

Ex 4.1

$$\frac{dB_t}{B_t} = 2 dt$$

From π and X_0
possible to reconstruct
q by self-financing

$$\frac{dS^1}{S^1} = 3 dt + dW_t^1 - 2 dW_t^2$$

$$\frac{dS^2}{S^2} = 5 dt - 2 dW_t^1 + 4 dW_t^2$$

$$H \approx (C) (\pi)$$

Bank, stocks

$\bar{\pi}^1$ farmer
and self-financing

$$\bar{\pi}^1 = \frac{X}{P}$$

$$\bar{\pi}^2 = \frac{X}{P}$$

means divide $\bar{\pi}$

$$\begin{aligned} & \bar{\pi}^1 = \frac{1}{2} \bar{\pi} + \frac{1}{2} \bar{\pi}^2 \\ & \bar{\pi}^2 = \frac{1}{2} \bar{\pi} - \frac{1}{2} \bar{\pi}^1 \\ & \text{mean } \bar{\pi} = \frac{1}{2} (\bar{\pi}^1 + \bar{\pi}^2) \\ & \text{variance } \bar{\pi} = \frac{1}{2} (\bar{\pi}^1 - \bar{\pi}^2)^2 \end{aligned}$$

What is Black-Scholes formula?

Price of a European call under the minimal replicating cost

Black-Scholes model for asset prices (i.e. constant interest rate and

$$C(T, K) = S \Phi(d_1) - K e^{-rT} \Phi(d_2)$$
$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

+
P
z
= S
+
F
T
(S)
+
X
S

$$= \frac{1}{2} \pi r^2 s$$

$$\begin{aligned}
 & \text{Left side: } \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = 0 \\
 & \text{Right side: } \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) = 0 \\
 & \text{Bottom boundary: } u = 0 \quad \text{and} \quad v = 0 \\
 & \text{Top boundary: } u = 0 \quad \text{and} \quad v = 0
 \end{aligned}$$

A hand-drawn diagram of a face in blue ink. The face has a wide, open mouth at the bottom, two eyes above it, and a nose in the center. Above the nose, there are two curved lines representing eyebrows. To the left of the nose, there is a vertical line with a small horizontal tick mark. To the right of the nose, there is another vertical line with a small horizontal tick mark. Above the top eye, there is a letter 'E'. Above the bottom eye, there is a letter 'S'. Below the top eye, there is a letter 'Z'. Below the bottom eye, there is a letter 'F'. To the left of the mouth, there is a letter 'H'. To the right of the mouth, there is a letter 'G'. Above the letter 'E', there is a circled number '2'. Above the letter 'S', there is a circled number '3'. Below the letter 'F', there is a circled number '1'. Below the letter 'G', there is a circled number '4'. Below the letter 'H', there is a circled number '5'. Below the letter 'Z', there is a circled number '6'. To the right of the face, there is a large, vertical bracket spanning from the top of the head down to the chin. Below the face, there is a vertical arrow pointing upwards. At the very bottom, there is a large, stylized letter 'U'.

$\exists X \exists^3 \forall^6 (G)$

Strict local minima's

Let X solve

$$dX = X^{\sim} dW$$

$$W = Bm$$

$$\begin{aligned} dX &= -\frac{(z^1 - 1)^2 + (z^2 - 1)^2}{(z^1 - 1)^2 + (z^2 - 1)^2} \cdot \frac{(z^1 - 1)^2 + (z^2 - 1)^2}{(z^1 - 1)^2 + (z^2 - 1)^2} \\ &\quad \times \left(\frac{(z^1 - 1)^2 + (z^2 - 1)^2}{(z^1 - 1)^2 + (z^2 - 1)^2} \right)^3 \\ &= \frac{1}{(z^1 - 1)^2 + (z^2 - 1)^2} \cdot \frac{1}{(z^1 - 1)^2 + (z^2 - 1)^2} \end{aligned}$$

$\boxed{Z = }$

local mart.

strict \rightarrow

X is

$$X = \lim_{n \rightarrow \infty} X_n$$

$E[X_n]$.

Local mart.

characteristic

$$\mu = \frac{1}{2} \mathbb{E}[B^2]$$

$$\nu = \int_0^\infty \mathbb{P}[X_t > s] ds$$

Quadratic variation

Local mart.

$$X = \lim_{n \rightarrow \infty} X_n$$

$$= \lim_{n \rightarrow \infty} \left((Z^{(1)} - Z^{(1)})^2 + (Z^{(2)} - Z^{(2)})^2 + \dots + (Z^{(n)} - Z^{(n)})^2 \right)$$

Suppose that X_t was a true martingale given a contradiction, then

$$dX_t = X_t (X_t dW_t) \quad (\text{X}_t \in \mathcal{G}_{t-5})$$

$$- \int_s^t X_s ds + \int_s^t X_s dw_s$$

?

$$\text{Fix } T > 0. \quad \text{Let } G := \frac{dQ}{dP} = X_T.$$

Girsanov theorem: $\tilde{W}_t = W_t - \int_0^t X_s ds$ is a Q-BM.

$$P_k X = \langle X \rangle P$$

Contract, un

$$\frac{dX}{dp} = \frac{X^2 - p^2}{X^2 + p^2}$$

- X " Y + z

11

11

(3)
e
l

1

$$\begin{array}{r}
 \frac{1}{x^3} x^4 + \\
 + \frac{1}{x^3} x^3 - \\
 \hline
 \end{array}$$

Summer

1. $\sin \theta = \frac{y}{r}$
2. $\cos \theta = \frac{x}{r}$

3(b) Sample

\mathcal{Z} adapted, measurable.

$$U_T = \mathcal{Z}_T$$

$$U_t = \max \{ \mathcal{Z}_t, \mathbb{E}(U_{t+1} | \mathcal{F}_t) \}$$

$$\mathcal{Z}_t = \left(\mathcal{Z}_0 + \mathcal{Z}_1 + \dots + \mathcal{Z}_t \right)$$

\mathcal{Z}_t are i.i.d.

$S_t = (S_0, \dots, S_t)$ generate filtration.

$$\mathcal{Z}_t = f(S_t)$$

f function $\Rightarrow U_t = V(t, S_t)$

Show

T

$$u_T = \mathbb{E}_T^+$$

$$f(S_T) = V(T, S) = f(S)$$

Suppose

$$u_{t+1} =$$

$$V(t+1, S_{t+1})$$

$$u_t = \max \left\{ \mathbb{E}_t^+ \left(u_{t+1} \mid \mathcal{F}_t \right) \right\}$$

$$= \max \left\{ f(S_t), \mathbb{E} \left[V(t+1, S_{t+1}) \mid \mathcal{F}_t \right] \right\}$$

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$$SV(t+1, S_{t+1}) \sim d\nu$$

$$M = \text{law of } \sum_{t=1}^{T+1}$$

$$u_t = V(t, S_t)$$

$$u_t = \max \left\{ f(S), SV(t+1, S_{t+1}) \sim d\nu \right\}$$

where

$$V(t, S) =$$

induction

Sample $\mathcal{F}(\omega)$

$$d\beta = \beta \alpha dt$$

$$\beta_t = e^{(a-b)t}$$

$$\begin{aligned} dS &= S(\alpha dt + (\alpha - b)S) \\ dV &= V(\alpha dt + cS) \end{aligned}$$

W, Z are indep BM.

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$$Z_t =$$

$$F(t, S_t, V_t)$$

$$F(T, S_t, V_t) = S$$

where:

$$\frac{dF}{dt} + S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 F}{\partial S^2} = 0$$

$$S_p = \frac{1}{2} \sigma^2 V^2 + C_P S^2$$

$$\frac{\partial F}{\partial S} =$$

$$(a - bV) \frac{\partial V}{\partial S}$$

$$S_n =$$

$$T -$$

$$t$$

If there exists a risk-central function α such that no other relative to β is risk-central then α and β are loc_{mar + s}

Claim: If α is risk-central then $\alpha \leq \beta$

Show

Surf + other stuff = P.

$$\int_0^t$$

$$O / B$$

+



$\int_0^t \frac{S}{B}$ is a martingale
with respect
to BM_S

(stochastic integral
for BM_S)

$$\begin{aligned} P &= e^{-rt} S + e^{-rt} \int_0^t e^{r u - \frac{1}{2} \sigma^2 u} dW_u \\ &= e^{-rt} S + e^{-rt} \left(\int_0^t e^{r u - \frac{1}{2} \sigma^2 u} du \right) P \\ &= e^{-rt} S + e^{-rt} \left(\int_0^t e^{r u - \frac{1}{2} \sigma^2 u} du \right) P \\ &= e^{-rt} S + e^{-rt} \left(\int_0^t e^{r u - \frac{1}{2} \sigma^2 u} du \right) P \\ &= e^{-rt} S - e^{-rt} \int_0^t e^{r u - \frac{1}{2} \sigma^2 u} du P \\ &= e^{-rt} \left(S - \int_0^t e^{r u - \frac{1}{2} \sigma^2 u} du \right) P \end{aligned}$$

ordinary
product
rule

D

is

Equivalent Martingale
à numériser

means relative
(local)

$$\text{means } \left(\frac{P_t}{N_t} \right)_t$$

means

means relative
 \Rightarrow non-neg
then
 $\Theta + I$
then
non
increasing

Non
increasing
means
local martingales

Non
increasing
means
supermartingales

Non
increasing
means
true martingales

Risk neutral: E from real life

$$E(X_3 | \mathcal{F}_t) = F_t = \frac{1}{T} \left(x_T + \sum_{\tau=t+1}^T (x_\tau - \bar{x}_{t+1:T}) \right)$$

$\bar{x}_{t+1:T} = \frac{1}{T-t} \sum_{\tau=t+1}^T x_\tau$

Ex 3.2 (c)

Health-Tariff-Mutation (1993)
continuous time.

$T_{\text{take}} < T$ forward rate curve as
given. (unlike
spot rate is given)

$$f_{t,T} = \frac{(1 + f_{t-1,T}) Z_{t,T}}{1 + f_{t,T}} \quad 1 \leq t \leq T$$

$$\mathbb{E} \left[\prod_{u=t+1}^T Z_{t,u} \right] = \frac{\mathbb{E} \left[F_{t-1} \right]}{\sigma_{t,T}^2 + \mu_{t,T}}$$

$\sigma_{t,T}, \mu_{t,T}$
means
 $\frac{1}{T-t}$

for $f \in \mathcal{N}(\omega)$, $\|f\|_{H^1}$ is defined as

$$E = \left(\sum_{u=t+1}^{\infty} M_{t,u} \right) + \left(\sum_{u=t+1}^{\infty} b_{t,u} \right)$$

$$\frac{1}{2} \left(\sum_{n=t+1}^T \sigma_{t,n}^2 \right) = \sum_{n=t+1}^{T-1} \left(\sum_{n=t+1}^T \sigma_{t,n}^2 \right) - \sum_{n=t+1}^{T-1} \left(\sum_{n=t+1}^T M_{t,n} \right)$$

$$\begin{aligned}
 & \sim \left(\begin{array}{c} T \\ i(y) \\ \hline -1 \end{array} \right) \sim \begin{array}{c} T^- \\ b^+ \\ -1 \end{array} \\
 & + \\
 & \sim \left(\begin{array}{c} T^- \\ b^+ \\ \hline + \end{array} \right) \quad \begin{array}{c} T^- \\ b^+ \\ \hline -1 \end{array} \\
 & \sim \left(\begin{array}{c} T^- \\ b^+ \\ \hline -1 \end{array} \right) \quad \left(\begin{array}{c} T^- \\ b^+ \\ \hline -1 \end{array} \right) \\
 & \sim \left(\begin{array}{c} T^- \\ b^+ \\ \hline -1 \end{array} \right) \quad \left(\begin{array}{c} T^- \\ b^+ \\ \hline -1 \end{array} \right)
 \end{aligned}$$

H^m :

$$\frac{d}{dt} f_{t,T} = M_{t,T} dt + \sigma_{t,T} dW_t$$

$$P_{t,T} = e^{-\int_t^T f_{s,T} ds - \frac{1}{2} \int_t^T f_{s,T}^2 ds}$$

It is not difficult to check condition (iii)

$$M_{t,T} = \frac{\partial}{\partial t} P_{t,T} = \frac{1}{P_{t,T}} \left(- \int_0^t r_s ds + \int_0^T b_{t,u} du \right)$$

ensures that

$$f_{t,T} = \left(- \int_0^t r_s ds + \int_0^T b_{t,u} du \right)_{0 \leq t \leq T}$$

is a local martingale

(means means in \vee is neutral
 \exists non blank) \Rightarrow no arrears

Ex 3 & 6 (c)

$$X_t = e^{\alpha t} X_0 + b \int_0^t e^{\alpha(t-s)} dW_s$$

$$dX = \alpha X dt + b dW$$

first of
 $\sum_s X_s + dt$

continuous.

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$$\xrightarrow{P} X_t(\omega)$$

$$\int_s^T X_e dt \xrightarrow{P} \lim_{n \rightarrow \infty} \sum_{i=1}^n X_{t_i} \Delta t_i$$

Riemann

$$\text{a.s.} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n X_{t_i} \Delta t_i$$

$$X_t = X_0 + a \int_0^t X_s ds + b W_t$$

$$E\left(\int_0^t \left(\int_0^s f(s,u) dW_u\right)^2 ds\right) < \infty$$

$$\frac{1}{2} E\left(\int_0^t \int_0^s X_s dX_s\right) = \frac{1}{2} \left(\int_0^t X_s dX_s - \int_0^t \int_0^s X_s dW_s \right)$$

sufficiently continuous
in (s,u) + random

$$= \frac{1}{2} \left(e^{a+} - e^{-a} \right) = \frac{1}{2} \sinh s$$

$$= \frac{1}{2} \left(e^{a+} - e^{-a} \right) x + b \int_0^t \left(e^{a(t-s)} - e^{-a(t-s)} \right) dW_s$$

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$$= \frac{1}{2} \left(e^{a+} - e^{-a} \right) x + b \int_0^t e^{a(t-s)} dW_s$$

A,B intervals

$$f = \frac{1}{2} A \times B$$

$$\begin{aligned} &= \int_0^t \int_0^s e^{a(s-u)} dW_u ds \\ &\approx \int_0^t \int_0^s e^{a(s-u)} dW_u ds = \int_0^t \int_0^s f(s,u) dW_u ds \\ &\approx \int_0^t \int_0^s f(s,u) dW_u ds = \int_0^t \int_0^s f(s,u) dW_u ds \\ &\Rightarrow x \sim \mathcal{N}(0, \int_0^t \int_0^s f(s,u)^2 ds) \end{aligned}$$