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Example sheet 3 - Michaelmas 2019

Problem 1. Let Z be a homogeneous Markov process taking values in a finite space E with transition probabilities

$$p_{z,y} = \mathbb{P}(Z_1 = y | Z_0 = z) \text{ for } z, y \in E.$$

Let $R : E \to (-1, \infty)$ and $F : \mathbb{Z}_+ \times E \to \mathbb{R}^d$ be functions such that

$$\sum_{y \in E} F(t+1, y) p_{z,y} = (1 + R(z)) F(t, z) \text{ for all } t \geq 0, z \in E.$$

Consider a market with a bank account with risk-free rate $r_t = R(Z_{t-1})$ and d stocks with prices $S_t^i = F^i(t, Z_t)$ for $i = 1, \ldots, d$.
(a) Show that the market has no arbitrage.
Now suppose that for each $z \in E$ that the set

$$\mathcal{Y}(z) = \{y \in E : p_{z,y} > 0\}$$

has exactly $d+1$ elements. Furthermore, suppose that for all $t \geq 0$ the $d+1$ functions $\{\mathbb{1}, F^1(t, \cdot), \ldots, F^d(t, \cdot)\}$ are linearly independent, where $\mathbb{1}(z) = 1$ for all $z \in E$.
(b) Show that the market is complete.
(c) Define the $d \times d$ matrix-valued function for $0 \leq t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Delta(t, z, y_0) = \big(F^i(t, y) - F^i(t, y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}, 1 \leq i \leq d\big).$$

Show that $\Delta(t, z, y_0)$ is invertible.
(d) For a function $f : E \to \mathbb{R}$ consider the $d \times 1$ vector defined for $z \in E$ and $y_0 \in \mathcal{Y}(z)$ by

$$\tilde{f}(z, y_0) = \big(f(y) - f(y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}\big).$$

Show that the $d \times 1$ vector $\Delta(t, z, y_0)^{-1} \tilde{f}(z, y_0)$ does not depend on y_0.
(e) Fix a function $g : E \to \mathbb{R}$ and let $V(T, z) = g(z)$ and

$$V(t, z) = \frac{1}{1 + R(z)} \sum_{y \in E} V(t+1, y) p_{z,y} \text{ for all } 0 \leq t \leq T-1, z \in E.$$

Define a $\mathbb{R}^d$-valued function for $0 \leq t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Pi(t, z) = \Delta(t, z, y_0)^{-1} \big(V(t, y) - V(t, y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}\big)$$

for any $y_0 \in \mathcal{Y}(z)$. (The choice of y_0 is irrelevant by part (d).) Finally, define a real-valued function by

$$\Phi(t, z) = V(t-1, z) - \Pi(t, z) \cdot F(t-1, z).$$

Show that the European claim with time T payout $\xi_T = g(S_T)$ is with unique no-arbitrage price $\xi_t = V(t, Z_t)$ and replicating strategy $H = (\phi, \pi)$ where

$$\phi_t = \frac{\Phi(t, Z_{t-1})}{B_{t-1}} \text{ and } \pi_t = \Phi(t, Z_{t-1})$$

Problem 2. Let $(\zeta_{t,T})_{1\le t<T}$ be a collection of positive random variables such that $\zeta_{t,T}$ is $\mathcal{F}_t$-measurable for all t and that

$$\mathbb{E}\left[\left(\prod_{u=t+1}^{T}\zeta_{t,u}\right)^{-1}|\mathcal{F}_{t-1}\right]=1$$

for all $1\le t<T$. Now given a non-random sequence $f_{0,T}>-1$ for $T>0$, let

$$1+f_{t,T}=(1+f_{t-1,T})\zeta_{t,T} \text{ for } 1\le t<T.$$

Let $r_t=f_{t-1,t}$ for $t\ge 1$ and $P_{t,T}=\left(\prod_{u=t+1}^{T}(1+f_{t,u})\right)^{-1}$ for $0\le t<T$.

Consider a market with a bank account with time t risk-free interest rate r_t and a collection of bonds such that $P_{t,T}$ is the time t of the bond of maturity T.

(a) Show that the market has no arbitrage.

(b) Use example sheet 2 problem 6 to show that the forward rate at time t for maturity T is given by $f_{t,T}$.

(c) Let $\zeta_{t,T}=\exp(\sigma_{t,T}\xi_t+\mu_{t,T})$ where $\sigma_{t,T}$ and $\mu_{t,T}$ are $\mathcal{F}_{t-1}$ measurable and ξ_t is $N(0,1)$ and independent of $\mathcal{F}_{t-1}$. Show that

$$\mu_{t,T}=\sigma_{t,T}\sum_{u=t+1}^{T-1}\sigma_{t,u}+\frac{1}{2}\sigma_{t,T}^2$$

Problem 3. Let g be a function on the integers, and define functions g' and g'' by the formulae

$$g'(x)=\tfrac{1}{2}[g(x+1)-g(x-1)] \text{ and } g''(x)=g(x+1)-2g(x)+g(x-1)$$

for all integers x.

Let $(x_t)_t$ be a sequence of integers with $x_t-x_{t-1}\in\{-1,0,1\}$ for each $t\ge 1$. Show that for all $t\ge 0$ we have

$$g(x_t)=g(x_0)+\sum_{s=1}^{t}g'(x_{s-1})(x_s-x_{s-1})+\frac{1}{2}\sum_{s=1}^{t}g''(x_{s-1})(x_s-x_{s-1})^2.$$

Problem 4. * Let $(S_t)_{t\ge 0}$ be a discrete-time martingale such that S_0 is an integer and for all $t\ge 1$ the increment S_t-S_{t-1} is valued in the set $\{-1,0,1\}$.

(a) Prove the identity

$$(S_T-K-1)^+-2(S_T-K)^++(S_T-K+1)^+=\mathbb{1}_{\{S_T=K\}}$$

for integers K and $T\ge 0$.

(b) Prove the identity

$$(S_T-K)^+=(S_0-K)^++\sum_{t=1}^{T}f(S_{t-1}-K)(S_t-S_{t-1})+\frac{1}{2}\sum_{t=1}^{K}\mathbf{1}_{\{S_T=K\}}(S_t-S_{t-1})^2$$

for integers K and $T\ge 1$, where f is defined by

$$f(x)=\mathbf{1}_{\{x>0\}}+\frac{1}{2}\mathbf{1}_{\{x=0\}}.$$

Let

$$C(T,K)=\mathbb{E}[(S_T-K)^+]$$

for integers K and $T \geq 0$ and

$$\sigma^2(T,K) = \mathrm{Var}(S_{T+1}|S_T = K)$$

for integers K and T such that $|K - S_0| \leq T$.
(c) Using parts (a) and (b), or otherwise, prove the identity

$$C(T+1,K) - C(T,K) = \frac{1}{2}\sigma^2(T,K)[C(T,K+1) - 2C(T,K) + C(T,K-1)]$$

for integers K and T such that $|K - S_0| \leq T$.
(d) Comment an application part (c) to finance.

Problem 5. Let f be a positive continuous (non-random) function and W a Brownian motion. Use Lévy's characterisation of Brownian motion to show that $\int_0^t f(s)dW_s$ is a normal random variable with mean zero and variance $\int_0^t f(s)^2 ds$.

Problem 6. * (Ornstein–Uhlenbeck process) Let W be a Brownian motion, and let

$$X_t = e^{at}x + b\int_0^t e^{a(t-s)}dW_s$$

for some $a, b, x \in \mathbb{R}$.
(a) Verify that $(X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = aX_t\, dt + b\, dW_t, \quad X_0 = x.$$

(b) Show that

$$X_t \sim N\left(e^{at}x, \frac{b^2}{2a}(e^{2at} - 1)\right).$$

(c) What is the distribution of the random variable $\int_0^T X_t\, dt$?

Problem 7. Let W be a Brownian motion. Show that if $Y_t = W_t^3 - 3tW_t$ then Y is a martingale (1) by hand, and (2) by Itô's formula.

Problem 8. (Heat equation) Let W be a scalar Brownian motion, and let $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function that satisfy the partial differential equation

$$\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0$$

with terminal condition

$$g(T,x) = G(x).$$

(a) Show that $(g(t,W_t))_{t\in[0,T]}$ is a local martingale.
(b) If the function g is bounded, deduce the formula

$$g(t,x) = \int_{-\infty}^{\infty} G(x + \sqrt{T-t}z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.$$

Problem 9. (Strictly local martingale) This is a technical exercise to exhibit a local martingale that is not a true martingale. Let $W = (W^1, W^2, W^3)$ be a three-dimensional Brownian motion and let $u = (1,0,0)$. It is a fact that $\mathbb{P}(W_t \neq u \text{ for all } t \geq 0) = 1$.
(a) Let $X_t = |W_t - u|^{-1}$. Use Itô's formula and Lévy's characterisation of Brownian motion to show that

$$dX_t = X_t^2 dZ_t, \quad X_0 = 1$$

where Z is a Brownian motion. In particular, show that X is a positive local martingale.
(b) By directly evaluating the integral or otherwise, show that

$$\mathbb{E}(X_t) = 2\Phi(t^{-1/2}) - 1$$

for all $t > 0$, where Φ is the distribution function of a standard normal random variable. Why does this imply that X is a strictly local martingale?

Problem 10. (strictly local martingales again) (a) Suppose that X is positive martingale with $X_0 = 1$. Fix $T > 0$ and let

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X_T.$$

Let $Y_t = 1/X_t$ for all $t \geq 0$. Show that $(Y_t)_{0 \leq t \leq T}$ is a positive martingale under $\mathbb{Q}$.
(b) Continuing from part (a), now suppose that X has dynamics

$$dX_t = X_t \sigma_t dW_t$$

where W is a Brownian motion under $\mathbb{P}$. Use Girsanov's theorem to show that there exists a $\mathbb{Q}$-Brownian motion $\hat{W}$ such that

$$dY_t = Y_t \sigma_t d\hat{W}_t$$

(c) Let X be a positive local martingale with $X_0 = 1$ and dynamics

$$dX_t = X_t^2 dW_t.$$

Our goal is to show that X is a strictly local martingale. For the sake of finding a contradiction, suppose X is a true martingale. In the notation of parts (a) and (b), show that

$$\mathbb{P}(Y_t > 0) = 1 \text{ but } \mathbb{Q}(Y_t > 0) = \Phi(t^{-1/2}).$$

Why does this contradict the assumption that X is a true martingale?