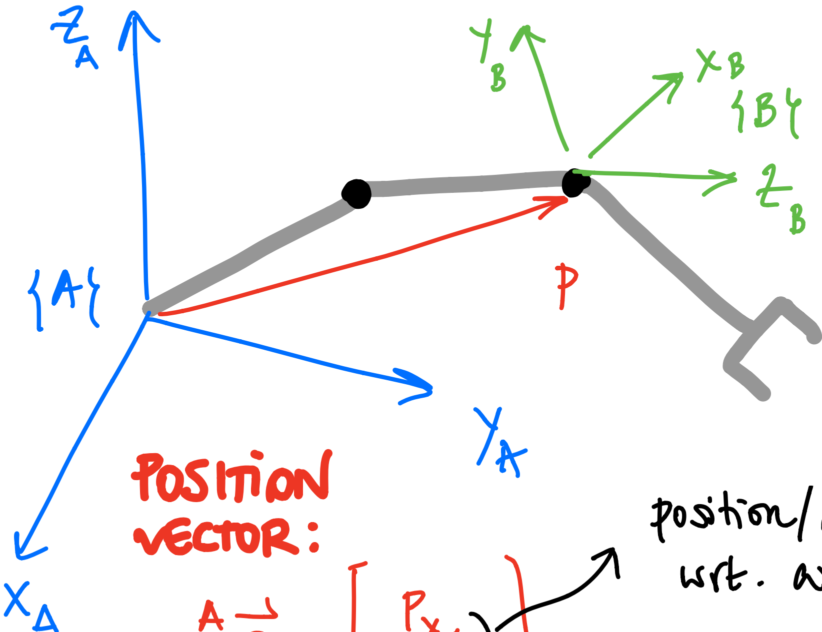
① Position vector



POSITION VECTOR:

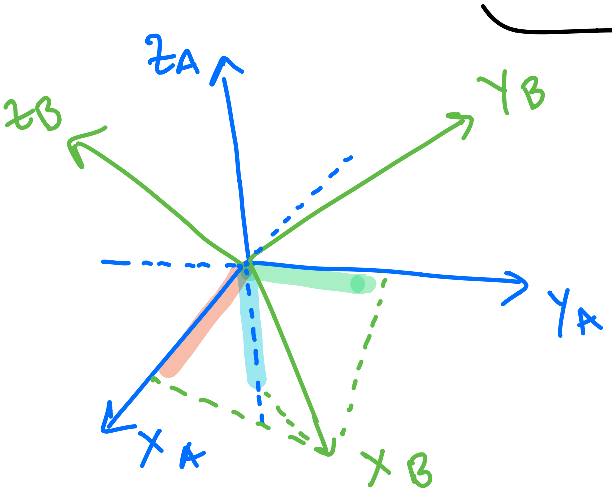
$$\underbrace{{}^{A}}_{\text{coordinate system}}\underbrace{\vec{P}}_{\text{point P}} = \underbrace{\begin{bmatrix} P_{X_A} \\ \hdashline P_{Y_A} \\ \hdashline P_{Z_A} \end{bmatrix}}_{3\times 1}$$

P_{X_A}: position/distance wrt. axis X_A of coordinate system A

② Orientation Matrix : description of an orthogonal frame wrt. another (reference) frame

Orientation of B relative to A

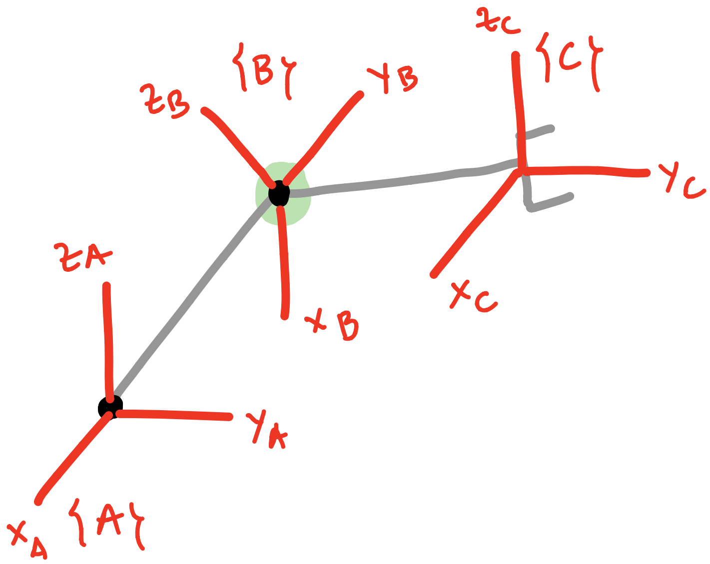
$${}^{A}\underset{\sim}{R}_{B} = \underbrace{{}^{A}_{B}\underset{\sim}{R}}_{\text{Craig notation}} = \left[{}^{A}\vec{X}_B \;\vdots\; {}^{A}\vec{Y}_B \;\vdots\; {}^{A}\vec{Z}_B \right] = \underbrace{\left[\begin{array}{c:c:c} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array}\right]}_{3\times 3}$$



$$\begin{array}{cc} & X_B \\ {}^{A}\vec{X}_B = & \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \end{array} \begin{array}{l} \\ X_A \\ Y_A \\ Z_A \end{array}$$

${}^{A}\vec{X}_B$ is { the projection of X_B on the basis of {A}, or
the vector X_B from {B} expressed in {A}

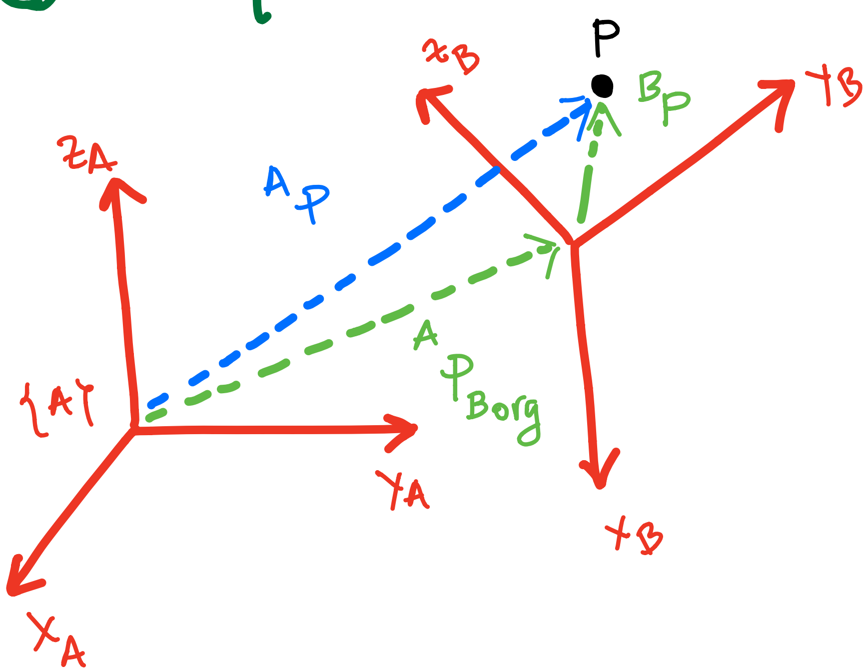
③ Frames = Coordinate systems



$$\{B\} = \{\, {}^{A}\underset{\sim}{R}_{B} ,\ {}^{A}\vec{P}_{Borg} \,\}$$

- $\{B\}$: frame B described relative to A
- ${}^{A}\underset{\sim}{R}_{B}$: orientation of frame B relative to A
- ${}^{A}\vec{P}_{Borg}$: origin position vector of frame B relative to A

④ Transformations



Given a position vector or point P expressed in B, we want to express it in A. That is an homogeneous transformation, which comprises a combination of translation & rotation

$$ {}^{A}\vec{P} = {}^{A}\underset{\sim}{H}_{B} \cdot {}^{B}\vec{P} $$

Mind the notation trick!

$$ {}^{A}\underset{\sim}{H}_{B} = \left[\begin{array}{ccc:c} & {}^{A}\underset{\sim}{R}_{B} & & {}^{A}\vec{P}_{B_{org}} \\ & 3\times3 & & 3\times1 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]_{4\times4} $$

We append '1' to $\vec{P}$ vectors so that sizes match

Note that the operation can be developed as follows:

$$ {}^{A}\vec{P} = {}^{A}\underset{\sim}{H}_{B} \cdot {}^{B}\vec{P} = \underbrace{{}^{A}\underset{\sim}{R}_{B} \cdot {}^{B}\vec{P}}_{} + \underbrace{{}^{A}\vec{P}_{B_{org}}}_{} $$

① First, we rotate ${}^{B}P$ so that the axes of {A} & {B} become parallel

② Then, we translate the origin of the old frame B to A

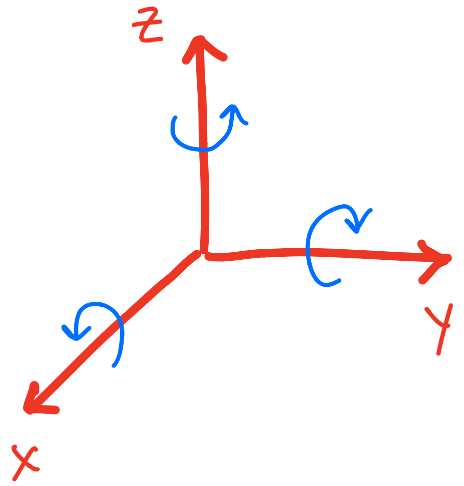
⑤ Rotation Matrices

A rotation matrix can be computed as a chain of rotations around axes X, Y, Z. Note that:

- The convention of clock-wise / counter clock-wise for rotation direction is important
- The order in which we apply axis rotations is important:

$$ X \to Y \to Z \neq X \to Z \to Y $$

- The basic counter clock-wise rotation matrices are:



Usually we rotate around the axes of the old frame, i.e., B, to go to the new one, i.e. A

$$R_x(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\vartheta & -s\vartheta \\ 0 & s\vartheta & c\vartheta \end{bmatrix}$$

$$R_y(\vartheta) = \begin{bmatrix} c\vartheta & 0 & s\vartheta \\ 0 & 1 & 0 \\ -s\vartheta & 0 & c\vartheta \end{bmatrix}$$

$$R_z(\vartheta) = \begin{bmatrix} c\vartheta & -s\vartheta & 0 \\ s\vartheta & c\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Two fundamental properties of rotation matrices:

1) $\underset{\sim}{R}^{-1} = \underset{\sim}{R}^{T}$: the inverse of a rot matrix is its transpose

2) $\det(\underset{\sim}{R}) = 1$: determinant is 1

⑥ Compositions of Transformations

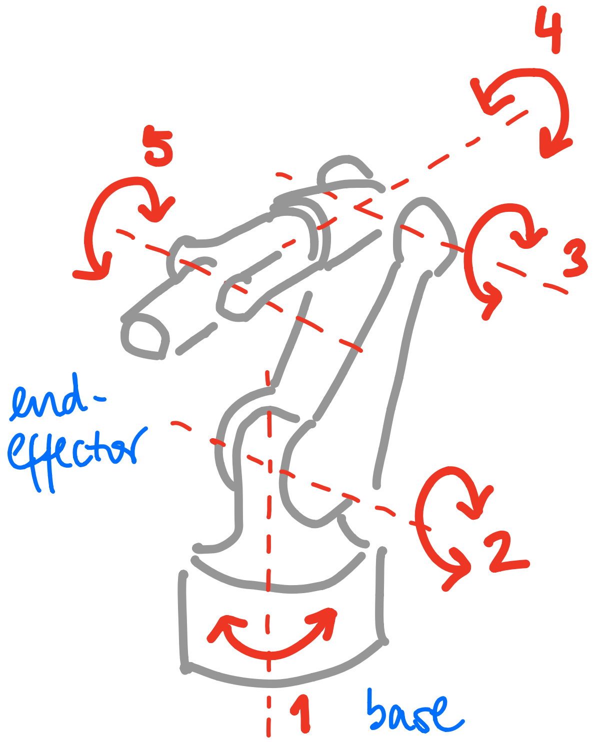
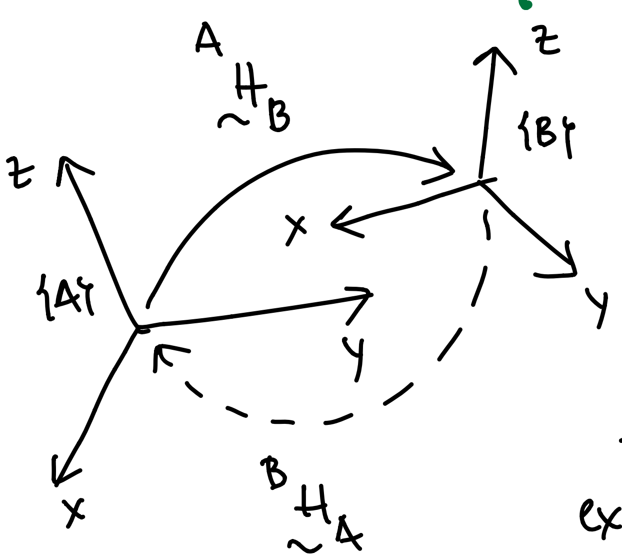


ABB robot with 5 joints

- We atach a frame to each joint
- The transformations between joints can be concatenated to get the transformation from the base frame to the endeffector

$${}^{5}\underset{\sim}{H}_{1} = {}^{5}\underset{\sim}{H}_{4} \cdot {}^{4}\underset{\sim}{H}_{3} \cdot {}^{3}\underset{\sim}{H}_{2} \cdot {}^{2}\underset{\sim}{H}_{1}$$

⑦ Inverse Transformation Matrix



Given ${}^{A}\underset{\sim}{H}_{B}$, which is ${}^{B}\underset{\sim}{H}_{A}$?

$$\boxed{\underbrace{{}^{B}\underset{\sim}{H}_{A}}_{\text{frame A expressed in B, or looked / described from B}} = \underbrace{\left({}^{A}\underset{\sim}{H}_{B}\right)^{-1}}_{\text{inverse of frame B described from A}}}$$

We can use Gaussian elimination for the inverse computation, or we can use this formula (only for 4x4):

$$
{}^{A}\underset{\sim}{H}_{B} = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \underset{\sim}{R} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
$$

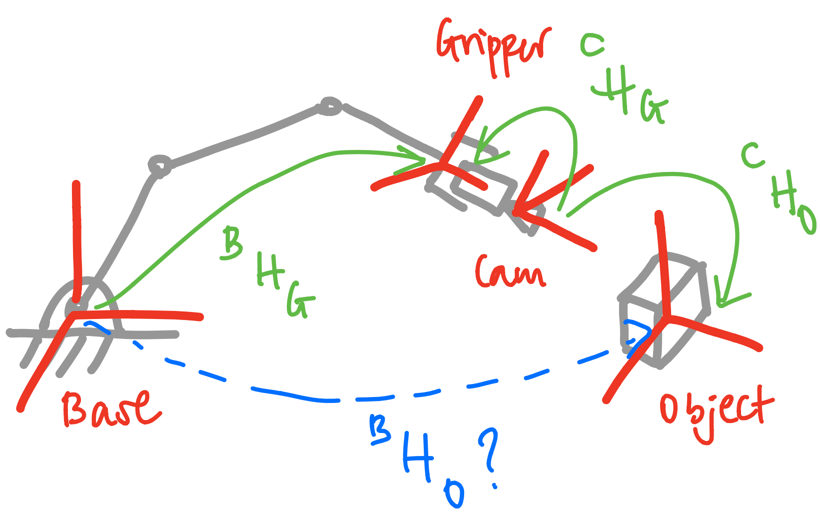
$\downarrow$ inv

transpose : R^T

$\boxed{-\underset{\sim}{R}\cdot\vec{d}}$

$$
{}^{B}\underset{\sim}{H}_{A} = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & -(d_1a_1 + d_2a_2 + d_3a_3) \\ b_1 & b_2 & b_3 & -(d_1b_1 + d_2b_2 + d_3b_3) \\ c_1 & c_2 & c_3 & -(d_1c_1 + d_2c_2 + d_3c_3) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

⑧ Example : Compositions



Given

- The frame of the gripper in the base: ${}^{B}H_{G}$
- The gripper in the camera: ${}^{C}H_{G}$
- The object in the camera: ${}^{C}H_{O}$

Which is the object in the base?

$$
{}^{B}\underset{\sim}{H}_{O} = {}^{B}H_{G} \, ({}^{C}H_{G})^{-1} \cdot {}^{C}H_{O}
$$

Important notes:

- Although we read from left to right, the transformations go from right to left!
- Note the convention tricks
  - The bottom rightmost and the upper leftmost yield the resulting transformation
  - All in-between frames must match in a chain

$$\left[\; {}^{5}H_{1} = {}^{5}H_{4}\,{}^{4}H_{3}\,{}^{3}H_{2}\,{}^{2}H_{1} \; \right]$$