A Faithful Distributed Implementation of Dual Decomposition and Average Consensus Algorithms

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Abstract— We consider large scale cost allocation problems and consensus seeking problems for multiple agents in which agents are suggested to collaborate in a distributed algorithm to find a solution. If agents are strategic and minimize their own individual cost rather than the global social cost, they are endowed with an incentive not to follow the intended algorithm, unless the tax/subsidy mechanism is carefully designed. Inspired by the classical Vickrey–Clarke–Groves mechanism and more recent algorithmic mechanism design theory, we propose a tax mechanism that incentivises agents to faithfully implement the intended algorithm. In particular, a new notion of asymptotic incentive compatibility is introduced to characterize a desirable property of such class of mechanisms. The proposed class of tax mechanisms provides a sequence of mechanisms that gives agents a diminishing incentive to deviate from suggested algorithm.

I. INTRODUCTION

A common difficulty in large scale optimization problems that arise in social, infrastructural, and communication networks is the heavy computational load that cannot be handled by a single computer. A practical solution algorithm for such problems should thus be parallelizable so that the computational load can be distributed over the network agents. Another potential challenge in such problems comes from the fact that nobody has access to the entire data defining the optimization problem, because this information is often private and localized within distributed agents. Hence, it is desirable that a solution algorithm allows agents to implement the algorithm without knowing other agents' private information.

In this paper, we consider two distributed algorithms that are both attractive in the above sense: the *dual decomposition* algorithm for cost allocation problems, and the *average consensus* algorithms for a consensus seeking. In the majority of the literature on these algorithms, it is assumed that the distributed agents are "benevolent" and blindly follow the intended algorithm. However, if the society involves rational and strategic agents, it is more realistic to assume that they behave in a selfish manner in an effort to minimize their individual cost rather than the global social cost. Hence in this paper, rather than *assuming* that agents are collaborative, we consider a mechanism by which rational agents are *incentivised* to follow the intended distributed algorithm.

Mechanism design theory ([1]–[3] to mention a few) concerns the question of how the society can make a preferable decision with the presence of strategic agents. The goal of a mechanism design is to suggest a social decision procedure (together with a tax/subsidy mechanism) that incentivises agents to follow the intended action. However, the framework of classical mechanism design theory is often insufficient to handle more complicated distributed algorithms.

It is only relatively recently that the discipline of algorithmic mechanism design [4] was recognized in computer science. Inspired by [5], we formulate a distributed mechanism design problem and suggest a tax mechanism that incentivises agents in a certain sense to follow the intended dual decomposition and the average consensus algorithm. Our tax mechanism can be seen as a generalization of the celebrated Vickrey–Clarke–Groves (VCG) mechanism.

By its nature, the two algorithms considered here are asymptotic algorithms: if terminated at some finite step, only an approximation to the optimal solution is obtained. It is recognized in [6], [7] that the VCG mechanism combined with such an approximated solution does not guarantee incentive compatibility in general. To circumvent this difficulty, we introduce a notion of *asymptotic incentive compatibility* for a sequence of mechanisms that provides agents a diminishing incentive to deviate from the intended algorithm.

Finally, with the present paper, we are able to relate our earlier study of real-time electricity pricing scheme [8], [9] to the mechanism design theory in more solid manner. Moreover, the framework of distributed mechanism design potentially allows us to implement the pricing scheme in [8], [9] in a smarter way: it allows distributed computations and allows agents not to disclose their private information.

II. A QUICK REVIEW OF MECHANISM DESIGN

A. Mechanisms

Consider a society E_N comprised of N agents. Besides these agents, the society also has a government who makes a social decision $x \in X \subset \mathbb{R}^{n_x}$. Each agent $i = 1, \ldots, N$ has private information $\theta_i \in \Theta_i$ called *type*. When a social decision x is made, the intrinsic cost $v_i(x; \theta_i)$ is charged to the i-th agent. The government desires to make a social decision x in such a way that the sum of individual costs is minimized. If the agents' types $\theta = (\theta_1, \ldots, \theta_N) \in \Theta_1 \times \cdots \times \Theta_N =: \Theta$ are available to the government, the desired social decision is given by

$$x(\theta) = \arg\min_{x \in X} \sum_{i=1}^{N} v_i(x; \theta_i).$$

A map $x : \Theta \to X$ is called a *decision rule*. A decision rule is said to be *efficient* if

$$\sum_{i=1}^{N} v_i(x(\theta); \theta_i) \le \sum_{i=1}^{N} v_i(x'; \theta_i)$$

for all $\theta \in \Theta$ and for all $x' \in X$. Since θ is private in reality, each agent is asked to report his type (denoted by $\hat{\theta}_i$) to the government. In this process, we assume that agents are strategic in that they are allowed to make an unfaithful report (i.e., $\hat{\theta}_i \ne \theta_i$). Notice that agent i has an incentive to do so if it will lead to an alternative social decision x' that costs less for himself, i.e., $v_i(x'; \theta_i) < v_i(x(\theta); \theta_i)$.

In order to encourage agents to be truthful, the government can introduce a monetary transfer function $t : \Theta \to \mathbb{R}^N$. This can be seen as a tax mechanism that the government imposes on each agent. We assume that the amount of tax imposed on each agent is determined by the government using the reported type $\hat{\theta}$:

$$t(\hat{\theta}) = (t_1(\hat{\theta}), \dots, t_N(\hat{\theta})).$$

Overall, agent i's net cost is given by

$$u_i(\hat{\theta}, \theta_i, x, t) = v_i(x(\hat{\theta}); \theta_i) + t_i(\hat{\theta})$$

and agents are expected to behave rationally to minimize this function. The pair $f = (x, t)$, $f : \Theta \mapsto X \times \mathbb{R}^N$ is called a *social choice function*.

The reporting process can be designed in an indirect manner using message functions $s_i : \Theta_i \to \Sigma_i$, where $\Sigma = \Sigma_1 \times \cdots \times \Sigma_N$ is the space of messages. Upon receiving "encoded" types $s(\theta) = (s_1(\theta_1), \dots, s_N(\theta_N))$, the government recovers the value of the social choice function using a "decoding" function $g : \Sigma \to X \times \mathbb{R}^N$ such that $(g \circ s)(\theta) = f(\theta)$ for every $\theta \in \Theta$. A *mechanism* is a triplet $M = (g, \Sigma, s)$ of an outcome function g, a space of messages Σ, and a particular encoding scheme $s : \Theta \to \Sigma$ such that $g \circ s = f$ (see Fig. 1). One can think of s as an intended encoding scheme that the government desires each agent to follow. A particular case with $\Sigma = \Theta$, $g = f$, and $s = Id$ (i.e., identity map) is called a *direct mechanism*, in which agents are asked to report their types θ_i to the government without encoding. A mechanism is said to be *dominant strategy incentive compatible* if implementing the suggested encoding scheme $s = (s_1, \dots, s_N)$ is a dominant strategy for each individual. If there exists such a dominant strategy incentive compatible mechanism $M = (g, \Sigma, s)$ such that $f = g \circ s$, f is said to be implemented by M in dominant strategies. Since the government announces a mechanism first and the agents react to it, the government and the agents are also referred to as the leader and the followers in what follows.

B. Fundamental results from mechanism theory

If a mechanism (g, Σ, s) implements a social choice function $f = (x, t)$ in dominant strategies, then a direct mechanism $(g \circ s, \Theta, Id)$ implements f in dominant strategies as well (revelation principle [3]). This suggests that a given social choice function is implemented by some mechanism M if and only if it is implemented by some direct mechanism
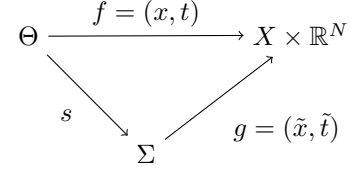

Fig. 1: A pair $f = (x, t)$ is called the social choice function while a pair $g = (\tilde{x}, \tilde{t})$ is called the outcome function

in dominant strategies.

If the government wishes to design a mechanism that implements a particular social decision rule x without introducing monetary transfer function (i.e., $t \equiv 0$), then the decision rule has to be quite trivial (Gibbard-Satterthwaite theorem [3]). This suggests that a tax (or subsidy) mechanism is almost necessary for the government to incentivise individuals to be truthful to make the right social decision x.

Let $x : \Theta \to X$ be an efficient social decision rule. It is elementary to prove, but valuable to realize that there exists a class of monetary transfer functions $t^{Groves} : \Theta \to \mathbb{R}^N$ such that a direct mechanism (f, Θ, Id) with the social choice function $f = (x, t^{Groves})$ is dominant strategy incentive compatible (see [3] for a great introduction). Such a class of monetary transfer functions is given by

$$x^{Groves}(\hat{\theta}) = \arg\min_{x \in X} \sum_{i=1}^{N} v_i(x; \hat{\theta}_i) \tag{1a}$$

$$t_i^{Groves}(\hat{\theta}) = k_i(\hat{\theta}_{-i}) + \sum_{j \ne i} v_j(x^{Groves}(\hat{\theta}); \hat{\theta}_j) \tag{1b}$$

for each $i = 1, \dots, N$, where $k_i : \prod_{j \ne i} \Theta_j \to \mathbb{R}$ is an arbitrary function that does not depend on $\hat{\theta}_i$. A mechanism obtained by the above scheme is referred to as a Groves' mechanism. It is known that the form of monetary transfer function (1b) is not only sufficient but also necessary in an appropriate sense, in order for x to be efficient and a mechanism (f, Θ, Id), $f = (x, t)$ is dominant strategy incentive compatible (Green-Laffont theorem).

C. VCG mechanisms

A particular choice of

$$k_i(\hat{\theta}_{-i}) = -\min_{x \in X} \sum_{j \ne i} v_j(x; \hat{\theta}_j)$$

yields a mechanism with some additional desirable properties. With this choice, the monetary transfer function becomes

$$t_i^{VCG}(\hat{\theta}) = \sum_{j \ne i} v_j(x^{Groves}(\hat{\theta}); \hat{\theta}_j) - \min_{x \in X} \sum_{j \ne i} v_j(x; \hat{\theta}_j). \tag{2}$$

Let E_{N-i} denote the society excluding the i-th agent. The first term in (2) corresponds to the total cost of $N - 1$ agents (excluding i) when an efficient social decision is made for E_N. The second term in (2) represents the minimum social cost achievable for E_{N-i}. Combined, (2) means that the tax imposed on the i-th agent is the same amount as the marginal contribution of the i-th agent to the rest of the society. The Groves mechanism with tax policy (2) is

called the Vickrey–Clarke–Groves (VCG) mechanism, which is known to be advantageous from the viewpoint of *budget balance* and *individual rationality* [2]. In order to compute (2), the government needs to determine social decisions that minimize the cost for each of E_N and $E_{N-i}, i = 1, \ldots, N$.

III. DISTRIBUTED MECHANISMS

A large body of mechanism design theory focuses on direct mechanisms. Although this is partially justified by the revelation principle, clearly there are a number of practical situations in which "indirect" mechanisms are preferable. For instance, indirect mechanisms allow distributed computations for large scale problems, while direct mechanisms require central computation by the government to determine the optimal social decision. Indirect mechanisms will also be advantageous for privacy preservation if they allow to find the optimal social decision without having individuals disclose their private information completely.

Previously, we considered Σ as the space of messages and $g : \Sigma \to X \times \mathbb{R}^N$ was viewed as a decoding scheme. From this section on, we want to consider more general computational interactions between the leader and the followers than mere encoding-decoding interactions. Specifically, we assume that the interaction between the leader and the followers occurs in multiple stages (indexed by $k = 1, 2, \ldots, n$). At each stage, the leader broadcasts his current computational output y_L^k to the followers. Also, we assume that each follower transmits his current computational output y_i^k directly to the leader (and possibly to the neighboring followers, depending on the communication topology) via secure channels. We assume that the leader can be modeled as a state-based computer with the internal state z_L^k, while the i-th follower can be modeled as a state-based computer with the internal state z_i^k. Given initial states z_L^0, z_i^0 and $y_L^0, y_i^0, i = 1, \ldots, N$, the state evolves according to:

$$z_i^k = G_{i,\theta_i}^k(z_i^{k-1}, y_L^{k-1}, \{y_j^{k-1}\}_{j \in N(i)}) \tag{3a}$$

$$y_i^k = H_{i,\theta_i}^k(z_i^k) \tag{3b}$$

$$z_L^k = G_L^k(z_L^{k-1}, y_1^k, \ldots, y_N^k) \tag{3c}$$

$$y_L^k = H_L^k(z_L^k) \tag{3d}$$

for $k = 1, 2, \ldots, n$. In the above, $\{y_j^{k-1}\}_{j \in N(i)}$ represents the outputs of the neighboring followers $N(i)$, and hence we are considering a communication topology as in Fig. 2. Finally, we require that

$$y_L^n = H_L^n(z_L^n) = (x, t) \in X \times \mathbb{R}^N$$

which will be the value of the social choice.

A strategy of the i-th follower is the sequence of functions in (3a) and (3b):

$$s_i(\theta_i) = \left\{ (G_{i,\theta_i}^k, H_{i,\theta_i}^k) : k = 1, 2, \ldots, n \right\}$$

parametrized by his type θ_i. On the other hand, the outcome function is defined by the sequence of functions in (3c) and (3d):

$$g = \left\{ (G_L^k, H_L^k) : k = 1, 2, \ldots, n \right\}.$$

We call $s_i(\cdot)$ a *slave* algorithm, which can be seen as a map from Θ_i to the space Σ_i of follower i's strategies. On
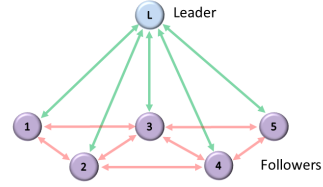

Fig. 2: Communication topology of distributed algorithms

the other hand, g is referred to as a *master* algorithm. Given an element in Σ, a master algorithm g determines a social choice (x, t). Hence, g can be a map from Σ to $X \times \mathbb{R}^N$ so that $(g \circ s)(\theta) = (x, t)$. This way, we can reuse the diagram of Fig. 1 with a generalized interpretation. To summarize:

- Σ_i is the space of all possible strategies that could be taken by followers involved in a game. Notice that the state space description of a dynamical system is not unique. Hence, we define Σ as a space of the equivalence classes of slave algorithms that have the same input-output behavior. A concrete description of the strategy space varies depending on the problem of interest.
- $s_i : \Theta_i \to \Sigma_i$ is a mapping that determines a strategy of the i-th follower with type θ_i. Each of these mappings for $i = 1, \ldots, N$ is referred to as a slave algorithm.
- A mapping $g : \Sigma \to X \times \mathbb{R}^N$, $g(s) = (\tilde{x}(s), \tilde{t}(s))$ is referred to as the master algorithm. A composition $a = g \circ s : \Theta \to \mathbb{R}^N$ is called a distributed algorithm.

With this re-interpretation, a mechanism (g, Σ, s) is defined as a triplet of a master algorithm g, followers' strategy space Σ, and a *suggested* slave algorithm s.

The notion of dominant strategy incentive compatibility is often too strong and hard to achieve. Hence, it is common to employ a weaker notion of incentive compatibility.

Definition 1: A mechanism (g, Σ, s) *implements* a social choice function f in *ex-post Nash equilibria* if

(1). $g \circ s = f$
(2). $\forall i, \forall \hat{s}_i \in \Sigma_i, \forall \theta \in \Theta$,

$$v_i(\tilde{x}_i \circ (s_i(\theta_i), s_{-i}(\theta_{-i})); \theta_i) + \tilde{t}_i \circ (s_i(\theta_i), s_{-i}(\theta_{-i}))$$
$$\leq v_i(\tilde{x}_i \circ (\hat{s}_i, s_{-i}(\theta_{-i})); \theta_i) + \tilde{t}_i \circ (\hat{s}_i, s_{-i}(\theta_{-i}))$$

where $s = (s_i, s_{-i})$. In this case, the mechanism is also said to be *incentive compatible*.

IV. DUAL DECOMPOSITION

Consider the following cost allocation problem.

$$\min \sum_{i=1}^{N} v_i(x_i; \theta_i) \tag{4a}$$

$$\text{s.t.} \quad Rx = c \tag{4b}$$

A vector $x = [x_1^T; \cdots; x_N^T]^T \in X$ is a concatenation of the social decision variables, and the domain X is defined by an affine constraint $Rx = c$. We assume that $R = [R_1 \cdots R_N]$ is full row rank.

Let $L(x, p) = \sum_{i=1}^{N} v_i(x_i; \theta_i) + p^T(Rx - c)$ be the

Algorithm 1 Distributed mechanism $M_n = (g^n, \Sigma^n, s^n)$

Input: Problem data, step size γ
Output: Social outputs $\tilde{x}, \tilde{t}_1, \ldots, \tilde{t}_N$
  (L) Initialize and broadcast p^0;
  for $k = 1, 2, \ldots, n$ **do**
    if $k \leq n - 1$ **then**
      (F) Find $\hat{x}_i^k = \arg\min_{x_i}(v_i(x_i; \theta_i) + p^{k-1^T} R_i x_i)$ and report it to the leader;
      (L) Update and broadcast $p^k = p^{k-1} + \gamma(R\hat{x}^k - c)$;
    else if $k = n$ **then**
      (L) Compute $\tilde{x} = \hat{x}^{k-1} - R(RR^T)^{-1}(R\hat{x}^{k-1} - c)$ and broadcast;
      (F) Compute and report $\hat{v}_i = v_i(\tilde{x}; \theta_i)$;
      (L) Determine social decision as $\tilde{x}$ and tax values as $\tilde{t}_i = \sum_{j \neq i} \hat{v}_i$;
    end if
  end for

Lagrangian of (4). The dual function is given by

$$g(p) = \inf_x L(x, p) = \sum_{i=1}^{N} \inf_{x_i}\left(v_i(x_i; \theta) + p^T R_i x_i\right) - p^T c,$$

and the dual problem is $\sup_p g(p)$. The primal-dual optimal solution (x^*, p^*) constitutes a saddle point of $L(x, p)$, and assuming strict convexity of $v_i(\cdot; \theta)$, the saddle point value L^* corresponds to the optimal value of (4). The following iteration is guaranteed to converge to (x^*, p^*)

$$\hat{x}_i^k = \arg\min_{\hat{x}_i}\left(v_i(\hat{x}_i, \theta_i) + p^{k-1^T} R_i \hat{x}_i\right) \quad (5a)$$

$$p^k = p^{k-1} + \gamma(R\hat{x}^k - c) \quad (5b)$$

if the step size γ is chosen to be sufficiently small [10].

Notice that the above algorithm has an attractive form for a distributed implementation since (5a) can be executed by the i-th follower and (5b) can be executed by the leader. By increasing the number of iterations, the intended social decision can be approximated with an arbitrary accuracy, provided that the followers faithfully implement (5a). In what follows, we consider what kind of side payment mechanism t_i suffices to incentivise followers to be faithful.

In a micro-economic interpretation, p is the price vector at which commodities are exchanged near the market equilibrium. In a pure competitive market (i.e., every follower has sufficiently small market power and the price cannot be affected by his sole action), it is natural for each follower to execute (5a) in an effort to minimize his own cost. However, many realistic markets are oligopoly, in which there exists a stakeholder who knows that his actions have certain effects on the market [11]. In this case, a stakeholder might be better off by deviating from (5a) and taking an alternative strategy.

V. Faithful implementation of dual decomposition

We are going to introduce a monetary transfer function t that incentivises followers to implement (5) regardless of their market power. A natural choice of $\tilde{t}_i$ inspired by the Groves mechanism is then

$$\tilde{t}_i := k_i(\hat{s}_{-i}) + \sum_{j \neq i} v_i(\tilde{x}_i; \theta_i) \quad (6)$$

where $k_i : \prod_{j \neq i} \Sigma_j \to \mathbb{R}$ is an arbitrary function that does not depend on the i-th follower's strategy. In what follows, we set $k_i = 0$ to simplify our discussion, but generalization to nonzero k_i is straightforward.

The proposed distributed algorithm is summarized in Algorithm 1. We require followers to report $\hat{x}_i^k$ at every iteration, and the value $v_i(\tilde{x}_i; \theta)$ in the end so that the leader is able to compute $\tilde{t}_i$. The algorithm is parametrized by the number of internal iterations $n \in \mathbb{N}$.

Algorithm 1 defines followers' strategies $s^n(\cdot)$ and an outcome function $g^n = (\tilde{x}^n, \tilde{t}^n)$. Note that every decision maker can be modeled as a state-based computer and their decision making process can be written in the form of (3). One can identify the i-th follower's decision making policy in Algorithm 1 with a set of mappings

$$s_i(\theta) = \left\{(G_{i,\theta_i}^k, H_{i,\theta_i}^k) : k = 1, 2, \ldots, n\right\}.$$

On the other hand, the i-th follower's strategy space Σ_i^n is the space of all alternative strategies that can be taken instead[1]. This way Algorithm 1 defines a sequence of mechanisms $\{M_n\}_{n \in \mathbb{N}}$, $M_n = (g^n, \Sigma^n, s^n)$, parametrized by the number of iterations n.

Notice that the payment t obtained in Algorithm 1 is only an approximation of the correct VCG payment because, as per (6), it is computed using the final value of $\tilde{x}$, as opposed to the optimal social decision. Hence generally it does not guarantee incentive compatibility in the sense of Definition 1. Such a fragile aspect of the VCG mechanism is considered in [6].

Definition 2: A sequence of mechanisms $\{M_n\}_{n \in \mathbb{N}}$, $M_n = (g^n, \Sigma^n, s^n)$ is said to *asymptotically implement* a social choice function f in ex-post Nash equilibria if for every $\delta_1 > 0, \delta_2 > 0$, there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$,

(1). $\|g^n \circ s^n - f\| < \delta_1$
(2). $\forall i, \forall \hat{s}_i^n \in \Sigma_i^n, \forall \theta \in \Theta$,

$$v_i\left(\tilde{x}_i^n \circ (s_i^n(\theta_i), s_{-i}^n(\theta_{-i})); \theta_i\right) + \tilde{t}_i^n \circ (s_i^n(\theta_i), s_{-i}^n(\theta_{-i}))$$
$$< v_i\left(\tilde{x}_i^n \circ (\hat{s}_i^n, s_{-i}^n(\theta_{-i})); \theta_i\right) + \tilde{t}_i^n \circ (\hat{s}_i^n, s_{-i}^n(\theta_{-i})) + \delta_2.$$

In this case, $\{M_n\}_{n \in \mathbb{N}}$ is said to be *asymptotically incentive compatible*.

Remark 1: For a fixed $n \in \mathbb{N}$, M_n is not incentive compatible. However, as $n \to \infty$, M_n provides every follower a diminishing incentive to deviate from the suggested slave algorithm.

Theorem 1: Assume that $v_i(\cdot; \theta_i), i = 1, 2, \ldots, N$ are strictly convex for every $\theta_i \in \Theta_i$, and define a social choice function $f = (x, t)$ by

$$x : \Theta \to X, x(\theta) = \arg\min_{x_1, \ldots, x_N} \sum_{i=1}^{N} v_i(x_i; \theta_i)$$

$$t : \Theta \to \mathbb{R}^n, t_i(\theta) = \sum_{j \neq i} v_j(x_j(\theta); \theta_j).$$

Then the sequence of mechanisms $\{M_n\}_{n \in \mathbb{N}}$ provided in Algorithm 1 asymptotically implements f in ex-post Nash

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[1]We assume that for all $\hat{s}^n \in \Sigma^n$ and $\theta \in \Theta$, the mechanism computes $g^n \circ \hat{s}^n(\theta)$ in finite time, since otherwise the outcome $(\tilde{x}, \tilde{t})$ is undefined.

Algorithm 2 Distributed mechanism $M'_n = (g^n, \Sigma^n, t^n)$ for asymptotically implementing the average consensus.

Input: Problem data, step size γ
Output: Social outputs $\tilde{x}, \tilde{t}_1, \ldots, \tilde{t}_N$
  (F) Initialize $x_1^0, \ldots, x_N^0$;
  (L) Initialize and broadcast p;
  for $k = 1, \ldots, n$ **do**
    if $k \leq n-1$ **then**
      (F) Calculate $x_i^k = \arg\min_{z \in \mathbb{R}} (z - \theta_i)^2 + p^{k-1^T} R_i z$ and transmit it to the leader;
      (F) Update $p_\ell^k = p_\ell^{k-1} + \gamma(x_i^k - x_j^k)$ for all $e_\ell = (i,j) \in \mathcal{E}$;
    else if $k = n$ **then**
      (L) Determine and broadcast the social decision $\tilde{x} = x_1^k \mathbf{1}$;
      (F) Calculate and transmit $v_i = (\tilde{x}_i - \theta_i)^2$;
      (L) Determine the tax values $\tilde{t}_i = \sum_{j \neq i} v_j$;
    end if
  end for

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equilibria.

Proof: By the convergence property of the dual decomposition algorithm, we have $\lim_{n \to \infty} \tilde{x}^n \circ s^n(\theta) = x(\theta)$. As a result, by the continuity of $v_i(\cdot, \theta_i)$, $\forall i = 1, \ldots, N$,

$$\lim_{n \to \infty} \tilde{t}_i^n \circ s^n(\theta) = \lim_{n \to \infty} \sum_{j \neq i} v_j(\tilde{x}_j^n \circ s^n(\theta); \theta_j)$$
$$= \sum_{j \neq i} v_j(x_j(\theta); \theta_j) = t_i(\theta).$$

This proves the first condition of Definition 2. To prove the second condition, suppose there exist i, $\delta_2 > 0$, $\theta \in \Theta$, a sequence of strategies $\hat{s}_i^n \in \Sigma_i$, and a subsequence $\{n_\ell\}_{\ell \in \mathbb{N}}$ such that

$$v_i(\tilde{x}^{n_\ell}(s_i^{n_\ell}(\theta_i), s_{-i}^{n_\ell}(\theta_{-i})); \theta_i) + \tilde{t}_i^{n_\ell}(s_i^{n_\ell}(\theta_i), s_{-i}^{n_\ell}(\theta_{-i}))$$
$$> v_i(\tilde{x}^{n_\ell}(\hat{s}_i^{n_\ell}, s_{-i}^{n_\ell}(\theta_{-i})); \theta_i) + \tilde{t}_i^{n_\ell}(\hat{s}_i^{n_\ell}, s_{-i}^{n_\ell}(\theta_{-i})) + \delta_2$$

for all $\ell \in \mathbb{N}$. This implies that

$$\sum_{i=1}^{N} v_i(\tilde{x}^{n_\ell}(s_i^{n_\ell}(\theta_i), s_{-i}^{n_\ell}(\theta_{-i})); \theta_i)$$
$$> \sum_{i=1}^{N} v_i(\tilde{x}^{n_\ell}(\hat{s}_i^{n_\ell}, s_{-i}^{n_\ell}(\theta_{-i})); \theta_i) + \delta_2.$$

for all $\ell \in \mathbb{N}$. Clearly, because of the projection in Algorithm 1, $\tilde{x}^{n_\ell}(\hat{s}_i^{n_\ell}(\theta_i), s_{-i}^{n_\ell}(\theta_{-i}))$ is feasible for all $\ell \in \mathbb{N}$. However, this contradicts with the fact that $\lim_{\ell \to \infty} \tilde{x}^{n_\ell}(s_i^{n_\ell}(\theta_i), s_{-i}^{n_\ell}(\theta_{-i})) = x(\theta)$ is a minimizer of the social cost function over the set of feasible variables. ∎

VI. FAITHFUL AVERAGE CONSENSUS IMPLEMENTATION

As an application of the approach presented above, we now consider average consensus seeking using dual decomposition when dealing with strategic agents. Let an undirected graph $\mathcal{G} = (\{1, \ldots, N\}, \mathcal{E})$, with vertex set $\{1, \ldots, N\}$ and edge set $\mathcal{E}$, be given to illustrate the communication links between the agents (see Fig. 2). Following [12], we can achieve the average consensus through solving the optimization problem

$$\min_{x \in \mathbb{R}^N} \quad \sum_{i=1}^{N} (x_i - \theta_i)^2, \tag{7a}$$
$$\text{s.t.} \quad x_i = x_j, \forall (i,j) \in \mathcal{E}, \tag{7b}$$

where $x_i \in \mathbb{R}$ denotes the decision variable of agent i, $1 \leq i \leq N$, and $\theta_i \in \Theta_i \subseteq \mathbb{R}$ is its type. Note that our assumption of considering scalar consensus problem is only in place to simplify the presentation and the results can be readily extended to higher dimensional cases using the same line of reasoning. Let us introduce the incidence matrix of $\mathcal{G}$. To do so, we need to assign arbitrary directions to the edges of $\mathcal{G}$. It is important to note that the underlying graph (specifically, the communication graph) is still an undirected graph. Let us define the incidence matrix $B(\mathcal{G}) \in \{-1, 0, +1\}^{N \times |\mathcal{E}|}$ so that $b_{ij}(\mathcal{G}) = 1$ if the edge $e_j \in \mathcal{E}$ leaves vertex i, $b_{ij}(\mathcal{G}) = -1$ if the edge $e_j \in \mathcal{E}$ enters vertex i, and $b_{ij}(\mathcal{G}) = 0$ otherwise. In the rest of the section, we assume that $\mathcal{G}$ is a tree. Using the incidence matrix, we can rewrite the optimization problem in (7) as

$$\min_{x \in \mathbb{R}^N} \quad \sum_{i=1}^{N} (x_i - \theta_i)^2, \tag{8a}$$
$$\text{s.t.} \quad Rx = 0, \tag{8b}$$

where $R = B(\mathcal{G})^\top$. Clearly, the optimization problem (8) is of the form discussed in (4) when substituting $v_i(x_i; \theta_i) = (x_i - \theta_i)^2$ for all $1 \leq i \leq N$. Noting that this optimization problem satisfies the Slater's condition, the duality gap is indeed zero and we can solve the problem using the dual decomposition [13, p. 226]. As a result, Algorithm 1 can be used to handle situation where, unlike in the classical literature (e.g., [12], [14], [15]), the agents engaged in the averaging process are strategic. However, we present two other algorithms which, unlike Algorithm 1, allow for direct communication between the followers and therefore, can be considered more desirable. Before stating the results, let us define the sequence of mechanisms $\{M'_n\}_{n \in \mathbb{N}}$, where each mechanism M'_n is introduced in Algorithm 2. Furthermore, note that Algorithm 2 suggests followers' strategies $s^n(\cdot)$ and a social output function $g^n = (\tilde{x}^n, \tilde{t}^n)$.

Proposition 1: Define a social choice function $f = (x, t)$ by

$$x : \Theta \to X, x(\theta) = \left(\frac{1}{N} \sum_{i=1}^{N} \theta_i\right) \mathbf{1}, \tag{9a}$$
$$t : \Theta \to \mathbb{R}^n, t_i(\theta) = \sum_{j \neq i} (x_j(\theta) - \theta_j)^2, \tag{9b}$$

where $\mathbf{1}$ denotes the vector of all ones in $\mathbb{R}^N$. Then the sequence of mechanisms $\{M'_n\}_{n \in \mathbb{N}}$ provided in Algorithm 2 asymptotically implements f in ex-post Nash equilibria.

Proof: The proof follows the same line of reasoning as in the proof of Theorem 1. ∎

Remark 2: As we have described in Algorithm 2, the agents need to solve $x_i = \arg\min_{z \in \mathbb{R}} (z - \theta_i)^2 + p^T R_i z$. This optimization problem has a explicit solution $x_i = \theta_i - 0.5 p^T R_i$. Therefore, at each iteration, the agents only need to apply a simple linear update rule, calculate the new cost, and send this information to the leader.

Note that $\{M'_n\}_{n \in \mathbb{N}}$ is not the only sequence of mechanisms that asymptotically implements the average consensus. In order to show this, we define the sequence of mechanisms

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Algorithm 3 Distributed mechanism $M_n'' = (g^n, \Sigma^n, t^n)$ for asymptotically implementing the average consensus.

Input: Problem data
Output: Social outputs $\tilde{x}, \tilde{t}_1, \ldots, \tilde{t}_N$
   (L) Set $\alpha \in (0, 1/d_{\max})$ (where $d_{\max}$ denotes the maximum degree of the vertices in $\mathcal{G}$) and broadcast it;
   (F) Initialize $z_i^0 = \theta_i$ for each $1 \le i \le N$;
   for $k = 1, \ldots, n$ **do**
       if $k \le n - 1$ **then**
           (F) Calculate $z_i^k = z_i^{k-1} + \alpha \sum_{j \in \mathcal{N}_i} (z_j^{k-1} - z_i^{k-1})$, where $\mathcal{N}_i$ is the set of all neighbors of vertex i in $\mathcal{G}$, and transmit it to the leader;
       else if $k = n$ **then**
           (L) Determine and broadcast the social decision $\tilde{x} = z_1^k \mathbf{1}$;
           (F) Compute and transmit $v_i = (\tilde{x}_i - \theta_i)^2$;
           (L) Determine the tax values $\tilde{t}_i = \sum_{j \ne i} v_j$;
       end if
   end for

$\{M_n''\}_{n \in \mathbb{N}}$ using Algorithm 3 and show that this sequence indeed asymptotically implements the average consensus. Again, Algorithm 3 suggests followers' strategies $s^n(\cdot)$ and a social output function $g^n = (\tilde{x}^n, \tilde{t}^n)$.

Proposition 2: Define a social choice function $f = (x, t)$ by (9). Then the sequence of mechanisms $\{M_n''\}_{n \in \mathbb{N}}$ provided in Algorithm 3 asymptotically implements f in ex-post Nash equilibria.

Proof: The proof follows the same line of reasoning as in the proof of Theorem 1. The only difference is that the convergence of the mechanism is guaranteed by [16]. ∎

VII. Conclusions

In this paper, we presented a framework for faithful implementation of distributed algorithms in a network of strategic agents. We introduced the notion of asymptotic incentive compatibility for a sequence of mechanisms, that is, this sequence provides every follower a diminishing incentive to deviate from the suggested slave algorithm. We proposed a tax mechanism, inspired by the classical VCG mechanisms, to asymptotically implements a social choice function in ex-post Nash equilibria.

In our earlier study [8], [9], we proposed a real-time electricity pricing scheme that incentivises strategic consumers/generators over the power grid to implement the socially optimal control action. We assumed that there was no private information so that leader could compute the socially optimal control. The motivation of introducing a payment mechanism in that scenario was not to induce followers to reveal faithful information as in the classical mechanism design problems but to induce them to take the intended control actions. In that paper, the payment mechanism was also strongly inspired by the VCG mechanism, but due to this difference, the connection to the classical mechanism design setting was not transparent.

Notice that the formulation of the distributed mechanism introduced in this paper contains both the classical mechanism design problem and the pricing scheme [8], [9] as special cases. The classical mechanism design problem corresponds to the single step case ($n = 1$) in (3), while

the pricing scheme in [8], [9] corresponds to the case where there is no private information (Θ is a trivial singleton set).

Moreover, the distributed mechanism for dual decomposition algorithm considered in this paper suggests a better implementation of the real-time pricing scheme. In our earlier study, the leader (the central computer) needs to solve a large scale optimal control problem in the centralized manner. Using the idea of dynamic dual decomposition (e.g., [17]) combined with the VCG-like tax mechanism (6), computation can be faithfully parallelized. Furthermore, since the current framework is built on non-singleton type space Θ, it allows strategic power generators/consumers to have private information. More details will be explored in our future work.

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