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A Warm-Started Homogeneous and Self-Dual Interior-Point Method for Linear Economic Model Predictive Control

Leo Emil Sokoler, Anders Skajaa, Gianluca Frison, Rasmus Halvgaard, and John Bagterp Jørgensen

Abstract— **In this paper, we present a warm-started homogenous and self-dual interior-point method (IPM) for the linear programs arising in economic model predictive control (MPC) of linear systems. To exploit the structure in the optimization problems, our algorithm utilizes a Riccati iteration procedure which is adapted to the non-standard system solved in homogenous and self-dual IPMs, and specifically tailored to economic MPC. Fast convergence is further achieved by means of a recent warm-starting strategy for homogenous and self-dual IPMs that has not previously been applied to MPC. We implement our algorithm in MATLAB and its performance is analyzed based on a smart grid power management case study. Closed loop simulations show that 1) our algorithm is significantly faster than state-of-the-art IPMs based on sparse linear algebra routines, and 2) warm-starting reduces the number of iterations by approximately 15-35%.**

I. INTRODUCTION

In conventional linear model predictive control (MPC), the control problem is formulated as a convex program that penalizes deviations between a desired set-point and the controlled output(s) [1]–[4]. Although this classical approach has become the standard way of formulating the control problem in MPC applications, recent studies show that for energy systems it is often more convenient to use an MPC-based controller with a pure economic objective function [5]–[9]. This variant of MPC, known as economic MPC, guarantees that the set-point is reached in the most profitable way (which is not necessarily the fastest). Applications where economic MPC have been applied to minimize operating costs include control of refrigeration systems [10], building climate control [11], [12], charging batteries in electric vehicles [13], as well as control of non-linear chemical processes [14]. Stability of economic MPC has been addressed in [8] and [6].

Aside the requirement of a predictive model, the main challenge in linear economic MPC is that a linear program has to be solved at each sampling instant. For large systems, the computation time of solving this optimization problem may render the method infeasible. To overcome this problem, our paper develops an efficient and reliable IPM for the linear programs arising in linear economic MPC. The proposed algorithm is a homogeneous and self-dual variant of Mehrotra's predictor-corrector method [15], [16] that exploits the following problem specific features:

- **Structure**: The optimal control problem solved in linear economic MPC can be posed as a highly structured linear program. We utilize this structure to speed-up the most time consuming numerical operations using a Riccati iteration procedure specifically tailored to economic MPC.
- **Warm-Start**: In MPC applications, the optimization problems solved at successive time steps are closely related. To take advantage of the solution from the previous sampling instant, we implement a recently developed warm-starting strategy for homogeneous and self-dual IPMs. This method does not introduce any additional significant computations and has been reported to reduce the number of iterations by 30-75% for an extensive amount of linear programs and quadratic conic problems [17].

Although Riccati based IPMs have been developed in [18]–[21] for set-point based MPC with ℓ_2-penalty, and ℓ_1-penalty in [22], these results are not directly applicable to the homogeneous and self-dual model, which has become widely adopted by state-of-the-art IPMs for linear programming. In our previous work [23], the main emphasis was to develop a Riccati iteration procedure specifically tailored to economic MPC, while the focus of this paper is to further improve our algorithm using the warm-starting strategy of [17].

We have organized the paper as follows. In Section II, we introduce the control problem solved in economic MPC. A homogeneous and self-dual IPM for solving this problem tailored to economic MPC is derived in Section III-IV, and warm-starting is discussed in Section V. Finally, section VI presents simulation results based on a MATLAB implementation of our algorithm denoted `LPempc`. Concluding remarks are given in Section VII.

II. PROBLEM DEFINITION

In linear economic MPC, the constrained optimal control problem solved at each sampling instant may be formulated as

$$\min_{u,x,\rho} \sum_{k=0}^{N-1} p_k^T u_k + q_{k+1}^T \rho_{k+1}, \tag{1a}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \qquad k = 0,1,\ldots,N-1, \tag{1b}$$

$$\underline{u}_k \le u_k \le \overline{u}_k, \qquad k = 0,1,\ldots,N-1, \tag{1c}$$

$$\Delta\underline{u}_k \le \Delta u_k \le \Delta\overline{u}_k, \qquad k = 0,1,\ldots,N-1, \tag{1d}$$

$$\underline{z}_k - \rho_k \le Cx_k \le \overline{z}_k + \rho_k, \qquad k = 1,2,\ldots,N, \tag{1e}$$

$$\rho_k \ge 0, \qquad k = 1,2,\ldots,N, \tag{1f}$$

where N is the length of the prediction horizon. The problem data are the state-space matrices, (A,B,C), defining the

The authors are with the Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark. L. E. Sokoler and A. Skajaa are also affiliated with DONG Energy, DK-2820 Gentofte, Denmark. Email: {leso, andsk, giaf, rhal, jbjo} at dtu.dk

linear system controlled, the initial state, x_0, the input limits, $(\underline{u}_k, \overline{u}_k)$, the input-rate limits, $(\Delta\underline{u}_k, \Delta\overline{u}_k)$, the output limits, $(\underline{z}_k, \overline{z}_k)$, the input prices, p_k, and the price for violating the output constraints q_k. As an example, in power systems p_k may be the cost of fuel and q_k may be the cost of not meeting the power demand.

The input-rate is defined in terms of the backward difference operator

$$\Delta u_k := u_k - u_{k-1}, \qquad k = 0, 1, \ldots, N-1.$$

By augmenting the state-space system such that

$$A := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad x_k := \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}, \quad B := \begin{bmatrix} B \\ I \end{bmatrix},$$
$$E := \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad C_{:} = \begin{bmatrix} C & 0 \end{bmatrix},$$

we can express (1d) as

$$\Delta\underline{u}_k \leq u_k - Dx_k \leq \Delta\overline{u}_k, \qquad k = 0, 1, \ldots, N-1$$

in which $D := \begin{bmatrix} 0 & I \end{bmatrix}$. This formulation simplifies computations in our Riccati iteration procedure considerably.

III. Homogeneous and Self-Dual Model

By aggregating the problem data into the structures $g \in \mathbb{R}^n$, $F \in \mathbb{R}^{m_E \times n}$, $b \in \mathbb{R}^{m_E}$, $H \in \mathbb{R}^{m_I \times n}$ and $c \in \mathbb{R}^{m_I}$, (1) can be written as the linear program

$$\min_t \ \{g^T t | Ft = b, Ht \leq c\}, \tag{2}$$

where $n := N(n_u + n_x + n_z)$, $m_E := Nn_x$ and $m_I := N(4n_u + 3n_z)$. As an example, consider the case for $N = 2$

$$t := \begin{bmatrix} u_0^T & x_1^T & \rho_1^T & u_1^T & x_2^T & \rho_2^T \end{bmatrix}^T,$$
$$g := \begin{bmatrix} p_0^T & 0 & q_1^T & p_1^T & 0 & q_2^T \end{bmatrix}^T,$$

and

$$\begin{bmatrix} F \mid b \end{bmatrix} := \left[\begin{array}{cccccc|c} B & -I & 0 & 0 & 0 & 0 & -Ax_0 \\ 0 & A & 0 & B & -I & 0 & 0 \end{array}\right],$$

$$\begin{bmatrix} H \mid c \end{bmatrix} := \left[\begin{array}{cccccc|c} I & 0 & 0 & 0 & 0 & 0 & \overline{u}_0 \\ 0 & 0 & 0 & I & 0 & 0 & \overline{u}_1 \\ -I & 0 & 0 & 0 & 0 & 0 & -\underline{u}_0 \\ 0 & 0 & 0 & -I & 0 & 0 & -\underline{u}_1 \\ I & 0 & 0 & 0 & 0 & 0 & \Delta\tilde{u}_0 \\ 0 & -D & 0 & I & 0 & 0 & \Delta\overline{u}_1 \\ -I & 0 & 0 & 0 & 0 & 0 & -\Delta\underline{u}_0 \\ 0 & D & 0 & -I & 0 & 0 & -\Delta\underline{u}_1 \\ 0 & C & -I & 0 & 0 & 0 & \overline{z}_1 \\ 0 & 0 & 0 & 0 & C & -I & \overline{z}_2 \\ 0 & -C & -I & 0 & 0 & 0 & -\underline{z}_1 \\ 0 & 0 & 0 & 0 & -C & -I & -\underline{z}_2 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 \end{array}\right],$$

in which we have defined

$$\Delta\tilde{u}_0 := \Delta\overline{u}_0 + Dx_0,$$
$$\Delta\underline{u}_0 := \Delta\underline{u}_0 + Dx_0.$$

Hence, the control problem (1) can be posed as a highly structured linear program.

The dual of the linear program (2) is

$$\max_{v,w} \ \{-b^T v - c^T w | - F^T v - H^T w = g, w \geq 0\}. \tag{3}$$

In homogeneous and self-dual IPMs, the solution to (2)-(3) is obtained by solving a related homogeneous and self-dual linear program [24]–[26]. Aside from an inherent ability to detect infeasibility, recent advances show that IPMs based on this approach can be warm-started efficiently [17].

If we introduce a new set of optimization variables $(\tilde{t}, \tilde{v}, \tilde{w}, \tilde{s})$, and the additional scalar variables $(\tilde{\tau}, \tilde{\kappa})$, the self-dual and homogeneous problem associated with (2)-(3), may be stated as the linear feasibility problem

$$\min_{\tilde{t},\tilde{v},\tilde{w},\tilde{s},\tilde{\tau},\tilde{\kappa}} \ 0, \tag{4a}$$
$$\text{s.t.} \ F^T\tilde{v} + H^T\tilde{w} + g\tilde{\tau} = 0, \tag{4b}$$
$$b\tilde{\tau} - F\tilde{t} = 0, \tag{4c}$$
$$c\tilde{\tau} - H\tilde{t} - \tilde{s} = 0, \tag{4d}$$
$$-g^T\tilde{t} - b^T\tilde{v} - c^T\tilde{w} + \tilde{\kappa} = 0, \tag{4e}$$
$$(\tilde{w}, \tilde{s}, \tilde{\tau}, \tilde{\kappa}) \geq 0, \tag{4f}$$

Proposition 1 shows that the solution to (2)-(3), can be obtained by solving (4).

Proposition 1: The linear feasibility problem (4) always has a strict complimentary solution $(\tilde{t}^*, \tilde{v}^*, \tilde{w}^*, \tilde{s}^*, \tilde{\tau}^*, \tilde{\kappa}^*)$ satisfying $\tilde{s}_j^*\tilde{w}_j^* = 0$ for $j = 1, 2, \ldots, m_I$ and $\tilde{\tau}^*\tilde{\kappa}^* = 0$. For such a solution, one of the following conditions hold

- **I**). $\tilde{\tau}^* > 0, \tilde{\kappa}^* = 0$: The scaled solution $(t^*, v^*, w^*, s^*) = (\tilde{t}^*, \tilde{v}^*, \tilde{w}^*, \tilde{s}^*)/\tilde{\tau}^*$ is a primal-dual optimal solution to (2)-(3).
- **II**). $\tilde{\tau}^* = 0, \tilde{\kappa}^* > 0$: The problem (2) is infeasible or unbounded; either $-b^T\tilde{v}^* - c^T\tilde{w}^* > 0$ (implies primal infeasibility), or $g^T\tilde{t}^* < 0$ (implies dual infeasibility).

Proof: See [26]. ∎

A. Interior Point Method

We now present a homogeneous and self-dual IPM for solving (4). For compact notation, we denote the optimization variables by $\phi := (\tilde{t}, \tilde{v}, \tilde{w}, \tilde{s}, \tilde{\tau}, \tilde{\kappa})$, and introduce superscript k to indicate a particular iteration number.

The necessary and sufficient optimality conditions for (4) are $(\tilde{w}, \tilde{s}, \tilde{\kappa}, \tilde{\tau}) \geq 0$ and

$$V(\phi) := \begin{bmatrix} F^T\tilde{v} + H^T\tilde{w} + g\tilde{\tau} \\ b\tilde{\tau} - F\tilde{t} \\ c\tilde{\tau} - H\tilde{t} - \tilde{s} \\ -g^T\tilde{t} - b^T\tilde{v} - c^T\tilde{w} + \tilde{\kappa} \\ \tilde{W}\tilde{S}\mathbf{1}_{m_I} \\ \tilde{\tau}\tilde{\kappa} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

W is a diagonal matrix with the elements of w on its diagonal, and similarly for S. Moreover, $\mathbf{1}_{m_I}$ is the column vector of all ones of size m_I.

To find a point satisfying the optimality conditions, we use a variant of Mehrotra's predictor-corrector method [15], [16]. The method tracks the central path $\mathscr{C}$, which connects

an initial point ϕ^0 satisfying $(\tilde{w}^0, \tilde{s}^0, \tilde{\kappa}^0, \tilde{\tau}^0) \geq 0$ to a strict complementary solution of (4), denoted ϕ^*. Formally, we can write the central path as

$$\mathscr{C} := \left\{ \phi | V(\phi) = \gamma r^0, (\tilde{w}, \tilde{s}, \tilde{\kappa}, \tilde{\tau}) \geq 0, \gamma \in [0,1] \right\}.$$

In this definition

$$r^0 := \left[V_1(\phi^0)^T \; V_2(\phi^0)^T \; V_3(\phi^0)^T \; V_4(\phi^0)^T \; \mu^0(\mathbf{1}_{m_I})^T \; \mu^0 \right]^T,$$

where $\mu^0 := ((\tilde{w}^0)^T \tilde{s}^0 + \tilde{\tau}^0 \tilde{\kappa}^0)/(m_I + 1)$ is a measure of the duality gap, and $V_i(\phi)$ is the i'th set of components of $V(\phi)$, defined as in (5).

The basic idea in IPMs is to generate a sequence of iterates along the central path $\{\phi^0, \phi^1, \ldots, \phi^k, \ldots, \phi^N\}$, such that $\phi^N \to \phi^*$ as $N \to \infty$. In Mehrotra's predictor-corrector method, the iterates are computed by repeating a two-step procedure. The first part of this procedure, known as the affine step, updates the value of γ and computes second-order correction terms. Secondly, a corrector step is determined and a new iterate is produced.

1) Affine Step: The affine direction $\Delta\phi^k_{\text{aff}}$ is obtained by solving the linear system

$$J_V(\phi^k)\Delta\phi^k_{\text{aff}} = -V(\phi^k), \tag{6}$$

which corresponds to the Newton direction for (5). The Jacobian of V evaluated at ϕ^k is

$$J_V(\phi^k) = \begin{bmatrix} 0 & F^T & H^T & 0 & g & 0 \\ -F & 0 & 0 & 0 & b & 0 \\ -H & 0 & 0 & -I & c & 0 \\ -g^T & -b^T & -c^T & 0 & 0 & 1 \\ 0 & 0 & \tilde{S}^k & \tilde{W}^k & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\kappa}^k & \tilde{\tau}^k \end{bmatrix}. \tag{7}$$

Given the solution to (6), we compute the maximum step length in the affine direction for the primal and dual variables, such that (4f) remains satisfied

$$\alpha^k_{\text{aff}} := \max\left\{ a_{\text{aff}} \in [0,1] | \begin{bmatrix} \tilde{w}^k \\ \tilde{\tau}^k \end{bmatrix} + a_{\text{aff}} \begin{bmatrix} \Delta\tilde{w}^k_{\text{aff}} \\ \Delta\tilde{\tau}^k_{\text{aff}} \end{bmatrix} \geq 0 \right\},$$
$$\beta^k_{\text{aff}} := \max\left\{ b_{\text{aff}} \in [0,1] | \begin{bmatrix} \tilde{s}^k \\ \tilde{\kappa}^k \end{bmatrix} + b_{\text{aff}} \begin{bmatrix} \Delta\tilde{s}^k_{\text{aff}} \\ \Delta\tilde{\kappa}^k_{\text{aff}} \end{bmatrix} \geq 0 \right\}.$$

Accordingly, affine variables are computed

$$\tilde{w}^k_{\text{aff}} := \tilde{w}^k + \alpha^k_{\text{aff}}\Delta\tilde{w}^k_{\text{aff}}, \qquad \tilde{s}^k_{\text{aff}} := \tilde{s}^k + \beta^k_{\text{aff}}\Delta\tilde{s}^k_{\text{aff}},$$
$$\tilde{\tau}^k_{\text{aff}} := \tilde{\tau}^k + \alpha^k_{\text{aff}}\Delta\tilde{\tau}^k_{\text{aff}}, \qquad \tilde{\kappa}^k_{\text{aff}} := \tilde{\kappa}^k + \beta^k_{\text{aff}}\Delta\tilde{\kappa}^k_{\text{aff}}.$$

The affine variables provide a measure of the relative reduction in the duality gap, in the affine direction. This information is used to update the centering parameter γ. For this purpose, we use the heuristic [16]

$$\gamma^k := \left[\frac{\mu^k_{\text{aff}}}{\mu^k} \right]^3 = \left[\frac{((\tilde{w}^k_{\text{aff}})^T \tilde{s}^k_{\text{aff}} + \tilde{\tau}^k_{\text{aff}} \tilde{\kappa}^k_{\text{aff}})}{((\tilde{w}^k)^T \tilde{s}^k + \tilde{\tau}^k \tilde{\kappa}^k)} \right]^3. \tag{8}$$

2) Predictor-Corrector Step: In the predictor-corrector step, we compute the search direction $\Delta\phi^k$ by solving (6) with a modified right hand side

$$J_V(\phi^k)\Delta\phi^k = -\bar{V}(\phi^k). \tag{9}$$

$\bar{V}_i(\phi) := (1-\gamma^k)V_i(\phi)$ for $i = 1,2,3,4$ and

$$\bar{V}_5(\phi^k) := V_5(\phi^k) + \Delta\tilde{W}^k_{\text{aff}}\Delta\tilde{S}^k_{\text{aff}}\mathbf{1}_{m_I} - \gamma^k\mu^k\mathbf{1}^T_{m_I},$$
$$\bar{V}_6(\phi^k) := V_6(\phi^k) + \Delta\tilde{\tau}^k_{\text{aff}}\Delta\tilde{\kappa}^k_{\text{aff}} - \gamma^k\mu^k.$$

The diagonal matrices $\Delta\tilde{W}^k_{\text{aff}}$ and $\Delta\tilde{S}^k_{\text{aff}}$ are defined in a similar way to W and S. Terms involving these matrices are, as well as $\Delta\tilde{\tau}^k_{\text{aff}}$ and $\Delta\tilde{\kappa}^k_{\text{aff}}$, included to compensate for linearization errors [16]. We also notice that by employing the heuristic (8), the search direction is forced towards the central path if $\mu^k_{\text{aff}} \approx \mu^k$, meaning that only a small step in the non-negative orthant $(\tilde{w}, \tilde{s}, \tilde{\kappa}, \tilde{\tau}) \geq 0$ is available in the affine direction.

3) Stopping Criteria: To classify a solution as optimal, we adopt the criteria proposed in [27]

$$\rho^k_E \leq \varepsilon_E, \qquad \rho^k_I \leq \varepsilon_I, \qquad \rho^k_D \leq \varepsilon_D, \qquad \rho^k_O \leq \varepsilon_O. \tag{10}$$

Moreover, the problem is considered to be infeasible if $\tilde{\tau}^k \leq \varepsilon_\tau \max(1, \tilde{\kappa}^k)$, and

$$\rho^k_E \leq \varepsilon_E, \qquad \rho^k_I \leq \varepsilon_I, \qquad \rho^k_D \leq \varepsilon_D, \qquad \rho^k_G \leq \varepsilon_G. \tag{11}$$

ε_τ, ε_E, ε_I, ε_D, ε_O and ε_G are small user-defined tolerances and

$$\rho_D := ||V_1(\phi)||_\infty / \max(1, ||\begin{bmatrix} H^T & F^T & g \end{bmatrix}||_\infty),$$
$$\rho_E := ||V_2(\phi)||_\infty / \max(1, ||\begin{bmatrix} F & b \end{bmatrix}||_\infty),$$
$$\rho_I := ||V_3(\phi)||_\infty / \max(1, ||\begin{bmatrix} H & I & c \end{bmatrix}||_\infty),$$
$$\rho_G := |L - \tilde{\kappa}| / \max(1, ||\begin{bmatrix} g^T & b^T & c^T & 1 \end{bmatrix}||_\infty),$$
$$\rho_O := |L| / (\tilde{\tau} + |-b^T\tilde{v} - c^T\tilde{w}|).$$

where $L := g^T\tilde{t} - (-b^T\tilde{v} - c^T\tilde{w})$ is the duality gap.

4) Algorithm: Algorithm 1 summarizes the homogeneous and self-dual IPM described in this paper. We use a damping parameter, ν, to keep the iterates well inside the interior of the non-negative orthant $(\tilde{w}, \tilde{s}, \tilde{\tau}, \tilde{\kappa}) \geq 0$, as they approach the solution. To speed-up numerical computations and reduce the storage requirements of `LPempc`, operations involving the structured matrices F and H are implemented as specialized linear algebra routines [28].

IV. Riccati Iteration Procedure

The main computational efforts in Algorithm 1 are the solution of the linear systems (6) and (9). In a generic form, we can write these operations as

$$F^T\Delta\tilde{v} + H^T\Delta\tilde{w} + g\Delta\tilde{\tau} = r_1, \tag{12a}$$
$$b\Delta\tilde{\tau} - F\Delta\tilde{t} = r_2, \tag{12b}$$
$$c\Delta\tilde{\tau} - H\Delta\tilde{t} - \Delta\tilde{s} = r_3, \tag{12c}$$
$$g^T\Delta\tilde{t} + b^T\Delta\tilde{v} + c^T\Delta\tilde{w} - \Delta\tilde{\kappa} = r_4, \tag{12d}$$
$$\tilde{W}\Delta\tilde{s} + \tilde{S}\Delta\tilde{w} = r_5, \tag{12e}$$
$$\tilde{\kappa}\Delta\tilde{\tau} + \tilde{\tau}\Delta\tilde{\kappa} = r_6. \tag{12f}$$

Algorithm 1 Homogeneous and self-dual IPM for (4)

Require: DATA (g,F,b,H,c); INITIAL POINT $(\tilde{t},\tilde{v},\tilde{w},\tilde{s},\tilde{\tau},\tilde{\kappa})$; PARAMETERS $\nu \in [0.95;0.999]$

```
// initialize
```
$\mu \leftarrow (\tilde{w}^T\tilde{s}+\tilde{\tau}\tilde{\kappa})/(m_I+1)$
while not CONVERGED **do**
```
  // affine step
```
$\Delta\phi_{\text{aff}} \leftarrow -J_V(\phi)^{-1}V(\phi)$

$\alpha_{\text{aff}} \leftarrow \max\left\{a_{\text{aff}} \in [0,1] | \begin{bmatrix}\tilde{w}\\ \tilde{\tau}\end{bmatrix} + a_{\text{aff}}\begin{bmatrix}\Delta\tilde{w}_{\text{aff}}\\ \Delta\tilde{\tau}_{\text{aff}}\end{bmatrix} \geq 0\right\}$

$\beta_{\text{aff}} \leftarrow \max\left\{b_{\text{aff}} \in [0,1] | \begin{bmatrix}\tilde{s}\\ \tilde{\kappa}\end{bmatrix} + b_{\text{aff}}\begin{bmatrix}\Delta\tilde{s}_{\text{aff}}\\ \Delta\tilde{\kappa}_{\text{aff}}\end{bmatrix} \geq 0\right\}$

$\tilde{s}_{\text{aff}} \leftarrow \tilde{s}+\beta_{\text{aff}}\Delta\tilde{s}_{\text{aff}}, \quad \tilde{\kappa}_{\text{aff}} \leftarrow \tilde{\kappa}+\tilde{\beta}_{\text{aff}}\Delta\tilde{\kappa}_{\text{aff}}$

$\tilde{w}_{\text{aff}} \leftarrow \tilde{w}+\alpha_{\text{aff}}\Delta\tilde{w}_{\text{aff}}, \quad \tilde{\tau}_{\text{aff}} \leftarrow \tilde{\tau}+\tilde{\alpha}_{\text{aff}}\Delta\tilde{\tau}_{\text{aff}}$

$\mu_{\text{aff}} \leftarrow (\tilde{w}_{\text{aff}}^T\tilde{s}_{\text{aff}}+\tilde{\tau}_{\text{aff}}\tilde{\kappa}_{\text{aff}})/(m_I+1)$

$\gamma \leftarrow (\mu_{\text{aff}}/\mu)^3$
```
  // predictor-corrector step
```
$\Delta\phi \leftarrow -J_V(\phi)^{-1}\bar{V}(\phi)$

$\alpha \leftarrow \max\left\{a \in [0,1] | \begin{bmatrix}\tilde{w}\\ \tilde{\tau}\end{bmatrix} + a\begin{bmatrix}\Delta\tilde{w}\\ \Delta\tilde{\tau}\end{bmatrix} \geq 0\right\}$

$\beta \leftarrow \max\left\{b \in [0,1] | \begin{bmatrix}\tilde{s}\\ \tilde{\kappa}\end{bmatrix} + b\begin{bmatrix}\Delta\tilde{s}\\ \Delta\tilde{\kappa}\end{bmatrix} \geq 0\right\}$

$\tilde{t} \leftarrow \tilde{t}+\nu\beta\Delta\tilde{t}, \quad \tilde{s} \leftarrow \tilde{s}+\nu\beta\Delta\tilde{s}, \quad \tilde{\kappa} \leftarrow \tilde{\kappa}+\nu\beta\Delta\tilde{\kappa}$

$\tilde{v} \leftarrow \tilde{v}+\nu\alpha\Delta\tilde{v}, \quad \tilde{w} \leftarrow \tilde{w}+\nu\alpha\Delta\tilde{w}, \quad \tilde{\tau} \leftarrow \tilde{\tau}+\nu\alpha\Delta\tilde{\tau}$

$\mu \leftarrow (\tilde{w}^T\tilde{s}+\tilde{\tau}\tilde{\kappa})/(m_I+1)$

end while

We notice that the system (12) is different from the system solved in standard IPMs. Consequently, existing Riccati iteration procedures for MPC cannot be applied directly. However, as shown in Proposition 2, the solution to (12) can be obtained by solving a reduced linear system and a number of computationally inexpensive operations.

Proposition 2: The solution to (12) can be computed as

$$\begin{bmatrix}0 & F^T & H^T\\ -F & 0 & 0\\ -H & 0 & \tilde{W}^{-1}\tilde{S}\end{bmatrix}\begin{bmatrix}f_1 & h_1\\ f_2 & h_2\\ f_3 & h_3\end{bmatrix} = \begin{bmatrix}r_1 & -g\\ r_2 & -b\\ r_3 & -c\end{bmatrix}, \qquad (13)$$

and subsequent computation of

$$\begin{aligned}\Delta\tilde{\tau} &= \frac{r_6-\tilde{\tau}(g^Tf_1+b^Tf_2+c^Tf_3)}{\tilde{\kappa}+\tilde{\tau}(g^Th_1+b^Th_2+c^Th_3)},\\ \Delta\tilde{t} &= f_1+h_1\Delta\tilde{\tau},\\ \Delta\tilde{v} &= f_2+h_2\Delta\tilde{\tau},\\ \Delta\tilde{w} &= f_3+h_3\Delta\tilde{\tau},\\ \Delta\tilde{\kappa} &= g^T\Delta\tilde{t}+b^T\Delta\tilde{v}+c^T\Delta\tilde{w}-r_4,\\ \Delta\tilde{s} &= \tilde{W}^{-1}(r_C-\tilde{S}\Delta\tilde{w}),\end{aligned}$$

where $r_3 := r_3+\tilde{W}^{-1}r_5$ and $r_6 := r_6+\tilde{\tau}r_4$.

Proof: See [27]. ∎

In the appendix, an efficient solution procedure for (13) using a Riccati iteration procedure specifically tailored to economic MPC is derived. The order of complexity of the proposed method is $O(N(n_u+n_x+n_z)^3)$. In comparison, the complexity of solving the system directly using sparse linear algebra routines is linear to quadratic in N. Finally, a general purpose dense solver yields the complexity $O(N^3(n_u+n_x+n_z)^3)$ which is two orders of magnitude larger in N compared to our approach.

V. WARM-STARTING

We apply the warm-starting strategy from [17] to pick an initial point for Algorithm 1. The main idea is to construct such a point by combining a guess of the solution (candidate point) with a standard cold starting point. The initial point is defined as

$$\begin{aligned}w^0 &= \lambda\bar{w}+(1-\lambda)\mathbf{1}_{m_I}, & s^0 &= \lambda\bar{s}+(1-\lambda)\mathbf{1}_{m_I},\\ t^0 &= \lambda\bar{t}, & v^0 &= \lambda\bar{v},\\ \tau^0 &= 1, & \kappa^0 &= (w^0)^Ts^0/N,\end{aligned}$$

where $(\bar{t},\bar{v},\bar{w},\bar{s})$ is the candidate point and $\lambda \in [0,1[$ is a tuning parameter. Notice that in case $\lambda = 0$, the initial point becomes the standard cold-starting point $\phi^0 = (0,0,\mathbf{1}_{m_I},\mathbf{1}_{m_I},1,1)$. Conversely, $\lambda = 1$ corresponds to using the candidate point as the initial solution. Since this point typically lies close to the boundary of the non-negative orthant $(\tilde{w},\tilde{s},\tilde{\kappa},\tilde{\tau}) \geq 0$, $\lambda = 1$ can lead to ill-conditioned linear systems and/or blocking of the search direction [29].

For MPC applications, a good choice of the candidate point at time k can be constructed using the solution from the previous time step. As an example consider the solution at time step $k = 0$, for $N = 3$

$$t := \begin{bmatrix}u_0^{*T} & x_1^{*T} & \rho_1^{*T} & u_1^{*T} & x_2^{*T} & \rho_2^{*T} & u_2^{*T} & x_3^{*T} & \rho_3^{*T}\end{bmatrix}^T.$$

In this case we use the following candidate point at time step $k = 1$

$$\bar{t} := \begin{bmatrix}u_1^{*T} & x_2^{*T} & \rho_2^{*T} & u_2^{*T} & x_3^{*T} & \rho_3^{*T} & u_2^{*T} & x_3^{*T} & \rho_3^{*T}\end{bmatrix}^T.$$

Similarly, we left-shift the slack variables, s, and the dual variables, v and w.

VI. CASE STUDY - SMART GRID POWER MANAGEMENT

In this section we compare LPempc against IPMs from the following software packages: Gurobi, SeDuMi and MOSEK. These state-of-the-art IPMs are mainly written in low-level language such as FORTRAN and C, and rely on sparse linear algebra that are specifically tailored to the solution of large-scale sparse linear and conic programs. We also include CPLEX in our comparison.

The tolerance parameters for LPempc in (10)-(11) are set to 10^{-8}. With these settings LPempc achieves the same solution accuracy as the other solvers using their default tolerance settings. We have performed our simulations on an Intel(R) Core(TM) i5-2520M CPU @ 2.50GHz with 4 GB RAM running a 64-bit Ubuntu 12.04.1 LTS operating system.

TABLE I

CASE STUDY PARAMETERS

	τ_i	p_k	$\underline{u}_k$	$\overline{u}_k$	$\Delta\underline{u}_k$	$\Delta\overline{u}_k$
Power Plant 1	90	100	0	200	-20	20
Power Plant 2	30	200	0	150	-40	40

In the case study, we represent a system of m power generating units by a collection of simple third order systems in the form

$$Y_i(s) = \frac{1}{(\tau_i s+1)^3} U_i(s), \qquad i = 1,2,\ldots,m. \tag{14}$$

where $U_i(s)$ is the amount of fuel fed to the i'th power unit and $Y_i(s)$ is its power production. In [30] the model (14) has been validated against actual measurement data.

The total production from the m power generating units is the sum

$$Z(s) = \sum_{i=1}^{m} Y_i(s) = \sum_{i=1}^{m} \frac{1}{(\tau_i s+1)^3} U_i(s). \tag{15}$$

In state space form, the system (14)-(15) can be written as

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + E\boldsymbol{d}_k, \qquad \boldsymbol{d}_k \sim N(0,R_d), \tag{16a}$$
$$\boldsymbol{y}_k = C_y\boldsymbol{x}_k + \boldsymbol{e}_k, \qquad \boldsymbol{e}_k \sim N(0,R_e), \tag{16b}$$
$$\boldsymbol{z}_k = C_z\boldsymbol{x}_k. \tag{16c}$$

$u_k \in \mathbb{R}^{n_u}$ is the amount of fuel fed to the power generating units, $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$ is the measured power production from each of the units and $\boldsymbol{z}_k \in \mathbb{R}^{n_z}$ is the total power production. Notice that we have introduced process noise $\boldsymbol{d}_k$, and measurement noise $\boldsymbol{e}_k$. To control the stochastic system (16), we use economic MPC based on the certainty equivalence principle. Hence, stochastic variables are replaced by estimates of their conditional mean value in (1). The estimates are computed using the Kalman filter.

A. Closed-Loop Simulation

The following results are generated using an example with two power generating units; a cheap/slow unit, and an expensive/fast unit. This can represent a common situation in the power industry where large thermal power plants often produce a majority of the electricity, while units with faster dynamics such as diesel generators are used only in critical peak periods.

In our example, the controller objective is to coordinate the most cost-efficient power production, respecting capacity constraints and a time-varying electricity demand. It is assumed that full information about the initial state is given $x_0 \sim (0,0)$, and that the penalty of violating the output constraints is $q_k = 10^4$ for all time steps. The system and controller parameters are listed in Table I. We set the prediction horizon to $N = 80$ time steps and use a sampling time of $T_s = 5$ seconds. Moreover, we let $R_d = R_e = \sigma I$. A closed-loop simulation with $\sigma = 1$ is depicted in Fig. 1. The plot illustrates how the work load is distributed among the power generating units. It can be read that the cheap unit accounts for the main load, while the expensive unit is activated only when faster dynamics are needed to satisfy the electricity demand. The figure also shows that warm-starting `LPempc` reduces the number of iterations needed to find a solution with the desired accuracy. On average the number of iterations is reduced by approximately 35%, which is significant as it is achieved without introducing any additional expensive computations. To tune λ, we have generated the box-plots depicted in Fig. 2. The case $\lambda = 0$ corresponds to a cold-start. Thus, in all cases warm-starting reduces the average number of iterations. We have chosen $\lambda = 0.99$ for our controller. For this value of λ, the initial point lies well inside the interior of the non-negative orthant, $(\tilde{w},\tilde{s},\tilde{\kappa},\tilde{\tau}) \geq 0$, while it still maintains characteristics of the candidate point. Fig 2 shows that for this value of λ the number of iterations is reduced even when the variance of the noise is increased significantly.

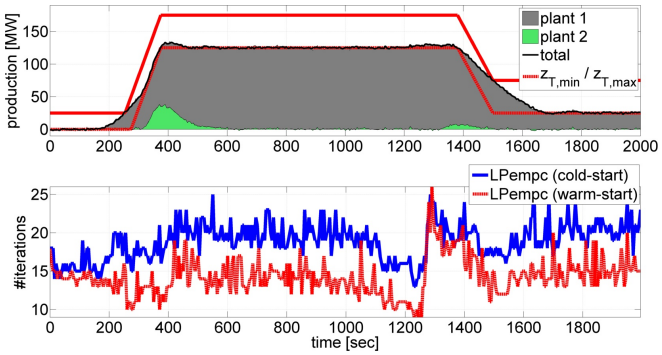

Fig. 1. Closed-loop simulation of a power system controlled by economic MPC. Warm-starting ($\lambda = 0.99$) yields a significant reduction in the number of iterations.

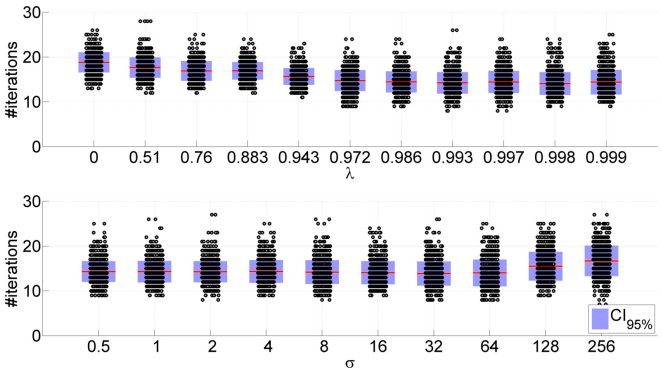

Fig. 2. Number of iterations needed to converge as a function of the tuning parameter λ, and the noise parameter σ . Each box-plot has been generated based on an entire closed-loop simulation. In the top plot we have fixed $\sigma = 1$, and in the bottom plot $\lambda = 0.99$.

Fig. 3 shows CPU-timings based on a closed-loop simulation with 10 power generating units and a prediction horizon of $N = 200$ time steps. In this simulation `LPempc` is up to an order of magnitude faster than `CPLEX`, `Gurobi`, `SeDuMi` and `MOSEK`, depending on the problem data. On average, `LPempc` is approximately 7 times faster than `Gurobi`, 5 times faster than `MOSEK` and 15 times faster than both `SeDuMi` and `CPLEX`. As demonstrated in [23], this difference grows with the problem size, as `LPempc` scales in a

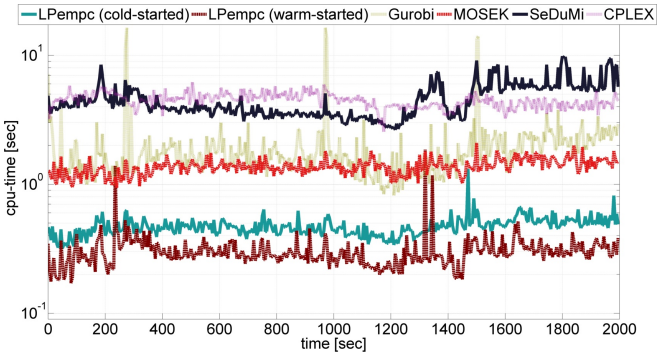

Fig. 3. CPU-time for solving (1) with 10 power generating units and a prediction horizon of 200 time steps.

favourable way.

VII. Conclusions

In this paper, we have developed an efficient IPM for linear economic MPC. The algorithm combines a homogeneous and self-dual model, and a Riccati iteration procedure specifically tailored to MPC. To speed up convergence, a recent warm-starting strategy for homogeneous and self-dual IPMs was implemented. Our simulations show that this strategy reduces the number of iterations by 10-35%, and that a MATLAB implementation of our algorithm, `LPempc`, is up to an order of magnitude faster than `CPLEX`, `Gurobi`, `SeDuMi` and `MOSEK`.

Appendix

Riccati Iteration Procedure for Economic MPC

Following the derivation in [23], we write the system (13) in the form

$$\begin{bmatrix} 0 & F^T & H^T \\ -F & 0 & 0 \\ -H & 0 & \tilde{W}^{-1}\tilde{S} \end{bmatrix} \begin{bmatrix} \tilde{t} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}. \tag{17}$$

where we have only included a single right hand side. We write the solution to this system as

$$\begin{aligned} \tilde{t} &= \begin{bmatrix} u_0^T & x_1^T & \rho_1^T & \dots & u_{N-1}^T & x_N^T & \rho_N^T \end{bmatrix}^T, \\ \tilde{v} &= \begin{bmatrix} \tilde{v}_0^T & \tilde{v}_1^T & \dots & \tilde{v}_{N-1}^T \end{bmatrix}^T, \\ \tilde{w} &= \begin{bmatrix} \eta^T & \lambda^T & \upsilon^T & \omega^T & \gamma^T & \zeta^T & \xi^T \end{bmatrix}^T. \end{aligned}$$

$\eta := \begin{bmatrix} \eta_0^T & \eta_1^T & \dots & \eta_{N-1}^T \end{bmatrix}^T$, and similarly for λ, υ, ω, γ, ζ and ξ. Accordingly, the right hand side is partitioned such that

$$\begin{aligned} \bar{r}_1 &= \begin{bmatrix} r_{u,0}^T & r_{x,1}^T & r_{w,1}^T & \dots & r_{u,N-1}^T & r_{x,N}^T & r_{w,N}^T \end{bmatrix}^T, \\ \bar{r}_2 &= \begin{bmatrix} R_{v,0}^T & R_{v,1}^T & \dots & R_{v,N-1}^T \end{bmatrix}^T, \\ \bar{r}_3 &= \begin{bmatrix} r_\eta^T & r_\lambda^T & r_\upsilon^T & r_\omega^T & r_\gamma^T & r_\zeta^T & r_\xi^T \end{bmatrix}^T. \end{aligned}$$

The diagonal matrix $\tilde{W}^{-1}\tilde{S}$ is written in terms of diagonal submatrices

$$\tilde{W}^{-1}\tilde{S} = \mathbf{diag}\left(\Sigma_\eta^T, \Sigma_\lambda^T, \Sigma_\upsilon^T, \Sigma_\omega^T, \Sigma_\gamma^T, \Sigma_\zeta^T, \Sigma_\xi^T\right)$$

For compact notation, we also introduce $\mathcal{N}_i := \{0+i, 1+i, \dots, N-1+i\}$.

Based on the notation above and the definition of F and H, the linear system of equations (17) can be stated in the form

$$\begin{aligned} \eta_i - \lambda_i + \upsilon_i - \omega_i + B^T\tilde{v}_i &= r_{u,i}, && i \in \mathcal{N}_0, \\ -u_i + \Sigma_{\eta,i}\eta_i &= r_{\eta,i}, && i \in \mathcal{N}_0, \\ u_i + \Sigma_{\lambda,i}\lambda_i &= r_{\lambda,i}, && i \in \mathcal{N}_0, \\ -u_i + Dx_i + \Sigma_{\upsilon,i}\upsilon_i &= r_{\upsilon,i}, && i \in \tilde{\mathcal{N}}_0, \\ u_i - Dx_i + \Sigma_{\omega,i}\omega_i &= r_{\omega,i}, && i \in \tilde{\mathcal{N}}_0, \\ x_{i+1} - Ax_i - Bu_i &= R_{v,i}, && i \in \tilde{\mathcal{N}}_0, \\ \rho_i - Cx_i + \Sigma_{\gamma,i}\gamma_i &= r_{\gamma,i}, && i \in \mathcal{N}_1, \\ \rho_i + Cx_i + \Sigma_{\zeta,i}\zeta_i &= r_{\zeta,i}, && i \in \mathcal{N}_1, \\ \rho_i + \Sigma_{\xi,i}\xi_i &= r_{\xi,i}, && i \in \mathcal{N}_1, \\ -\gamma_i - \zeta_i - \xi_i &= r_{w,i}, && i \in \mathcal{N}_1, \\ -\tilde{v}_i + C^T(\gamma_{i+1} - \zeta_{i+1}) + A^T\tilde{v}_i & \\ \quad + D^T(\omega_i - \upsilon_i) &= r_{x,i}, && i \in \tilde{\mathcal{N}}_0, \end{aligned}$$

with $\tilde{\mathcal{N}}_0 := \mathcal{N}_0 \setminus 0$ and the special cases

$$\begin{aligned} -u_0 + \Sigma_{\upsilon,0}\upsilon_0 &= r_{\upsilon,0}, \\ u_0 + \Sigma_{\omega,0}\omega_0 &= r_{\omega,0}, \\ x_1 - Bu_0 &= R_{v,0}, \\ -\tilde{v}_{N-1} + C^T(\gamma_N - \zeta_N) &= r_{x,N}. \end{aligned}$$

By eliminating η, λ, υ, ω, γ, ζ and ξ we get

$$\begin{aligned} B^T\tilde{v}_0 + U_0u_0 &= R_{u,0}, && && (18a) \\ B^T\tilde{v}_i + U_iu_i + G_ix_i &= R_{u,i}, && i \in \tilde{\mathcal{N}}_0, && (18b) \\ -x_1 + Bu_0 &= R_{v,0}, && && (18c) \\ -x_{i+1} + Ax_i + Bu_i &= R_{v,i}, && i \in \tilde{\mathcal{N}}_0, && (18d) \\ W_i\rho_i + M_i^Tx_i &= R_{w,i}, && i \in \mathcal{N}_1, && (18e) \\ -\tilde{v}_{i-1} + M_i\rho_i + \bar{X}_ix_i + G_i^Tu_i + A^T\tilde{v}_i &= \bar{R}_{x,i}, && i \in \tilde{\mathcal{N}}_0, && (18f) \\ -\tilde{v}_{N-1} + M_N\rho_N + \bar{X}_Nx_N &= \bar{R}_{x,N}, && && (18g) \end{aligned}$$

where we have defined

$$\begin{aligned} U_i &:= \Sigma_{\eta,i}^{-1} + \Sigma_{\lambda,i}^{-1} + \Sigma_{\omega,i}^{-1} + \Sigma_{\upsilon,i}^{-1}, && i \in \mathcal{N}_0, \\ G_i &:= -(\Sigma_{\omega,i}^{-1} + \Sigma_{\upsilon,i}^{-1})D, && i \in \tilde{\mathcal{N}}_0, \\ W_i &:= \Sigma_{\zeta,i}^{-1} + \Sigma_{\xi,i}^{-1} + \Sigma_{\gamma,i}^{-1}, && i \in \mathcal{N}_1, \\ M_i &:= C^T(\Sigma_{\zeta,i}^{-1} - \Sigma_{\gamma,i}^{-1}), && i \in \mathcal{N}_1, \\ \bar{X}_i &:= C^T(\Sigma_{\zeta,i}^{-1} + \Sigma_{\upsilon,i}^{-1})C + D^T(\Sigma_{\gamma,i}^{-1} + \Sigma_{\omega,i}^{-1})D, && i \in \tilde{\mathcal{N}}_0, \\ \bar{X}_N &:= C^T(\Sigma_{\zeta,N}^{-1} + \Sigma_{\upsilon,N}^{-1})C. \end{aligned}$$

Furthermore

$$\begin{aligned} R_{u,i} &:= r_{u,i} + \bar{r}_{\lambda,i} + \bar{r}_{\omega,i} - \bar{r}_{\eta,i} - \bar{r}_{\upsilon,i}, && i \in \mathcal{N}_0, \\ R_{v,i} &:= -R_{v,i}, && i \in \mathcal{N}_0, \\ R_{w,i} &:= r_{w,i-1} + \bar{r}_{\zeta,i-1} + \bar{r}_{\xi,i} + \bar{r}_{\gamma,i}, && i \in \mathcal{N}_1, \\ \bar{R}_{x,i} &:= r_{x,i} + C^T(\bar{r}_{\zeta,i} - \bar{r}_{\gamma,i}) + D^T(\bar{r}_{\upsilon,i} - \bar{r}_{\omega,i}), && i \in \tilde{\mathcal{N}}_0, \\ \bar{R}_{x,N} &:= r_{x,N} + C^T(\bar{r}_{\zeta,N} - \bar{r}_{\gamma,N}). \end{aligned}$$

$\bar{r}_{\lambda,i} := \Sigma_{\lambda,i}^{-1} r_{\lambda,i}$, and similarly for $\bar{r}_{\omega,i}$, $\bar{r}_{\eta,i}$, $\bar{r}_{\upsilon,i}$, $\bar{r}_{\zeta,i}$, $\bar{r}_{\xi,i}$ and $\bar{r}_{\gamma,i}$. Solving (18e) for $\tilde{w}$ gives

$$\rho_i = W_i^{-1}(R_{w,i} - M_i^T x_i), \qquad i \in \mathscr{N}_1. \tag{19}$$

Substituting back into (18) yields the equations

$$\begin{aligned}
B^T\tilde{v}_0 + U_0 u_0 &= R_{u,0} \\
B^T\tilde{v}_i + U_i u_i + G_i x_i &= R_{u,i}, \qquad i \in \tilde{\mathscr{N}_0} \\
-x_1 + Bu_0 &= R_{v,0} \\
-x_{i+1} + Ax_i + Bu_i &= R_{v,i}, \qquad i \in \tilde{\mathscr{N}_0} \\
-\tilde{v}_{i-1} + X_i x_i + G_i^T u_i + A^T\tilde{v}_i &= R_{x,i}, \qquad i \in \tilde{\mathscr{N}_0} \\
-\tilde{v}_{N-1} + X_N x_N &= R_{x,N}
\end{aligned}$$

where $X_i := \bar{X}_i - M_i W_i^{-1} M_i^T$ and $R_{x,i} := \bar{R}_{x,i} - M_i W_i^{-1} R_{w,i}$. This set of equations can be solved efficiently by a Riccati iteration procedure [18]–[22]. Thus, by utilizing the structure of (1) we have reduced (13) to a smaller system which can be solved efficiently using a recursive approach.

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