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On Indigenous Random Consensus and Averaging Dynamics

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Abstract— **We study indigenously evolving random averaging dynamics, i.e., random averaging dynamics whose evolution depends on the history of the random dynamics itself. Such dynamical processes find applications in, e.g., models of distributed learning of comparative adjectives in Linguistics, asymmetric state-dependent random gossiping in Computer Science, Hegselmann-Krause opinion dynamics with link-failure and/or random observation radius in Social Sciences, to name just a few. We introduce a novel supermartingale technique to analyze such history-dependent random dynamics. Using this new tool, we show that an adapted random averaging dynamics converges under general conditions and provide a characterization for the asymptotic behavior of such dynamics.**

I. Introduction

Recently, averaging dynamics have gained a lot of attention due to their role in many distributed engineering problems and modeling of many complex behaviors such as distributed computation [1], [2], [3], distributed optimization [1], [4], distributed estimation [5], [6], distributed rendezvous [7], and opinion dynamics [8].

Many of the works in this field have been focused on the study of deterministic averaging dynamics [9], [10], [11], [12], [13], [14], [15]. To address practical issues such as link-failure and random disturbances in the links, there has been an increasing interest in the study of random averaging dynamics [16], [17], [18], [19], [20], [21], [22], [23]. With the exception of [23], a common feature in almost all of those works is that the randomness is caused by some exogenous disturbance, i.e. the disturbance does not depend on the history of the process but rather is imposed on the process by some external source of error. In [23], averaging dynamics for adapted processes have been studied and some necessary conditions are developed for convergence of those random dynamics to consensus. However, many random models such as various random variants of Hegselmann-Krause dynamics do not satisfy those conditions and stability of such random dynamics has remained widely open.

In this paper, we study random adapted averaging dynamics and show that under a general condition such processes are stable. We provide extensions of some of the results discussed in [24], [15] (for deterministic dynamics) and [25] (for random independent dynamics). Note that due to the space limitations, the proof of Theorem 2 is omitted and will appear elsewhere.

The structure of this paper is as follows: in Section II, we motivate our study by two practical question from the fields of Computer Science, and Social Science. In Section III, we provide our main result and discuss its proof in Section IV. Finally, in Section V, we conclude our paper and discuss some directions for further study in this domain.

II. Motivation

In this section, we provide two motivating examples for our study of random indigenous averaging dynamics. The first example concerns the study of the celebrated Hegselmann-Krause opinion dynamics under random confidence levels and the second example, is on the study of asymmetric gossip algorithm for time-varying networks. To the best of the authors' knowledge, the study of such dynamics cannot be done using the previously known tools for the study of deterministic and random averaging dynamics.

A. *Hegselmann-Krause Opinion Dynamics with Random Confidence Levels:*

Consider the Krause-Hegselmann dynamics as discussed in [8]. In this model, we have a society $[m] = \{1, \ldots, m\}$ of m agents. At time 0, any agent $i \in [m]$ has an initial opinion $x_i(0) \in \mathbb{R}$ about an issue such as presidential candidate. From this time onward, each agent averages out her opinion with agents with similar beliefs in the society. More precisely, let $\mathcal{N}_i(x, \epsilon) = \{j \in [m] \mid \|x_i - x_j\| \leq \epsilon\}$. Then,

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(x(k),\epsilon)} \frac{1}{|\mathcal{N}_i(x(k), \epsilon)|} x_j(k).$$

In this model, $x_i(k)$ is referred to as the opinion of the ith agent at time k and the vector $x(k)$ is referred to as the opinion profile of the society at time k. This model for the evolution of opinions in a society was originally introduced in [8], and later it was used for distributed rendezvous in a robotic network [7]. In this model, $\epsilon > 0$ is called the confidence level and is assumed to be fixed and homogeneous. Using the tools developed in [26] and [27], one can analyze such dynamics, even in the presence of time-varying confidence levels but yet, fixed throughout the society.

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Now, consider the case where at time k the confidence level of each agent is drawn from a distribution $\mathcal{E}(k)$ independent of the other agents. In this case, none of the tools in [26] and [27] can address the stability of such dynamics because in both of the aforementioned analyses, it is essential that if agent i averages her opinion with agent j's opinion, agent j also averages her opinion with the opinion of agent i, i.e. there is mutual respect in this society. However, if the confidence level of the agents in this society are drawn randomly and independent of each other from an arbitrary distribution, there is a chance that this does not happen. Note that such dynamics is not time-independent or stationary. Using the upcoming results, we will show that such dynamics in fact converges, no matter what the sequence of the confidence distributions $\mathcal{E}(k)$ is.

B. Asymmetric Indigenous Gossiping

Here, we discuss an extension of the asymmetric gossip algorithm as discussed in [28] to time-varying networks. In the original gossip algorithm [29], [2], we have a set $[m]$ of m agents and each of them has an initial scalar $x_i(0)$. At each discrete time instant $k = 0, 1, 2, \ldots$, nature picks two agents i, j with some probability $P_{ij} > 0$ from a connected graph $G = ([m], E)$ (i.e. $\{i, j\} \in E$). Then the two agents set:

$$x_i(k+1) = x_j(k+1) = \frac{1}{2}(x_i(k) + x_j(k)), \quad (1)$$

and for the other agents their value remain unchanged. Now, suppose that at each time instant $k \geq 0$, nature picks an ordered pair of agents $(i(k), j(k))$ randomly but dependent on the history of the process. One interesting practical case of such a selection scheme (for example in multi-hop wireless network) is that nature picks $i(k) \in [m]$ and then, picks $j(k) \in \mathcal{N}_{i,\epsilon}(x(k))$ uniformly. Then, agent $i(k)$ sends her value $x_i(k)$ to agent $j(k)$ and agent $j(k)$ updates her value as

$$x_{j(k)}(k+1) = (1 - \alpha(k))x_{j(k)}(k) + \alpha(k)x_{i(k)}(k)), \quad (2)$$

where $\alpha(k)$ is a random variable in $[0, \frac{1}{2}]$.

Again due to the indigenous random nature of this dynamics, none of the previously known approaches to analyze such dynamics applies here. In Section III, we show that if for all $k \geq 0$ and $i, j \in [m]$

$$\begin{aligned}&\mathsf{Pr}\left((i(k), j(k)) = (i, j) \mid \mathcal{F}_k\right)\\ &\qquad \geq \alpha \mathsf{Pr}\left((i(k), j(k)) = (j, i) \mid \mathcal{F}_k\right),\end{aligned}$$

where $\mathcal{F}_k$ is the history of the random evolution up to time k, and also $\alpha(k)$ is independent of $(i(k), j(k))$, then the random dynamics (2) is convergent almost surely.

III. Averaging Dynamics for Adapted Processes

In this section, we present the main result of this paper. We start our discussion by reviewing some notations that will be used throughout the rest of this paper. Then, we present the main result of the paper and discuss its implications on the study of the motivating examples discussed above.

Let $(\Omega, \mathcal{M}, \mathsf{Pr}\,())$ be a probability space and $\{\mathcal{F}_k\}$ be a filtration for $(\Omega, \mathcal{M})$. Also for any $k \in \mathbb{Z}^+$, let $W(k) : \Omega \to S^m$ be a measurable random stochastic matrix where S^m is the set of stochastic matrices in $\mathbb{R}^{m\times m}$ (which are non-negative matrices where elements of each row add up to one). We refer to such a sequence of random matrices as a *random stochastic matrix process*. If $W(k)$ is measurable with respect to $\mathcal{F}_k$, we say that $\{W(k)\}$ is an adapted process (with respect to $\{\mathcal{F}_k\}$). Furthermore, if $\{\mathcal{F}_k\}$ is a sequence of independent sigma-fields, we say that $\{W(k)\}$ is an independent random stochastic matrix process. Finally, for a matrix W and non-trivial subsets $S, T \subset [m]$ (i.e. $S, T \neq \emptyset$ and $S, T \neq [m]$), let:

$$W_{ST} = \sum_{i\in S} \sum_{j\in T} W_{ij}.$$

Our main focus in this paper is to study the dynamics:

$$x(k+1) = W(k+1)x(k), \quad \text{for } k \geq 0, \quad (3)$$

where $x(0)$ is a random vector measurable with respect to $\mathcal{F}_0$ and $\{W(k)\}$ is adapted to $\{\mathcal{F}_k\}$ and provide convergence result for such dynamics. As highlighted in [24], [27], [15], for different models and types of averaging dynamics, the main idea for the convergence of such dynamics is that there is a *balancedness* between nodes in averaging dynamics and our goal is to extend those results for the case of the adapted random dynamics (3). For this, as defined in [25], we say that a random stochastic matrix process $\{W(k)\}$ is balanced if there exists an $\alpha > 0$, independent of time k, with

$$\mathsf{E}[W_{\bar{S}S}(k+1) \mid \mathcal{F}_k] \geq \alpha \mathsf{E}[W_{\bar{S}S}(k+1) \mid \mathcal{F}_k],$$

for all non-trivial $S \subset [m]$ and all $k \geq 0$ where $\bar{S} = [m] \setminus S$ is the complement of the set S (with respect to $[m]$).

The main result of this paper is as follows:

Theorem 1: For any balanced adapted random stochastic matrix process $\{W(k)\}$, such that for all $i \in [m]$ and $k \geq 0$, we have $W_{ii}(k) \geq \gamma > 0$ almost surely, the dynamics (3) converges almost surely.

The proof of this result is presented in Section IV. But before that let us revisit the applications mentioned in Section II. In fact, it is not hard to see that both are examples of dynamics system (3). For example, in the case of the Hegselmann-Krause dynamics with random confidence interval, if $e_1(k), \ldots, e_m(k)$ are the confidence levels of the m agents at time k which are drawn from the distribution $\mathscr{E}(k)$, then this dynamics can be viewed as the dynamics (3) driven by $\{W(k)\}$ defined by:

$$W_{ij}(k+1) = \begin{cases} \frac{1}{|\mathcal{N}_i(x(k),e_i(k))|}, & \text{if } j \in \mathcal{N}_i(x(k),e_i(k)) \\ 0, & \text{otherwise.} \end{cases}$$

Note that in this case, regardless of the realizations $e_1(k),\ldots,e_m(k)$, we have $1 \leq |\mathcal{N}_i(x(k),e_i(k))| \leq m$ as $i \in \mathcal{N}_i(x(k),e_i(k))$ almost surely. Therefore, it follows that $W_{ii}(k) \geq \frac{1}{m}$. Also, note that since $e_i(k), e_j(k)$ have the same distribution, it follows that $P(i \in \mathcal{N}_j(x(k),e_j(k)) = P(j \in \mathcal{N}_i(x(k),e_i(k))$ and hence,

$$\mathsf{E}[W_{ij}(k+1) \mid \mathcal{F}_k] \geq \frac{1}{m}\mathsf{E}[W_{ji}(k+1) \mid \mathcal{F}_k].$$

By summing up both sides of the above equation for $i \in S$ and $j \in \bar{T}$, it follows that the matrix process generated by the Hegselmann-Krause opinion dynamics with random confidence interval is a balanced process. Therefore, by Theorem 1 it follows that the Hegselmann-Krause opinion dynamics with random confidence interval is convergent.

Similarly, it can be shown that for asymmetric indigenous gossiping the dynamics (2) is an instance of the dynamics (3) and also $\mathsf{E}[W_{ij}(k+1) \mid \mathcal{F}_k] \geq \alpha\mathsf{E}[W_{ji}(k+1) \mid \mathcal{F}_k]$. In addition $W_{ii}(k) \geq \frac{1}{2}$ for all $i \in [m]$ and $k \geq 0$ and hence, by the above discussion, it falls in the scope of the random processed addressed by Theorem 1. Therefore, the asymmetric indigenous gossip algorithm is convergent almost surely.

IV. Proof of the Main Theorem

In this section, we provide a proof of Theorem 1. Before discussing the proof, let us introduce some notations that will be used subsequently. We say that a random set $S : \Omega \to \mathscr{P}([m])$ is a random subset of $[m] = \{1,\ldots,m\}$ if S is measurable with respect to $([m], \mathscr{P}([m])$ where $\mathscr{P}([m])$ is the set of all subsets of $[m]$. Moreover, we say that a sequence $\{S(k)\}$ of random subsets is adapted to $\{\mathcal{F}_k\}$ if $S(k)$ is measurable with respect to $\mathcal{F}_k$.

In our development an object, *regular sequence*, plays a central role. As defined in [30], in the deterministic setting, a sequence $\{S(k)\}$ of subsets of $[m]$ is called regular if $|S(k)| = |S(0)| \geq 1$ for all k, i.e. the cardinality of $S(k)$ does not change with time. We say that $\{S(k)\}$ is an adapted regular sequence if $S(k)$ is measurable with respect to $\mathcal{F}_k$ and also $|S(k)| = \ell$ almost surely for some $\ell \in [m]$. It should be clear that in a deterministic setting, i.e. the case that $\mathcal{F}_k = \{\emptyset, \Omega\}$ for all $k \geq 0$, the two definitions coincide.

Now, let us extend the notion of balanced asymmetry [15] to adapted processes as follows.

Definition 1: We say that an adapted stochastic matrix process $\{W(k)\}$ is *balanced asymmetry* if for any regular adapted sequence $\{S(k)\}$, we have:

$$\begin{aligned}&\mathsf{E}\big[W_{\bar{S}(k+1)S(k)}(k+1) \mid \mathcal{F}_k\big] \\ &\qquad \geq \alpha\mathsf{E}\big[W_{S(k+1)\bar{S}(k)}(k+1) \mid \mathcal{F}_k\big]\end{aligned} \tag{4}$$

for some $\alpha > 0$.

Note that if $\{S(k)\}$ is a regular adapted process $\{\bar{S}(k)\}$ is also a regular adapted process. Therefore, considering this fact, it immediately follows that $\alpha \leq 1$ in (1).

The major challenging step towards proving Theorem 1 is to show that the balanced processes described in the statement of the result, i.e. balanced matrix processes such that $W_{ii}(k) \geq \gamma$ for all $i \in [m]$ and $k \geq 0$ almost surely, are balanced asymmetry.

Theorem 2: Let $\{W(k)\}$ be an adapted matrix process. Assume that $W_{ii}(k) \geq \gamma$ almost surely for some $\gamma \in (0,1]$ and all $i \in [m]$ and any $k \geq 0$. Furthermore, suppose that

$$\mathsf{E}[W_{\bar{S}S}(k+1) \mid \mathcal{F}_k] \geq a\mathsf{E}[W_{S\bar{S}}(k+1) \mid \mathcal{F}_k], \tag{5}$$

for all $S \subset [m]$ and $k \geq 0$. Then $\{W(k)\}$ is balanced asymmetric with coefficient $\alpha = \frac{a\gamma}{4m}$.

The next step is to show that any random averaging dynamics generated by any balanced asymmetric adapted process is convergent up to a random permutation. The proof technique is based on the proof technique in [31], and its modification in [15] and the developed machinery above.

Theorem 3: Let $\{W(k)\}$ be an adapted matrix process that is balanced asymmetric with coefficient α and let $\{x(k)\}$ be a dynamics generated by $\{W(k)\}$.

a. Let $z(t)$ be a random vector formed by an ordering of the entries of $x(t)$, i.e. $z_1(t) \leq z_2(t) \leq \cdots \leq z_m(t)$ and for any $\omega \in \Omega$, there exists a permutation $\pi(\omega) : [m] \to [m]$ such that $x_{\pi(\omega,i)} = z_i(t)$. Then, $\lim_{t\to\infty} z(t) = z(\infty)$ exists almost surely.

b. Consider the infinite flow event

$$\Omega^\infty = \{\omega \in \Omega \mid \sum_{t=}^{\infty} W_{\bar{S}(k+1)S(k)}(k) = \infty$$
$$\text{for any adapted regular sequence } \{S(k)\}\}.$$

Then, on Ω^∞, we almost surely have $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$, i.e. agents reach consensus. As a result, on Ω^∞, we almost surely have $\lim_{t\to\infty} x(t) = c\mathbf{1}$ for a random variable c.

c. Suppose that $W_{ii}(k) \geq \gamma > 0$ almost surely for all $i \in [m]$ and $k \geq 0$. Then, $\lim_{t\to\infty} x(t)$ exists almost surely.

Proof: a. Fix an $\ell \in [m]$. Let $S_\ell(k) : \Omega \to \mathscr{P}([m])$ be the index of the lower ℓ entries of $x(k)$, i.e. $S_\ell(k) = \{\pi^{-1}(\omega,1),\ldots,\pi^{-1}(\omega,\ell)\}$ (π is the random permutation defined in the statement of theorem). Note that $S(k)$ is measurable with respect to $\mathcal{F}_k$. Now, let

$$V_\ell(k) = \sum_{i=1}^{\ell} \beta^i z_i(k) = \sum_{i\in S(k)} \beta^{\pi^{-1}(i)} x_i(k),$$

where $\beta = \frac{\alpha}{2}$. Note that $V_\ell(k)$ is measurable with respect to $\mathcal{F}_k$ and is bounded almost surely. Using some algebraic steps and the fact that $W(k)$ is stochastic almost surely, as shown in Eq. (31) and Eq. (32) in [15], it follows that almost surely:

$$V_\ell(k+1) - V_\ell(k) \geq \sum_{p=1}^{\ell-1} \Big(\beta^p W_{S_p(k+1)\bar{S}_p(k)}(k+1) - \alpha^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \Big) (z_{p+1}(k) - z_p(k)).$$

Applying conditional expectation on both sides of the above inequality and using the balanced asymmetric property of $\{W(k)\}$, it follows that:

$$\begin{aligned}
&\mathsf{E}[V_\ell(k+1) - V_\ell(k) \mid \mathcal{F}_k] \\
&\geq \sum_{p=1}^{\ell-1} \mathsf{E}[\beta^p W_{S_p(k+1)\bar{S}_p(k)}(k+1) \\
&- \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \mid \mathcal{F}_k](z_{p+1}(k) - z_p(k)) \\
&\geq \sum_{p=1}^{\ell-1} \mathsf{E}\Big[2\beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \\
&- \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \mid \mathcal{F}_k\Big](z_{p+1}(k) - z_p(k)) \\
&= \sum_{p=1}^{\ell-1} \mathsf{E}\Big[\beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \mid \mathcal{F}_k\Big] \\
&\qquad \times (z_{p+1}(k) - z_p(k)). \qquad (6)
\end{aligned}$$

From Doob's martingale convergence theorem, one can immediately imply that $V_\ell(k)$ is convergent almost surely for any $\ell \in [m]$. But note that $z_\ell(k) = \beta^{-\ell-1}(V_{\ell+1}(k) - V_\ell(k))$ and hence, it follows that $\lim_{k\to\infty} z(k) = z(\infty)$ exists almost surely.

b. Since the process $\{V_\ell(k)\}$ is bounded almost surely, from (6), it follows that:

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{m-1} \sum_{p=1}^{\ell-1} \mathsf{E}\Big[\beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \mid \mathcal{F}_k\Big] \times (z_{p+1}(k) - z_p(k)) < \infty.$$

But since $\beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1)(z_{p+1}(k) - z_p(k)) \leq d(x(0))$ is bounded almost surely, from the dominated convergence theorem for conditional expectations ([32], page 262), it follows that

$$\mathsf{E}\Big[\sum_{\substack{k\geq 0 \\ \ell\in[m-1]}} \sum_{p=1}^{\ell-1} \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \times (z_{p+1}(k) - z_p(k))\Big] < \infty$$

and hence, we almost surely have

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{m-1} \sum_{p=1}^{\ell-1} \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \times (z_{p+1}(k) - z_p(k)) < \infty.$$

Now, if for some $\omega \in \Omega^\infty$, and some $i \in [m]$, we have $\lim_{k\to\infty}(z_i(k) - z_{i-1}(k)) = z_i(\infty) - z_{i-1}(\infty) > 0$, then since $\sum_{k=1}^{\infty} W_{\bar{S}_i(k+1)S_{i-1}(k)}(k+1) = \infty$ on Ω^∞, it follows that $\sum_{k=1}^{\infty} W_{\bar{S}_i(k+1)S_{i-1}(k)}(k+1)(z_i(k) - z_i(k)) = \infty$. But since

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{m-1} \sum_{p=1}^{\ell-1} \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \times (z_{p+1}(k) - z_p(k)) < \infty$$

it follows that for almost all points in Ω^∞, we have $z_i(\infty) - z_{i-1}(\infty) = 0$. This implies that the dynamics $\{x(k)\}$ admits weak consensus almost surely on Ω^∞ and, moreover, from Theorem 1 in [33] it follows that $\{x(k)\}$ is convergent to the consensus line on Ω^∞.

c. Suppose that for all $k \geq 0$ and $i \in [m]$, $W_{ii}(k) \geq \gamma$ almost surely and suppose that on a set $\Omega' \subset \Omega$, we have that $\lim_{k\to\infty} x(k)$ does not exist. Without loss of generality, we may assume that there exists $i \in [m]$ such that $\lim_{k\to\infty} x_i(k)$ does not exists on the set Ω' (otherwise, we can restrict ourself on such a set). First notice that $\lim_{k\to\infty} z(k) \neq c\mathbf{1}$ on Ω', otherwise, as in the previous case, this implies that $\lim_{k\to\infty} x(k) = c\mathbf{1}$. Now, fix an $\omega \in \Omega'$ and consider the corresponding sample path of the dynamics. Let $\{a_1, \ldots, a_q\} = \{z_1(\infty), \ldots, z_m(\infty)\}$ with $a_1 < \ldots < a_q$ be the distinct values of the entries of $z(\infty)$ for the sample point ω ($q \leq m$). Note that, for any $\epsilon \leq \frac{1}{4}\min_{1\leq p<q}(a_{p+1} - a_p)$, there exists a time instance $T_\epsilon \geq 0$ such that for $k \geq T_\epsilon$, $x_i(k)$ is at the ϵ-neighborhood of one of the points in $\{a_1, \ldots, a_q\}$. This point is unique because $\epsilon \leq \frac{1}{4}\min_{1\leq p<q}(a_{p+1} - a_p)$. Let the index of that point be $p(k)$, i.e. $|x_i(k) - a_{p(k)}| < \epsilon$ for $k > T_\epsilon$. But since $\lim_{k\to\infty} x_i(k)$ does not exists, it follows that there is a sequence of the increasing time instances $k_1 < k_2 < \ldots$ such that $p(k_t) \not= p(k_t + 1)$. This implies that $S(k_t+1) \not= S(k_t)$ for some ℓ, as defined in part a. and also, $z_{\ell+1}(k_t) - z_\ell(k_t) \geq \frac{1}{4}\min_{1\leq p<q}(a_{p+1} - a_p)$. But since $W_{ii}(k_t+1) \geq \gamma$ almost sure for all i, it follows that $W_{\bar{S}(k_t+1)S(k_t)}(k_t+1) \geq \gamma$, and hence,

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{m-1} \sum_{p=1}^{\ell-1} \beta^{p+1} W_{\bar{S}_p(k+1)S_p(k)}(k+1) \times (z_{p+1}(k) - z_p(k)) = \infty,$$

for the sample point $\omega \in \Omega'$. But based on part b. this happens almost never, and hence, it follows that $\mathsf{Pr}(\Omega') = 0$ and hence, the result follows. ■

Using Theorem 2 and Theorem 3, one can immediately conclude the proof of Theorem 1: by Theorem 2, balanced random matrix processes $\{W(k)\}$ with the property that $W_{ii}(k) \geq \gamma$ almost surely for all $k \geq 0$ and $i \in [m]$ are balanced asymmetric and based on part c. of Theorem 3, it follows that the dynamics system (3) is convergence almost surely for such random matrix processes.

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V. CONCLUSION

Motivated by two problems in social sciences and computer science, we studied indigenous random averaging dynamics. We showed that under a general condition, such dynamics are convergent almost surely. To do, this we generalized notions of balanced and balanced asymmetric from deterministic dynamics to adapted processed. We showed that under the general property of balanced asymmetry, random averaging dynamics converge up to a permutation almost surely.

There are many directions for further study worth mentioning. One of the related theoretical challenges, which has important practical implications, is to characterize the rate of convergence of such random dynamics. Another interesting direction to explore is to use the tools and results developed here to study multi-agent learning dynamics which is a subject of our current research.

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