AE 352: Final Project

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We will consider a 3-link planar robot arm with revolute joints:

The x and y positions of the end-effector can be defined as:

$$x = a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) + a_3\cos(\theta_1+\theta_2+\theta_3)$$

$$y = a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) + a_3\sin(\theta_1+\theta_2+\theta_3)$$

Derivation of the Jacobian relating the end effector velocity to the world frame.

$$J = \begin{bmatrix} z_0^0 \times (O_3^0 - O_0^0) & z_1^0 \times (O_3^0 - O_1^0) & z_2^0 \times (O_3^0 - O_2^0) \\ z_0^0 & z_1^0 & z_2^0 \end{bmatrix}$$

$$T_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_1\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_1\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2\sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) + a_3\cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) + a_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix} a_2c_{12} + a_3c_{123} \\ a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix} a_3c_{123} \\ a_3s_{123} \\ 0 \end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$J = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$
\begin{aligned}
T &= \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix}\begin{bmatrix}H_{11} & H_{12} & H_{13}\\ H_{12} & H_{22} & H_{23}\\ H_{13} & H_{23} & H_{33}\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}\\
H &= \sum_{i=1}^{n}\left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i}\right)\\
T &= \sum_{i=1}^{n}\left(m_i \dot q^T J_{v_i}^T J_{v_i}\dot q + \dot q^T J_{\omega_i}^T I_i J_{\omega_i}\dot q\right)\\
U &= \sum_{i=1}^{n} m_i g_i y_i
\end{aligned}
$$

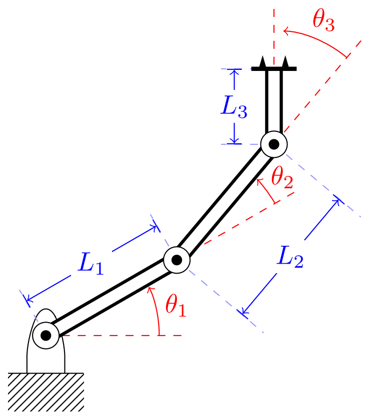
$$
\begin{aligned}
x_1 &= \frac{a_1}{2}c_1\\
y_1 &= \frac{a_1}{2}s_1\\
J_{v_1} &= \begin{bmatrix}\frac{\partial x_1}{\partial\theta_1} & \frac{\partial x_1}{\partial\theta_2} & \frac{\partial x_1}{\partial\theta_3}\\ \frac{\partial y_1}{\partial\theta_1} & \frac{\partial y_1}{\partial\theta_2} & \frac{\partial y_1}{\partial\theta_3}\\ \frac{\partial z_1}{\partial\theta_1} & \frac{\partial z_1}{\partial\theta_2} & \frac{\partial z_1}{\partial\theta_3}\end{bmatrix}\\
J_{v_1} &= \begin{bmatrix}-\frac{a_1}{2}s_1 & 0 & 0\\ \frac{a_1}{2}c_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\\
J_{\omega_1} &= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
U_1 = m\frac{a_1}{2}s_1
$$

$$
\begin{aligned}
x_2 &= a_1c_1 + \frac{a_2}{2}c_{12}\\
y_2 &= a_1s_1 + \frac{a_2}{2}s_{12}\\
J_{v_2} &= \begin{bmatrix}\frac{\partial x_2}{\partial\theta_1} & \frac{\partial x_2}{\partial\theta_2} & \frac{\partial x_2}{\partial\theta_3}\\ \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3}\\ \frac{\partial z_2}{\partial\theta_1} & \frac{\partial z_2}{\partial\theta_2} & \frac{\partial z_2}{\partial\theta_3}\end{bmatrix}\\
J_{v_2} &= \begin{bmatrix}-a_1s_1 - \frac{a_2}{2}s_{12} & -\frac{a_2}{2}s_{12} & 0\\ a_1c_1 + \frac{a_2}{2}c_{12} & \frac{a_2}{2}c_{12} & 0\\ 0 & 0 & 0\end{bmatrix}\\
J_{\omega_2} &= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}
\end{aligned}
$$

$$
U_2 = m(a_1s_1 + \frac{a_2}{2}s_{12})
$$

Figure 1: A 3-Link Planar Robot Arm



$$
\begin{aligned}
x_3 &= a_1 c_1 + a_2 c_{12} + \frac{a_3}{2} c_{123} \\
y_3 &= a_1 s_1 + a_2 s_{12} + \frac{a_3}{2} s_{123} \\
J_{v_3} &= \begin{bmatrix} \frac{\partial x_3}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_3} \\ \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial z_3}{\partial \theta_1} & \frac{\partial z_3}{\partial \theta_2} & \frac{\partial z_3}{\partial \theta_3} \end{bmatrix} \\
J_{v_3} &= \begin{bmatrix} -a_1 s_1 - \frac{a_2}{2} s_{12} - a_3 s_{123} & -\frac{a_2}{2} s_{12} - \frac{a_3}{2} s_{123} & -\frac{a_3}{2} s_{123} \\ a_1 c_1 + \frac{a_2}{2} c_{12} + \frac{a_3}{2} c_{123} & \frac{a_2}{2} c_{12} + \frac{a_3}{2} c_{123} & \frac{a_3}{2} c_{123} \\ 0 & 0 & 0 \end{bmatrix} \\
J_{\omega_3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
U_3 = m_3 (a_1 s_1 + a_2 s_{12} + \frac{a_3}{2} s_{123})
$$

$$
I_{\text{cuboid}} = \begin{bmatrix} \frac{1}{12} m (y^2 + z^2) & 0 & 0 \\ 0 & \frac{1}{12} m (x^2 + z^2) & 0 \\ 0 & 0 & \frac{1}{12} m (x^2 + y^2) \end{bmatrix} \tag{1}
$$