Fenchel's Theorem

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Overview

Basics

Let $\vec{\alpha}(s) = (x(s), y(s), z(s))$ where $s \in [0, L]$ be a unit-speed curve in $\mathbb{R}^3$. Then, we have
$\vec{t}(s) = \vec{\alpha}'(s) = (x'(s), y'(s), z'(s))$.

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Consider mapping $\vec{t}(s)$ on the interval $[0, L]$ to the unit sphere, S^2. We define the **tangent indicatrix** as $\Gamma = \{\vec{t}(s) : s \in [0, L]\} \subseteq S^2$, which is the curve generated by the mapping.

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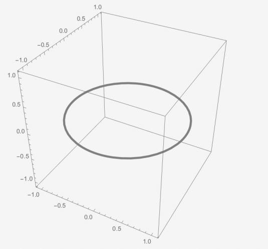
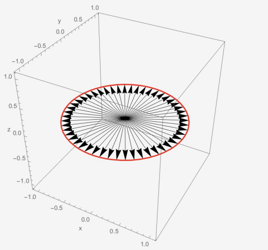
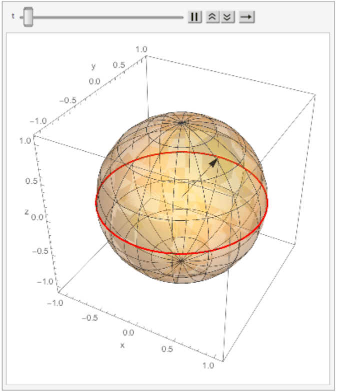
For a plane curve C, the **rotation index** γ tells us how many times $\vec{t}$ wraps around the unit circle.

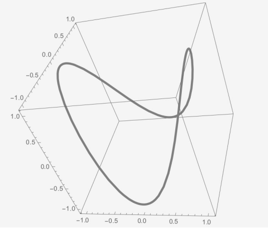
Tangent Indicatrix for the unit circle

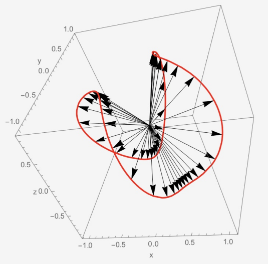


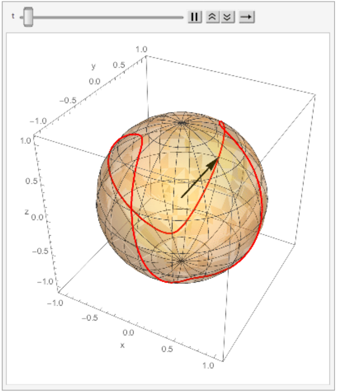




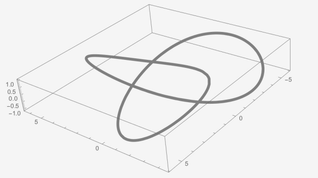
Tangent Indicatrix for a curve sitting on a saddle

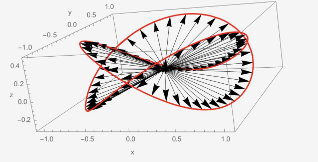


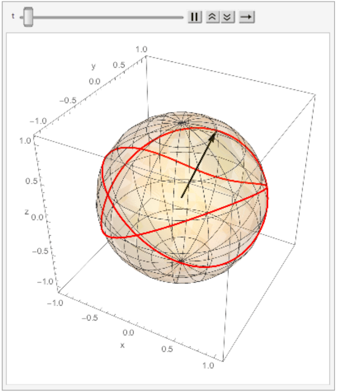




Tangent Indicatrix for a trefoil







Total Curvature

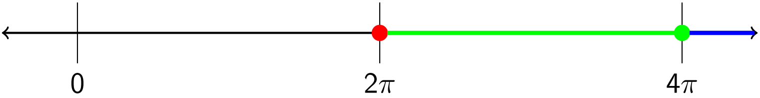
For a curve $\vec{\alpha}(s)$ with curvature $k(s)$, the total curvature is

$$\int_a^b k(s)ds = \ell(\Gamma)$$

So, the length of the tangent indicatrix encodes the total curvature. Furthermore, for a plane curve, this measures how many times the curve "wraps around" the unit circle S^1.

Goal: Characterize Total Curvature of Closed Curves

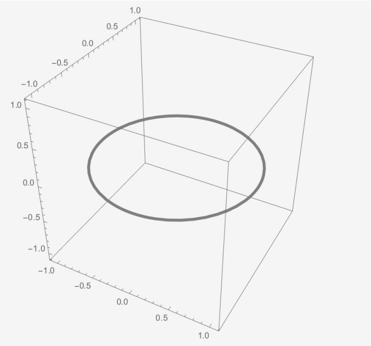
For a closed curve, the following image defines its total curvature:

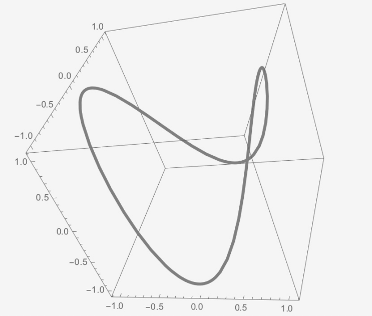


On $(0, 2\pi)$: $\vec{\alpha}$ is *not closed*
At 2π: $\vec{\alpha}$ is a *convex plane curve* (Fenchel's Theorem, 1929)
On $(2\pi, 4\pi]$: $\vec{\alpha}$ is an *unknotted curve*
Greater than 4π: $\vec{\alpha}$ may be a *knotted or unknotted curve* (Fáry-Milnor Theorem, 1949)

Examples of Total Curvatures

Total curvature for each curve:





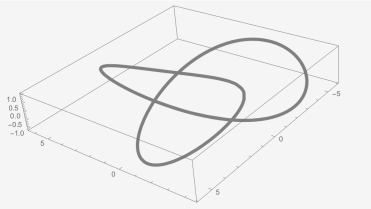
unit circle

$6.28319... = 2\pi$

curve on saddle

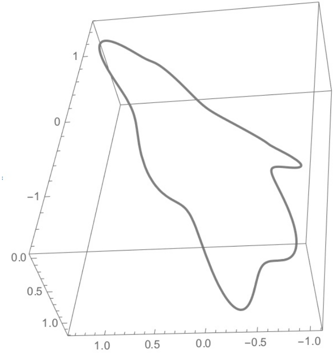
$7.88373 \approx 2.51\pi$

Examples of Total Curvature



curve on trefoil

$13.03524 \approx 4.15\pi$



a random unknotted curve

$21.8507 \approx 6.96\pi$

Structure of Proofs

Lemma 2.15

Lemma 2.13

$+$

Lemma 2.14

$\Downarrow$

Fenchel's Theorem

Preliminary Lemmas and Theorems

Theorem (2.10 Turning Angle)

The rotation index of a simple closed planar curve ± 1

Theorem (2.12)

A closed planar curve C is convex iff it is simple and the signed curvature k does not change sign

$$\Downarrow$$

Lemma (2.15)

A simple closed planar curve is convex if and only if $\ell(\Gamma) = 2\pi$, *where* Γ *is the tangent indicatrix*

Theorem 2.10 Turning Angle

Theorem (2.10 Turning Angle)

The **rotation index** *of a simple closed planar curve* ± 1

rotation index γ: $\int_0^L k_s(s)\ ds = 2\pi\gamma$

- Step 1: Define $h(s,t)$: maps each **pair of points** to a **unit-length vector**
- Step 2: Define $\bar{\theta}(s,t)$: maps each **pair of values of t** on the curve to the **angle from** $x-$**axis to** $h(s,t)$
- Step 3: decompose $\int_0^L k_s(s)\ ds = \theta(L) - \theta(0)$.

Theorem 2.10 Turning Angle

Step 1

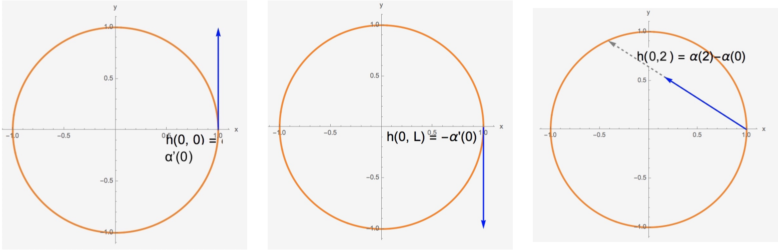
Define $h(s,t)$: $h(s,t) = \begin{cases} \vec{\mathbf{t}}(s), & s = t \\ -\vec{\mathbf{t}}(s), & s = 0, t = L \\ \frac{\vec{\alpha}(t) - \vec{\alpha}(s)}{|\vec{\alpha}(t) - \vec{\alpha}(s)|}, & \text{otherwise} \end{cases}$

maps each **pair of values** from domain of the curve α on the curve to a **unit-length vector**:

Theorem 2.10 Turning Angle

Example: $\alpha(x) = (\cos(x), \sin(x)),\ x \in [0, 2\pi]$

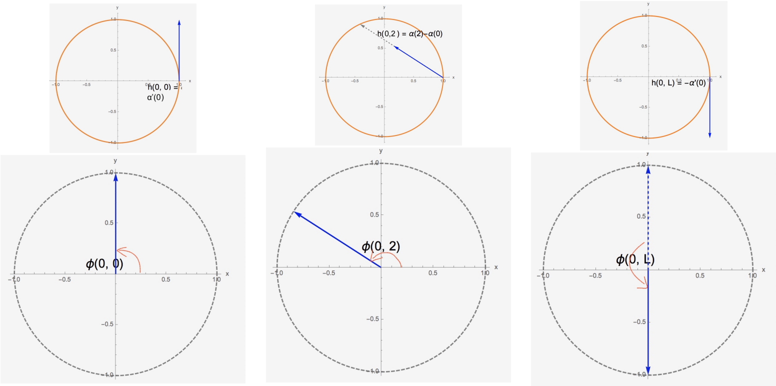
- $(s, s) \to \vec{\mathbf{t}}(s)$
- $(0, L) \to -\vec{\mathbf{t}}(0)$
- $(s, t) \to$ the vector denoting the direction from $\alpha(s)$ to $\alpha(t)$



Theorem 2.10 Turning Angle

Step 2

Define $\phi(s,t)$: maps each **pair of values from the domain** of the curve to the **angle** from $x-$axis to $h(s,t)$



Theorem 2.10 Turning Angle

Step 3

Decompose formula

$$\int_0^L k_s(s)\ ds = \theta(L) - \theta(0),$$
$$\theta : \text{the angle between } x\text{−axis and } \vec{\mathbf{t}}(s)$$

$$\begin{aligned}\int_0^L k_s(s)\ ds &= \theta(L) - \theta(0)\\ &= \text{angle of } h(L, L) - \text{angle of } h(0, 0)\end{aligned}$$

Theorem 2.10 Turning Angle

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Theorem 2.10 Turning Angle

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Decompose formula

$$\int_0^L k_s(s)\ ds = \theta(L) - \theta(0),$$

$$\theta : \text{the angle between } x\text{−axis and } \vec{\mathbf{t}}(s)$$

$$\begin{aligned}
\int_0^L k_s(s)\ ds &= \theta(L) - \theta(0) \\
&= \text{angle of } h(L, L) + \text{angle of } h(0, 0) \\
&= \phi(L, L) - \phi(0, 0) \\
&= \phi(L, L) - \phi(0, 0) + (\phi(0, L) - \phi(0, L)) \\
&= (\phi(0, L) - \phi(0, 0)) + (\phi(L, L) - \phi(0, L))
\end{aligned}$$

Theorem 2.10 Turning Angle

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$$\int_0^L k_s(s)\ ds = \theta(L) - \theta(0),$$
θ : the angle between $x-$axis and $\vec{\mathbf{t}}(s)$

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\int_0^L k_s(s)\ ds &= \theta(L) - \theta(0) \\
&= \text{angle of } h(L, L) + \text{angle of } h(0, 0) \\
&= \phi(L, L) - \phi(0, 0) \\
&= \phi(L, L) - \phi(0, 0) + (\phi(0, L) - \phi(0, L)) \\
&= (\phi(0, L) - \phi(0, 0)) + (\phi(L, L) - \phi(0, L)) \\
&= \text{angle change from } h(0, 0) \to h(0, L) + \\
&\quad \text{angle change from } h(0, L) \to h(L, L)
\end{aligned}$$

Theorem 2.10 Turning Angle

$$\int_0^L k_s(s)\ ds = \text{angle}(h(0,0), h(0,L)) + \text{angle}(h(0,L) \to h(L,L))$$

$$\pi + \pi$$

$$(-\pi) + (-\pi)$$

$$\Downarrow$$

$$\int_0^L k_s(s)\ ds = \pm 2\pi$$

Recall that $\int_0^L k_s(s)\ ds = 2\pi\gamma$, hence $\gamma = \pm 1$.

Theorem 2.12

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A closed planar curve C is **convex** iff it is **simple** and the signed curvature k_s does not change sign

- convex: A planar curve $\mathcal{C}$ is convex if $\mathcal{C}$ lies entirely in the closed half-plane determined by $\vec{t}(s)$ for all $s \in [0, L]$.
  In other words, if we draw a **tangent line** at any point on the curve, the **entire curve** stays on **one side** of the tangent line
- simple: A curve $\mathcal{C}$ is simple if $\mathcal{C}$ does not cross itself

The simplest example of a convex plane curve

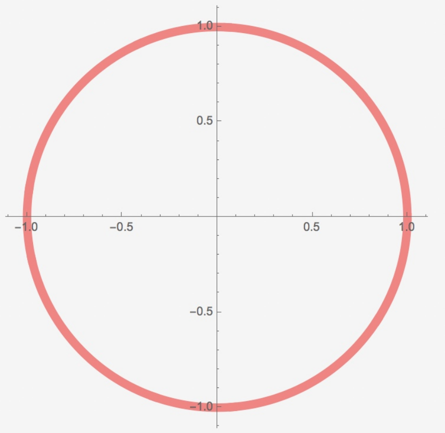

Figure: A convex curve!

Theorem 2.12

- $\Longrightarrow$ convex $\rightarrow$ simple && k_s does not change sign
- $\Longleftarrow$ simple && k_s does not change sign $\rightarrow$ convex

Sketch of a proof $\Longrightarrow$

○ Assume the curve is convex

- What happens if the curve is **not** simple?
- What happens if k_s changes sign?

Sketch of a proof $\Longleftarrow$

○ Assume the curve is simple, and k_s does not change sign

- What happens if the curve is **not** convex?

Lemma 2.15

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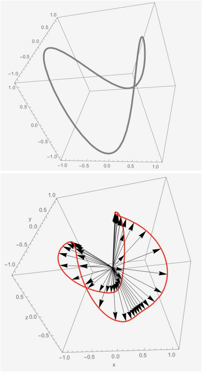
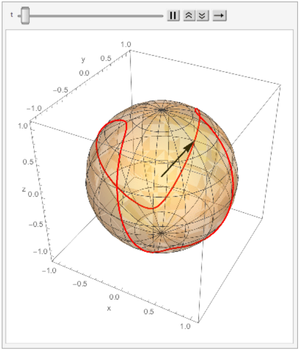
A **simple closed planar** curve is **convex** if and only if $\ell(\Gamma) = 2\pi$, where Γ is the tangent indicatrix

Here it comes again: Tangent Indicatrix

$$\int_0^L k(s)ds = \ell(\Gamma)$$





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A **simple closed planar** curve is **convex** if and only if $\ell(\Gamma) = 2\pi$, where Γ is the tangent indicatrix

- $\Longrightarrow$ convex $\rightarrow$ $\ell(\Gamma) = 2\pi$
- $\Longleftarrow$ $\ell(\Gamma) = 2\pi \rightarrow$ convex

Lemma 2.15

- $\Longrightarrow$ Assume the curve $\mathcal{C}$ is convex

Proof.

$$\mathcal{C} \text{ is convex} \Longrightarrow k_s \textbf{ does not change sign}, \quad \mathcal{C} \textbf{ is simple} \quad [2.12]$$

$$\text{can always orient } \mathcal{C} \Longrightarrow k_s(s) \geq 0, \ \forall s, \mathbf{k}(\mathbf{s}) = \mathbf{k_s}(\mathbf{s})$$

$$\Longrightarrow \ell(\Gamma) = \int_0^L k(s)ds = \int_\mathbf{0}^\mathbf{L} \mathbf{k_s}(\mathbf{s})\mathbf{ds}$$

$$\int_0^L k_s(s)ds = \theta(L) - \theta(0) \Longrightarrow \ell(\Gamma) = \theta(L) - \theta(0)$$

$$\theta(L) - \theta(0) = 2\pi\gamma \Longrightarrow \ell(\Gamma) = \mathbf{2}\pi\gamma$$

$$\mathcal{C} \text{ is simple, closed, planar} \Longrightarrow \gamma = \pm 1 \Longrightarrow \ell(\Gamma) = \pm\mathbf{2}\pi \quad [2.10]$$

$$\ell(\Gamma) \geq 0 \Longrightarrow \ell(\Gamma) = \mathbf{2}\pi$$

$\square$

Lemma 2.15

- $\Longleftarrow$ Assume $\ell(\Gamma) = 2\pi$

Proof.

$$\mathcal{C} \text{ is simple, closed, planar} \Longrightarrow \gamma = \pm 1 \quad [2.10]$$

$$\text{can always orient } \mathcal{C} \Longrightarrow \gamma = \mathbf{1}$$

$$\int_0^L k_s(s)ds = \theta(L) - \theta(0) = 2\pi\gamma \Longrightarrow \boldsymbol{\int_0^L k_s(s)ds = 2\pi}$$

$$\ell(\Gamma) = 2\pi \Longrightarrow \ell(\Gamma) = \int_0^L k_s(s)ds$$

$$\ell(\Gamma) = \int_0^L k(s)ds \Longrightarrow \int_0^L k(s)ds = \int_0^L k_s(s)ds$$

$$\Longrightarrow k_s \textbf{ does not change sign}$$

$$\text{also, } \mathcal{C} \text{ is simple} \Longrightarrow \mathcal{C} \textbf{ is convex} \quad [2.12]$$

$\square$

Preliminary Lemmas and Theorems

Lemma 2.13

The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. Furthermore, it lies in a closed hemisphere if and only if C is planar.

Claim 1: The tangent indicatrix Γ of a simple closed curve does not lie in an open hemisphere.
Claim 2: Γ lies in a closed hemisphere if and only if C is planar.

Claim 1

Lemma 2.13

The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. Furthermore, it lies in a closed hemisphere if and only if C is planar.

- Suppose Γ lies in the northern hemisphere. Then, $z'(s) \geq 0$ for all $s \in [0, L]$.

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- Suppose Γ lies in the northern hemisphere. Then, $z'(s) \geq 0$ for all $s \in [0, L]$.
- The curve C is closed, so $z(L) - z(0) = \int_0^L z'(s)ds = 0$.

Claim 1

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- Suppose Γ lies in the northern hemisphere. Then, $z'(s) \geq 0$ for all $s \in [0, L]$.
- The curve C is closed, so $z(L) - z(0) = \int_0^L z'(s)ds = 0$.
- Thus, $z'(s)$ cannot be strictly positive, so Γ cannot lie in an open hemisphere.

Claim 2

Lemma 2.13

The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. **Furthermore, it lies in a closed hemisphere if and only if C is planar.**

- Suppose Γ lies in the closed, northern hemisphere.

Claim 2

Lemma 2.13

The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. **Furthermore, it lies in a closed hemisphere if and only if C is planar.**

- Suppose Γ lies in the closed, northern hemisphere.
- Necessarily, $z'(s) \geq 0$ and $\int_0^L z'(s)ds = 0$. Therefore, $z'(s) \equiv 0$ and C lies in a plane where z is constant.

Claim 2

Lemma 2.13

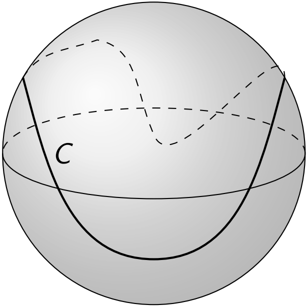
The tangent indicatrix Γ of a simple closed curve C does not lie in an open hemisphere of S^2. **Furthermore, it lies in a closed hemisphere if and only if C is planar.**

- Suppose Γ lies in the closed, northern hemisphere.
- Necessarily, $z'(s) \geq 0$ and $\int_0^L z'(s)ds = 0$. Therefore, $z'(s) \equiv 0$ and C lies in a plane where z is constant.
- Suppose C is planar. Then, Γ lies on a great circle and therefore lies in a closed hemisphere.

Preliminary Lemmas and Theorems

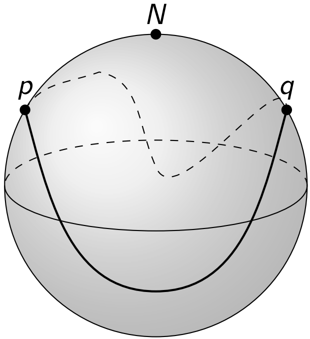
Lemma (2.14)

Let C be a closed curve on S^2. If $\ell(C) < 2\pi$, then C lies in an open hemisphere of S^2. If $\ell(C) = 2\pi$, then C lies in a closed hemisphere of S^2.



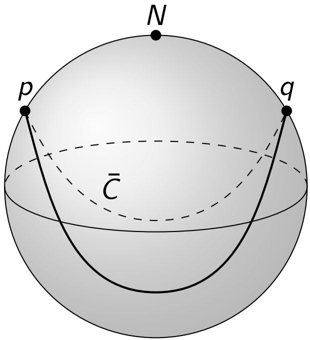
Step 1

Choose points p, q on C such that the curves $C_1 = pq$ and $C_2 = qp$ have the same length. Let N be the midpoint of the shorter geodesic connecting p and q.



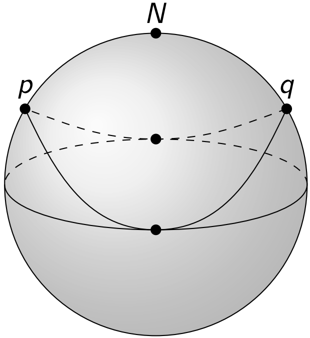
Step 2

Suppose C intersects the equator at some point. Let $\bar{C}_1$ be a rotation of C_1 by π around N. Let $\bar{C} = C_1 \cup \bar{C}_1$. Notice that $\ell(C) = \ell(\bar{C})$.



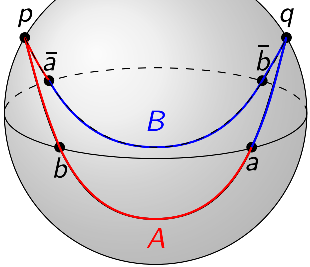
Check in

$\bar{C}$ can intersect the equator without crossing into the southern hemisphere. In this case, we know that $\ell(\bar{C}) \geq 2\pi$ so $\ell(C) \geq 2\pi$. Now, we want to show that if $\bar{C}$ *does* cross into the southern hemisphere that $\ell(C) > 2\pi$.



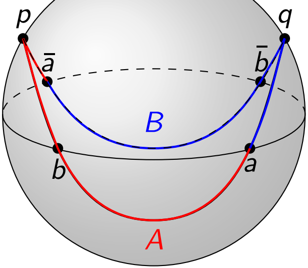
Step 3

Let a, b be the intersections of C_1 with the equator and $\bar{a}, \bar{b}$ be the intersections of $\bar{C}_1$ with the equator such that $a, \bar{a}$ and $b, \bar{b}$ are antipodal. Then, $\ell(\bar{C}) = \ell(C_1) + \ell(\bar{C}_1) = \ell(A) + \ell(B)$.



Claim: The length from a to $\bar{a}$ along $\bar{C}$ is greater than π.

Reasoning: $\ell(a\bar{a})$ along A is longer than $\ell(a\bar{a})$ along the equator geodesic. Since $a, \bar{a}$ are antipodal, the distance between them along the geodesic is π. Hence, $\ell(A), \ell(B) > \pi$, so $\ell(A) + \ell(B) > 2\pi$.



Lemma 2.14: Let C be a closed curve on S^2. If $\ell(C) < 2\pi$ then C lies in an open hemisphere. If $\ell(C) = 2\pi$ then C lies in a closed hemisphere of S^2.

If $\ell(C) < 2\pi$ then C lies in an open hemisphere.
We have shown that if C intersects the equator, then $\ell(C) \geq 2\pi$ regardless of whether C enters the southern hemisphere.

If $\ell(C) = 2\pi$ then C lies in a closed hemisphere of S^2.
We have shown that if C intersects the equator then $\ell(C) \geq 2\pi$. Furthermore, if C enters the southern hemisphere then $\ell(C) > 2\pi$. Therefore, $\ell(C) = 2\pi$ exactly when C lies in a closed hemisphere of S^2.

Fenchel's Theorem

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The total curvature of a simple closed curve C is greater than or equal to 2π, with equality if and only if the curve is plane convex.

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Good news! We've already done the hard work to prove Fenchel's theorem. Now we just need to put the pieces together.

Fenchel's Theorem

The total curvature of a simple closed curve C is greater than or equal to 2π, with equality if and only if the curve is plane convex.

Recall that we calculating the total curvature is the same as determining the length of the tangent indicatrix Γ associated with C.

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By **Lemma 2.13**, Γ cannot lie in an open hemisphere.

Fenchel's Theorem

The total curvature of a simple closed curve C is greater than or equal to 2π, with equality if and only if the curve is plane convex.

Recall that we calculating the total curvature is the same as determining the length of the tangent indicatrix Γ associated with C.

By **Lemma 2.13**, Γ cannot lie in an open hemisphere. Furthermore, by **Lemma 2.14**, $\ell(\Gamma) = \int_0^L k(s)ds \geq 2\pi$.

Fenchel's Theorem

The total curvature of a simple closed curve C is greater than or equal to 2π, **with equality if and only if the curve is plane convex.**

($\Longrightarrow$) Suppose $\ell(\Gamma) = 2\pi$. By **Lemma 2.14**, Γ lies in a closed hemisphere. Hence, Γ is planar (**Lemma 2.13**). Finally, **Lemma 2.15** guarantees that C is convex.

Fenchel's Theorem

The total curvature of a simple closed curve C is greater than or equal to 2π, **with equality if and only if the curve is plane convex.**

($\Longrightarrow$) Suppose $\ell(\Gamma) = 2\pi$. By **Lemma 2.14**, Γ lies in a closed hemisphere. Hence, Γ is planar (**Lemma 2.13**). Finally, **Lemma 2.15** guarantees that C is convex.

($\Longleftarrow$) Suppose C is simple closed planar and convex. Then, by **Lemma 2.15**, $\ell(\Gamma) = \int_0^L kds = 2\pi$.

References

Charles M. Evans (2010)
Curves, Knots, and Total Curvature

The End