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On the Exactness of Semidefinite Relaxation for Nonlinear Optimization over Graphs: Part I

Somayeh Sojoudi* and Javad Lavaei+
*Department of Computing and Mathematical Sciences, California Institute of Technology
+Department of Electrical Engineering, Columbia University

***Abstract*— This work is concerned with finding a global optimization technique for a broad class of nonlinear optimization problems, including quadratic and polynomial optimizations. The main objective of this two-part paper is to investigate how the (hidden) structure of a given real/complex-valued optimization makes the problem easy to solve. To this end, three conic relaxations are proposed. Necessary and sufficient conditions are derived for the exactness of each of these relaxations, and it is shown that these conditions are satisfied if the optimization is highly structured. More precisely, the structure of the optimization is mapped into a generalized weighted graph, where each edge is associated with a weight set extracted from the coefficients of the optimization. In the real-valued case, it is shown that the relaxations are all exact if each weight set is sign definite and in addition a condition is satisfied for each cycle of the graph. It is also proved that if some of these conditions are violated, the relaxations still provide a low-rank solution for weakly cyclic graphs. In the complex-valued case, the notion of "sign definite complex sets" is introduced for complex weight sets. It is then shown that the relaxations are exact if each weight set is sign definite (with respect to complex numbers) and the graph is acyclic. In part II of the paper, the complex-valued case is further studied for cyclic graphs and moreover the application of this two-part paper in power system is thoroughly discussed.**

I. Introduction

Several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program (QCQP) as a special case, are nonlinear/non-convex and NP-hard in the worst case. The paper [1] provides a survey on the computational complexity of optimizing various classes of continuous functions over some simple constraint sets. Due to the complexity of such problems, several convex relaxations based on linear matrix inequality (LMI), semidefinite programming (SDP), and second-order cone programming (SOCP) have gained popularity [2], [3]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value. The paper [4] shows how SDP relaxation can be used to find better approximations for maximum cut (MAX CUT) and maximum 2-satisfiability (MAX 2SAT) problems. Another approach is proposed in [5] to solve the max-3-cut problem via complex SDP. The approaches in [4] and [5] have been generalized in several papers, including [6]–[10].

The SDP relaxation converts an optimization with a vector variable to a convex optimization with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank (e.g., rank-1) solution for the SDP relaxation. Several papers have studied the existence of a low-rank solution to matrix optimizations with linear and LMI constraints [11], [12]. The papers [13] and [14] provide an upper bound on the lowest rank of all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [15] to find a rank-1 solution whenever the number of constraints is small. This technique is extended in [16] to the complex SDP problem. The paper [17] derives a polynomial-time algorithm for finding an approximate low-rank solution.

The present paper is motivated by the fact that real-world optimization problems are highly structured in many ways and their structures could in principle help reduce the computational complexity. For example, transmission lines and transformers used in power networks are passive devices, and as a result optimization problems defined over electrical power networks have certain structures which distinguish them from abstract optimizations with random coefficients. The high-level objective of this paper is to understand how the computational complexity of a given nonlinear optimization is related to its (hidden) structure. This work is concerned with a broad class of nonlinear real/complex optimization problems, including QCQP. The main feature of this class is that the argument of each objective and constraint function is quadratic (as opposed to linear) with respect to the optimization variable and the goal is to use three conic relaxations (SDP, reduced SDP and SOCP) to convexify the argument of the optimization.

In this work, the structure of the nonlinear optimization is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization (e.g., the coefficients). This generalized weighted graph captures both the sparsity of the optimization and possible patterns in the coefficients. In Part I of the paper, it is shown that the proposed relaxations are exact for real-valued optimizations, provided a set of conditions is satisfied. These conditions need each weight set to be sign definite and each cycle of the graph to have an even number of positive weight sets. It is also shown that if some of these conditions are not satisfied, the SDP relaxation is guaranteed to have a rank-2 solution for weakly cyclic graphs, from which an approximate rank-1 solution may be recovered. To study the complex-valued case, the notion of "sign-definite complex weight sets" is introduced and it is

Emails: sojoudi@caltech.edu and lavaei@ee.columbia.edu

then proved that the relaxations are exact for a complex optimization if the graph is acyclic with sign definite weight sets (with respect to complex numbers). In Part II of the paper, the complex case is further studied for general graphs. In addition, Part II discusses the application of this work in optimization for power systems and shows that a broad class of energy optimizations can be convexified due to the physics of power networks. The results of this paper extend the recent works on energy optimization [18]–[23] and general quadratic optimization [24], [25].

In the next section, we formally state the optimization problem and then survey two related works. The main contributions of the paper are outlined in Section II-D, where the plan for the rest of the paper is also given.

II. Problem Statement and Contributions

Before introducing the problem, essential notations and definitions will be provided first.

A. Notations and Definitions

Notation 1: In this work, scalars, vectors, and matrices will be shown by lowercase, bold lowercase, and uppercase letters (e.g., x, $\mathbf{x}$, and X). Furthermore, x_i denotes the i^{th} entry of a vector $\mathbf{x}$, and X_{ij} denotes the $(i,j)^{\text{th}}$ entry of a matrix X.

Notation 2: $\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}^n$, and $\mathcal{H}^n$ denote the sets of real numbers, complex numbers, $n \times n$ symmetric matrices, and $n \times n$ Hermitian matrices, respectively.

Notation 3: Re$\{M\}$, Im$\{M\}$, M^H, Rank$\{M\}$, and Trace$\{M\}$ denote the real part, imaginary part, conjugate transpose, rank, and trace of a given scalar/matrix M, respectively. The notation $M \succeq 0$ means that M is symmetric/Hermitian and positive semidefinite.

Notation 4: The symbol $\measuredangle(x)$ represents the phase of a complex number x. The imaginary unit is denoted as "i", while "i" is used for indexing.

Notation 5: Given an undirected graph $\mathcal{G}$, the notation $i \in \mathcal{G}$ means that i is a vertex of $\mathcal{G}$. Moreover, the notation $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of $\mathcal{G}$ and besides $i < j$.

Notation 6: Given a set $\mathcal{T}$, $|\mathcal{T}|$ denotes its cardinality. Given a graph $\mathcal{G}$, $|\mathcal{G}|$ shows the number of its vertices. Given a number (vector) $\mathbf{x}$, $|\mathbf{x}|$ denotes its absolute value (2-norm).

Definition 1: A finite set $\mathcal{T} \subset \mathcal{R}$ is said to be *sign definite with respect to* $\mathcal{R}$ if its elements are either all negative or all nonnegative. $\mathcal{T}$ is called *negative* if its elements are negative and is called *positive* if its elements are nonnegative.

Definition 2: A finite set $\mathcal{T} \subset \mathcal{C}$ is said to be *sign definite with respect to* $\mathcal{C}$ if when the sets $\mathcal{T}$ and $-\mathcal{T}$ are mapped into two collections of points in $\mathcal{R}^2$, then there exists a line separating the two sets (the elements of the sets are allowed to lie on the line).

To illustrate Definition 2, consider a complex set $\mathcal{T}$ with four elements, whose corresponding points are labeled as 1, 2, 3, and 4 in Figure 1(a). The points corresponding to $-\mathcal{T}$ are labeled as 1', 2', 3', and 4' in the same picture. Since there exists a line separating x's (elements of $\mathcal{T}$) from o's (elements of $-\mathcal{T}$), the set $\mathcal{T}$ is sign definite. In contrast, if the elements of $\mathcal{T}$ are distributed according to Figure 1(b), the set will no longer be sign definite. Note that Definition 2 is inspired by the fact that a real set $\mathcal{T}$ is sign definite with respect to $\mathcal{R}$ if $\mathcal{T}$ and $-\mathcal{T}$ are separable via a point (on the horizontal axis).

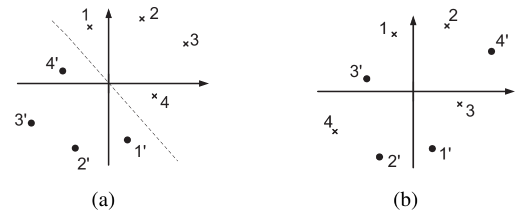

Fig. 1. In Figure (a), there exists a line separating x's (elements of $\mathcal{T}$) from o's (elements of $-\mathcal{T}$) so the set $\mathcal{T}$ is sign definite. In Figure (b), this is not the case.

Definition 3: Given a graph $\mathcal{G}$, a cycle space is the set of all possible cycles in the graph. An arbitrary basis for this cycle space is called a "cycle basis".

Definition 4: In this work, a graph $\mathcal{G}$ is called weakly cyclic if every edge of the graph belongs to at most one cycle in $\mathcal{G}$ (i.e., the cycles of $\mathcal{G}$ are all edge-disjoint).

Definition 5: Consider a graph $\mathcal{G}$, a subgraph $\mathcal{G}_s$ of this graph, and a matrix $X \in \mathcal{C}^{|\mathcal{G}|\times|\mathcal{G}|}$. Define $X\{\mathcal{G}_s\}$ as a submatrix of X obtained by choosing every row and column of X whose index belongs to the vertex set of $\mathcal{G}_s$. For instance, $X\{(i,j)\}$, where $(i,j) \in \mathcal{G}$, has rows i,j and columns i,j of X.

B. Problem Statement

Consider an undirected graph $\mathcal{G}$ with n vertices (nodes), where each edge $(i,j) \in \mathcal{G}$ has been assigned a nonzero edge weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ with k real/complex numbers (note that the superscripts in the weights are not exponents). This graph is called a *generalized weighted graph* as every edge is associated with a set of weights as opposed to a single weight. Consider an unknown vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}$ belonging to $\mathcal{D}^n$, where $\mathcal{D}$ is either $\mathcal{R}$ or $\mathcal{C}$. For every $i \in \mathcal{G}$, x_i is a variable associated with node i of the graph $\mathcal{G}$. Define:

$$\mathbf{y} = \left\{|x_i|^2 \;\middle|\; \forall i \in \mathcal{G}\right\},$$
$$\mathbf{z} = \left\{\text{Re}\{c_{ij}^{(t)} x_i x_j^H\} \;\middle|\; \forall (i,j) \in \mathcal{G},\ t \in \{1,...,k\}\right\}$$

Note that according to Notation 5, $(i,j) \in \mathcal{G}$ means that (i,j) is an edge of the graph and that $i < j$. The sets $\mathbf{y}$ and $\mathbf{z}$ can be regarded as two vectors, where

- $\mathbf{y}$ collects the quadratic terms $|x_i|^2$'s (one term for each vertex).
- $\mathbf{z}$ collects the cross terms Re$\{c_{ij}^{(t)} x_i x_j^H\}$'s ($k$ terms for each edge).

Although the above formulation deals with Re$\{c_{ij}^{(t)} x_i x_j^H\}$ whenever $(i,j) \in \mathcal{G}$, it can handle terms of the form Re$\{\alpha x_j x_i^H\}$ and Im$\{\alpha x_i x_j^H\}$ for a complex weight α. This can be carried out using the following transformations:

$$\text{Re}\{\alpha x_j x_i^H\} = \text{Re}\{(\alpha^H) x_i x_j^H\},$$
$$\text{Im}\{\alpha x_i x_j^H\} = \text{Re}\{(-\alpha \text{i}) x_i x_j^H\}$$

This work is concerned with the following optimization:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{D}^n} \quad & f_0(\mathbf{y},\mathbf{z}) \\ \text{subject to} \quad & f_j(\mathbf{y},\mathbf{z}) \leq 0, \quad j=1,2,...,m \end{aligned} \tag{1}$$

for given functions $f_0, ..., f_m$. The computational complexity of the above optimization depends in part on the structure of the functions f_j's. Regardless of these functions, Optimization (1) is intrinsically hard to solve (NP-hard in the worst case) because $\mathbf{y}$ and $\mathbf{z}$ are both nonlinear functions of $\mathbf{x}$. The objective is to convexify the second-order nonlinearity embedded in $\mathbf{y}$ and $\mathbf{z}$. To this end, notice that there exist two linear functions $l_1 : \mathcal{C}^{n\times n} \to \mathcal{R}^n$ and $l_2 : \mathcal{C}^{n\times n} \to \mathcal{R}^{k\tau}$ such that $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$, where τ denotes the number of edges in $\mathcal{G}$. Motivated by the above observation, if $\mathbf{x}\mathbf{x}^H$ is replaced by a new matrix variable X, then $\mathbf{y}$ and $\mathbf{z}$ both become linear in X. This implies that the non-convexity induced by the quadratic terms $\text{Re}\{c_{ij}^{(t)} x_i x_j\}$'s and $|x_i|^2$'s all disappear if Optimization (1) is reformulated in terms of X. However, the optimal solution X may not be decomposable as $\mathbf{x}\mathbf{x}^H$ unless some additional constraints are imposed on X. It is straightforward to verify that Optimization (1) is equivalent to

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{2a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j=1,...,m \tag{2b}$$
$$X \succeq 0, \tag{2c}$$
$$\text{Rank}\{X\} = 1 \tag{2d}$$

where there is an implicit constraint that $X \in \mathcal{S}^n$ if $\mathcal{D} = \mathcal{R}$ and $X \in \mathcal{H}^n$ if $\mathcal{D} = \mathcal{C}$. To reduce the computational complexity of the above problem, two actions can be taken: (i) removing the non-convex constraint (2d), and (ii) relaxing the convex, but computationally-expensive, constraint (2c) to a set of simpler constraints on certain low-order submatrices of X. Based on this methodology, three relaxations will be proposed for Optimization (1) next.

SDP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{3a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \quad j=1,...,m \tag{3b}$$
$$X \succeq 0 \tag{3c}$$

Reduced SDP relaxation: Choose a set of cycles $\mathcal{O}_1, ..., \mathcal{O}_p$ in the graph $\mathcal{G}$ such that they form a cycle basis. Let Ω denote the set of all subgraphs $\mathcal{O}_1, ..., \mathcal{O}_p$ as well as all edges of $\mathcal{G}$ that do not belong to any cycle in the graph (i.e., bridge edges). The reduced SDP relaxation is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{4a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j=1,...,m \tag{4b}$$
$$X\{\mathcal{G}_s\} \succeq 0, \qquad \forall \mathcal{G}_s \in \Omega \tag{4c}$$

SOCP relaxation: This optimization is defined as

$$\min_{X} \quad f_0(l_1(X), l_2(X)) \tag{5a}$$
$$\text{s.t.} \quad f_j(l_1(X), l_2(X)) \leq 0, \qquad j=1,...,m \tag{5b}$$
$$X\{(i,j)\} \succeq 0, \qquad \forall (i,j) \in \mathcal{G} \tag{5c}$$

The reason why the above optimization is called an SOCP problem is that the condition $X\{(i,j)\} \succeq 0$ can be replaced by the linear and norm constraints

$$X_{ii}, X_{jj} \geq 0, \quad X_{ii} + X_{jj} \geq \left\| \begin{bmatrix} X_{ii} & X_{jj} & \sqrt{2}X_{ij} \end{bmatrix} \right\|$$

The above SDP, reduced SDP, and SOCP relaxations are targeted at the non-convexity caused by the nonlinear relationship between $\mathbf{x}$ and $(\mathbf{y}, \mathbf{z})$. Note that these optimizations are convex relaxations only when the functions $f_0, ..., f_m$ are all convex. If any of these functions is non-convex, additional relaxations may be needed to convexify the SDP, reduced SDP, or SOCP optimization. Define f^*, f^*_{SDP}, $f^*_{\text{r-SDP}}$, and f^*_{SOCP} as the optimal solutions of Optimizations (2), (3), (4), and (5), respectively. By comparing the feasible sets of these optimizations, it can be concluded that

$$f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^* \tag{6}$$

Given a particular optimization of the form (1), if any of the above inequalities for f^* turns into an equality, the associated relaxation may be able to find the solution of the original optimization. In this case, it is said that the relaxation is "tight" or "exact". The objective of this paper is to relate the exactness of the proposed relaxations to the topology of the graph $\mathcal{G}$ and its weight sets $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$'s.

It is noteworthy that the aforementioned problem formulation can be easily generalized in two directions:

- *Allowance of weight sets with different cardinalities:* The above problem formulation assumes that every edge weight set has k elements. However, if the weight sets have different sizes, the trivial weight 0 can be added to each set multiple times in such a way that all expanded sets reach the same cardinality.
- *Inclusion of linear terms in* $\mathbf{x}$*:* Optimization 1 is formulated in $\mathbf{x}\mathbf{x}^H$ with no linear term in $\mathbf{x}$. This issue can be fixed by defining an expanded vector $\tilde{\mathbf{x}}$ as $\begin{bmatrix} 1 & \mathbf{x}^H \end{bmatrix}^H$. Then, the matrix $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H$ needs to be replaced by a new matrix variable $\tilde{X}$ under the constraint $\tilde{X}_{11} = 1$.

C. Related Work

Consider the QCQP optimization:

$$\min_{\mathbf{x}\in\mathcal{D}^n} \quad \mathbf{x}^H M_1 \mathbf{x} \tag{7a}$$
$$\text{s.t.} \quad \mathbf{x}^H M_j \mathbf{x} \leq 0, \quad j=2,...,k \tag{7b}$$

for given matrices $M_1, ..., M_k \in \mathcal{H}^n$. This problem is a special case of Optimization (1), where its generalized weighted graph $\mathcal{G}$ has two properties:

- Given two nodes $i, j \in \{1, ..., n\}$ such that $i < j$, there exists an edge between nodes i and j if and only if the

(i,j) off-diagonal entry of at least one of the matrices $M_1, ..., M_k$ is nonzero.

- For every $(i,j) \in \mathcal{G}$, the weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ is the union of the $(i,j)^{\text{th}}$ entries of $M_1, ..., M_k$.

Due to the relation $\mathbf{x}^H M_i \mathbf{x} = \text{Trace}\{M_i \mathbf{x}\mathbf{x}^H\}$ for $i = 1, ..., k$, the SDP relaxation of Optimization (7) turns out to be

$$\min_{X} \quad \text{Trace}\{M_1 X\} \tag{8a}$$

$$\text{s.t.} \quad \text{Trace}\{M_j X\} \leq 0, \quad j = 2, ..., k \tag{8b}$$

$$X \succeq 0 \tag{8c}$$

The SOCP relaxation of Optimization (7) is obtained by replacing the constraint (8c) with $X\{(i,j)\} \succeq 0,\ (i,j) \in \mathcal{G}$. The relationship between Optimization (7) and its relaxations have been studied in two special cases in the literature:

- Consider the case $D = \mathcal{R}$. It has been shown in [24] that $f^*_{\text{SOCP}} = f^*_{\text{SDP}} = f^*$ if $-M_0, ..., -M_k$ are all Metzler matrices. This implies that the proposed relaxations are all exact, independent of the topology of $\mathcal{G}$, as long as the set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ is negative for all $(i,j) \in \mathcal{G}$.
- Consider the case $D = \mathcal{C}$. It has been shown in the recent work [25] that $f^*_{\text{SDP}} = f^*$ if three conditions hold:
  1) $\mathcal{G}$ is a tree graph.
  2) M_1 is a positive semidefinite matrix.
  3) For every $(i,j) \in \mathcal{G}$, the origin $(0,0)$ is not an interior point of the convex hull of the 2-d polytope induced by the weight set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$.

It can be shown that Condition (3) implies that the complex set $\{c_{ij}^{(1)}, c_{ij}^{(2)}, ..., c_{ij}^{(k)}\}$ is sign definite (see Definition 2). The above results suggest that the polynomial-time solvability of certain classes of QCQP problems might be inferred from weak properties of their underlying generalized weighted graphs.

D. Contributions

Throughout this paper, we assume that $f_j(\mathbf{y}, \mathbf{z})$ is monotonic in every entry of $\mathbf{z}$ for $j = 0, 1, ..., m$ (but possibly non-convex in $\mathbf{y}$ and $\mathbf{z}$). With no loss of generality, suppose that $f_j(\mathbf{y}, \mathbf{z})$ is an increasing function with respect to all entries of $\mathbf{z}$ (to ensure this property, it may be needed to change the sign of some edge weights and then redefine the functions). A few of the results to be developed in this work do not need this assumption, in which cases the name of the function f_j will be changed to g_j to avoid any confusion in the assumptions.

In section III, we derive necessary and sufficient conditions for the exactness of the SDP, reduced SDP, and SOCP relaxations in both real and complex cases. In Section IV, we consider the real-valued case $\mathcal{D} = \mathcal{R}$ and show that these relaxations are all tight, provided each weight set is sign definite with respect to $\mathcal{R}$ and that each cycle of the graph has an even number of edges with positive weight sets. These conditions are naturally satisfied for: (i) an cyclic graph $\mathcal{G}$ with arbitrary sign definite edge sets, (ii) a bipartite graph $\mathcal{G}$ with positive weight sets, and (iii) an arbitrary graph $\mathcal{G}$ with negative weight sets. It is also shown that if the SDP relaxation is not exact, it still has a low rank (rank-2) solution for: (i) an acyclic graph $\mathcal{G}$ (with potentially indefinite weight sets), and (ii) a weakly-cyclic bipartite graph $\mathcal{G}$ with sign definite edge sets. In section V, we consider the complex-valued case $\mathcal{D} = \mathcal{C}$ under the assumption that each edge set $\{c_{ij}^{(1)}, ..., c_{ij}^{(k)}\}$ is sign definite with respect to $\mathcal{C}$. This assumption is trivially met if $k \leq 2$ or each weight set contains only real (or imaginary) numbers. It is shown that the proposed relaxations are all exact if $\mathcal{G}$ is acyclic. In section VI, illustrative examples are provided.

III. SDP, Reduced-SDP, and SOCP Relaxations

In this section, the objective is to derive necessary and sufficient conditions for the exactness of the SDP, reduced-SDP, and SOCP Relaxations. For every $r \in \{1, 2, ..., p\}$, let $\vec{\mathcal{O}}_r$ denote a directed cycle corresponding to $\mathcal{O}_r$, meaning that all edges of the undirected cycle $\mathcal{O}_r$ has been oriented consistently.

Theorem 1: The following statements hold true in both real and complex cases $\mathcal{D} = \mathcal{R}$ and $\mathcal{D} = \mathcal{C}$:

i) The SDP relaxation is exact (i.e., $f^*_{\text{SDP}} = f^*$) if and only if it has a rank-1 solution X^*.

ii) The reduced SDP relaxation is exact (i.e., $f^*_{\text{r-SDP}} = f^*$) if and only if it has a solution X^* such that

$$\text{Rank}\{X^*\{\mathcal{G}_s\}\} = 1, \qquad \forall \mathcal{G}_s \in \Omega \tag{9}$$

iii) The SOCP relaxation is exact (i.e., $f^*_{\text{SOCP}} = f^*$) if and only if it has a solution X^* such that

$$\text{Rank}\{X^*\{(i,j)\}\} = 1, \qquad \forall (i,j) \in \mathcal{G}$$

and that

$$\sum \measuredangle X^*_{ij} = 0, \qquad \forall r \in \{1, 2, ..., p\} \tag{10}$$

where the sum is taken over all directed edges (i,j) of the oriented cycle $\vec{\mathcal{O}}_r$. Moreover, the same result holds even if the condition (10) is replaced by (9).

Proof: The proof has been moved to [26]. ■

Theorem 1 provides necessary and sufficient conditions for the exactness of the SDP, reduced SDP, and SOCP relaxations. As mentioned before, one can write:

$$f^*_{\text{SOCP}} \leq f^*_{\text{r-SDP}} \leq f^*_{\text{SDP}} \leq f^*$$

Using the matrix completion theorem, two conclusions can be made [27]:

- If $\mathcal{G}$ is an acyclic graph, then the relation $f^*_{\text{SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}}$ holds, independently of whether or not $f^*_{\text{SDP}} = f^*$.
- Consider an expansion of the graph $\mathcal{G}$ by connecting all vertices inside each cycle $\mathcal{O}_r$ to each other for $r = 1, 2, ..., p$. The relation $f^*_{\text{r-SDP}} = f^*_{\text{SDP}}$ holds (independently of whether or not $f^*_{\text{SDP}} = f^*$) if every maximal clique (complete subgraph) of the expanded graph corresponds to a single edge of $\mathcal{G}$ or one of the cycles $\mathcal{O}_1, ..., \mathcal{O}_p$. This mild condition is met for weakly cyclic graphs as well as a broad class of planar graphs.

IV. Real-valued Optimization

In this section, Optimization (1) will be studied in the real-valued case (i.e., $\mathcal{D}=\mathcal{R}$). Since $\mathbf{x}\in\mathcal{R}^n$, one can write:

$$\text{Re}\left\{c_{ij}^{(t)}x_i x_j^H\right\} = \text{Re}\left\{\text{Re}\{c_{ij}^{(t)}\}x_i x_j^H\right\}$$

for all $(i,j)\in\mathcal{G}$ and $t\in\{1,...,k\}$. Hence, changing the complex weight $c_{ij}^{(t)}$ to $\text{Re}\{c_{ij}^{(t)}\}$ does not affect the optimization. Therefore, with no loss of generality, assume that the edge weights are all real numbers. For every edge $(i,j)\in\mathcal{G}$, define the edge sign σ_{ij} as follows:

$$\sigma_{ij} = \begin{cases} 1 & \text{if} \quad c_{ij}^{(1)},...,c_{ij}^{(k)}\geq 0 \\ -1 & \text{if} \quad c_{ij}^{(1)},...,c_{ij}^{(k)}\leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

By convention, we define $\sigma_{ij}=-1$ if $c_{ij}^{(1)}=\cdots=c_{ij}^{(k)}=0$.

Theorem 2: The relations $f^*_{\text{SOCP}}=f^*_{\text{r-SDP}}=f^*_{\text{SDP}}=f^*$ hold for Optimization (1) in the real-valued case $\mathcal{D}=\mathcal{R}$ if

$$\sigma_{ij}\neq 0, \qquad \forall (i,j)\in\mathcal{G} \tag{12a}$$

$$\prod_{(i,j)\in\mathcal{O}_r}\sigma_{ij}=(-1)^{|\mathcal{O}_r|}, \qquad \forall r\in\{1,...,p\} \tag{12b}$$

Proof: In light of the relation $f^*_{\text{SOCP}}\leq f^*_{\text{r-SDP}}\leq f^*_{\text{SDP}}\leq f^*$, it suffices to prove that $f^*\leq f^*_{\text{SOCP}}$. Consider an arbitrary feasible point X of Optimizations (5). It is enough to show the existence of a feasible point $\mathbf{x}$ for Optimization (1) with the property that the objective value of this optimization at $\mathbf{x}$ is lower than or equal to the objective value of the SOCP relaxation at the point X. For this purpose, choose an arbitrary spanning tree $\mathcal{T}$ of the graph $\mathcal{G}$. A set of ±1 numbers $\sigma_1,\sigma_2,...,\sigma_n$ can be iteratively assigned to the vertices of this tree in such a way that $\sigma_i\sigma_j=-\sigma_{ij}$ for every $(i,j)\in\mathcal{T}$ (this is because of (12a)). Now, it can be deduced from (12b) that

$$\sigma_i\sigma_j=-\sigma_{ij}, \qquad \forall (i,j)\in\mathcal{G}$$

Corresponding to the feasible point X of the SOCP relaxation, define the vector $\mathbf{x}$ as

$$\left[\begin{array}{cccc}\sigma_1\sqrt{X_{11}} & \sigma_2\sqrt{X_{22}} & \cdots & \sigma_n\sqrt{X_{nn}}\end{array}\right]^H$$

(note that $X_{11},...,X_{nn}\geq 0$ due to the condition $X\succeq 0$). Observe that

$$|x_i|^2=X_{ii}, \qquad i=1,...,n \tag{13}$$

On the other hand, (5c) yields

$$X_{ij}\leq\sqrt{X_{ii}}\sqrt{X_{jj}}, \qquad \forall (i,j)\in\mathcal{G}$$

and therefore

$$\begin{aligned} c_{ij}^{(t)}X_{ij} &\geq -|c_{ij}^{(t)}|\sqrt{X_{ii}}\sqrt{X_{jj}} = -c_{ij}^{(t)}\sigma_{ij}\sqrt{X_{ii}}\sqrt{X_{jj}} \\ &= c_{ij}^{(t)}\sigma_i\sigma_j\sqrt{X_{ii}}\sqrt{X_{jj}} = c_{ij}^{(t)}x_i x_j, \quad \forall (i,j)\in\mathcal{G} \end{aligned} \tag{14}$$

for $t=1,...,k$. It can be concluded from (13) and (14) that

$$l_1\left(\mathbf{x}\mathbf{x}^H\right)=l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right)\leq l_2(X)$$

Hence, since $f_0(\cdot,\cdot)$ is increasing in its second vector argument, one can write:

$$f_j(\mathbf{y},\mathbf{z})\leq f_j(l_1(X),l_2(X))$$

for $j=0,1,...,m$, where $\mathbf{y}=l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z}=l_2\left(\mathbf{x}\mathbf{x}^H\right)$. This implies that $\mathbf{x}$ is a feasible point of Optimization (1) whose corresponding objective value is smaller than or equal to the objective value for the feasible point X of Optimization (5). This proves the claim $f^*\leq f^*_{\text{SOCP}}$ and thus completes the proof. ■

Condition (12a) ensures that each edge weight set is sign definite. Theorem 2 states that the SDP, reduced SDP, and SOCP relaxations are exact for the original optimization (1) under the above sign definite condition, provided that each cycle in the cycle basis has an even number of edges with positive signs. This holds true in three important special cases, as explained below.

Corollary 1: The relations $f^*_{\text{SOCP}}=f^*_{\text{r-SDP}}=f^*_{\text{SDP}}=f^*$ hold for Optimization (1) in the case $\mathcal{D}=\mathcal{R}$ if one of the following happens:

1) $\mathcal{G}$ is acyclic with arbitrary sign definite edge sets (with respect to $\mathcal{R}$).

2) $\mathcal{G}$ is bipartite with positive weight sets.

3) $\mathcal{G}$ is arbitrary with negative weight sets.

Proof: The proof follows immediately from Theorem 2 by noting that a bipartite graph has no odd cycle. ■

Assume that the edge sets of the graph $\mathcal{G}$ are all sign definite. Corollary 1 implies a trade-off between the topology and the edge signs σ_{ij}'s. On one extreme, the edge signs could be arbitrary as long as the graph has a very sparse topology. On the other extreme, the graph topology could be arbitrary (sparse or dense) as long as the edge signs are all negative.

The following theorem proves that if σ_{ij}'s are zero, Optimization (1) becomes NP-hard even for an acyclic graph $\mathcal{G}$.

Theorem 3: Finding an optimal solution of Optimization (1) is an NP-hard problem for an acyclic $\mathcal{G}$ with sign-indefinite weight sets (even if $k=2$).

Proof: The proof has been omitted due to space restrictions and may be found in [26]. ■

Theorem 3 states that optimization over a very sparse generalized weighted graph (acyclic graph with only two elements in each weight set) is still hard unless the weight sets are sign definite. However, it will be shown in the next subsection that the SDP relaxation always has a rank 1 or 2 solution for this type of graph, from which an exact or approximate solution of the original problem may be recovered.

A. Low-Rank Solution for SDP Relaxation

Suppose that the conditions stated in Theorem 2 do not hold. The SDP relaxation may still be exact (depending on the coefficients of Optimization (1)), in which case the relaxation has a rank-1 solution X^*. A question arises as to whether the rank of X^* is yet small whenever the relaxation is inexact. The objective of this subsection is to address this problem in two important scenarios.

Given the graph $\mathcal{G}$ and the parameters $\mathbf{x}, \mathbf{y}, \mathbf{z}$ introduced earlier, consider the optimization

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{R}^n} \quad & g_0(\mathbf{y},\mathbf{z}) \\ \text{s.t.} \quad & g_j(\mathbf{y},\mathbf{z}) \leq 0, \quad j=1,2,...,m \end{aligned} \tag{15}$$

for arbitrary functions $g_i(\cdot,\cdot)$, $i=0,1,...,m$. The difference between the above optimization and (1) is that the functions $g_i(\cdot,\cdot)$'s may not be increasing in $\mathbf{z}$. In line with the technique used in Section II for the non-convex optimization (1), an SDP relaxation can be defined for the above optimization. As expected, this relaxation may not have a rank-1 solution, in which case the relaxation is not exact. Nevertheless, it is beneficial to find out how small the rank of an optimal solution of this relaxation could be. This problem will be addressed next for an acyclic graph $\mathcal{G}$.

Theorem 4: Assume that the graph $\mathcal{G}$ is acyclic. The SDP relaxation for Optimization (15) always has a solution X^* whose rank is at most 2.

Proof: The SDP relaxation for Optimization (15) is as follows:

$$\min_{X\in\mathcal{S}^n} \quad g_0(l_1(X), l_2(X)) \tag{16a}$$

$$\text{s.t.} \quad g_j(l_1(X), l_2(X)) \leq 0, \quad j=1,...,m \tag{16b}$$

$$X \succeq 0 \tag{16c}$$

This is indeed a real-valued SDP relaxation. One can consider a complex-valued SDP relaxation as

$$\min_{\tilde{X}\in\mathcal{H}^n} \quad g_0(l_1(\tilde{X}), l_2(\tilde{X})) \tag{17a}$$

$$\text{s.t.} \quad g_i(l_1(\tilde{X}), l_2(\tilde{X})) \leq 0, \quad j=1,...,m \tag{17b}$$

$$\tilde{X} \succeq 0 \tag{17c}$$

where its matrix variable, denoted as $\tilde{X}$, is complex. Observe that $l_1(\tilde{X}) = l_1(\text{Re}\{\tilde{X}\})$ and $l_2(\tilde{X}) = l_2(\text{Re}\{\tilde{X}\})$ for every arbitrary Hermitian matrix $\tilde{X}$, due to the fact that the edge weights of the graph $\mathcal{G}$ are all real. This implies that the real and complex SDP relaxations have the same optimal objective value (note that $\text{Re}\{\tilde{X}\} \succeq 0$ if $\tilde{X} \succeq 0$). In particular, if $\tilde{X}^*$ denotes an optimal solution of the complex SDP relaxation, $\text{Re}\{\tilde{X}^*\}$ will be an optimal solution of the real SDP relaxation. As shown in Theorem 1 of Part II of this paper, Optimization (17) has a rank-1 solution $\tilde{X}^*$. Therefore, $\tilde{X}^*$ can be decomposed as $(\tilde{\mathbf{x}}^*)(\tilde{\mathbf{x}}^*)^H$ for some complex vector $\tilde{\mathbf{x}}^*$. Now, one can write:

$$\text{Re}\{\tilde{X}^*\} = \text{Re}\{\tilde{\mathbf{x}}^*\}\text{Re}\{\tilde{\mathbf{x}}^*\}^H + \text{Im}\{\tilde{\mathbf{x}}^*\}\text{Im}\{\tilde{\mathbf{x}}^*\}^H$$

Hence, $\text{Re}\{\tilde{X}^*\}$ is a real-valued matrix with rank at most 2 (as it is the sum of two rank-1 matrices), which is also a solution of the real SDP relaxation. This completes the proof. ■

Theorem 4 states that the SDP relaxation of the general optimization (15) always has a rank 1 or 2 solution if its sparsity can be captured by an acyclic graph. This result makes no assumptions on the monotonicity of the functions $g_j(\cdot,\cdot)$'s. The SDP relaxation for Optimization (15) may not have a unique solution. Hence, if a sample of this optimization is solved numerically, the obtained solution may be high rank, in which case the low-rank solution X^* is hidden and needs to be recovered (following the constructive proof of the theorem).

Theorem 4 studies the SDP relaxation for only acyclic graphs. Partial results will be provided below for cyclic graphs.

Theorem 5: Assume that $\mathcal{G}$ is a weakly-cyclic bipartite graph, and that

$$\sigma_{ij} \neq 0 \qquad \forall (i,j) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_p$$

The SDP relaxation (3) for Optimization (1) in the real-valued case $\mathcal{D} = \mathcal{R}$ has a solution X^* whose rank is at most 2.

Proof: The proof is based on Theorem 4 of Part II of this paper. The details may be found in [26]. ■

There are several applications, where the goal is to find a low-rank positive semidefinite matrix X satisfying a set of constraints (such as linear matrix inequalities). Theorems 4 and 5 provide sufficient conditions under which the feasibility problem:

$$\begin{aligned} & f_j(l_1(X), l_2(X)) \leq 0, \quad j=1,...,m \\ & X \succeq 0, \end{aligned} \tag{18}$$

has a low rank solution, where the rank does not depend on the size of the problem.

V. Complex-valued Optimization

Consider the case where each edge weight set is complex and sign definite with respect to $\mathcal{C}$.

Theorem 6: The relations $f^*_{\text{SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold in the complex-valued case $\mathcal{D} = \mathcal{C}$, provided that the graph $\mathcal{G}$ is acyclic and the weight set $\{c^{(1)}_{ij}, c^{(2)}_{ij}, ..., c^{(k)}_{ij}\}$ is sign definite for all $(i,j) \in \mathcal{G}$.

Proof: The decomposition technique developed in [20] will be deployed to prove this theorem. Similar to Theorem 2, it is enough to show that $f^* \leq f^*_{\text{SOCP}}$. To this end, consider an arbitrary feasible solution of optimization (5), denoted as X. Given an edge $(i,j) \in \mathcal{G}$, since the set $\{c^{(1)}_{ij}, c^{(2)}_{ij}, ..., c^{(k)}_{ij}\}$ is sign definite, it follows from the hyperplane separation theorem that there exists a nonzero real vector $(\alpha_{ij}, \beta_{ij})$ such that

$$\text{Re}\{c^{(t)}_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\} = \text{Re}\{c^{(t)}_{ij}\}\alpha_{ij} - \text{Im}\{c^{(t)}_{ij}\}\beta_{ij} \leq 0 \tag{19}$$

for every $t \in \{1,2,...,k\}$. On the other hand, (5c) yields

$$|X_{ij}| \leq \sqrt{X_{ii}}\sqrt{X_{jj}}, \qquad \forall (i,j) \in \mathcal{G} \tag{20}$$

Consider the function

$$\left|X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\right|^2 - X_{ii}X_{jj}$$

in which γ_{ij} is an unknown real number. This function is negative at $\gamma = 0$ (because of (20)) and positive at $\gamma = +\infty$. Hence, due to the continuity of this function, there exists a positive number γ_{ij} such that

$$\left|X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\text{i})\right|^2 = X_{ii}X_{jj} \tag{21}$$

Define θ_{ij} as the phase of the complex number $X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})$. A set of angles $\{\theta_1, \theta_2, ..., \theta_n\}$ can be found iteratively by exploiting the tree topology of the graph $\mathcal{G}$ in such a way that

$$\theta_i - \theta_j = \theta_{ij}, \quad \forall (i,j) \in \mathcal{G} \tag{22}$$

Define the vector $\mathbf{x}$ as

$$\left[\begin{array}{cccc} \sqrt{X_{11}}e^{-\theta_1 \mathrm{i}} & \sqrt{X_{22}}e^{-\theta_2 \mathrm{i}} & \cdots & \sqrt{X_{nn}}e^{-\theta_n \mathrm{i}} \end{array} \right]^H \tag{23}$$

Using (19), (21) and (22), one can write:

$$\begin{aligned} \mathrm{Re}\{c_{ij}^{(t)} x_i x_j^H\} &= \mathrm{Re}\left\{ c_{ij}^{(t)} \sqrt{X_{ii}}\sqrt{X_{jj}} e^{(\theta_i - \theta_j)\mathrm{i}} \right\} \\ &= \mathrm{Re}\left\{ c_{ij}^{(t)} \sqrt{X_{ii}}\sqrt{X_{jj}} e^{\theta_{ij}\mathrm{i}} \right\} \\ &= \mathrm{Re}\{ c_{ij}^{(t)} (X_{ij} + \gamma_{ij}(\alpha_{ij} + \beta_{ij}\mathrm{i})) \} \\ &= \mathrm{Re}\{ c_{ij}^{(t)} X_{ij} \} + \gamma_{ij} \mathrm{Re}\{ c_{ij}^{(t)} (\alpha_{ij} + \beta_{ij}\mathrm{i}) \} \\ &\leq \mathrm{Re}\{ c_{ij}^{(t)} X_{ij} \} \end{aligned}$$

for every $t \in \{1, 2, ..., k\}$. Having shown the above relation, the rest of the proof is in line with the proof of Theorem 2. More precisely, the above inequality implies that

$$l_1\left(\mathbf{x}\mathbf{x}^H\right) = l_1(X), \qquad l_2\left(\mathbf{x}\mathbf{x}^H\right) \leq l_2(X)$$

and therefore

$$f_j(\mathbf{y}, \mathbf{z}) \leq f_i(l_1(X), l_2(X)), \qquad j = 0, 1, ..., m$$

where $\mathbf{y} = l_1\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{z} = l_2\left(\mathbf{x}\mathbf{x}^H\right)$. Hence, $\mathbf{x}$ is a feasible point of Optimization (1) whose corresponding objective value is smaller than or equal to the objective value for the feasible point X of Optimization (5). Consequently, $f^* \leq f^*_{\text{SOCP}}$. This completes the proof. ∎

The quadratically-constrained quadratic optimization (7) is a special case of optimization (1). Hence, the SDP relaxation is tight for this QCQP problem if $\mathcal{G}$ is acyclic with sign definite weight sets. This result improves upon the result developed in [25] by removing the assumption $M_0 \succeq 0$.

Corollary 2: The relations $f^*_{\text{SOCP}} = f^*_{\text{r-SDP}} = f^*_{\text{SDP}} = f^*$ hold in the complex-valued case $\mathcal{D} = \mathcal{C}$ if the graph $\mathcal{G}$ is acyclic and $k \leq 2$.

Proof: The proof is an immediate consequence of Theorem 6 and the fact that every complex set with one or two elements is sign definite. ∎

Corollary 2 states that Optimization (1) in the complex-valued case can be solved through three relaxations if its structure can be captured by an acyclic graph with at most two weights on each of its edges.

The cases that have been studied so far have the property that whenever the SDP relaxation is exact, the reduced SDP and SOCP relaxations are also exact simultaneously. However, Part II of this paper will consider important cases for which this is no longer true. Some of the results to be developed in Part II of the paper are as follows:

i) The reduced SDP and SDP relaxations are exact if $\mathcal{G}$ is bipartite and weakly cyclic with positive or negative real weight sets.
ii) The reduced SDP and SDP relaxations are exact if $\mathcal{G}$ is a weakly cyclic graph with homogeneous imaginary weight sets.
iii) The SDP relaxation is exact if the graph $\mathcal{G}$ can be decomposed as a union of edge-disjoint subgraphs in an acyclic way with the property that each subgraph has a good structure for which the SDP relaxation is always exact.

In part II of the paper, it will also be discussed how the results on complex optimization can be applied to power systems.

VI. Examples

Example 1: Consider the problem of minimizing the bivariate polynomial

$$f_0(x_1, x_2) = x_1^4 + a x_2^2 + b x_1^2 x_2 + c x_1 x_2 \tag{24}$$

with the real-valued variables x_1 and x_2, where the parameters $a, b, c \in \mathcal{R}$ are known. In order to find the global minimum of this optimization, the standard convex optimization technique cannot readily be used due to the non-convexity of $f(x_1, x_2)$ for generic values of a, b, and c. To address this issue, the above unconstrained minimization problem will be converted to a constrained quadratic optimization. More precisely, the problem of minimizing $f_0(x_1, x_2)$ can be reformulated in terms of x_1, x_2 and two auxiliary variables x_3, x_4 as:

$$\min_{\mathbf{x} \in \mathcal{R}^4} \quad x_3^2 + a x_2^2 + b x_3 x_2 + c x_1 x_2 \tag{25a}$$

$$\text{s.t.} \quad x_1^2 - x_3 x_4 = 0, \tag{25b}$$

$$x_4^2 - 1 = 0, \quad x_4 \geq 0 \tag{25c}$$

where $\mathbf{x} = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \right]^H$. The above optimization can be recast as follows:

$$\min_{\mathbf{x} \in \mathcal{R}^4, X \in \mathcal{S}^4} \quad X_{33} + a X_{22} + b X_{23} + c X_{12} \tag{26a}$$

$$\text{s.t.} \quad X_{11} - X_{34} \leq 0, \quad X_{44} - 1 = 0 \tag{26b}$$

and subject to the additional constraint $X = \mathbf{x}\mathbf{x}^H$. Note that $X_{11} - X_{34} \leq 0$ should have been $X_{11} - X_{34} = 0$, but this modification does not change the solution. To eliminate the non-convexity induced by the constraint $X = \mathbf{x}\mathbf{x}^H$, one can use an SDP relaxation obtained by replacing the constraint $X = \mathbf{x}\mathbf{x}^H$ with the convex SDP constraint $X = X^H \succeq 0$. To understand the exactness of this relaxation, the weighted graph $\mathcal{G}$ capturing the structure of the optimization under study should be constructed. This graph is depicted in Figure 2. Due to Corollary 1, since $\mathcal{G}$ is acyclic, the SDP relaxation is exact for all values of a, b, c. Note that this does not imply that every solution X of the SDP relaxation has rank 1. However, there is a simple systematic procedure for recovering a rank-1 solution from an arbitrary optimal solution of this relaxation. Note also that one can use an SOCP relaxation instead.

Now, assume that a set of constraints

$$f_j(x_1, x_2) = x_1^4 + a_j x_2^2 + b_j x_1^2 x_2 + c_j x_1 x_2 \leq d_j \quad j = 1, ..., m$$

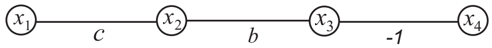

Fig. 2. Graph $\mathcal{G}$ corresponding to minimization of $f_0(x_1, x_2)$ given in (24).

has been added to Optimization (24) for given coefficients a_j, b_j, c_j, d_j. In this case, the graph $\mathcal{G}$ depicted in Figure 2 needs to be modified by replacing its edge sets $\{b\}$ and $\{c\}$ with $\{b, b_1, ..., b_m\}$ and $\{c, c_1, ..., c_m\}$, respectively. Due to Corollary 1, the SDP relaxation corresponding to the new optimization is exact as long as the sets $\{c, c_1, ..., c_m\}$ and $\{b, b_1, ..., b_m\}$ are both sign definite. Moreover, in light of Theorem 4, if these sets are not sign definite, then the SDP relaxation will still have a low rank (rank 1 or 2) solution.

Example 2: Consider the optimization

$$\min_{\mathbf{x} \in \mathcal{C}^n} \quad \mathbf{x}^H M_0 \mathbf{x}$$
$$\text{s.t.} \quad \mathbf{x}^H M_j \mathbf{x} \leq 0, \quad j = 1, 2, ..., m$$

where $M_0, ..., M_m$ are symmetric real matrices, while $\mathbf{x}$ is an unknown complex vector. Similar to what was done in Example 1, a generalized weighted graph $\mathcal{G}$ can be constructed for this optimization. Regardless of the edge weights, as long as the graph $\mathcal{G}$ is acyclic, the SDP, reduced SDP, and SOCP relaxations are all tight (see Theorem 6). The reason is that every real set is sign definite with respect to $\mathcal{C}$ according to Definition 2. As a result, the above class of optimization problems is polynomial-time solvable.

VII. Conclusions

This work deals with three conic relaxations for a broad class of nonlinear real/complex optimization problems, where the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable. Several types of optimizations, including polynomial optimization, can be cast as the problem under study. To explore the exactness of the proposed relaxations, the structure of the optimization is mapped into a generalized weighted graph with a weight set assigned to each edge. In the case of real-valued optimization, it is shown that the relaxations are exact if a set of conditions are satisfied, which depend on some weak properties of the underlying generalized weighted graph. A similar result is derived in the complex-valued case after introducing the notion of "sign-definite complex weight sets", under the assumption that the graph is acyclic. In Part II of the paper, the complex case is further studied for general graphs. Moreover, the application of this two-part paper in solving energy optimizations for power systems is spelled out. In particular, it is shown that the weight sets are sign definite for power networks due to the passivity of transmission lines, and this makes a broad class of energy optimizations easy to solve.

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