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Evolutionary Computation for Model Order Reduction with Parametric Generalised SPA

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***Abstract*— In this paper evolutionary computation algorithms are applied to select optimal parameters in model order reduction for linear systems. In particular a parameterized set of reduced order model is obtained by using a Parametric Generalised Singular Perturbation Approximation of a balanced realization. The optimization algorithm is then used to select the parameter set that minimize a suitable performance index. Numerical examples are reported in comparison with other model order reduction methods.**

I. Introduction

Model order reduction is very relevant to obtain simpler models for simulations of complex systems and for the design of lower order controllers [3]. Several approaches have been proposed in literature for the approximation of linear dynamical systems. Balanced realizations introduced by Moore [2] and then extended by many other researchers [3] are based on a basis transformation which allows to separate the state variables selecting those difficult to control and to observe. The reduced models are then usually obtained by using Direct Truncation (DT) or Singular Perturbation Approximation (SPA) [4]. In [5] and [6] the authors introduced the Parametric Generalised Singular Perturbation Approximation (PGSPA) as a generalization of DT and SPA for reducing the order of a balanced system. PGSPA represents a further generalization of Generalised Singular Perturbation Approximation (GSPA) introduced in [7] and further studied in [8], [9] and [10], by proposing a parameterization which allow to weight separately each of the state variables that can be neglected. Moreover it is shown that most of the properties of direct truncation valid for the balanced realizations are still true. In particular the a priori bound on the H_∞ norm of the error is the same as direct truncation for the whole range of variation of the parameters. One open problem was the choice of such free parameters in order to satisfy a desired criterion.

In this paper the optimal choice of the parameters is tackled by using a genetic algorithm for optimization of a suitable performance index. This problem is typically a non-convex one thus justifying the adoption of evolutionary algorithms.

In particular the H_∞ norm of the error between the original system and the reduced one is considered in the paper. Non-Convex optimization algorithms have found several applications in controller and model reduction. For example in [12] randomized algorithms have been used for reduced order H_∞ controller design. Genetic algorithms [11] have been proposed in [13], [14], [23] and [24]. In [18] it has also been shown that a genetic algorithm works better than a randomized algorithm. Such an aspect open new perspectives to genetic algorithm as a valid alternative to randomized algorithms for the solution of many control problems [19].

In the next sections a short introduction to balanced realizations and PGSPA will be reported. Then the proposed approach will be described and validated through numerical examples.

II. PGSPA of Internally Balanced Realizations

Given a minimal realization S of an asymptotically stable matrix transfer function:

$$G(s) = C\left(sI - A\right)^{-1} B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

where $A \in \Re^{n\times n}, B \in \Re^{n\times m}, C \in \Re^{p\times n}$ and $D \in \Re^{p\times m}$; the realization is said to be *internally balanced*, when the so-called *Controllability and Observability Gramians*:

$$AH_C + H_C A^T + BB^T = 0 \tag{2}$$

$$A^T H_O + H_O A + C^T C = 0 \tag{3}$$

are equal and diagonal:

$$H_C = H_O = \Sigma = diag\left\{\sigma_1 I_{k_1}, \sigma_2 I_2, \ldots, \sigma_f I_{k_f}\right\} \tag{4}$$

where $\sigma_1 > \sigma_2 > \ldots \sigma_f > 0$ are the *Hankel singular values* of the system with multiplicity $k_1, k_2, \ldots, k_f$ respectively and $n = k_1 + k_2 + \ldots + k_f$ is the dimension of the state vector.
Consider now a partition of such realization:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

where $x_1 \in \Re^r, x_2 \in \Re^{n-r}$ with $r = k_1 + k_2 + \ldots + k_a$; $a < f$, $u(t) \in \Re^m$ is the control vector and $y(t) \in \Re^p$ is the output vector.

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In order to parameterize the reduced order systems the Parametric Generalised Singular Perturbation Approximation has been introduced [6].

PGSPA consists in applying the basic approximation equation:

$$\dot{x}_2 = Px_2 \qquad (6)$$

where P is defined as:

$$P = diag\left\{p_{a+1}I_{k_{a+1}}, p_{a+2}I_{k_{a+2}}, \dots, p_f I_{k_f}\right\} \qquad (7)$$

with

$$p_{a+1}, p_{a+2}, \dots p_f \geq 0 .$$

In the general case the parameters of the approximation are f-a, but in most systems when the singular values are distinct i.e. $k_1 = k_2 = \ldots = k_f = 1$, the number of parameters correspond to n-r that is the number of the states eliminated.

It can observed that when:

$$p_{a+1} = p_{a+2} = \ldots = p_f = 0 \qquad (8)$$

the SPA is obtained, while when:

$$p_{a+1} = p_{a+2} = \ldots = p_f \Rightarrow \infty \qquad (9)$$

the DT is obtained.

The reduced order model is then:

$$\begin{aligned} &G_P(s) = \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI_r - A_{11} & -A_{12} \\ -A_{21} & P - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D = \\ &= \overline{C}(P)\left(sI_r - \overline{A}(P)\right)^{-1}\overline{B}(P) + \overline{D}(P) := \\ &:= \left[\begin{array}{c|c} \overline{A}(P) & \overline{B}(P) \\ \hline \overline{C}(P) & \overline{D}(P) \end{array}\right] = S_P \end{aligned} \qquad (10)$$

where:

$$\begin{cases} \overline{A}(P) = A_{11} + A_{12}(P - A_{22})^{-1}A_{21} \\ \overline{B}(P) = B_1 + A_{12}(P - A_{22})^{-1}B_2 \\ \overline{C}(P) = C_1 + C_2(P - A_{22})^{-1}A_{21} \\ \overline{D}(P) = D + C_2(P - A_{22})^{-1}B_2 \end{cases} \qquad (11)$$

Remark 1: Observe that if

$$p_{a+1} = p_{a+2} = \ldots = p_f = \sigma_0$$

then

$$G_P(\sigma_0) = G(\sigma_0)$$

i.e. the approximated model and the full order one are equal in $s = \sigma_0$.

Remark 2: The PGSPA can be viewed as an upper LFT representation [21], as shown in Figure 1.

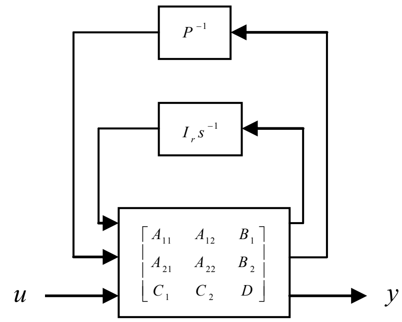

Figure 1. Upper LFT representation of the PGSPA.

The following Theorem, demonstrated in [6], guarantees that the classical error bound of DT and SPA introduced in [17] and [4] are still valid for PGSPA.

Theorem 1: Given an internally balanced realization S of an asymptotically stable system with transfer matrix $G(s)$ and $f-a$ parameters $p_{a+1} \geq 0$, $p_{a+2} \geq 0, \ldots, p_f \geq 0$, then the reduced order system S_P computed with PGSPA is asymptotically stable and satisfies the bound:

$$\left\| G(s) - G_P(s) \right\|_\infty \leq 2\sum_{i=a+1}^{f} \sigma_i \qquad (12)$$

Proof: See [6].

Remark 3 . It should be observed that this method represents a wide generalization of the classical balanced truncation, since when all Hankel singular values are distinct there are now $f-a$ free parameters that can be chosen while maintaining the main properties of the other reduction methods.

Remark 4. Such an extension can be also applied to the other reduction method based on balancing with equivalent results, such as closed-loop balanced realizations, fractional balanced realizations, bounded real balancing, stochastic balancing, etc.

This parameterization gives the designer a wider class of approximated models, where the parameters can be selected by using a suitable performance index and optimization strategy.
In this paper therefore we propose to minimize the index:

$$J(P) = \left\| G(s) - G_P(s) \right\|_\infty \qquad (13)$$

by using a genetic algorithm.

III. Approximation By Using Genetic Algorithms

As it was previously observed, when the Hankel Singular values are distinct, the number of parameters that have to be selected is equal to the number of state variables approximated. In the case of a large scale system or an high order controller, this number could be high. Such a fact from one side increases the degrees of freedom in the selection of the reduced order system suitable for our optimization criteria, but from the other make the optimization procedure harder.

Among optimization methods for non-convex problems Genetic algorithms belong to the wider class of evolutionary algorithms [11], [22]. The optimization problems are solved using an approach inspired by natural evolution.

In this paper a classical genetic algorithm has been used with the following main optimization parameters:

- Initial population initialized with uniformly distributed random function;
- Stop criteria: Maximum number of generations=300;
- Double Vector codification of individuals;
- Scattered Cross-Over function with crossover fraction=0.8;
- Mutation performed by adding a random number taken from a gaussian distribution to each entry of the parent vector;
- Selection is performed with a stochastic uniform distribution;
- The number of elite individuals selected to survive to the next generation is 2.

The following examples show a comparison between the reduction obtained with classical methods and that obtained by using the proposed approach.

A. Example 1.

Consider a 7th-order system whose transfer function is:

$$G(s) = \frac{n(s)}{d(s)} \tag{13}$$

where:

$n(s) = s^7 - 5.4333\, s^6 + 16.6225\, s^5 - 48.8318\, s^4 + 67.9157\, s^3 + 137.5551\, s^2 - 44.5699\, s + 767.7631$

and

$d(s) = s^7 + 2.5667\, s^6 + 20.4010\, s^5 + 35.9485\, s^4 + 118.8027\, s^3 + 114.4154\, s^2 + 154.9417\, s + 69.7966.$

G(s) has been obtained by computing an internally open loop balanced system whose Hankel singular values have been randomly chosen. A reduction to a second order system was performed following several different methods and the results are reported in TABLE I. The table reports also the theoretically lower achievable bound [17].

In particular for the optimization with the genetic algorithm 300 generations of 40 individuals each have been used. In Figure 2. the value of the optimized H_∞ norm of the error with respect to the generations is reported.

B. Example 2.

A 12th-order model of a flexible structure is considered:

$$G(s) = \sum_{i=1}^{n/2} \frac{b_i}{s^2 + 2\xi_i \omega_{n_i} s + \omega_{n_i}^2} \tag{14}$$

The parameters of which are given in TABLE II.

Also in this case several different reduction methods were performed. The comparison of the H_∞ norm of the error for the different methods and for different reduction orders is reported in TABLE III. The frequency domain errors between *G(s)* and four different fourth order reduced models are shown in Figure 3. Also in this case it can be observed how the genetic algorithms quickly find parameters values which improve the approximation and always get the best approximation error.

TABLE I. RESULTS FOR EXAMPLE 1

Method	*H_∞ norm of the error*
DT	8.6076
SPA	10.6384
Hankel	8.2949
Genetic PGSPA	6.015
Lower bound σ_3	5

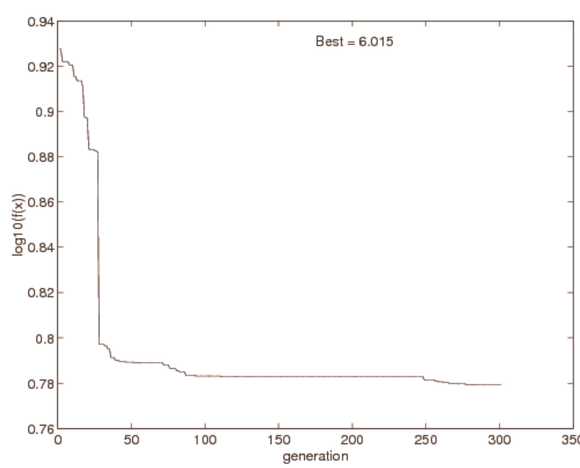

Figure 2. $Log_{10}(J)$ versus the number of generations for example 1.

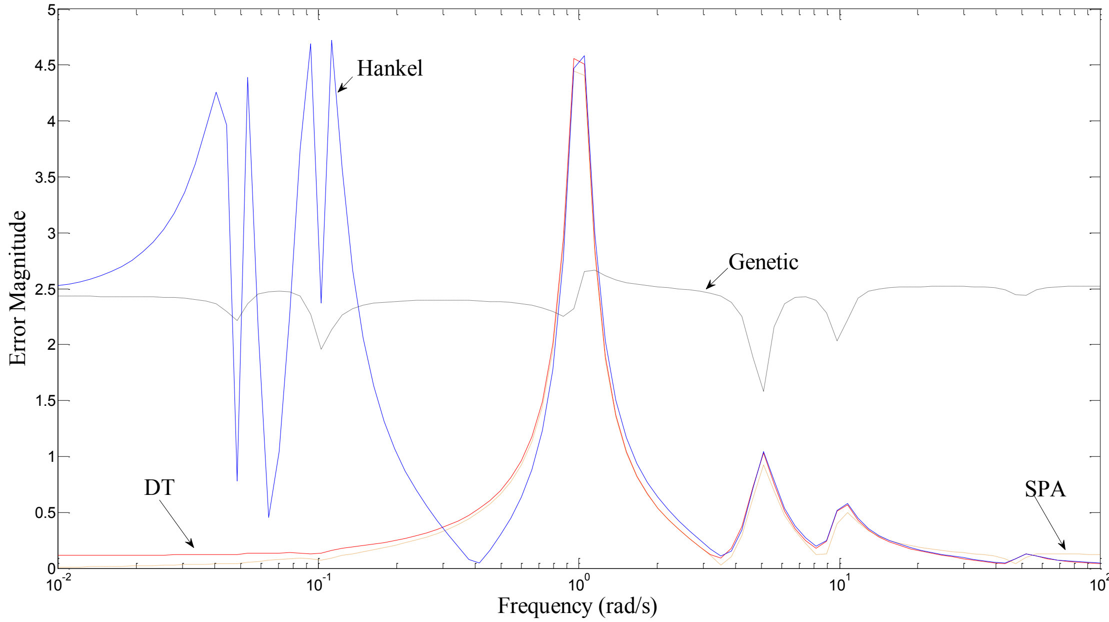

Figure 3. Frequency errors for different 4th order reduced models for example 1.

TABLE II. PARAMETERS FOR EXAMPLE 2.

i	b_i	ξ_i	ω_{n_i}
1	1	0.1	0.05
2	1	0.1	0.1
3	1	0.1	1
4	1	0.1	5
5	1	0.1	10
6	1	0.1	50

TABLE III. H_∞ NORM OF THE ERROR FOR DIFFERENT METHODS FOR EXAMPLE 2.

r	*DT*	*SPA*	*Optimal Hankel*	*Genetic PGSPA*	*Lower bound* σ_{r+1}
8	0.4790	0.4501	0.4756	0.3468	0.2409
7	0.9976	0.9712	0.9590	0.7798	0.4793
6	1.0116	0.9712	1.0134	0.5124	0.4801
5	4.9071	4.8848	5.0608	4.1044	2.4435
4	5.0012	4.8849	5.0022	2.6851	2.4817
3	45.3597	45.3123	44.3448	43.5632	22.6802
2	48.1208	45.6235	47.7886	30.9543	24.1961

C. Example 3.

This example, taken from [20], considers the reduction of a model of a building. The building is the Los Angeles University Hospital, with eight floors, each with three degrees of freedom - two displacements in x and y and one rotation. A 48-state model was adopted to represent the input-output relationship for any one of these displacements. In particular each state represents a displacement or its velocity.

The results of the reduction to different order model are reported in TABLE IV. In particular for the optimization with the genetic algorithm for the 10th order reduced model, 51 generations of 20 individuals each have been used. The values of the optimized parameters p_i are reported in Figure 4. Also in this case the genetic algorithm always gets the best solution.

TABLE IV. H_∞ NORM OF THE ERROR FOR DIFFERENT METHODS FOR EXAMPLE 3.

Order of the reduced model	DT	SPA	Genetic PGSPA	*Lower bound* σ_{r+1}
15	0.000449	0.000462	0.000399	0.000212
10	0.000603	0.000529	0.000482	0.000273
5	0.001575	0.001579	0.00126	0.000703

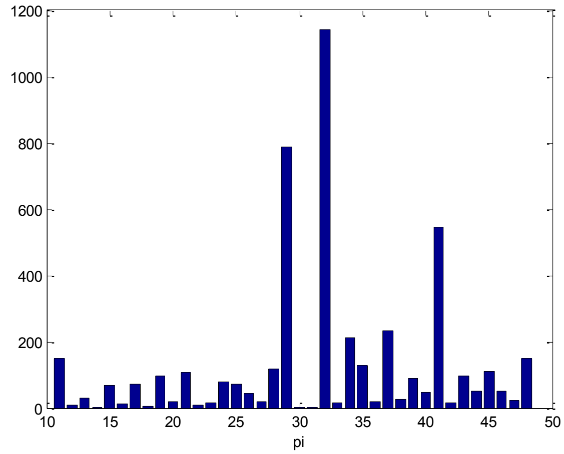

Figure 4. Values of the p_i parameters for the 10th order reduced model for Example 3..

IV. CONCLUSION

In the paper it was shown how the adoption of a Genetic algorithm with a Parametric Generalised Singular Perturbation Approximation can be a useful tool for the solution of model reduction problems.

Three examples have been shown applied to balanced realizations, but the same techniques can be used to other balancing methods as closed-loop balancing, bounded real balancing, multiplicative-model reduction, etc. Further studies are in progress to apply such a method to weighted model reduction procedures in order to solve Controller reduction problems and to compare the results with other approaches.

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