

## AE 352: Homework 6

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1.      **a.**

**i.**

$$T = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}kx^2$$

**ii.**

$$L = T - V$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

**iii.**

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(m\dot{x}) + kx &= 0 \\ m\ddot{x} + kx &= 0\end{aligned}$$

$$m\ddot{x} + kx = 0$$

**b.**

**i.**

$$T = \frac{1}{2}m\dot{x}^2$$

$$V = \frac{1}{2}k(x^2 + 1^2)$$

**ii.**

$$L = T - V$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x^2 + 1^2)$$

iii.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(m\dot{x}) + kx &= 0 \\ m\ddot{x} + kx &= 0\end{aligned}$$

$$m\ddot{x} + kx = 0$$

c.

i.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$V = \frac{1}{2}k((x-1)^2 + y^2) + \frac{1}{2}k((x+1)^2 + y^2)$$

ii.

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k((x-1)^2 + y^2) - \frac{1}{2}k((x+1)^2 + y^2)$$

iii.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(m\dot{x}) + k(x-1) + k(x+1) &= 0 \\ m\ddot{x} + k(x-1) + k(x+1) &= 0 \\ m\ddot{x} + 2kx &= 0 \\ \frac{d}{dt}(m\dot{y}) + ky + ky &= 0 \\ m\ddot{y} + 2ky &= 0\end{aligned}$$

$$m\ddot{x} + 2kx = 0$$

$$m\ddot{y} + 2ky = 0$$

2. a.

$$\begin{aligned}
x &= L \sin(\theta) \cos(\phi) \\
\dot{x} &= L \dot{\theta} \cos(\theta) \cos(\phi) - L \dot{\phi} \sin(\theta) \sin(\phi) \\
y &= -L \cos(\theta) \\
\dot{y} &= L \dot{\theta} \sin(\theta) \\
z &= L \sin(\theta) \sin(\phi) \\
\dot{z} &= L \dot{\theta} \cos(\theta) \sin(\phi) + L \dot{\phi} \sin(\theta) \cos(\phi) \\
T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
T &= \frac{1}{2} m ((L \dot{\theta} \cos(\theta) \cos(\phi) - L \dot{\phi} \sin(\theta) \sin(\phi))^2 + (L \dot{\theta} \sin(\theta))^2 + (L \dot{\theta} \cos(\theta) \sin(\phi) + L \dot{\phi} \sin(\theta) \cos(\phi))^2) \\
&\text{or} \\
T &= \frac{1}{2} m L^2 \sin^2(\theta) \dot{\phi}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 \\
V &= -mgL \cos(\theta)
\end{aligned}$$

$$T = \frac{1}{2} m L^2 \sin^2(\theta) \dot{\phi}^2 + \frac{1}{2} m L^2 \dot{\theta}^2$$

$$V = -mgL \cos(\theta)$$

b.

$$L = T - V$$

$$L = \frac{1}{2} m L^2 \sin^2(\theta) \dot{\phi}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 + mgL \cos(\theta)$$

c.

$$\begin{aligned}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt} (m L^2 \dot{\theta}) - (m L^2 \sin(\theta) \cos(\theta) \dot{\phi}^2 - mgL \sin(\theta)) &= 0 \\
m L^2 \ddot{\theta} - m L^2 \sin(\theta) \cos(\theta) \dot{\phi}^2 + mgL \sin(\theta) &= 0 \\
\ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\phi}^2 + \frac{g}{L} \sin(\theta) &= 0 \\
\frac{d}{dt} (m L^2 \sin^2(\theta) \dot{\phi}) &= 0 \\
2m L^2 \sin(\theta) \cos(\theta) \dot{\phi} + m L^2 \sin^2(\theta) \ddot{\phi} &= 0 \\
\sin(\theta)^2 \ddot{\phi} + 2 \sin(\theta) \cos(\theta) \dot{\phi} &= 0
\end{aligned}$$

$$\ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\phi}^2 + \frac{g}{L} \sin(\theta) = 0$$

$$\sin(\theta)^2 \ddot{\phi} + 2 \sin(\theta) \cos(\theta) \dot{\phi} = 0$$

3. a.

i.

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
\vec{F} &= m\ddot{x} \\
\vec{F} &= m \frac{d^2}{dt^2} (R \sin(\theta)) \\
\vec{F} &= m \frac{d}{dt} (R \dot{\theta} \cos(\theta)) \\
\vec{F} &= m(R\ddot{\theta} \cos(\theta) - R\dot{\theta}^2 \sin(\theta)) \\
0 &= m(R\ddot{\theta} \cos(\theta) - R\dot{\theta}^2 \sin(\theta)) \\
R\dot{\theta}^2 \sin(\theta) &= R\ddot{\theta} \cos(\theta) \\
\dot{\theta}^2 &= \cot(\theta) \ddot{\theta} \\
\text{now we will consider the y-direction} \\
\vec{F} &= m\ddot{y} \\
\vec{F} &= m \frac{d^2}{dt^2} (-R \cos(\theta)) \\
\vec{F} &= m \frac{d}{dt} (R \dot{\theta} \sin(\theta)) \\
\vec{F} &= m(R\ddot{\theta} \sin(\theta) + R\dot{\theta}^2 \cos(\theta)) \\
-mg &= m(R\ddot{\theta} \sin(\theta) + R\dot{\theta}^2 \cos(\theta)) \\
-g &= R\ddot{\theta} \sin(\theta) + R \cos(\theta) \cot(\theta) \ddot{\theta} \\
-g \sin(\theta) &= R\ddot{\theta} \sin(\theta)^2 + R \cos(\theta)^2 \ddot{\theta} \\
\ddot{\theta} &= -\frac{g}{R} \sin(\theta)
\end{aligned}$$

$$\ddot{\theta} = -\frac{g}{R} \sin(\theta)$$

ii.

$$\begin{aligned}
x &= R \sin(\theta) \\
\dot{x} &= R \dot{\theta} \cos(\theta) \\
y &= -R \cos(\theta) \\
\dot{y} &= R \dot{\theta} \sin(\theta) \\
T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \\
T &= \frac{1}{2} m (R^2 \dot{\theta}^2 \cos(\theta)^2 + R^2 \dot{\theta}^2 \sin(\theta)^2) \\
T &= \frac{1}{2} m (R^2 \dot{\theta}^2) \\
V &= mgR(1 - \cos(\theta))
\end{aligned}$$

$$T = \frac{1}{2} m (R^2 \dot{\theta}^2)$$

$$V = mgR(1 - \cos(\theta))$$

iii.

$$L = T - V$$

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR(1 - \cos(\theta))$$

iv.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(mR^2\dot{\theta}) + mgR\sin(\theta) &= 0 \\ mR^2\ddot{\theta} + mgR\sin(\theta) &= 0 \\ R\ddot{\theta} + g\sin(\theta) &= 0\end{aligned}$$

$$\ddot{\theta} = -\frac{g}{R}\sin(\theta)$$

b.

i. in this problem,  $X$  will denote the position of the large block  $M$ . Lower case  $x$  will denote the position of the the smaller mass  $m$ .

$$\begin{aligned}x &= R\sin(\theta) \\ \dot{x} &= R\dot{\theta}\cos(\theta) \\ y &= -R\cos(\theta) \\ \dot{y} &= R\dot{\theta}\sin(\theta) \\ T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x} + \dot{X})^2 + \frac{1}{2}m\dot{y}^2 \\ T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 \\ V &= mgR(1 - \cos(\theta))\end{aligned}$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2$$

$$V = mgR(1 - \cos(\theta))$$

ii.

$$\begin{aligned}L &= T - V \\ L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R\dot{\theta}\cos(\theta) + \dot{X})^2 + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 - mgR(1 - \cos(\theta)) \\ L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(R^2\dot{\theta}^2\cos(\theta)^2 + 2R\dot{X}\dot{\theta}\cos(\theta) + \dot{X}^2) + \frac{1}{2}mR^2\dot{\theta}^2\sin(\theta)^2 - mgR(1 - \cos(\theta))\end{aligned}$$

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{X}\dot{\theta}\cos(\theta) - mgR(1 - \cos(\theta))$$

iii.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(M\dot{X} + m\dot{X} + mR\dot{\theta}\cos(\theta)) &= 0 \\ M\ddot{X} + m\ddot{X} + mR\ddot{\theta}\cos(\theta) - mR\dot{\theta}^2\sin(\theta) &= 0\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt}(mR^2\dot{\theta} + mR\dot{X}\cos(\theta)) - (-mR\dot{X}\dot{\theta}\sin(\theta) - mgR\sin(\theta)) &= 0 \\ mR^2\ddot{\theta} + mR\ddot{X}\cos(\theta) - mR\dot{X}\dot{\theta}\sin(\theta) + mR\dot{X}\dot{\theta}\sin(\theta) + mgR\sin(\theta) &= 0 \\ mR^2\ddot{\theta} + mR\ddot{X}\cos(\theta) + mgR\sin(\theta) &= 0 \\ R\ddot{\theta} + \ddot{X}\cos(\theta) + g\sin(\theta) &= 0\end{aligned}$$

$$M\ddot{X} + m\ddot{X} + mR\ddot{\theta}\cos(\theta) - mR\dot{\theta}^2\sin(\theta) = 0$$

$$R\ddot{\theta} + \ddot{X}\cos(\theta) + g\sin(\theta) = 0$$

iv. Please see the attached MATLAB code at the end of the document.

Figure 1:  $x$  vs time, short time scale

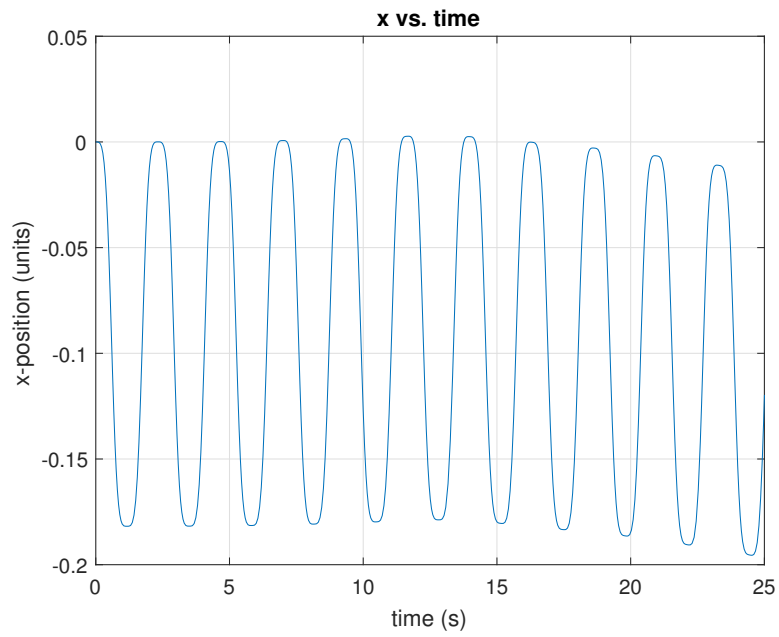


Figure 2:  $\theta$  vs time, short time scale

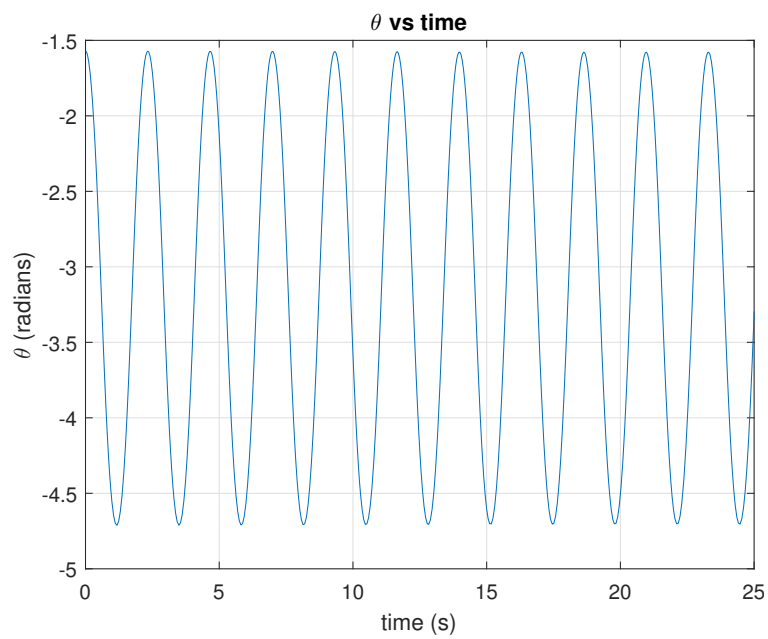


Figure 3:  $x$  vs time, Long time scale

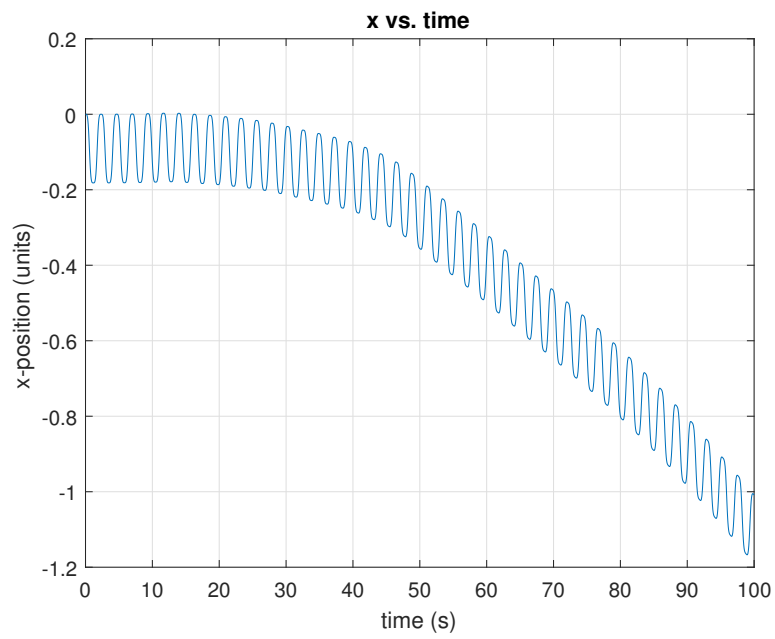
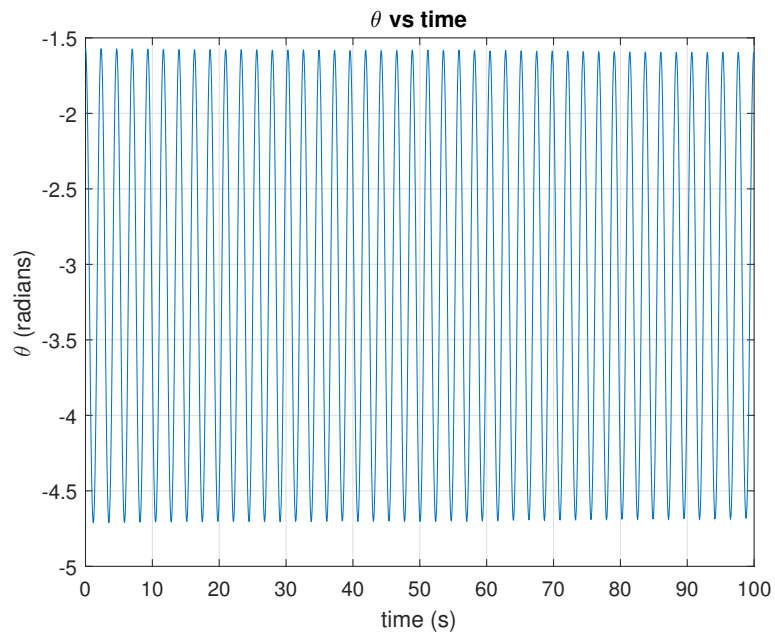


Figure 4:  $\theta$  vs time, long time scale





v. Please see the attached MATLAB code at the end of the document. The collision point can be found by inspecting Figure 6. The two small masses collide at roughly 0.0377 radians or 2.16°.

$$\begin{aligned}
x &= R \sin(\theta) \\
\dot{x} &= R \dot{\theta} \cos(\theta) \\
y &= -R \cos(\theta) \\
\dot{y} &= R \dot{\theta} \sin(\theta) \\
T &= \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m_1 (\dot{x}_1 + \dot{X})^2 + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 (\dot{x}_2 + \dot{X})^2 + \frac{1}{2} m_2 \dot{y}_2^2 \\
T &= \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m_1 (R \dot{\theta}_1 \cos(\theta_1) + \dot{X})^2 + \frac{1}{2} m_1 R^2 \dot{\theta}_1^2 \sin^2(\theta_1) + \frac{1}{2} m_2 (R \dot{\theta}_2 \cos(\theta_2) + \dot{X})^2 + \frac{1}{2} m_2 R^2 \dot{\theta}_2^2 \sin^2(\theta_2) \\
V &= m_1 g R (1 - \cos(\theta_1)) + m_2 g R (1 - \cos(\theta_2)) \\
L &= \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m_1 \dot{X}^2 + \frac{1}{2} m_1 R^2 \dot{\theta}_1^2 + m_1 R \dot{X} \dot{\theta}_1 \cos(\theta_1) - m_1 g R (1 - \cos(\theta_1)) \\
&\quad + \frac{1}{2} m_2 \dot{X}^2 + \frac{1}{2} m_2 R^2 \dot{\theta}_2^2 + m_2 R \dot{X} \dot{\theta}_2 \cos(\theta_2) - m_2 g R (1 - \cos(\theta_2))
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt} (M \dot{X} + m_1 \dot{X} + m_2 \dot{X} + m_1 R \dot{\theta}_1 \cos(\theta_1) + m_2 R \dot{\theta}_2 \cos(\theta_2)) &= 0 \\
M \ddot{X} + m_1 \ddot{X} + m_2 \ddot{X} + m_1 R \ddot{\theta}_1 \cos(\theta_1) - m_1 R \dot{\theta}_1^2 \sin(\theta_1) \\
&\quad + m_2 R \ddot{\theta}_2 \cos(\theta_2) - m_2 R \dot{\theta}_2^2 \sin(\theta_2) = 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt} (m_1 R^2 \dot{\theta}_1 + m_1 R \dot{X} \cos(\theta_1)) - (-m_1 R \dot{X} \dot{\theta}_1 \sin(\theta_1) - m_1 g R \sin(\theta_1)) &= 0 \\
m_1 R^2 \ddot{\theta}_1 + m_1 R \ddot{X} \cos(\theta_1) - m_1 R \dot{X} \dot{\theta}_1 \sin(\theta_1) + m_1 R \dot{X} \dot{\theta}_1 \sin(\theta_1) + m_1 g R \sin(\theta_1) &= 0 \\
m_1 R^2 \ddot{\theta}_1 + m_1 R \ddot{X} \cos(\theta_1) + m_1 g R \sin(\theta_1) &= 0 \\
R \ddot{\theta}_1 + \ddot{X} \cos(\theta_1) + g \sin(\theta_1) &= 0
\end{aligned}$$

The equations are identical for the second mass  $m_2$ . The subscripts simply change.

Figure 5:  $x$  vs time, short time scale

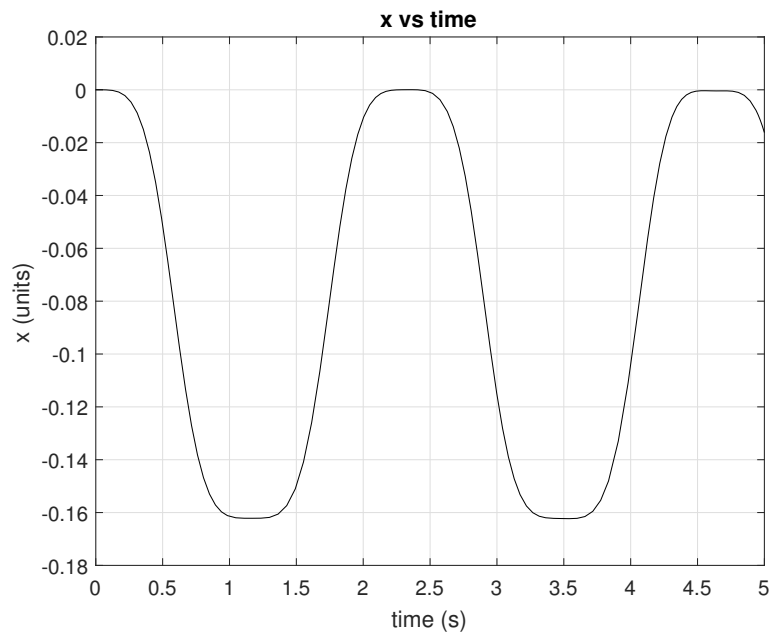
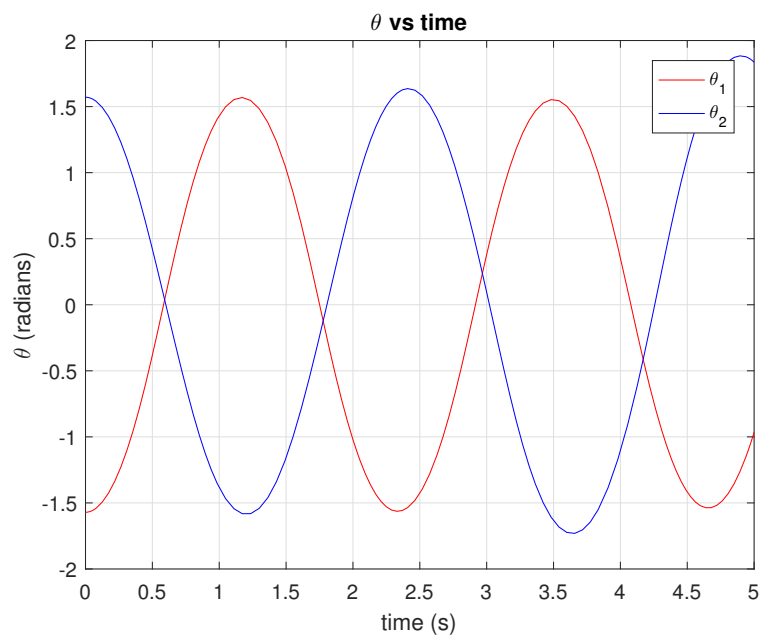


Figure 6:  $\theta$  vs time, short time scale



vi. I think that the Lagrangian method is significantly more systematic. Every time one attempts to use it, the same approach is used, find the kinetic and potential energies and the Lagrangian and then apply the Lagrangian equation, even if different coordinate systems are being used. In contrast, the Newtonian method lacks a straightforward set of steps to be followed when it is used.

## MATLAB Code

```

1 %% AE 352 HW 6
2 % Problem 3
3 % Author: Max Feinberg
4 % Simulates the motion of a ball rolling in a hemisphere
5 function main
6 clear all; close all; clear figure; clc;
7
8 t0 = 0;           % initial time
9 tf = 100;         % final time
10 y0 = [0; 0; -1*pi/2; 0; 0; ]; % initial conditions
11
12 [t,y] = ode45(@diff1,[t0 tf],y0);
13
14 figure(1)
15 plot(t(1:550),y(1:550,1));
16 xlabel('time (s)');
17 ylabel('x-position (units)');
18 title('x vs. time');
19 xlim([0,25]);
20 grid on
21 print -depsc xShort
22
23 figure(2)
24 plot(t,y(:,1));
25 xlabel('time (s)');
26 ylabel('x-position (units)');
27 title('x vs. time');
28 grid on
29
30 print -depsc xLong
31
32
33 figure(3)
34 plot(t(1:550),y(1:550,3));
35 xlabel('time (s)');
36 ylabel('\theta (radians)');
37 title('\theta vs time');
38 xlim([0, 25]);
39 grid on
40
41 print -depsc thetaShort
42
43 figure(4)
44 plot(t,y(:,3));
45 xlabel('time (s)');
46 ylabel('\theta (radians)');
47 title('\theta vs time');
48 grid on
49
50 print -depsc thetaLong
51
52 y0_EC = [0; 0; -1*pi/2; 0; pi/2; 0];
53 [u,z] = ode45(@diff2,[t0 5],y0_EC);
54
55
56 figure(5)
57 plot(u,z(:,1), 'k-');
58 xlabel('time (s)');
59 ylabel('x (units)');
60 title('x vs time');
61 grid on
62 print -depsc xEC
63
64 figure(6)
65 plot(u,z(:,3), 'r-', u,z(:,5), 'b-');
66 xlabel('time (s)');
67 ylabel('\theta (radians)');
68 title('\theta vs time');
69 legend('\theta_1', '\theta_2');
70 grid on
71
72 print -depsc thetaEC
73 end
74

```

```

75 function state = diff1(t,y)
76 g = -9.81;% m/s^2
77 m = 10;    % kg
78 M = 100;   % kg
79 R = 1;     % units
80 % =====
81 ddot_X = (R*y(4)^2*m*sin(y(3))+m*g*sin(y(3))*cos(y(3))) ...
82           /(M+m-m*cos(y(3))^2*R);
83 state = [y(2); ddot_X; y(4); (-1 * ddot_X*cos(y(3))-g*sin(y(3)))/R];
84 end
85
86 function state = diff2(u,y)
87 g = -9.81;% m/s^2
88 m1 = 10;   % kg
89 m2 = 1;    % kg
90 M = 100;   % kg
91 R = 1;     % units
92 % =====
93 ddot_X = (R*y(4)^2*m1*sin(y(3))+R*y(6)^2*m2*sin(y(5)) ...
94           - m1*g*sin(y(3))*cos(y(3))-m2*g*sin(y(5))*cos(y(5))) ...
95           /(M+m1+m2-m1*cos(y(3))^2*R - m2*cos(y(5))^2*R);
96 state = [y(2); ddot_X; y(4); (-1 * ddot_X*cos(y(3))+g*sin(y(3)))/R; y(6);...
97           (-1 * ddot_X*cos(y(5))+g*sin(y(5)))/R];
98 end

```