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Computation of Lower bounds for a Multiple Depot, Multiple Vehicle Routing Problem With Motion Constraints

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Abstract— **In this paper, the problem of planning paths for a collection of vehicles passing through a set of targets is considered. Each vehicle starts at a specified location (called a depot) and it is required that each target be on the path of at least one vehicle. Every vehicle has a motion constraint and the path of each vehicle must satisfy that constraint. In this article, we developed a method to compute lower bounds to this path planning problem by relaxing some of the constraints and posing it as a standard multiple traveling salesmen problem. For those problem instances where the distance between every pair of targets is at least 4 units, another method is developed to compute a lower bound using the convexity property of the length of such paths. The proposed bounds are numerically corroborated.**

I. INTRODUCTION

A multiple depot vehicle routing problem may be stated as the following: for a given a set of targets and a set of vehicles, each vehicle located at its depot, the objective is to find the path for each of the vehicles such that every target is visited at least once by one of the vehicles, each path satisfies the motion constraints of the vehicle and the sum of the length of the paths is the minimum. The motion of each of the vehicles must satisfy a given constraint. The motion constraint considered here is that the yaw rate of the vehicle at any instant along its path is upper bounded by a constant. We refer to a vehicle satisfying this yaw rate constraint as the Dubins vehicle, and the path planning problem as the Multiple Depot Multi-Vehicle Dubins Traveling Salesman Problem (**MDMVDTSP**). Problems of this type arise naturally in military and civil applications where UAVs are used for border surveillance, forest fire monitoring, weather monitoring etc.

To solve MDMVDTSP, one has to identify the targets to be visited by each vehicle, find the heading angle for each of the vehicle at their assigned targets and the sequence in which the targets must be visited. Once the heading angles are known for any two adjacent targets along the tour, the result in [1] can be used to determine the path with shortest length. Routing problems of this genre were earlier studied in [2], [3], [4], [5] and [6]. References [2] and [3] provide an approximate solution guaranteed to be within a constant factor of optimum. In [4], a two step approach is prescribed to solve a Multi Depot, Multiple TSP. The sequence of targets to be visited is first determined as a solution of a combinatorial problem and the heading angle at each target is later computed using dynamic programming. Multiple Depot routing problem without the yaw rate constraints is considered in [5]. The problem is formulated as a minimum cost constrained forest problem subject to side constraints. A lower bound is computed by dualizing these side constraints. The work in [6] deals with a Heterogeneous, Multi Depot, Multiple UAV Routing Problem and is solved by transforming it into a standard Asymmetric TSP.

There are currently no algorithms that can either find an optimal solution or a good lower bound for MDMVDTSP. Lower bounds are important because they can be used to corroborate the quality of the solutions produced by the heuristics or the approximation algorithms.

Currently, there are two ways for obtaining lower bounds for the MDMVDTSP. If the motion constraints are relaxed, this path planning problem reduces to the Euclidean Multiple Traveling Salesman Problem (EMTSP) and the solution to this is a lower bound to the MDMVDTSP. In [6], MDMVDTSP is posed as a one in a set TSP and provides a transformation method which converts the one-in-a-set TSP into a standard TSP. A lower bound to the resulting TSP serves as a lower bound to the MDMVDTSP. However, the transformation method as prescribed in [6] does not provide a good lower bound for every instance of the MDMVDTSP. A method to compute a lower bound for a single vehicle routing problem with motion constraints is provided in [7]. The result in this article is a generalization of this method for the multiple vehicle case.

We consider a Lagrangian relaxation obtained by removing some of the constraints in the MDMVDTSP and penalizing them in the objective whenever they are violated. Using the weak duality theorem, it follows that the cost of the solution to the Lagrangian relaxation is a lower bound to the optimal cost of the MDMVDTSP. The Lagrangian relaxation is posed as an asymmetric Multiple Traveling Salesman Problem (MTSP) where the cost of traveling each edge is computed by solving a variational problem. Using the transformation method provided in [8], the MTSP is transformed into a single vehicle TSP. This could be solved using the Lin-Kernighan Helsgaun (LKH) heuristic [9]. The LKH heuristic also gives a tight lower bound which serves as a lower bound to the MDMVDTSP. Simulation results seem to corroborate that the proposed method produces a lower bound better than the transformation method in almost all instances. Additionally, this lower bound is a significant improvement when compared to the solution of EMTSP.

For a given set of targets such that every pair of targets in the set are atleast 4 units apart, it was proved in [10] that, the length of the path is convex in the heading angles at each target. Using the convexity, another method is developed to compute a lower bound for the MDMVDTSP when the targets satisfy the distance condition.

This paper is organized in the following format. The problem statement is presented in section II. In section III, computation of a lower bound from Lagrangian relaxation and subgradient optimization is explained. In section IV, the transformation method used in [6] is introduced and computing lower bound using convexity is presented in section V. Numerical results and the comparison with the existing results are presented in section VI. Conclusions are provided in section VII.

II. PROBLEM FORMULATION

Let $V = \{1,2,...n\}$ be the set of given targets and $\theta = \{\theta_1, \theta_2, ...\theta_n\}$ be the set of headings at each target. Let $D = \{1,2,..m\}$ be the set of vehicles, each located initially at its depot. Let $N = V \cup D$ and let E denote the set of all the edges joining any two vertices in N. Let $\mathscr{F}$ represent the set of all feasible solutions, such that in each solution, every target in V is visited atleast once by one of the vehicles in M. Let x_{ij} be a binary decision variable which equals 1 if there is an edge from i to j in the tour and equals 0 otherwise. Let X be the matrix of decision variables, whose entry in the i^{th} row and j^{th} column is x_{ij}. We will say that $X \in \mathscr{F}$, if the matrix corresponds to one of the feasible solutions. The MDMVDTSP can be stated as the following:

$$\min_{\theta,X} \sum_{(i,j)\in E} d_{ij}(\theta_i,\theta_j)x_{ij} \tag{1}$$

subject to:

$$X \in \mathscr{F}, \tag{2}$$

where $d_{ij}(\theta_i, \theta_j)$ is the length of the shortest path of the Dubins vehicle starting from vertex i at (x_i, y_i) with a heading θ_i traveling to vertex j at (x_j, y_j) with a heading θ_j. The length d_{ij} can be expressed in terms of the kinematics of the Dubins vehicle as:

$$d_{ij}(\theta_i,\theta_j) = \min_{u_{ij}} \quad t_{ij}, \tag{3}$$

subject to:

$$\dot{\zeta}_{ij} = \cos\theta_{ij}, \quad \dot{\eta}_{ij} = \sin\theta_{ij}, \quad \dot{\theta}_{ij} = u_{ij}, \quad |u_{ij}| \le \Omega, \tag{4}$$
$$\zeta_{ij}(0) = x_i, \quad \eta_{ij}(0) = y_i, \tag{5}$$
$$\zeta_{ij}(t_{ij}) = x_j, \quad \eta_{ij}(t_{ij}) = y_j, \tag{6}$$
$$\theta_{ij}(0) = \theta_i, \quad \theta_{ij}(t_{ij}) = \theta_j. \tag{7}$$

Here, ζ_{ij} and η_{ij} are the position coordinates of the Dubins vehicle in x and y directions, $\theta_{ij}(t)$ is the heading angle of the vehicle at time t and t_{ij} is the time at which the vehicle reaches vertex j. The term $\dot{\theta}_{ij}$ is the yaw rate of the vehicle and it is upper bounded by Ω. When the minimum turning radius of a UAV equals to 1, the term Ω equals 1. Vehicles with different turning radii can be modelled by changing the corresponding value of Ω.

Suppose the desired tour contains the edges (i, j) and (j, k) in the path of one of the vehicles, the arriving (final) heading of the vehicle traveling from target i to target j should be equal to the departure (initial) heading of the vehicle while traveling from target j to target k. We do not know which target precedes others in the desired tour, but we know that there is only one incoming and outgoing edge incident on target j. So, we can make use of the binary variables x_{ij} to formally state the heading angle constraint as follows:

$$\sum_{i:(i,j)\in E} \theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \theta_{jk}(0)x_{jk} = 0. \quad \forall j \in N. \tag{8}$$

In general, equations (7) and (8) are difficult constraints to deal with. The domain of θ_{ij} is cylindrical and one has to identify 0 and 2π as one and the same. We will pose these domain constraints using sines and cosines of the angles θ_{ij} as shown below:

$$\cos\theta_{ij}(0) = \cos\theta_i, \quad \sin\theta_{ij}(0) = \sin\theta_i, \tag{9}$$
$$\cos\theta_{ij}(t_{ij}) = \cos\theta_j, \quad \sin\theta_{ij}(t_{ij}) = \sin\theta_j. \tag{10}$$

In summary, the constraint on the heading angles (8) at each target j can be re-stated in terms of the sines and cosines as:

$$\sum_{i:(i,j)\in E} \cos\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \cos\theta_{jk}(0)x_{jk} = 0, \tag{11}$$
$$\sum_{i:(i,j)\in E} \sin\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \sin\theta_{jk}(0)x_{jk} = 0, \quad \forall j \in N. \tag{12}$$

III. MAIN RESULT

A. Computation of lower bounds

To compute a lower bound, the idea is to relax the constraints (11) and (12) and penalize the objective function (1) whenever the constraints are violated via a set of penalty (dual) variables. Let the penalty variable corresponding to the angle constraint of target j in (11) is α_j and the penalty variable corresponding to the angle constraint of target j in (12) is β_j. Let $\Pi = [\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n]$, where $\alpha_j, \beta_j \in \Re \; \forall j = 1...n$. Then, a lower bound can be obtained for any set of penalty variables by solving the following Lagrangian relaxation:

$$\begin{aligned} L(\Pi) = \min_{\theta,X} & \sum_{(i,j)\in E} d_{ij}(\theta_i,\theta_j)x_{ij} \\ & - \sum_{j\in N} \alpha_j \left[\sum_{i:(i,j)\in E} \cos\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \cos\theta_{jk}(0)x_{jk} \right] \\ & - \sum_{j\in N} \beta_j \left[\sum_{i:(i,j)\in E} \sin\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \sin\theta_{jk}(0)x_{jk} \right], \end{aligned} \tag{13}$$

subject to:

$$X \in \mathscr{F}.$$

By replacing $\cos\theta_{ij}(t_{ij})$ and $\cos\theta_{jk}(0)$ with $\cos\theta_j$, and replacing $\sin\theta_{ij}(t_{ij})$ and $\sin\theta_{jk}(0)$ with $\sin\theta_j$ and simplifying the objective in (13), $L(\Pi)$ can be written as

$$L(\Pi) = \min_{\theta,X} \sum_{(i,j)\in E} [d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i]x_{ij}. \tag{14}$$

Now, for any set of penalty variables, note that $L(\Pi) \ge J(\Pi)$

where

$$J(\Pi) = \min_X \sum_{(i,j)\in E} \min_\Theta [d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i]x_{ij}. \quad (15)$$

Theorem: For any given Π, the solution to the minimization problem with objective $J(\Pi)$ shown in equation (15), subject to the constraints in (2) is a lower bound to the MDMVDTSP (1,2,11,12).

Proof: Clearly $L(\Pi)$ in (13) is the Lagrangian relaxation of the MDMVDTSP defined by the objective in (1) and the constraints in (2,11,12). The weak duality theorem states that for a minimization problem, the cost of the Lagrangian relaxation for any set of penalty variables is at most equal to the optimal cost of the MDMVDTSP. Therefore, $L(\Pi)$ is a lower bound to the MDMVDTSP. One can also note that that $J(\Pi)$ is at most equal to $L(\Pi)$. Therefore, for any given Π, the solution to (15) is a lower bound to the MDMVDTSP. ■

Consider the following variational problem

$$v_{ij}(\alpha_i,\alpha_j,\beta_i,\beta_j) = \min_{\theta_i,\theta_j} d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i. \quad (16)$$

where $d_{ij}(\theta_i,\theta_j)$ is given by equations (3) to (6) and (9) to (10). $d_{ij}(\theta_i,\theta_j)$ is the minimum Dubins distance required by the vehicle to travel from the configuration (x_i,y_i,θ_i) to (x_j,y_j,θ_j) and can be calculated using the result from Dubins [1]. Given the values of $\alpha_i,\alpha_j,\beta_i,\beta_j$, one can compute v_{ij} as follows: Discretize the allowable values of the heading angle and obtain a discrete set of heading angles $\Phi = \{\phi_1,\phi_2,...\phi_d\}$. We assume that this discrete set of heading angles is the same for all the targets, *i.e.* $\Phi_i = \Phi, \forall i \in N$. Therefore, for every pair of $\theta_i,\theta_j \in \Phi$, the value of $d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i$ can be easily computed using the Dubins result. The minimum of all these values each corresponding to a pair of $\theta_i,\theta_j \in \Phi$ is v_{ij}.

Now, given $\Pi = (\alpha_1,.....\alpha_n,\beta_1,.....\beta_n)$, v_{ij} can be computed for every edge $(i,j)\in E$. The minimization problem in (15) then reduces to the following:

$$J(\Pi) = \min_{X\in\mathscr{F}} \sum_{(i,j)\in E} v_{ij}(\alpha_i,\alpha_j,\beta_i,\beta_j)x_{ij}, \quad (17)$$

This is an asymmetric Multiple Depot Multiple Traveling Salesmen Problem (MDMTSP) where the weight of each edge (i,j) is v_{ij}. Oberlin et. al. presented a method in [8] to transform the MDMTSP into an asymmetric traveling salesman problem(ATSP). The resulting ATSP may be solved using the LKH heuristic and it gives a tight lower bound for the ATSP, which is also a lower bound to the MDMVDTSP.

For any given Π, $J(\Pi)$ is a lower bound to the MDMVDTSP, the best lower bound can be found by maximizing $J(\Pi)$ for all Π, *i.e.*, by solving $J^* = \max_\Pi J(\Pi)$. Since $J(\Pi)$ is a combination of a finite number of linear functions, it is concave in Π. Therefore, we use a subgradient optimization technique to solve for J^*. Please refer to [11] and [12] to understand the details of subgradient optimization.

1) Implementation details for the MDMVDTSP: An important part of the subgradient optimization technique is to find a direction of the subgradient and step size at each iteration k. With respect to the MDMVDTSP, as constraints (11) and (12) are relaxed, one can chose the following as the subgradient:

$$s^k = \begin{bmatrix} \sum_{i:(i,j)\in E}\cos\theta^k_{ij}(t_{ij})x^k_{ij} - \sum_{k:(j,k)\in E}\cos\theta^k_{jk}(0)x^k_{jk} \\ \sum_{i:(i,j)\in E}\sin\theta^k_{ij}(t_{ij})x^k_{ij} - \sum_{k:(j,k)\in E}\sin\theta^k_{jk}(0)x^k_{jk} \end{bmatrix}. \quad (18)$$

Here, x^k_{ij} and θ^k_{ij} are the solutions for the problems defined in (17) and (16) respectively in the k^{th} iteration. For the simulations in this article, we chose the step size (δ^k) to be 2 initially and reduced it by a factor of 2 after every 50 iterations. This iterative procedure is outlined as below:

1) Initialize $k := 0$. Choose the initial penalty vector Π^0 with all the penalty variables to be equal to zero. Let ε be a small number and k_{max} be the maximum number of iterations allowed for the subgradient procedure.
2) Compute v_{ij} as shown in (16), for all $(i,j)\in E$.
3) Transform the MDMTSP into ATSP using the result from [8].
4) Solve the ATSP using the LKH heuristic. Set the lower bound, $J(\Pi^k)$, during the k^{th} iteration to be equal to the lower bound computed using the LKH heuristic.
5) If $k > 1$ and $\frac{J(\Pi^{k+1})-J(\Pi^k)}{J(\Pi^k)} \le \varepsilon$ stop the iterative procedure and output the best lower bound.
6) Compute $\Pi^{k+1} = \Pi^k + \delta^k s^k$, where s^k is given by the equation (18). If $k \ge k_{max}$, stop the iterative procedure and output the best lower bound; else, set $k = k+1$ and return to step 2.

IV. One in a Set Transformation

To corroborate the performance of the proposed technique, we also solve the MDMVDTSP using the method in Oberlin et. al [6]. In [6], the authors solve the MDMVDTSP (where the choice of the heading angle at each target is restricted to a discrete set) by transforming the MDMVDTSP into an ATSP. They replicate each target m times such that each of the m replications correspond to a possible heading angle. Now, the MDMVDTSP is posed as a problem of finding a subtour for each vehicle such that exactly one copy of each target is visited once and the total distance traveled by the vehicle is a minimum. This problem is then transformed into an ATSP using the method presented by Noon and Bean in [13]. One can solve this ATSP using the LKH heuristic. This algorithm readily gives an approximate solution to the MDMVDTSP, which is an upper bound to the optimum. Also it provides a lower bound which can be used to compare with the lower bounds obtained using the method proposed in this article.

V. Lower Bound Using Convexity

Let $p_1,....p_n$ be a sequence of targets on a plane. For $1 < i < n$, p_i is called a sharp turn if the angle $\angle(p_{i-1},p_i,p_{i+1})$ is acute and one of the p_i's neighbours is within a distance

4 units from the segment joining p_i to its other neighbour. Let $\theta_1, \theta_2....\theta_n$ be the headings at each target, and let $\mathscr{C}$ is a mapping of the angles vector $(\theta_1, \theta_2....\theta_n)$ to the length of the shortest Dubins path, with a minimum turning radius of 1 unit, visiting the targets in the given order. It was proved in [10], when there are no sharp turns, all global minima of $\mathscr{C}$ is strictly convex over its lifted domain in $\Re^n$. The shortest distance (d_{ij}) between two targets i and j is a function of the headings at the targets (θ_i, θ_j). The partial derivatives of d_{ij} were given in [10] as:

$$\frac{\partial d_{ij}(\theta_i,\theta_j)}{\partial \theta_i} = \pm(1-cos\alpha_i), \tag{19}$$

$$\frac{\partial d_{ij}(\theta_i,\theta_j)}{\partial \theta_j} = \pm(1-cos\alpha_j), \tag{20}$$

where α_i and α_j are the length of the circular arcs in the vehicle's path at target i and j respectively. We attempt to compute a lower bound to the single vehicle DTSP and MDMVDTSP using the convexity of $\mathscr{C}$, when the distance between every pair of targets is atleast 4 units.

To find the optimal solution for an instance of DTSP, one has to find the optimal sequence of targets to be visited and the optimal headings at each target. Let T^* be the optimal sequence of targets, $\theta^*(T^*)$ be the optimal headings at each target and $C(\theta^*(T^*))$ be the cost of the optimal solution. Since this is hard to solve, one can restrict the values of allowable headings at each target to a discrete set Φ_d and pose the DTSP as a one-in-a-set TSP. This was explained in detail in section IV. In general, the optimal sequence and headings of the one-in-a-set TSP can be different from the optimal solution of the DTSP. Let T^1, $\theta^1(T^1)$ be the optimal sequence of targets and headings and $C(\theta^1(T^1))$ be the optimal cost for the corresponding one-in-a-set TSP.

Let $\Phi_d = \{\phi^1, \phi^2, ...\phi^d\}$ be the set of discrete values of headings allowed at each target, where $\phi^{i+1} - \phi^i = \delta, i = 1,2,..,d-1$. Let Θ be the set of vectors of size n, defined as $\Theta = \{\theta : \theta_i \in \Phi_d, i = 1,2,..,n\}$.

Theorem: $C(\theta^1(T^1)) - 4n\delta$ is a lower bound to the optimal solution of DTSP, $C(\theta^*(T^*))$.
Proof: For a given sequence T of targets, there are two cases. In the first case, the headings at each target are restricted to a discrete set Φ_d and in the second case, there are no restrictions on the headings. We use two different notations for the optimal solutions of these two cases. Let $\hat{\theta}(T)$ be the optimal vector of headings corresponding to the case where headings at each target are restricted and $\theta^0(T)$ be the vector of optimal headings for the unrestricted case.

Similarly for the optimal sequence T^* of targets , let $\theta^2(T^*)$ be the optimal vector of headings corresponding to the restricted (or discrete) case and $\theta^*(T^*)$ be the vector of optimal headings for the unrestricted case.

For the given sequence T of targets, $C(\theta(T))$ is convex in $\theta(T)$ and therefore

$$\begin{aligned} C(\theta^0(T)) &\geq C(\hat{\theta}(T)) + \nabla_\theta C(\hat{\theta}(T)) \cdot (\theta^0(T) - \hat{\theta}(T)) \\ &\geq C(\hat{\theta}(T)) + \\ \sum_{(i,j):x_{ij}\in T} &\left[\frac{\partial d_{ij}(\theta_i,\theta_j)}{\partial \theta_i}(\theta_i^0 - \hat{\theta}_i) + \frac{\partial d_{ij}(\theta_i,\theta_j)}{\partial \theta_j}(\theta_j^0 - \hat{\theta}_j)\right] x_{ij} \end{aligned} \tag{21}$$

From equations (19) and (20), one can see that the maximum value of $\frac{\partial d_{ij}}{\partial \theta_i}$ and $\frac{\partial d_{ij}}{\partial \theta_j}$ is 2 and the maximum value of $(\theta_i^0 - \hat{\theta}_i)$ is δ. The inequality in (21) becomes

$$C(\theta^0(T)) \geq C(\hat{\theta}(T)) - \sum_{(i,j):x_{ij}\in T} (2\delta + 2\delta) x_{ij}$$

$$C(\theta^0(T)) \geq C(\hat{\theta}(T)) - 4n\delta \tag{22}$$

For the sequence T^*, we can write the inequality similar to (22) as:

$$C(\theta^*(T^*)) \geq C(\theta^2(T^*)) - 4n\delta. \tag{23}$$

Here the elements of the vector $\theta^2(T^*)$ belongs to the discrete set Φ_d and therefore $(T^*, \theta^2(T^*))$ is a feasible solution to the one-in-a-set TSP. Since $(T^1, \theta^1(T^1))$ is the optimal solution to the one-in-a-set TSP, we can write

$$C(\theta^1(T^1)) \leq C(\theta^2(T^*)). \tag{24}$$

From inequalities (23) and (24)

$$C(\theta^1(T^1)) - 4n\delta \leq C(\theta^*(T^*)). \tag{25}$$

■

For the sake of conserving space, we have provided only the proof for the single vehicle case. The same argument holds for the multiple vehicle case too.

VI. NUMERICAL RESULTS

The lower bound for the MDMVDTSP are computed using the proposed method and one in a set transformation method for several instances with 10, 20, 30 and 40 targets. The number of depots in each of these instances is 3. The lower bounds are computed for different values of the minimum turning radius (ρ) and listed in Table I. The first column indicates the the number of targets in an instance. The second column refers to the optimal solution of the corresponding EMTSP, which is a lower bound to the MDMVDTSP. The third and fourth columns refers to the lower bound calculated using the proposed method and the one in a set transformation method. the fifth and sixth columns refers to the percentage improvement of lower bound using these two methods when compared with the optimal solution of the EMTSP. The seventh and eighth columns refer to the time taken for the calculation using the two methods.

The proposed method performs better than the transformation method in all the instances except one instance. In most of the instances, the transformation method produced either negative value for the lower bound or a value less than the solution of EMTSP. That is because this method relies on modifying the cost of traveling for some of the edges by a large constant(generally called big-*M*). Hence the lower bound depends on the value chosen for M. Also we can observe that with higher turning radius the improvement in the lower bound is higher. With turning radius 5, the average improvement is 46%, which is very significant.

The average computation time required by the proposed method and the transformation method are shown in Fig. 1. We can clearly see that the computation time required by the proposed method is significantly less compared to the transformation method. The transformation method transforms the MDMVDTSP into an asymmetric TSP with a very large number of nodes and hence the high computation time. For one of the instances, a dual solution and a heuristic solution are shown in Fig. 2. The path in blue color is the dual solution and the path in the red color is the feasible solution with solution quality of 1.25.

For the instances where the targets satisfy the distance condition, the lower bounds are computed using the convexity as explained in section V. These are compared with the bounds from the Lagrangian relaxation method and transformation method in Table II for the single vehicle DTSP and the MDMVDTSP. This method produced better bounds than the other two methods in 8 out of 10 instances for the single vehicle case and all the instances for the multiple vehicles case.

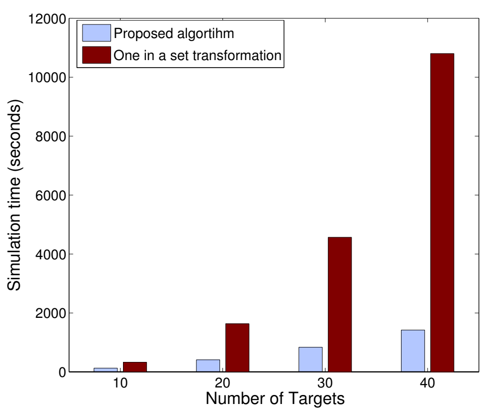

Fig. 1. Computation Time Comparison.

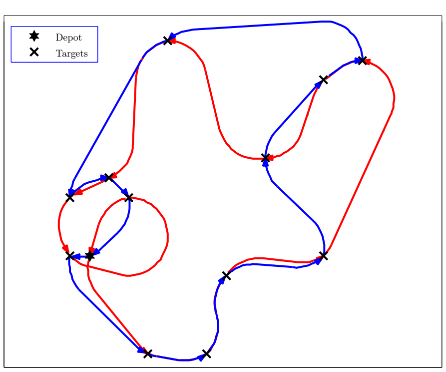

Fig. 2. Dual Solution Vs Heuristic Solution for one of the vehicles in a collection

TABLE II
LOWER BOUNDS COMPARISON FOR THE MDMVDTSP

Lower bounds for the single vehicle DTSP				
# Targets	ETSP[a]	Lag[b]	OST[c]	Conv[d]
10	106.67	108.19	-380.14	107.94
10	116.48	118.43	-177.20	116.88
20	163.63	167.57	13.49	169.64
20	175.64	177.61	-1512.35	182.80
30	211.10	212.94	-445.53	222.69
30	197.61	200.87	-6.73	202.49
40	237.20	240.97	83.49	246.78
40	236.89	240.64	-3134.59	242.70
50	271.71	275.83	-315.26	289.11
50	272.24	276.53	63.58	283.95
Lower bounds for the MDMVDTSP with 3 vehicles				
10	100.58	103.01	-81.11	125.78
10	125.17	125.30	-1290.98	134.91
20	167.93	169.74	29.45	180.42
20	165.44	168.31	-3271.64	169.32
30	210.28	214.04	-3137.66	245.38
30	213.71	216.04	-3859.22	242.82
40	240.79	245.95	14.68	257.80
40	239.24	242.99	24.64	262.75

[a]Optimal solution of Euclidean TSP
[b]Lower bound using proposed Lagrangian algorithm
[c]Lower bound using transformation method
[d]Lower bound computed using convexity

VII. CONCLUSIONS

In this paper, we provided a method to compute a lower bound for a multiple depot multiple vehicle routing problem with motion constraints. This method is explained in detail for an UAV modelled as a Dubins vehicle. The lower bounds computed using this method are compared with the solution of EMTSP and lower bound calculated using the transformation method from [6]. The proposed method produced lower bounds better than the bounds from other existing methods and required significantly less computation time than the transformation method. For certain instances as explained in Section V, using the convexity property of the shortest path, lower bounds are computed and this produced better bounds than the Lagrangian relaxation method.

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TABLE I

LOWER BOUNDS COMPARISON OF MDMVDTSP WITH 3 DEPOTS

Lower bound comparison with minimum turning radius = 2							
# Targets	**Euc.**[a]	**Lag**[b]	**OST**[c]	**%Diff. (Lag)**[d]	**%Diff. (OST)** [e]	**Sim. time (Lag)**[f]	**Sim. time (OST)** [g]
10	51.88	59.37	-4.84	14.43	-109.34	129	302
10	49.82	57.71	3.62	15.83	-92.73	131	310
20	77.40	87.94	25.15	13.63	-67.51	408	1517
20	68.96	78.31	61.86	13.57	-10.29	404	1425
30	96.06	106.02	42.69	10.37	-55.55	824	4557
30	79.53	97.63	59.21	22.76	-25.54	821	4095
40	101.44	114.59	125.43	12.96	23.65	1439	13001
40	102.35	116.37	71.81	13.70	-29.84	1363	10859
Mean				**14.66**	**-45.89**	**690**	**4508**
Lower bound comparison with minimum turning radius = 3							
10	51.87	66.09	47.60	27.41	-8.23	134	332
10	49.82	64.46	38.98	29.38	-21.76	129	330
20	77.40	94.51	34.99	22.11	-54.79	402	1482
20	68.96	83.17	72.12	20.62	4.58	406	1761
30	96.06	113.76	77.11	18.43	-19.72	822	5126
30	79.53	108.96	-68.11	37.00	-185.64	823	4273
40	101.44	125.22	100.13	23.44	-1.29	1443	13110
40	102.35	126.33	25.41	23.43	-75.17	1404	10714
Mean				**25.23**	**-45.25**	**776**	**5257**
Lower bound comparison with minimum turning radius = 4							
10	51.88	73.10	-36.91	40.91	-171.14	134	323
10	49.82	69.42	65.12	39.34	30.69	132	417
20	77.40	100.48	-3.67	29.83	-104.74	403	1624
20	68.96	89.01	70.05	29.08	1.58	412	1658
30	96.06	120.82	-66.88	25.78	-169.63	823	4717
30	79.53	121.16	104.24	52.35	31.07	832	4331
40	101.44	134.50	-201.44	32.59	-298.58	1427	9105
40	102.35	140.85	-1.00	37.61	-100.98	1417	9943
Mean				**35.94**	**-97.72**	**697**	**4015**
Lower bound comparison with minimum turning radius = 5							
10	62.11	80.18	53.09	29.09	-14.52	130	327
10	47.59	81.73	-108.10	71.74	-327.17	130	309
20	78.72	106.02	87.33	34.68	10.94	435	1975
20	64.67	92.30	-100.46	42.72	-255.34	434	1631
30	84.60	126.61	-189.03	49.66	-323.44	881	4560
30	84.34	129.03	-163.85	52.98	-294.27	880	4882
40	106.21	142.43	45.62	34.11	-57.05	1433	10415
40	93.60	147.56	-90.68	57.64	-196.87	1446	9271
Mean				**46.58**	**-182.22**	**721**	**4171**

[a] Optimal solution of Eucliden MDMTSP
[b] Lower bound from proposed Lagrangian algorithm
[c] Lower bound from one-in-a-set transformation
[d] % Difference with proposed Lagrangian algorithm
[e] % Difference with one-in-a-set transformation
[f] Simulation time (seconds) for proposed Lagrangian algorithm
[g] Simulation time (seconds) for one-in-a-set transformation

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