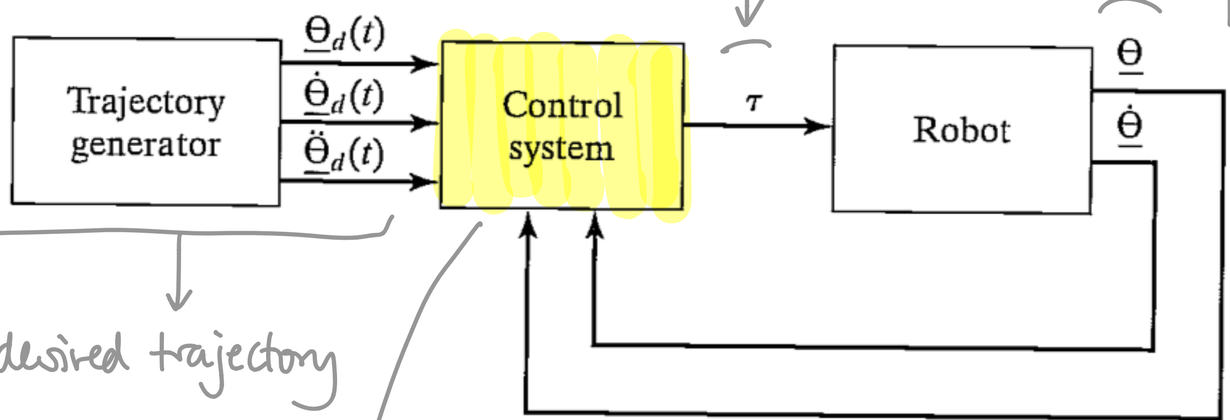
9. Linear Control

Linear control can be applied to lineal differential equations. Although that is usually not the case, linear control is often a valid approximation.

9.1. Feedback and Closed Loop Control

we apply torques, $N \times 1$

we measure at each joint, $N \times 1$



desired trajectory

FIGURE 9.1: High-level block diagram of a robot-control system.

Craig, 3rd Ed., p.263

The control system computes the $\vec{\tau}$ necessary for the desired $\vec{\theta}$

We are given θ_d, $\vec{\dot{\theta}}_d$, $\vec{\ddot{\theta}}_d$ by the motion planer. In a perfect world without disturbances and perfect models, we could operate in open loop and use only the equations of motion for $\vec{\tau}$:

$$\vec{\tau} = \underset{\sim}{M}(\vec{\theta}_d)\vec{\ddot{\theta}}_d + \underset{\sim}{V}(\vec{\theta}_d, \vec{\dot{\theta}}_d) + \underset{\sim}{G}(\vec{\theta}_d)$$

However, real world requires closed loop control: servo error is used to apply the torques, i.e., the difference between desired and measured signals:

$$\left.\begin{aligned} e &= \theta_d - \theta_m \\ \dot{e} &= \dot{\theta}_d - \dot{\theta}_m \end{aligned}\right\} \longrightarrow \text{the control systems computes them: } \vec{\tau} = \vec{\tau}(e, \dot{e})$$

A closed-loop control system can be:

- STABLE: error e is minimized
- UNSTABLE: error e is increased

Control can be approached as:

- MIMO: multiple-input, multiple output, ie. Nx1 vectors, as above
- SISO: single-input, single-output, ie. N independent blocks, one for each of the N joints
  - This is the most common
  - It's an approximation, since joint effects are not decoupled!

9.2. Second Order Linear Systems

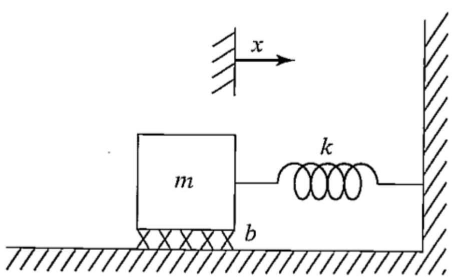


FIGURE 9.2: Spring–mass system with friction.
Craig, 3rd Ed., p. 265

Forces acting on the system:

$$\underbrace{m}_{\text{mass}}\ddot{x} + \underbrace{b}_{\text{viscous friction}}\dot{x} + \underbrace{k}_{\text{spring stiffness coefficient}}x = 0$$

Laplace Transformation

We can use the Laplace transformation to solve such linear differential equations:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \,, \quad s = \sigma + j\omega \,, \quad j = \sqrt{-1}$$

Steps:

1. We apply $\mathcal{L}$ (using tables),
2. operate in s (frequency domain), algebraically
3. Apply the inverse of Laplace $\mathcal{L}^{-1}$ (using tables)

Some Laplace transformations:

	$\mathcal{L}[f(t)]$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - s \cdot f(0) - f'(0)$
a	a/s
$\sin(at)$	$a/(s^2+a^2)$
$\cos(at)$	$s/(s^2+a^2)$
e^{at}	$1/(s-a)$
$e^{at}\sin(bt)$	$b/((s-a)^2+b^2)$
$e^{at}\cos(bt)$	$(s-a)/((s-a)^2+b^2)$

Solving the second order linear diff. eq. with Laplace

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$$

$$\Big\downarrow \; \mathcal{L} \text{ with init. conds } = 0$$

CHARACTERISTIC EQUATION:

$$ms^2X + bsX + kX = 0 \longmapsto \boxed{ms^2 + bs + k = 0}$$

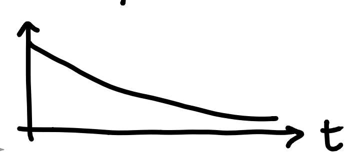
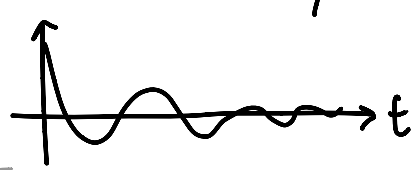
The solutions of the characteristic equation are (the roots):

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m}$$

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

The location of the poles s_1 & s_2 in the Re–Im plane dictates the nature of motion!

Basically, 3 types of responses

1. Real and unequal roots	$b^2 > 4mk$	friction/damping dominates, slow decrease, overdamped	$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ 
2. Complex roots	$b^2 < 4mk$	stiffness dominates, oscillation, underdamped	$x(t) = re^{\lambda t}\cos(\mu t - \delta)$ 
3. Real and equal roots	$b^2 = 4mk$	friction and stiffness balanced, fastest decay, critically damped	$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_2 t}$ 

Important: note that roots with NEGATIVE REAL part generate decaying responses $\Rightarrow$ STABLE! A natural passive system is always stable!

Harmonic Oscillator

The harmonic oscillator has the form of the damped spring-mass system, but it's set in a general form with its natural frequency and damping ratio:

$$m\ddot{x} + b\dot{x} + Kx = 0 \quad \rightarrow \quad \boxed{\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0}$$

$\omega_0 = \sqrt{\frac{K}{m}}$: Natural frequency

$\zeta = \frac{b}{2\sqrt{mK}}$: Damping ratio

$x(t) = A e^{-\zeta\omega_0 t} \cdot \sin\left(\sqrt{1-\zeta^2}\,\omega_0 t + \varphi\right)$

- $\zeta = 0$: simple harmonic oscillator
- $\zeta > 1$: overdamped
- $\zeta = 1$: critically damped
- $\zeta < 1$: underdamped

$\rightarrow$ damped harmonic oscillator

9.3. Control of Second Order Systems

In control, the goal is to obtain a system response as close as possible to a reference value. We achieve that by artificially modifying the m, b, K coefficients of the system! BUT: when we do that, we need to determine whether the system is still STABLE

Position Regulation Control

We have the 2nd order system with sensors (for $x, \dot{x}, \ddot{x}$) and an actuator which applies force f; we want to know the value of f to obtain a desired response, which consists in a fixed position even under disturbances.

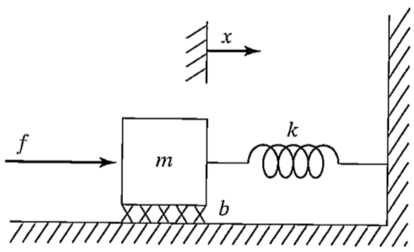


FIGURE 9.6: A damped spring–mass system with an actuator.

Craig, 3rd Ed., p. 271

$$m\ddot{x} + b\dot{x} + kx = f$$

$$\left[f = -k_p x - k_v \dot{x} \right]$$

that is a control law we propose, ie.: f is proportional to x & $\dot{x}$, which we measure

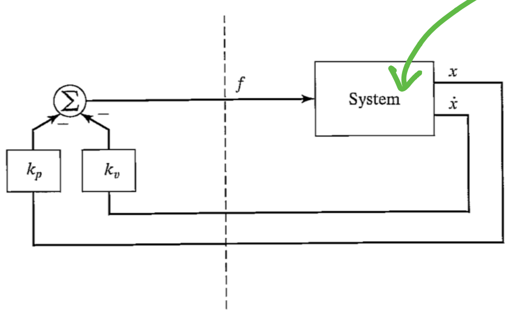


FIGURE 9.7: A closed-loop control system. The control computer (to the left of the dashed line) reads sensor input and writes actuator output commands.

Craig, 3rd Ed., p. 272

Re-arrangin terms:

$$m\ddot{x} + \underbrace{(b + k_v)}_{b'}\dot{x} + \underbrace{(k + k_p)}_{k'}x = 0$$

Now, we have the equation of the harmonic oscillator. We can choose k_p & k_v as we wish, eg., for the system to be critically damped:

$$\zeta = 1 = \frac{b'}{2\sqrt{mk'}} \Rightarrow b + k_v = 2\sqrt{m(k + k_p)}$$

We can now analyze the stability or the response as in 9.2

9.4. Control Law Partitioning

Usually, for more complicated systems control law partitioning is done: we partition the controller into 2 parts:

1) model-based portion: system params m, b, k appear only here!
2) servo portion: independent from m, b, k!

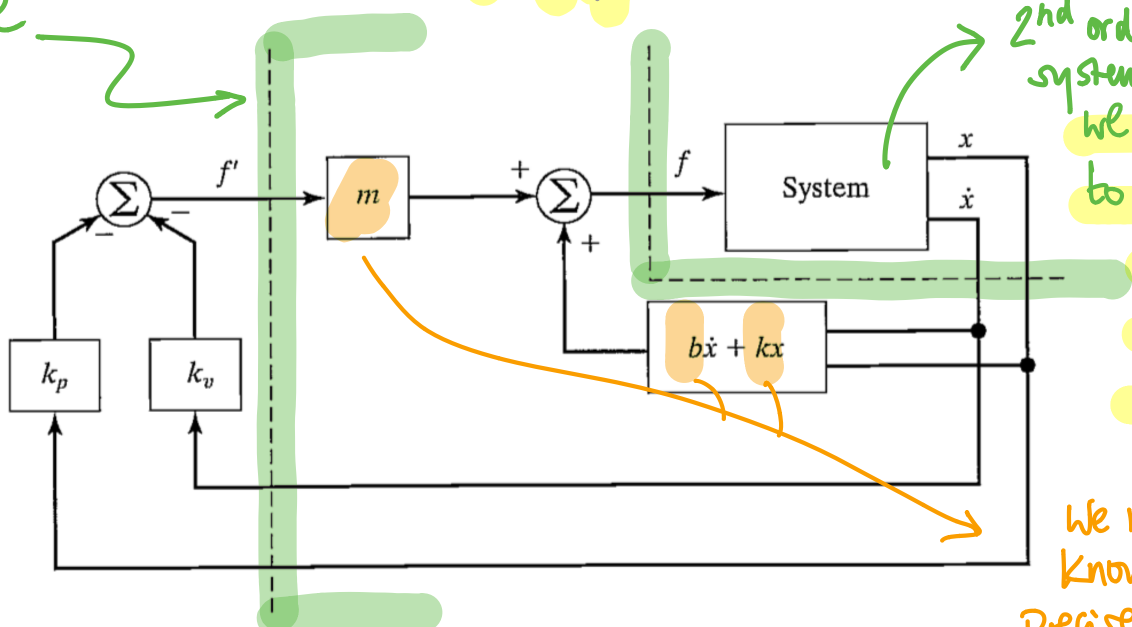
$$m\ddot{x} + b\dot{x} + kx = f = \alpha f' + \beta$$

$$= m\,\underbrace{f'}_{=\ddot{x}} + b\dot{x} + kx$$

Position regulation structure

2nd order linear system: we apply f to it and read with sensors $x, \dot{x}$

eg. of motion of a unit mass



We need to know m, k, b precisely!

FIGURE 9.8: A closed-loop control system employing the partitioned control method.

Craig, 3rd Ed., p. 274

With this re-arrangement, we apply the same structure as before but with f', and we get:

critically damped:

$$\boxed{\ddot{x} = f'} = -k_v\dot{x} - k_p x \Rightarrow \ddot{x} + k_v\dot{x} + k_p x = 0 \longrightarrow k_v = 2\sqrt{k_p}$$

Very importantly, the k_v and k_p don't depend on k, b, m anymore to have a critically damped system!

9.5. Trajectory-Following Control

Instead of maintaining x at a value, we design the control system so that x follows a desired $x_d(t)$.

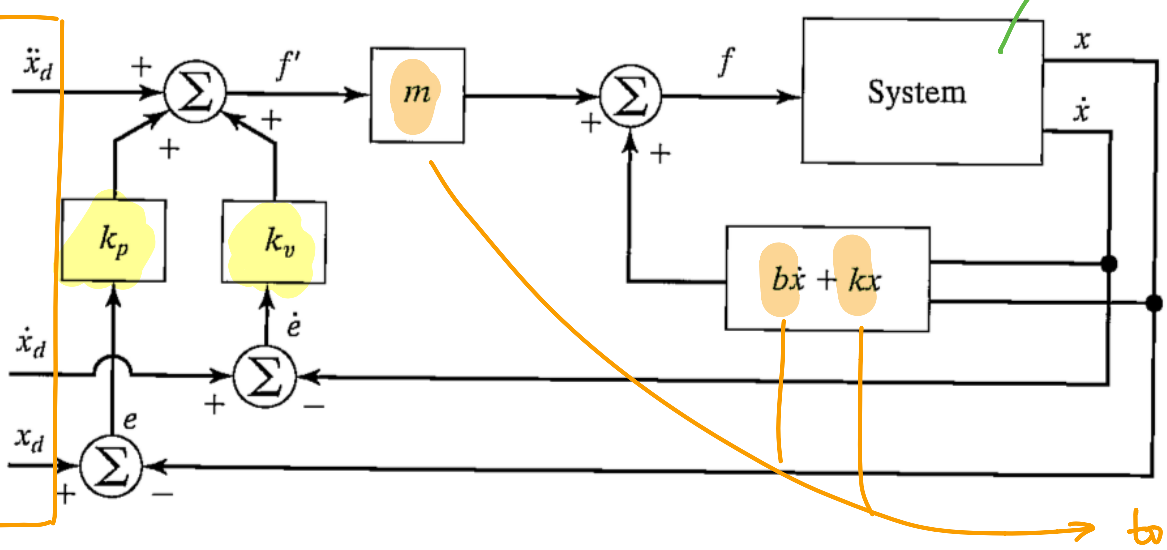
Servo error: $e = x_d(t) - x(t)$

Using the control law partitioning scheme, we have:

$$\ddot{e} + k_v \dot{e} + k_p e = 0$$

This control law scheme using the error spaces makes possible for a 2nd order linear system to follow a trajectory with the desired response - usually, the critically damped is used, ie: $k_v = 2\sqrt{k_p}$

Signal of the desired position $x_d(t)$



2nd order linear system: we apply f to it and read its $x, \dot{x}$

We need to know the system (m, b, k) very precisely!

FIGURE 9.9: A trajectory-following controller for the system in Fig. 9.6.

Craig, 3rd Ed., p. 275

9.6. Disturbance Rejection During Trajectory Following

Here, the trajectory following control is updated to cope with external disturbances or noise. Our system has an additional input: f_{dist}

$\ddot{e} + k_v \dot{e} + k_p e = f_{dist}$ → if f_{dist} is bounded e will be also

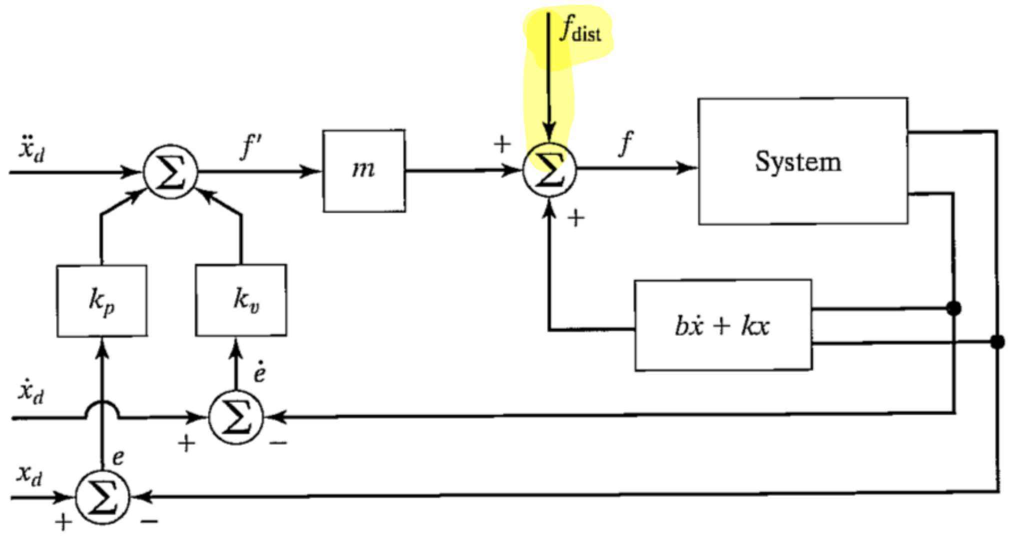


FIGURE 9.10: A trajectory-following control system with a disturbance acting.

Craig, 3rd Ed., p. 276

Steady-state error

In the case f_{dist} = const and we reach a steady state (all derivatives ∅):

$k_p e = f_{dist} \implies e = \dfrac{f_{dist}}{k_p}$: steady state error

We see that with $k_p \uparrow \implies e \downarrow$ BUT we need to still be stable, so we cannot choose high k_p values.

Therefore, in order to decrease the steady-state error, we add an integral term: PID controller

PID controller (trajectory following example)

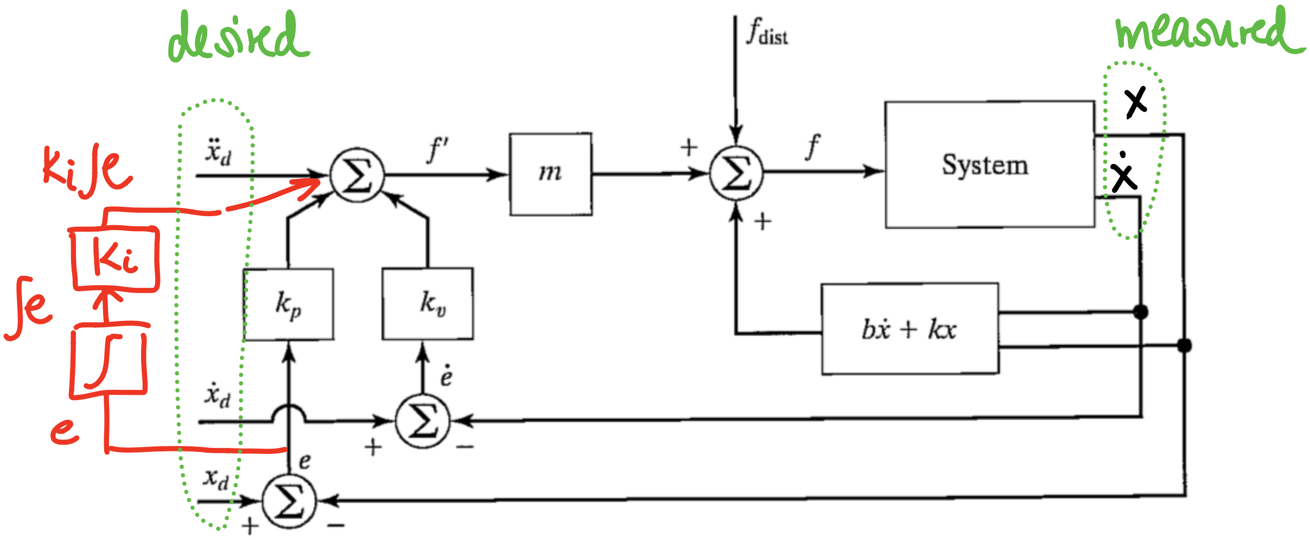


FIGURE 9.10: A trajectory-following control system with a disturbance acting.

Craig, 3rd Ed., p. 276

In order to cope with the steady state error, we add an integral term that cancels it:

$$\ddot{e} + k_v \dot{e} + k_p e + k_i \int e\,dt = f_{dist}$$

$$\dddot{e} + k_v \ddot{e} + k_p \dot{e} + k_i e = \dot{f}_{dist}$$

If system steady, all derivatives $\emptyset$: $k_i e = \dot{f}_{dist} = 0$

$e = 0$

Often K_i is kept small so that our 3rd order system is close to the 2nd order system.

NOTE: I have a more thorough analysis of P, PD, PI, PID controllers in my control notes — it's worth a look there...

9.7. Continuous vs. Discrete Time Control

We have wrongly assumed that we measure, compute and act instantaneously in the continuous domain... Instead, we work with stair-cased functions that have a delay and a sampling frequency!

Some points to consider: (rules of thumb)

- Computation frequency should be much faster than the natural frequency of the system
- Servo rate/frequency is often the limiting factor
- Tracking/Measuring frequency must be at least 2x that of reference input computation
- Sampling period should be at least 10x shorter than the autocorrelation time of noise or repetitive disturbances (autocorrelation: usually, noise is not really random, but it has some periodicity)
- Aliased signals should lead to small energy values
- Sampling rate should be at least 2x the natural frequency of structural resonances that appear as vibrations due to the finite stiffness of the structure.

9.8. Modeling and Control of a Single Joint

We model a simplified version on a DC motor. We have two important constants and mappings between i/v and $\tau/\dot{\theta}$:

- **Motor-torque**:

  torque ripple is usually neglected, considering we can control the torque only with current

$$\boxed{\underbrace{\tau_m}_{\text{torque}} = \underbrace{k_m}_{\text{constant}} \cdot \underbrace{i_a}_{\text{armature current}}}$$

- **Back EMF** (electromotive force):

$$\boxed{\underbrace{v_{emf}}_{\text{voltage generated when motor rotates}} = \underbrace{k_e}_{\text{constant}} \cdot \underbrace{\dot{\theta}_m}_{\text{angular vel of motor}}}$$

Additionally, we model the electric circuit and the mechanical parts:

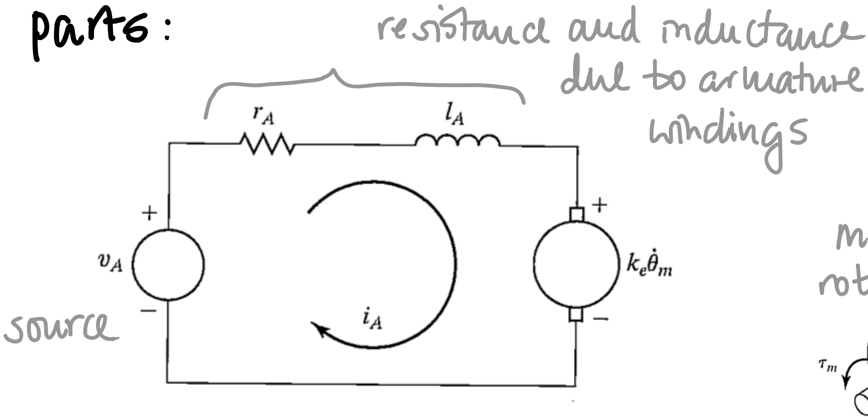


FIGURE 9.11: The armature circuit of a DC torque motor.

Craig, 3rd Ed., p. 279

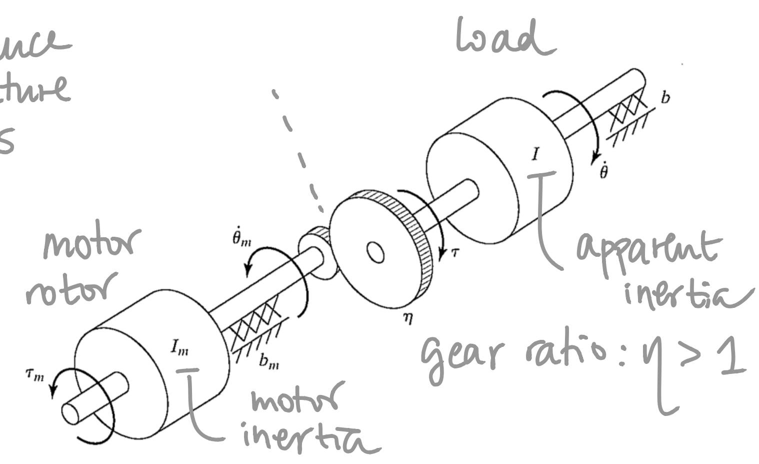


FIGURE 9.12: Mechanical model of a DC torque motor connected through gearing to an inertial load.

Craig, 3rd Ed., p. 280

$$\left[\underbrace{L_a \frac{di_a}{dt}} + r_a i_a = v_a - k_e \dot{\theta}_m\right]$$

↳ this term is often neglected

$$\left[\tau = \eta \tau_m\right], \quad \left[\dot{\theta} = \frac{1}{\eta}\dot{\theta}_m\right]$$

$$\left[\tau_m = I_m \ddot{\theta}_m + b_m \dot{\theta}_m + \frac{1}{\eta}(I\ddot{\theta} + b\dot{\theta})\right]$$

If we develop the equations of the mechanical model:

$$\left[\underbrace{\tau}_{\substack{\text{Torque}\\\text{at the}\\\text{load}\\\text{side}}} = \underbrace{(I + \eta^2 I_m)}_{\substack{\text{effective inertia}\\\text{seen on link side}\\\text{or output}}} \ddot{\theta} + \underbrace{(b + \eta^2 b_m)}_{\substack{\text{effective}\\\text{damping}}} \dot{\theta} \right]$$

We can also express τ_m in motor rotor variables

if $\eta >> 1 \Rightarrow$ inertia is high!

if $\eta = 1 \Rightarrow$ direct drive (no gears)

Notes on the models of a DC motor:

- The (apparent) inertia I of a joint varies with the load and the configuration of the robot; we usually define a range in which it moves: $[I_{min}, I_{max}]$
- Torque ripple is usually neglected and we consider true: $\tau_m = k_m i_a$
- The effect of the inductance is also often neglected
- We consider the joints and links are not flexible (or infinitely rigid); that is ok if the system is sufficiently stiff, so that its resonance natural frequencies are very high. Good rule of thumb:

$$\underbrace{\omega_n}_{\text{our closed loop natural frequency}} \leq \frac{1}{2} \underbrace{\omega_{res}}_{\text{lowest structural resonance}}$$

notes in the book on how to approximate it

Control of a Single Joint : Summary

Assumptions:

1. Motor inductance l_a neglected
2. With high gearing, effective inertia is considered constant:

$$\underbrace{I_{max}}_{\text{we take the upper bound of the region of possible inertias}} + \underbrace{\eta^2}_{\text{gear ratio; if } \eta \gg 1 \text{, the effect of } I_m \text{ is higher}} I_m$$

3. Structural flexibilities are neglected, except lowest structural resonance ω_{res} is used to set the servo gains k_p, k_v

Control with a partitioned structure:

$$\left.\begin{aligned} \alpha &= I_{max} + \eta^2 I_m \\ \beta &= (b + \eta^2 b_m)\dot{\theta} \\ \tau' &= \ddot{\theta}_d + k_v\dot{e} + k_p e \end{aligned}\right\} \xrightarrow{\text{closed loop}} \ddot{e} + k_v\dot{e} + k_p e = \tau_{dist}$$

$\downarrow$ chosen gains

$$k_p = \omega_n^2 = \frac{1}{4}\omega_{res}^2$$

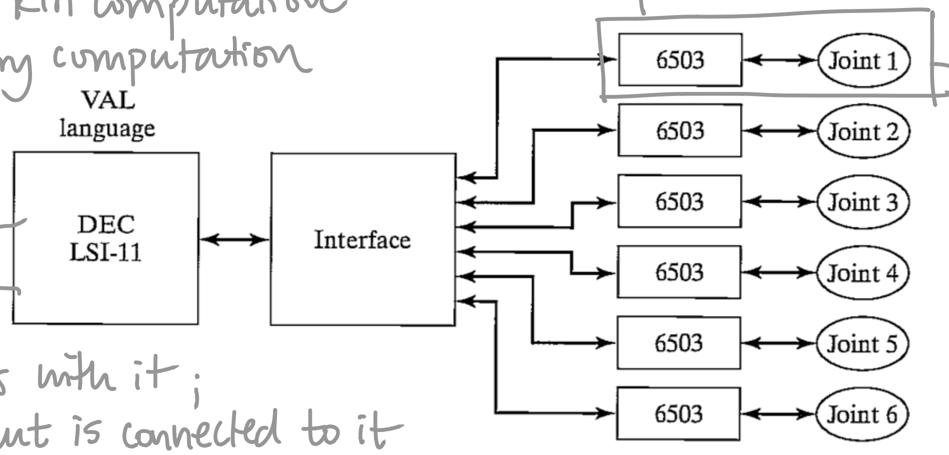
$$k_v = 2\sqrt{k_p} = \omega_{res}$$

ω_{res}? Notes on how to roughly estimate it with lumped masses given in Craig.

9.9. Architecture of an Industrial Robot Controller (PUMA 560)

computer sending a desired pose every 28 ms, which requires:
- inverse kin computation
- trajectory computation

each joint has a dedicated processor for the PID control



user interfaces with it;
the teach pendant is connected to it

FIGURE 9.14: Hierarchical computer architecture of the PUMA 560 robot-control system.

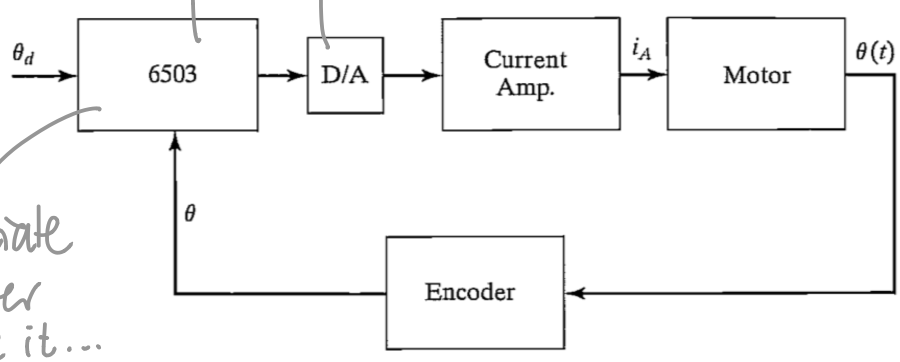
Craig, 3rd Ed., p. 284

Rockwell 6503 microprocessor for the control

digital to analog converter

desired

measured



we differentiate θ inside, better to measure it...

FIGURE 9.15: Functional blocks of the joint-control system of the PUMA 560.

Craig, 3rd Ed., p. 285