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Cotangent space

In differential geometry, the **cotangent space** is a vector space associated with a point x on a smooth (or differentiable) manifold $\mathcal{M}$; one can define a cotangent space for every point on a smooth manifold. Typically, the cotangent space, $T_x^*\mathcal{M}$ is defined as the dual space of the tangent space at x, $T_x\mathcal{M}$, although there are more direct definitions (see below). The elements of the cotangent space are called **cotangent vectors** or **tangent covectors**.

Properties

All cotangent spaces at points on a connected manifold have the same dimension, equal to the dimension of the manifold. All the cotangent spaces of a manifold can be "glued together" (i.e. unioned and endowed with a topology) to form a new differentiable manifold of twice the dimension, the cotangent bundle of the manifold.

The tangent space and the cotangent space at a point are both real vector spaces of the same dimension and therefore isomorphic to each other via many possible isomorphisms. The introduction of a Riemannian metric or a symplectic form gives rise to a natural isomorphism between the tangent space and the cotangent space at a point, associating to any tangent covector a canonical tangent vector.

Formal definitions

Definition as linear functionals

Let $\mathcal{M}$ be a smooth manifold and let x be a point in $\mathcal{M}$. Let $T_x\mathcal{M}$ be the tangent space at x. Then the cotangent space at *x* is defined as the dual space of $T_x\mathcal{M}$:

$$T_x^*\mathcal{M} = (T_x\mathcal{M})^*$$

Concretely, elements of the cotangent space are linear functionals on $T_x\mathcal{M}$. That is, every element $\alpha \in T_x^*\mathcal{M}$ is a linear map

$$\alpha : T_x\mathcal{M} \to F$$

where F is the underlying field of the vector space being considered, for example, the field of real numbers. The elements of $T_x^*\mathcal{M}$ are called cotangent vectors.

Alternative definition

In some cases, one might like to have a direct definition of the cotangent space without reference to the tangent space. Such a definition can be formulated in terms of equivalence classes of smooth functions on $\mathcal{M}$. Informally, we will say that two smooth functions *f* and *g* are equivalent at a point x if they have the same first-order behavior near x, analogous to their linear Taylor polynomials; two functions f and g have the same first order behavior near x if and only if the derivative of the function $f - g$ vanishes at x. The cotangent space will then consist of all the possible first-order behaviors of a function near x.

Let $\mathcal{M}$ be a smooth manifold and let x be a point in $\mathcal{M}$. Let I_x be the ideal of all functions in $C^\infty(\mathcal{M})$ vanishing at x, and let I_x^2 be the set of functions of the form $\sum_i f_i g_i$, where $f_i, g_i \in I_x$. Then I_x and I_x^2 are both real vector spaces and the cotangent space can be defined as the quotient space $T_x^*\mathcal{M} = I_x/I_x^2$ by showing that the two spaces are isomorphic to each other.

This formulation is analogous to the construction of the cotangent space to define the Zariski tangent space in algebraic geometry. The construction also generalizes to locally ringed spaces.

The differential of a function

Let M be a smooth manifold and let $f \in C^\infty(M)$ be a smooth function. The differential of f at a point x is the map

$$\mathrm{d}f_x(X_x) = X_x(f)$$

where X_x is a tangent vector at x, thought of as a derivation. That is $X(f) = \mathcal{L}_X f$ is the Lie derivative of f in the direction X, and one has $\mathrm{d}f(X) = X(f)$. Equivalently, we can think of tangent vectors as tangents to curves, and write

$$\mathrm{d}f_x(\gamma'(0)) = (f \circ \gamma)'(0)$$

In either case, $\mathrm{d}f_x$ is a linear map on T_xM and hence it is a tangent covector at x.

We can then define the differential map $\mathrm{d} : C^\infty(M) \to T_x^*(M)$ at a point x as the map which sends f to $\mathrm{d}f_x$. Properties of the differential map include:

1. d is a linear map: $\mathrm{d}(af + bg) = a\mathrm{d}f + b\mathrm{d}g$ for constants a and b,
2. $\mathrm{d}(fg)_x = f(x)\mathrm{d}g_x + g(x)\mathrm{d}f_x$

The differential map provides the link between the two alternate definitions of the cotangent space given above. Since for all $f \in I_x^2$ there exist $g_i, h_i \in I_x$ such that $f = \sum_i g_i h_i$, we have, $\mathrm{d}f_x = \sum_i \mathrm{d}(g_i h_i)_x = \sum_i (g_i(x)\mathrm{d}(h_i)_x + \mathrm{d}(g_i)_x h_i(x)) = \sum_i (0\mathrm{d}(h_i)_x + \mathrm{d}(g_i)_x 0) = 0$ i.e. All function in I_x^2 have differential zero, it follows that for every two functions $f \in I_x^2$, $g \in I_x$, we have $\mathrm{d}(f + g) = \mathrm{d}(g)$. We can now construct an isomorphism between $T_x^*\mathcal{M}$ and I_x/I_x^2 by sending linear maps α to the corresponding cosets $\alpha + I_x^2$. Since there is a unique linear map for a given kernel and slope, this is an isomorphism, establishing the equivalence of the two definitions.

The pullback of a smooth map

Just as every differentiable map $f : M \to N$ between manifolds induces a linear map (called the *pushforward* or *derivative*) between the tangent spaces

$$f_*\colon T_xM \to T_{f(x)}N$$

every such map induces a linear map (called the *pullback*) between the cotangent spaces, only this time in the reverse direction:

$$f^*\colon T^*_{f(x)}N \to T^*_xM.$$

The pullback is naturally defined as the dual (or transpose) of the pushforward. Unraveling the definition, this means the following:

$$(f^*\theta)(X_x) = \theta(f_*X_x),$$

where $\theta \in T^*_{f(x)}N$ and $X_x \in T_xM$. Note carefully where everything lives.

If we define tangent covectors in terms of equivalence classes of smooth maps vanishing at a point then the definition of the pullback is even more straightforward. Let g be a smooth function on N vanishing at $f(x)$. Then the pullback of the covector determined by g (denoted $\mathrm{d}g$) is given by

$$f^*\mathrm{d}g = \mathrm{d}(g \circ f).$$

That is, it is the equivalence class of functions on M vanishing at x determined by $g \circ f$.

Exterior powers

The k-th exterior power of the cotangent space, denoted $\Lambda^k(T^*_x\mathcal{M})$, is another important object in differential geometry. Vectors in the k-th exterior power, or more precisely sections of the k-th exterior power of the cotangent bundle, are called differential k-forms. They can be thought of as alternating, multilinear maps on k tangent vectors. For this reason, tangent covectors are frequently called *one-forms*.

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