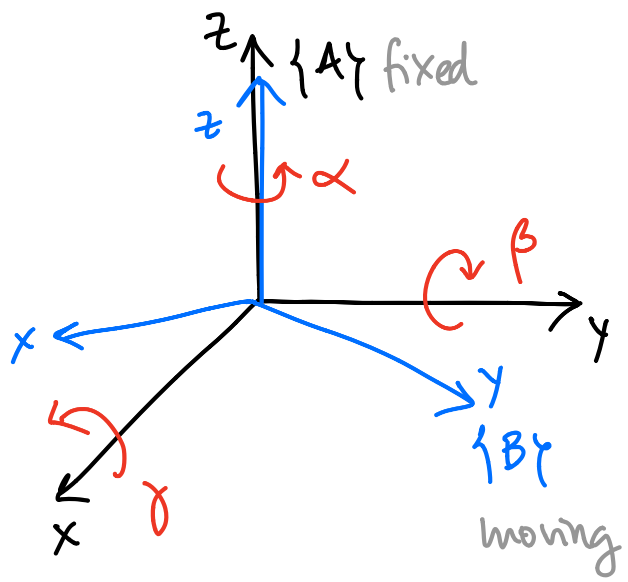
① Fixed Angle Representation



- When using a fixed angle representation, each rotation takes place about a fixed (not moving) reference frame
- We rotate a frame {B} using the axes of a fixed frame {A}

$$ {}^{A}_{B}\underset{\sim}{R}_{xyz}(\gamma, \beta, \alpha) = \underbrace{R_z(\alpha)} R_y(\beta) R_x(\gamma) $$

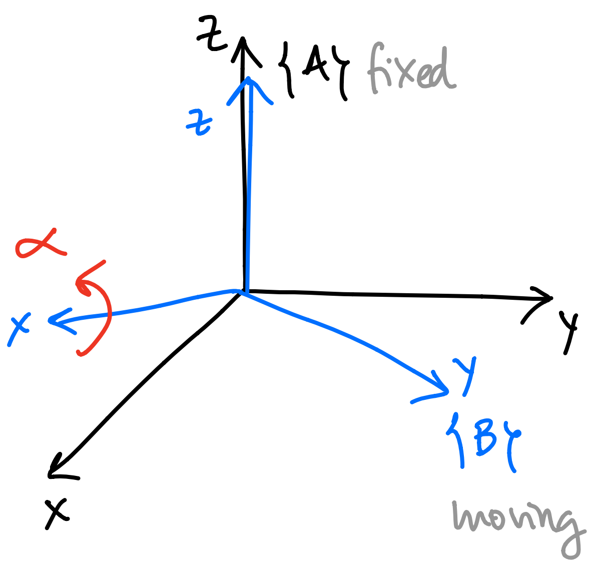
x, y, z

we apply backwards

Note that each of the elementary $R_x(\theta)$, $R_y(\theta)$, $R_z(\theta)$ are the same for fixed/moving

rotation of α around axis z of fixed frame {A}

② Euler Angle Representation (Moving)



- With this representation, each rotation takes place about the axis of the moving system

$$ {}^{A}_{B}R_{x'y'z'}(\gamma, \beta, \alpha) = R_x(\alpha)\, R_y(\beta)\, R_z(\gamma) $$

x', y', z'

same order

x': Euler angle representation, moving axes

Note that each of the elementary $R_x(\theta), R_y(\theta), R_z(\theta)$ are the same for fixed/moving

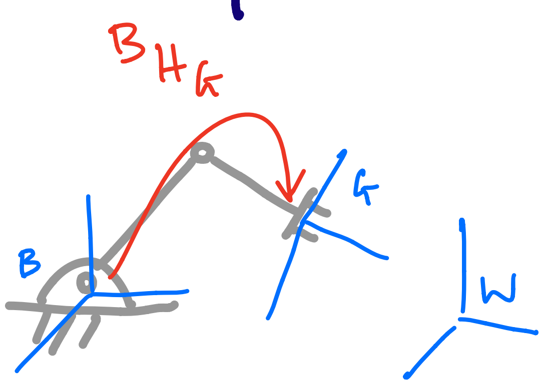
③ Comparing rotation representations

All representations yield the same rotation. We need 3 values for describing the orientation, but the order in which they're applied is relevant – that's why we have rotation representation conventions.

There is a set of 24 angle representation conventions, depending on:

- fixed / moving (Euler)
- order: XYZ, ZYZ, ZYX ...

④ Example: Find Euler angles



$$ {}^{W}H_G = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad {}^{W}H_B = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Euler angles Z-X-Z of ${}^{B}H_G$?

$$
{}^{B}H_{G} = \left({}^{W}H_{B}\right)^{-1} \cdot {}^{W}H_{G} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\left[\begin{array}{c|c} R^{T} & -R^{T}\cdot d \\ \hline 0\,0\,0 & 1 \end{array}\right]} \cdot \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \left[\begin{array}{ccc|c} 0 & 0 & -1 & 8 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

The order used here is not relevant

$$
R_{z'x'z'}(\alpha, \beta, \gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma) =
$$

Euler = moving

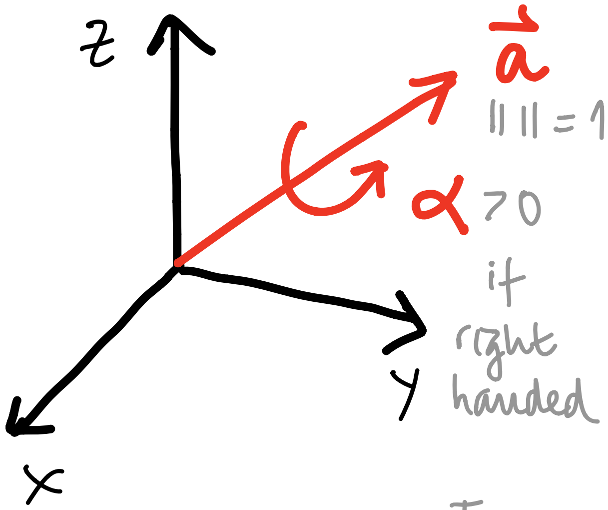
Same order zyz

$$
= \underbrace{\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{bmatrix}}_{R_y} \underbrace{\begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z} =
$$

$$
= \dots
$$

By comparing the developed matrix with the values, we can isolate/solve for α, β, γ.

⑤ Angle-Axis Representation



$\vec{a} = (a_x, a_y, a_z)^T,$
α

→ 4 variables

It is possible to represent a rotation/orientation with an axis $\vec{a}$ and an angle α around it.

Interpretation: we attach $\vec{a}$ to a frame/object and rotate α around it.

$$\underset{\sim}{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \rightleftarrows \begin{cases} \boxed{\alpha = \operatorname{acos}\left(\dfrac{r_{11}+r_{22}+r_{33}-1}{2}\right)} \\ \boxed{\vec{a} = \dfrac{1}{2\sin\alpha} \begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}} \end{cases}$$

the inverse transformation is also possible, look at the Wikipedia!

Note that it is undefined for $\alpha = 0, 180$.

6) Unit Quaternions = Euler parameters

Unit quaternions are 4D constructs that describe rotations as a vector:

$$\vec{q} = (q_1, q_2, q_3, q_4)^T, \quad \|\vec{q}\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = 1$$

They are related to the angle-axis representation:

$$\vec{a} = (a_x, a_y, a_z)^T, \quad \alpha$$

$$q_1 = a_x \cdot \sin(\alpha/2)$$

$$q_2 = a_y \cdot \sin(\alpha/2)$$

$$q_3 = a_z \cdot \sin(\alpha/2)$$

$$q_4 = \cos(\alpha/2)$$

7) Rodrigues' Formula

A rotation in 3D space belongs to the rotation group of all possible rotations in Euclidean space $\mathbb{R}^3$, denoted as $SO(3)$. Exponential maps can be used to compute rotations in that group. The Rodrigues' formula shows how to rotate a vector around an axis without computing its exponential map.

Thus, it can be used to rotate the 3 axes of of frame without computing the rotation matrix!

Given

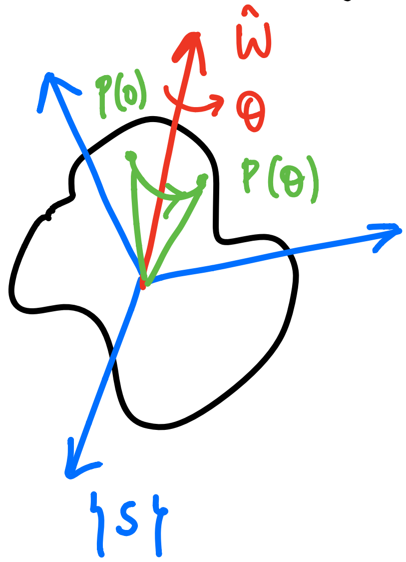
- A vector $\vec{v} \in \mathbb{R}^3$
- A unit vector $\vec{k}$ of rotation
- An angle θ for rotation ($\theta > 0$ right handed)

The rotated vector $\vec{v}_{rot}$ is:

$$\boxed{\vec{v}_{rot} = \vec{v} \cdot \cos\theta + (\vec{k} \times \vec{v}) \cdot \sin\theta + \vec{k} \cdot (\vec{k} \cdot \vec{v})(1 - \cos\theta)}$$

8 Exponential Representation of rotations

This representation is related to the axis-angle representation and the Rodrigues formula. In fact, probably the intuition of the angle-axis representation is the basis.



- We consider a point p described in a fixed frame $\{s\}$
- We define an axis $\hat{\omega}$ and an angle θ; we rotate p about $\hat{\omega}$ an angle θ

– The derivative of P is

$$\dot{P} = \underbrace{\hat{\omega} \times P}_{\text{cross prod.}} = \underbrace{[\hat{\omega}]P}_{\text{matrix}} \quad , \quad \boxed{[\hat{\omega}] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}}$$

skew-symmetric matrix operator: [ ]

$$\hat{\omega} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

– We notice that is a differential equation for which the solution is given by the exponential:

$$\dot{x}(t) = a\,x(t) \longrightarrow x(t) = e^{at} \cdot x(0)$$

– Thus, the matrix exponential is defined and applied for describing rotations:

$$\dot{p}(t) = [\hat{\omega}]\,P(t) \longrightarrow P(t) = e^{[\hat{\omega}]t}\,P(0)$$

$$\longrightarrow \boxed{\begin{aligned} P(\theta) &= e^{[\hat{\omega}]\theta}\,P(0) \\ Rot(\hat{\omega},\theta) &= e^{[\hat{\omega}]\theta} \end{aligned}}$$

rotation of θ about $\hat{\omega}$

– The Rodrigues formula is derived from that idea, making use of the Taylor expansion of the exponential: rearranging terms, the exponential is decomposed in sin & cos.

- Terminology:
  - $\hat{\omega}\theta \in \mathbb{R}^3$ represents the exponential coordinates of a rotation matrix $\underset{\sim}{R} \in SO(3)$

    $\underset{\sim}{R}$: 3x3

    $SO(3)$: Lie group which contains all rotations

  - $[\hat{\omega}]\theta \in so(3)$

    $so(3)$: Lie algebra (small letters, not caps)

    is the matrix logarithm of the rotation $\underset{\sim}{R}$ and it represents the angular velocity. If we integrate it for a unit of time it rotates a frame from $\underset{\sim}{I}$ to $\underset{\sim}{R}$.

  - the homogeneous transformation matrix: $\underset{\sim}{H}_{4\times4} \in SE(3)$

    $SE(3)$: special Euclidean group