STAT391, Lecture 2

Relationships between interest rate dynamics

Again we are given the triplet $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the usual conditions. Furthermore W is a Brownian motion w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

We consider the following three dynamics

1. The short rate dynamics
$$dr_t = a_t dt + b_t dW_t. \tag{0.1}$$
Here a_t and b_t are adapted processes and chosen so that r_t given by (0.1) is well defined.

2. The bond price dynamics
$$dP(t,T) = m(t,T)P(t,T)dt + v(t,T)P(t,T)dW_t, \quad t \leq T. \tag{0.2}$$
Here the terminal time T remains fixed. Furthermore $m(t,T)$ and $v(t,T)$ are adapted processes (w.r.t. running time t) and chosen so that $P(t,T)$ given by (0.2) is well defined.

3. The forward rate dynamics
$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad t \leq T \tag{0.3}$$
Here again the terminal time T remains fixed and $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes (w.r.t. running time t) and chosen so that $f(t,T)$ given by (0.3) is well defined.

In (0.1) we can for example let $a_t = \theta(t) - ar_t$ and $b_t = \sigma$ giving the Hull and White model (See Exercise 1, Homework 1)

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t.$$

We then have the following important result

Theorem 0.1 *Assume that m, v, α and σ in (0.2) and (0.3) are continuously differentiable w.r.t. time to maturity T, and that they are sufficiently regular for the Fubini theorem and the stochastic Fubini theorem to hold as well as the necessary differentiations under the integral sign. Let $m_T(t,T) = \frac{\partial}{\partial T}m(t,T)$ and similarly with $v_T(t,T)$, $\alpha_T(t,T)$ and $\sigma_T(t,T)$. Then we have the following.*

a) If $P(t,T)$ satisfies (0.2) *then*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t,$$

where

$$\begin{aligned}
\alpha(t,T) &= v_T(t,T)v(t,T) - m_T(t,T), \\
\sigma(t,T) &= -v_T(t,T)
\end{aligned}$$

b) If $f(t,T)$ satisfies (0.3) *then*

$$dr_t = a_t dt + b_t dW_t,$$

where

$$\begin{aligned}
a_t &= f_T(t,t) + \alpha(t,t), \\
b_t &= \sigma(t,t)
\end{aligned}$$

c) If $f(t,T)$ satisfies (0.3) *then*

$$dP(t,T) = m(t,T)P(t,T)dt + v(t,T)P(t,T)dW_t,$$

where

$$\begin{aligned}
m(t,T) &= r_t + A(t,T) + \frac{1}{2}S^2(t,T), \\
v(t,T) &= S(t,T)
\end{aligned}$$

and

$$\begin{aligned}
A(t,T) &= -\int_t^T \alpha(t,s)ds, \\
S(t,T) &= -\int_t^T \sigma(t,s)ds.
\end{aligned}$$

Proof We first prove part a. Set

$$Y_t = \log P(t,T) = -\int_t^T f(t,s)ds.$$

Itô's formula then gives

$$\begin{aligned}
dY_s &= \frac{1}{P(s,T)}dP(s,T) - \frac{1}{2}\frac{1}{P^2(s,T)}(dP(s,T))^2 \\
&= a(s,T)ds + v(s,T)dW_s
\end{aligned}$$

where

$$a(s,T) = m(s,T) - \frac{1}{2}v^2(s,T).$$

Note that

$$a_T(t,T) = m_T(t,T) - v_T(t,T)v(t,T) = -\alpha(t,T).$$

Now integrate the expression for dY_s from $s=0$ to t. This gives

$$Y_t - Y_0 = -\int_t^T f(t,s)ds + \int_0^T f(0,s)ds = \int_0^t a(s,T)ds + \int_0^t v(s,T)dW_s.$$

Taking the partial derivative $\frac{\partial}{\partial T}$ on both sides, and changing derivation and integration gives

$$-f(t,T) + f(0,T) = \int_0^t a_T(s,T)ds + \int_0^t v_T(s,T)dW_s.$$

Finally take the differential w.r.t. t on both sides. This gives

$$-df(t,T) = a_T(t,T)dt + v_T(t,T)dW_t.$$

The result follows from the above expression for $a_T(t,T)$.

Now part b. Integrating the expresssion

$$df(s,t) = \alpha(s,t)ds + \sigma(s,t)dW_s$$

from $s=0$ to t and using that $f(t,t) = r_t$ gives,

$$r_t = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s.$$

Using the relations

$$\begin{aligned}
\alpha(s,t) &= \alpha(s,s) + \int_s^t \alpha_T(s,u)du, \\
\sigma(s,t) &= \sigma(s,s) + \int_s^t (\sigma_T(s,u)du
\end{aligned}$$

gives

$$\begin{aligned}
r_t &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \alpha_T(s,u)duds \\
&\quad + \int_0^t \sigma(s,s)dW_s + \int_0^t \int_s^t \sigma_T(s,u)dudW_s \\
&= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \int_0^u \alpha_T(s,u)dsdu \\
&\quad + \int_0^t \sigma(s,s)dW_s + \int_0^t \int_0^u \sigma_T(s,u)dW_sdu,
\end{aligned}$$

where we used the Fubini theorem and the stochastic Fubini theorem. Taking the differential w.r.t. t gives

$$\begin{aligned} dr_t &= f_T(0,t)dt + \alpha(t,t)dt + \left(\int_0^t \alpha_T(s,t)ds\right)dt \\ &\quad + \sigma(t,t)dW_t + \left(\int_0^t \sigma_T(s,t)dW_s\right)dt. \end{aligned} \tag{0.4}$$

From the relation

$$f(t,u) = f(0,u) + \int_0^t \alpha(s,u)ds + \int_0^t \sigma(s,u)dW_s$$

we get by taking the partial derivative w.r.t u

$$f_T(t,u) = f_T(0,u) + \int_0^t \alpha_T(s,u)ds + \int_0^t \sigma_T(s,u)dW_s.$$

In particular setting $u = t$ this becomes

$$f_T(t,t) = f_T(0,t) + \int_0^t \alpha_T(s,t)ds + \int_0^t \sigma_T(s,t)dW_s.$$

We therefore get from (0.4)

$$dr_t = (f_T(t,t) + \alpha(t,t))dt + \sigma(t,t)dW_t$$

and the result is proved.

Now the proof of part c. As in the proof of part a, let $Y_t = \log P(t,T)$. Then again using the Fubini and stochastic Fubini theorems we get

$$\begin{aligned} Y_t &= -\int_t^T f(t,u)du \\ &= -\int_t^T \left(\int_{s=0}^t df(s,u) + f(0,u)\right)du \\ &= -\int_t^T f(0,u)du - \int_{s=0}^t \int_{u=t}^T du df(s,u) \\ &= -\int_0^T f(0,u)du - \int_{s=0}^t \int_{u=s}^T du df(s,u) + \int_0^t f(0,u)du + \int_{s=0}^t \int_{u=s}^t du df(s,u) \\ &= Y_0 - \int_{s=0}^t \int_{u=s}^T (\alpha(s,u)duds + \sigma(s,u)dudW_s) \\ &\quad + \int_0^t f(0,u)du + \int_{s=0}^t \int_{u=s}^t \alpha(s,u)duds + \int_{s=0}^t \int_{u=s}^t \sigma(s,u)dudW_s \\ &= Y_0 + \int_0^t A(s,T)ds + \int_0^t S(s,T)dW_s \\ &\quad + \int_0^t \left(f(0,u) + \int_0^u \alpha(s,u)ds + \int_0^u \sigma(s,u)dW_s\right)du \\ &= Y_0 + \int_0^t A(s,T)ds + \int_0^t S(s,T)dW_s + \int_0^t r_u du. \end{aligned}$$

Here we used that

$$Y_0 = -\int_0^T f(0,u)du$$

and that

$$r_u = f(u,u) = f(0,u) + \int_0^u \alpha(s,u)ds + \int_0^u \sigma(s,u)dW_s.$$

To finish, note that $P(t,T) = f(Y_t)$ where $f(x) = e^x$. Since $f'(x) = f''(x) = f(x)$, we get by Itô's formula

$$\begin{aligned} dP(t,T) &= e^{Y_t}dY_t + \frac{1}{2}e^{Y_t}(dY_t)^2 \\ &= P(t,T)(r_t + A(t,T) + \frac{1}{2}S^2(t,T))dt + P(t,T)S(t,T)dW_t. \end{aligned}$$

This finishes the proof.

Short rate models

The Vasicek approach to non-arbitrage pricing

We will assume that the short rate follows the SDE

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t,$$

where μ and σ are chosen so that this equation has a unique solution.

The aim is to price a bond $P(t,T)$ in a consistent and arbitrage-free way. Assume at time t the price can be written as

$$P(t,T) = F(t,r_t,T) = F^T(t,r_t)$$

for all t and T with $t \leq T$, and for F^T a $C^{1,2}$ function. Obviously we must have $F^T(T,r_T) = 1$. By Itô's formula

$$\begin{aligned} dF^T(t,r_t) &= (F_t^T(t,r_t) + \mu(t,r_t)F_x^T(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)F_{xx}^T(t,r_t))dt \\ &\quad +\sigma(t,r_t)F_x^T(t,r_t)dW_t \\ &= \alpha_t^T F^T(t,r_t)dt + \sigma_t^T F^T(t,r_t)dW_t, \end{aligned}$$

where

$$\alpha_t^T = \frac{F_t^T(t,r_t) + \mu(t,r_t)F_x^T(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)F_{xx}^T(t,r_t)}{F^T(t,r_t)} \tag{0.5}$$

and

$$\sigma_t^T = \frac{\sigma(t,r_t)F_x^T(t,r_t)}{F^T(t,r_t)}. \tag{0.6}$$

Consider a self-financing portfolio with relative value u_t^S in S-bonds and u_t^T in T-bonds so that

$$u_t^S + u_t^T = 1. \tag{0.7}$$

Then the value of the portfolio, V_t, satisfies (due to self-financing)

$$\begin{aligned}
\frac{dV_t}{V_t} &= u_t^S \frac{dF^S(t,r_t)}{F^S(t,r_t)} + u_t^T \frac{dF^T(t,r_t)}{F^T(t,r_t)} \\
&= (u_t^S \alpha_t^S + u_t^T \alpha_t^T)dt + (u_t^S \sigma_t^S + u_t^T \sigma_t^T)dW_t.
\end{aligned}$$

In order for this portfolio to be risk free, it is necessary that

$$u_t^S \sigma_t^S + u_t^T \sigma_t^T = 0. \tag{0.8}$$

But if it is risk free, it must earn the risk free rate of return, i.e.

$$u_t^S \alpha_t^S + u_t^T \alpha_t^T = r_t. \tag{0.9}$$

Solving (0.7) and (0.8) for u_t^S and u_t^T gives

$$\begin{aligned}
u_t^T &= -\frac{\sigma_t^S}{\sigma_t^T - \sigma_t^S} \\
u_t^S &= \frac{\sigma_t^T}{\sigma_t^T - \sigma_t^S}
\end{aligned}$$

Inserting these solutions into (0.9) then yields

$$\frac{\alpha_t^S \sigma_t^T - \alpha_t^T \sigma_t^S}{\sigma_t^T - \sigma_t^S} = r_t.$$

Multiplying both sides by $\sigma_t^T - \sigma_t^S$ and moving terms to the other side gives

$$(\alpha_t^S - r_t)\sigma_t^T = (\alpha_t^T - r_t)\sigma_t^S$$

Finally dividing both sides by $\sigma_t^T \sigma_t^S$ yields the fundamental relation

$$\frac{\alpha_t^S - r_t}{\sigma_t^S} = \frac{\alpha_t^T - r_t}{\sigma_t^T}.$$

Since the left hand side only depends on the time to maturity S, while the right hand only depends on T, and S and T are arbitrary, it follows that they must both be independent of S and T. Therefore we can define the *Market price of risk* as

$$\lambda_t = \lambda(t, r_t) = \frac{\alpha_t^T - r_t}{\sigma_t^T}. \tag{0.10}$$

and this is independent of time to maturity.

From (0.10) and (0.6) we have

$$\alpha_t^T F^T = (r_t + \lambda_t \sigma_t^T) F^T = r_t F^T + \lambda_t \sigma F_x^T, \tag{0.11}$$

where for simplicity we wrote σ for $\sigma(t, r_t)$ and F^T for $F^T(t, r_t)$. However, from (0.5) we get

$$\alpha_t^T F^T = F_t^T + \mu F_x^T + \frac{1}{2}\sigma^2 F_{xx}^T. \tag{0.12}$$

Equating (0.11) and (0.12) easily gives

$$F_t^T(t, r_t) + \frac{1}{2}\sigma^2(t, r_t) F_{xx}^T(t, r_t) + (\mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t)) F_x^T(t, r_t) - r_t F^T(t, r_t) = 0$$

with boundary condition

$$F^T(T, r_T) = 1.$$

Therefore we must solve the PDE

$$F_t^T(t, r) + \frac{1}{2}\sigma^2(t, r) F_{xx}^T(t, r) + (\mu(t, r) - \lambda(t, r)\sigma(t, r)) F_x^T(t, r) - r F^T(t, r) = 0 \tag{0.13}$$

over $[0, T)$ and the relevant interval for r. The boundary condition is

$$F^T(T, r) = 1.$$

If instead we are pricing a derivative which at time of expiration T depends on r_T only, i.e. its value is of the form $\Phi(r_T)$, we would have the same PDE, only the boundary condition would change to

$$F^T(T, r) = \Phi(r).$$

From the Feynman-Kac formula it follows that $F^T(t, r)$ can be written as

$$F^T(t, r) = \tilde{E}^{t,r}\left[e^{-\int_t^T r_s ds}\right], \tag{0.14}$$

where under the measure $\tilde{P}$,

$$dr_t = (\mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t))dt + \sigma(t, r_t) d\tilde{W}_t \tag{0.15}$$

and $\tilde{W}$ is a $\tilde{P}$ Brownian motion. More exactly

$$\tilde{W}_t = W_t + \int_0^t \lambda(s, r_s) ds.$$

If we instead priced a derivative with value at expiration T equal to $\Phi(r_T)$, its value at time t would be

$$\Pi(t, r) = \tilde{E}^{t,r}\left[\Phi(r_T) e^{-\int_t^T r_s ds}\right]. \tag{0.16}$$

Equivalent to (0.14) we have

$$P(t,T) = \tilde{E}\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right].$$

Let

$$Z(t,T) = B_t^{-1} P(t,T) = e^{-\int_0^t r_s ds} \tilde{E}\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = \tilde{E}\left[e^{-\int_0^T r_s ds} \middle| \mathcal{F}_t \right],$$

hence Z is a $\tilde{P}$ martingale w.r.t. $\{\mathcal{F}_t\}_{t \geq 0}$. Informally, we therefore get by the Itô representation theorem (informally since we do not know whether $\tilde{W}$ generates $\{\mathcal{F}_t\}_{t \geq 0}$),

$$Z(t,T) = P(0,T) + \int_0^t \tilde{h}_s^T d\tilde{W}_s = P(0,T) + \int_0^t h_s^T Z(s,T) d\tilde{W}_s$$

since $Z(0,T) = P(0,T)$ and $Z(t,T) > 0$. From $P(t,T) = B_t Z(t,T)$ we get by using the differential rule for a product

$$dP(t,T) = r_t B_t Z(t,T)dt + B_t dZ(t,T) = r_t P(t,T)dt + h_t^T P(t,T) d\tilde{W}_t. \qquad (0.17)$$

Thus under the measure $\tilde{P}$, $P(t,T)$ has drift equal to the risk free short rate r_t. Under the original measure we have then

$$dP(t,T) = (r_t + h_t^T \lambda(t, r_t)) P(t,T)dt + h_t^T P(t,T) dW_t.$$