) \le 0$ *for all* $i \to j$.
(ii) $r_{ij}/x_{ij} = r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$.
(iii) $r_{ij}/x_{ij} \ge r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$, *and* $\hat{P}_{ij}(\overline{p}) \le 0$ *for all* $i \to j$.
(iv) $r_{ij}/x_{ij} \le r_{jk}/x_{jk}$ *for all* $i \to j$, $j \to k$, *and* $\hat{Q}_{ij}(\overline{q}) \le 0$ *for all* $i \to j$.

In distribution networks, the constraints $|V_i| \le \overline{V}_i$ cannot be ignored, especially with distributed generators making the voltages likely exceed $\overline{V}$.

To summarize, all sufficient conditions in literature that guarantee the exactness of SOCP require removing some of the constraints. In fact, SOCP is in general not exact [26].

III. A MODIFIED OPF PROBLEM

In this section, we first provide a sufficient condition under which SOCP is exact, and then modify OPF' accordingly to satisfy this condition.

A. A sufficient condition

The sufficient condition is built on a linear approximation of the power flow in "the worst case". First define the linear approximation. Define

$$S_{ij} := P_{ij} + \mathbf{i}Q_{ij} := (W_{ii} - W_{ij}) y_{ij}^H \tag{8}$$

as the sending-end power flow from bus i to bus j for $i \to j$, then $\hat{S}_{ij}(s)$ defined in (7) is a linear approximation of S_{ij} (linear in s). Let $S := (S_{ij}, i \to j)$ denote the collection of power flow on all lines. Also define

$$\hat{W}_{ii}(s) := W_{00} + 2 \sum_{(j,k)\in\mathcal{P}_i} \mathrm{Re}\left(z_{jk}^H \hat{S}_{jk}(s)\right)$$

for every $i \in \mathcal{N}^+$ and every power injection s, then $\hat{W}_{ii}(s)$ is a linear approximation of $W_{ii} = |V_i|^2$ (linear in s).

The linear approximations $\hat{W}_{ii}(s)$ and $\hat{S}_{ij}(s)$ are upper bounds on W_{ii} and S_{ij}, as stated in Lemma 1. To state the lemma, let the operator $\leq$ denote componentwise, e.g., for two complex numbers $a, b \in \mathbb{C}$,

$$a \leq b \overset{\text{def}}{\iff} \mathrm{Re}(a) \leq \mathrm{Re}(b) \text{ and } \mathrm{Im}(a) \leq \mathrm{Im}(b).$$

Lemma 1 *If* (s, S, W, s_0) *satisfies* (6a), (6d), (6e) *and* (8), *then* $S_{ij} \leq \hat{S}_{ij}(s)$ *for* $i \to j$ *and* $W_{ii} \leq \hat{W}_{ii}(s)$ *for* $i \in \mathcal{N}^+$.

Lemma 1 is proved in an accompanying technical report [26].

The linear approximations $\hat{S}_{ij}(s)$ and $\hat{W}_{ii}(s)$ are close to S_{ij} and W_{ii} in practice. It can be verified that $\hat{S}_{ij}$, $\hat{W}_{ii}$ solve

$$\begin{aligned} \hat{S}_{jk} &= s_j + \sum_{i:\, i \to j} \hat{S}_{ij}, & j \to k; \\ \hat{W}_{jj} &= \hat{W}_{ii} - 2\mathrm{Re}(z_{ij}^H \hat{S}_{ij}), & i \to j, \end{aligned}$$

which is called *Linear DistFlow model* in the literature and known to approximate the exact power flow model well. In fact, the Linear DistFlow model has been used to study the optimal placement and sizing of shunt capacitors [27], [28], to reconfigure distribution networks [29], and to control reactive power injections for voltage regulation [30].

The sufficient condition we derive for the exactness of SOCP is based on the linear approximation $\hat{S}_{ij}(p + \mathbf{i}q) = \hat{P}_{ij}(p) + \mathbf{i}\hat{Q}_{ij}(q)$ of the power flow, in the case where power injection is maximized. To state the condition, assume that $\mathcal{S}_i \subseteq \{s \mid \mathrm{Re}(s) \leq \overline{p}_i,\ \mathrm{Im}(s) \leq \overline{q}_i\}$ for $i \in \mathcal{N}^+$, define $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$, let $a_0^1 = 1$, $a_0^2 = 0$, $a_0^3 = 0$, $a_0^4 = 1$, and define

$$\begin{aligned} a_i^1 &:= \prod_{(j,k) \in \mathcal{P}_i} \left(1 - \frac{2r_{jk}\hat{P}_{jk}^+(\overline{p})}{\underline{V}_j^2}\right), \\ a_i^2 &:= \sum_{(j,k) \in \mathcal{P}_i} \frac{2r_{jk}\hat{Q}_{jk}^+(\overline{q})}{\underline{V}_j^2}, \\ a_i^3 &:= \sum_{(j,k) \in \mathcal{P}_i} \frac{2x_{jk}\hat{P}_{jk}^+(\overline{p})}{\underline{V}_j^2}, \\ a_i^4 &:= \prod_{(j,k) \in \mathcal{P}_i} \left(1 - \frac{2x_{jk}\hat{Q}_{jk}^+(\overline{q})}{\underline{V}_j^2}\right) \end{aligned}$$

for $i \in \mathcal{N}^+$.

Lemma 2 *Assume that* $\mathcal{S}_i \subseteq \{s \mid \mathrm{Re}(s) \leq \overline{p}_i,\ \mathrm{Im}(s) \leq \overline{q}_i\}$ *for* $i \in \mathcal{N}^+$, *then SOCP is exact, provided that every solution* $w = (s, W, s_0)$ *of SOCP satisfies* $\hat{W}_{ii}(s) \leq \overline{V}_i^2$ *for* $i \in \mathcal{N}^+$, *and*

$$a_j^1 r_{ij} > a_j^2 x_{ij},\ a_j^3 r_{ij} < a_j^4 x_{ij}, \qquad i \to j. \tag{9}$$

Lemma 2 is proved in an accompanying technical report [26].

The condition $\hat{W}_{ii}(s) \leq \overline{V}_i^2$ depends on solutions of SOCP, and cannot be checked before solving SOCP. This shortcoming motivates us to modify OPF' in Section III-B.

B. A modified OPF' problem

One can impose additional constraints

$$\hat{W}_{ii}(s) \leq \overline{V}_i^2, \quad i \in \mathcal{N}^+ \tag{10}$$

on the power injection s, so that the condition $\hat{W}_{ii}(s) \leq \overline{V}_i^2$ in Lemma 2 holds automatically. Note that the constraints in (6c) and (10) can be combined as

$$\underline{V}_i^2 \leq W_{ii},\ \hat{W}_{ii}(s) \leq \overline{V}_i^2, \qquad i \in \mathcal{N}^+ \tag{11}$$

since $W_{ii} \leq \hat{W}_{ii}(s)$ according to Lemma 1.

To summarize, the modified OPF' problem is

$$\begin{aligned} \textbf{OPF'-m:}\ \min\ & \sum_{i \in \mathcal{N}} \mathrm{Re}(s_i) \\ \text{over}\ & s, W, s_0 \\ \text{s.t.}\ & \text{(6a), (6b), (6d), (6e), (6f), (11).} \end{aligned}$$

Note that a modification is necessary for an exact SOCP, since SOCP is in general not exact. Remarkably, with the proposed modification, the feasible sets of OPF'-m and OPF' are close since $\hat{W}_{ii}(s)$ is close to W_{ii} in practice. This is justified by the empirical studies in Section IV-A.

The corresponding relaxation for OPF'-m is

$$\begin{aligned} \textbf{SOCP-m:}\ \min\ & \sum_{i \in \mathcal{N}} \mathrm{Re}(s_i) \\ \text{over}\ & s, W, s_0 \\ \text{s.t.}\ & \text{(6a), (6b), (6d), (6e), (11).} \end{aligned}$$

The main contribution of this paper is to provide a sufficient condition for the exactness of SOCP-m, that can be checked in priori and does not require removing any of the constraints. The sufficient condition is given in Theorem 1, which directly follows from Lemma 2.

Theorem 1 *Assume that* $\mathcal{S}_i \subseteq \{s \mid \mathrm{Re}(s) \leq \overline{p}_i,\ \mathrm{Im}(s) \leq \overline{q}_i\}$ *for* $i \in \mathcal{N}^+$, *then SOCP-m is exact if* (9) *holds.*

Condition (9) can be checked without solving SOCP-m since it does not depend on the solutions of SOCP-m. In fact, $(a_j^k)_{j \in \mathcal{N}, k=1,2,3,4}$ are functions of $(r, x, \overline{p}, \overline{q}, \underline{V})$ that can be computed efficiently in $O(n)$ time, therefore the complexity of checking Condition (9) is $O(n)$.

Condition (9) requires $\overline{p}$ and $\overline{q}$ be "small". Fix $(r, x, \underline{V})$, then (9) is a condition on $(\overline{p}, \overline{q})$. It can be verified that if $(\overline{p}, \overline{q}) \leq (\overline{p}', \overline{q}')$ where the operator $\leq$ denotes componentwise, then

$$\text{(9) holds for } (\overline{p}', \overline{q}') \ \Rightarrow\ \text{(9) holds for } (\overline{p}, \overline{q}),$$

i.e., the smaller power injections, the more likely (9) holds. It can also be verified that if $(\overline{p}, \overline{q}) = (0, 0)$, then (9) holds. Hence, if $(\overline{p}, \overline{q}) \leq (0, 0)$, e.g., there is no distributed generation, then (9) holds.

As will be seen in the empirical studies in Section IV-B, (9) holds for three test networks, even those with high penetration of distributed generation, i.e., big $(\overline{p}, \overline{q})$. Hence, we expect (9) to hold widely in practice.

Theorem 1 holds for more general objective functions. In particular, the objective function in (4) can be generalized to $f(L(\ell), s)$ where the function $f(x, y) : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$ is strictly increasing in x. This includes generation costs of the form $\sum_{i\in\mathcal{N}} f_i(\mathrm{Re}(s_i))$ where f_0 is strictly increasing.

Theorem 2 *SOCP/SOCP-m has at most one solution if it is exact.*

Theorem 2 is proved in an accompanying technical report [26]. It holds for more general objective functions and power injection constraints. In particular, the objective function in (4) can be generalized to any convex function, and the power injection constraints in (2) can be generalized to $s \in \mathcal{S}$ where $\mathcal{S}$ is an arbitrary convex set.

IV. Case Studies

In this section we show that the feasible sets of OPF' and OPF'-m are close, and that condition (9) holds, for the IEEE 13-bus network and two real-world networks.

The IEEE 13-bus network is modified from [31] to satisfy the power flow physical laws (1), as detailed in [26]. The real-world networks, a 47-bus one and a 56-bus one, come from Southern California Edison (SCE), a utility company [16], [32]. These networks have increasing penetration of distributed generation (DG), as listed in Table I.

A. *Feasible sets of OPF' and OPF'-m are similar.*

We show that OPF'-m eliminates some feasible points of OPF', that are close to the voltage upper bounds, for all three networks.

To state the results, define the following quantities. It is claimed in [33] that given $s = p + \mathbf{i}q$, there exists a unique voltage $V(s)$ near the nominal value that satisfies the power flow constraint (1) for tree networks. Define

$$\varepsilon(s) := \max_{i\in\mathcal{N}^+} \hat{W}_{ii}(s) - |V_i(s)|^2$$

as the maximum deviation from $\hat{W}_{ii}(s)$ to $W_{ii}(s) = |V_i(s)|^2$ over $i \in \mathcal{N}^+$. It follows from Lemma 1 that $\hat{W}_{ii}(s) \geq W_{ii}(s)$, therefore $\varepsilon(s) \geq 0$. Further define

$$\varepsilon := \max_{s_i\in\mathcal{S}_i} \varepsilon(s)$$

as the maximum deviation from $\hat{W}_{ii}(s)$ to $W_{ii}(s)$ over s, i.

The value ε serves as a measure for the difference between the feasible sets of OPF' and OPF'-m for the following reason. Consider OPF' with stricter voltage upper bound constraints $W_{ii} \leq \overline{V}_i^2 - \varepsilon$

$$\begin{aligned} \textbf{OPF'-}\varepsilon\textbf{:}\ \min \quad & \sum_{i\in\mathcal{N}} \mathrm{Re}(s_i) \\ \text{over} \quad & s, W, s_0 \\ \text{s.t.} \quad & (6a), (6b), (6d), (6e), (6f); \\ & \underline{V}_i^2 \leq W_{ii} \leq \overline{V}_i^2 - \varepsilon, \quad i \in \mathcal{N}^+. \end{aligned}$$

Then it follows from

$$W_{ii}(s) \leq \overline{V}_i^2 - \varepsilon \implies \hat{W}_{ii}(s) \leq \overline{V}_i^2, \qquad i \in \mathcal{N}^+$$

that the feasible set $\mathcal{F}_{\text{OPF'-}\varepsilon}$ of OPF'-ε is contained in the feasible set $\mathcal{F}_{\text{OPF'-m}}$ of OPF'-m. Hence,

$$\mathcal{F}_{\text{OPF'-}\varepsilon} \subseteq \mathcal{F}_{\text{OPF'-m}} \subseteq \mathcal{F}_{\text{OPF'}},$$

as illustrated in Fig. 2.

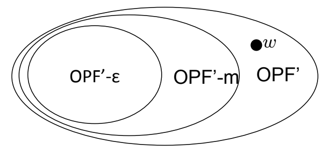

Fig. 2. Feasible sets of OPF'-ε, OPF'-m, and OPF'. The point w is feasible for OPF' but infeasible for OPF'-m.

If ε is small, then $\mathcal{F}_{\text{OPF'-m}}$ and $\mathcal{F}_{\text{OPF'}}$ are similar. Moreover, any point w that is feasible for OPF' but infeasible OPF'-m is close to the voltage upper bound since $W_{ii} > \overline{V}_i^2 - \varepsilon$ for some $i \in \mathcal{N}^+$. Such points are perhaps undesirable for robust operation.

The quantity ε takes relatively small values for all three networks. To evaluate ϵ, we assume that $V_0 = 1$, and that $\overline{V}_i = 1.05$, $\underline{V}_i = 0.95$ for $i \in \mathcal{N}^+$. For the IEEE network, we further assume that $\overline{p} = \underline{p}$, $\overline{q} = \underline{q}$, and that they equal the values specified in [31]. For the SCE networks, we further assume that all loads draw peak spot apparent power at power factor 0.97, that all shunt capacitors are switched on, and that distributed generators generate real power at their nameplate capacities with zero reactive power. The values of ε are summarized in Table I. For instance, $\varepsilon = 0.0043$ for the IEEE 13-bus network, in which case the voltage constraints are $0.9025 \leq W_{ii} \leq 1.1025$ for OPF' and $0.9025 \leq W_{ii} \leq 1.0982$ for OPF'-ε.

TABLE I

Closeness of OPF'-m and OPF'

	DG penetration	ε
IEEE 13-bus	0%	0.0043
SCE 47-bus	56.6%	0.0036
SCE 56-bus	130.4%	0.0106

B. *Condition* (9) *holds in all test networks.*

We have checked that (9) holds for all three networks, in the worst case where power injections are maximized:

- for a load bus i, we set $(\overline{p}_i, \overline{q}_i) = (0, 0)$ while they are negative in practice.
- for a shunt capacitor bus i, we set $\overline{p}_i = 0$ and $\overline{q}_i$ to equal to its nameplate capacity.
- for a distributed generator bus i, we set $\overline{q}_i = 0$ and $\overline{p}_i$ to equal to its nameplate capacity. In practice, $\overline{p}_i$ is usually smaller.

Note that (9) is more difficult to satisfy as $(\overline{p}, \overline{q})$ increases, and that $(\overline{p}_i, \overline{q}_i)$ is artificially enlarged for all buses, (9) holds for all three networks. Furthermore, the SCE 56-bus network has 130.4% DG penetration, which is difficult for (9) to be satisfied. Therefore, we expect (9) to hold more widely in practice.

V. Conclusion

We have proved that the SOCP relaxation for the OPF problem is exact under a prior checkable condition (9), after imposing additional constraints on power injections. Condition (9) holds for three test networks, an IEEE 13-bus network and two real-world networks with high penetration of distributed generation. The additional constraints on power injections eliminate feasible points of the OPF problem that are close to the voltage upper bounds.

There remains many open questions on finding the global optimum of the OPF problem: is the convex relaxation for the OPF problem in mesh networks exact? Is there an exact convex relaxation for the OPF problem in unbalanced three-phase networks? If the SOCP relaxation is not exact, then how to recover a "good" solution of the OPF problem?

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