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A Non-Progressive Model of Innovation Diffusion in Social Networks

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Abstract— **In this paper we deal with the diffusion of innovation in social network, presenting a non progressive instance of the well known Linear Threshold model. Each individual in the social network is influenced by the behaviour of its neighbours, and at each steps it decides either to adopt, abandon or maintain the innovation by following a threshold mechanism. We assume that the innovation is incepted in the network by a seed set of individuals which are assumed to maintain the innovation independently of the state of their neighbours for a finite time. Here we describe in details the evolution of the system depending on different initial conditions, and in particular we focus on how the system dynamics are influenced by the cohesive sub-groups of the network.**

I. Introduction

Social Network Analysis is an area that is attracting the interest of researchers coming from different communities, such as economists, sociologists, anthropologists, computer scientists and control engineers. Many mathematical models have been proposed to describe typical phenomena of social networks such as the opinion diffusion, social influence or epidemic diseases. Most of the models proposed in literature use graphs to represent a social networks: nodes represent individuals while edges represent the connections between individuals. In our paper the terms *node* and *individual* are used interchangeably.

Here we focus on the diffusion of innovation in social networks, i.e., the process by which individuals are persuaded to adopt an innovation influenced by the behaviours of their social contacts. Since the 40's, many mathematical models on the diffusion of innovation has been proposed ([1], [2]): the Linear Threshold Model, the *Independent cascade model* ([3]) and epidemic models such as SIS and SIR ([4], [5]). Many of these models are based on the *threshold effect*: an individual adopts a behaviour if a certain ratio of its social contacts have already adopted it, differently from the epidemic models in which a node adopts a behaviour with a certain probability if at least one of its neighbours has adopted it. Threshold models are more suitable to describe social influence phenomena and individual behaviours, while epidemic models are more used for mass behaviours. Examples of threshold models are presented in [6], [7], [8].

In this paper we consider the *Linear Threshold Model*, which was introduced in [6] and has been widely studied in literature ([9], [10], [11]).

In ([11]) an instance of the Linear Threshold Model has been discussed, in which the innovation spreads in the network starting from seed set, i.e., a set of individuals who have originally adopted the innovation. When a node adopts the innovation we say that it becomes *active*, otherwise is said to be *inactive*. A threshold value is assigned to each node, and a non-active node becomes active as soon as the fraction of its neighbours which are active is greater than or equal its threshold. It is assumed that a node can switch its state from inactive to active but cannot switch it from active to inactive. This model can be used to represent systems in which the adoption of a innovation is permanent and in the literature is called *progressive* ([12], [13]).

In many cases, however, the progressive model is not suitable to correctly describe the spread of innovation, as habits may change: an individual who votes for a party for a period can decide to change its preference, a person who eats every day at the same restaurant can be persuaded to change of venue. Moreover, the influence pattern in real networks is usually time-varying, as the human connections are subjected to changes: friendships can become stronger or weaker due to the passing of time, new connections can be setted up and old connections can be removed. All these changes in the network can influence the spread of the innovation, and in such systems an individual who has adopted the innovation can be persuaded to abandon it. Such types of mechanisms can be described using *non-progressive* models, in which each individual periodically updates its state by looking at its neighbours, deciding either to be active or inactive.

In this paper we present a non-progressive instance of the linear threshold model which can be considered as a generalization of the model presented in [11]. We assume that the innovation is incepted in the network by a seed set, and the seed nodes are supposed to maintain the innovation for a finite time - the seeding time -, after which they start to update their state by following the same rules adopted by all the other nodes in the network.

An important part of our discussion in this paper is focused on the social cohesion, which is considered a key aspect to understand collective behaviours in social networks. Many definitions of cohesiveness have been proposed in literature, and good surveys can be found in [14], [15]. Here we use

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two types of cohesive subgroups, namely the cohesive and persistent sets, to characterize the system.

We characterize the system evolution in two different phases: during and after the seeding time. We show that during the seeding time the system behaves as in the progressive model in [11]. The main contribution of our work is the analysis of the system evolution after the seeding time, which represent the main difference between our model and the previously ones presented in literature, as in this phase non-progressive mechanisms may occur. We use cohesive groups to characterize some conditions under which such mechanisms take place.

The paper is organized as follows. In section III we introduce the *non-progressive linear threshold model*, formalizing the used notation, the main assumptions and the adopting conditions. In section IV we define and characterize the *persistent sets* with respect of the presented model. Finally, in section V we analyse how the innovation spreads in a social network according to the non-progressive linear threshold model, and we confirm the analytical results through some numerical examples.

II. Background

Let us represent a social network with a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where each node $i \in \mathcal{V}$ represents an individual and each edge $(i, j) \in \mathcal{E}$ denotes that node i influences node j. We denote as $n = |\mathcal{V}|$ the number of individuals in the network. No self-loops are allowed, thus $(i, i) \notin \mathcal{E},\ \forall i \in \mathcal{V}$. For all nodes $i \in \mathcal{V}$ we denote as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{V}\}$ the set of the *in-neighbours*. A weight $w_{ij} \in [0, 1]$ is associated to each edge $(i, j) \in \mathcal{E}$ and denotes how much node i influences node j. We assume that for all $i \in \mathcal{V}$ it holds: $\sum_{j \in \mathcal{N}_i} w_{j,i} = 1$.

III. Non-progressive linear threshold model

In this section we introduce a non-progressive instance of the linear threshold model. Firstly we list the assumptions on which the model is based, then we define the update rule. For the rest of the paper we refers to this model as the *non-progressive linear threshold model*.

A. System description

A threshold value $\lambda_i \in [0, 1]$ is associated to all nodes $i \in \mathcal{V}$. We assume that the independent variable time t belongs to $\mathbb{N}$. The innovation spreads in the network starting from a seed set ϕ_0, i.e., a set of individuals are active at time $t = 0$. We assume that all the nodes in ϕ_0 are active for a time interval $t \in [0, T_s]$, independently of the state of their neighbours, then for $t > T_s$ they update their state following the same rule as the rest of the nodes. We call T_s the *seeding time*.

We assume that:

- the topology of the network is static and all the connections and the influence weights are known;
- the thresholds $\lambda_i, \forall i \in \mathcal{V}$ are static and known;
- a node can be more influenced by some neighbours than others, thus for each node the weights of the in-edges may be different.

B. Update rule

Let Φ_t be the set of active nodes at time t. In the non-progressive linear threshold model the nodes update their states at time t according to the following equation:

$$\Phi_t = \begin{cases} \phi_0 & t = 0 \\ \phi_0 \bigcup \{i \mid \sum_{j \in (\mathcal{N}_i \cap \Phi_{t-1})} w_{ji} \geq \lambda_i\}, & t \in [1, T_s] \\ \{i \mid \sum_{j \in (\mathcal{N}_i \cap \Phi_{t-1})} w_{ji} \geq \lambda_i\}, & t > T_s \end{cases} \tag{1}$$

In other words, after the seeding time a node is active at time t if the sum of the weights of the in-edges coming from active neighbours at time $t-1$ is greater than or equal to its threshold. Differently from the progressive model, in which a node maintains the innovation indefinitely once adopted, in the non-progressive model a node can switch its state from inactive to active and vice versa.

Additional notation that will be used in the rest of the paper is the following.

- $\phi_t^+ = \Phi_t \setminus \Phi_{t-1}$, i.e., the set of nodes which become active at time t;
- $\phi_t^- = \Phi_{t-1} \setminus \Phi_t$, i.e., the set of nodes which become inactive at time t;
- $\Phi^* = \lim_{t \to +\infty} \Phi_t$ denotes if it exists, the set of final adopters.

Note that the set Φ^* does not always exist. The existence of this set will be discussed in section V.

IV. Cohesive and Persistent Sets

In this section we define two types of cohesive groups in the non-progressive linear threshold model, which are useful to analyse the spread of innovation in the network. We firstly adapt to our model the concept of *cohesive sets* as presented in [11]. Then we introduce the idea of *persistent sets*, which describe a different type of coherence with respect to cohesive sets.

Definition 1 (Cohesive set ([11])) *A set X is cohesive if for all nodes $i \in X$ the sum of the weights of the in-edges coming from nodes which are not in X is lower than their threshold λ_i, i.e.:*

$$\forall i \in X, \qquad \sum_{j \in (\mathcal{N}_i \cap X)} w_{ji} > 1 - \lambda_i. \tag{2}$$

An important property of a cohesive set, proved in [11], is that if none of the nodes within the set is active at time t, then none of them can become active for all $t' > t$. In Figure 1 the sets $\{1, 2, 3\}$ and $\{8, 9\}$ are cohesive, while $\{4, 5, 6, 7\}$ is not cohesive.

Definition 2 (Persistent set) *A set X is persistent if for all nodes $i \in X$ the sum of the weights of the in-edges coming from nodes within X is greater than or equal their threshold λ_i, i.e.:*

$$\forall i \in X, \qquad \sum_{j \in (\mathcal{N}_i \cap X)} w_{ji} \geq \lambda_i. \tag{3}$$

The following theorem points out the reason why such type of sets are important in the non-progressive linear threshold model.

Theorem IV.1 *Let X be a persistent set. If at time t' all the nodes in X are active, then they remain active for all $t > t'$.*

Proof: If all nodes in X are active at time t', i.e., $X \subseteq \Phi_{t'}$, from (3) follows that

$$\forall i \in X, \qquad \sum_{j \in (\mathcal{N}_i \cap \Phi_{t'})} w_{ji} \geq \sum_{j \in (\mathcal{N}_i \cap X)} w_{ji} \geq \lambda_i.$$

hence $X \subseteq \Phi_{t'+1}$. The result follows by recursion. □

Property IV.2 *Let X_1 and X_2 be two persistent sets. The set $X_1 \cup X_2$ is a persistent set as well.*

Proof: As X_1 is persistent, each node i in X_1 satisfies equation (3). As $X_1 \subseteq X_1 \cup X_2$ it holds for $k = 1, 2$:

$$i \in X_k, \qquad \sum_{j \in (\mathcal{N}_i \cap (X_1 \cup X_2))} w_{ji} \geq \sum_{j \in (\mathcal{N}_i \cap X_k)} w_{ji} \geq \lambda_i.$$

Thus all the nodes in $X_1 \cup X_2$ satisfy equation (3), i.e., $X_1 \cup X_2$ is a persistent set. □

In Figure 1 the sets $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$ are persistent, while $\{3, 4\}$ is not persistent. We conclude this section by observing that a set can be both cohesive and persistent, e.g., the set $\{1, 2, 3\}$.

V. SYSTEM'S DYNAMIC

The purpose of this section is to characterize how the innovation spreads in the network according to the non-progressive model. We analyse separately two different phases of the evolution in the network:

- during the seeding time, i.e. for $0 \leq t \leq T_s$;
- after the seeding time, i.e., for $t > T_s$.

We pay particular attention to the evolution of the innovation after the seeding time: which are the nodes that are able to hold their states active after T_s?

We use the following definitions to describe the evolution of the innovation in the network according to the presented model.

Definition 3 (Progressive evolution) *The diffusion of the innovation in the network is* progressive *(or* non-decreasing*) during a time interval $[t_1, t_2]$ if:*

$$\forall t \in [t_1, t_2], \qquad \phi_t^- = \emptyset.$$

In other words, for all $t \in [t_1, t_2]$ all active nodes $i \in \Phi_{t-1}$ remain active at time t. If $t_1 = t_2 = t'$, we said that the evolution is progressive in t' if $\phi_{t'}^- = \emptyset$.

Definition 4 (Non-progressive evolution) *The diffusion of the innovation in the network is* non-progressive *during a time interval $[t_1, t_2]$ if:*

$$\exists t \in [t_1, t_2], \qquad \phi_t^- \neq \emptyset.$$

In other words, during the time interval $t \in [t_1, t_2]$ there is at least a node which becomes inactive.

Definition 5 (Degressive evolution) *The diffusion of the innovation in the network is degressive (or* non-increasing*) during a time interval $[t_1, t_2]$ if:*

$$\forall t \in [t_1, t_2], \qquad \phi_t^+ = \emptyset.$$

Definition 6 (Periodic evolution) *The diffusion of the innovation in the network is periodic after time t if there exist a $T > 0 \in \mathbb{N}$ such that:*

$$\forall k \in \mathbb{N}, t' \geq t \qquad \Phi_t' = \Phi_{t'+kT}.$$

where T is the period of the evolution.

The definitions of progressive and degressive follow the usual definitions in literature. Note that an evolution can be both progressive and degressive if the set of active nodes is constant. In the following parts we prove analytically the following results:

(a) during the seeding time the system has a progressive evolution;
(b) after the seeding time the evolution of the system is progressive if Φ_{T_s} is persistent, otherwise is non-progressive;
(c) if T_s is sufficiently large (larger than a parameter T_d called diffusion time and introduced in the following section) two results holds: a) the set of final adopters Φ^* exists and is the maximal persistent set in Φ_{T_s}; b) if Φ_{T_s} is not persistent the system has a degressive evolution for $t > T_s$.

Examples of evolutions, including a case in which the system has a periodic evolution, are given in the final subsection.

A. Evolution during the seeding time: $0 \leq t \leq T_s$

In this part we prove that in the non-progressive model, according to the assumptions made so far, during the seeding time $[0, T_s]$ the system has a *progressive evolution*.

Theorem V.1 *The evolution of a social network with seed set ϕ_0 and seeding time T_s is progressive in the time interval $[0, T_s]$.*

Proof: We prove the statement by induction on the time step t, assuming $T_s \geq 1$ (if $T_s = 0$ the result is trivial).

(base step) At time step $t = 1$, the evolution is progressive because by equation (1) $\Phi_0 = \phi_0 \subseteq \Phi_1$, hence $\phi_1^- = \emptyset$.

(inductive step) Assume that at time step $t-1$ (where $t \in [2, T_s]$) the evolution is progressive: we now show that the evolution is also progressive at time step t thus completing the proof.

Observe that the assumption $\phi_{t-1}^- = \emptyset$ implies $\Phi_{t-2} \subseteq \Phi_{t-1}$, hence for all $i \in \mathcal{V}$ holds:

$$\mathcal{N}_i \cap \Phi_{t-2} \subseteq \mathcal{N}_i \cap \Phi_{t-1}.$$

By (1) this implies that $\Phi_{t-1} \subseteq \Phi_t$, hence $\phi_t^- = \emptyset$. □

The previous analysis also points out that as long as the nodes of the seed set are active, no node in the network can become inactive, i.e., during the seed time a node, which is not in the seed set, adopts the innovation as soon as the sum of the weights of the in-edges coming from active nodes is greater than or equal its threshold value, and maintains it.

This behaviour is also typical of the progressive instance of the linear threshold model presented in [11]. Differently from our model, the progressive in [11] assumes that all in-edges at each node have the same weight, i.e., for all $i \in \mathcal{V}$ it holds:

$$w_{ji} = \frac{1}{|\mathcal{N}_i|}, \quad \forall j \in \mathcal{N}_i.$$

In the progressive model an inactive node i adopt the innovation at time t if at time $t-1$ it holds:

$$\sum_{j \in (\mathcal{N}_i \cap \Phi_{t-1})} w_{ji} = \frac{|\Phi_{t-1} \bigcap \mathcal{N}_i|}{|\mathcal{N}_i|} \geq \lambda_i \tag{4}$$

According to the previous equation, also in the progressive model a node adopts the innovation as soon as the sum of the weights of the in-edges coming from active nodes is above its threshold value, but differently from our non-progressive model an individual is assumed to never abandon the innovation once adopted. Thus we can claim that the non-progressive linear threshold model represents a generalization of the progressive model. In particular, the evolution of the progressive model corresponds to the evolution of the non-progressive model in case of $T_s \to \infty$.

We can exploit this similarity even further. We know from [11] that the progressive model reaches in a finite time a steady state where the set of active nodes remains constant and is:

$$\hat{\Phi}^* = \mathcal{V} - \mathcal{M}$$

, where $\mathcal{M}$ denotes the maximal cohesive set in the complement of the seed set.

Motivated by this, we define a parameter, the diffusion time, which will play an important role in the analysis of the evolution of the non-progressive model as will be shown in the following sections.

Definition 7 (Diffusion Time T_d) *For T_s sufficiently large the innovation spreads in the network until a time $T_d \leq T_s$ such that $\Phi_{T_d} = \Phi_{T_d+1} = \cdots = \Phi_{T_s}$. The parameter T_d is the* diffusion time *of the network.*

B. Evolution after the seeding time: $t > T_s$

At time $T_s + 1$ some nodes in the seed set may become inactive, as they may not satisfy equation (1). If that happens, at time $T_s + 2$ some active nodes connected to the seed set may become inactive, etc. Such a tendency to abandon the innovation leads to a *non-progressive evolution*.

In this section we characterize the evolution of our model after the seeding time and also present some particular results that hold in the special case $T_s < T_d$.

Lemma V.1 *Consider a social network with seeding time T_s. If there exists a time step $\bar{t} > T_s$ such that the evolution in $\bar{t}$ is progressive, then the evolution is also progressive for all $t > \bar{t}$.*

Proof: Observe that the assumption $\phi^-_{\bar{t}} = \emptyset$ implies $\Phi_{\bar{t}-1} \subseteq \Phi_{\bar{t}}$, hence for all $i \in \mathcal{V}$ holds

$$\mathcal{N}_i \cap \Phi_{\bar{t}-1} \subseteq \mathcal{N}_i \cap \Phi_{\bar{t}}.$$

By (1) this implies that $\Phi_{\bar{t}} \subseteq \Phi_{\bar{t}+1}$, hence $\phi^-_{\bar{t}+1} = \emptyset$. The result follows by recursion. □

The following theorem fixes the conditions under which the evolution of the system remains progressive for $t > T_s$.

Theorem V.2 *Consider a social network with seed set ϕ_0 and seeding time T_s. The evolution of the network is progressive for all $t > 0$ if and only if Φ_{T_s} is persistent.*

Proof: We prove separately the if and only if parts.

(if) For $0 \leq t \leq T_s$ it has been shown in Theorem V.1 that the network has a progressive evolution. If Φ_{T_s} is persistent, Theorem IV.1 implies that the evolution at time step $T_s + 1$ is progressive. From Lemma V.1 one concludes that the evolution is also progressive for all time steps $t > T_s + 1$.

(only if) If Φ_{T_s} is not persistent, by Definition 2 there exists a node $i \in \Phi_{T_s}$ such that $\sum_{j \in (\mathcal{N}_i \cap \Phi_{T_s})} w_{ji} < \lambda_i$. By (1) if follows that node i becomes inactive at step $T_s + 1$, hence the network has a non-progressive evolution. □

The following corollary points out that to determine if the system has a progressive evolution after T_s it is sufficient to determine if all nodes in the seed set remain active at time $T_s + 1$.

Corollary V.1 *The evolution of a social network with seed set ϕ_0 and a seed time T_s is progressive for all $t > 0$ if and only if at time $T_s + 1$ it holds: $\phi_0 \cap \phi^-_{T_s+1} = \emptyset$.*

Proof: Since $\phi_0 \cap \phi^-_{T_s+1} = \emptyset$ it holds

$$\phi_0 \subseteq \left\{ i \mid \textstyle\sum_{j \in (\mathcal{N}_i \cap \Phi_{T_s})} w_{ji} \geq \lambda_i \right\}$$

hence

$$\begin{aligned} &\phi_0 \bigcup \left\{ i \mid \textstyle\sum_{j \in (\mathcal{N}_i \cap \Phi_{T_s-1})} w_{ji} \geq \lambda_i \right\} \\ &\subseteq \left\{ i \mid \textstyle\sum_{j \in (\mathcal{N}_i \cap \Phi_{T_s})} w_{ji} \geq \lambda_i \right\} \end{aligned}$$

and by (1) this implies that $\Phi_{T_s+1} \subseteq \Phi_{T_s}$. The result follows from Lemma V.1. □

The following theorem points out a sufficient condition on the structure on the seed set under which the evolution of the system is progressive.

Theorem V.3 *Consider a social network with seed set ϕ_0 and seeding time T_s. If ϕ_0 is persistent, the evolution of the network is progressive for all $t > 0$.*

Proof: To prove this statement is sufficient to prove that if ϕ_0 is persistent, then Φ_{T_s} is persistent as well. We can consider Φ_{T_s} as:

$$\Phi_{T_s} = \phi_0 + \phi^+_1 + \phi^+_2 + \dots \phi^+_{T_s}$$

. Since ϕ_0 is persistent, it holds:

$$\phi_0 \in \Phi_{T_s+1}.$$

Since all the nodes in ϕ_0 are active at time T_s+1, it holds:

$$\phi_1^+ \in \Phi_{T_s+1}.$$

Using the same argument we can observe that:

$$\phi_2^+ \in \Phi_{T_s+1};\ \dots\ ;\phi_{T_s}^+ \in \Phi_{T_s+1}$$

. Thus it follows that:

$$\phi_{T_s+1}^- = \emptyset$$

and from Corollary V.1 it follows that the evolution is progressive for $t > 0$. □

We now present some results that apply to the special case in which $T_s \geq T_d$. If this condition holds, the progressive evolution during the seeding time reaches a steady state and $\Phi_{T_d} = \Phi_{T_d+1} = \dots = \Phi_{T_s}$.

Next theorem points out which are the nodes that remain active for all $t > T_d$.

Theorem V.4 *Let ϕ_0 be a seed set of a social network with a seed time T_s and diffusion time $T_d < T_s$. If $\Phi_{T_s} = \Phi_{T_d}$ is not persistent, then the system has a degressive evolution for $t > T_s$.*

Proof: The proof is based on verifying the following two facts.

(a) Firstly, we prove that if Φ_{T_s} is non-persistent, then $\phi_{T_s+1}^+ = \emptyset$ and $\phi_{T_s+1}^- \neq \emptyset$. Observe that if $\Phi_{T_s} = \Phi_{T_d}$ is not persistent it follows from Theorem V.3 that $\phi_{T_s+1}^- \neq \emptyset$. Moreover, as $T_s > T_d$, it holds that $\mathcal{V} - \Phi_{T_s} = \mathcal{M}$, where $\mathcal{M}$ is the maximal cohesive subset of the complement of the seed set. Thus no nodes can adopt the innovation at time T_s+1, i.e., $\phi_{T_s+1}^+ = \emptyset$.

(b) Secondly we prove that for all $t > T_s+1$ it holds $\phi_t^+ = \emptyset$. At time T_s+1 it holds $\Phi_{T_s+1} \subseteq \Phi_{T_s}$, thus according to equation (1) it holds $\phi_{T_s+2}^+ = \emptyset$. By the iteration of the same argument, for all $t > T_s+1$ it is: $\Phi_t \subseteq \Phi_{t-1} \quad \Leftrightarrow \quad \phi_{t+1}^+ = \emptyset$ □

Theorem V.5 *Let ϕ_0 be a seed set of a social network with seed time T_s and diffusion time $T_d < T_s$. The set Φ^* of active nodes for $t \to \infty$ is the maximal persistent set contained in Φ_{T_s} and is reached at time $T_f \leq T_s + |\Phi_{T_s}| - |\Phi^*|$.*

Proof: If the set of active nodes at step t is not persistent, there is at least one node in Φ_t that becomes inactive at step $t+1$. This, since the evolution is degressive according to Theorem V.4, the number of active nodes decreases at each step until the system reaches a persistent set of active nodes Φ^*, which is the maximal persistent set contained in Φ_{T_s}. The steady state is achieved from T_s in a number of steps which is at maximum $|\Phi_{T_s}| - |\Phi^*|$, thus:

$$T_f \leq T_s + |\Phi_{T_s}| - |\Phi^*|$$

. □

C. Some examples

In this section we consider social networks with seeding time T_s smaller than the diffusion time T_d because in this case several types of evolutions are possible, as opposed the networks with $T_s \geq T_d$ that we have shown can only admit degressive evolutions after the seeding time. We illustrate three different scenarios separately through examples.

Example V.6 (Scenario 1: progressive evolution) *Consider the network in Fig. 1 with seed set $\phi_0 = \{1, 2\}$ and seeding time $T_s = 2$. The diffusion time for the considered network is $T_d = 4$. As it is shown in Fig. 2, the evolution of the system is progressive. According to Theorem V.3 the progressive evolution can be predicted by observing that $\Phi_{T_S} = \Phi_2$ is a persistent set, as all the nodes that belong to it satisfy equation* (3)*. The set of final adopters exists and is $\Phi^* = \{1, 2, 3, 4, 5, 6, 7\}$.*

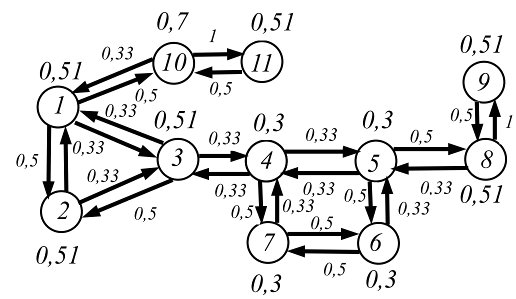

Fig. 1: Network in scenario 1.

t	Φ_t	ϕ_t^+	ϕ_t^-
0	$\{1, 2\}$		
1	$\{1, 2, 3\}$	$\{3\}$	$\emptyset$
2	$\{1, 2, 3, 4\}$	$\{4\}$	$\emptyset$
3	$\{1, 2, 3, 4, 5, 7\}$	$\{5, 7\}$	$\emptyset$
4	$\{1, 2, 3, 4, 5, 6, 7\}$	$\{6\}$	$\emptyset$
5	$\{1, 2, 3, 4, 5, 6, 7\}$	$\emptyset$	$\emptyset$

Fig. 2: Evolution in scenario 1.

Example V.7 (Scenario 2: non-progressive evolution) *Consider the network in Fig. 3 with seed set $\phi_0 = \{1, 3\}$ and seeding time $T_s = 1$. The diffusion time for the considered network is $T_d = 3$. As it is shown in Fig. 4, the evolution of the system is non-progressive. The set of final adopters exists and is $\Phi^* = \emptyset$.*

The numerical results confirm the analytical result obtained in Theorem V.3: as the set Φ_{T_s} is non-persistent, the system has a non-progressive evolution.

The next example represent a case in which the evolution of the system is periodic after T_s. This is a particular, but interesting, case of non-progressive evolutions but so far

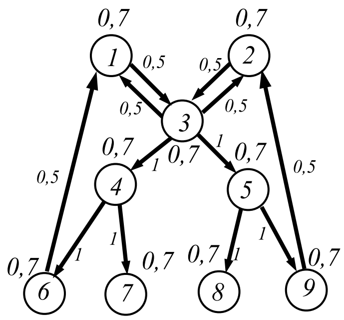

Fig. 3: Network in scenario 2.

t	Φ_t	ϕ_t^+	ϕ_t^-
0	$\{1,2\}$		
1	$\{1,2,3\}$	$\{3\}$	$\emptyset$
2	$\{3,4,5\}$	$\{4,5\}$	$\{1,2\}$
3	$\{4,5,6,7,8,9\}$	$\{6,7,8,9\}$	$\{3\}$
4	$\{6,7,8,9\}$	$\emptyset$	$\{4,5\}$
5	$\emptyset$	$\emptyset$	$\{6,7,8,9\}$

Fig. 4: Evolution in scenario 2.

we have not found any analytical characterization of this behavior.

Example V.8 (Scenario 3: periodic evolution.) *Consider the network in Fig. 5 with seed set $\phi_0 = \{1,3\}$ and seeding time $T_s = 1$. The diffusion time for the considered network is $T_d = 2$. As it is shown in Fig. 6, the evolution of the system is non-progressive after T_s, as the set Φ_{T_s} is non-persistent. Moreover, the system has a periodic evolution with period $T = 2$ from $t = 2$. In this case the set Φ^* cannot be defined.*

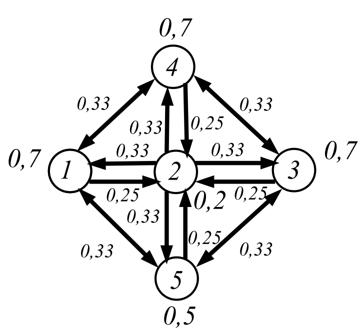

Fig. 5: Network in scenario 3.

VI. Conclusions

In this paper we have presented a non-progressive instance of the linear threshold model, in which the diffusion of the innovation starts from a seed set whose nodes are assumed to maintain the innovation for a finite time. We

t	Φ_t		
0	$\{1,3\}$	ϕ_t^+	ϕ_t^-
1	$\{1,2,3,5\}$	$\{2,5\}$	$\emptyset$
2	$\{2,4,5\}$	$\{4\}$	$\{1,3\}$
3	$\{1,2,3\}$	$\{1,3\}$	$\{4,5\}$
4	$\{2,4,5\}$	$\{4,5\}$	$\{1,3\}$
5	$\{1,2,3\}$	$\{1,3\}$	$\{4,5\}$
6	$\{2,4,5\}$	$\{4,5\}$	$\{1,3\}$

Fig. 6: Evolution in scenario 3.

characterized analytically the conditions under which the system has a progressive, non-progressive and degressive evolution. This model represents a first step in the analysis of non-progressive mechanisms dealing with the linear threshold model. In our future works we want to extend the presented model by exploring other mechanisms which can lead the network to a non-progressive evolution, such as changes in the network topology or in the influence weights. Furthermore we also plan to characterize the set of final adopters when $T_s < T_d$ and to find some conditions on the graph structure to characterize the evolution on the network.

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