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Gradient Based Projection Method for Constrained Optimization

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Abstract— **We introduce a continuous-time gradient based optimization scheme for the convex programming problem. The dynamics of the optimization parameter are described by a continuous projection of the map's gradient. Its mechanics parallel that of an augmented steepest descent method with the exception that it behaves as an interior point method. The projection affects the flow field as if subject to an interior point barrier function. Under mild assumptions the optimization trajectories are shown to stay entirely within the feasible region and converge to the constrained optimum. The approach also simultaneously solves the Lagrangian dual problem even though the dynamics are not governed by it.**

I. INTRODUCTION

In the study of constrained minimization there are two general forms of the nonlinear programming (NLP) problem. The Equality Constrained Problem (ECP) and the Inequality Constrained Problem (ICP) of form

$$\inf_{\hat{\theta}} f(\hat{\theta}) \text{ s.t. } g(\hat{\theta}) \leq 0 \tag{1}$$

The concept of having a dynamical system solve optimization problems has been around for some time. The simplest approach for an unconstrained optimization problem is the method of steepest descent. The systems evolves in the direction of the negative gradient and asymptotically converges to a minimizing point (assuming it exists). This will not work for constrained problems where the critical point lies outside the feasible region. The optimization trajectory would evolve into an infeasible region as the final solution would violate the imposed constraints.

A common thematic approach for dealing with constraints is to eliminate them entirely and capture their intent elsewise. Hard constraints are the original constraints of an ICP and must be necessarily satisfied. Soft constraints on the other hand, are costs functions that favor certain solutions over others. The use of interior-point barrier functions force optimization trajectories away from hard constraints and are only defined inside the strictly feasible region. This is normally implemented with a logarithmic barrier function. The barrier function blows up as it approaches very near the boundary and creates a sort of 'force field' that counteracts the optimizers intent to exit the feasible set [1]. Unfortunately the hessian varies rapidly at the boundary and is subject to being ill conditioned, so it is not usually successful when used in practice with an algorithm such as the Newton Method. Alternatively, exterior-point penalty functions can be used to force optimization trajectories inward toward the feasible region when soft constraints are permissible. Exterior penalty functions are defined at least on a neighborhood of the feasible set. The desired characteristics of a penalty function are non-negativity, strict convexity, that it goes to infinity as infeasibility increases and zero valued for feasible points. Additionally desirable traits include smoothness and sharply increased function growth as infeasibility occurs. At the expense of complicating the cost function further, hybrid approaches which combine the two aforementioned ideas have been used in [2], [1] for example. This is beneficial as the barrier-penalty function is defined in both the feasible and infeasible region. Design of these barrier-penalty functions requires careful attention to guarantee continuous derivatives. The interior barrier, exterior penalty, and hybrid approaches are akin to eliminating constraints and reformulating (1) with an augmented cost function.

We can also eliminate the constraints by formulating a problem of dual variables:

$$L(\hat{\theta}, \lambda) = f(\hat{\theta}) + \sum_{i}^{m} \lambda_i g_i(\hat{\theta}) \tag{2}$$

The ECP in (2) is known as the Lagrangian dual form and is the most common formulation of the dual problem. The optimal values of the primal (original) and dual problems need not be equal in general. The difference between primal and dual solutions is referred to as the duality gap. However, in convex optimization problems the duality gap is zero assuming certain constraint qualifications have been met. While systems of equations can be constructed and used to solve (2), the procedure can quickly become cumbersome. As a result computational methods for solving problems of (2) naturally became of great interest in the late 1950's and early 1960's. The use of a dynamical system to solve (1) which involved the evolution of both λ and $\hat{\theta}$ from (2) began with K. J. Arrow et. al [3]. Approaches of this form are often referred to as Lagrangian methods.

Continuous optimization refers the ability of the objective function to assume real values as opposed to discrete. Most computational methods for continuous optimization are actually presented in discrete time because the field of convex optimization has been emphatically focused on increasing the efficiency of computational algorithms. [4], [5], [6], [7], [8], [9], [10], [11]. These resources are rich with information towards the mathematical discussion of convex optimization, but algorithmically speaking narrow their focus to improving convergence time for known objective functions.

Differential equations that solve a NLP problem can be combined with a numerical scheme for solving ODEs to provide a numerical scheme for solving the NLPs. As seen in [12], a general framework for developing a family of dynamical systems that solve (1) is presented as well as how

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to select the stepsize for the corresponding ODEs. It is advantageous to work in continuous-time because learning from the successes and limitations of interior point methods in this way positions us to develop notions of stability and convergence for continuous-time extremum seeking (ES) schemes subject to constraints. Additionally powerful mathematical tools are available in continuous-time and ultimately the scheme can be adopted to discrete-time, while the converse is not necessarily true. Continuous-time Lagrangian dynamical systems which solve the problem in (1) have been proposed in [13]. These dynamical systems while proven to converge to the constrained optimum, do permit excursions outside the feasible set. While new approaches to these methods seek to improve convergence properties, their requirement on knowledge of the gradient and hessian of the Lagrangian function are concerning. They may be difficult to estimate in practice, sensitive to noise, or partial to becoming ill conditioned in an ES scheme that has no knowledge of the map. This potentially poses problems to being effectively adapted to non-model based methods.

Lagrangian methods where the objective function is unknown have been studied in [14], [15]. In [14] the map is unknown and not necessarily convex. While convergence is not proven, the only possible points of convergence are shown to be the constrained local minima. Evolution of the λ parameter may cause transients that strongly violate the feasibility region.

Here we propose a continuous-time dynamical system which solves the convex optimization problem of (1) and has full knowledge of the objective function a priori. The optimization parameter evolves along a simple projection of the vector field. This avoids over complication of the objective function and its potentially ill effects on the vector field near the boundary. In effect, it has similarities to an interior-point barrier function and guarantees the optimization trajectories are at all times feasible. Although the equivalent barrier function in this approach cannot be written in a simple analytic expression, as a log barrier function might be. The remainder of the paper is structured as follows. In section II we introduce the necessary notation and mathematical preliminaries. Section III demonstrates the trajectories of our algorithm exist and do not leave the feasible set. The possible equilibria are characterized and the algorithm is shown to converge. In section IV we provide some numerical simulations to give a graphical interpretation of the how the algorithm looks when it is realized. Section V discusses adaptations to the feasible set and its effects on the constrained optimum. Finally, we conclude with a summary and outlook for future development.

II. BACKGROUND

We will make use of the following notation and definitions actively throughout the paper:

Let two points $\hat{\theta}_1, \hat{\theta}_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$ be given. Their convex combination is as follows

$$\hat{\theta} = \lambda\hat{\theta}_1 + (1-\lambda)\hat{\theta}_2 \tag{3}$$

We refer to the set $\Pi \in \mathbb{R}^n$ as convex, if all convex combinations of any two points $\hat{\theta}_1, \hat{\theta}_2 \in \Pi$ are again in Π.

A function $f : \Pi \to \mathbb{R}$ defined on a convex set Π is called convex if for all $\hat{\theta}_1, \hat{\theta}_2 \in \Pi$ and $0 \leq \lambda \leq 1$ one has

$$f\big(\lambda\hat{\theta}_1 + (1-\lambda)\,\hat{\theta}_2\big) \leq \lambda f(\hat{\theta}_1) + (1-\lambda) f(\hat{\theta}_2) \tag{4}$$

f is referred to as a strictly convex function if the $\leq$ relation in (4) is just $<$.

Assumption 1: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be C^1.

Assumption 2: $f(\hat{\theta}) : \mathbb{R}^n \to \mathbb{R}$, $g(\hat{\theta}) : \mathbb{R}^n \to \mathbb{R}$. f and g are convex functions.

Assumption 1 and Assumption 2 should be considered to be true throughout the remainder of the paper. We shall express the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ as follows: $\nabla f(\hat{\theta}) = \left(\frac{\partial f(\hat{\theta})}{\partial \hat{\theta}}\right)^T = \left[\frac{\partial f(\hat{\theta})}{\partial \hat{\theta}_1}, \dots, \frac{\partial f(\hat{\theta})}{\partial \hat{\theta}_n}\right]^T$. And the norm of a vector $\hat{\theta} \in \mathbb{R}^n$ as $||\hat{\theta}|| = \sqrt{\sum_{i=1}^n \hat{\theta}_i^2}$. $B_\delta(\hat{\theta}_1)$ is the open ball of radius δ centered about $\hat{\theta}_1$. $B_\delta(\hat{\theta}_1) = \{\hat{\theta}_2 \in \mathbb{R}^n : ||\hat{\theta}_1 - \hat{\theta}_2|| < \delta\}$. We shall refer to $\hat{\theta}$ as a stationary point of an at least C^1 function $f(\hat{\theta})$, such that $\nabla f(\hat{\theta}) = 0$.

Let Π be the convex set of all $\hat{\theta}$ which satisfy the constraint imposed by $g(\hat{\theta}) \leq 0$.

$$\Pi = \left\{\hat{\theta} \in \mathbb{R}^n \mid g(\hat{\theta}) \leq 0 \right\} \tag{5}$$

The boundary of Π is described by the set of points $\partial\Pi = \{\hat{\theta} \in \Pi \mid g(\hat{\theta}) = 0 \}$. ∇g represents an outward normal vector from the set at $\hat{\theta} \in \partial\Pi$. The interior is simply, $\mathring{\Pi} = \Pi \backslash \partial\Pi$, where the inequality $g(\hat{\theta}) < 0$ is strictly upheld. The boundary may in fact not be expressible by a single equation. In the instance that $g(\hat{\theta}) = 0$ is an implicit function, we require a few treatments. Let $g(\hat{\theta})$ be piecewise continuous described by $g(\hat{\theta}) = g_i(\hat{\theta}), i = [1, ..., m]$. Each $g_i(\hat{\theta})$ must be convex and differentiable as previously assumed. When multiple $g_i(\hat{\theta}) = 0$ for a single $\hat{\theta}$ we require all the corresponding $\nabla g_i(\hat{\theta})$ be equal for that $\hat{\theta}$. This is of concern where the segments of $g(\hat{\theta})$ are patched together and the formed boundary should be smooth. Let $Ind(\hat{\theta}) = \{\arg_i g_i(\hat{\theta}) = 0, \forall\ \hat{\theta}\ s.t.\ g_i(\hat{\theta}) = 0\}$. Then $\nabla g_i(\hat{\theta}) = \nabla g_j(\hat{\theta}),\ \forall\ i \neq j \in Ind(\hat{\theta})$. When constructed properly it should satisfy some regularity condition. Slater's Condition is a specific type of constraint qualification which requires the existence of a point that simultaneously satisfies each constraint strictly. In this framework, the existence of a Slater point is equivalent to a non-empty interior $\mathring{\Pi} \neq \emptyset$. Now our formulation is more aligned with the general expression (2). The following conditions are often referred to as the Karush-Kuhn-Tucker (KKT) conditions.

Theorem 1: Suppose Slater's Condition and Assumptions 1 and 2 are satisfied. Then $\hat{\theta}$ is a solution of (1) if and only if there $\exists\ \lambda \in \mathbb{R}_+^m$ such that the following three conditions are satisfied:

$$\nabla f(\hat{\theta}) + \sum_{i=1}^m \lambda \nabla g_i(\hat{\theta}) = 0 \tag{6}$$

$$\lambda_i g_i(\hat{\theta}) = 0, i = 1, ..., m \tag{7}$$

$$g_i(\hat{\theta}) \leq 0, i = 1, ..., m \tag{8}$$

The above conditions guarantee: positivity of the multipliers, stationarity, complementarity, and primal feasibility. Stationarity is a necessary condition for a critical point of the Lagrangian function. The restriction of the Lagrangian multipliers to a positive cone guarantees the feasibility of the dual problem. And primal feasibility implies the constrained solution satisfies each constraint at the same time. The constraints are considered active when $g(\hat{\theta}) = 0$, inactive for $g(\hat{\theta}) < 0$ and violated when $g(\hat{\theta}) > 0$. Finally, complementary slackness requires that if a constraint is not active, then the corresponding multiplier is zero. This captures the notion that inactive constraints are not of importance so long as their feasibility conditions are still satisfied.

Consider another convex set which is the union of (5) and an $O(\epsilon)$ boundary layer around it.

$$\Pi_\epsilon = \left\{\hat{\theta} \in \mathbb{R}^n \mid g(\hat{\theta}) \leq \epsilon\right\} \tag{9}$$

The extended optimization problem of (1) is then as follows

$$\inf f(\hat{\theta}) \text{ s.t. } \hat{\theta} \in \Pi_\epsilon \tag{10}$$

$f(\hat{\theta})$ need not be globally convex so long as it is assumed to be locally convex over our extended set.

$\hat{\theta}_c^*$ is the set of arguments which minimizes the constrained problem in (10)

$$\hat{\theta}_c^* = \arg\min_{\hat{\theta}} f(\hat{\theta}), \ \hat{\theta} \in \Pi_\epsilon \tag{11}$$

Remark 1: Although $f(\hat{\theta})$ is assumed to be locally or globally convex, it need not actually permit a minimum. Because we do not restrict Π_ϵ to be a compact set, we must guarantee the problem is formulated in such a manner that (10) is feasible. If $f(\hat{\theta})$ permits a minimum then (10) is necessarily feasible. In either case, we will always assume $\hat{\theta}_c^* \neq \emptyset$.

The following projection operator has been adapted from [17].

$$\text{Proj}\{\xi\} = \begin{cases} \xi, & \hat{\theta} \in \mathring{\Pi} \text{ or } D(\hat{\theta}) \leq 0 \\ \left(I - c(\hat{\theta}) \frac{\nabla g(\hat{\theta}) \nabla g(\hat{\theta})^T}{\nabla g(\hat{\theta})^T \nabla g(\hat{\theta})}\right) \xi, & \begin{array}{c}\hat{\theta} \in \Pi_\epsilon \backslash \mathring{\Pi} \text{ and} \\ D(\hat{\theta}) > 0\end{array} \end{cases} \tag{12}$$

$$c(\hat{\theta}) = \min\left\{1, \frac{g(\hat{\theta})}{\epsilon}\right\} \tag{13}$$

Where $D(\hat{\theta}) = \nabla g(\hat{\theta})^T \xi$ is a condition that describes the directional relationship of ∇g and ξ at $\hat{\theta}$. A similar projection applied only at the boundary of Π creates a discontinuous vector field. The projection in (12) is desirable because it achieves a continuous transition of the unmodified vector field from inside Π all the way up to $\partial\Pi_\epsilon$. Proj{} is not necessarily differentiable however, because of a likely kink at $\hat{\theta} \in \partial\Pi$.

Remark 2: It use is useful to note that $c(\hat{\theta}) = 0 \ \forall \ \hat{\theta} \in \partial\Pi$ and $c(\hat{\theta}) = 1 \ \forall \ \hat{\theta} \in \partial\Pi_\epsilon$

III. MAIN RESULTS

We propose the following dynamical system to solve the convex and constrained minimization problem in (10)

$$\dot{\hat{\theta}} = \text{Proj}\{-k\nabla f(\hat{\theta})\}, \ \hat{\theta}(0) \in \Pi_\epsilon \tag{14}$$

$k \in \mathbb{R}$ s.t. $0 < k < \infty$. While not presented here, the maximal problem would follow similarly for a finite $k < 0$.

A. Existence and Feasibility of Solutions

Lemma 2: The vector field, $\text{Proj}\{-k\nabla f\}$, points to either the inside of Π_ϵ or is tangential to the hyperplane of $\partial\Pi_\epsilon$ for all $\hat{\theta} \in \partial\Pi_\epsilon$.

Proof: Using (12) and some minor manipulation it is clear that

$$\nabla g(\hat{\theta})^T \text{Proj}\{-k\nabla f(\hat{\theta})\} = \begin{cases} D(\hat{\theta}), & \hat{\theta} \in \mathring{\Pi} \quad \text{or } D(\hat{\theta}) \leq 0 \\ \left(1 - c(\hat{\theta})\right) D(\hat{\theta}), & \hat{\theta} \in \Pi_\epsilon \backslash \mathring{\Pi} \text{ and } D(\hat{\theta}) > 0 \end{cases} \tag{15}$$

Then by using $\hat{\theta} \in \partial\Pi_\epsilon \to c(\hat{\theta}) = 1$, we recognize that $\nabla g(\hat{\theta})^T \text{Proj}\{-k\nabla f(\hat{\theta})\} \leq 0$ ■

Remark 3: Since $\hat{\theta}(0) \in \Pi_\epsilon$, it follows that $\hat{\theta}(t) \in \Pi_\epsilon$ so long as the solution exists. Because $\text{Proj}\{-k\nabla f\}$ is continuous and k is finite, solutions exist so long as the gradient is bounded, $||\nabla f(\hat{\theta})|| < L, \ L \in \mathbb{R}_+$, over Π_ϵ.

B. Equilibria

Fermat's Theorem famously states that a necessary condition to occur at a minimum of a differentiable function is that the derivative shall be equal the zero vector. And as a corollary, global extrema of a function f on a domain only occur at boundaries, non-differentiable points, and stationary points. Based on Assumption 1 and Assumption 2, we shall then expect our constrained optimum to occur in the strict interior of the set and be a stationary point of f, or appear on the boundary with no requirement of stationarity.

1) Equilibria Conditions for Non-stationary Points of $f(\hat{\theta})$:

Lemma 3: Assume $\hat{\theta} \in \Pi_\epsilon$ is not a stationary point of $f(\hat{\theta})$. If $\hat{\theta}$ is an equilibrium of (14), then $\exists \ \lambda \in \mathbb{R}_+ \ s.t.$

$$\nabla f(\hat{\theta}) + \lambda \nabla g(\hat{\theta}) = 0, \tag{16}$$

Proof: Because $\hat{\theta}$ is not stationary, $\nabla f(\hat{\theta}) \neq 0$. It follows that $\text{Proj}\{-k\nabla f(\hat{\theta})\} = 0$ only for $\hat{\theta} \in \Pi_\epsilon \backslash \mathring{\Pi}$ and $-\nabla g(\hat{\theta})^T k \nabla f(\hat{\theta}) > 0$. Then by (12) we have $\left(-I + c(\hat{\theta}) \frac{\nabla g(\hat{\theta}) \nabla g(\hat{\theta})^T}{\nabla g(\hat{\theta})^T \nabla g(\hat{\theta})}\right) k \nabla f(\hat{\theta}) = 0$. By rearranging terms we have $\nabla f(\hat{\theta}) = c(\hat{\theta}) \left(\frac{\nabla g(\hat{\theta}) \nabla g(\hat{\theta})^T}{\nabla g(\hat{\theta})^T \nabla g(\hat{\theta})}\right) \nabla f(\hat{\theta})$. We move the scalar c and group the terms to have $\nabla g(\hat{\theta}) \left(c(\hat{\theta}) \frac{\nabla g(\hat{\theta})^T \nabla f(\hat{\theta})}{\nabla g(\hat{\theta})^T \nabla g(\hat{\theta})}\right) = -\nabla g(\hat{\theta}) \lambda(\hat{\theta})$.

$$\lambda(\hat{\theta}) = -c(\hat{\theta}) \frac{\nabla g(\hat{\theta})^T \nabla f(\hat{\theta})}{\nabla g(\hat{\theta})^T \nabla g(\hat{\theta})} \in \mathbb{R} \tag{17}$$

For (16) to hold we know that $\nabla g(\hat{\theta})$ and $\nabla f(\hat{\theta})$ must be (anti)parallel. With that in mind and inspection of (17), (16) only holds for $|c(\hat{\theta})| = 1$. And $c(\hat{\theta}) = [0,1] \to c(\hat{\theta}) = 1$. Here $D(\hat{\theta}) > 0 \to \nabla g(\hat{\theta})^T \nabla f(\hat{\theta}) < 0$. Then by informed inspection of (17) we see $\lambda > 0$ ∎

Remark 4: $\nabla g(\hat{\theta})$ and $\nabla f(\hat{\theta})$ are in fact antiparallel for an equilibrium of (14) that is not a stationary point of $f(\hat{\theta})$

Corollary 4: If $\hat{\theta}$ is an equilibrium of (14) and is not a stationary point of $f(\hat{\theta})$, then $\hat{\theta} \in \partial\Pi_\epsilon$.

Proof: As stated in the proof of Lemma 3, c=1. By inspection of (13), $g(\hat{\theta}) = \epsilon \to \hat{\theta} \in \partial\Pi_\epsilon$ ∎

Lemma 5: Assume $\hat{\theta} \in \Pi_\epsilon$ is an equilibrium of (14) and is not a stationary point of $f(\hat{\theta})$, then $\hat{\theta} \in \hat{\theta}_c^*$

Proof: By Lemma 3 the first and second KKT conditions are immediately satisfied. Our expanded set includes points satisfying $g(\hat{\theta}) \leq \epsilon$ as feasible, the last condition is true by assumption. Because then all the KKT conditions have been satisfied, $f(\hat{\theta})$ is a solution of (1) and therefore (10). Then $\hat{\theta} \in \hat{\theta}_c^*$ ∎

Remark 5: (16) looks exactly like the Lagrangian optimization condition for a minimum subject to equality constraints and it is a natural extension to assume Lemma 5. In fact it is necessity for the structure of our dynamic system to preserve the KKT conditions at its equilibria, otherwise it cannot converge to an optimum.

2) Equilibria Conditions for Stationary Points of $f(\hat{\theta})$:

Proposition 6: Stationary points of convex functions are global minimum.

Proof: Stationary points of convex functions are local minima. Local minima of a convex function are global extremum. ∎

Lemma 7: If $\hat{\theta}$ is equilibrium of (14) and a stationary point of $f(\hat{\theta})$, then $\hat{\theta} \in \hat{\theta}_c^*$

Proof: A stationary point in dimensions higher than n=1 is not enough to categorize the point as a local maxima or minimum (as with inflection points for example). However, because $f(\hat{\theta})$ is assumed to be convex, $\hat{\theta}$ is necessarily a local minima. Then $\hat{\theta} \in \hat{\theta}_c^*$. ∎

Theorem 8: If $\hat{\theta}$ is an equilibrium of (14), then $\hat{\theta} \in \hat{\theta}_c^*$

Proof: The union of stationary and non-stationary points of the domain is the domain itself. So if $\hat{\theta}$ is an equilibrium, by Lemma 5 and Lemma 7 $\to \hat{\theta} \in \hat{\theta}_c^*$ ∎

3) Set-valued Equilibria: We have previously characterized the conditions for an equilibrium of (14). We will further show the set of equilibria, $\hat{\theta}_c^*$, is an attractor that is a single connected component of Π_ϵ. If $\hat{\theta}_c^* \subseteq \partial\Pi_\epsilon$, then it is a flat subset of dimension n-1. If $g(\hat{\theta})$ or the level sets of $f(\hat{\theta})$ are strictly convex then $|\hat{\theta}_c^*| = 1$ and the solution is a singleton.

Proposition 9: Let $f(\hat{\theta})$ be a function defined on a convex set $\Pi_\epsilon \in \mathbb{R}^n$. The function $f(\hat{\theta})$ is convex if and only if the function $\phi(\lambda) = f(\hat{\theta} + \lambda s)$ is convex on the interval $\lambda = [0,1]$ for all $\hat{\theta}, \hat{\theta} + s \in \Pi_\epsilon$

Proof: Start by assuming f is convex and let $\lambda_1, \lambda_2, \alpha \in [0,1]$. We start from (4) and write $\phi(\alpha\lambda_1 + (1-\alpha)\lambda_2) = f(\hat{\theta} + [\alpha\lambda_1 + (1-\alpha)\lambda_2]s)$. By grouping the terms differently and noting the convexity of f, we recognize $f(\alpha[\hat{\theta} + \lambda_1 s] + (1-\alpha)[\hat{\theta} + \lambda_2 s]) \leq \alpha f(\hat{\theta} + \lambda_1 s) + (1-\alpha) f(\hat{\theta} + \lambda_2 s) = \alpha\phi(\lambda_1) + (1-\alpha)\phi(\lambda_2)$. So ϕ is convex. Alternatively, start with the assumption $\phi(\lambda)$ is convex over $\lambda \in [0,1]$ and $\hat{\theta}_1, \hat{\theta}_1 + s \in \Pi_\epsilon,\ s = \hat{\theta}_2 - \hat{\theta}_1$, and $0 \leq \alpha \leq 1$. Then $f(\alpha\hat{\theta}_2 + (1-\alpha)\hat{\theta}_1) = f(\hat{\theta}_1 + \alpha(\hat{\theta}_2 - \hat{\theta}_1)) = \phi(\alpha)$. Next we introducte phantom terms and use the convexity of ϕ to recognize $\phi(\alpha 1 + (1-\alpha)0) \leq \alpha\phi(1) + (1-\alpha)\phi(0) = \alpha f(\hat{\theta}_2) + (1-\alpha) f(\hat{\theta}_1)$. ∎

Lemma 10: $\hat{\theta}_c^*$ is connected

Proof: Assume $\hat{\theta}_c^*$ is not connected. Then let $\hat{\theta}_1$ and $\hat{\theta}_2$ belong to disjoint partitions of $\hat{\theta}_c^*$. Because $\hat{\theta}_1$ and $\hat{\theta}_2$ are members of $\hat{\theta}_c^*$, $f(\hat{\theta}_1) = f(\hat{\theta}_2)$. Let $s = \hat{\theta}_1 - \hat{\theta}_2$. We describe the line segment between $\hat{\theta}_1$ and $\hat{\theta}_2$ by $\phi(\lambda) = \hat{\theta}_1 + \lambda s,\ \lambda = [0,1]$. Because Π_ϵ is a convex set, the entirety of the parametric line segment, $\phi(\lambda)$, lies within Π_ϵ. By assumption that $\hat{\theta}_c^*$ is disconnected, $\exists\lambda \in [0,1]\ s.t.\ \phi(\lambda) \notin \hat{\theta}_c^*$.Then $f(\hat{\theta}_1) < f\big(\phi(\lambda)\big)$ and $f(\hat{\theta}_2) < f\big(\phi(\lambda)\big)$. As we have restricted $f(\hat{\theta})$ to the line segment $\phi(\lambda)$, $f(\hat{\theta})$ is convex iff $f\big(\phi(\lambda)\big)$ is convex. Since it is clear that $\phi(\lambda)$ is not convex, neither must $f(\hat{\theta})$ be, and we reach a contradiction. $\hat{\theta}_c^*$ is connected. ∎

Definition 1: Let a convex function $f : \Pi \to \mathbb{R}$ defined on the convex set Π be given. Let $\alpha \in \mathbb{R}$ be an arbitrary number. The set $D_\alpha = \Big\{\hat{\theta} \in \Pi \mid f(\hat{\theta}) \leq \alpha\Big\}$ is called a sublevel set of the function f.

Proposition 11: Let $f(\hat{\theta})$ be a convex function over the convex set Π_ϵ, then for all $\alpha \in \mathbb{R}$ the sublevel set D_α is a convex set.

Proof: Let $\hat{\theta}_1, \hat{\theta}_2 \in D_\alpha$ and $0 \leq \lambda \leq 1$. Then we have $f(\hat{\theta}_1) \leq \alpha, f(\hat{\theta}_2) \leq \alpha$ and we write the following, $f(\lambda\hat{\theta}_1 + (1-\lambda)\hat{\theta}_2) \leq \lambda f(\hat{\theta}_1) + (1-\lambda) f(\hat{\theta}_2) \leq \lambda\alpha + (1-\lambda)\alpha = \alpha$. ∎

The sublevel sets can possibly be empty, but the empty set is a convex set. The first inequality follows from convexity of $f(\hat{\theta})$.

Lemma 12: $\hat{\theta}_c^* \subset \partial\Pi_\epsilon$ is either a singleton or a flat hyperplane of dimension n-1, perpendicular to ∇g

Proof: Let $\hat{\theta}_{eq} \in \hat{\theta}_c^*$ and $\hat{\theta}_{eq} \in \partial\Pi_\epsilon$. Let $\alpha = f(\hat{\theta}_{eq})$, any other $\hat{\theta} \in \hat{\theta}_c^*$ must belong to the sublevel set D_α. In fact, since α is the infimum of Π_ϵ, it is just a level set. Since a sublevel set of a convex function is convex itself, D_α must be convex. $g(\hat{\theta})$ is convex by assumption. As stated in Remark 4, $\nabla g(\hat{\theta}_{eq})$ and $-\nabla f(\hat{\theta}_{eq})$ are parallel. Let $\hat{\theta}_{eq} \in$ the hyperplane perpendicular to $\nabla g(\hat{\theta}_{eq})$ and $\nabla f(\hat{\theta}_{eq})$. Because $\nabla g(\hat{\theta}_{eq})$ is an outward normal vector of $g(\hat{\theta})$, and $g(\hat{\theta})$ is convex, $g(\hat{\theta}) \leq \epsilon$ must lie entirely on one side of the hyperplane or on the hyperplane itself. The same holds for the level set of D_α but on the other side. Because the hyperplane partitions the n-space into halves, the only possible intersection of D_α and $g(\hat{\theta}) \leq \epsilon$ is the hyperplane. If either D_α or $g(\hat{\theta})$ is locally strictly convex about $\hat{\theta}_{eq}$, only a single point would reside on the hyperplane and the solution $\hat{\theta}_c^*$ would be a singleton. ∎

C. Convergence of Trajectories

Theorem 13: If $\hat{\theta}(0) \in \Pi_\epsilon$, then $\hat{\theta}(t) \to \theta_{eq} \in \hat{\theta}_c^*$ as $t \to \infty$ and $\hat{\theta}(t) \in \Pi_\epsilon\ \forall\ t \geq 0$.

Theorem 13 says that (14) will asymptotically converge to the constrained optimum of Π_ϵ without leaving Π_ϵ

Proof: Lemma 2 guarantees the trajectories of $\hat{\theta}$ stay in Π_ϵ so we have $\hat{\theta}$ is forward invariant w.r.t. Π_ϵ. We also have that $\hat{\theta}_c^* \in \Pi_\epsilon$ is positively invariant along $\dot{\hat{\theta}}$. Now let us define a candidate Lyapunov functon. $V(\hat{\theta}) = \frac{1}{|k|}\Big(f(\hat{\theta}) - f(\theta_{eq})\Big)$

It easily follows from $f(\hat{\theta}) \geq f(\theta_{eq})$ that $\frac{1}{|k|}\Big(f(\hat{\theta}) - f(\theta_{eq})\Big) \geq 0$. We have V positive along $\dot{\hat{\theta}}$ exluding $\hat{\theta}_c^*$, where it is zero valued. Before we consider the derivative $\dot{V}$ it will be useful to note the following

$$\text{Proj}\{k\nabla f(\hat{\theta})\} = |k|\text{Proj}\{\text{sgn}(k)\nabla f(\hat{\theta})\} \tag{18}$$

We start with $\dot{V} = \frac{1}{|k|}\nabla f(\hat{\theta})^T\dot{\hat{\theta}}$ and substitute in (14). Now with $\frac{1}{|k|}\nabla f(\hat{\theta})^T\text{Proj}\{-k\nabla f(\hat{\theta})\}$ we use (18) and arrive at $\nabla f(\hat{\theta})^T\text{Proj}\{-\nabla f(\hat{\theta})\}$. We restate more clearly:

$$\dot{V} = \begin{cases} -\nabla f^T\nabla f, & \hat{\theta} \in \mathring{\Pi} \quad \text{ or } \quad D \leq 0 \\ -\nabla f^T\nabla f + c(\hat{\theta})\frac{\nabla f^T\nabla g\nabla g^T\nabla f}{\nabla g^T\nabla g}, & \hat{\theta} \in \Pi_\epsilon\backslash\mathring{\Pi} \text{ and } D > 0 \end{cases}$$

The term $-\nabla f^T\nabla f$ is clearly negative for $\nabla f \neq 0$, but the other projection output requires more analysis. We start by recognizing $c(\hat{\theta})\frac{\nabla f^T\nabla g\nabla g^T\nabla f}{\nabla g^T\nabla g}$ is $c(\hat{\theta})\frac{\nabla f^T\nabla g}{||\nabla g||}\frac{\nabla g^T\nabla f}{||\nabla g||}$, the squared scalar projection of ∇f onto the unit vector $\frac{\nabla g}{||\nabla g||}$. Because a scalar projection is upper bounded by a vector projecting onto itself and $0 \leq c(\hat{\theta}) \leq 1$, we see that $c(\hat{\theta})\frac{\nabla f^T\nabla g}{||\nabla g||}\frac{\nabla g^T\nabla f}{||\nabla g||} \leq c(\hat{\theta})||\nabla f||^2 \leq \nabla f^T\nabla f$. Because we have shown that $c(\hat{\theta})\frac{\nabla f^T\nabla g\nabla g^T\nabla f}{\nabla g^T\nabla g} \leq \nabla f^T\nabla f$ it follows that $-\nabla f^T\nabla f + c(\hat{\theta})\frac{\nabla f^T\nabla g\nabla g^T\nabla f}{\nabla g^T\nabla g} \leq 0$.

This however is only zero valued for an equilibrium of (14). And by Theorem 8, we recognize that the equilibrium belongs to $\hat{\theta}_c^*$. It is finally clear $\dot{V}$ is strictly negative valued in Π_ϵ excluding $\hat{\theta}_c^*$, where it is zero valued. When $\hat{\theta}_c^*$ is a singleton, this reduces to the definition of Lyapunov asymptotic stabilty of an equilibrium point. Otherwise, shown by the existence of a positive definite function V and a negative definite $\dot{V}$ in a neighborhood of $\hat{\theta}_c^*$ excluding $\hat{\theta}_c^*$ itself, $\hat{\theta}_c^*$ is an asympotically stable invariant set with regards to (14) and initial conditons in Π_ϵ. Simply stated, $f(\hat{\theta}) \to f(\theta_c)$ and accordingly $\hat{\theta} \to \theta_c \in \hat{\theta}_c^*$ ■

IV. Simulation

The blue curves in Figure 1 represent the original and epsilon boundaries for two similar elliptical constraint sets. The red, green, and black trajectories represent the gradient based projection method for their respective sets. And the magenta trajectory is the unconstrained method of steepest descent starting in either set. From the simulation it is apparent the trajectories are highly subject the orientation of the feasible set over the objective function. Similar feasible sets have very different convergence characteristics depending on the objective function to which they are related. The projection is utilized only in the ϵ-boundary layer, and its primary purpose is to provide continuity of the vector field from the interior to its boundaries. Trajectories then have a tendency to drift towards, and then along the constraints as the objective function is minimized. The convergence rate of the steepest descent method is already only asymptotic, and the indirect trajectories towards the optimum also leave something to be desired.

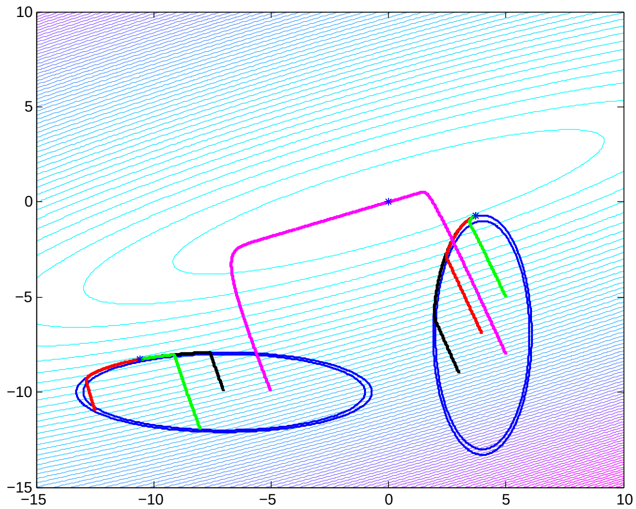

Fig. 1. Method of steepest descent (pink) and gradient based projection method (red/green/black)

V. Effects of Set Augmentiation

Given a subset of Π of $\mathbb{R}^n$, we define the erosion $e_r(\Pi)$ of Π by the radius δ as the set of points of Π at distance $\geq \delta$ from the complement Π^C of Π. So we have

$$e_\delta(\Pi) = \Pi \setminus \bigcup_{\hat{\theta}\in\Pi^C} B_\delta(\hat{\theta}) \tag{19}$$

And the dilation as

$$d_\delta(\Pi) = \Pi \cup \bigcup_{\hat{\theta}\in\Pi} B_\delta(\hat{\theta}) \tag{20}$$

Consider now Π_{smooth} to be another convex set achieved through two different possible augmentations of Π. The exterior augmentation $\Pi_\epsilon = d_\epsilon(\Pi)$ and the interior augmentation $\Pi_\epsilon = d_\epsilon(e_\epsilon(\Pi))$. Π_{smooth} allows for the original constraint boundary to be non-smooth as the dilation has a smoothing effect and preserves the set convexity. Then letting $g = 0$ describe this new boundary and creating an $O(\epsilon)$ boundary layer around that. It is important to note that dilation of Π cannot be used in place of the $O(\epsilon)$ boundary layer, because in general not all convex sets are resilient to dilation or erosion. That is to say, we need the preservation of ∇g from the original constraint throughout the boundary layer. The dilation does not guarantee that the 'shape' of the boundary is preserved. This brings up an important realization of the KKT condition, which is a firmly rooted geometrical interpretation. The distance from the original constrained optimum to the new constrained optimum may be further than $O(\epsilon)$ away.

In Figure 2, the solid and dotted blue regions represent the original and expanded constrained sets. The purple contour

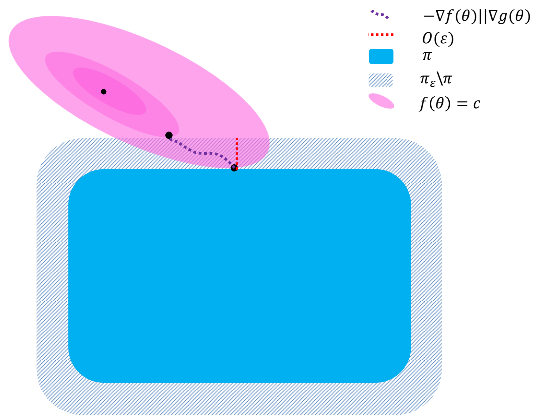

Fig. 2. Caricature of how constrained the optimal parameter can change with the expanded problem

regions represent some level sets of the convex function f. The points of tangency between the blue and purple regions represent the constrained optimal solutions. The dashed dark purple path represents the sequence of points where $-\nabla f$ and ∇g are in the same direction and should satisfy (16). These are not equilibria though because $c \neq 1$. While a λ for (16) exists, is not the value of λ in the projection . This example is meant to illustrate that the extended constrained optimum is not bounded to be within $O(\epsilon)$ of the original constrained optimum.

VI. Conclusions

We proposed a projection operator to act on the vector field of a convex set. Under suitable assumptions of the map and the constraints we present a continuous-time dynamical system which solves the problem in (10). The evolution of the continuous optimization parameter only permits feasible trajectories subject to its convex constraints. The only possible equilibria of the algorithm are the set of constrained optimum and the algorithm converges asymptotically to it. Further, if the constrained optimum are not local minimum of the objective function, they are either a flat hyperplane of dimension n-1 or a singleton. Because we have an expression for λ in (17) we also simultaneously solve the Lagrangian problem in (2).

The projection is simple to construct as a point wise evaluation. But the equivalent barrier function would be difficult to construct analytically. This is the opposite approach to something like a logarithmic barrier function. There the function is defined everywhere ahead of time, and the gradient is subject to becoming ill conditioned as the boundary is approached. The structure of a log barrier function would persist for a given set of constraints even if the objective function were modified. In our case, the corresponding barrier function would be specific to the combination of the objective and constraint functions. In future work we hope to expand on the foundation laid out in this paper by replacing the gradient descent method with an extremum seeking technique that has no requirement on knowledge of the map.

This leads to immediate consequences on the successes of this gradient based approach. By limiting the information available to just a scalar evaluation of the objective function, the optimization parameters requires excitation to acquire enough information about its neighborhood. Because of the necessary excitation, the optimization trajectories cannot step in the direction of a decreasing gradient at all times.The problem will lend itself towards tools of convergence, stability, and averaging. The limitation to be an interior point method also raises issues with notions of convergence and stability. The optimizer cannot converge to a typical limit cycle about the peak value if it is on the boundary. It may be possible to prove or disprove the existence of a periodic orbit in such a case. If no limit cycle is possible, it will presumably still remain within some boundable neighborhood of the optimal boundary point. And ultimately, how it might be possible to tie in the benefits of the projection scheme with the nominal structure of an ES approach.

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