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On Optimal Operation of Storage Devices under Stochastic Market Prices

Mahdi Kefayati and Ross Baldick

Abstract—**Recent results in the literature have demonstrated the value of energy storage assets in increasing the efficiency of power grid. As operation of most assets in the grid is governed by their generated economic value to their owner, it is imperative to study the optimal operation of storage assets in response to pricing signals from the market.**

In this work we consider the problem of optimal operation of a generic energy storage device under stochastic energy prices. We show that under certain conditions, the optimal policy for operating the storage asset follows an extended threshold form and can be obtained in a computationally efficient manner.

I. Introduction

The need to maintain continuous supply demand balance in electric grid, and the increased variability and stochasticity in demand, and more recently supply due to intermittent renewables has exacerbated the need for energy storage in the grid. This need has resulted in a trend in development of storage assets at different levels of the grid and in different sizes.

As these assets become mainstream, a key question that needs to be addressed is how to efficiently operate them. For most of the current assets, efficient and optimal operation is defined with respect to the asset owners objective. For example, in some cases, the storage is used to displace the energy generated by renewables from the time they are more abundant to the time the are more needed, i.e. high demand times. The time scale in such cases can vary from a day to a season. In other cases, such assets are primarily geared towards grid stabilization and provision of ancillary services, and some newer applications involves firming intermittent sources. At micro grid and consumer level, storage assets have been used to increase reliability and more recently, shaping the load profile to reduce the overall energy bill; e.g. by reducing demand charges.

What ultimately drives operation of an asset; however, is the economic value it provides to its operator. Hence, it is conceivable for the owner to operate the asset to maximize its economic benefit. This economic value depends on the structure of the underlying energy market. In other words, the optimal operation of the asset, and hence its impact on the overall power system is directly influenced by the incentives provided by the market.

The authors are with the ECE department of the University of Texas at Austin, 1 University Station Austin TX 78712. kefayati@utexas.edu, baldick@ece.utexas.edu

In most jurisdictions, the grid is operated as an open market in which power is traded while grid reliability and supply demand balance is maintained by the Independent System Operator (ISO). The incentives in such markets, also known as the restructured electricity markets, are mainly prices that are formed as a result of market clearance based on the bids submitted by participants. Given the stochastic nature of demand and most renewable supplies, market participants, particularly smaller ones, effectively respond to market prices. Hence, for a storage asset operator, optimal operation of the asset is equivalent to optimal response to real-time stochastic prices and this is our focus in this paper.

Potential benefits and applications of the storage assets in power systems has been investigated in the past by various authors. In [1], the authors aim at giving a broad perspective of the value of energy storage in power systems, one that is most suitable for assessment of long term benefits and investment purposes. Many authors have proposed optimizing the storage operation to stabilize the combined output of an intermittent generation source, like a wind farm, and the storage asset. For example in [2], Korpaas *et al.* investigate optimal operation of a combination of a storage asset and a wind farm and propose using the storage asset to meet the output schedule which has been formed based on a forecast model. Along the same lines, Teleke [3] considered control strategies for smoothing wind farm output using battery storage. Denholm and Sioshansi [4], examined co-locating wind farms with Compressed Air Energy Storage (CASE) to assess the value of energy storage. In [5], Divya and Østergaard investigate various applications of battery storage in power systems. They discuss firming wind power as well as using battery for energy arbitrage between high and low price times.

In this paper, we present our preliminary results on optimal operation of the energy storage device as an independent asset in response to stochastic real-time market prices. We first present our generic model for an storage asset and then formulate the optimal operation problem as a dynamic program whose objective is to maximize the net present value of the profit achieved by operating the asset. Our main contribution in this work is to give the exact optimal operation policy under fairly generally distributed stochastic prices and demonstrate that under some conditions, the optimal policy can be computed and stored in a computationally efficient manner. Our approach is not limited to grid scale storage devices and can be applied to micro grid or even consumer

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level storage assets, e.g. V2G, if they are under the same pricing structure.

The rest of this paper is organized as follows: In Section II we present our generic storage asset model. Section III is dedicated to presenting our main result on optimal operation of the storage assets. We conclude the paper and comment on our future directions in Section IV.

II. System Model

In this section, we first introduce our general model for energy storage assets and the objective of the optimization problem. We then make some simplifying assumption and prepare the ground for our results on optimal operation of the proposed model in the proceeding section.

We consider an energy storage asset which is being operated under real-time prices where the asset owner aims at maximizing its profit through buying and selling energy over time. We assume the storage operator participates as a price taker; i.e. its decisions does not impact the prices. Considering a discrete time setup, at each time slot, the storage operator needs to decide how much energy to buy or sell given the price of energy over that time slot. Note that due to lack of market power assumption, obtaining the optimal policy for energy trading is equivalent to finding the optimal bidding strategy.

We assume a Markovian model for prices whose statistics are available to the operator. Following a similar model as in [6], we assume stochastic prices evolving as:

$$\pi_t = \lambda_t(\pi_{t-1}) + \epsilon_t, \tag{1}$$

where ϵ_t is the random variable modeling price innovations and $\lambda_t(\bullet)$ is modeling the inter-stage correlation of prices and seasonality. We assume $\epsilon_t \sim F_t(\bullet)$ to be independent with respect to t, $\lambda_t(\bullet)$ to be monotone to avoid some technicalities, and define $\theta_t \triangleq \pi_{t-1}$ for notational convenience. Note that the price innovation statistics is assumed to be completely general with arbitrary mean and distribution.

Our objective is to find the optimal policy using which the storage operator, or operator in short, maximizes its expected profit from operating the storage asset. To this end, we need to model the dynamics of the storage asset. Storage assets come in wide variety depending on their size and technology. Examples range from batteries based on different chemistries and in various sizes to flywheels, thermal storages, pumped hydro assets and Compressed Air Energy Storage (CAES). Although the dynamics of each of these storage technologies are different, they share some common fundamentals. From grid's perspective, all of these assets have a State of Charge (SoC) which is basically the amount of energy stored in them. For efficient operation of the asset, its SoC is bounded to a specific range. Moreover, the operator can control the flow of energy in and out of the asset and similar to SoC, for efficient operation of the asset, there are upper and lower bounds on the rate of energy flow. The operator pays for flows going in, i.e. consuming electricity and is paid for flows out.

We formalize this model as follows. Let us denote the state of charge by x_t and the total energy transfered in/out of the storage asset at time t by u_t, then the dynamics of the asset can be described as:

$$\begin{aligned} &x_{t+1} = x_t - (\eta^+(u_t)^+ + \eta^-(u_t)^-), \\ &\underline{u} \le u_t \le \overline{u}, \\ &\underline{x} \le x_t \le \overline{x}, \end{aligned} \tag{2}$$

where η^- and η^+ capture inefficiencies in energy storage and retrieval respectively; and, $\underline{x}$, $\overline{x}$, $\underline{u}$ and $\overline{u}$ represent the upper and lower bounds on feasible SoCs and energy flow per time slot respectively. Also, we define: $(\bullet)^+ \triangleq \max\{0, \bullet\}$ and $(\bullet)^- \triangleq \min\{0, \bullet\}$. Note that for a physical system where losses are always non-negative, hence:

$$0 \le \eta^- \le 1 \le \eta^+. \tag{3}$$

Moreover, the bi-directional flow of energy implies that:

$$\underline{u} \le 0 \le \overline{u}. \tag{4}$$

At each time step, the utility of the operator is given by its energy sales to the grid:

$$g_t(x_t, u_t, \pi_t) = \pi_t u_t, \tag{5}$$

and final stage utility at stage T, which captures the effect of remaining energy in the storage device is given by:

$$g_T(x_T) = m_T(x_t - \underline{x}). \tag{6}$$

where m_T is the price for energy remained in the storage asset after final stage. The objective is to minimize the total expected net present value of the profits:

$$J_0^*(x_0, \theta_0) = \max \mathbb{E}_{\pi_t}[\sum_{t=0}^{T-1} a(t) g_t(x_t, u_t, \pi_t) + a(T) g_T(x_T)], \tag{7}$$

where $a(t) = \prod_{t'<t} \alpha_{t'}$; α_t is the discount factor; and the maximization is over all policies that admit a feasible u_t, denoted by $u_t(x_t, \pi_t)$ by abuse of notation. Here, we consider a finite horizon model due to the non-stationary statistics of prices and the infinite horizon case can be approximated by picking a large enough T.

III. Optimal Operation of Storage Assets

In this section, we state our main result which provides a closed form optimal policy and value function for the no-loss case and then show that under price independence assumption, the optimal policy can be computed and stored computationally efficiently. We then discuss the technical challenges with the lossy case and shed some light on sufficient conditions on tractable cases.

For the sake of brevity and notational clarity, let us define some notation and make some simplifying assumptions. We define $a \wedge b \triangleq \min\{a, b\}$ and $a \vee b \triangleq \max\{a, b\}$. Also, we assume symmetry in the rate limits in and out of the storage device from the grid perspective, i.e.:

$$\underline{u} = -\overline{u}. \tag{8}$$

Also, defining $n_P = \lfloor \frac{\overline{x}-\underline{x}}{\overline{u}} \rfloor$, we assume:

$$n_P \in \mathbb{Z}, \tag{9}$$

which implies that state space range is divisible by the per time step limit of energy transfer. Moreover, without loss of generality, we assume:

$$\underline{x} = 0. \tag{10}$$

Finally, for the ideal case, we assume:

$$\eta^- = \eta^+ = 1. \tag{11}$$

Now we are ready to present our first result, which gives the optimal operation policy under correlated prices:

Theorem 1. *Consider the system described in (1-11).*

(a) The optimal value function is continuous, piecewise linear and concave with $n_P + 1$ pieces given by:

$$J_0^*(x,\theta) = \sum_{j=0}^{n_P} m_0^j(\theta)[(x-j\bar{u})^+ \wedge \bar{u}] + c_0(\theta)\bar{u}, \tag{12}$$

where $m_t^i(\theta_t)$ is given by the following backward recursion:

$$m_t^i(\theta_t) = \mathbb{E}_{\epsilon_t}[M(\theta_t, \epsilon_t)], \tag{13}$$

where,

$$M(\theta,\epsilon) = \begin{cases} \tilde{m}_{t+1}^{i+1}(\pi) & \pi < \tilde{m}_{t+1}^{i+1} \\ \pi & \tilde{m}_{t+1}^{i+1} \le \pi < \tilde{m}_{t+1}^{i-1} \\ \tilde{m}_{t+1}^{i-1}(\pi) & \tilde{m}_{t+1}^{i-1} \le \pi \end{cases}, \tag{14}$$

$\pi \triangleq \lambda_t(\theta) + \epsilon$, $\tilde{m}_\tau^i \triangleq \alpha_{\tau-1} m_\tau^i$, $m_T^i = m_T$, $\forall i$, $m_t^0 = +\infty$ and $m_t^{n_P} = -\infty, \forall t$, and m_t^i is the extended solution to the following fixed point equation:

$$\mu = \tilde{m}_t^i(\mu), \tag{15}$$

and, $c_t(\theta_t)$ is given by:

$$c_t(\theta_t) = \mathbb{E}_{\epsilon_t}[C(\theta_t, \epsilon_t)], \tag{16}$$

where,

$$C(\theta,\epsilon) = \alpha_t c_{t+1}(\pi) + \begin{cases} \tilde{m}_{t+1}^i(\pi) - \pi & \pi \le \tilde{m}_{t+1}^i \\ \pi - \tilde{m}_{t+1}^{i-1}(\pi) & \tilde{m}_{t+1}^{i-1} \le \pi \end{cases}. \tag{17}$$

(b) The optimal policy is given by:

$$u_t^*(x,\pi_t) = \begin{cases} \underline{u} & \pi_t \le \tilde{m}_t^{i+1}, \\ x - (i+1)\overline{u} & \tilde{m}_t^{i+1} < \pi_t \le \tilde{m}_t^i, \\ x - i\overline{u} & \tilde{m}_t^i < \pi_t \le \tilde{m}_t^{i-1}, \\ \overline{u} & \tilde{m}_t^{i-1} < \pi_t. \end{cases}, \tag{18}$$

where

$$i = \lfloor \frac{x}{\overline{u}} \rfloor. \tag{19}$$

The proof is provided in the appendix.

Theorem 1 demonstrates that the optimal operation policy follows a multi-threshold form in the sense that for any given x and its corresponding i can be found and the thresholds can be obtained. Moreover, the number of these thresholds are limited to n_P which is basically the minimum number of time steps needed to span the SoC state space.

Moreover, one main result of this theorem is the structure of the optimal value function. Equation (12) basically shows how the price state and SoC state interact. In particular, the optimal value function is piecewise linear in SoC state and the price statistics only affect the coefficients of the pieces.

Intuitively, these coefficients correspond to the expected marginal net present value of the storage at different SoCs. That is, at given SoC, x, these coefficients represent the expected marginal change in the profit of operation given the price statistics forward, adjusted by the discount rate. This marginal profit involves both selling and buying energy forward in time. $c_t(\theta)$, on the other hand, is the constant profit obtained from the operation at stage t independent of x, or basically the base profit.

Since we have assumed that the storage operator acts as a price taker, it is not hard to see that the optimal policy obtained also corresponds to the optimal bidding strategy by the operator. Therefore, the piecewise linear structure of the optimal value function means that using piecewise liner marginal cost specification which is common in many energy markets does not impact optimality of the bidding for the storage operator, and Theorem 1 indeed gives the optimal bidding strategy under assumptions given in (1-11).

Although Theorem 1 simplifies the structure of the value function by decomposing the role of SoC and prices and giving $O(n_P)$ fixed point calculations for SoC recursions, price expectation recursions still pose a computational challenge due to correlations. Taking this lead, in the next theorem, we show that there recursions can be much simplified and the optimal policy can be computed and stored in a computationally efficiently under price independence assumption.

Theorem 2. *Consider the system specified by (1-11) and further assume $\lambda_t(\theta) = 0$ for all t, then, the optimal value function given in Theorem 1 simplifies to:*

$$J_0^*(x) = \sum_{j=0}^{n_P} m_0^j[(x-j\bar{u})^+ \wedge \bar{u}] + c_0\bar{u}, \tag{20}$$

where m_t^i is given by:

$$m_t^i = \tilde{m}_{t+1}^{i+1} - G_t(\tilde{m}_{t+1}^{i-1}, \tilde{m}_{t+1}^{i+1}), \tag{21}$$

and c_t is given by:

$$c_t = \alpha_t c_{t+1} + G_t(-\infty, \tilde{m}_{t+1}^i) + \bar{G}_t(\tilde{m}_{t+1}^{i-1}, \infty), \tag{22}$$

in which,

$$G_t(z,z') \triangleq \int_z^{z'} F_t(\zeta)\, d\zeta, \tag{23}$$

$$\bar{G}_t(z,z') \triangleq \int_z^{z'} (1 - F_t(\zeta))\, d\zeta, \tag{24}$$

and $\tilde{m}_\tau^i \triangleq \alpha_{\tau-1} m_\tau^i$. The optimal policy remains the same as given in Theorem 1.

The proof is provided in the appendix.

According to this result, the description of the optimal value function, and hence optimal policy, which essentially consists of m_t^i and c_t can be stored in vector of n_P+1 elements per time step and hence is $\Theta(n_P)$. For any given T, the total size of the optimal policy description is then would be $\Theta(Tn_P)$. Computationally, assuming $G(\bullet,\bullet)$ and $\bar{G}(\bullet,\bullet)$ are $\Theta(1)$, we have the same result computationally. Moreover, the per stage computation can be parallelized into n_P+1 independent computations for higher efficiency.

IV. Conclusion and Future Directions

We proposed a generic model for energy storage assets and formulated the profit maximization problem for the asset owner under stochastic prices. We also obtained the exact optimal policy for operation of such assets under no-loss assumption and further demonstrated that if prices are not correlated, the optimal policy and value function can be computed and stored in a computationally efficient manner.

Due to the limited space, we could not present the extended results on the impact of employing such optimal operation policies in the grid as well as numerical examples that would demonstrate the value of the storage asset more vividly. Expanding our results to more general cases and full demonstration of their impact is left to future work. Moreover, we believe that there is a synergy between the model we adopted here and flexible loads and we plan to present our results on the more general model that encompasses flexible assets, including flexible loads, generators and storage in our future work.

Appendix I – Proof of Theorem 1

We prove the result by establishing that the proposed optimal value function satisfies the Bellman equation and deriving the optimal policy on the way. We assume that the correlation function is well behaved and to avoid technicalities, assume existence of expected values, cdfs and integrals throughout.

Since the stage and final cost functions are assumed concave, linear in fact, and the dynamics is linear establishing concavity of the optimal value function is straightforward, hence we skip it here for brevity. A similar argument can be found in the proof of Proposition 1 in [7].

To establish part (a), we basically use backward induction on t, that is, we assume that the optimal value function has the desired form as in Equation (12) for $t+1$:

$$J^*_{t+1}(x,\theta)=\sum_{j=0}^{n_P} m^j_{t+1}(\theta)[(x-j\bar{u})^+\wedge\bar{u}]+c_{t+1}(\theta)\bar{u}, \quad (25)$$

and then plug it into the Bellman equation to show its validity for t. Note that this form is valid for $t=T$ by simply setting $m^j_T(\theta)=m_T$, $\forall j$.

Using Bellman equation, we need to show:

$$J^*_t(x_t,\theta_t)=\mathbb{E}_{\epsilon_t}[\max_u\{g_t(x_t,u)+\alpha_t J^*_{t+1}(x_{t+1},\theta_{t+1})\}], \quad (26)$$

Using the dynamics equations, (1) and (2), (26) is transformed to:

$$J^*_t(x_t,\theta_t)=\mathbb{E}_{\epsilon_t}[\max_u\{\pi_t u+\alpha_t J^*_{t+1}(x_t-u,\pi_t)\}], \quad (27)$$

noting that $\theta_{t+1}=\pi_t$ by definition.

Since $\underline{x}\le x_t\le\overline{x}$, there exists some $i\le n_P$ such that $i\bar{u}\le x_t\le(i+1)\bar{u}$. Hence, we can rewrite Equation (25) as:

$$J^*_t(x_t,\theta)=\sum_{j=1}^{i-1} m^j_t(\theta)\bar{u}+m^i_t(\theta)[x_t-i\bar{u}]+c_t(\theta)\bar{u}. \quad (28)$$

Since $-\bar{u}\le u\le\bar{u}$, $(i-1)\bar{u}\le x_t-u\le(i+2)\bar{u}$, hence invoking a similar expansion on $J^*_{t+1}(x_t-u,\pi_t)$ and some rearrangement, we have:

$$\begin{aligned}
J^*_{t+1}(x_t-u,\pi_t)=&\sum_{j=1}^{i-2} m^j_{t+1}(\pi_t)\bar{u}+c_{t+1}(\pi_t)\bar{u}\\
&+m^{i-1}_{t+1}(\pi_t)[(x_t-u-(i-1)\bar{u})^+\wedge\bar{u}]\\
&+m^{i}_{t+1}(\pi_t)[(x_t-u-i\bar{u})^+\wedge\bar{u}]\\
&+m^{i+1}_{t+1}(\pi_t)(x_t-u-(i+1)\bar{u})^+,\\
\overset{(a)}{=}&\sum_{j=1}^{i-1} m^j_{t+1}(\pi_t)\bar{u}+m^i_{t+1}(\pi_t)\tilde{x}_t\\
&+c_{t+1}(\pi_t)\bar{u}\\
&-m^{i-1}_{t+1}(\pi_t)[(u-\tilde{x}_t)^+]\\
&+m^{i}_{t+1}(\pi_t)[((\tilde{x}_t-u)^+\wedge\bar{u})-\tilde{x}_t]\\
&+m^{i+1}_{t+1}(\pi_t)(\tilde{x}_t-u-\bar{u})^+\\
=&\sum_{j=1}^{i-1} m^j_{t+1}(\pi_t)\bar{u}+m^i_{t+1}(\pi_t)\tilde{x}_t\\
&+c_{t+1}(\pi_t)\bar{u}\\
&-m^{i-1}_{t+1}(\pi_t)(u-\tilde{x}_t)^+\\
&+m^{i}_{t+1}(\pi_t)[((\tilde{x}_t-u)^+\wedge\bar{u})-\tilde{x}_t]\\
&+m^{i+1}_{t+1}(\pi_t)(\tilde{x}_t-u-\bar{u})^+
\end{aligned} \quad (29)$$

where $\tilde{x}_t\triangleq x_t-i\bar{u}$. Now, substituting (29) in (27), we proceed as depicted in (31) where, (a) is obtained by using the fact that $-(\bullet)^+=(\bullet)^-$ and substitution and (b) is obtained by using the equality:

$$-u\equiv[(\tilde{x}_t-u)^-]+[((\tilde{x}_t-u)^+\wedge\bar{u})-\tilde{x}_t]+(\tilde{x}_t-u-\bar{u})^+, \quad (30)$$

and refactoring. Equality (30) can be verified to be valid for $-\bar{u}\le u\le\bar{u}$ and $0\le\tilde{x}_t\le\bar{u}$ by simple conditioning and inspection.

Now, note that the optimization is only on the first three terms in (31) since the sum is independent of u. Moreover, concavity of the value function, implies that m^j_τ is increasing in j; i.e. $m^j_\tau\le m^{j'}_\tau$, $\forall j\ge j'$. This implies that the price dependent coefficients of the form $(\pi_t-\alpha_t m^j_{t+1})$ are ordered as:

$$\pi_t-\alpha_t m^{i-1}_{t+1}(\pi_t)\le\pi_t-\alpha_t m^{i}_{t+1}(\pi_t)\le\pi_t-\alpha_t m^{i+1}_{t+1}(\pi_t). \quad (32)$$

$$
\begin{aligned}
J_t^*(x_t,\theta_t)&=\mathbb{E}_{\epsilon_t}\left[\max_u \left\{\pi_t u+\alpha_t J_{t+1}^*(x_t-u,\pi_t)\right\}\right]\\
&\overset{(a)}{=} \mathbb{E}_{\epsilon_t}\left[\max_u \left\{\pi_t u+\alpha_t\left(\sum_{j=1}^{i-1} m_{t+1}^j(\pi_t)\bar{u}+m_{t+1}^i(\pi_t)\tilde{x}_t+c_{t+1}(\pi_t)\bar{u}\right.\right.\right.\\
&\qquad\left.\left.\left. m_{t+1}^{i-1}(\pi_t)(\tilde{x}_t-u)^-+m_{t+1}^i(\pi_t)[((\tilde{x}_t-u)^+\wedge\bar{u})-\tilde{x}_t]+m_{t+1}^{i+1}(\pi_t)(\tilde{x}_t-u-\bar{u})^+\right)\right\}\right]\\
&\overset{(b)}{=} \mathbb{E}_{\epsilon_t}\left[\max_u \{(\tilde{m}_{t+1}^{i-1}(\pi_t)-\pi_t)(\tilde{x}_t-u)^-+(\tilde{m}_{t+1}^i(\pi_t)-\pi_t)(((\tilde{x}_t-u)^+\wedge\bar{u})-\tilde{x}_t)\right.\\
&\qquad\left.+(\tilde{m}_{t+1}^{i+1}(\pi_t)-\pi_t)(\tilde{x}_t-u-\bar{u})^+\}+\left(\sum_{j=1}^{i-1}\tilde{m}_{t+1}^j(\pi_t)\bar{u}+\tilde{m}_{t+1}^i(\pi_t)\tilde{x}_t+\alpha_t c_{t+1}(\pi_t)\bar{u}\right)\right]
\end{aligned}
\tag{31}
$$

Since the optimization problem at hand is piecewise linear in u, and the ordering of the coefficients as in (32), only four cases can happen depending on π_t, resulting in different signs of the coefficients. Let us label the events corresponding to each of these cases as:

$$e_1 \quad : \quad \pi_t\le\alpha_t m_{t+1}^{i+1}(\pi_t) \tag{33}$$
$$e_2 \quad : \quad \alpha_t m_{t+1}^{i+1}(\pi_t)\le\pi_t\le\alpha_t m_{t+1}^{i}(\pi_t) \tag{34}$$
$$e_3 \quad : \quad \alpha_t m_{t+1}^{i}(\pi_t)\le\pi_t\le\alpha_t m_{t-1}^{i-1}(\pi_t) \tag{35}$$
$$e_4 \quad : \quad \alpha_t m_{t+1}^{i-1}(\pi_t)\le\pi_t \tag{36}$$

In each case, the optimal u can be easily obtained by inspecting the corresponding term in the optimization problem in (31):

- Under e_1, all the coefficients are positive and hence all the terms involving u need to be maximized. This is achieved by minimizing u and setting $u^*=\underline{u}=-\bar{u}$.
- Under e_2, only the first two coefficients are positive and hence $u^*=\tilde{x}_t-\bar{u}$.
- Under e_3, only the first coefficient is positive and hence $u^*=\bar{u}-\tilde{x}_t$.
- Under e_4, all the coefficient are negative and hence all terms should be minimized. This is achieved by setting $u^*=\bar{u}$.

Therefore, using the definition of $\tilde{x}_t$, the optimal policy can be summarized as:

$$
u^*(x_t,\pi_t)=\begin{cases}-\bar{u} & \text{if} \quad e_1,\\ x_t-(i+1)\bar{u} & \text{if} \quad e_2,\\ x_t-i\bar{u} & \text{if} \quad e_3,\\ \bar{u} & \text{if} \quad e_4.\end{cases}
$$

The conditions defining e_k are not explicit. Therefore, to achieve the desired form in (18), we need to make the conditions explicit. Given the assumptions on $\lambda_t(\bullet)$, this is achieved by simply calculating the extended fixed point of the corresponding m_{t+1}^j, as defined in (15).

Now we can plug the optimal policy and find the desired optimal value function, $J_t^*(x_t,\theta_t)$. To this end, we decompose the optimal value function to its conditionals over the partition formed by $\{e_k\}$:

$$
\begin{aligned}
J_t^*(x_t,\theta_t)=&J_t^*(x_t,\theta_t|e_1)\mathbb{P}\{e_1\}+J_t^*(x_t,\theta_t|e_2)\mathbb{P}\{e_2\}\\
&+J_t^*(x_t,\theta_t|e_3)\mathbb{P}\{e_3\}+J_t^*(x_t,\theta_t|e_4)\mathbb{P}\{e_4\},
\end{aligned}
\tag{37}
$$

where,

$$
\begin{aligned}
J_t^*(x,\theta_t|e_1)=&\sum_{j=1}^{i-1}\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^j(\pi_t)|e_1]\bar{u}\\
&+\alpha_t\mathbb{E}_{\epsilon_t}[c_{t+1}(\pi_t)|e_1]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^i(\pi_t)-\pi_t|e_1]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i+1}(\pi_t)|e_1][x-i\bar{u}],\\
J_t^*(x,\theta_t|e_2)=&\sum_{j=1}^{i-1}\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^j(\pi_t)|e_2]\bar{u}\\
&+\alpha_t\mathbb{E}_{\epsilon_t}[c_{t+1}(\pi_t)|e_2]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^i(\pi_t)-\pi_t|e_2]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\pi_t|e_2][x-i\bar{u}],\\
J_t^*(x,\theta_t|e_3)=&\sum_{j=1}^{i-1}\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^j(\pi_t)|e_2]\bar{u}\\
&+\alpha_t\mathbb{E}_{\epsilon_t}[c_{t+1}(\pi_t)|e_3]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\pi_t|e_3][x-i\bar{u}],\\
J_t^*(x,\theta_t|e_4)=&\sum_{j=1}^{i-1}\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^j(\pi_t)|e_4]\bar{u}\\
&+\alpha_t\mathbb{E}_{\epsilon_t}[c_{t+1}(\pi_t)|e_4]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\pi_t-\tilde{m}_{t+1}^{i-1}(\pi_t)|e_4]\bar{u}\\
&+\mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i-1}(\pi_t)|e_4][x-i\bar{u}].
\end{aligned}
\tag{38}
$$

Combining these results with (37) we conclude the desired form as claimed in (28) and equivalently (12). Now, what remains is to obtain recursions for $m_t^i(\theta_t)$ and $c_t(\theta_t)$, which are basically coefficients of $x - i\bar{u}$ and $\bar{u}$. To this end, we use the definitions of $M(\theta, \epsilon)$ and $C(\theta, \epsilon)$ as given in (14) and (17) respectively. Using Equations (37) and (38):

$$\begin{aligned} m_t^i(\theta_t) =& \mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i+1}(\lambda_t(\theta_t) + \epsilon_t)|e_1]\mathbb{P}\{e_1\} \\ &+ \mathbb{E}_{\epsilon_t}[\lambda_t(\theta_t) + \epsilon_t | e_2 \cup e_3]\mathbb{P}\{e_2 \cup e_3\} \\ &+ \mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i-1}(\lambda_t(\theta_t) + \epsilon_t)|e_4]\mathbb{P}\{e_4\} \\ =& \mathbb{E}_{\epsilon_t}[M(\theta_t, \epsilon_t)]. \end{aligned} \tag{39}$$

Similarly:

$$\begin{aligned} c_t(\theta_t) =& \mathbb{E}_{\epsilon_t}[\alpha_t c_{t+1}(\pi_t)] \\ &+ \mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i}(\pi_t) - \pi_t | e_1 \cup e_2]\mathbb{P}\{e_1 \cup e_2\} \\ &+ \mathbb{E}_{\epsilon_t}[\pi_t - \tilde{m}_{t+1}^{i-1}(\pi_t)|e_4]\mathbb{P}\{e_4\} \\ =& \mathbb{E}_{\epsilon_t}[C(\theta_t, \epsilon_t)]. \end{aligned} \tag{40}$$

■

APPENDIX II – PROOF OF THEOREM 2

Under price independence assumption, i.e. $\lambda_t(\bullet) = 0, \forall t$, the state space only consists of SoC. Moreover, there is no need for calculation of fixed points since the conditions defining events e_k automatically become explicit. More simplification in the corresponding recursions can be made as well, which is mainly what is claimed by this theorem.

To this end, we treat the corresponding recursions for m_t^i and c_t directly by expanding the corresponding expectations noting that the price independence assumption results in $\pi_t = \epsilon_t$. Using (39), we have:

$$m_t^i = \tilde{m}_{t+1}^{i+1}\mathbb{P}\{e_1\} + \underbrace{\mathbb{E}_{\epsilon_t}[\epsilon_t | e_2 \cup e_3]\mathbb{P}\{e_2 \cup e_3\}}_{A} + \tilde{m}_{t+1}^{i-1}\mathbb{P}\{e_4\}. \tag{41}$$

Now, by definition of the conditional expected value and using integration by parts:

$$\begin{aligned} A &= \int_{\tilde{m}_{t+1}^{i+1}}^{\tilde{m}_{t+1}^{i-1}} \zeta \, dF_t(\zeta) \\ &= \tilde{m}_{t+1}^{i-1}F_t(\tilde{m}_{t+1}^{i-1}) - \tilde{m}_{t+1}^{i+1}F_t(\tilde{m}_{t+1}^{i+1}) - \int_{\tilde{m}_{t+1}^{i+1}}^{\tilde{m}_{t+1}^{i-1}} F_t(\zeta)\, d\zeta \\ &= \tilde{m}_{t+1}^{i-1}\mathbb{P}\{\bar{e}_4\} - \tilde{m}_{t+1}^{i+1}\mathbb{P}\{e_1\} - G_t(\tilde{m}_{t+1}^{i+1}, \tilde{m}_{t+1}^{i-1}), \end{aligned}$$

where $\bar{e}_k$ is the complement of e_k. Plugging for A in (41):

$$m_t^i = \tilde{m}_{t+1}^{i-1} - G_t(\tilde{m}_{t+1}^{i+1}, \tilde{m}_{t+1}^{i-1}).$$

Similarly, using (40), we have:

$$\begin{aligned} c_t =& \alpha_t c_{t+1} + \mathbb{E}_{\epsilon_t}[\tilde{m}_{t+1}^{i} - \epsilon_t | e_1 \cup e_2]\mathbb{P}\{e_1 \cup e_2\} \\ &+ \mathbb{E}_{\epsilon_t}[\epsilon_t - \tilde{m}_{t+1}^{i-1}|e_4]\mathbb{P}\{e_4\} \\ =& \alpha_t c_{t+1} + \tilde{m}_{t+1}^{i}\mathbb{P}\{e_1 \cup e_2\} - \tilde{m}_{t+1}^{i-1}\mathbb{P}\{e_4\} \\ &- \underbrace{\mathbb{E}_{\epsilon_t}[\epsilon_t | e_1 \cup e_2]\mathbb{P}\{e_1 \cup e_2\}}_{A'} + \underbrace{\mathbb{E}_{\epsilon_t}[\epsilon_t | e_4]\mathbb{P}\{e_4\}}_{B'} \end{aligned} \tag{42}$$

Similar to A:

$$\begin{aligned} A' &= \int_{-\infty}^{\tilde{m}_{t+1}^{i}} \zeta \, dF_t(\zeta) \\ &= \tilde{m}_{t+1}^{i}F_t(\tilde{m}_{t+1}^{i}) + \int_{-\infty}^{\tilde{m}_{t+1}^{i}} F_t(\zeta)\, d\zeta \\ &= \tilde{m}_{t+1}^{i}\mathbb{P}\{e_1 \cup e_2\} - G_t(-\infty, \tilde{m}_{t+1}^{i}), \end{aligned}$$

$$\begin{aligned} B' &= \int_{\tilde{m}_{t+1}^{i-1}}^{\infty} \zeta \, dF_t(\zeta) \\ &= \tilde{m}_{t+1}^{i-1}(1 - F_t(\tilde{m}_{t+1}^{i-1})) + \int_{\tilde{m}_{t+1}^{i-1}}^{\infty} (1 - F_t(\zeta))\, d\zeta \\ &= \tilde{m}_{t+1}^{i-1}\mathbb{P}\{e_4\} + \bar{G}_t(\tilde{m}_{t+1}^{i-1}, \infty), \end{aligned}$$

Now using A' and B':

$$c_t = \alpha_t c_{t+1} + G_t(-\infty, \tilde{m}_{t+1}^{i}) + \bar{G}_t(\tilde{m}_{t+1}^{i-1}, \infty),$$

which is the desired result. ■

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