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Large Gain Stability and Adaptive Expansion Estimation in Extremum Seeking

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Abstract— **Convergence of Extremum Seeking (ES) algorithms has been established in the limit of small gains. Using averaging theory and contraction analysis, we propose a framework for computing explicit bounds on the departure of ES schemes from their ideal dominant-order average dynamics. The bounds remain valid for possibly large gains. This framework allows us to establish stability and to estimate convergence rates and it opens the way to selecting "optimal" finite gains for ES schemes. Moreover, it constitutes a powerful aid in the design of efficient Perturbation Based ES. We extend this study by providing a simple technique inspired by adaptive control for estimating the cost function derivatives in Numerical Optimization based ES.**

I. INTRODUCTION

Extremum Seeking (ES) is a special class of algorithms designed to optimize dynamic systems [1]. Typically, traditional unconstrained optimization algorithms are designed to find the minimum of a cost function $c : \mathbf{R}^N \to \mathbf{R}$ through a sequence of evaluations $\{c(x_n)\}$. In the kind of problem addressed by ES, the map parameters are continuous functions of time $x(t) \in \mathbf{R}$ and the map output is continuously measured, possibly through a dynamic system $c([z],x)$ where $[z]$ represents the possible internal dynamics [2].

ES can be traced back to [3] and was an active field until the third quarter of the 20th century [4]. It has regained much attention since the early 2000's, in part after the theoretical progresses in [5], [6], where stability of an ES scheme was proven. ES has known a rapid development in the last decade and its range of applications is expanding rapidly with the generalization of large-scale and low-cost autonomous dynamic systems and robots [7], [8].

Design and stability analyses of Perturbation based ES (PBES) schemes have been presented for several classes of systems that follow a similar structure [9]:

- A particular gradient related method is to be mimicked. By introducing an oscillatory perturbation, or exploration signal, $v(t)$ to the input and measuring the correlation with the system's output, the gradient and perhaps the Hessian [10] are estimated. Those estimates are used to define a direction of search,
- The loop is closed by drifting the operating point x on a slow timescale along the search direction. Typically, the slow time dynamics x_{av} of x reduces to a gradient $d_t x_{av} = -\eta \nabla c(x_{av})$ or Newton descent $-\eta H^{-1} \nabla c$, with η some positive constant.

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- From there, averaging theory and singular perturbation are invoked and it is shown that if the gains used in the design of the PBES scheme are small enough, x converges to a limit cycle in neighborhood of the optimal point x^* [6].

By doing so, the stability of ES is only proven in the limit of infinitely small gains. However, the speed of convergence is driven by the amplitude of those small gains. For practical applications it is therefore important to know how large they can be to provide a satisfactory search speed while still maintaining stability and search precision. Although some basic work and scaling remarks were done to address this problem [6], [9], [11]–[13], no explicit solution has been provided yet, let alone for nonlinear systems.

We show that those limitations can be overcome if two quantitative tools –averaging theory up to higher orders and singular perturbation theory revisited by contraction theory [14]–[16]– are used. The basic idea is to bound or estimate the departure of the real system from its ideal descent and to build an auxiliary optimization problem called meta-optimization that will select the finite gains so that the errors remain below a fixed bound. Just as the small gain stability has been proven for specific algorithms (as opposed to the whole class of PBES algorithms), the same goes for the finite gain theory.

In section II we summarize the averaging theory as presented in Chapter 3 of [17] and discuss its application to finite order averaging before stating the results from the contraction analysis of singularly perturbed systems [16] and showing how they can be combined for the analysis of PBES schemes. In section III we apply those quantitative tools to the maximization of a one dimension map with the very simple ES scheme presented in section 4.1 of [6]. This study is extended to the optimization of a black box dynamic system. For the study of a scheme with filtering, the reader is referred to [18]. We close section III by illustrating on the simple 2-dimensional case how higher order averaging can be used as a qualitative design tool to pinpoint and possibly avoid high order, undesired terms. Section IV is dedicated to the presentation of a simple and elegant way to estimate the cost function derivatives in Numerical Optimization based ES (NOES) with an algorithm inspired by adaptive control.

II. THEORY

A. Averaging

1) Fundamental relation of averaging: Let $f,w : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be smooth functions and ∇ the derivation operator with respect to the first variable. We denote $\mathscr{D}_w h = \nabla h \cdot w$

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the spatial Lie derivative and $\mathscr{L}_w h = \mathscr{D}_w h - \mathscr{D}_h w$ the spatial Lie bracket. In addition, we define a non autonomous Lie bracket $\tilde{\mathscr{L}}$ such that $\tilde{\mathscr{L}}_w = \mathscr{L}_w(\cdot) - \partial_t w$ and for $p \geq 2$, $\tilde{\mathscr{L}}^p = \mathscr{L}\tilde{\mathscr{L}}^{p-1}$. It is recalled that the exponential of a differential operator $\mathscr{A}$ is defined by

$$e^{\mathscr{A}} x = x + \mathscr{A}x + \frac{\mathscr{A}^2 f(x)}{2!} + \frac{\mathscr{A}^3 f(x)}{3!} + \dots$$

The present study makes heavy use of the theory of averaging from [17] Chapter 3 and [19], Annex C. For convenience, the main results are summarized below.

Lemma 1 *Let*

$$\dot{x} = \varepsilon f(x,t) \tag{1}$$

be a dynamic system. The dynamics of the variable y defined by the implicit, near identity transformation $x = U(y,t)$ with

$$U(y,t) = e^{\mathscr{D}_{w(y,t)}} y \tag{2}$$

is given by $\dot{y} = g_\infty(y,t)$ with

$$g_\infty(y,t) = \varepsilon e^{\tilde{\mathscr{L}}_{w(y,t)}} f(y,t). \tag{3}$$

Simply put, this theorem means that given a variable change $x = U(y,t)$ (the introduction of the generator function w can be thought of as an indirect way to define U), the dynamics for the new variable takes the simple form given by equation (3).

2) Interpretation when f and w are written as series:

Assumption *$f(x,t)$ is T-periodic in t. The time average operation is written $\frac{1}{2\pi}\int_0^T f(x,y)dt = \overline{f}$.*

If $\varepsilon f(x,t) = \sum_{i=1}^{\infty} \varepsilon^i f_i(x,t)$ and $w(y,t) = \sum_{i=1}^{\infty} \varepsilon^i w_i(y,t)$, where ε is a given scalar, the previous relations must hold at all orders in ε. Collecting the terms of same order together, the previous expressions become

$$x = y + \sum_{i=1}^{\infty} \varepsilon^i u_i(y,t) \quad \text{and} \quad g_\infty(y,t) = \sum_{i=1}^{\infty} \varepsilon^i g_i(y,t).$$

Theorem 1 *It is possible to construct $\{w_i\}$ such that $\{g_i\}$ are independent of t. We use the shortcut:*

$$g_i(y,t) = g_i(y)$$

In other words, the non autonomous system (1) can be transformed into an autonomous system by an appropriate change of variable (2).

Indeed the g_i's are built by expanding the $e^{\tilde{\mathscr{L}}_w}$ operator in equation (2). It is important to note that their expressions can be computed algorithmically at each order by recursion. Moreover, expanding equations (2) and (3) we notice that they are polynomials of f_j's up to order i, w_j's and their space derivatives up to order $i-1$ and $\partial_t w_j$ up to order i. More precisely, g_i can be rewritten

$$g_i(y,t) = -\partial_t w_i + E_i(y,t)$$

where $E_i(y,t)$ contains f_j, $j \leq i$, w_j, $j \leq i-1$ and their derivatives. In other words, if the assumption holds, by choosing

$$w_i(y,t) = \int_0^t E_i(y,t) - \overline{E}_i dt + K_i(y), \tag{4}$$

$\{g\}_i = \{\overline{E}_i\}$ is independent of time at each order. The integration constants K_i are free and can be chosen by the designer. By construction, the periodicity of f propagates to w and U. Therefore if the K_i's are chosen to be 0, y and x coincide at $t = 0[T]$. A probably better choice, used in the present article, is to set K_i so that $\overline{u}_i = 0$ for $i \geq 1$, leading to $\overline{x} = y$.

3) Finite order averaging: The previous sections provide an algorithmic way to build a change of variables that transforms the non autonomous, periodically driven, dynamic system into an autonomous one. For practical applications however the change of variables can only be performed up to a finite order n. From the knowledge of $\{w_i\}_{1 \leq i \leq n}$ an approximate change of variable $x = U_n(y,t)$ can be constructed. Reinjecting this into equation (1) and assuming that U remains invertible, the forcing $\dot{y} = g(y,t)$ can be computed:

$$g(y,t) = [\partial_y U]^{-1}(f(U(y,t),t) - \partial_t U). \tag{5}$$

By construction, the change of variable U ensures that g is autonomous up to order n. The above equation can therefore be rewritten

$$\dot{y} = g_n(y) + R_g(y,t)$$

where g_n is the truncation of g_∞ and $R_g(y,t) = O(\varepsilon^{n+1})$ is a higher order forcing not canceled by the order n change of variables.

B. Singular Perturbation with Contraction

1) Contraction theory [14]: Consider the system $\dot{x} = f(x,t)$. It is said to be contracting if all trajectories converge exponentially towards each other. A necessary condition for contraction is that there exists a metric $\Theta(x,t)$ such that $\Theta^T\Theta$ is uniformly positive definite and $\beta > 0$ such that

$$F = \dot{\Theta}\Theta^{-1} + \Theta \nabla f \Theta^{-1} \preccurlyeq -\beta I.$$

β is called the contraction rate. We also define χ as a bound on the condition number of Θ. A useful result from contraction theory is the robustness lemma

Lemma 2 *if f is contracting with rate β and R is a bounded perturbation such that $\dot{y} = f(y) + R(t)$, $y(t)$ converges to a $|R|/\kappa$ neighborhood of $x(t)$ where $\kappa = \beta/\chi$. We say that the system is κ-robust.*

2) Singular perturbation [16]: Consider the dynamic system

$$\begin{aligned} \mu \dot{z} &= g(x,z) \\ \dot{x} &= f(x,z,t). \end{aligned} \tag{6}$$

Lemma 3 *Assume that the fast system $\mu\dot{z} = g(x_0(t),z)$ is partially λ/μ-robust with respect to z (the $x-z$ coupling has been replaced by an external forcing). Write $\gamma(x)$ its equilibrium, assume that there exists $d > 0$ such that*

$|\partial_x \gamma(x) f(x,z,t)| \leq \Gamma$. *Assume also that f is Lipschitz in z with constant α. Assume eventually that $z(t=0) = \gamma(x(t=0))$ and let x_s be the solution of the reduced, slow singular perturbation system $\dot{x}_s = f(x_s, \gamma(x_s), t)$ (y is frozen to its equilibrium state). Then,*

$$|\dot{x} - \dot{x}_\gamma| \leq \frac{\Gamma \alpha \mu}{\lambda}.$$

C. Utilizing higher order averaging and singular perturbations with contraction in PBES

The typical PBES scheme is pictured in Fig. 1. The input x of a black box dynamic system (f_{bb}, f_{bb}) of internal state z is to be optimized. The ES algorithm (f_{es}, h_{es}) of internal state ξ reads the output of the black box system and optimizes x in real time. ES schemes usually depend on a set of small parameters (gains and cutoff frequencies), we assume that those parameters have been rescaled so that the system depends on a parameter of possibly small amplitude ε and a set of parameters $r = [r_1 \dots r_q]$ of order unity. For instance, in system (10), the small parameters are $[a, \eta]$. We choose $a \hat{=} \varepsilon$ as our small parameter. η is rescaled according to $\eta = pa^2$ and p becomes the parameter of order unity.

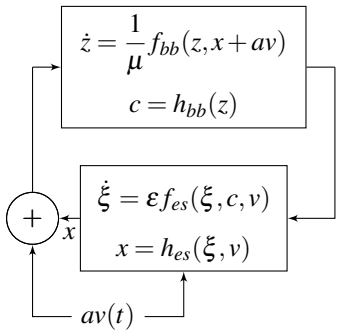

Fig. 1. General PBES scheme.

The general state equation is

$$\frac{d}{dt}\begin{bmatrix} z \\ \xi \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu} f_{bb}(z, h_{es}(\xi, v) + av) \\ \varepsilon f_{es}(\xi, h_{bb}(z), v)] \end{bmatrix} \tag{7}$$

which can be put in the form of equation (6) with a minor change of variables. If the black box system is contracting and satisfies the hypotheses of lemma 3, it can be approximated by its steady state input-output map written $c(\cdot)$ as a convenient shortcut. The simplified system is

$$\frac{d}{dt}\xi_s = \varepsilon f_{es}(\xi_s, c(h_{es}(\xi_s, u) + au), u) \tag{8}$$

and amounts to the optimization of a static map. The error between the reduced system ξ_s and the real system can be bounded with $\dot{\xi} - \dot{\xi}_s = R_s$ with $|R_s| \leq \alpha \Gamma \mu / \lambda$.

In turn, if v is T-periodic, system (8) is in the form where the averaging machinery can be applied to construct the transformation $\xi_s = U(y,t)$ and $\dot{y} = g_n(y) + R_{av}(y,t)$. We decompose further g_n into its dominant order term g_0, by design a descent direction, and its higher order terms δg_n that represent the unwanted departures from the descent scheme foreseen by the averaging procedure

While it is constructed from the slow system ξ_s, the transformation U can be applied to ξ. A straightforward calculation brings the following decomposition. It is one of the main results of the present study:

Fundamental Decomposition

$$\begin{aligned} \xi &= y + \delta U(y,t) \\ \dot{y} &= g_0(y) + \delta g_n(y) + R_{av}(y,t) + R_{sp}(y,z,t) \end{aligned} \tag{9}$$

$\delta U = O(\varepsilon)$ is the non identity part of the coordinate transformation, g_0 gathers the dominant order, ideal terms (typically the gradient or Newton descent), $\delta g_n = o_\varepsilon(g_0)$ the modeled deviation of the average system from the ideal descent, up to order n in ε, $R_{av} = O(\varepsilon^{n+1})$ is the higher order dynamics that is left from the n^{th} order averaging and $R_{sp} = [\partial_y U]^{-1} R_c$ is the error induced by the fast dynamics of the black box. If bounds are known on c and its derivatives (gathered in the notation $[\![c]\!] = (\|c\|, \|c'\| \dots)$), each of those higher terms can be bounded.

Using this decomposition, a relatively simple optimization can be run to select μ, ε and the order unity parameters r_i to maximize the search speed while keeping the error terms below acceptable predetermined bounds. Several particular possible meta-optimizations and useful relaxations are presented in the next section.

III. Applications

A. 1 state, 1-D system

To illustrate this technique, we apply the procedure to the most simple 1-state ES scheme from [6] and work it through step by step. For ease of computation, we assume the exploration signal to be sinusoidal $v(t) = \sin t$, although the same analysis can be carried out with any periodic signal. The state equation for this ES scheme is:

$$\dot{x} = -\eta c(x + a \sin t) \sin t. \tag{10}$$

Qualitatively, if η is small, the right hand side is small so that x is varying slowly and its long term evolution is given by the short time average of the right hand side, which reduces to $\dot{x}_{av} = -a\eta c'(x_{av})/2$ if a is also small. The system therefore mimics a gradient descent scheme of c at rate $a\eta/2$. Here, η and a are parameters set up by the practitioner. In the cited literature, they were assumed to be small parameters, enough so that the approximations that make x approximately driven by ∇c hold. It is important however to note that the descent speed is driven by $a\eta$, and that those parameters should therefore be as large as possible. It appears that the amplitudes of a and η constitute a trade-off between speed and accuracy. Our interest is to use averaging theory to quantify the system's departure from the ideal gradient descent and compute what finite gains are acceptable to make the search perform at a set precision while maximizing search speed.

Before moving further, we take advantage of this example to discuss the parametrization of PBES. Averaging theory, as presented in the previous section and in the reference literature, is parametrized with one small parameter only.

In the case of PBES, there are several small parameters so that the averaging analysis cannot be performed as is. One way to bypass this theoretical limitation is to parametrize each parameter as a function of the averaging parameter ε: $\eta = \eta(\varepsilon)$ and $a = a(\varepsilon)$. We require a and η to be of class $\mathscr{K}_\infty$. The most natural choice is to use power laws, $a = \varepsilon^{m_1}$ and $\eta = p\varepsilon^{m_2}$, $m_1, m_2 \in \mathbf{N}^*$. This way, we separate in η the "magnitude" part ε^{m_2}, and its fine tuning value p. Then, to the reparametrized system

$$\dot{x} = -\eta(\varepsilon)c\left(x + a(\varepsilon)\sin t\right)\sin t$$

corresponds an averaged system in coordinate y defined by $x = U(y,t)$ such that

$$\dot{y} = \sum_{i=1}^{n} \varepsilon^i g_i(y) + R_{av}(y,t)$$

where the g_i are sums and products of h, η and a up the i^{th} derivative. If $m_1 = m_2 = 1$, the averaged system to the dominant order is

$$\begin{aligned} \dot{y} &= -\frac{a\eta}{2}c'(y) - \frac{1}{16}\left(a\eta^3\mathscr{L}_c^2 c' + \eta a^3 c^{(3)}\right) + R_{av} \\ x &= y + \eta\sin(t)c(y) + R_u \end{aligned}$$

where $R_{av} = O(\varepsilon^4)$ and $R_u = O(\varepsilon^2)$ are higher order terms in a, η that can be computed explicitly by expanding equation (5). The term $\frac{a\eta}{2}c'$ is the ideal gradient descent that the scheme is intended to reproduce. The middle term in the equation for y was noted δg_n in the previous section. For a given function c, the speed or search is proportional to $a\eta$. Optimizing for the gains implies therefore some sort of maximization of $a\eta$.

The dynamics for y is in the form of the general dynamic nonlinear system with noise $R = \delta g_n + R_{av}$ considered in the robustness lemma. If c is κ-robust, lemma 2 can be applied to show that y follows the ideal trajectory $\dot{z} = \frac{a\eta}{2}h'(z)$ with an error that is at most $\delta_1 = 2\frac{\|R_{av}\|+\|\delta g_n\|}{a\eta\kappa}$. Similarly, $|x - y| \leq \delta_2 = \eta\|c'\| + \|R_u\|$. Assuming that an estimate on $[\![c]\!]$ (and therefore on R_i) is known, the gains can be chosen to maximize the search speed while keeping δ_1 and δ_2 below some tolerance error. In practice, it is possible to relax the constraints slightly by only considering the dominant error terms. There are several possible strategies and some are listed below:

1) The guaranteed meta-optimization consists in bounding the distance from the real system to the ideal system. A simple triangular inequality shows that $|x - z| \leq \delta_1 + \delta_2$ so that the problem reduces to:

$$\max_{\eta,a\geq 0} \ \eta a \quad \text{s.t.} \ \sum \delta_i \leq \Delta.$$

This is the safest optimization, as it guarantees convergence and the errors bounds (x will converge to a Δ-neighborhood of x^*). However, the higher order remainders are often complex so that, even if they can be bounded provided that bounds $[\![c]\!]$ are known, the bound is unlikely to be tight, leading to overly conservative regimes. Also, the error due to the average system and the error due to the oscillation of the real system may not be of the same importance for the user, so that it might be fruitful to separate them in the constraints. Those two drawbacks motivate the next two strategies

2) Splitting up the error from the average system (the DC error) δ_1 and the oscillatory error δ_2, the meta-optimization becomes:

$$\max_{\eta,a\geq 0} \ \eta a \quad \text{s.t.} \ \delta_i \leq \Delta_i.$$

3) Neglecting the highest order of the remainder greatly simplifies the bound expressions. For instance, in the case where $\eta = p\varepsilon$ and $a = \varepsilon$, the 1-D problem reduces to

$$\begin{aligned} &\max_{\eta,a\geq 0} \ \eta a \\ &\quad \text{s.t.} \ \frac{2}{a\eta\kappa}\left\{\frac{1}{16}(\eta a^3\|h^{(3)}\| + a\eta^3\|\mathscr{L}_h^2 h'\|)\right\} \leq \Delta_1 \\ &\quad \text{s.t.} \ \eta\|h\| \leq \Delta_2. \end{aligned}$$

In that case, since a truncation is made, a more aggressive search is recommended by the meta-optimization. Formal guarantee of convergence is lost. It is therefore wise to check, after such an meta-optimization, that the neglected terms are indeed negligible.

4) The previous strategies are well suited when the objective is to locate the optimal point (such as when performing source/pollutant tracking for instance). In other situations it might be important to keep the real system $x + a\sin t$ close to x^* at all times. The extension of strategy 1) is

$$\max_{\eta,a\geq 0} \ \eta a \quad \text{s.t.} \quad a + \frac{2\|R_{av}\|}{a\eta\kappa} + \|R_u\| \leq \Delta.$$

Remarks:

1) In the case $a = \varepsilon$ and $\eta = p\varepsilon$, the terms in ηa^3 and $a\eta^3$ in the constraint for y have the same order. However, if $\eta = p\varepsilon^{m_2}$, $a = \varepsilon^{m_1}$ and $m \neq n$, the two terms won't necessarily have the same order. Therefore, for more general m_1, m_2, the symmetry may be broken leading to a problem with monomial constraints

$$\max_{\eta,a\geq 0} \ \eta a \qquad \text{s.t.} \quad \eta^{p_1}a^{q_1} \leq K_1 \quad \text{and} \quad \eta^{p_2}a^{q_2} \leq K_2$$

It has a finite solution if $p_1/q_1 < 1 < p_2/q_2$ which is $a = K_2^{\frac{p_1}{q_2p_1-q_1p_2}}K_1^{\frac{-p_2}{q_2p_1-q_1p_2}}$ and $\eta = K_2^{\frac{-q_1}{q_2p_1-q_1p_2}}K_1^{\frac{q_2}{q_2p_1-q_1p_2}}$.

2) Assume this monomial form for a and η and $m_1 < m_2$. then, in the first constraint from case 3), the first term of the first constraint is dominant. Assuming that the expansion remains the same as in the $m_i = 1$ and that both constraints are active brings $\eta = \Delta_2/\|c\|$ and $a = \sqrt{\frac{8\Delta_1\kappa}{\|c^{(3)}\|}}$.

3) Two consistency checks should be performed to ensure meaningful results:

- Are the higher order terms indeed small?
- Is p close to unity? Otherwise, the parameters m_1, m_2 chosen to perform the expansion may not be the appropriate ones, *e.g.* by increasing m_2 if p is too large. Constructing guidelines to select m_i's is still an active subject of research.

4) Obviously, if some rough estimates on $[\![c]\!]$ are known, they can be used to compute tighter bounds on δg_n and R_i.

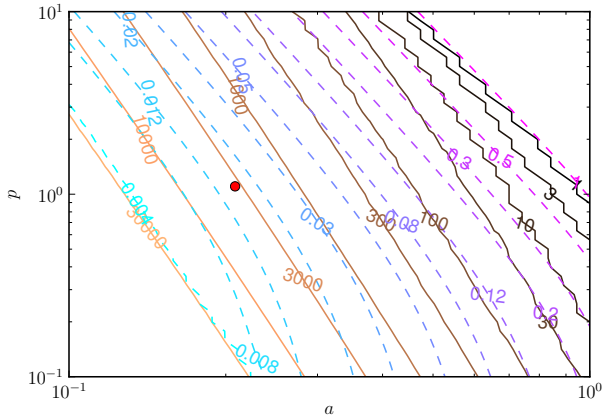

Fig. 2. Performance map of ES as a function of a and p (log-log scale). The full lines indicate the speed of search, the dashed lines represent the the error $|x-x^*|$. The red dot is the result from the meta optimization.

B. One state case, fully worked example

Let's consider the optimization of $c(x) = -\cos(x) + x^3/6$, starting from $x = 1$. It is chosen as a toy example for all of its derivatives are bounded by 1. The cubic term is added to break the third order symmetry at the minimum $x^* = 0$. Of course, 0 is not a global minimum, but for a, η small enough, it is still the attractor.

We solve here the meta-optimization problem 3). The constraints require us to have estimates for the bounds on c, c', c'', $c^{(3)}$ and κ. We take $\|c^{(i)}\| \hat{=} \max_{x\in[-1,1]} |c^{(i)}(x)|$ and $\kappa = c''(0)$. We also set $\Delta_i = 0.01$ and start with $\eta = p\varepsilon$, $a = \varepsilon$. It gives the numerical solution $\eta = 0.01$, $a = 0.207$ Therefore $p \approx 0.05 \ll 1$. To bring p closer to unity, we repeat the averaging (it has to be done up to order 7) with $\eta = p\varepsilon^3$, $a = \varepsilon$. The constraint for y simplifies to $\frac{a^2\|c^{(3)}\|}{8\kappa} \leq \Delta_1$. This time, as stated in the remarks, the problem can be solved analytically:

$$a = \sqrt{\frac{8\Delta_1\kappa}{\|c^{(3)}\|}}, \quad \eta = \frac{\Delta_2}{\|c\|},$$

which brings with the numeric values discussed before $\eta = 0.01$ and $a = 0.209$ ($p = 1.09$). The reason why the result has not changed much is because the term in $a\eta^3$ found when $m_i = 1$ in the first constraint is small already. The result is illustrated in Fig. 2. The operation point suggested by the procedure is close to but different from the optimal point for this very map, which is expected.

C. Optimization of a dynamic system

In this section we show how the singular perturbation theory allows to set the frequency of the exploration signal. For simplicity, we will consider again the 1-state ES, extended with a dynamic system:

$$\frac{d}{dt}\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} -z + x \\ kc(z)\sin\omega t \end{bmatrix}. \tag{11}$$

In the present article we illustrate the technique with a first order filter for ease of computation but point out that the technique applies to any nonlinear, contracting system straightforwardly. To put system (11) in form (7), we change the timescale $\tau = \omega t$ and introduce $y = x + a\sin\tau$ and the reduced parameter $\eta = k/\omega$:

$$\frac{d}{d\tau}\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}(-z+y) \\ \eta c(z)\sin\tau + a\cos\tau \end{bmatrix}. \tag{12}$$

Lemma 3 can be applied with $\Gamma = \|c\|\eta + a$, $\lambda = 1$, $\mu = \omega$, $\alpha = \|c'\|$, to get $|d_\tau y - d_\tau y_s| \leq \eta\omega\|c'\|(\eta\|c\| + a)$ which can be transformed back into the x,t variables:

$$|\dot{x} - \dot{x}_s| \leq \eta\omega^2\|c'\|(\eta\|c\| + a).$$

We form the meta-optimization problem by relaxing $\|c'\|$ into $|c'(x)|$. This gives an approximation for the worst case speed at the dominant order

$$|g_0| - |\dot{x} - \dot{x}_s| \approx \left(\frac{a\eta\omega}{2} - \eta\omega^2(\eta\|c\| + a)\right)|c'|$$

which defines the objective. Keeping the constraints on δ_i from the previous part unchanged, the meta-optimization problem can be written as:

$$\max_{\eta,a,\omega\geq 0} \left(\frac{a\eta\omega}{2} - \eta\omega^2(\eta|c| + a)\right)$$
$$\text{s.t. } \frac{1}{8\kappa}a^2\|c^{(3)}\| \leq \Delta_1 \text{ and } \eta\|c\| \leq \Delta_2$$

which gives $\omega = 0.48$ and keeps a and η unchanged.

D. Averaging aided design: ES in two dimensions as an illustration of channels interaction

Practical optimization problems are multi-dimensional. An ES specific added complixity arises from the error resulting from the coupling between directions during the estimation of c's derivatives. In this section, we show that higher order averaging analysis can be used as an ES design tool as it allows to pinpoint detrimental behaviors and quantify such errors. The result provided here is a 2-particle coupling that is similar to the 3-particle coupling from [10], [11]. For simplicity, we limit ourselves to the bi-dimensional case where $c(x_1, x_2)$ is to be minimized:

$$\dot{x}_1 = \eta v_1 h(x_1 + ad_1, x_2 + av_2)$$
$$\dot{x}_2 = \eta v_2 h(x_1 + ad_1, x_2 + av_2).$$

We start with a qualitative presentation. In order to estimate the derivative of h with respect to each of the variables, the correlation between the exploration signal v_i and the output c is measured. For instance for small gains the average equation for x_1 should reduce to:

$$\dot{x}_{av,1} \approx a\eta\left\{\overline{v_1 v_1}\partial_{x_1}c + \overline{v_1 v_2}\partial_{x_2}c\right\}.$$

This qualitative analysis suggests that as long as $\overline{v_1 v_2} = 0$, the dynamics should reduce to the desired $\dot{\bar{x}}_i \approx -a\eta\overline{v_i v_i}\partial_{x_i}c$. This advocates for the possibility to use both sines and cosines for the exploration, which is *a priori* beneficial as it would allow a bandwidth twice smaller for a given number of channels.

However, the averaged equations for $v_1 = \cos t$ and $v_2 = \sin t$, $a = \varepsilon$, $\eta = p\varepsilon$ are:

$$\dot{y}_1 = \tfrac{1}{2} a\eta \partial_{x_1} h - \tfrac{1}{2}\eta^2 h \partial_{x_2} h + O(\varepsilon^3)$$
$$\dot{y}_2 = \tfrac{1}{2} a\eta \partial_{x_2} h + \tfrac{1}{2}\eta^2 h \partial_{x_1} h + O(\varepsilon^3).$$

Although the term in $a\eta$ is the expected gradient, the term in η^2 is a precessing term, which, if dominant, drives the system on level sets instead of along the gradient. These terms arise from the parasitic propagation of $\frac{1}{|s|} c(x) \sin t$ into the cosine channel, and vice-versa regarding the sine channel. Adding a high pass filter as in Fig. 2 of [5] cancels out this detrimental component. The strength of higher order averaging resides in quantifying the usefulness of the high pass filter. It also allows to advocate against the use of both sines and cosines if this section's simple ES scheme is to be used.

IV. NOES: A WAY TO BALANCE EXPLORATION AND EXPLOITATION

PBES schemes can be decomposed into two functional parts: one devoted to exploration and one to exploitation. Exploration is performed by injecting the exploration signal $v(t)$ into the system and measuring the correlations to estimate the derivatives of c. Exploitation entails forming a descent direction d (typically $d = -J$ or $d = -H^{-1}J$) and moving along the descent direction $\dot{x} = \eta d$.

A fundamental drawback of PBES schemes relies in the fact that exploration and exploitation happen at the same time. In fact, for PBES to work, exploitation speed (driven by ηd) has to be much smaller than exploration speed (driven by $a\omega$). A direct consequence is that exploration is performed over and over again to update information already known and changing only slowly.

Recently, emphasis has been put on integrating ES into the well developed framework of nonlinear optimization (NLO) [8]. Although NLO and ES share many characteristics, they differ on important aspects, some of which are discussed below. In NLO, function evaluations are measured sequentially. In many black box problems, the optimization speed is proportional to the number of function evaluations. The cost function is evaluated at discrete points and evaluating k points is largely independent on their relative location. In ES, the cost function is evaluated at all times along the system trajectory so that the number of function evaluations is not a relevant concept. A direct consequence is that the stepping strategy between derivative evaluations can be much richer than for NLO. In particular, line minimization comes for free and continuous versions of the classical forward step length selection schemes are straightforward (e.g. Armijo or line search).

While finding simple criteria to assess the performance of ES algorithms is a non trivial task and subject of active research, it is speculated that the curve length of $x + av$ in the parameter space should be a key quantity, particularly in applications such as source seeking or swarm dynamics. Likewise, the ratio between the length of the curve described by x_{av} and that of the real operating point $x + av$ is probably important in quantifying the balance between exploration and exploitation.

A. Estimating c and its derivatives with an adaptive control like algorithm

While [8] covers several aspects of NOES, they assume at all times that the derivatives of c are known, which is a strong assumption. We present here a very simple way to estimate the function value, gradient and Hessian of c at any given point, based on an adaptive control like algorithm. We hope that this will help complete the work in [8] and bridge the gap between NLO and NOES.

Let x be an operating point and $v(t)$ an exploration signal. Let's assume that for $|v| \leq K$, $c(x+av) = c(x) + a\nabla cv + \frac{1}{2}a^2 v^T \nabla^2 cv + R$ where R represents the higher order derivatives and maybe noise. We assume that the only quantity accessible to measurement is $c(x+av)$ and we build an estimate $\hat{c} = \hat{c}_0 + aJv + \frac{1}{2}a^2 v^T H v$, whose parameters are updated following

$$\begin{aligned} \dot{\hat{c}}_0 &= -\alpha e \\ \dot{J} &= -\beta e v \\ \dot{H} &= -\delta e v v^T \end{aligned} \tag{13}$$

with $e = \hat{c} - c(x+av)$. If we define the error for each component by $\tilde{c} = \hat{c}_0 - c(x)$, $\tilde{J}_i = J_i - \partial_i c$ *etc.* and gather them in a composite vector $C = [\tilde{c}, \dots, \tilde{J}_i, \dots, \tilde{H}_{ij}, \dots]$ along with a composite exploration signal $V(t) = [1, \dots, v_i, \dots, v_i v_j, \dots]^T$, the error can be rewritten $e = V^T C + R$ and equation (13) simplifies into

$$\dot{F} = -D\left\{VV^T C - VR\right\}$$

with $D = \mathrm{diag}(\alpha, \beta, \dots, \delta, \dots)$ which is the standard equation of adaptive systems. If $R = 0$ and V is persistently exciting, the system is exponentially converging to 0 and a bound on the speed of convergence can be obtained [20], [21]. Of course, this method can be simplified so as to estimate only the first derivative.

As an example, we present the optimization of the Rosenbrock function with a NOES scheme based on conjugate gradient and compare it with a PBES method with filtering.

B. Conjugate Gradient NOES

1) Exploration phase: We set $v = [\cos t, \sin t]^T$, $a = 0.01$ and learn $\hat{c}_0$ and J for 10 periods with $\alpha = \beta = 1$. Their average over the last period is used as the estimate of c and its gradient in the exploitation phase. A typical exploration phase is shown in the inset of Fig. 3.

2) Exploitation phase: With an estimate of the gradient any first order NLO scheme can be implemented. The particularity of ES is that during stepping the objective measurement is continuous time. This potentially allows for advanced stepping schemes. Also, an interesting property is that line search can be performed for free by moving along the descent direction until cost measurement stops decreasing: given a descent direction, $\dot{x} = \eta d_k$ as long as $d/dt(c) \leq 0$.

The conjugate gradient method is therefore particularly interesting in ES as it relies on line minimization. In the

present example, we use $d_k = -J_k$ when k is even and the Polak-Ribière direction when k is odd. Starting form $x_0 = [-1.9, -0]^T$, it converges to a a-ball of the minimum in 28 steps.

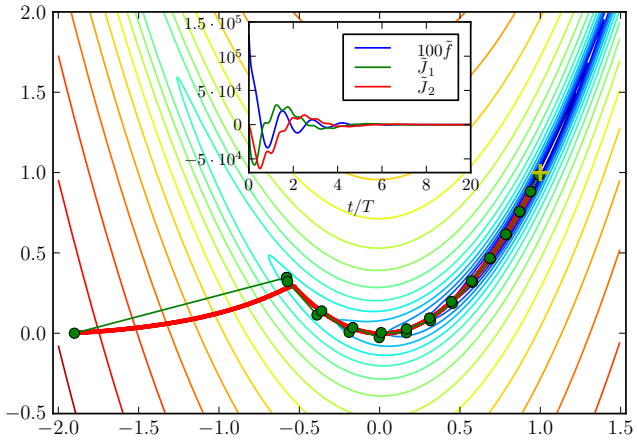

Fig. 3. Red: PBES of the Rosenbrock function. Green: NOES. Inset: adaptation of $\hat{c}_0$ and J during the first exploration phase. Yellow cross: minimizer of the Rosenbrock function.

C. Comparison with PBES

To compare with NOES we use a PBES scheme with a high and low pass filterings inspired by Fig. 2 of [5]. The gradient J is estimated by correlating $v = [\cos(t), \sin(t)]$ and $c(x + av)$, with high and low pass with $\omega_h = \omega_l = 1/5$. x is updated following $\dot{x} = -\eta J/|J|$ with $\eta = 10^{-3}$ and $a = 10^{-2}$ (the same value as for the NOES example). For a fair comparison, the gains were set by hand, beyond the guarantee of stability, to provide for a PBES search as fast as possible. The trajectory of both algorithms is provided in Fig. 3. After 10^3 periods, PBES has traveled a distance of about 60 length units (u) and is still 0.1u away from the minimum. NOES on the contrary reaches the 0.01u ball in about 21u (17 devoted to exploration and 3.7 to exploitation). This preliminary result suggests that NOES might be much more efficient than PBES in many cases. Moreover, it is inherently more stable than PBES as in the exploration phase, the system's trajectory is not a state but an input. Although NOES seems much less sensitive to the gains than PBES, the authors believe that following the spirit of meta-optimization of the present paper, a similar work could be performed in NOES, particularly in the exploration phase.

V. Conclusion

In this paper, we have shown how contraction, singular perturbation and modern averaging theories can help bring qualitative and quantitative insights into ES. Specifically we have shown methods for selecting the finite gains of ES schemes optimally. Although the present study used periodic exploration signals, our techniques should generalize to stochastic PB and NO ES [22]. The next natural research direction, constrained ES, should also benefit from our approach, e.g as in [23] be easily implemented in NOES where the trajectories are given as inputs.

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References

[1] Y. Tan, W. Moase, C. Manzie, D. Nešić, and I. Mareels, "Extremum seeking from 1922 to 2010," in *Proceedings of the 29th Chinese Control Conference*, 2010, pp. 14–26.

[2] K. B. Ariyur and M. Krstic, *Real-time optimization by extremum-seeking control*. Wiley-Interscience, 2003.

[3] M. Leblanc, "Sur l'électrification des chemins de fer au moyen de courants alternatifs de fréquence élevée," *Revue générale de l'électricité*, vol. 12, pp. 275–277, 1922.

[4] J. Sternby, "Extremum control systems - An area for adaptive control?" in *Joint Automatic Control Conference*, 1980.

[5] M. Krstić and H.-h. Wang, "Stability of extremum seeking feedback for general nonlinear dynamic systems," *Automatica*, vol. 36, pp. 595–601, 2000.

[6] Y. Tan, D. Nešić, and I. Mareels, "On non-local stability properties of extremum seeking control," *Automatica*, vol. 42, no. 6, pp. 889–903, Jun. 2006.

[7] R. Leyva, C. Alonso, I. Queinnec, D. Lagrange, and I. Martínez-Salamera, "MPPT of photovoltaic systems using extremum-seeking control," *IEEE TRANSACTIONS ON AEROSPACE AND ELECTRONIC SYSTEMS*, vol. 42, no. 1, pp. 249–257, 2006.

[8] C. Zhang and R. Ordóñez, *Extremum Seeking Control and Applications*. Springer, 2012.

[9] D. Nešić, Y. Tan, W. Moase, and C. Manzie, "A Unifying Approach to Extremum Seeking : Adaptive Schemes Based on Estimation of Derivatives," in *49th IEEE Conference on Decision and Control*, 2010, pp. 4625–4630.

[10] A. Ghaffari, M. Krstić, and D. Nešić, "Multivariable Newton-based extremum seeking," *Automatica*, vol. 48, no. 8, pp. 1759–1767, Aug. 2012.

[11] M. Rotea, "Analysis of multivariable extremum seeking algorithms," in *Proceedings of the Control Conference Chicago, Illinois*, no. June, 2000, pp. 433–437.

[12] A. Teel and D. Popović, "Solving smooth and nonsmoooth multivariable extremum seeking problems by the methods of nonlinear programming," in *Proceedings of the American Control Conference*, 2001, pp. 2394–2399.

[13] M. Krstić, "Performance improvement and limitations in extremum seeking control," *Systems & Control Letters*, vol. 39, pp. 313–326, 2000.

[14] W. Lohmiller and J.-J. E. Slotine, "On Contraction Analysis for Nonlinear Systems," *Automatica*, vol. 34, no. 6, pp. 683–696, Jun. 1998.

[15] W. Wang and J.-J. E. Slotine, "On Partial Contraction Analysis for Coupled Nonlinear Oscillators." *Biological cybernetics*, vol. 92, no. 1, pp. 38–53, Jan. 2005.

[16] D. Del Vecchio and J.-J. E. Slotine, "A Contraction Theory Approach to Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 752–757, 2013.

[17] J.A. Sanders F. Verhulst J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, 2nd ed. Springer London, Limited, 2007, vol. 59.

[18] G. Bousquet and J.-J. E. Slotine, "Large Gain Stability of Extremum Seeking and Higher-Order Averaging Theory," *arXiv:1303.4474v1*, 2013.

[19] J. Murdock, *Normal Forms and Unfoldings for Local Dynamical systems*. Springer, 2003.

[20] K. Narendra and A. Annaswamy, *Stable adaptive systems*. Dover Publications, 2012.

[21] J.-J. E. Slotine and W. Li, *Applied nonlinear control*. Prentice hall New Jersey, 1991.

[22] S.-J. Liu and M. Krstić, "Stochastic Averaging in Continuous Time and Its Applications to Extremum Seeking," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2235–2250, 2010.

[23] J. Soto and J.-J. E. Slotine, "Nonlinear Contraction Tools for Constrained Optimization," *arXiv:1206.1838v1*, 2012.