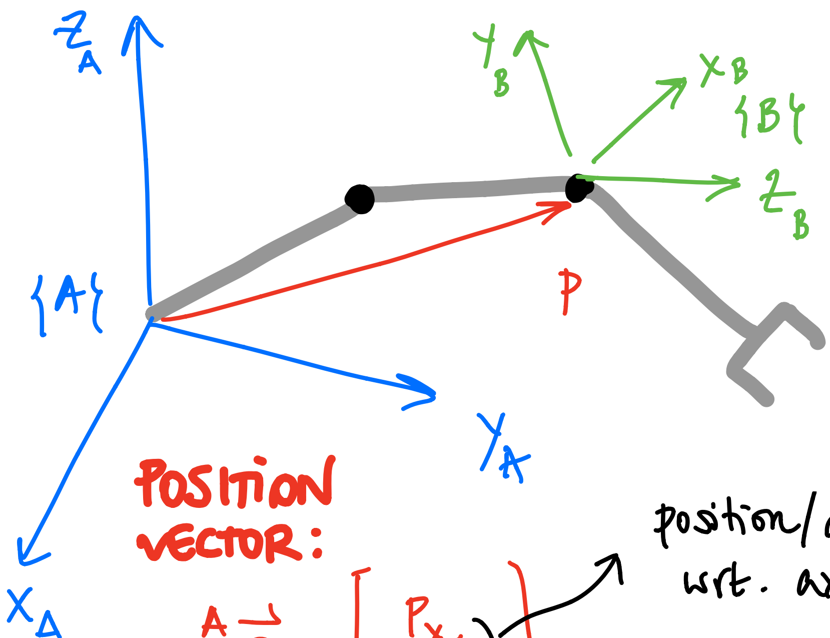
① Position vector



POSITION VECTOR:

$${}^{A}\vec{P} = \underbrace{\begin{bmatrix} P_{x_A} \\ P_{y_A} \\ P_{z_A} \end{bmatrix}}_{3\times1}$$

coordinate system

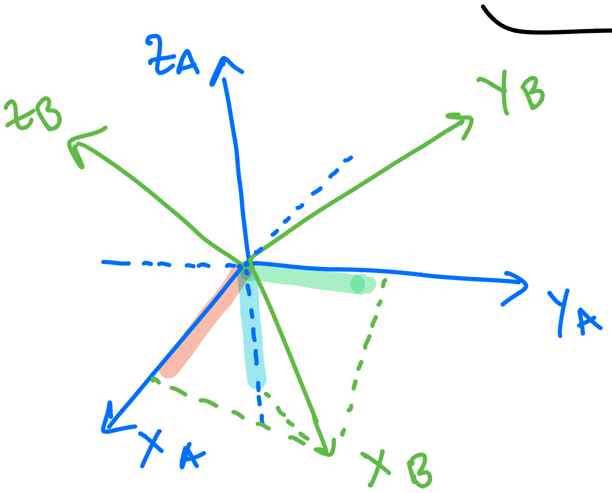
point P

position/distance wrt. axis X_A of coordinate system A

② Orientation Matrix : description of an orthogonal frame wrt. another (reference) frame

Orientation of B relative to A

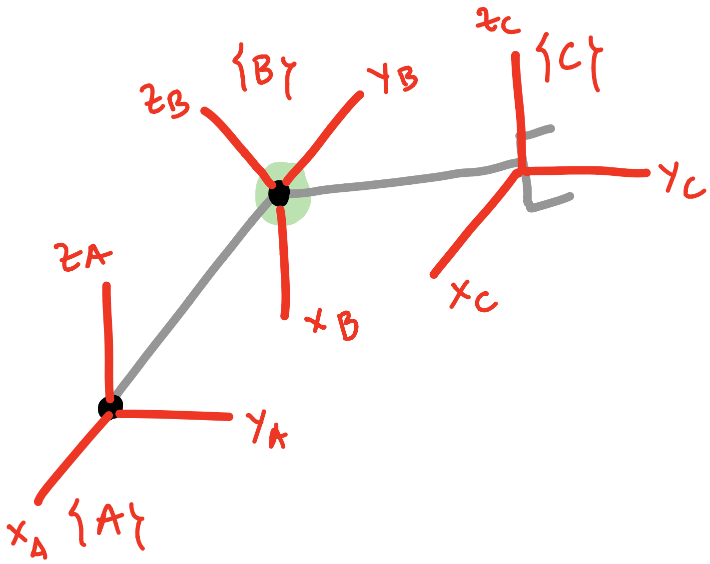
$${}^{A}\underset{\sim}{R}_{B} = \overbrace{{}^{A}_{B}\underset{\sim}{R}}^{\text{Craig notation}} = \left[{}^{A}\vec{X}_{B} \;\vdots\; {}^{A}\vec{Y}_{B} \;\vdots\; {}^{A}\vec{Z}_{B} \right] = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{3\times3}$$



$${}^{A}\vec{X}_{B} = \begin{array}{c} X_B \\ \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \end{array} \begin{array}{l} \\ X_A \\ Y_A \\ Z_A \end{array}$$

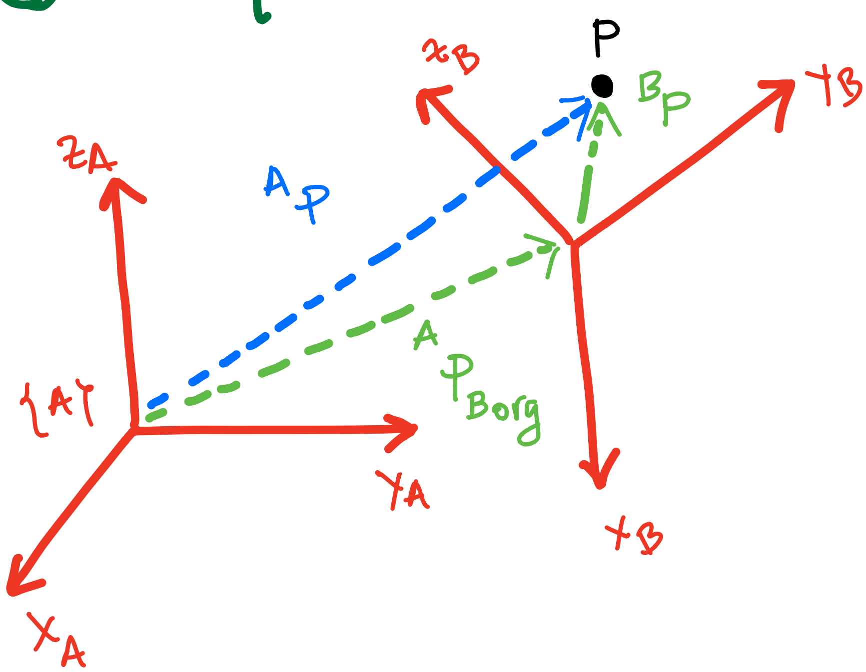
${}^{A}\vec{X}_B$ is the projection of X_B on the basis of $\{A\}$, or
the vector X_B from $\{B\}$ expressed in $\{A\}$

③ Frames = Coordinate systems



$$\underbrace{\{B\}}_{\text{frame B described relative to A}} = \left\{ \underbrace{{}^{A}\underset{\sim}{R}_{B}}_{\text{orientation of frame B relative to A}} ,\ \underbrace{{}^{A}\vec{P}_{Borg}}_{\text{origin position vector of frame B relative to A}} \right\}$$

④ Transformations



Given a position vector or point P expressed in B, we want to express it in A. That is an homogeneous transformation, which comprises a combination of translation & rotation

$${}^{A}\vec{P} = \underset{\sim}{H}{}^{A}_{B} \cdot {}^{B}\vec{P}$$

Mind the notation trick!

$$\underset{\sim}{H}{}^{A}_{B} = \left[\begin{array}{c|c} \underset{\sim}{R}{}^{A}_{B} \; {\scriptstyle 3\times3} & {}^{A}\vec{P}_{Borg} \; {\scriptstyle 3\times1} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right]_{4\times4}$$

We append '1' to $\vec{P}$ vectors so that sizes match

Note that the operation can be developed as follows:

$${}^{A}\vec{P} = \underset{\sim}{H}{}^{A}_{B} \cdot {}^{B}\vec{P} = \underbrace{\underset{\sim}{R}{}^{A}_{B} \cdot {}^{B}\vec{P}}_{①} + \underbrace{{}^{A}\vec{P}_{Borg}}_{②}$$

① First, we rotate ${}^{B}P$ so that the axes of {A} & {B} become parallel

② Then, we translate the origin of the old frame B to A

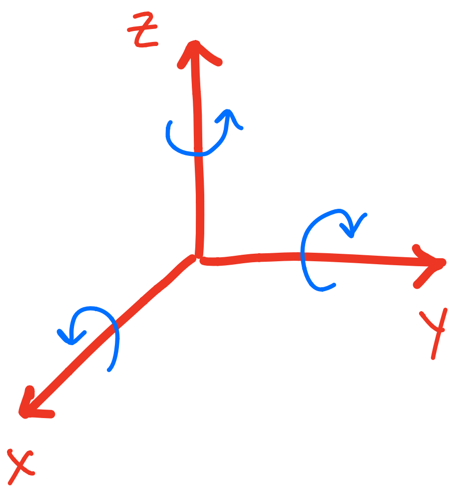
⑤ Rotation Matrices

A rotation matrix can be computed as a chain of rotations around axes X, Y, Z. Note that:

- The convention of clock-wise / counter clock-wise for rotation direction is important
- The order in which we apply axis rotations is important:

$$X \to Y \to Z \neq X \to Z \to Y$$

- The basic counter clock-wise rotation matrices are:



Usually we rotate around the axes of the old frame, i.e., B, to go to the new one, i.e. A

$$R_x(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\vartheta & -s\vartheta \\ 0 & s\vartheta & c\vartheta \end{bmatrix}$$

$$R_y(\vartheta) = \begin{bmatrix} c\vartheta & 0 & s\vartheta \\ 0 & 1 & 0 \\ -s\vartheta & 0 & c\vartheta \end{bmatrix}$$

$$R_z(\vartheta) = \begin{bmatrix} c\vartheta & -s\vartheta & 0 \\ s\vartheta & c\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Two fundamental properties of rotation matrices:

1) $\underset{\sim}{R}^{-1} = \underset{\sim}{R}^T$: the inverse of a rot matrix is its transpose

2) $\det(\underset{\sim}{R}) = 1$: determinant is 1

⑥ Compositions of Transformations

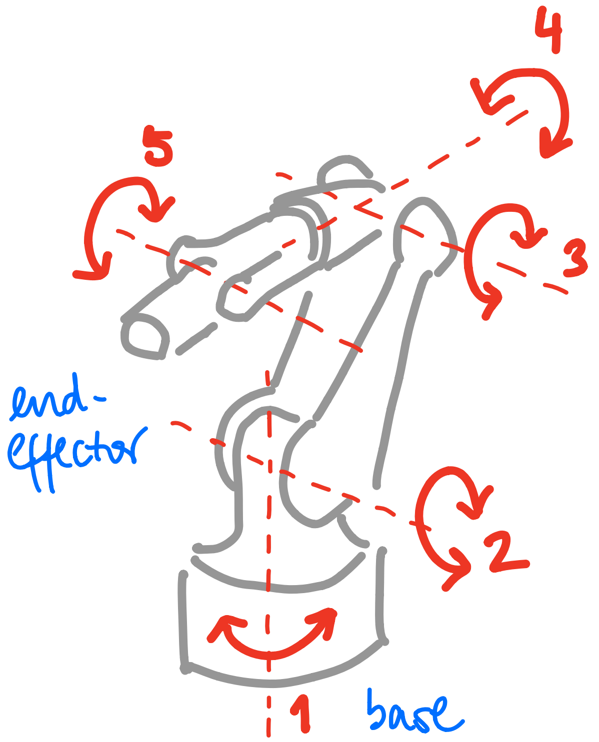
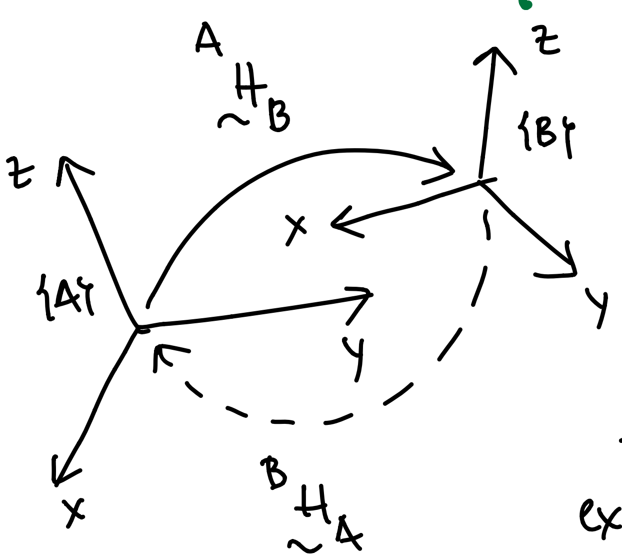


ABB robot with 5 joints

- We atach a frame to each joint
- The transformations between joints can be concatenated to get the transformation from the base frame to the endeffector

$$ {}^{5}\underset{\sim 1}{H} = {}^{5}\underset{\sim 4}{H} \cdot {}^{4}\underset{\sim 3}{H} \cdot {}^{3}\underset{\sim 2}{H} \cdot {}^{2}\underset{\sim 1}{H} $$

⑦ Inverse Transformation Matrix



Given ${}^{A}\underset{\sim B}{H}$, which is ${}^{B}\underset{\sim A}{H}$?

$$ \boxed{ {}^{B}\underset{\sim A}{H} = \left({}^{A}\underset{\sim B}{H} \right)^{-1} } $$

frame A expressed in B, or looked/described from B

inverse of frame B described from A

We can use Gaussian elimination for the inverse computation, or we can use this formula (only for 4x4):

$$\begin{array}{cc} & \begin{array}{cccc} \quad \underset{\sim}{R} & & & \vec{d} \end{array} \\ {}^A\underset{\sim}{H}_B = & \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

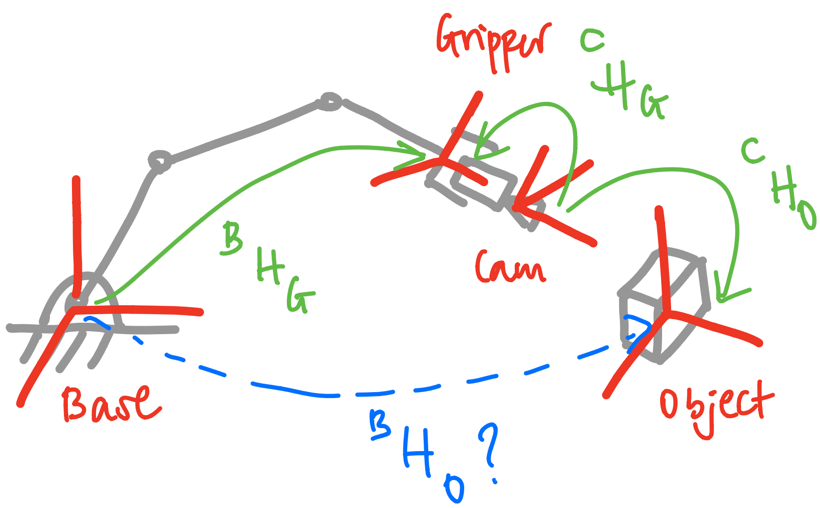
$\downarrow$ inv

transpose : R^T

$-\underset{\sim}{R}^T \cdot \vec{d}$

$${}^B\underset{\sim}{H}_A = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & -(d_1 a_1 + d_2 a_2 + d_3 a_3) \\ b_1 & b_2 & b_3 & -(d_1 b_1 + d_2 b_2 + d_3 b_3) \\ c_1 & c_2 & c_3 & -(d_1 c_1 + d_2 c_2 + d_3 c_3) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{R}^T & -\underset{\sim}{R}^T\vec{d} \\ \hline 0\,0\,0 & 1 \end{array}\right]$$

⑧ Example : Compositions



Given

- The frame of the gripper in the base: ${}^B H_G$
- The gripper in the camera: ${}^C H_G$
- The object in the camera: ${}^C H_O$

Which is the object in the base?

$${}^B\underset{\sim}{H}_O = {}^B H_G \left({}^C H_G\right)^{-1} \cdot {}^C H_O$$

Important notes:

- Although we read from left to right, the transformations go from right to left!
- Note the convention tricks
  - The bottom rightmost and the upper leftmost yield the resulting transformation
  - All in-between frames must match in a chain

$$ {}^{5}H_{1} = {}^{5}H_{4}\,{}^{4}H_{3}\,{}^{3}H_{2}\,{}^{2}H_{1} $$