AE 352: Homework 7

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1. **a.**

$$\begin{aligned}
J &= \int_0^m r^2 \ dm \\
dm &= \rho dV = \rho L 2\pi r dr \\
J &= 2\pi\rho L \int_0^r r^3 \ dr \\
J &= 2\pi\rho L \frac{r^4}{4} \\
\rho &= \frac{m}{\pi r^2 L} \\
J &= \frac{1}{2} m r^2
\end{aligned}$$

$$\boxed{J = \frac{1}{2} m r^2}$$

b.

$$\begin{aligned}
(R + r)\theta &= r(\theta + \phi) \\
R\theta &= r\phi
\end{aligned}$$

$$\boxed{R\theta = r\phi}$$

c.

i.

A.

$$\begin{aligned}
\sum F &= m\ddot{x} \\
m\ddot{x} &= -mg\sin(\theta) - T
\end{aligned}$$

$$\boxed{m\ddot{x} = -mg\sin(\theta) - T}$$

B.

$$\begin{aligned}\sum \tau &= J\alpha \\ J\ddot{\phi} &= Tr \\ \frac{1}{2}mr^2\ddot{\phi} &= Tr\end{aligned}$$

$$\boxed{\ddot{\phi} = 2\frac{T}{mr}}$$

C.

$$\begin{aligned}\frac{1}{2}mr^2\ddot{\phi} &= Tr \\ R\ddot{\theta} &= r\ddot{\phi} \\ r\ddot{\phi} &= 2\frac{T}{m} \\ \frac{R\ddot{\theta}}{\ddot{\phi}}\ddot{\phi} &= 2\frac{T}{m} \\ \ddot{\theta} &= 2\frac{T}{mR}\end{aligned}$$

$$\boxed{\ddot{\theta} = 2\frac{T}{mR}}$$

D.

$$\begin{aligned}\ddot{\theta} &= 2\frac{T}{mR} \\ T &= \frac{1}{2}mR\ddot{\theta} \\ m\ddot{x} &= -mg\sin(\theta) - \frac{1}{2}mR\ddot{\theta} \\ mR\ddot{\theta} &= -mg\sin(\theta) - \frac{1}{2}mR\ddot{\theta} \\ \frac{3}{2}mR\ddot{\theta} &= -mg\sin(\theta) \\ \ddot{\theta} &= -\frac{2g}{3R}\sin(\theta)\end{aligned}$$

$$\boxed{\ddot{\theta} = -\frac{2g}{3R}\sin(\theta)}$$

ii.

A.

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\omega^2 \\ T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(\frac{1}{2}mr^2)(\frac{\dot{x}}{r})^2 \\ T &= \frac{3}{4}m\dot{x}^2 \\ x &= r\phi \\ \dot{x} &= r\dot{\phi} \\ R\dot{\theta} &= r\dot{\phi} \\ \dot{x} &= R\dot{\theta} \\ T &= \frac{3}{4}mR^2\dot{\theta}^2 \end{aligned}$$

$$\boxed{T = \frac{3}{4}mR^2\dot{\theta}^2}$$

B.

$$\begin{aligned} V &= mgh \\ V &= mgR(1-\cos(\theta)) \end{aligned}$$

$$\boxed{V = mgR(1-\cos(\theta))}$$

C.

$$\begin{aligned} L &= T - V \\ L &= \frac{3}{4}mR^2\dot{\theta}^2 - mgR(1-\cos(\theta)) \end{aligned}$$

$$\boxed{L = \frac{3}{4}mR^2\dot{\theta}^2 - mgR(1-\cos(\theta))}$$

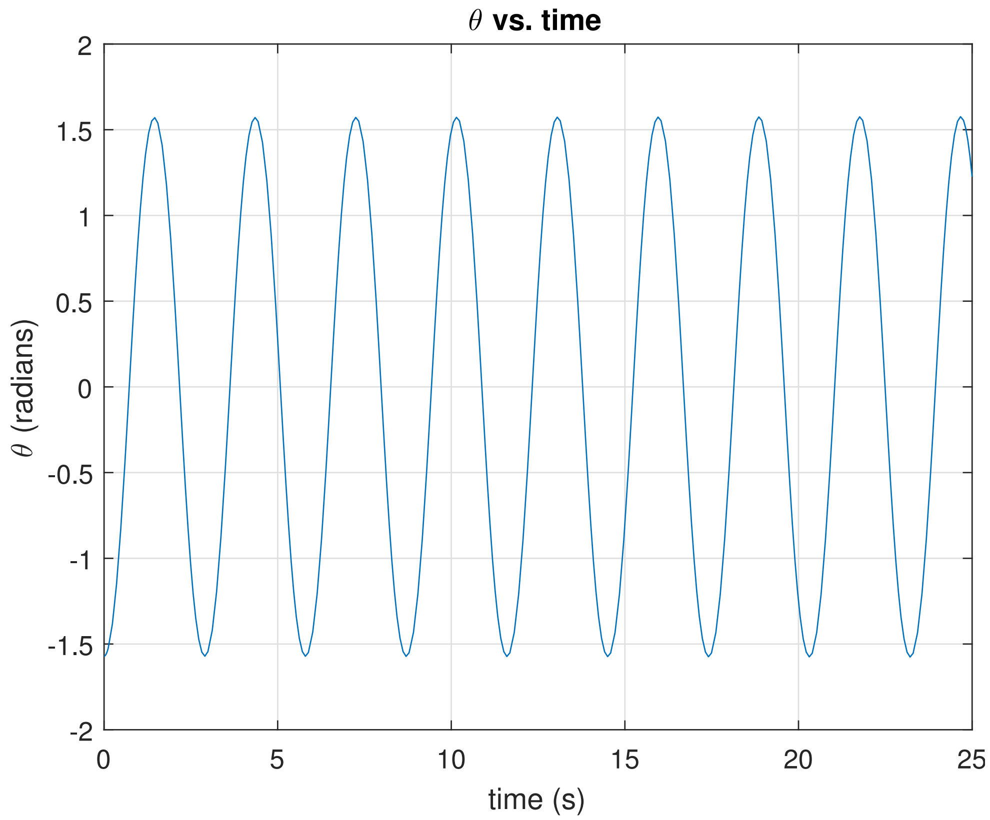
D.

$$\begin{aligned} \frac{d}{dt}(\frac{3}{2}mR^2\dot{\theta}) + (mgR\sin(\theta)) &= 0 \\ \frac{3}{2}mR^2\ddot{\theta} + mgR\sin(\theta) &= 0 \\ \frac{3}{2}mR\ddot{\theta} &= -mg\sin(\theta) \\ \ddot{\theta} &= -\frac{2g}{3R}\sin(\theta) \end{aligned}$$

$$\boxed{\ddot{\theta} = -\frac{2g}{3R}\sin(\theta)}$$

d. See the attached MATLAB code.

Figure 1: A graph of θ vs. Time produced by numerically solving the differential equations found above.



e.

$$
\begin{aligned}
f &= \frac{1}{2\pi}\sqrt{\frac{2g}{3R}} \\
T &= 2\pi\sqrt{\frac{3R}{2g}}
\end{aligned}
$$

$$\boxed{T = 2\pi\sqrt{\frac{3R}{2g}}\ \mathrm{s}}$$

2. **a.**

$$
\begin{aligned}
J &= \int_0^L r^2\, dm \\
J &= \int_0^L r^2 \frac{m}{L}\, dr \\
J &= \frac{1}{12} mL^2
\end{aligned}
$$

$$\boxed{J = \frac{1}{12} mL^2}$$

b.

$$z = \frac{L}{2}\cos(\theta)$$

$$\boxed{z = \frac{L}{2}\cos(\theta)}$$

c.

i.

A.

$$\begin{aligned} \sum F &= m\ddot{z} \\ m\ddot{z} &= N - mg \end{aligned}$$

$$\boxed{m\ddot{z} = N - mg}$$

B.

$$\begin{aligned} m\ddot{z} &= N - mg \\ \dot{z} &= -\frac{L}{2}\dot{\theta}\sin(\theta) \\ \ddot{z} &= -\frac{L}{2}\ddot{\theta}\sin(\theta) - \frac{L}{2}\dot{\theta}^2\cos(\theta) \\ m\big(-\frac{L}{2}\ddot{\theta}\sin(\theta) - \frac{L}{2}\dot{\theta}^2\cos(\theta)\big) &= N - mg \end{aligned}$$

$$\boxed{-mL\ddot{\theta}\sin(\theta) = 2N - 2mg + mL\dot{\theta}^2\cos(\theta)}$$

C.

$$\begin{aligned} J\ddot{\theta} &= \frac{NL}{2}\sin(\theta) \\ \frac{1}{12}mL^2\ddot{\theta} &= \frac{NL}{2}\sin(\theta) \end{aligned}$$

$$\boxed{mL\ddot{\theta} = 6N\sin(\theta)}$$

D.

$$\begin{aligned}
\frac{1}{12}mL^2\ddot{\theta} &= \frac{NL}{2}\sin(\theta) \\
\frac{1}{6}mL\ddot{\theta} &= N\sin(\theta) \\
N &= mg + m\big(-\frac{L}{2}\ddot{\theta}\sin(\theta) - \frac{L}{2}\dot{\theta}^2\cos(\theta)\big) \\
\frac{1}{6}mL\ddot{\theta} &= \big(mg + m\big(-\frac{L}{2}\ddot{\theta}\sin(\theta) - \frac{L}{2}\dot{\theta}^2\cos(\theta)\big)\big)\sin(\theta) \\
\frac{1}{6}mL\ddot{\theta} &= mg\sin(\theta) + m\big(-\frac{L}{2}\ddot{\theta}\sin(\theta) - \frac{L}{2}\dot{\theta}^2\cos(\theta)\big)\sin(\theta) \\
\frac{1}{6}mL\ddot{\theta} &= mg\sin(\theta) - \frac{L}{2}m\ddot{\theta}\sin(\theta)^2 - \frac{L}{2}m\dot{\theta}^2\cos(\theta)\sin(\theta) \\
\frac{1}{6}\ddot{\theta} &= \frac{g}{L}\sin(\theta) - \frac{1}{2}\ddot{\theta}\sin(\theta)^2 - \frac{1}{2}\dot{\theta}^2\cos(\theta)\sin(\theta) \\
\ddot{\theta} &= 6\frac{g}{L}\sin(\theta) - 3\ddot{\theta}\sin(\theta)^2 - 3\dot{\theta}^2\cos(\theta)\sin(\theta) \\
\ddot{\theta} + 3\ddot{\theta}\sin(\theta)^2 &= 6\frac{g}{L}\sin(\theta) - 3\dot{\theta}^2\cos(\theta)\sin(\theta) \\
\ddot{\theta}(1 + 3\sin(\theta)^2) &= 6\frac{g}{L}\sin(\theta) - 3\dot{\theta}^2\cos(\theta)\sin(\theta)
\end{aligned}$$

$$\boxed{\ddot{\theta}(1 + 3\sin(\theta)^2) = 6\frac{g}{L}\sin(\theta) - 3\dot{\theta}^2\cos(\theta)\sin(\theta)}$$

ii.

A.

$$\begin{aligned}
T &= \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J\omega^2 \\
T &= \frac{1}{2}m(\frac{L}{2}\dot{\theta}\sin(\theta))^2 + \frac{1}{2}(\frac{1}{12}mL^2)(\dot{\theta})^2 \\
T &= \frac{1}{8}mL^2\dot{\theta}^2\sin(\theta)^2 + \frac{1}{24}mL^2\dot{\theta}^2
\end{aligned}$$

$$\boxed{T = \frac{1}{8}mL^2\dot{\theta}^2\sin(\theta)^2 + \frac{1}{24}mL^2\dot{\theta}^2}$$

B.

$$\begin{aligned}
V &= mgz \\
V &= \frac{1}{2}mgL\cos(\theta)
\end{aligned}$$

$$\boxed{V = \frac{1}{2}mgL\cos(\theta)}$$

C. NOTE: the L on the left side of the equation denotes the Lagrangian while the L on the right side of the equation denotes the length of the stick.

$$\begin{aligned}
L &= T - V \\
L &= \frac{1}{8}mL^2\dot{\theta}^2\sin(\theta)^2 + \frac{1}{24}mL^2\dot{\theta}^2 - \frac{1}{2}mgL\cos(\theta)
\end{aligned}$$

$$\boxed{L = \frac{1}{8}mL^2\dot{\theta}^2\sin(\theta)^2 + \frac{1}{24}mL^2\dot{\theta}^2 - \frac{1}{2}mgL\cos(\theta)}$$

D.

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} &= 0 \\
\frac{d}{dt}(\frac{1}{4}mL^2\dot{\theta}\sin(\theta)^2 + \frac{1}{12}mL^2\dot{\theta}) - \frac{1}{4}mL^2\dot{\theta}^2\sin(\theta)\cos(\theta) - \frac{1}{2}mgL\sin(\theta) &= 0 \\
\frac{1}{4}mL^2\ddot{\theta}\sin(\theta)^2 + \frac{1}{4}mL^2\dot{\theta}^2\sin(\theta)\cos(\theta) + \frac{1}{12}mL^2\ddot{\theta} - \frac{1}{2}mgL\sin(\theta) &= 0 \\
3\ddot{\theta}\sin(\theta)^2 + 3\dot{\theta}^2\sin(\theta)\cos(\theta) + \ddot{\theta} - 6\frac{g}{L}\sin(\theta) &= 0 \\
\ddot{\theta}(1 + 3\sin(\theta)^2) &= 6\frac{g}{L}\sin(\theta) - 3\dot{\theta}^2\sin(\theta)\cos(\theta)
\end{aligned}$$

$$\boxed{\ddot{\theta}(1 + 3\sin(\theta)^2) = 6\frac{g}{L}\sin(\theta) - 3\dot{\theta}^2\sin(\theta)\cos(\theta)}$$

c. See the attached MATLAB code.

Figure 2: A graph of θ vs Time for a sticking falling, short time scale

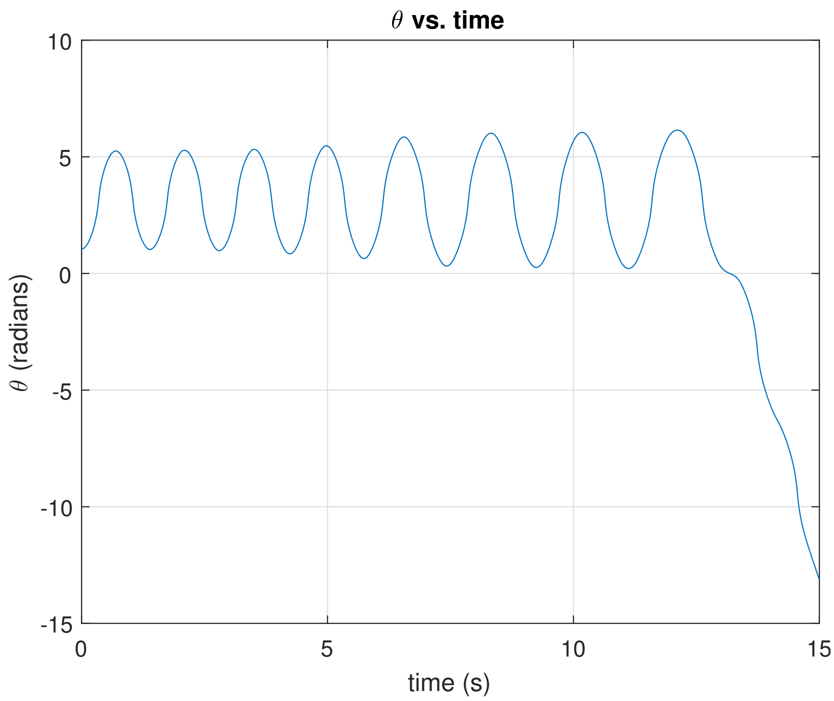
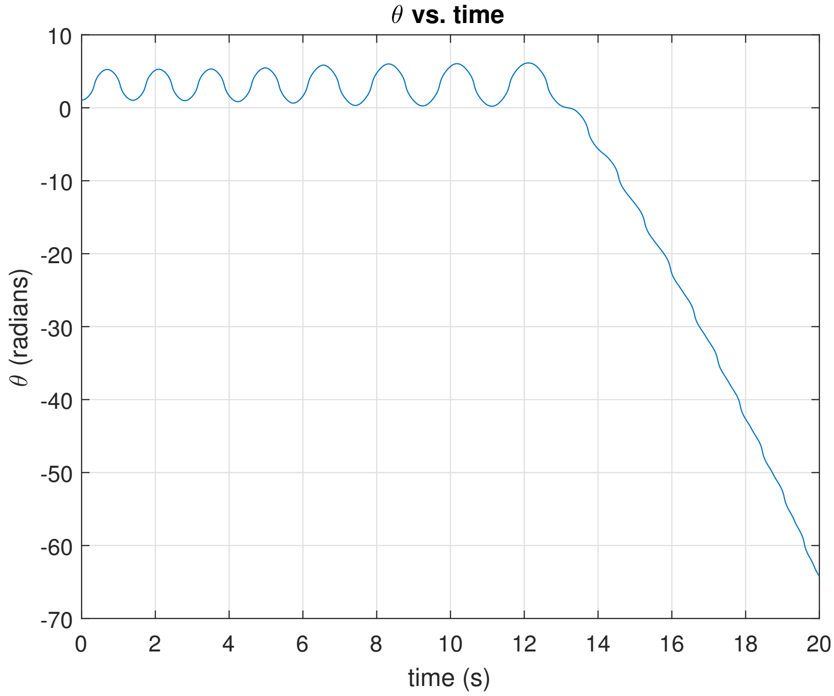


Figure 3: A graph of θ vs Time for a sticking falling, long time scale



3. **a.**

$$
\begin{aligned}
J &= \int_0^m r^2 \, dm \\
dm &= \rho dV = \rho L 2\pi r dr \\
J &= 2\pi\rho L \int_0^r r^3 \, dr \\
J &= 2\pi\rho L \frac{r^4}{4} \\
\rho &= \frac{m}{\pi r^2 L} \\
J &= \frac{1}{2} m r^2
\end{aligned}
$$

$$\boxed{J = \frac{1}{2} m r^2}$$

b.

$$
\begin{aligned}
s &= r\theta \\
x &= r\theta
\end{aligned}
$$

$$\boxed{x = r\theta}$$

c.

$$\begin{aligned} m\ddot{x}\Delta t &= m\dot{x} \\ m\ddot{x}\Delta t &= F\Delta t \end{aligned}$$

$$\boxed{m\ddot{x}\Delta t = F\Delta t}$$

d.

$$\begin{aligned} J\ddot{\theta}\Delta t &= F(h-r)\Delta t \\ (\frac{1}{2}mr^2)\ddot{\theta}\Delta t &= F(h-r)\Delta t \\ \frac{1}{2}mr^2\ddot{\theta}\Delta t &= F(h-r)\Delta t \end{aligned}$$

$$\boxed{\frac{1}{2}mr^2\ddot{\theta}\Delta t = F(h-r)\Delta t}$$

e.

$$\begin{aligned} \ddot{x} &= r\ddot{\theta} \\ m\ddot{x}\Delta t &= F\Delta t \\ mr\ddot{\theta} &= F \\ \frac{1}{2}mr^2\ddot{\theta}\Delta t &= F(h-r)\Delta t \\ \frac{1}{2}mr^2\ddot{\theta} &= (mr\ddot{\theta})(h-r) \\ \frac{1}{2}r &= h-r \\ h &= \frac{3}{2}r \end{aligned}$$

$$\boxed{h = \frac{3}{2}r}$$

f.

Due to the fact that the moment of inertia for a sphere is slightly smaller than the moment of inertia for a cylinder, a billiard player must strike the ball just slightly lower than where they should hit it if it were a cylinder while still hitting above the center of mass. This makes the "sweet spot" for rolling without slipping very difficult to estimate, making billiard ball games difficult.

MATLAB Code

```
%%  AE 352 HW 7
%   Problem 1-2
%   Author: Max Feinberg
%   Simulates the motion of a cylinder rolling in a hemisphere
%   and a stick falling
function main
clear all; close all; clear figure; clc; %stand clears

R = 1; % Ramp Radius
r = 0.2; % cylinder radius
g = 9.8; % gravity

t0 = 0;          % initial time
tf = 25;         % final time
phi_0 = -1*pi/2*R/r;
y0 = [-1*pi/2; 0; phi_0; 0];    % initial conditions

[t,y] = ode45(@diff1,[t0 tf],y0); % Use ode45 to solve

figure(1) % Plot theta vs t
plot(t,y(:,1));
xlabel('time (s)');
ylabel('\theta (radians)');
title('\theta vs. time');
grid on
print -depsc P1Theta

theta0 = [pi/3; 0;];
[u,v] = ode45(@diff2,[0 20],theta0); % simulate stick falling

figure(2)
plot(u,v(:,1));
xlabel('time (s)');
ylabel('\theta (radians)');
title('\theta vs. time');
grid on
print -depsc P2Theta

end

function state = diff1(t,y)
R = 1;
r = 0.2;
g = 9.8;

state = [y(2); -1*2*g/(3*R)*sin(y(1)); y(4); -2*g/(3*r)*sin(y(1))];
end

function state = diff2(u,v)
l = 2;
g = 9.8;

state = [v(2); (6*g*l*sin(v(1))-3*v(2)^2*sin(v(1))*cos(v(1)))/ (1 + 3 * sin(v(1))^2)];
end
```