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Example sheet 3 - Michaelmas 2019

Problem 1. Let Z be a homogeneous Markov process taking values in a finite space E with transition probabilities

$$p_{z,y} = \mathbb{P}(Z_1 = y | Z_0 = z) \text{ for } z, y \in E.$$

Let $R : E \to (-1, \infty)$ and $F : \mathbb{Z}_+ \times E \to \mathbb{R}^d$ be functions such that

$$\sum_{y \in E} F(t+1, y) p_{z,y} = (1 + R(z)) F(t, z) \text{ for all } t \geq 0, z \in E.$$

Consider a market with a bank account with risk-free rate $r_t = R(Z_{t-1})$ and d stocks with prices $S^i_t = F^i(t, Z_t)$ for $i = 1, \ldots, d$.
(a) Show that the market has no arbitrage.
Now suppose that for each $z \in E$ that the set

$$\mathcal{Y}(z) = \{y \in E : p_{z,y} > 0\}$$

has exactly $d+1$ elements. Furthermore, suppose that for all $t \geq 0$ the $d+1$ functions $\{\mathbb{1}, F^1(t, \cdot), \ldots, F^d(t, \cdot)\}$ are linearly independent, where $\mathbb{1}(z) = 1$ for all $z \in E$.
(b) Show that the market is complete.
(c) Define the $d \times d$ matrix-valued function for $0 \leq t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Delta(t, z, y_0) = \big(F^i(t, y) - F^i(t, y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}, 1 \leq i \leq d\big).$$

Show that $\Delta(t, z, y_0)$ is invertible.
(d) For a function $f : E \to \mathbb{R}$ consider the $d \times 1$ vector defined for $z \in E$ and $y_0 \in \mathcal{Y}(z)$ by

$$\tilde{f}(z, y_0) = \big(f(y) - f(y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}\big).$$

Show that the $d \times 1$ vector $\Delta(t, z, y_0)^{-1} \tilde{f}(z, y_0)$ does not depend on y_0.
(e) Fix a function $g : E \to \mathbb{R}$ and let $V(T, z) = g(z)$ and

$$V(t, z) = \frac{1}{1 + R(z)} \sum_{y \in E} V(t+1, y) p_{z,y} \text{ for all } 0 \leq t \leq T-1, z \in E.$$

Define a $\mathbb{R}^d$-valued function for $0 \leq t \leq T, z \in E, y_0 \in \mathcal{Y}(z)$ by

$$\Pi(t, z) = \Delta(t, z, y_0)^{-1} \big(V(t, y) - V(t, y_0) : y \in \mathcal{Y}(z) \backslash \{y_0\}\big)$$

for any $y_0 \in \mathcal{Y}(z)$. (The choice of y_0 is irrelevant by part (d).) Finally, define a real-valued function by

$$\Phi(t, z) = V(t-1, z) - \Pi(t, z) \cdot F(t-1, z).$$

Show that the European claim with time T payout $\xi_T = g(S_T)$ is with unique no-arbitrage price $\xi_t = V(t, Z_t)$ and replicating strategy $H = (\phi, \pi)$ where

$$\phi_t = \frac{\Phi(t, Z_{t-1})}{B_{t-1}} \text{ and } \pi_t = \Phi(t, Z_{t-1})$$

Solution 1. (a) Note by the Markov property that

$$\begin{aligned}\mathbb{E}\left(\frac{S_{t+1}}{B_{t+1}}|\mathcal{F}_t\right) &= \frac{1}{B_{t+1}}\mathbb{E}\left(F(t+1, Z_{t+1})|Z_t\right)\\ &= \frac{1}{B_{t+1}}(1+R(Z_t))F(t,Z_t)\\ &= \frac{S_t}{B_t}\end{aligned}$$

so the measure $\mathbb{P}$ is risk-neutral. This implies that there is no arbitrage.
(b) Claim: for every $t \geq 0$ and any $\mathcal{F}_t$ measurable random variable ξ_t, there exists a $\mathcal{F}_{t-1}$-measurable random vector H_t valued in $\mathbb{R}^{1+d}$ such that $H_t \cdot P_t = \xi_t$, where $P_t = (B_t, S_t)$.

Proof that the claim implies completeness: Given a claim with time T payout ξ_T, let H_T be $\mathcal{F}_{T-1}$-measurable and such that $H_T \cdot P_T = \xi_T$, and for $t < T$ let H_t be $\mathcal{F}_{t-1}$-measurable and such that

$$H_t \cdot P_t = H_{t+1} \cdot P_t.$$

This strategy is previsible and self-financing by construction, and replicates ξ_T.

To prove the claim: conditional on $\mathcal{F}_{t-1} = \sigma(Z_1, \ldots, Z_{t-1})$ the random variable Z_t takes $d+1$ values, say $y_1, \ldots, y_{d+1}$ (by the Markov property). On the other hand, an $\mathcal{F}_t$-measurable random variable is of the form $\xi_t = G(Z_1, \ldots, Z_t)$, so conditional on $\mathcal{F}_{t-1}$, the random variable ξ_t can take only $d+1$ values, say $x_1, \ldots x_{d+1}$. So we have to solve the $d+1$ equations

$$\phi_t B_t + \sum_{i=1}^{d} \pi_t^i F^i(t, y_j) = x_j$$

in $d+1$ unknowns $\phi_t, \pi_t^1, \ldots, \pi_t^d$. By the assumed linear independence, the $(1+d) \times (1+d)$ matrix

$$A = \begin{pmatrix} B_t & F^1(t,y_1) & \cdots & F^d(t,y_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_t & F^1(t,y_{d+1}) & \cdots & F^d(t,y_{d+1}) \end{pmatrix}$$

has rank $d+1$. Hence A is invertible, and there exists a unique solution to the system of equations.
(c) By linear algebra, we have for any $1 \leq j \leq d+1$ that

$$\begin{aligned}\det A = \det &\begin{pmatrix} 0 & F^1(t,y_1) - F^1(t,y_j) & \cdots & F^d(t,y_1) - F^d(t,y_j) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & F^1(t,y_{j-1}) - F^1(t,y_j) & \cdots & F^d(t,y_{j-1}) - F^d(t,y_j)) \\ B_t & F^1(t,y_j) & \cdots & F^d(t,y_j)) \\ 0 & F^1(t,y_{j+1}) - F^1(t,y_j) & \cdots & F^d(t,y_{j+1} - F^d(t,y_j)) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & F^1(t,y_{d+1}) - F^1(t,y_j) & \cdots & F^d(t,y_{d+1} - F^d(t,y_j)) \end{pmatrix}\\ = (-1)^{j+1} &B_t \det \Delta(t, Z_{t-1}, y_j)\end{aligned}$$

From part (b), the determinant of A is not zero, and hence neither is the determinant of $\Delta(t, Z_{t-1}, y_j)$.

(d) From part (b), the system of equations

$$a_0 + \sum_{i=1}^{d} a_i F^i(t, y_j) = f(y_j)$$

has a unique solution. By subtracting the row corresponding to y_0 from every equation yields the system of d equations in d unknowns

$$\sum_{i=1}^{d} a_i(F^i(t, y_j) - F^i(t, y_0)) = f(y_j) - f(y_0)$$

The unique solution is $(a_1, \ldots, a_d)^\top = \Delta^{-1}\tilde{f}$ which does not depend on the choice of y_0.
(e) Let $(\hat{\phi}_t, \hat{\pi}_t)$ be the unique $\mathcal{F}_{t-1}$-measurable solution to

$$\hat{\phi}_t B_t + \hat{\pi}_t \cdot F(t, Z_t) = V(t, Z_t)$$

. From (d) we know that $\hat{\pi}_t = \pi_t$. Now divide by $1 + r_t$ and compute the expected value of both sides of the displayed equation conditional on $\mathcal{F}_{t-1}$. We have

$$\hat{\phi}_t B_{t-1} + \hat{\pi}_t \cdot F(t - t, Z_{t-1}) = V(t - 1, Z_{t-1})$$

Hence $\hat{\phi}_t = \phi_t$.

Problem 2. Let $(\zeta_{t,T})_{1 \le t < T}$ be a collection of positive random variables such that $\zeta_{t,T}$ is $\mathcal{F}_t$-measurable for all t and that

$$\mathbb{E}\left[\left(\prod_{u=t+1}^{T} \zeta_{t,u}\right)^{-1} |\mathcal{F}_{t-1}\right] = 1$$

for all $1 \le t < T$. Now given a non-random sequence $f_{0,T} > -1$ for $T > 0$, let

$$1 + f_{t,T} = (1 + f_{t-1,T})\zeta_{t,T} \text{ for } 1 \le t < T.$$

Let $r_t = f_{t-1,t}$ for $t \ge 1$ and $P_{t,T} = \left(\prod_{u=t+1}^{T}(1 + f_{t,u})\right)^{-1}$ for $0 \le t < T$.

Consider a market with a bank account with time t risk-free interest rate r_t and a collection of bonds such that $P_{t,T}$ is the time t of the bond of maturity T.
(a) Show that the market has no arbitrage.
(b) Use example sheet 2 problem 6 to show that the forward rate at time t for maturity T is given by $f_{t,T}$.
(c) Let $\zeta_{t,T} = \exp(\sigma_{t,T}\xi_t + \mu_{t,T})$ where $\sigma_{t,T}$ and $\mu_{t,T}$ are $\mathcal{F}_{t-1}$ measurable and ξ_t is $N(0, 1)$ and independent of $\mathcal{F}_{t-1}$. Show that

$$\mu_{t,T} = \sigma_{t,T} \sum_{u=t+1}^{T-1} \sigma_{t,u} + \frac{1}{2}\sigma_{t,T}^2$$

Solution 2. (a) Note

$$\frac{P_{t,T}}{B_t} = \frac{P_{t-1,T}}{B_{t-1}} \left(\prod_{u=t+1}^{T} \zeta_{t,T}\right)^{-1}$$

and hence $(P_{t,T}/B_t)_{0\le t\le T}$ is a martingale by example sheet 1 problem 3. Hence $\mathbb{P}$ is a risk-neutral measure and hence their is no arbitrage by the 1FTAP.
(b) We know that the forward rate is $\frac{P_{t,T-1}}{P_{t,T}} - 1$ which simplifies to $f_{t,T}$.
(c) Using the moment generating function of the standard normal we have

$$\begin{aligned}1 &= \mathbb{E}\left[\exp\left(-\sum_{u=t+1}^{T}(\sigma_{t,u}\xi_t + \mu_{t,u})\right)|\mathcal{F}_{t-1}\right]\\ &= \exp\left(\frac{1}{2}\left(\sum_{u=t+1}^{T}\sigma_{t,u}\right)^2 - \sum_{u=t+1}^{T}\mu_{t,u}\right)\end{aligned}$$

The conclusion follows from solving for $u_{t,T}$.

Problem 3. Let g be a function on the integers, and define functions g' and g'' by the formulae

$$g'(x) = \tfrac{1}{2}[g(x+1) - g(x-1)] \text{ and } g''(x) = g(x+1) - 2g(x) + g(x-1)$$

for all integers x.

Let $(x_t)_t$ be a sequence of integers with $x_t - x_{t-1} \in \{-1, 0, 1\}$ for each $t \ge 1$. Show that for all $t \ge 0$ we have

$$g(x_t) = g(x_0) + \sum_{s=1}^{t} g'(x_{s-1})(x_s - x_{s-1}) + \frac{1}{2}\sum_{s=1}^{t} g''(x_{s-1})(x_s - x_{s-1})^2.$$

Solution 3. It is sufficient to check that

$$g(x_t) = g(x_{t-1}) + g'(x_{t-1})(x_t - x_{t-1}) + \frac{1}{2}g''(x_{t-1})(x_t - x_{t-1})^2,$$

since then the identity would be proven by induction.

Suppose $x_t - x_{t-t} = \varepsilon$, so the right-hand side becomes

$$\tfrac{\varepsilon(\varepsilon+1)}{2}g(x_{t-1}+1) + \tfrac{\varepsilon(\varepsilon-1)}{2}g(x_{t-1}-1) + (1-\varepsilon^2)g(x_{t-1})$$

It is a simple matter to check that this expression yields $g(x_{t-1}+\varepsilon)$ in the three cases $\varepsilon = -1, 0, 1$.

Problem 4. * Let $(S_t)_{t\ge 0}$ be a discrete-time martingale such that S_0 is an integer and for all $t \ge 1$ the increment $S_t - S_{t-1}$ is valued in the set $\{-1, 0, 1\}$.
(a) Prove the identity

$$(S_T - K - 1)^+ - 2(S_T - K)^+ + (S_T - K + 1)^+ = \mathbb{1}_{\{S_T = K\}}$$

for integers K and $T \ge 0$.
(b) Prove the identity

$$(S_T - K)^+ = (S_0 - K)^+ + \sum_{t=1}^{T} f(S_{t-1} - K)(S_t - S_{t-1}) + \frac{1}{2}\sum_{t=1}^{T}\mathbf{1}_{\{S_t = K\}}(S_t - S_{t-1})^2$$

for integers K and $T \ge 1$, where f is defined by

$$f(x) = \mathbf{1}_{\{x>0\}} + \frac{1}{2}\mathbf{1}_{\{x=0\}}.$$

Let

$$C(T, K) = \mathbb{E}[(S_T - K)^+]$$

for integers K and $T \geq 0$ and

$$\sigma^2(T, K) = \text{Var}(S_{T+1}|S_T = K)$$

for integers K and T such that $|K - S_0| \leq T$.

(c) Using parts (a) and (b), or otherwise, prove the identity

$$C(T+1, K) - C(T, K) = \frac{1}{2}\sigma^2(T, K)[C(T, K+1) - 2C(T, K) + C(T, K-1)]$$

for integers K and T such that $|K - S_0| \leq T$.

(d) Comment an application part (c) to finance.

Solution 4. (a) Let $g(a) = (a+1)^+ - 2a^+ + (a-1)^+$. Check: if $a \geq 1$ then $g(a) = (a+1) - 2a + (a-1) = 0$. If $a = 0$ then $g(a) = (a+1) - 2a + 0 = 1$. And if $a \leq -1$ then $g(a) = 0 - 20 + 0 = 0$.

(b) This is follows from Problem 3 above.

(c) Computing expectations of (b) yields

$$C(T+1, K) - C(T, K) = \frac{1}{2}\mathbb{E}[\mathbf{1}_{\{S_T = K\}}(S_{T+1} - S_T)^2]$$

using the assumption that S is a martingale to eliminate the first term. Again by the martingale property $\mathbb{E}[(S_{T+1} - S_T)^2|\mathcal{F}_T] = \text{Var}(S_{T+1}|\mathcal{F}_T)$ so the right-hand side becomes

$$\frac{1}{2}\mathbb{P}(S_T = K)\sigma^2(T, K)$$

by using the tower property. Finally, compute the expectation of (a) to yield the identity.

(d) Consider a market consisting of cash, a stock with price process S and a family of call options of strikes and maturities. There are at least two uses of the equation from part (c): The first is to compute the initial call prices in terms of the dynamic parameters of S. Alternatively, given the quoted prices of calls at time 0, use the equation to solve for $\sigma^2(T, K)$ and thereby work out the dynamics of S.

Problem 5. Let f be a positive continuous (non-random) function and W a Brownian motion. Use Lévy's characterisation of Brownian motion to show that $\int_0^t f(s)dW_s$ is a normal random variable with mean zero and variance $\int_0^t f(s)^2 ds$.

Solution 5. Let $F(t) = \int_0^t f(s)^2 ds$. Note that F is strictly increasing and continuous. Let

$$Z_u = \int_0^{F^{-1}(u)} f(s)dW_s.$$

Note Z is a continuous local martingale in the filtration $(\mathcal{F}_{F^{-1}(u)})_{u \geq 0}$ with quadratic variation

$$\langle Z \rangle_u = \int_0^{F^{-1}(u)} f(s)^2 ds = u.$$

Hence Z is a Brownian motion. Therefore,

$$\int_0^t f(s)dW_s = Z_{F(t)} \sim N(0, F(t))$$

as desired.

Problem 6. * (Ornstein–Uhlenbeck process) Let W be a Brownian motion, and let

$$X_t = e^{at}x + b\int_0^t e^{a(t-s)}dW_s$$

for some $a, b, x \in \mathbb{R}$.
(a) Verify that $(X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = aX_t\ dt + b\ dW_t, \quad X_0 = x.$$

(b) Show that

$$X_t \sim N\left(e^{at}x, \frac{b^2}{2a}(e^{2at}-1)\right).$$

(c) What is the distribution of the random variable $\int_0^T X_t\ dt$?

Solution 6. (a) Since

$$X_t = e^{at}\left(x + b\int_0^t e^{-as}\ dW_s\right)$$

we can apply Itô's formula

$$\begin{aligned} dX_t &= e^{at}\left(be^{-at}dW_t\right) + \left(x + b\int_0^t e^{-as}\ dW_s\right)ae^{at}dt \\ &= b\ dW_t + aX_t\ dt \end{aligned}$$

(b) Since

$$\int_0^t (e^{a(t-s)})^2 ds = \frac{e^{2at}-1}{2a}$$

this part follows from Problem 2.
(c) Method 1: Note that by rearranging the stochastic differential equation we have

$$\int_0^T X_t\ dt = \frac{1}{a}(X_T - x - bW_T)$$

and hence $\int_0^T X_t\ dt$ is normally distributed with mean $(e^{aT}-1)x/a$. To compute the variance, first note that

$$\begin{aligned} \mathrm{Cov}(X_T, W_T) &= \mathrm{Cov}\left(b\int_0^T e^{a(T-t)}, dW_t \int_0^T dW_t\right) \\ &= b\int_0^T e^{a(T-t)}dt \\ &= \frac{b}{a}(e^{aT}-1) \end{aligned}$$

by Itô's isometry. Hence

$$\begin{aligned} \mathrm{Var}\left(\int_0^T X_t\ dt\right) &= \frac{1}{a^2}\left(\mathrm{Var}(X_T) - 2b\ \mathrm{Cov}(X_T, W_T) + b^2\mathrm{Var}(W_T)\right) \\ &= \frac{b^2}{2a^3}(e^{2aT} - 4e^{aT} + 3 + 2aT). \end{aligned}$$

Method 2:

$$\begin{aligned}\int_0^T X_t\, dt &= \int_0^T e^{aT} x\, dt + \int_0^T \int_0^t e^{a(t-s)} b\, dW_s\, dt \\ &= \int_0^T e^{aT} x\, dt + \int_0^T \int_s^T e^{a(t-s)} b\, dt\, dW_s \\ &= \frac{e^{aT}-1}{a} x + \int_0^T \frac{e^{a(T-s)}-1}{a} b\, dW_s\end{aligned}$$

Hence $\int_0^T X_t\, dt$ is normally distributed with mean $(e^{aT}-1)x/a$ and variance

$$\frac{b^2}{a^2}\int_0^T (e^{a(T-s)}-1)^2 ds = \frac{b^2}{2a^3}(e^{2aT} - 4e^{aT} + 3 + 2aT)$$

This calculation is useful in the study of the Vasicek interest rate model.

Problem 7. Let W be a Brownian motion. Show that if $Y_t = W_t^3 - 3tW_t$ then Y is a martingale (1) by hand, and (2) by Itô's formula.

Solution 7. (1) By hand: Since Gaussian random variables have finite moments of all orders, Y is integrable. Indeed, we have

$$\mathbb{E}(|Y_t|) \le \mathbb{E}(|W_t^3|) + 3t\mathbb{E}(|W_t|) = Ct^{3/2} < \infty$$

where $C = 5\sqrt{2/\pi}$. Therefore, using the independence of the increments of W we have

$$\begin{aligned}\mathbb{E}(Y_t|\mathcal{F}_s) =&\mathbb{E}(W_t^3 - 3tW_t|\mathcal{F}_s) \\ =&\mathbb{E}[(W_t - W_s + W_s)^3 - 3t(W_t - W_s + W_s)|\mathcal{F}_s] \\ =&\mathbb{E}[(W_t - W_s)^3] + 3\mathbb{E}[(W_t - W_s)^2]W_s + 3\mathbb{E}(W_t - W_s)W_s^2 + W_s^3 \\ &- 3t\mathbb{E}(W_t - W_s) - tW_s \\ =&0 + 3(t-s)W_s + 0 + W_s^3 + 0 - tW_s \\ =&Y_s\end{aligned}$$

for $0 \le s < t$.

(2) By Itô's rule:

$$\begin{aligned}dY_t &= d(W_t^3 - 3tW_t) \\ &= (3W_t^2\, dW_t + 3W_t\, dt) - 3(t\, dW_t + W_t\, dt) \\ &= 3(W_t^2 - t)dW_t\end{aligned}$$

and hence Y is a local martingale. Recall that if $\mathbb{E}\left(\int_0^t \alpha_s^2\, ds\right) < \infty$ for all $t \ge 0$ then the process $\left(\int_0^t \alpha_s\, dW_s\right)_{t\ge 0}$ is a martingale. Again, it's clear that the integrand is square integrable in this case since Gaussian random variables have finite moments of all orders. But, just to be explicit,

$$\mathbb{E}\int_0^t [3(W_s^2 - s)]^2 ds = 9\int_0^t \mathbb{E}(W_s^4 - 2W_s^2 s + s^2)ds = 9\int_0^t 2s^2\, ds = 6t^3 < \infty$$

and hence $(Y_t)_{t\ge 0}$ is a martingale.

Problem 8. (Heat equation) Let W be a scalar Brownian motion, and let $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a smooth function that satisfy the partial differential equation

$$\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}=0$$

with terminal condition

$$g(T,x)=G(x).$$

(a) Show that $(g(t,W_t))_{t\in[0,T]}$ is a local martingale.
(b) If the function g is bounded, deduce the formula

$$g(t,x)=\int_{-\infty}^{\infty}G(x+\sqrt{T-t}z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.$$

Solution 8. (a) By Itô's formula:

$$\begin{aligned} dg(t,W_t) &= \left(\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right)dt+\frac{\partial g}{\partial x}dW_t \\ &= \frac{\partial g}{\partial x}dW_t \end{aligned}$$

and hence $(g(t,W_t))_{t\in[0,T]}$ is a local martingale.
(b) Recall a bounded local martingale is a true martingale. In particular, by the independence of the increments of Brownian motion, we have

$$\begin{aligned} g(t,W_t) &= \mathbb{E}[g(T,W_T)|\mathcal{F}_t] \\ &= \mathbb{E}[G(W_T)|\mathcal{F}_t] \\ &= \mathbb{E}[G(W_t+W_T-W_t)|\mathcal{F}_t] \\ &= \int_{-\infty}^{\infty}G(W_t+\sqrt{T-t}z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \end{aligned}$$

since $W_T-W_t\sim N(0,T-t)$. Since the above formula holds identically, we have the desired integral representation of the solution of the heat equation.

Problem 9. (Strictly local martingale) This is a technical exercise to exhibit a local martingale that is not a true martingale. Let $W=(W^1,W^2,W^3)$ be a three-dimensional Brownian motion and let $u=(1,0,0)$. It is a fact that $\mathbb{P}(W_t\neq u \text{ for all } t\geq 0)=1$.
(a) Let $X_t=|W_t-u|^{-1}$. Use Itô's formula and Lévy's characterisation of Brownian motion to show that

$$dX_t=X_t^2dZ_t,\quad X_0=1$$

where Z is a Brownian motion. In particular, show that X is a positive local martingale.
(b) By directly evaluating the integral or otherwise, show that

$$\mathbb{E}(X_t)=2\Phi(t^{-1/2})-1$$

for all $t>0$, where Φ is the distribution function of a standard normal random variable. Why does this imply that X is a strictly local martingale?

Solution 9. (a) Let $f(x_1,x_2,x_3)=((x_1-1)^2+x_2^2+x_3^2)^{-1/2}$ so that

$$\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\right)=-[f(x_1,x_2,x_3)]^3\ (x_1-1,x_2,x_3)$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}+\frac{\partial^2 f}{\partial x_3^2} &= -f^3+3f^5(x_1-1)^2+-f^3+3f^5x_2^2-f^3+3f^5x_3^2\\
&=0.
\end{aligned}
$$

In particular, Itô's formula yields

$$dX_t = -X_t^3\ [(W_t^1-1)dW_t^1+W_t^2dW_t^2+W_t^3dW_t^3].$$

Since X can be written as a stochastic integral of a three dimensional Brownian motion, it is a local martingale. Now let Z be the local martingale such that $Z_0=0$ and

$$dZ_t = -X_t[(W_t^1-1)dW_t^1+W_t^2dW_t^2+W_t^3dW_t^3].$$

Since

$$
\begin{aligned}
d\langle Z\rangle_t &= X_t^2[(W_t^1-1)^2+(W_t^2)^2+(W_t^3)^2]dt\\
&= dt
\end{aligned}
$$

by construction, the process Z is a Brownian motion by Lévy's characterisation theorem.
(b) Switch to spherical coordinates:

$$
\begin{aligned}
\mathbb{E}(X_t) &= (2\pi)^{-3/2}\iiint \frac{e^{-x_1^2/2-x_2^2/2-x_3^2/2}}{\sqrt{(\sqrt{t}x_1-1)^2+tx_2^2+tx_3^2}}dx_1\ dx_2\ dx_3\\
&= (2\pi)^{-3/2}\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{r^2\sin\theta e^{-r^2/2}}{\sqrt{tr^2-2\sqrt{t}\cos\theta+1}}d\phi\ d\theta\ dr\\
&= (2\pi)^{-1/2}\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\frac{r^2\sin\theta e^{-r^2/2}}{\sqrt{tr^2-2\sqrt{t}\cos\theta+1}}d\theta\ dr\\
&= (2\pi t)^{-1/2}\int_{r=0}^{\infty} re^{-r^2/2}\sqrt{tr^2-2\sqrt{t}\cos\theta+1}\Big|_{\theta=0}^{\pi}\ dr\\
&= (2\pi t)^{-1/2}\int_{r=0}^{\infty} 2(r\mathbb{1}_{\{r>t^{-1/2}\}}+\sqrt{t}r^2\mathbb{1}_{\{r\le t^{-1/2}\}})e^{-r^2/2}dr\\
&= 2\int_0^{t^{-1/2}}\frac{e^{-r^2/2}}{\sqrt{2\pi}}dr
\end{aligned}
$$

Note that $\mathbb{E}(X_t)<X_0$ for all $t>0$, so X is a strictly local martingale.

Problem 10. (strictly local martingales again) (a) Suppose that X is positive martingale with $X_0=1$. Fix $T>0$ and let

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X_T.$$

Let $Y_t = 1/X_t$ for all $t\ge 0$. Show that $(Y_t)_{0\le t\le T}$ is a positive martingale under $\mathbb{Q}$.
(b) Continuing from part (a), now suppose that X has dynamics

$$dX_t = X_t\sigma_t dW_t$$

where W is a Brownian motion under $\mathbb{P}$. Use Girsanov's theorem to show that there exists a $\mathbb{Q}$-Brownian motion $\hat{W}$ such that

$$dY_t = Y_t\sigma_t d\hat{W}_t$$

(c) Let X be a positive local martingale with $X_0 = 1$ and dynamics

$$dX_t = X_t^2 dW_t.$$

Our goal is to show that X is a strictly local martingale. For the sake of finding a contradiction, suppose X is a true martingale. In the notation of parts (a) and (b), show that

$$\mathbb{P}(Y_t > 0) = 1 \text{ but } \mathbb{Q}(Y_t > 0) = \Phi(t^{-1/2}).$$

Why does this contradict the assumption that X is a true martingale?

Solution 10. (a) Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and

$$\mathbb{P}(X_t > 0 \text{ for all } t) = 1 \text{ then } \mathbb{Q}(Y_t > 0 \text{ for all } t) = 1.$$

Now to show that Y is a $\mathbb{Q}$-martingale, note that

$$\mathbb{E}^{\mathbb{Q}}(Y_T|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}(X_T Y_T|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{Q}}(X_T|\mathcal{F}_t)} = \frac{1}{X_t}$$

(b) By Itô's formula,

$$\begin{aligned} dY_t &= dX_t^{-1} \\ &= -X_t^{-2} dX_t + X_t^{-3} d\langle X\rangle_t \\ &= -Y_t \sigma_t (dW_t - \sigma_t dt) \end{aligned}$$

Now by Girsanov's theorem, the process $d\check{W}_t = dW_t - \sigma_t dt$ defines a $\mathbb{Q}$ Brownian motion. And of course $\hat{W} = -\check{W}$ is a Brownian motion also.
(c) Now assuming X is a true martingale, then Girsanov's theorem applies and hence

$$dY_t = Y_t \sigma_t d\hat{W}_t = d\hat{W}_t$$

since $\sigma_t = X_t$. Hence $\mathbb{Q}(Y_t > 0) = \mathbb{Q}(\hat{W}_t > -1) = \Phi(t^{-1/2}) < 1$. But since

$$\mathbb{P}(Y_t > 0) = \mathbb{P}(X_t > 0) = 1.$$

Therefore $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent after all.