

# AE 352: Final Project

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We will consider a 3-link planar robot arm with revolute joints:

The x and y positions of the end-effector can be defined as:

$$x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) + a_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Derivation of the Jacobian relating the end effector velocity to the world frame.

$$\begin{aligned}
 J &= \begin{bmatrix} z_0^0 \times (O_3^0 - O_0^0) & z_1^0 \times (O_3^0 - O_1^0) & z_2^0 \times (O_3^0 - O_2^0) \\ z_0^0 & z_1^0 & z_2^0 \end{bmatrix} \\
 T_1^0 &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_1 \cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_1 \sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 T_2^1 &= \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2 \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2 \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 T_3^2 &= \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3 \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3 \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 T_2^0 &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 & a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 T_3^0 &= \begin{bmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & 0 & a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & 0 & a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) + a_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 J &= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c_{12} + a_3 c_{123} \\ a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_3 c_{123} \\ a_3 s_{123} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \\
 J &= \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

$$T = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$H = \sum_{i=1}^n \left( m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i} \right)$$

$$T = \sum_{i=1}^n \left( m_i \dot{q}^T J_{v_i}^T J_{v_i} \dot{q} + \dot{q}^T J_{\omega_i}^T I_i J_{\omega_i} \dot{q} \right)$$

$$U = \sum_{i=1}^n m_i g_i y_i$$

$$\begin{aligned} x_1 &= \frac{a_1}{2} c_1 \\ y_1 &= \frac{a_1}{2} s_1 \\ J_{v_1} &= \begin{bmatrix} \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} & \frac{\partial x_1}{\partial \theta_3} \\ \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \frac{\partial y_1}{\partial \theta_3} \\ \frac{\partial z_1}{\partial \theta_1} & \frac{\partial z_1}{\partial \theta_2} & \frac{\partial z_1}{\partial \theta_3} \end{bmatrix} \\ J_{v_1} &= \begin{bmatrix} -\frac{a_1}{2} s_1 & 0 & 0 \\ \frac{a_1}{2} c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ J_{\omega_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$U_1 = m \frac{a_1}{2} s_1$$

$$\begin{aligned} x_2 &= a_1 c_1 + \frac{a_2}{2} c_{12} \\ y_2 &= a_1 s_1 + \frac{a_2}{2} s_{12} \\ J_{v_2} &= \begin{bmatrix} \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} & \frac{\partial x_2}{\partial \theta_3} \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial z_2}{\partial \theta_1} & \frac{\partial z_2}{\partial \theta_2} & \frac{\partial z_2}{\partial \theta_3} \end{bmatrix} \\ J_{v_2} &= \begin{bmatrix} -a_1 s_1 - \frac{a_2}{2} s_{12} & -\frac{a_2}{2} s_{12} & 0 \\ a_1 c_1 + \frac{a_2}{2} c_{12} & \frac{a_2}{2} c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ J_{\omega_2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$U_2 = m \left( a_1 s_1 + \frac{a_2}{2} s_{12} \right)$$

$$\begin{aligned}
x_3 &= a_1 c_1 + a_2 c_{12} + \frac{a_3}{2} c_{123} \\
y_3 &= a_1 s_1 + a_2 s_{12} + \frac{a_3}{2} s_{123} \\
J_{v_3} &= \begin{bmatrix} \frac{\partial x_3}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_3} \\ \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial z_3}{\partial \theta_1} & \frac{\partial z_3}{\partial \theta_2} & \frac{\partial z_3}{\partial \theta_3} \end{bmatrix} \\
J_{v_3} &= \begin{bmatrix} -a_1 s_1 - \frac{a_2}{2} s_{12} - \frac{a_3}{2} s_{123} & -\frac{a_2}{2} s_{12} - \frac{a_3}{2} s_{123} & -\frac{a_3}{2} s_{123} \\ a_1 c_1 + \frac{a_2}{2} c_{12} + \frac{a_3}{2} c_{123} & \frac{a_2}{2} c_{12} + \frac{a_3}{2} c_{123} & \frac{a_3}{2} c_{123} \\ 0 & 0 & 0 \end{bmatrix} \\
J_{\omega_3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

$$U_3 = m_3(a_1 s_1 + a_2 s_{12} + \frac{a_3}{2} s_{123})$$

We will begin by considering the simple case of a two-link robot arm first:

$$\begin{aligned}
T &= \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
T &= \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2} a_2^2 + a_1 a_2 c_2) + I_2 & m_2(\frac{1}{4} a_2^2 + \frac{1}{2} a_1 a_2 c_2) + I_2 \\ m_2(\frac{1}{4} a_2^2 + \frac{1}{2} a_1 a_2 c_2) + I_2 & \frac{1}{4} m_2 a_2^2 + I_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
T &= \frac{1}{2} \left[ \dot{\theta}_1 \left( \frac{1}{4} m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2} a_2^2 + a_1 a_2 c_2) + I_2 \right) + \dot{\theta}_2 \left( m_2(\frac{1}{4} a_2^2 + \frac{1}{2} a_1 a_2 c_2) + I_2 \right) \right]^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
T &= \frac{1}{2} \left( \dot{\theta}_1^2 \left( \frac{1}{4} m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2} a_2^2 + a_1 a_2 c_2) + I_2 \right) + \dot{\theta}_1 \dot{\theta}_2 \left( m_2(\frac{1}{4} a_2^2 + \frac{1}{2} a_1 a_2 c_2) + I_2 \right) \right. \\
&\quad \left. + \dot{\theta}_1 \dot{\theta}_2 \left( m_2(\frac{1}{4} a_2^2 + \frac{1}{2} a_1 a_2 c_2) + I_2 \right) + \dot{\theta}_2^2 \left( \frac{1}{4} m_2 a_2^2 + I_2 \right) \right) \\
U &= \frac{1}{2} m_1 g a_1 s_1 + m_2 g (a_1 s_1 + \frac{1}{2} a_2 s_{12})
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} &= \tau_i \\
\frac{\partial L}{\partial q_1} &= -\frac{\partial U}{\partial q_1} = -\left(\frac{1}{2}m_1 a_1 g c_1 + m_2 g \left(\frac{1}{2}a_2 c_{12} + a_1 c_1\right)\right) \\
\frac{\partial L}{\partial \dot{q}_1} &= H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} &= H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 + \frac{\partial H_{11}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_2^2 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} &= \left(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\right)\ddot{\theta}_1 + \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right)\ddot{\theta}_2 \\
&\quad - m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_2^2 \\
\tau_1 &= \left(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\right)\ddot{\theta}_1 + \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right)\ddot{\theta}_2 + \\
&\quad - m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_2^2 + \frac{1}{2}m_1 a_1 g c_1 + m_2 g \left(\frac{1}{2}a_2 c_{12} + a_1 c_1\right) \\
\frac{\partial L}{\partial q_2} &= -\frac{\partial U}{\partial q_2} = -\frac{1}{2}m_2 a_2 g c_{12} \\
\frac{\partial L}{\partial \dot{q}_2} &= H_{22}\dot{\theta}_2 + H_{12}\dot{\theta}_1 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} &= H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + \frac{dH_{22}}{dt}\dot{\theta}_1\dot{\theta}_2 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} &= H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} &= \left(\frac{1}{4}m_2 a_2^2 + I_2\right)\ddot{\theta}_2 + \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right)\ddot{\theta}_1 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_1 \dot{\theta}_2 \\
\tau_2 &= \left(\frac{1}{4}m_2 a_2^2 + I_2\right)\ddot{\theta}_2 + \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right)\ddot{\theta}_1 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2}m_2 a_2 g c_{12}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} \left(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\right) & \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right) \\ \left(m_2\left(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2\right) + I_2\right) & \left(\frac{1}{4}m_2 a_2^2 + I_2\right) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\
&+ \begin{bmatrix} -m_2 s_2 & -\frac{1}{2}m_2 a_1 a_2 s_1 \\ -\frac{1}{2}m_2 a_1 a_2 s_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}m_1 a_1 g c_1 + m_2 g \left(\frac{1}{2}a_2 c_{12} + a_1 c_1\right) \\ \frac{1}{2}m_2 a_2 g c_{12} \end{bmatrix}
\end{aligned}$$

$$I_{\text{cuboid}} = \begin{bmatrix} \frac{1}{12}m(y^2 + z^2) & 0 & 0 \\ 0 & \frac{1}{12}m(x^2 + z^2) & 0 \\ 0 & 0 & \frac{1}{12}m(x^2 + y^2) \end{bmatrix} \quad (1)$$

$\mathbb{Q}$

$$\begin{aligned}
\mathbf{r}_i &= \mathbf{r}_i(q_1, \dots, q_n) \\
d\mathbf{r}_i &= \frac{\partial \mathbf{r}_i}{\partial q_1} dq_1 + \dots + \frac{\partial \mathbf{r}_i}{\partial q_n} dq_n \\
\mathbf{r}_i &= \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \\
d\mathbf{r}_i &= \begin{bmatrix} \frac{\partial x_i}{\partial q_1} dq_1 + \dots + \frac{\partial x_i}{\partial q_n} dq_n \\ \frac{\partial y_i}{\partial q_1} dq_1 + \dots + \frac{\partial y_i}{\partial q_n} dq_n \\ \frac{\partial z_i}{\partial q_1} dq_1 + \dots + \frac{\partial z_i}{\partial q_n} dq_n \end{bmatrix} \\
d\mathbf{r}_i &= \begin{bmatrix} \frac{\partial x_i}{\partial q_1} dq_1 & \dots & \frac{\partial x_i}{\partial q_n} dq_n \\ \frac{\partial y_i}{\partial q_1} dq_1 & \dots & \frac{\partial y_i}{\partial q_n} dq_n \\ \frac{\partial z_i}{\partial q_1} dq_1 & \dots & \frac{\partial z_i}{\partial q_n} dq_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
d\mathbf{r}_i &= \mathbb{J}_i d\mathbf{q} \\
\frac{d\mathbf{r}_i}{dt} &= \mathbb{J}_i \frac{d\mathbf{q}}{dt} \\
\mathbf{v}_i &= \mathbb{J}_i \dot{\mathbf{q}} \\
T &= \frac{1}{2} \sum_{i=1}^n (m_i v_i^2 + I_i w_i^2) \\
T &= \frac{1}{2} \sum_{i=1}^n (m_i v_i^T v_i + w_i^T \mathbb{I}_i w_i) \\
T &= \frac{1}{2} \sum_{i=1}^n (m_i \dot{\mathbf{q}}^T (\mathbb{J}_i^L)^T \mathbb{J}_i^L \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\mathbb{J}_i^A)^T \mathbb{I}_i \mathbb{J}_i^A \dot{\mathbf{q}}) \\
T &= \frac{1}{2} \dot{\mathbf{q}}^T \sum_{i=1}^n (m_i (\mathbb{J}_i^L)^T \mathbb{J}_i^L + (\mathbb{J}_i^A)^T \mathbb{I}_i \mathbb{J}_i^A) \dot{\mathbf{q}} \\
T &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbb{H} \dot{\mathbf{q}} \\
T &= \frac{1}{2} \begin{bmatrix} \dot{q}_1 & \dots & \dot{q}_n \end{bmatrix} \begin{bmatrix} H_{11} & \dots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \dots & H_{nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \\
T &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n H_{ij} \dot{q}_i \dot{q}_j \\
\frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^n H_{ij} \dot{q}_j \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) &= \sum_{j=1}^n \left( \frac{dH_{ij}}{dt} \dot{q}_j + H_{ij} \ddot{q}_j \right) \\
H_{ij} &= H_{ij}(q_1, \dots, q_n) \\
\frac{dH_{ij}}{dt} &= \sum_{k=1}^n \frac{\partial H_{ij}}{\partial q_k} \frac{dq_k}{dt} = \sum_{k=1}^n \frac{\partial H_{ij}}{\partial q_k} \dot{q}_k \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) &= \sum_{j=1}^n \left( \sum_{k=1}^n \frac{\partial H_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j + H_{ij} \ddot{q}_j \right)
\end{aligned}$$

Figure 1: A 3-Link Planar Robot Arm

