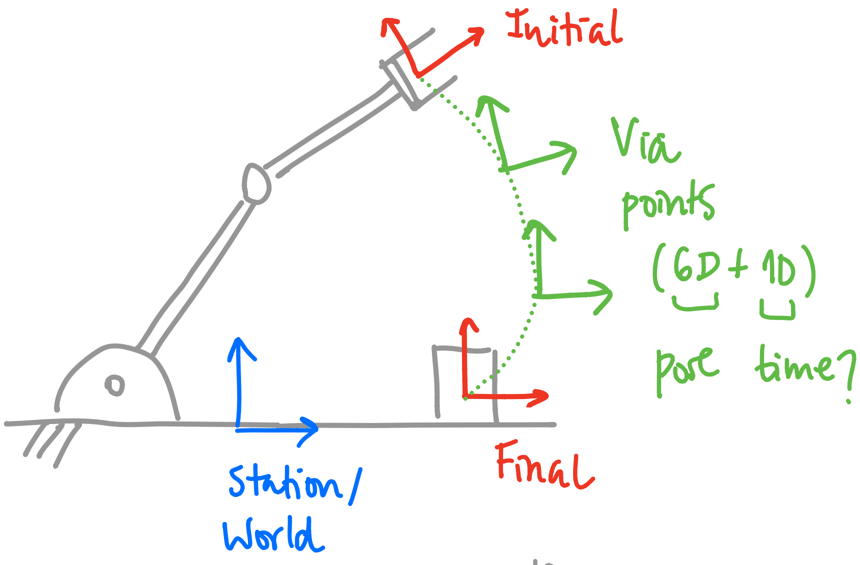


We want to compute the trajectories that join some initial and final poses (often the tool defined wrt a world frame), which additionally go smoothly through some via points

Continuous function + continuous first derivative

The complete trajectory is often called path, and to schemes are used for computing it: joint-scheme & Cartesian-scheme.

1 Path Generation with Joint-Schemes

Joints schemes are the easiest to compute and have the least issues: we set via points expressed in joint angles and these values are interpolated, usually with cubics. Singularities do not matter.

$$\vec{\theta}(t) \begin{cases} \vec{\theta}(t=0) = \vec{\theta}_0 = (\theta_{01}, \theta_{02}, \theta_{03}, \theta_{04}, \theta_{05}, \theta_{06})^T & \text{START} \\ \vec{\theta}(t=t_f) = \vec{\theta}_f = (\theta_{f1}, \dots, \theta_{f6})^T & \text{END / FINAL} \end{cases}$$

We treat each joint angle independently, and start defining function constraints so that we have a smooth interpolation.

$$\boxed{\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3}$$

We want to obtain the value of the coefficients a_i

Conditions:

$\theta(t=0)=\theta_0$

$\theta(t=t_f)=\theta_f$

$\dot\theta(t=0)=0$

$\dot\theta(t=t_f)=0$

smooth start and end

Derivatives:

$\dot\theta(t) = a_1 + 2a_2 t + 3a_3 t^2$

$\ddot\theta(t) = 2a_2 + 6a_3 t$

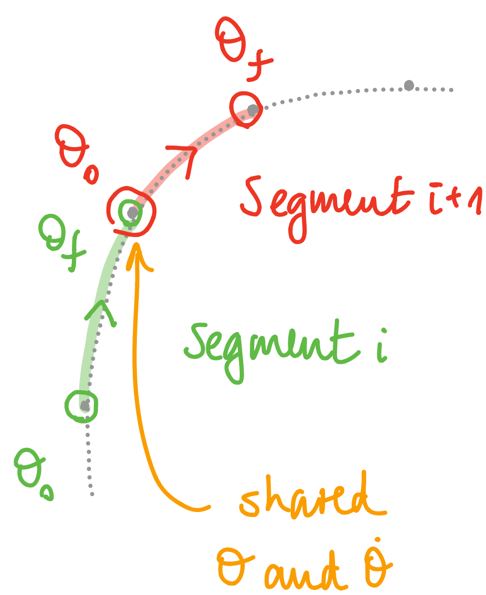
Coefficients after replacing the conditions in the $\theta, \dot\theta, \ddot\theta$ functions:

$$\boxed{\begin{aligned} &a_0 = \theta_0 , \quad a_1 = 0 \\ &a_2 = \frac{3}{2t_f^2}(\theta_f - \theta_0) \quad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) \end{aligned}}$$

Cubic Polynomials of Joint Values with Via Points

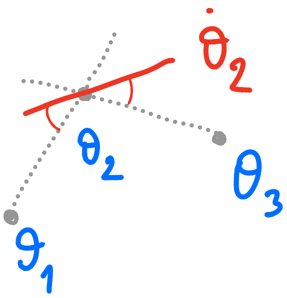
We apply the same equations as before for segments constituted by the paths between via points, but now we have a non-zero $\dot\theta_0$ & $\dot\theta_f$; therefore:

$$a_0 = \theta_0$$
$$a_1 = \dot\theta_0$$
$$a_2 = \frac{3}{2t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot\theta_0 - \frac{1}{t_f}\dot\theta_f$$
$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot\theta_f - \dot\theta_0)$$



We have 2 options to choose $\dot{\theta}_0$ and $\dot{\theta}_f$:

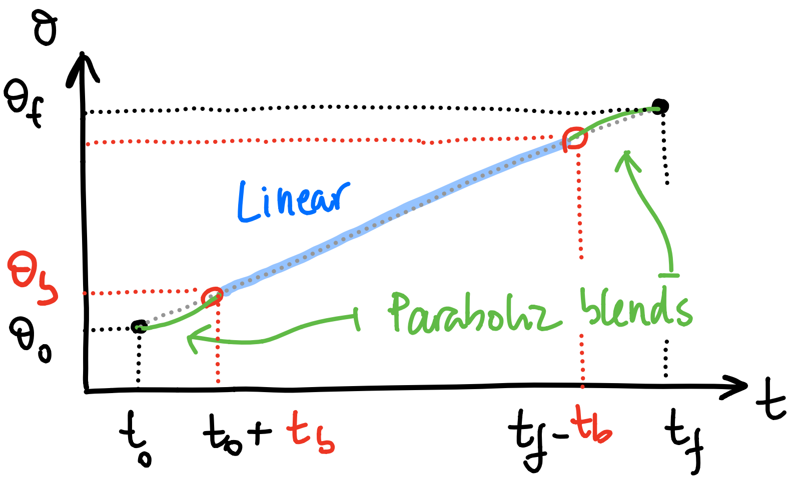
1) The user specifies the Cartesian velocities and we convert them to Joint space using the Jacobian



2) A heuristic automatically selects them either to
   - ensure a sensible tangent between via point lines
   - ensure a continuous acceleration at the via points

     a 5th order polynomial is required for that

Linear Functions with Parabolic Blends



We can similarly define θ interpolations which are linear except for at small regions at the start (0) and end (f) points, where a parabolic blend is forced: $\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta}\, t_b^2$

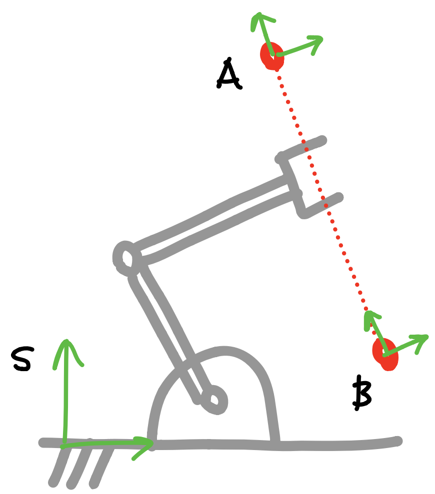
Via points can be considered, too.

② Path Generation with Cartesian-Schemes

Using a joint-scheme, the endeffector won't follow a known trajectory except for the via points. If the in-between trajectory is relevant, we can use Cartesian interpolation: we interpolate the path in cartesian space and then apply the inverse kinematics on all its points. That's:

- computationally more expensive
- riskier, because we might fall into singularities

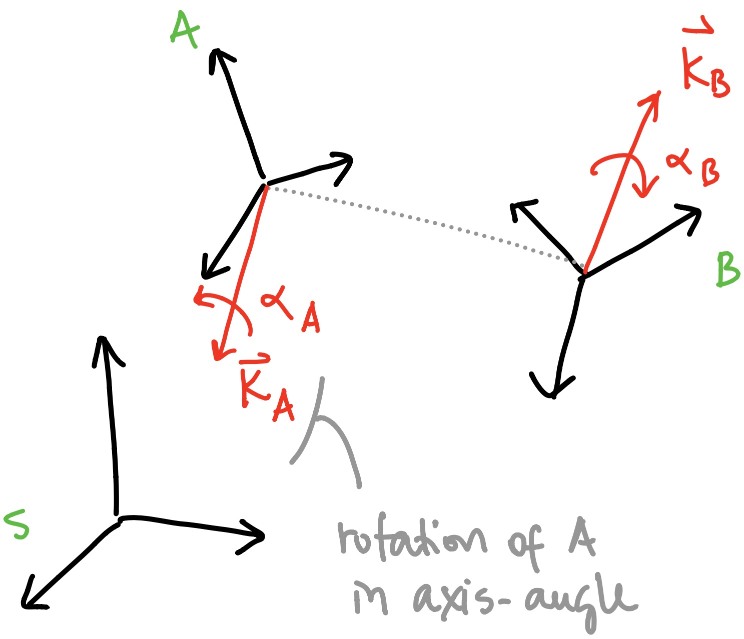
Cartesian Straight-Line Motion



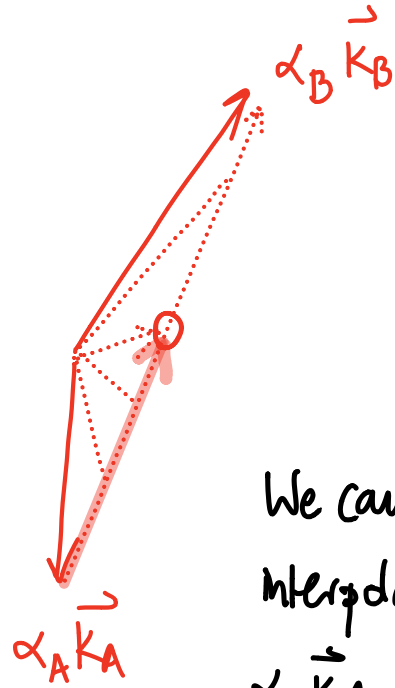
We want to interpolate a line in Cartesian space without cheating. Given $\underset{\sim}{H}$:

- all three components of the position can be interpolated linearly
- but not rotation components!
  - ↳ for rotation, we use the angle-axis representation

$$
{}^S\underset{\sim}{H}_A = \begin{bmatrix} {}^S\underset{\sim}{R}_A & {}^S\vec{P}_{AORG} \\ 0\;0\;0 & 1 \end{bmatrix} \longrightarrow \underbrace{{}^S\vec{X}_A}_{\text{generalized position with axis-angle}} = \underbrace{\Big[\, \underbrace{{}^S\vec{P}_{AORG}^{\,T}}_{x,y,z \text{ coords of A origin wrt S}} ,\ \overbrace{\underbrace{\alpha}_{\text{angle}} \cdot \underbrace{\vec{K}^{T}}_{\text{axis}}}^{{}^S\underset{\sim}{R}_A} \,\Big]^T}_{6\times 1}
$$



If we go from A to B in a straight line rotations can change also linearly



We can linearly interpolate from $\alpha_A \vec{K}_A$ to $\alpha_B \vec{K}_B$, as with positions

If we use the generalized position vector $\vec{X} = (\vec{P}, \alpha \vec{K})$ with the axis-angle representation, we can interpolate as in the joint-scheme! However, if we use blends, the blend time for all degrees of freedom must be the same!

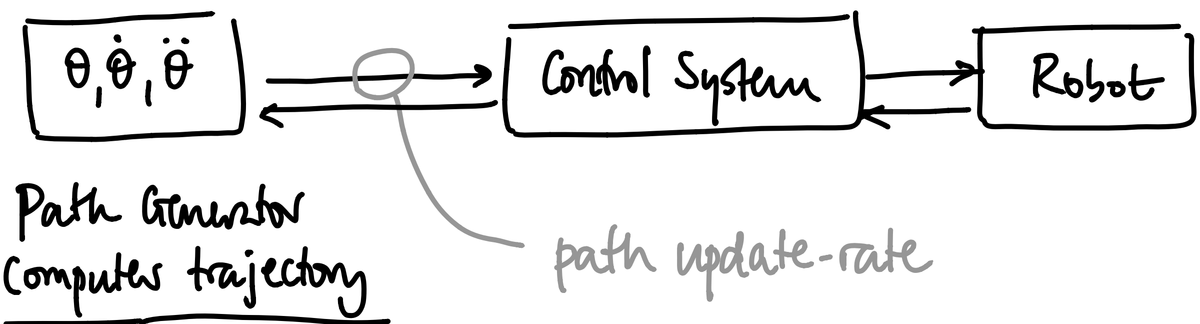
Problems with Cartesian Paths

1) Intermediate points might be unreachable, although start & end/final points are reachable.
2) We might approach to singularities, which lead to high joint rates.

3) start and goal/final points might be reachable in several configurations.

Therefore, the standard way to go is to use a joint-scheme for path generation — start, goal and via points can be defined in Cartesian space, but these are converted to joint space with the inv. kin. and then joint values interpolated.

③ Path Generation at Run Time



↳ The formulas/approaches explained in Section 1 are used. If cartesian schemes followed, Section 2 is used. However, the inverse kinematics converts each Cartesian pose to joint angle values. Usually, θ is differentiated to obtain $\dot{\theta}$ and $\ddot{\theta}$ instead of using the Jacobian ...