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Extremum Seeking for Multi-Population Games

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Abstract— **This paper introduces novel schemes of continuous deterministic extremum seeking controllers based on multi-population games, designed for the solution of multi-constrained optimization problems on dynamical systems with a multi-agent system (MAS) structure. In this way, we consider different cluster of agents, interacting by means of different cost functions, which in general depend on the states of all the other agents. The agents of the same cluster aim to simultaneously maximize their common cost function, whose mathematical form is unknown, and which is only accessible by measurements. The optimization is carried out under different types of constraints: coupled or decoupled among clusters, describing multi-resource allocation problems and market share competition problems. The implementation of the algorithms is illustrated via simulations.**

I. INTRODUCTION

Extremum seeking control (ESC) has emerged as an adaptive control designed to achieve real-time optimization of general dynamical systems whose mathematical form is unknown or poorly known. The first rigorous stability analysis of ESC was presented in [1], based on classic averaging and singular perturbation theory. After this, alternative extremum seeking schemes have been studied [2][3]. One of the areas in which recently ESC has been shown to be a feasible algorithm, is in the area of learning in games. For instance, in [4][5], different extremum seeking schemes were shown to satisfactorily converge to a Nash equilibrium in a classic non-cooperative game, where the payoffs of the players were unknown and only available by measurements. However, even though the application and design of extremum seeking controllers to solve these classical game theoretical problems have been addressed in recent years, the area of extremum seeking control for evolutionary games, remains unexplored.

Evolutionary game theory studies the dynamic process that emerges in the context of population games [6]. In this type of games, several entities who play different types of roles in a society, interact in a strategic way by means of common cost functions [7]. The entities in the same role are characterized by an equal cost function, which in general depends on the distribution of the entities on the other roles. Since the population of entities in a role, and the amount of roles, are maintained constant, the evolution of the proportion of entities implementing each role occurs under a natural type of constraints. In this sense, the framework of population games characterizes a wide number of engineering applications, e.g., congestion control systems, water and power distribution systems, wireless networked systems, urban traffic systems, and building temperature control systems. Based on the fact that most of these engineering systems have an optimal operation point, and run under dynamical and unpredictable operational environments, control algorithms capable of converging to an optimal equilibrium in an on-line and robust fashion are required. From this perspective, the motivation for the study of extremum seeking controllers in this type of engineering systems appears naturally.

In this paper we introduce novel schemes of extremum seeking controllers, based on the general framework of *multi-populations games*. In [8] an ESC based on single-population games was presented, designed to perform multi-variable optimization of multiple-input-single-output (MISO) dynamical systems under a *single* linear constraint on the control variables. However, our results in the present paper are much general in the sense that not only generalize the case for *multiple* linear constraints for MISO systems, but also consider the on-line multi-constrained optimization of dynamical MAS comprised of different clusters of agents, which compete to simultaneously maximize their own cost function by controlling their individual states. We consider constraints that describe naturally multi-resource allocation problems, and market share competition problems in MAS. We assume that every agent has internal unknown but asymptotically stable dynamics, which directly affect the cost function of each cluster of agents. We correlate the stability and optimality properties of the equilibrium point of the extremum seeking controllers with the concepts of Nash equilibrium (NE) and evolutionarily stable states (ESS). We show that the weaker equilibrium concept of coupled-ESS [9] is enough to guarantee semi-global practical asymptotical stability of the closed loop system in the positive orthant.

The rest of this paper is organized as follows. Section II introduces some preliminaries on population games, Section III describes the mathematical structure of the problems under study. Sections IV and V present our main results for ESC for multi-population games. Section VI illustrates the results with some numerical examples, and finally Section VII ends with some conclusions.

II. PRELIMINARIES

Let $\mathbb{N}_{>0}$ be the set of positive integers, and $\mathbb{R}^n$ the set of real numbers in the n-dimension, where $n \in \mathbb{N}_{>0}$. Let $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ be the set of non-negative real numbers, and positive real numbers, respectively, in the n-dimension. We use the acronym GAS and SPA when we speak of global asymptotic stability [10], and semi-global practical asymptotical stability [2].

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A. Population games

We consider a society of multiple populations of entities, where the set $\mathcal{P} = \{1, \ldots, M\}$ is defined as the social set of populations. Each of the entities of each population is preprogrammed to play the i^{th} pure strategy from a social set of pure strategies $\mathcal{H} = \{1, \ldots, N\}$, where $i \in \mathcal{H}$. The set of pure strategies that are present in the j^{th} population is given by $\mathcal{H}_j \subseteq \mathcal{H}$, for all $j \in \mathcal{P}$. We denote $N_j = card(\mathcal{H}_j)$ as the cardinality of the set $\mathcal{H}_j$. Also, we define the set $\mathcal{P}_i$ as the set of populations where the i^{th} strategy from the set $\mathcal{H}$ is present, where $\mathcal{P}_i \subseteq \mathcal{P}$, and we define $M_i = card(\mathcal{P}_i)$. We also define the quantity $\Omega = \sum_{j\in\mathcal{P}} N_j = \sum_{i\in\mathcal{H}} M_i$.

Let $p_{ij}(t) \geq 0$ be the amount of individuals playing the i^{th} strategy at the j^{th} population at a given time t. Then, the amount of entities of the entire society associated *to the j^{th} population* is given by $\sum_{i\in\mathcal{H}_j} p_{ij}(t) = P_j(t)$, for all $j \in \mathcal{P}$, while the amount of entities of the entire society associated *to the i^{th} strategy* is given by $\sum_{j\in\mathcal{P}_i} p_{ij}(t) = S_i(t)$, for all $i \in \mathcal{H}$. The total amount of entities in the entire society is given by $\sum_{j\in\mathcal{P}} \sum_{i\in\mathcal{H}_j} p_{ij} = \sum_{i\in\mathcal{H}} \sum_{j\in\mathcal{P}_i} p_{ij} = \mathcal{T}$. Based on this framework, we define two types of society states: i) normalized with respect to P_j, for every population $j \in \mathcal{P}$; and ii) normalized with respect to S_i, for each strategy $i \in \mathcal{H}$.

1) Social state w.r.t P_j: Define $y_{ij}(t) = \frac{p_{ij}(t)}{P_j} Y_j$ as the proportion of entities playing the i^{th} strategy in the j^{th} population, where $Y_j \in \mathbb{R}_{>0}$, and where we have assumed that P_j is constant, for every $j \in \mathcal{P}$. According to this, for the j^{th} population, the society state is defined as $y_j(t) \in \mathbb{R}^{N_j}_{\geq 0}$, and the overall society state is given by $y = [y_1(t)^\top, \ldots, y_j(t)^\top, \ldots, y_M(t)^\top]^\top$, where $y \in \mathbb{R}^{\Omega}_{\geq 0}$. Now, by its definition, every society state associated to the j^{th} population is confined to a simplex Δ_{Y_j}, given by $\Delta_{Y_j} = \{y_{ij}(t) \in \mathbb{R}_{\geq 0} : \sum_{i\in\mathcal{H}_j} y_{ij}(t) = Y_j\}$, for all $j \in \mathcal{P}$. Associated to the i^{th} strategy in the j^{th} population, there exists a continuously differentiable function $f_{ij}(\cdot) : \Theta \mapsto \mathbb{R}$, where $\Theta = \times_{j\in\mathcal{P}} \Delta_{Y_j}$ represents the cartesian product of the M different manifolds Δ_{Y_j}. The function $f_{ij}(\cdot)$ represents the payoff or fitness perceived by the proportion of entities of the j^{th} population implementing the i^{th} strategy in the society. This function plays an important role in the evolution of the society state y. In general, this evolution can be ruled by a family of different type of dynamics (i.e., evolutionary dynamics) [7]. In this paper, however, we will restrict our attention to the well known replicator dynamics (RDs) [6], which for the multi-population setting are given by,

$$\dot{y}_{ij} = \beta_j y_{ij} \left(f_{ij}(y) - \frac{1}{Y_j} \sum_{i\in\mathcal{H}_j} y_{ij} f_{ij}(y) \right) \tag{1}$$

for all $i \in \mathcal{H}_j$, and $j \in \mathcal{P}$, where $\beta_j \in \mathbb{R}_{>0}$. Note that in (1), the evolution of the society states y_j, associated to the j^{th} population, depends also on the society states of the other populations by means of the payoff functions $f_{ij}(\cdot)$. Also, note that the simplex Δ_{Y_j} is invariant under de dynamics (1) for every population $j \in \mathcal{P}$, i.e., if $y_{ij}(t_0) \in \Delta_{Y_j}$, then $y_{ij}(t) \in \Delta_{Y_j}$, for all $t \geq t_0$, for all $j \in \mathcal{P}$. According to this, the simplexes Δ_{Y_j} can be seen as *decoupled constraints* between populations, i.e., the constraint Δ_{Y_j} is defined individually for every population and depends only on the state y_j, for every $j \in \mathcal{P}$.

2) Social state w.r.t S_i: Define now, $y_{ij}(t) = \frac{p_{ij}(t)}{S_i} Y^i$ as the proportion of entities in the j^{th} population playing the i^{th} strategy, where $Y^i \in \mathbb{R}_{>0}$, and where we have assumed that S_i is constant, i.e., the number of entities playing each strategy does not vary along time, for every $i \in \mathcal{H}$. Then, for the i^{th} strategy, the society state is defined as $y_i(t) \in \mathbb{R}^{M_i}_{\geq 0}$, and the overall society state as $y = [y_1(t)^\top, \ldots, y_i(t)^\top, \ldots, y_N(t)^\top]^\top$, where $y \in \mathbb{R}^{\Omega}_{\geq 0}$. In this case, by its definition, the society state $y_i(t)$ related to the i^{th} strategy is confined to a simplex Δ_{Y^i}, where $\Delta_{Y^i} = \{y_{ij}(t) \in \mathbb{R}_{\geq 0} : \sum_{j\in\mathcal{P}_i} y_{ij}(t) = Y^i\}$, for all $i \in \mathcal{H}$. And in this case the RDs are given by,

$$\dot{y}_{ij} = \beta_i y_{ij} \left(f_{ij}(y) - \frac{1}{Y^i} \sum_{j\in\mathcal{P}_i} y_{ij} f_{ij}(y) \right) \tag{2}$$

for all $j \in \mathcal{P}_i$, and $i \in \mathcal{H}$, where $\beta_i \in \mathbb{R}_{>0}$. The payoff function for each strategy in each population is now defined as $f_{ij}(\cdot) : \Xi \mapsto \mathbb{R}$, where $\Xi = \times_{i\in\mathcal{H}} \Delta_{Y^i}$ represents the cartesian product of the N different manifolds Δ_{Y^i}. Again, the simplex Δ_{Y^i} is invariant under the dynamics (2), i.e, if $y_{ij}(t_0) \in \Delta_{Y^i}$, then $y_{ij}(t) \in \Delta_{Y^i}$, for all $t \geq t_0$, for all $i \in \mathcal{H}$. Note that in this case, the simplexes Δ_{Y^i} can be seen as *coupled constraints* between different populations, i.e., the constraint set Δ_{Y^i} is defined with respect to the i^{th} state of the different populations in which the strategy $i \in \mathcal{H}$ is present.

B. Shahshahani Gradients for Multiple Populations

Based on results on biological systems and differential geometry [11], and using the Shahshahani metric (SM) [12], it is possible to see systems (1) and (2), as gradient systems with respect to a potential function associated to the entire society, or to each population.

The following Lemmas are adapted from [11] [12][13].

Lemma 2.1: Based on the SM, the RDs given by (1) are a gradient system with respect to a potential function $J_j(\cdot) : \mathbb{R}^{\Omega}_{>0} \mapsto \mathbb{R}$, in Δ_{X_j}, for each population $j \in \mathcal{P}$.

Lemma 2.2: Based on the SM, the RDs given by (2) are a gradient system with respect to a potential function $J_j(\cdot) : \mathbb{R}^{\Omega}_{>0} \mapsto \mathbb{R}$, in Δ_{X_i}, for each strategy $i \in \mathcal{H}$.

The main stability concept associated to population games is the evolutionarily stable state (ESS).

Definition 2.1: Define y_j as the social state associated to the j^{th} population, and y as the overall social state. For each of the j^{th} populations, a state y_j^* is an ESS if $y_j^* \cdot f(y) > y_j \cdot f(y)$, for all $y \in \Theta$, holds [9].

Then, an ESS is necessarily a NE, but a NE is not necessarily an ESS. However, for multi-population games there exists a weaker stability notion called a "coupled-ESS" [9].

Definition 2.2: A state y^* is a "coupled-ESS" if $\sum_{j=1}^{M} \left(y_j^* \cdot f(y) \right) > \sum_{j=1}^{M} \left(y_j \cdot f(y) \right)$, holds for all $y \in \Theta$. Note that if y_j is an ESS for every population $j \in \mathcal{P}$, the social state y^* is also a coupled-ESS, but the converse is not true.

III. PROBLEM STATEMENT

Consider a multi-agent system comprised of Ω agents, divided into M different clusters. Assume that the j^{th} cluster has N_j different agents, for all $j \in \mathcal{P}$, where $\mathcal{P}$ is the set of clusters and $M = card(\mathcal{P})$. The i^{th} agent in the j^{th} cluster, controls its own state $x_{ij} \in \mathbb{R}_{\geq 0}$, where $i \in \mathcal{H}_j$ for all $j \in \mathcal{P}$. Therefore, the vector of states of the MAS is given by $x \in \mathbb{R}^{\Omega}_{>0}$. At the same time, each agent has intrinsic internal dynamics $\dot{\theta}_{ij} = g_{ij}(x, \theta)$, that in general depends on the overall states x and θ, where $\theta \in \mathbb{R}^{\Omega}$, and where for purposes of clarity we have assume that the internal state $\theta_{ij} \in \mathbb{R}$. Each of the agents of the j^{th} cluster aims to maximize an unknown cost function $J_j(x, \theta)$, for all $j \in \mathcal{P}$, which in general depends on the overall states θ and x. In the present paper, we assume that each agent has full information of the states x of the MAS. Based on this structure we aim to solve the following two different problems:

$$\max \ J_j(x, \theta) \ \ s.t. \ x \in \Theta, \ \ \dot{\theta}_{ij} = g_{ij}(x, \theta). \tag{3}$$

where $\Theta = \times_{j \in \mathcal{P}} \Delta_{X_j}$, for all $j \in \mathcal{P}$, and where the constrained set for the j^{th} cluster is given by $\Delta_{X_j} = \{x_{ij} \in \mathbb{R}_{\geq 0} : \sum_{i \in \mathcal{H}_j} x_{ij} = X_j\}$, and

$$\max \ J_j(x, \theta) \ \ s.t. \ x \in \Xi, \ \ \dot{\theta}_{ij} = g_{ij}(x, \theta). \tag{4}$$

where $\Xi = \times_{i \in \mathcal{H}} \Delta_{X^i}$, for all $i \in \mathcal{H}$, and where the constrained set for the i^{th} strategy is given by $\Delta_{X^i} = \{x_{ij} \in \mathbb{R}_{\geq 0} : \sum_{j \in \mathcal{P}_i} x_{ij} = X^i\}$.

Problem (3) describes a dynamical multi-resource allocation problem, where multiple fixed resources X_j, for every $j \in \mathcal{P}$, must be optimally allocated among all the agents of the M clusters of the MAS. On the other hand, problem (4) describes a multi-market share competition game, where the i^{th} resource X^i is shared by M_i multiple clusters in the MAS, for all $i \in \mathcal{H}$, aiming to maximize their own cost function. Both problems must be solved based on the fact that the cost functions of each population are unknown and only accessible by measurements. Furthermore, they must be solved without the exact knowledge of the functions $g_{ij}(\cdot)$.

To guarantee the existence of an extremum seeking solution with respect to the input-to-output mapping associated to each agent of every cluster, we need the following assumption on the dynamics of each agent.

Assumption 3.1: For all $i \in \mathcal{H}_j$, and all $j \in \mathcal{P}$, the function $g_{ij}(\cdot, \cdot) : \mathbb{R}^{\Omega}_{\geq 0} \times \mathbb{R}^{\Omega} \mapsto \mathbb{R}$ is continuously differentiable, and satisfies that $g_{ij}(x, \theta^*) = 0$, if and only if $\theta^*_{ij} = l_{ij}(x)$, where $l_{ij}(\cdot) : \mathbb{R}^{\Omega}_{\geq 0} \mapsto \mathbb{R}$, and for each $x \in \mathbb{R}^{\Omega}_{\geq 0}$, the equilibrium $\theta^*_{ij} = l_{ij}(x)$ is GAS.

We now introduce extremum seeking controllers for multi-population games, designed to solve problems (3) and (4).

IV. EXTREMUM SEEKING FOR CLUSTERS WITH MULTIPLE DECOUPLED CONSTRAINTS

To analyze problem (3), we first consider the case when the dynamics associated to the state θ are negligible, i.e., $J_j(\cdot) : \mathbb{R}^{\Omega}_{\geq 0} \mapsto \mathbb{R}$ is a direct mapping of the overall state of actions x. After this, we generalize our results for the general dynamic problem (3).

A. Static Maps Analysis

From now on in this section, we will refer to the point x_j as a feasible point if $x_j \in \Delta_{X_j}$, for all $j \in \mathcal{P}$, i.e., $x \in \Theta$. In order to solve problem (3) when $J_j(\cdot)$ is a static map, we need the following assumption.

Assumption 4.1: Let us define the vector $\nabla J_j(\cdot)$, as the gradient vector of the j^{th} cost function associated to the j^{th} cluster, with respect to the states under control by the agents of the same cluster. Then, there exists a feasible isolated point x^*_j that satisfies $\sum_{j \in \mathcal{P}} \left(\nabla J_j(x) \cdot x^*_j\right) > \sum_{j \in \mathcal{P}} \left(\nabla J_j(x) \cdot x_j\right)$, for all feasible $x \neq x^*$.

Now, consider the following extremum seeking dynamics for the i^{th} agent of the j^{th} cluster,

$$\begin{aligned} \dot{\hat{x}}_{ij} = k\hat{x}_{ij}\bigg(&\frac{J_j(x)\mu_{ij}(t)}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj} + C \dots \\ &\dots - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}(J_j(x)\mu_{kj}(t) + C)\bigg) \end{aligned} \tag{5}$$

where $x_{ij} = \hat{x}_{ij} + a\mu_{ij}(t)$, for all $i \in \mathcal{H}_j$, and $j \in \mathcal{P}$. The dither signal is defined as $\mu_{ij}(t) = \sin(\bar{\omega}\omega_{ij})$, where $\bar{\omega} \in \mathbb{N}_{>0}$, and ω_{ij} is different for every agent $i \in \mathcal{H}_j$, in every cluster $j \in \mathcal{P}$. Also, the auxiliary variable $\hat{x}_{ij} \in \mathbb{R}_{\geq 0}$ has been introduced for every state x_{ij}. The adaptation gain for all the agents of the society is given by $k = 2\epsilon\bar{\omega}$. The constants $\epsilon \in \mathbb{R}_{>0}$, $a \in \mathbb{R}_{>0}$, and $C \in \mathbb{R}_{\geq 0}$ satisfy $0 < a, \frac{1}{C}, \epsilon \ll 1$. Note that in (5), the summations are defined over the states associated to all the agents present in the j^{th} cluster. Rewriting system (5) in the $\tau = \bar{\omega}t$ scale, leads to the following dynamics for the i^{th} agent of the j^{th} cluster,

$$\begin{aligned} \frac{\partial \hat{x}_{ij}}{\partial \tau} = 2\epsilon\hat{x}_{ij}\bigg(&\frac{J_j(x)\mu_{ij}(\tau)}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj} + C \dots \\ &\dots - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}(J_j(x)\mu_{kj}(\tau) + C)\bigg) \end{aligned} \tag{6}$$

Defining the dither vector $\mu(t) \in \mathbb{R}^{\Omega}$ as $\mu(t) = [\mu_1(t)^\top, \dots, \mu_j(t)^\top, \dots, \mu_M(t)^\top]^\top$, whose components are given $\mu_j(t) \in \mathbb{R}^{N_j}$, and $x = \hat{x} + a\mu(\tau)$, where $\hat{x} = [\hat{x}_1^\top, \dots, \hat{x}_j^\top, \dots, \hat{x}_M^\top]^\top$, $\hat{x} \in \mathbb{R}^{\Omega}_{\geq 0}$ and $\hat{x}_j \in \mathbb{R}^{N_j}_{\geq 0}$, for all $j \in \mathcal{P}$, replacing x in $J_j(\cdot)$, considering a as a small constant, and expanding $J_j(\cdot)$ in its Taylor series with respect to $\hat{x}$ [3], leads to the following dynamics for the i^{th} agent of the j^{th} cluster,

$$\begin{aligned} \frac{\partial \hat{x}_{ij}}{\partial \tau} = 2\epsilon\hat{x}_{ij}\Big(&\big(J_j(\hat{x}) + a\mu(\tau)^\top \nabla J_j(\hat{x}) \dots \\ &\dots + O(a^2)\big)\frac{\mu_{ij}(\tau)}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{ij} + C \dots \\ &\dots - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}\big(\big(J_j(\hat{x}) + a\mu(\tau)^\top \nabla J_j(\hat{x}) \dots \\ &\dots + O(a^2)\big)\mu_{kj}(\tau) + C\big)\Big) \end{aligned} \tag{7}$$

For sufficiently small ϵ, the non-autonomous system (7) can be approximated by its average autonomous system [10]. This leads to the following average dynamics for the i^{th}

agent of the j^{th} cluster,

$$\begin{aligned}\frac{\partial \hat{x}_{ij}^A}{\partial \tau} =\ & \epsilon \hat{x}_{ij}^A \Big(\frac{1}{X_j} \Big(a \frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{ij}^A} + O(a^2) \Big) \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}^A \dots \\ & \dots + 2C - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \Big(a \frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{kj}^A} + O(a^2) + 2C \Big) \hat{x}_{kj}^A \Big)\end{aligned} \tag{8}$$

for all $i \in \mathcal{H}_j$ and $j \in \mathcal{P}$. To analyze the dynamics (8) consider the introduction of an auxiliary variable $z_j^A = (\sum_{k \in \mathcal{H}_j} \hat{x}_{kj}^A - X_j)C$, for every cluster $j \in \mathcal{P}$, which generates the following extended system in the $s = \epsilon\tau$-scale,

$$\begin{aligned}\frac{\partial \hat{x}_{ij}^A}{\partial s} =\ & \hat{x}_{ij}^A \Big(\Big(\frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{ij}^A} + O(a^2) \Big) \Big(\frac{\delta z_j^A}{X_j} + 1 \Big) \dots \\ & \dots - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \Big(\frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{kj}^A} + O(a^2) \Big) \hat{x}_{kj}^A \Big) - \frac{2}{X_j} \hat{x}_{ij}^A z_j^A\end{aligned} \tag{9}$$

$$\delta \frac{\partial z_j^A}{\partial s} = -\frac{2}{X_j} z_j^A \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}^A \tag{10}$$

and note that for sufficiently small δ, system (10) moves in faster time scale than system (9), for all $j \in \mathcal{P}$, and considering $\hat{x}_{ij}^A$ as a frozen variable, for every $i \in \mathcal{H}_j$ and every $j \in \mathcal{P}$, the equilibrium point $z_j^* = 0$ will be asymptotically stable if $\sum_{k \in \mathcal{H}_j} \hat{x}_{kj}^A > 0$. Thus, in the positive orthant $\mathbb{R}_{>0}^{N_j}$, the manifold $\Delta_{X_j} = \{\hat{x}_{ij}^A \in \mathbb{R}_{\geq 0} : \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}^A = X_j\}$, is asymptotically stable for every cluster $j \in \mathcal{P}$. To analyze the slower dynamics (9), we set z_j to its equilibrium $z_j^* = 0$, and as $\hat{x}_{ij}^A$ is now confined to the manifold Δ_{X_j}, the trajectories of $\hat{x}_{ij}^A(t)$ are bounded by X_j, for all $j \in \mathcal{P}$. Therefore, we obtain the "reduced" dynamics for the i^{th} agent of the j^{th} cluster,

$$\frac{\partial \hat{x}_{ij}^A}{\partial s} = a \hat{x}_{ij}^A \left(\frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{ij}^A} - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \frac{\partial J_j(\hat{x}^A)}{\partial \hat{x}_{kj}^A} \hat{x}_{kj}^A \right) + O(a^2) \tag{11}$$

for all $i \in \mathcal{H}_j$, and $j \in \mathcal{P}$. System (11) is an approximate Shahshahani gradient (1) with respect to the j^{th} population, which under Assumption 4.1 has a unique equilibrium point $\hat{x}^{*A} \in \hat{\Theta}^A$, where $\hat{\Theta}^A = \times_{j \in \mathcal{P}} \hat{\Delta}_{X_j}^A$, and $\hat{\Delta}_{X_j}^A = \{\hat{x}_{ij}^A \in \mathbb{R}_{\geq 0} : \sum_{i \in \mathcal{H}_j} \hat{x}_{ij}^A = X_j\}$, for all $j \in \mathcal{P}$. From Lemma 2.1, this equilibrium point maximizes $J_j(x)$ for all $j \in \mathcal{P}$, i.e., $\hat{x}^{*A}$ is an NE, and a coupled-ESS. In order to analyze the stability of this point, define the error state as $\tilde{\hat{x}}_{ij}^A = \hat{x}_{ij}^A - \hat{x}_{ij}^{A*}$, and consider the Lyapunov function $V(\tilde{\hat{x}}^A) = -\sum_{j \in \mathcal{P}} \big(\sum_{i \in \mathcal{H}_j} \hat{x}_{ij}^{A*} \ln \big(\frac{\tilde{\hat{x}}_{ij}^A}{\hat{x}_{ij}^{A*}} + 1 \big) \big)$, which is equal to zero when $\tilde{\hat{x}}^A = 0$, and positive definite everywhere else in $\hat{\Theta}^A$ [6]. The derivative of V along the trajectories of the system (11), is given by, $\dot{V} = a \sum_{j \in \mathcal{P}} \big(\sum_{i \in \mathcal{H}_j} \tilde{\hat{x}}_{ij}^A \frac{\partial J(\hat{x}^A)}{\partial \hat{x}_{ij}^A} \big) + O(a^2)$, which under Assumption 4.1, and sufficiently small values of a, will be negative definite outside of a ball around the origin $\tilde{\hat{x}}_{ij}^A = 0$ in $\hat{\Theta}^A$. Hence we conclude that the system (11) will be SPA stable in the parameter a. By the SPA stability of system (11), and the AS stability in $\mathbb{R}_{>0}^{N_j}$ of the dynamics (10), for every $j \in \mathcal{P}$, the average system (8) will be SPA stable with respect to the small parameters (ϵ, a, $1/C$), for every $j \in \mathcal{P}$ in $\mathbb{R}_{>0}$. Hence, as $x_{ij} = \hat{x}_{ij} + a\mu_{ij}(t)$, by an appropriate tuning of the parameters ϵ, a, and $1/C$, the state $x(t)$ can be arbitrarily regulated from an initial condition in $\mathbb{R}_{>0}^{\Omega}$, to an arbitrary small neighborhood of the coupled-ESS x^*.

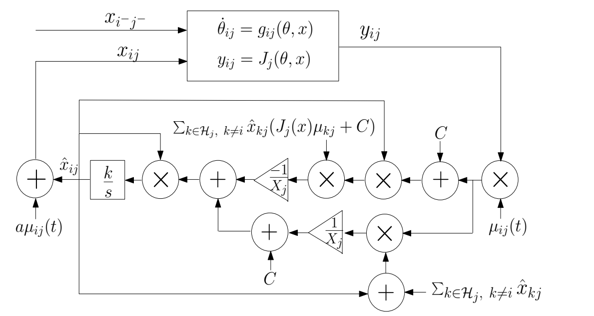

Fig. 1: Scheme for the ESC of the i^{th} agent of the j^{th} cluster.

B. Dynamical Systems Analysis

Now, consider the entire dynamical system (3) under the ESC for multi-population games. The dynamics of the i^{th} agent of the j^{th} cluster in the $\tau = \bar{\omega}t$ scale are given by,

$$\bar{\omega} \frac{\partial \theta_{ij}}{\partial \tau} = g_{ij}(\hat{x} + a\mu(\tau), \theta) \tag{12}$$

$$\begin{aligned}\frac{\partial \hat{x}_{ij}}{\partial \tau} &= 2\epsilon \hat{x}_{ij} \Big(\frac{J_j(x,\theta)\mu_{ij}(\tau)}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj} + C \dots \\ & \dots - \frac{1}{X_j} \sum_{k \in \mathcal{H}_j} \hat{x}_{kj}(J_j(\theta)\mu_{kj}(\tau) + C) \Big)\end{aligned} \tag{13}$$

for all $i \in \mathcal{H}_j$ and $j \in \mathcal{P}$. System (12)-(13) is in the standard singular perturbation form, where for a small $\bar{\omega}$, system (12) evolves in a faster time scale than (13), and by Assumption 3.1, the associated boundary layer dynamics will be globally asymptotically stable. Defining $l_j = [l_{1j}, \dots, l_{ij}, \dots, l_{N_j j}]^\top$, for all $j \in \mathcal{P}$, and $l(\cdot) = [l_1^\top, \dots, l_j^\top, \dots, l_M^\top]^\top$, substituting $\theta_{ij}^* = l_{ij}(x)$ in (13) and letting $J_j(x) := J_j(x, l(x))$, the reduced dynamics for the i^{th} agent of the j^{th} cluster are obtained, which will be equal to system (5), which under Assumption 4.1 was was shown to be SPA stable with respect to a, ϵ, and $1/C_j$, for all $j \in \mathcal{P}$ in $\mathbb{R}_{>0}^{\Omega}$. Using Lemma 1 in [2], the original system (12)-(13) will be SPA stable with respect to $\bar{\omega}$, a, ϵ, and $1/C_j$, in the time scale t, for all $j \in \mathcal{P}$. We summarize with the following theorem.

Theorem 4.1: Consider the general problem (3) in a MAS with M clusters and Ω agents, where the N_j agents of the j^{th} cluster are characterized by a common cost function $J_j(\cdot,\cdot) : \mathbb{R}_{\geq 0}^{\Omega} \times \mathbb{R}^{\Omega} \mapsto \mathbb{R}$, for every $j \in \mathcal{P}$. Under Assumptions 3.1 and 4.1, the closed loop system given by (12)-(13), is SPA stable with respect to the parameters $\bar{\omega}$, a, ϵ, and $1/C$, in $\mathbb{R}_{>0}^{\Omega}$, for every agent $i \in \mathcal{H}_j$, in every cluster $j \in \mathcal{P}$.

V. EXTREMUM SEEKING FOR CLUSTERS WITH MULTIPLE COUPLED CONSTRAINTS

We follow a similar analysis as in Section IV to analyze the extremum seeking dynamics that solve the general problem (4), where the states of agents of different clusters are coupled by a constrained set. Aiming to avoid redundancy, we present only the main steps.

A. Static Maps Analysis

We now refer to the point x_i as a feasible point if $x_i \in \Delta_{X^i}$, for all $i \in \mathcal{H}$, i.e., $x \in \Xi$. In this case we need the following assumption on the cost functions $J_j(\cdot)$.

Assumption 5.1: Let us define the vector $\nabla_i J$ as the vector whose entries are the derivatives of the different cost functions $J_j(\cdot)$, for all $j \in \mathcal{P}$, with respect to the i^{th} fixed state in the different clusters (e.g., $\nabla_i J = [\frac{\partial J_1}{\partial x_{i1}}, \ldots, \frac{\partial J_j}{\partial x_{ij}}, \ldots]^\top$). Then, there exists a feasible isolated point x_i^*, such that $\sum_{i=1}^N \left(\nabla_i J(x) \cdot x_i^*\right) > \sum_{i=1}^N \left(\nabla_i J(x) \cdot x_i\right)$, for all feasible $x \neq x^*$.

To solve problem (4) when $J_j(\cdot)$ are static maps from the actions x, we introduce the following ESC dynamics for the i^{th} agent of the j^{th} cluster,

$$\begin{aligned}\dot{\hat{x}}_{ij} = k\hat{x}_{ij}\Bigg(&\frac{J_j(x)\mu_{ij}(t)}{X^i}\sum_{k\in\mathcal{P}_i}\hat{x}_{ik} + C\ldots\\ &\ldots - \frac{1}{X^i}\sum_{k\in\mathcal{P}_i}\hat{x}_{ik}(J_k(x)\mu_{ik}(t)+C)\Bigg) \qquad (14)\end{aligned}$$

for all $j \in \mathcal{P}$, and $i \in \mathcal{H}$, where $x_{ij} = \hat{x}_{ij} + a\mu_{ij}(t)$, the dither signal $\mu_{ij}(t)$, and the parameters a , k, and C are defined as in Section IV. Note that in this case, the summation terms in (14) are defined over the i^{th} agents of all the clusters $j \in \mathcal{P}_i$, in the MAS.

Under Lemma 2.3, Assumption (5.1), and using the Lyapunov function, $V = -\sum_{i\in\mathcal{H}}\left(\sum_{j\in\mathcal{P}_i}\hat{x}_{ij}^{A*}\ln\left(\frac{\tilde{\hat{x}}_{ij}^A}{\hat{x}_{ij}^A}+1\right)\right)$ the dynamics (14) will be SPA stable with respect to the parameters ϵ, a, and $1/C$, in $\mathbb{R}^\Omega_{>0}$. for all $i \in \mathcal{H}$.

B. Dynamical Systems Analysis

For the complete dynamical system in (4), the closed loop system with the ESC will be given by,

$$\dot{\theta}_{ij} = g_{ij}(\hat{x} + a\mu(t), \theta) \qquad (15)$$

$$\begin{aligned}\dot{\hat{x}}_{ij} = k\hat{x}_{ij}\Bigg(&\frac{J_j(x,\theta)\mu_{ij}(t)}{X^i}\sum_{k\in\mathcal{P}_i}\hat{x}_{ik} + C\ldots\\ &\ldots - \frac{1}{X^i}\sum_{k\in\mathcal{P}_i}\hat{x}_{ik}(J_k(\theta)\mu_{ik}(t)+C)\Bigg) \qquad (16)\end{aligned}$$

which in the τ scale is again in the singular perturbation form, with the parameter $\bar{\omega}$ acting as a small constant. Under Assumption 3.1 the dynamics (15) are GAS, and replacing the equilibrium point of (15) in (16) leads to the reduced system (14), which is SPA stable with respect to a, ϵ, and $1/C$ in $\mathbb{R}^\Omega_{>0}$. Following the same procedure as in Section IV-B, the entire closed loop system (15)-(16) will be SPA stable with respect to the parameters a, ϵ, $1/C$, and $\bar{\omega}$, in $\mathbb{R}^\Omega_{>0}$.

Theorem 5.1: Consider the general problem (4) in a MAS with M clusters and Ω agents, where the N_j agents of the j^{th} cluster are characterized by a common cost function $J_j(\cdot) : \mathbb{R}^\Omega_{>0} \times \mathbb{R}^\Omega \mapsto \mathbb{R}$, for every $j \in \mathcal{P}$. Under Assumptions (3.1) and (5.1), the extremum seeking control given by (15)-(16), is SPA stable with respect to the parameters $\bar{\omega}$, a, ϵ, and $1/C$, in $\mathbb{R}^\Omega_{>0}$, for all the agents $i \in \mathcal{H}$, and clusters $j \in \mathcal{P}$.

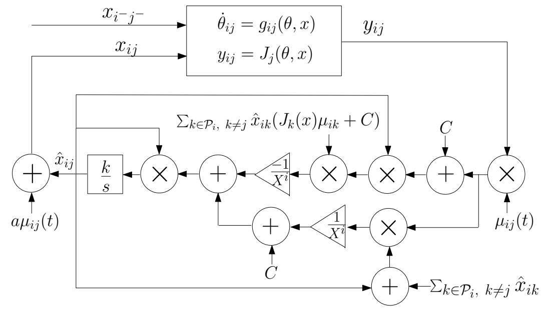

Fig. 2: Scheme for the ESC of the i^{th} agent of the j^{th} cluster.

Remark 5.1: Similarly as in [8], inequality constraints in the sets Δ_{X_j} and Δ_{X^i} can be addressed by introducing fictitious agents into the clusters, i.e., slack variables in $\hat{x}_j$. Note that the asymptotic stability in $\mathbb{R}^{N_j}_{>0}$ of the manifolds Δ_{X_j}, for every $j \in \mathcal{P}$, shown in Section IV, and in $\mathbb{R}^{M_i}_{>0}$ of Δ_{X^i}, for every agent $i \in \mathcal{H}$, shown in Section V, may be critical for the implementation of extremum seeking controllers to optimize dynamical systems under constraints. Additionally, note that the mathematical structure of the problems (3) and (4) when the agents have full information of x is analogous to a non-cooperative game, where each cluster act as a player aiming to optimize its own cost function, and where the actions of the players (i.e., x_{ij}) are constrained to evolve on the sets Δ_{X_j} or Δ_{X^i}. However, we retain the MAS structure presented in this paper, and also used in [14], to evidence the fact that the general framework of population games can also be used to design distributed multi-constrained extremum seeking controllers for MAS, where the agents have only local information of the overall system (e.g., a not strongly connected network). Thus, under simmilar assumptions than those presented in this paper, we could solve the distributed extremum seeking problem where each agent shares only information with its neighboring agents, using the same presented population games framework. In that situation the extremum seeking dynamics will be formulated in terms of the adjacency matrix associated to the network of the MAS.

VI. NUMERICAL EXAMPLE

Consider an electrical power distribution system comprised of two microgrids, where each microgrid must satisfy with its own distributed generators (DGs) a given demanded power in a specific geographical zone. Without loss of generality we assume that the cost functions of the microgrids have the quadratic structure $J_1(x) = x^\top Q_1 x + b_1 x$, and $J_2(x) = x^\top Q_2 x + b_2 x$, which is common in power distribution systems [15]. Here, we have defined $x = [x_{11},\ x_{21},\ x_{12},\ x_{22}]^\top$ as the vector of powers dispatched by the DGs of the microgrids, where x_{ij} stands for the power dispatched by the i^{th} DG, in the j^{th} microgrid, for $i = \{1,\ 2\}$, and $j = \{1,\ 2\}$. The matrices Q_j are defined as, $Q_1 = \begin{bmatrix} -2 & -1 & 1 & 3\\ 1 & -3 & 2 & 4\\ 1 & 2 & 3 & 4\\ 2 & 3 & 6 & 1\end{bmatrix}$, and $Q_2 = \begin{bmatrix} 2 & 1 & 1 & 3\\ 1 & -3 & 2 & 5\\ -1 & 2 & 3 & 4\\ -2 & 3 & 6 & 1\end{bmatrix}$ and the vectors b_j as $b_1 = [1,\ 1,\ 1,\ 1]$ and $b_2 = [1,\ 1,\ 1,\ 1]$. In the

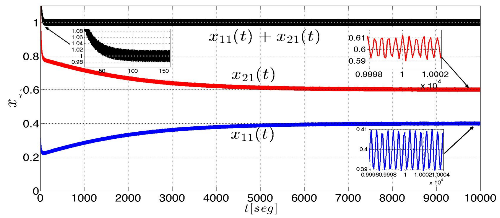

Fig. 3: Dispatch of DGs for microgrid 1.

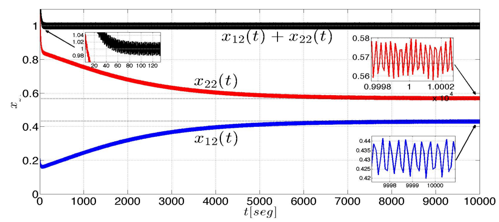

Fig. 4: Dispatch of DGs for microgrid 2.

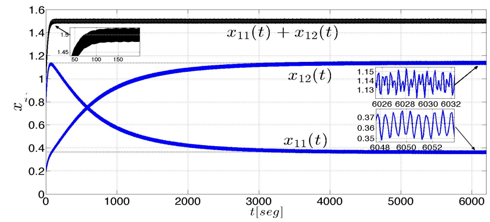

Fig. 5: Dispatch of DGs for zone 1.

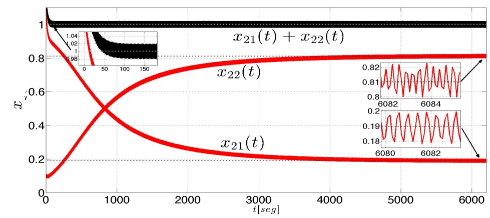

Fig. 6: Dispatch of DGs for zone 2.

first scenario, we assume that both microgrids must satisfy a demand of 1 MW of electric power, i.e., $x_{11}+x_{21} = 1$ MW, $x_{12}+x_{22} = 1$ MW, and $x_{ij} \geq 0$, which represent the feasible sets. This problem has a unique optimal solution given by $x^* = [0.4,\ 0.6,\ 0.4334,\ 0.5667]^\top$ MW. Figures 3 and 4 show the trajectories along time of the powers dispatched by each DG, in each microgrid. The initial conditions are given as $\hat{x}(0) = [0.3,\ 1.1,\ 0.2,\ 1.1]^\top$ MW, such that $\hat{x} \notin \hat{\Theta}$ but $\hat{x} \in \mathbb{R}^{\Omega}_{>0}$. The simulation parameters are $a = 0.01$, $k = 0.1$, $C = 5$, $\bar{\omega} = 10$, $\omega_{11} = 1$, $\omega_{12} = 1.5$, $\omega_{21} = 1.8$, and $\omega_{22} = 2$. The inset in Figures 3 and 4, shows the power dispatched by the microgrids converging to the feasible sets in a faster time scale. Then, the cost functions are optimized under the two simultaneous constraints.

Now, consider the scenario when the two microgrids must satisfy given demanded powers in different geographical zones by sharing DGs. Therefore, in this case we define the feasible sets as $x_{11} + x_{12} = 1.5\ MW$, and $x_{21} + x_{22} = 1\ MW$, $x_{ij} \geq 0$. To illustrate the results we consider the matrices $Q_1 = \begin{bmatrix} -2 & -1 & 1 & 3 \\ 1 & -3 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 6 & 1 \end{bmatrix}$, and $Q_2 = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 2 & 5 \\ 4 & 2 & -3 & 4 \\ 2 & 3 & 6 & -1 \end{bmatrix}$ and vectors $b_1 = [1,\ 5,\ 3,\ 2]$, $b_2 = [1,\ 2,\ 2,\ 1]$, which generate a nontrivial solution. In this case, the optimal dispatch is given by $x^* = [0.364,\ 0.1897,\ 1.1358,\ 0.8102]^\top\ MW$. Figures 5 and 6 shows the evolution of the dispatched powers along time, using the same simulation parameters, and converging to the optimal solution that maximizes $J_1(x)$ and $J_2(x)$, under the two simultaneous constraints.

VII. CONCLUSIONS

We have introduced novel extremum seeking controllers, based on evolutionary game theory ideas that emerge in general multi-populations games. The algorithms proposed allows the on-line optimization of MAS modeled as population games, where different groups of agents characterized by the same cost function, compete aiming to maximize their cost functions, under multiple coupled and decoupled constraints. The extremum seeking controllers are suitable for application in constrained optimization of networked systems.

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