Tangent bundle

A **tangent bundle** is the collection of all of the tangent spaces for all points on a manifold, structured in a way that it forms a new manifold itself. Formally, in differential geometry, the tangent bundle of a differentiable manifold M is a manifold TM which assembles all the tangent vectors in M. As a set, it is given by the disjoint union[note 1] of the tangent spaces of M. That is,

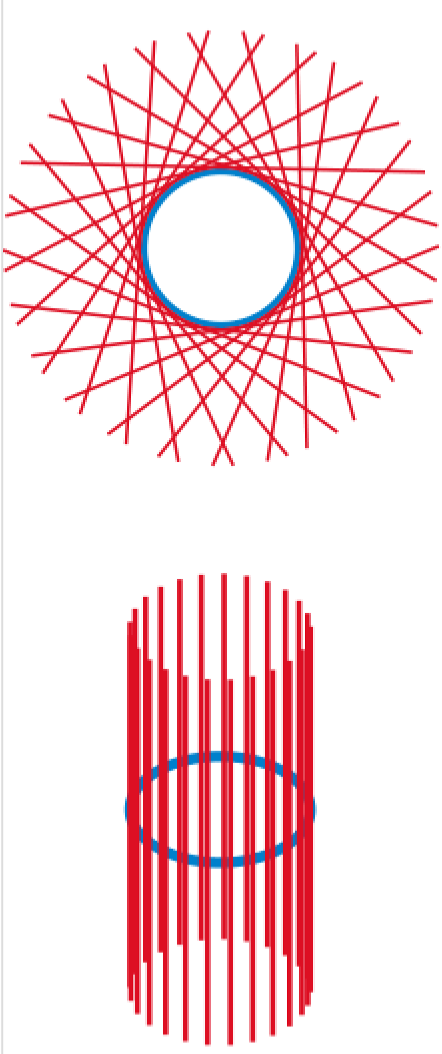
$$\begin{aligned} TM &= \bigsqcup_{x\in M} T_xM \\ &= \bigcup_{x\in M} \{x\}\times T_xM \\ &= \bigcup_{x\in M} \{(x,y)\mid y\in T_xM\} \\ &= \{(x,y)\mid x\in M,\, y\in T_xM\} \end{aligned}$$

where T_xM denotes the tangent space to M at the point x. So, an element of TM can be thought of as a pair (x,v), where x is a point in M and v is a tangent vector to M at x.

There is a natural projection

$$\pi : TM \twoheadrightarrow M$$

defined by $\pi(x,v)=x$. This projection maps each element of the tangent space T_xM to the single point x.



Informally, the tangent bundle of a manifold (which in this case is a circle) is obtained by considering all the tangent spaces (top), and joining them together in a smooth and non-overlapping manner (bottom).[note 1]

The tangent bundle comes equipped with a natural topology (described in a section below). With this topology, the tangent bundle to a manifold is the prototypical example of a vector bundle (which is a fiber bundle whose fibers are vector spaces). A section of TM is a vector field on M, and the dual bundle to TM is the cotangent bundle, which is the disjoint union of the cotangent spaces of M. By definition, a manifold M is parallelizable if and only if the tangent bundle is trivial. By definition, a manifold M is framed if and only if the tangent bundle TM is stably trivial, meaning that for some trivial bundle E the Whitney sum $TM\oplus E$ is trivial. For example, the n-dimensional sphere S^n is framed for all n, but parallelizable only for $n = 1, 3, 7$ (by results of Bott-Milnor and Kervaire).

Role

One of the main roles of the tangent bundle is to provide a domain and range for the derivative of a smooth function. Namely, if $f : M \to N$ is a smooth function, with M and N smooth manifolds, its derivative is a smooth function $Df : TM \to TN$.

Topology and smooth structure

The tangent bundle comes equipped with a natural topology (*not* the disjoint union topology) and smooth structure so as to make it into a manifold in its own right. The dimension of TM is twice the dimension of M.

Each tangent space of an *n*-dimensional manifold is an *n*-dimensional vector space. If U is an open contractible subset of M, then there is a diffeomorphism $TU \to U \times \mathbb{R}^n$ which restricts to a linear isomorphism from each tangent space T_xU to $\{x\} \times \mathbb{R}^n$. As a manifold, however, TM is not always diffeomorphic to the product manifold $M \times \mathbb{R}^n$. When it is of the form $M \times \mathbb{R}^n$, then the tangent bundle is said to be *trivial*. Trivial tangent bundles usually occur for manifolds equipped with a 'compatible group structure'; for instance, in the case where the manifold is a Lie group. The tangent bundle of the unit circle is trivial because it is a Lie group (under multiplication and its natural differential structure). It is not true however that all spaces with trivial tangent bundles are Lie groups; manifolds which have a trivial tangent bundle are called parallelizable. Just as manifolds are locally modeled on Euclidean space, tangent bundles are locally modeled on $U \times \mathbb{R}^n$, where U is an open subset of Euclidean space.

If *M* is a smooth *n*-dimensional manifold, then it comes equipped with an atlas of charts (U_α, ϕ_α), where U_α is an open set in M and

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

is a diffeomorphism. These local coordinates on U_α give rise to an isomorphism $T_xM \to \mathbb{R}^n$ for all $x \in U_\alpha$. We may then define a map

$$\widetilde{\phi}_\alpha : \pi^{-1}\left(U_\alpha\right) \to \mathbb{R}^{2n}$$

by

$$\widetilde{\phi}_\alpha\left(x, v^i\partial_i\right) = \left(\phi_\alpha(x), v^1, \cdots, v^n\right)$$

We use these maps to define the topology and smooth structure on TM. A subset A of TM is open if and only if

$$\widetilde{\phi}_\alpha\left(A \cap \pi^{-1}\left(U_\alpha\right)\right)$$

is open in $\mathbb{R}^{2n}$ for each α. These maps are homeomorphisms between open subsets of TM and $\mathbb{R}^{2n}$ and therefore serve as charts for the smooth structure on TM. The transition functions on chart overlaps $\pi^{-1}\left(U_\alpha \cap U_\beta\right)$ are induced by the Jacobian matrices of the associated coordinate transformation and are therefore smooth maps between open subsets of $\mathbb{R}^{2n}$.

The tangent bundle is an example of a more general construction called a vector bundle (which is itself a specific kind of fiber bundle). Explicitly, the tangent bundle to an n-dimensional manifold M may be defined as a rank n vector bundle over M whose transition functions are given by the Jacobian of the associated coordinate transformations.

Examples

The simplest example is that of $\mathbb{R}^n$. In this case the tangent bundle is trivial: each $T_x\mathbb{R}^n$ is canonically isomorphic to $T_0\mathbb{R}^n$ via the map $\mathbb{R}^n \to \mathbb{R}^n$ which subtracts x, giving a diffeomorphism $T\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.

Another simple example is the unit circle, S^1 (see picture above). The tangent bundle of the circle is also trivial and isomorphic to $S^1 \times \mathbb{R}$. Geometrically, this is a cylinder of infinite height.

The only tangent bundles that can be readily visualized are those of the real line $\mathbb{R}$ and the unit circle S^1, both of which are trivial. For 2-dimensional manifolds the tangent bundle is 4-dimensional and hence difficult to visualize.

A simple example of a nontrivial tangent bundle is that of the unit sphere S^2: this tangent bundle is nontrivial as a consequence of the hairy ball theorem. Therefore, the sphere is not parallelizable.

Vector fields

A smooth assignment of a tangent vector to each point of a manifold is called a **vector field**. Specifically, a vector field on a manifold M is a smooth map

$$V\colon M \to TM$$

such that $V(x) = (x, V_x)$ with $V_x \in T_xM$ for every $x \in M$. In the language of fiber bundles, such a map is called a *section*. A vector field on M is therefore a section of the tangent bundle of M.

The set of all vector fields on M is denoted by $\Gamma(TM)$. Vector fields can be added together pointwise

$$(V + W)_x = V_x + W_x$$

and multiplied by smooth functions on *M*

$$(fV)_x = f(x)V_x$$

to get other vector fields. The set of all vector fields $\Gamma(TM)$ then takes on the structure of a module over the commutative algebra of smooth functions on *M*, denoted $C^\infty(M)$.

A local vector field on M is a *local section* of the tangent bundle. That is, a local vector field is defined only on some open set $U \subset M$ and assigns to each point of U a vector in the associated tangent space. The set of local vector fields on M forms a structure known as a sheaf of real vector spaces on M.

The above construction applies equally well to the cotangent bundle – the differential 1-forms on M are precisely the sections of the cotangent bundle $\omega \in \Gamma(T^*M)$, $\omega : M \to T^*M$ that associate to each point $x \in M$ a 1-covector $\omega_x \in T^*_xM$, which map tangent vectors to real numbers: $\omega_x : T_xM \to \mathbb{R}$.

Equivalently, a differential 1-form $\omega \in \Gamma(T^*M)$ maps a smooth vector field $X \in \Gamma(TM)$ to a smooth function $\omega(X) \in C^\infty(M)$.

Higher-order tangent bundles

Since the tangent bundle TM is itself a smooth manifold, the second-order tangent bundle can be defined via repeated application of the tangent bundle construction:

$$T^2M = T(TM).$$

In general, the kth order tangent bundle T^kM can be defined recursively as $T\left(T^{k-1}M\right)$.

A smooth map $f : M \to N$ has an induced derivative, for which the tangent bundle is the appropriate domain and range $Df : TM \to TN$. Similarly, higher-order tangent bundles provide the domain and range for higher-order derivatives $D^kf : T^kM \to T^kN$.

A distinct but related construction are the jet bundles on a manifold, which are bundles consisting of jets.

Canonical vector field on tangent bundle

On every tangent bundle TM, considered as a manifold itself, one can define a **canonical vector field** $V : TM \to T^2M$ as the diagonal map on the tangent space at each point. This is possible because the tangent space of a vector space *W* is naturally a product, $TW \cong W \times W$, since the vector space itself is flat, and thus has a natural diagonal map $W \to TW$ given by $w \mapsto (w, w)$ under this product structure. Applying this product structure to the tangent space at each point and globalizing yields the canonical vector field. Informally, although the manifold M is curved, each tangent space at a point x, $T_xM \approx \mathbb{R}^n$, is flat, so the tangent bundle manifold TM is locally a product of a curved M and a flat $\mathbb{R}^n$. Thus the tangent bundle of the tangent bundle is locally (using $\approx$ for "choice of coordinates" and $\cong$ for "natural identification"):

$$T(TM) \approx T(M \times \mathbb{R}^n) \cong TM \times T(\mathbb{R}^n) \cong TM \times (\mathbb{R}^n \times \mathbb{R}^n)$$

and the map $TTM \to TM$ is the projection onto the first coordinates:

$$(TM \to M) \times (\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n).$$

Splitting the first map via the zero section and the second map by the diagonal yields the canonical vector field.

If (x, v) are local coordinates for TM, the vector field has the expression

$$V = \sum_i v^i \left.\frac{\partial}{\partial v^i}\right|_{(x,v)}.$$

More concisely, $(x, v) \mapsto (x, v, 0, v)$ – the first pair of coordinates do not change because it is the section of a bundle and these are just the point in the base space: the last pair of coordinates are the section itself. This expression for the vector field depends only on v, not on x, as only the tangent directions can be naturally identified.

Alternatively, consider the scalar multiplication function:

$$\begin{cases} \mathbb{R} \times TM \to TM \\ (t, v) \longmapsto tv \end{cases}$$

The derivative of this function with respect to the variable $\mathbb{R}$ at time $t = 1$ is a function $V : TM \to T^2M$, which is an alternative description of the canonical vector field.

The existence of such a vector field on TM is analogous to the canonical one-form on the cotangent bundle. Sometimes V is also called the **Liouville vector field**, or **radial vector field**. Using V one can characterize the tangent bundle. Essentially, V can be characterized using 4 axioms, and if a manifold has a vector field satisfying these axioms, then the manifold is a tangent bundle and the vector field is the canonical vector field on it. See for example, De León et al.

Lifts

There are various ways to lift objects on M into objects on TM. For example, if γ is a curve in M, then γ' (the tangent of γ) is a curve in TM. In contrast, without further assumptions on M (say, a Riemannian metric), there is no similar lift into the cotangent bundle.

The *vertical lift* of a function $f : M \to \mathbb{R}$ is the function $f^\vee : TM \to \mathbb{R}$ defined by $f^\vee = f \circ \pi$, where $\pi : TM \to M$ is the canonical projection.

See also

- Pushforward (differential)
- Unit tangent bundle
- Cotangent bundle
- Frame bundle
- Musical isomorphism

Notes

1. The disjoint union ensures that for any two points x_1 and x_2 of manifold M the tangent spaces T_1 and T_2 have no common vector. This is graphically illustrated in the accompanying picture for tangent bundle of circle S^1, see Examples section: all tangents to a circle lie in the plane of the circle. In order to make them disjoint it is necessary to align them in a plane perpendicular to the plane of the circle.

References

- Lee, Jeffrey M. (2009), *Manifolds and Differential Geometry*, Graduate Studies in Mathematics, vol. 107, Providence: American Mathematical Society. ISBN 978-0-8218-4815-9
- Lee, John M. (2012). *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Vol. 218. doi:10.1007/978-1-4419-9982-5 (https://doi.org/10.1007%2F978-1-4419-9982-5). ISBN 978-1-4419-9981-8.

- Jürgen Jost, *Riemannian Geometry and Geometric Analysis*, (2002) Springer-Verlag, Berlin. ISBN 3-540-42627-2
- Ralph Abraham and Jerrold E. Marsden, *Foundations of Mechanics*, (1978) Benjamin-Cummings, London. ISBN 0-8053-0102-X
- León, M. De; Merino, E.; Oubiña, J. A.; Salgado, M. (1994). "A characterization of tangent and stable tangent bundles" (http://archive.numdam.org/article/AIHPA_1994__61_1_1_0.pdf) (PDF). *Annales de l'I.H.P.: Physique Théorique*. **61** (1): 1–15.
- Gudmundsson, Sigmundur; Kappos, Elias (2002). "On the geometry of tangent bundles". *Expositiones Mathematicae*. **20**: 1–41. doi:10.1016/S0723-0869(02)80027-5 (https://doi.org/10.1016%2FS0723-0869%2802%2980027-5).

External links

- "Tangent bundle" (https://www.encyclopediaofmath.org/index.php?title=Tangent_bundle), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Wolfram MathWorld: Tangent Bundle (http://mathworld.wolfram.com/TangentBundle.html)
- PlanetMath: Tangent Bundle (https://planetmath.org/tangentbundle)