AE 352: Homework 6

Max A. Feinberg

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1. **a.**

i.

$$\boxed{T = \frac{1}{2}m\dot{x}^2}$$

$$\boxed{V = \frac{1}{2}kx^2}$$

ii.

$$L = T - V$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2}$$

iii.

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\big(m\dot{x}\big) + kx &= 0 \\
m\ddot{x} + kx &= 0
\end{aligned}$$

$$\boxed{m\ddot{x} + kx = 0}$$

b.

i.

$$\boxed{T = \frac{1}{2}m\dot{x}^2}$$

$$\boxed{V = \frac{1}{2}k\big(x^2 + 1^2\big)}$$

ii.

$$L = T - V$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\big(x^2 + 1^2\big)}$$

iii.

$$\begin{aligned}
\frac{d}{dt}\big(\frac{\partial L}{\partial \dot{q}_i}\big) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\big(m\dot{x}\big) + kx &= 0 \\
m\ddot{x} + kx &= 0
\end{aligned}$$

$$\boxed{m\ddot{x} + kx = 0}$$

c.

i.

$$\boxed{T = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big)}$$

$$\boxed{V = \frac{1}{2}k\big((x-1)^2 + y^2\big) + \frac{1}{2}k\big((x+1)^2 + y^2\big)}$$

ii.

$$L = T - V$$

$$\boxed{L = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) - \frac{1}{2}k\big((x-1)^2 + y^2\big) - \frac{1}{2}k\big((x+1)^2 + y^2\big)}$$

iii.

$$\begin{aligned}
\frac{d}{dt}\big(\frac{\partial L}{\partial \dot{q}_i}\big) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\big(m\dot{x}\big) + k(x-1) + k(x+1) &= 0 \\
m\ddot{x} + k(x-1) + k(x+1) &= 0 \\
m\ddot{x} + 2kx &= 0 \\
\frac{d}{dt}\big(m\dot{y}\big) + ky + ky &= 0 \\
m\ddot{y} + 2ky &= 0
\end{aligned}$$

$$\boxed{m\ddot{x} + 2kx = 0}$$

$$\boxed{m\ddot{y} + 2ky = 0}$$

2. **a.**

$$
\begin{aligned}
x &= L\sin(\theta)\cos(\phi) \\
\dot{x} &= L\dot{\theta}\cos(\theta)\cos(\phi) - L\dot{\phi}\sin(\theta)\sin(\phi) \\
y &= -L\cos(\theta) \\
\dot{y} &= L\dot{\theta}\sin(\theta) \\
z &= L\sin(\theta)\sin(\phi) \\
\dot{z} &= L\dot{\theta}\cos(\theta)\sin(\phi) + L\dot{\phi}\sin(\theta)\cos(\phi) \\
T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \\
T &= \frac{1}{2}m((L\dot{\theta}\cos(\theta)\cos(\phi) - L\dot{\phi}\sin(\theta)\sin(\phi))^2 + (L\dot{\theta}\sin(\theta))^2 + (L\dot{\theta}\cos(\theta)\sin(\phi) + L\dot{\phi}\sin(\theta)\cos(\phi))^2) \\
&\mathit{or} \\
T &= \frac{1}{2}mL^2\sin(\theta)^2\dot{\phi}^2 + \frac{1}{2}mL^2\dot{\theta}^2 \\
V &= -mgL\cos(\theta)
\end{aligned}
$$

$$\boxed{T = \frac{1}{2}mL^2\sin(\theta)^2\dot{\phi}^2 + \frac{1}{2}mL^2\dot{\theta}^2}$$

$$\boxed{V = -mgL\cos(\theta)}$$

b.

$$L = T - V$$

$$\boxed{L = \frac{1}{2}mL^2\sin(\theta)^2\dot{\phi}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos(\theta)}$$

c.

$$
\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}(mL^2\dot{\theta}) - (mL^2\sin(\theta)\cos(\theta)\dot{\phi}^2 - mgL\sin(\theta)) &= 0 \\
mL^2\ddot{\theta} - mL^2\sin(\theta)\cos(\theta)\dot{\phi}^2 + mgL\sin(\theta) &= 0 \\
\ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}^2 + \frac{g}{L}\sin(\theta) &= 0 \\
\frac{d}{dt}(mL^2\sin(\theta)^2\dot{\phi}) &= 0 \\
2mL^2\sin(\theta)\cos(\theta)\dot{\phi} + mL^2\sin(\theta)^2\ddot{\phi} &= 0 \\
\sin(\theta)^2\ddot{\phi} + 2\sin(\theta)\cos(\theta)\dot{\phi} &= 0
\end{aligned}
$$

$$\boxed{\ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}^2 + \frac{g}{L}\sin(\theta) = 0}$$

$$\boxed{\sin(\theta)^2\ddot{\phi} + 2\sin(\theta)\cos(\theta)\dot{\phi} = 0}$$

3. **a.**

i.

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
\vec{F} &= m\ddot{x} \\
\vec{F} &= m\frac{d^2}{dt^2}(Rsin(\theta)) \\
\vec{F} &= m\frac{d}{dt}(R\dot{\theta}\cos(\theta)) \\
\vec{F} &= m(R\ddot{\theta}\cos(\theta) - R\dot{\theta}^2\sin(\theta)) \\
0 &= m(R\ddot{\theta}\cos(\theta) - R\dot{\theta}^2\sin(\theta)) \\
R\dot{\theta}^2\sin(\theta) &= R\ddot{\theta}\cos(\theta) \\
\dot{\theta}^2 &= \cot(\theta)\ddot{\theta} \\
\text{now} & \text{ we will consider the y-direction} \\
\vec{F} &= m\ddot{y} \\
\vec{F} &= m\frac{d^2}{dt^2}(-Rcos(\theta)) \\
\vec{F} &= m\frac{d}{dt}(R\dot{\theta}\sin(\theta)) \\
\vec{F} &= m(R\ddot{\theta}\sin(\theta) + R\dot{\theta}^2\cos(\theta)) \\
-mg &= m(R\ddot{\theta}\sin(\theta) + R\dot{\theta}^2\cos(\theta)) \\
-g &= R\ddot{\theta}\sin(\theta) + R\cos(\theta)\cot(\theta)\ddot{\theta} \\
-g\sin(\theta) &= R\ddot{\theta}\sin(\theta)^2 + R\cos(\theta)^2\ddot{\theta}) \\
\ddot{\theta} &= -\frac{g}{R}\sin(\theta)
\end{aligned}
$$

$$\boxed{\ddot{\theta} = -\frac{g}{R}\sin(\theta)}$$

ii.

$$
\begin{aligned}
x &= R\sin(\theta) \\
\dot{x} &= R\dot{\theta}\cos(\theta) \\
y &= -R\cos(\theta) \\
\dot{y} &= R\dot{\theta}\sin(\theta) \\
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
T &= \frac{1}{2}m(R^2\dot{\theta}^2\cos(\theta)^2 + R^2\dot{\theta}^2\sin(\theta)^2) \\
T &= \frac{1}{2}m(R^2\dot{\theta}^2) \\
V &= mgR(1 - \cos(\theta))
\end{aligned}
$$

$$\boxed{T = \frac{1}{2}m(R^2\dot{\theta}^2)}$$

$$\boxed{V = mgR(1 - \cos(\theta))}$$

iii.

$$L = T - V$$

$$\boxed{L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR(1 - \cos(\theta))}$$

iv.

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}(mR^2\dot{\theta}) + mgR\sin(\theta) &= 0 \\
mR^2\ddot{\theta} + mgR\sin(\theta) &= 0 \\
R\ddot{\theta} + g\sin(\theta) &= 0
\end{aligned}$$

$$\boxed{\ddot{\theta} = -\frac{g}{R}\sin(\theta)}$$

b.

i. in this problem, X will denote the position of the large block M. Lower case x will denote the position of the the smaller mass m.

$$\begin{aligned}
x &= R\sin(\theta) \\
\dot{x} &= R\dot{\theta}\cos(\theta) \\
y &= -R\cos(\theta) \\
\dot{y} &= R\dot{\theta}\sin(\theta) \\
T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x} + \dot{X})^2 + \frac{1}{2}m\dot{y}^2 \\
T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 \\
V &= mgR(1 - \cos(\theta))
\end{aligned}$$

$$\boxed{T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2}$$

$$\boxed{V = mgR(1 - \cos(\theta))}$$

ii.

$$\begin{aligned}
L &= T - V \\
L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 - mgR(1 - \cos(\theta)) \\
L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R^2\dot{\theta}^2\cos(\theta)^2 + 2R\dot{X}\dot{\theta}\cos(\theta) + \dot{X}^2) + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 - mgR(1 - \cos(\theta))
\end{aligned}$$

$$\boxed{L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{X}\dot{\theta}\cos(\theta) - mgR(1-\cos(\theta))}$$

iii.

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}(M\dot{X} + m\dot{X} + mR\dot{\theta}\cos(\theta)) &= 0 \\
M\ddot{X} + m\ddot{X} + mR\ddot{\theta}\cos(\theta) - mR\dot{\theta}^2\sin(\theta) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}(mR^2\dot{\theta} + mR\dot{X}\cos(\theta)) - (-mR\dot{X}\dot{\theta}\sin(\theta) - mgR\sin(\theta)) &= 0 \\
mR^2\ddot{\theta} + mR\ddot{X}\cos(\theta) - mR\dot{X}\dot{\theta}\sin(\theta) + mR\dot{X}\dot{\theta}\sin(\theta) + mgR\sin(\theta) &= 0 \\
mR^2\ddot{\theta} + mR\ddot{X}\cos(\theta) + mgR\sin(\theta) &= 0 \\
R\ddot{\theta} + \ddot{X}\cos(\theta) + g\sin(\theta) &= 0
\end{aligned}$$

$$\boxed{M\ddot{X} + m\ddot{X} + mR\ddot{\theta}\cos(\theta) - mR\dot{\theta}^2\sin(\theta) = 0}$$

$$\boxed{R\ddot{\theta} + \ddot{X}\cos(\theta) + g\sin(\theta) = 0}$$

iv. Please see the attached MATLAB code at the end of the document.

Figure 1: x vs time, short time scale

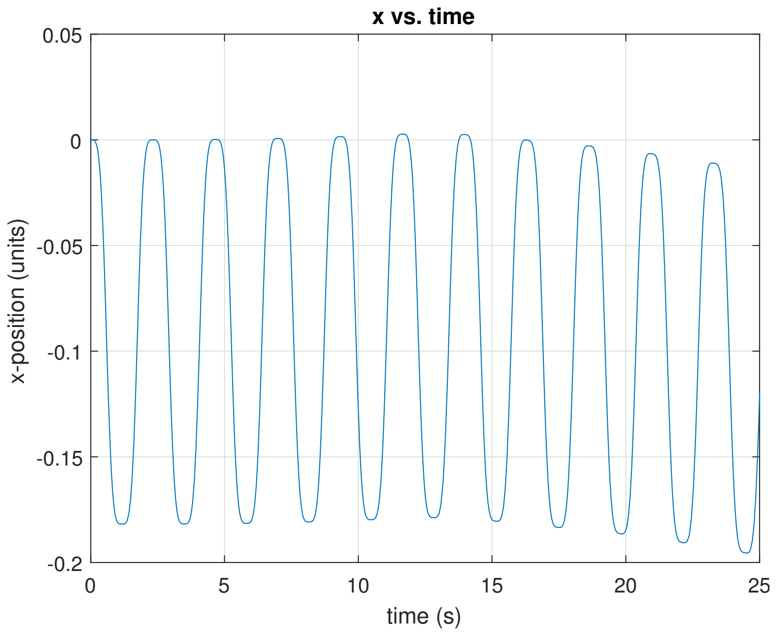


Figure 2: θ vs time, short time scale

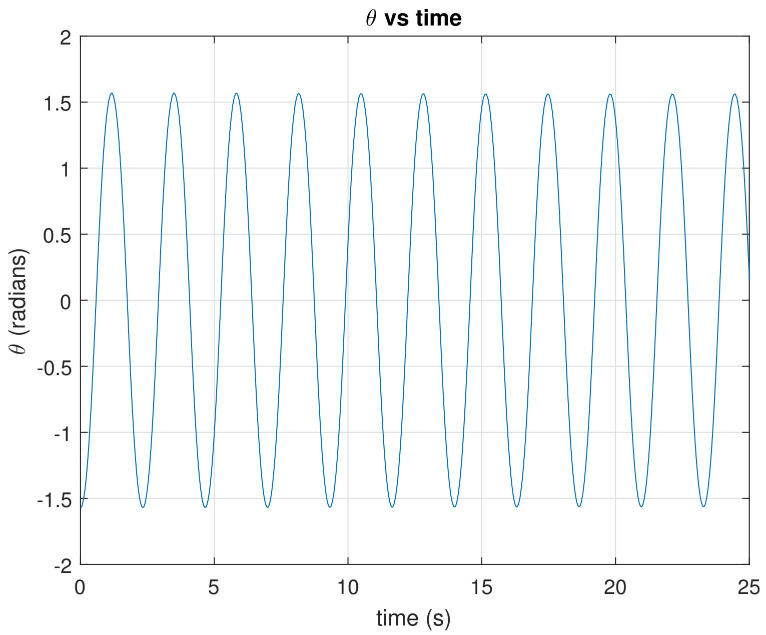


Figure 3: x vs time, Long time scale

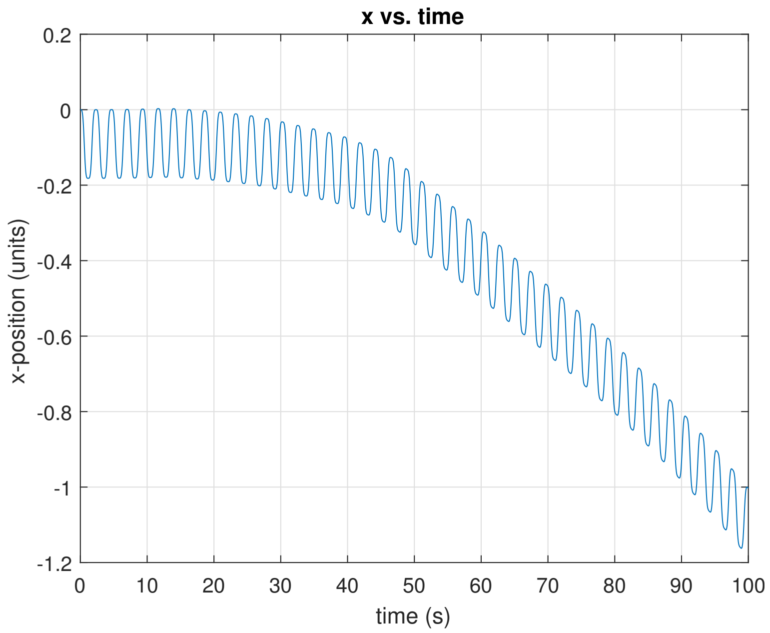
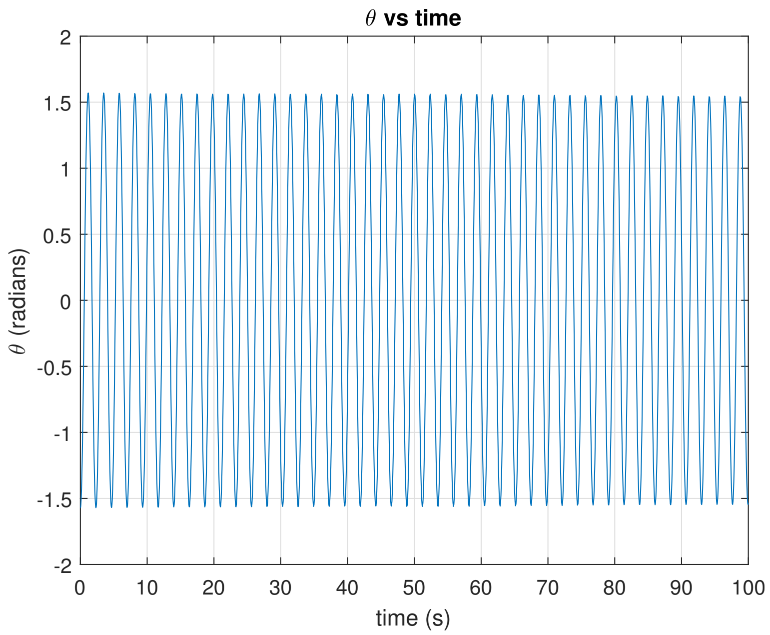


Figure 4: θ vs time, long time scale



v. Please see the attached MATLAB code at the end of the document. The collision point can be found by inspecting Figure 6. The two small masses collide at roughly 0.0377 radians or 2.16°.

$$\begin{aligned}
x &= R\sin(\theta)\\
\dot{x} &= R\dot{\theta}\cos(\theta)\\
y &= -R\cos(\theta)\\
\dot{y} &= R\dot{\theta}\sin(\theta)\\
T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m_1(\dot{x}_1+\dot{X})^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2(\dot{x}_2+\dot{X})^2 + \frac{1}{2}m_2\dot{y}_2^2\\
T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m_1(R\dot{\theta}_1\cos(\theta_1)+\dot{X})^2 + \frac{1}{2}m_1R^2\dot{\theta}_1^2\sin(\theta_1)^2 + \frac{1}{2}m_2(R\dot{\theta}_2\cos(\theta_2)+\dot{X})^2 + \frac{1}{2}m_2R^2\dot{\theta}_2^2\sin(\theta_2)^2\\
V &= m_1gR(1-\cos(\theta_1)) + m_2gR(1-\cos(\theta_2))\\
L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_1R^2\dot{\theta}_1^2 + m_1R\dot{X}\dot{\theta}_1\cos(\theta_1) - m_1gR(1-\cos(\theta_1))\\
&\quad +\frac{1}{2}m_2\dot{X}^2 + \frac{1}{2}m_2R^2\dot{\theta}_2^2 + m_2R\dot{X}\dot{\theta}_2\cos(\theta_2) - m_2gR(1-\cos(\theta_2))
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0\\
\frac{d}{dt}(M\dot{X} + m_1\dot{X} + m_2\dot{X} + m_1R\dot{\theta}_1\cos(\theta_1) + m_2R\dot{\theta}_2\cos(\theta_2)) &= 0\\
M\ddot{X} + m_1\ddot{X} + m_2\ddot{X} + m_1R\ddot{\theta}_1\cos(\theta_1) - m_1R\dot{\theta}_1^2\sin(\theta_1)&\\
+m_2R\ddot{\theta}_2\cos(\theta_2) - m_2R\dot{\theta}_2^2\sin(\theta_2) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0\\
\frac{d}{dt}(m_1R^2\dot{\theta}_1 + m_1R\dot{X}\cos(\theta_1)) - (-m_1R\dot{X}\dot{\theta}_1\sin(\theta_1) - m_1gR\sin(\theta_1)) &= 0\\
m_1R^2\ddot{\theta}_1 + m_1R\ddot{X}\cos(\theta)_1 - m_1R\dot{X}\dot{\theta}_1\sin(\theta_1) + m_1R\dot{X}\dot{\theta}_1\sin(\theta_1) + m_1gR\sin(\theta_1) &= 0\\
m_1R^2\ddot{\theta}_1 + m_1R\ddot{X}\cos(\theta_1) + m_1gR\sin(\theta_1) &= 0\\
R\ddot{\theta}_1 + \ddot{X}\cos(\theta_1) + g\sin(\theta_1) &= 0
\end{aligned}$$

The equations are identical for the second mass m_2. The subscripts simply change.

Figure 5: x vs time, short time scale

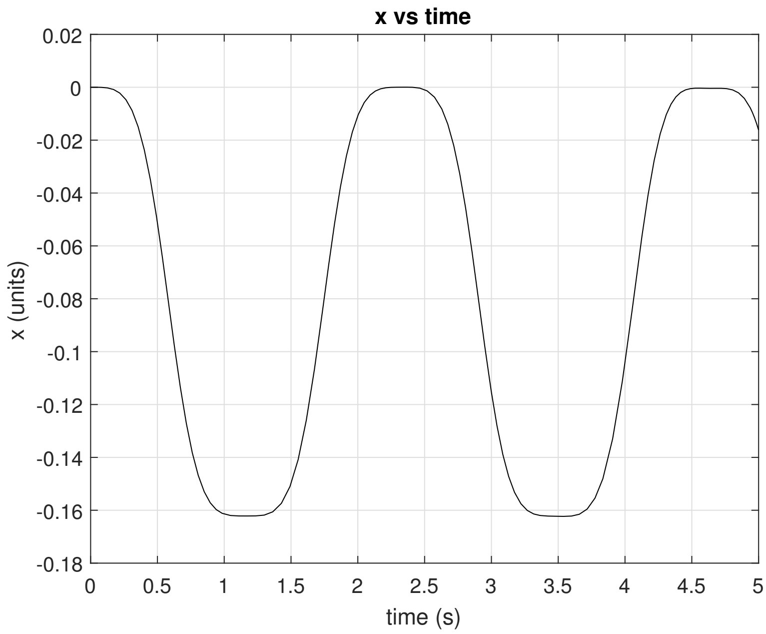
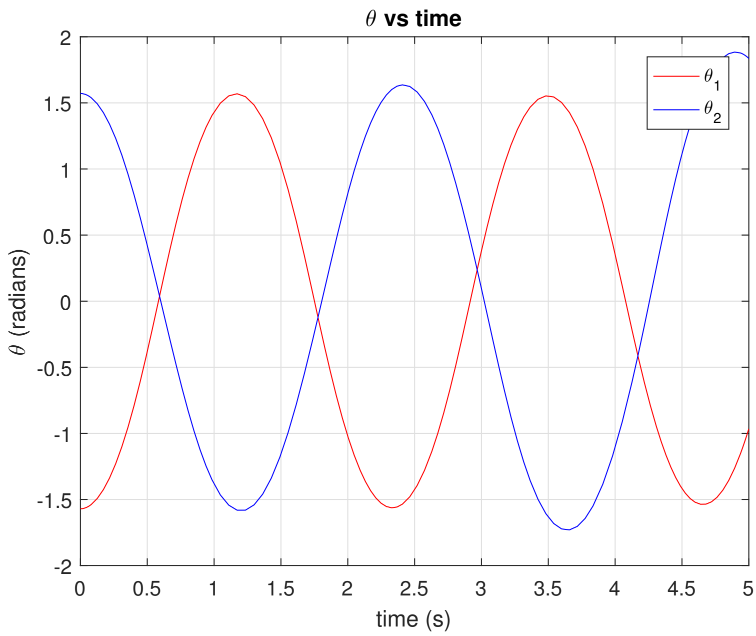


Figure 6: θ vs time, short time scale



vi. I think that the Lagrangian method is significantly more systematic. Every time one attempts to use it, the same approach is used, find the kinetic and potential energies and the Lagrangian and then apply the Lagrangian equation, even if different coordinate systems are being used. In contrast, the Newtonian method lacks a straightforward set of steps to be followed when it is used.

MATLAB Code

```
%%  AE 352 HW 6
%   Problem 3
%   Author: Max Feinberg
%   Simulates the motion of a ball rolling in a hemisphere
function main
clear all; close all; clear figure; clc;

t0 = 0;          % initial time
tf = 100;         % final time
y0 = [0; 0; -1*pi/2; 0; ];   % initial conditions

[t,y] = ode45(@diff1,[t0 tf],y0);

figure(1)
plot(t(1:550),y(1:550,1));
xlabel('time (s)');
ylabel('x-position (units)');
title('x vs. time');
xlim([0,25]);
grid on
print -depsc xShort

figure(2)
plot(t,y(:,1));
xlabel('time (s)');
ylabel('x-position (units)');
title('x vs. time');
grid on

print -depsc xLong


figure(3)
plot(t(1:550),y(1:550,3));
xlabel('time (s)');
ylabel('\theta (radians)');
title('\theta vs time');
xlim([0, 25]);
grid on

print -depsc thetaShort

figure(4)
plot(t,y(:,3));
xlabel('time (s)');
ylabel('\theta (radians)');
title('\theta vs time');
grid on

print -depsc thetaLong

y0_EC = [0; 0; -1*pi/2; 0; pi/2; 0];
[u,z] = ode45(@diff2,[t0 5],y0_EC);


figure(5)
plot(u,z(:,1), 'k-');
xlabel('time (s)');
ylabel('x (units)');
title('x vs time');
grid on
print -depsc xEC

figure(6)
plot(u,z(:,3), 'r-', u,z(:,5), 'b-');
xlabel('time (s)');
ylabel('\theta (radians)');
title('\theta vs time');
legend('\theta_1', '\theta_2');
grid on

print -depsc thetaEC
end

```

```
function state = diff1(t,y)
g = -9.81;% m/s^2
m = 10;    % kg
M = 100;   % kg
R = 1;     % units
% ==================
ddot_X = (R*y(4)^2*m*sin(y(3))-m*g*sin(y(3))*cos(y(3))) ...
           /(M+m-m*cos(y(3))^2*R);
state = [y(2); ddot_X; y(4); (-1 * ddot_X*cos(y(3))+g*sin(y(3)))/R];
end

function state = diff2(u,y)
g = -9.81;% m/s^2
m1 = 10;  % kg
m2 = 1;   % kg
M = 100;  % kg
R = 1;    % units
% ==================
ddot_X = (R*y(4)^2*m1*sin(y(3))+R*y(6)^2*m2*sin(y(5)) ...
           - m1*g*sin(y(3))*cos(y(3))-m2*g*sin(y(5))*cos(y(5))) ...
           /(M+m1+m2-m1*cos(y(3))^2*R - m2*cos(y(5))^2*R );
state = [y(2); ddot_X; y(4); (-1 * ddot_X*cos(y(3))+g*sin(y(3)))/R; y(6);...
           (-1 * ddot_X*cos(y(5))+g*sin(y(5)))/R];
end
```