AE 352: Final Project

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We will consider a 3-link planar robot arm with revolute joints:

The x and y positions of the end-effector can be defined as:

$$x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) + a_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Derivation of the Jacobian relating the end effector velocity to the world frame.

$$\begin{aligned}
J &= \begin{bmatrix} z_0^0 \times (O_3^0 - O_0^0) & z_1^0 \times (O_3^0 - O_1^0) & z_2^0 \times (O_3^0 - O_2^0) \\ z_0^0 & z_1^0 & z_2^0 \end{bmatrix} \\
T_1^0 &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_1\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_1\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_2^1 &= \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2\sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_3^2 &= \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_2^0 &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_3^0 &= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) + a_3\cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) + a_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
J &= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2c_{12} + a_3c_{123} \\ a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_3c_{123} \\ a_3s_{123} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \\
J &= \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
T &= \frac{1}{2}\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3\end{bmatrix}\begin{bmatrix}H_{11} & H_{12} & H_{13}\\ H_{12} & H_{22} & H_{23}\\ H_{13} & H_{23} & H_{33}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix}\\
H &= \sum_{i=1}^{n}\left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i}\right)\\
T &= \sum_{i=1}^{n}\left(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T I_i J_{\omega_i}\dot{q}\right)\\
U &= \sum_{i=1}^{n} m_i g_i y_i
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{a_1}{2}c_1\\
y_1 &= \frac{a_1}{2}s_1\\
J_{v_1} &= \begin{bmatrix}\frac{\partial x_1}{\partial\theta_1} & \frac{\partial x_1}{\partial\theta_2} & \frac{\partial x_1}{\partial\theta_3}\\ \frac{\partial y_1}{\partial\theta_1} & \frac{\partial y_1}{\partial\theta_2} & \frac{\partial y_1}{\partial\theta_3}\\ \frac{\partial z_1}{\partial\theta_1} & \frac{\partial z_1}{\partial\theta_2} & \frac{\partial z_1}{\partial\theta_3}\end{bmatrix}\\
J_{v_1} &= \begin{bmatrix}-\frac{a_1}{2}s_1 & 0 & 0\\ \frac{a_1}{2}c_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\\
J_{\omega_1} &= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
U_1 = m\frac{a_1}{2}s_1
$$

$$
\begin{aligned}
x_2 &= a_1c_1 + \frac{a_2}{2}c_{12}\\
y_2 &= a_1s_1 + \frac{a_2}{2}s_{12}\\
J_{v_2} &= \begin{bmatrix}\frac{\partial x_2}{\partial\theta_1} & \frac{\partial x_2}{\partial\theta_2} & \frac{\partial x_2}{\partial\theta_3}\\ \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3}\\ \frac{\partial z_2}{\partial\theta_1} & \frac{\partial z_2}{\partial\theta_2} & \frac{\partial z_2}{\partial\theta_3}\end{bmatrix}\\
J_{v_2} &= \begin{bmatrix}-a_1s_1 - \frac{a_2}{2}s_{12} & -\frac{a_2}{2}s_{12} & 0\\ a_1c_1 + \frac{a_2}{2}c_{12} & \frac{a_2}{2}c_{12} & 0\\ 0 & 0 & 0\end{bmatrix}\\
J_{\omega_2} &= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}
\end{aligned}
$$

$$
U_2 = m(a_1s_1 + \frac{a_2}{2}s_{12})
$$

$$
\begin{aligned}
x_3 &= a_1c_1 + a_2c_{12} + \frac{a_3}{2}c_{123} \\
y_3 &= a_1s_1 + a_2s_{12} + \frac{a_3}{2}s_{123} \\
J_{v_3} &= \begin{bmatrix} \frac{\partial x_3}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_3} \\ \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial z_3}{\partial \theta_1} & \frac{\partial z_3}{\partial \theta_2} & \frac{\partial z_3}{\partial \theta_3} \end{bmatrix} \\
J_{v_3} &= \begin{bmatrix} -a_1s_1 - \frac{a_2}{2}s_{12} - a_3s_{123} & -\frac{a_2}{2}s_{12} - \frac{a_3}{2}s_{123} & -\frac{a_3}{2}s_{123} \\ a_1c_1 + \frac{a_2}{2}c_{12} + \frac{a_3}{2}c_{123} & \frac{a_2}{2}c_{12} + \frac{a_3}{2}c_{123} & \frac{a_3}{2}c_{123} \\ 0 & 0 & 0 \end{bmatrix} \\
J_{\omega_3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
U_3 = m_3(a_1s_1 + a_2s_{12} + \frac{a_3}{2}s_{123})
$$

We will begin by considering the simple case of a two-link robot arm first:

$$
\begin{aligned}
T &= \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2\end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22}\end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix} \\
T &= \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2\end{bmatrix}\begin{bmatrix} \frac{1}{4}m_1a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1a_2c_2) + I_2 & m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2 \\ m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2 & \frac{1}{4}m_2a_2^2 + I_2 \end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix} \\
T &= \frac{1}{2}\begin{bmatrix} \dot\theta_1\big(\frac{1}{4}m_1a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1a_2c_2) + I_2\big) + \dot\theta_2\big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2\big) \\ \dot\theta_1\big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2\big) + \dot\theta_2\big(\frac{1}{4}m_2a_2^2 + I_2\big) \end{bmatrix}^T \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix} \\
T &= \frac{1}{2}\Big(\dot\theta_1^2\big(\frac{1}{4}m_1a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1a_2c_2) + I_2\big) + \dot\theta_1\dot\theta_2\big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2\big) \\
&\quad + \dot\theta_1\dot\theta_2\big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1a_2c_2) + I_2\big) + \dot\theta_2^2\big(\frac{1}{4}m_2a_2^2 + I_2\big)\Big) \\
U &= \frac{1}{2}m_1ga_1s_1 + m_2g(a_1s_1 + \frac{1}{2}a_2s_{12})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} &= \tau_i \\
\frac{\partial L}{\partial q_1} &= -\frac{\partial U}{\partial q_1} = -(\frac{1}{2}m_1 a_1 g c_1 + m_2 g(\frac{1}{2}a_2 c_{12} + a_1 c_1)) \\
\frac{\partial L}{\partial \dot{q}_1} &= H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} &= H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 + \frac{\partial H_{11}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_2^2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} &= \big(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\big)\ddot{\theta}_1 + \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big)\ddot{\theta}_2 \\
& -m_2 s_2 \dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_2^2 \\
\tau_1 &= \big(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\big)\ddot{\theta}_1 + \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big)\ddot{\theta}_2 + \\
& -m_2 s_2 \dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_2^2 + \frac{1}{2}m_1 a_1 g c_1 + m_2 g(\frac{1}{2}a_2 c_{12} + a_1 c_1) \\
\frac{\partial L}{\partial q_2} &= -\frac{\partial U}{\partial q_2} = -\frac{1}{2}m_2 a_2 g c_{12} \\
\frac{\partial L}{\partial \dot{q}_2} &= H_{22}\dot{\theta}_2 + H_{12}\dot{\theta}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} &= H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + \frac{dH_{22}}{dt}\dot{\theta}_1\dot{\theta}_2 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} &= H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} &= \big(\frac{1}{4}m_2 a_2^2 + I_2\big)\ddot{\theta}_2 + \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big)\ddot{\theta}_1 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_1\dot{\theta}_2 \\
\tau_2 &= \big(\frac{1}{4}m_2 a_2^2 + I_2\big)\ddot{\theta}_2 + \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big)\ddot{\theta}_1 - \frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2 a_2 g c_{12}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} &= \begin{bmatrix} \big(\frac{1}{4}m_1 a_1^2 + I_1 + m_2(a_1^2 + \frac{1}{2}a_2^2 + a_1 a_2 c_2) + I_2\big) & \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big) \\ \big(m_2(\frac{1}{4}a_2^2 + \frac{1}{2}a_1 a_2 c_2) + I_2\big) & \big(\frac{1}{4}m_2 a_2^2 + I_2\big) \end{bmatrix} \begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} \\
& + \begin{bmatrix} -m_2 s_2 & -\frac{1}{2}m_2 a_1 a_2 s_1 \\ -\frac{1}{2}m_2 a_1 a_2 s_1 \dot{\theta}_1\dot{\theta}_2 & 0 \end{bmatrix} \begin{bmatrix}\dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix} \frac{1}{2}m_1 a_1 g c_1 + m_2 g(\frac{1}{2}a_2 c_{12} + a_1 c_1) \\ \frac{1}{2}m_2 a_2 g c_{12} \end{bmatrix}
\end{aligned}
$$

$$
I_{\text{cuboid}} = \begin{bmatrix} \frac{1}{12}m(y^2 + z^2) & 0 & 0 \\ 0 & \frac{1}{12}m(x^2 + z^2) & 0 \\ 0 & 0 & \frac{1}{12}m(x^2 + y^2) \end{bmatrix} \tag{1}
$$

$\mathbb{Q}$

$$
\begin{aligned}
\mathbf{r}_i &= \mathbf{r}_i(q_1,\ldots,q_n) \\
d\mathbf{r}_i &= \frac{\partial \mathbf{r}_i}{\partial q_1}dq_1 + \cdots + \frac{\partial \mathbf{r}_i}{\partial q_n}dq_n \\
\mathbf{r}_i &= \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \\
d\mathbf{r}_i &= \begin{bmatrix} \frac{\partial x_i}{\partial q_1}dq_1 + \cdots + \frac{\partial x_i}{\partial q_n}dq_n \\ \frac{\partial y_i}{\partial q_1}dq_1 + \cdots + \frac{\partial y_i}{\partial q_n}dq_n \\ \frac{\partial z_i}{\partial q_1}dq_1 + \cdots + \frac{\partial z_i}{\partial q_n}dq_n \end{bmatrix} \\
d\mathbf{r}_i &= \begin{bmatrix} \frac{\partial x_i}{\partial q_1}dq_1 & \ldots & \frac{\partial x_i}{\partial q_n}dq_n \\ \frac{\partial y_i}{\partial q_1}dq_1 & \ldots & \frac{\partial y_i}{\partial q_n}dq_n \\ \frac{\partial z_i}{\partial q_1}dq_1 & \ldots & \frac{\partial z_i}{\partial q_n}dq_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
d\mathbf{r}_i &= \mathbb{J}_i d\mathbf{q} \\
\frac{d\mathbf{r}_i}{dt} &= \mathbb{J}_i \frac{d\mathbf{q}}{dt} \\
\mathbf{v}_i &= \mathbb{J}_i \dot{\mathbf{q}} \\
T &= \frac{1}{2}\sum_{i=1}^{n}(m_i v_i^2 + I_i w_i^2) \\
T &= \frac{1}{2}\sum_{i=1}^{n}(m_i v_i^T v_i + w_i^T \mathbb{I}_i w_i) \\
T &= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{\mathbf{q}}^T (\mathbb{J}_i^L)^T \mathbb{J}_i^L \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\mathbb{J}_i^A)^T \mathbb{I}_i \mathbb{J}_i^A \dot{\mathbf{q}}) \\
T &= \frac{1}{2}\dot{\mathbf{q}}^T \sum_{i=1}^{n}(m_i (\mathbb{J}_i^L)^T \mathbb{J}_i^L + (\mathbb{J}_i^A)^T \mathbb{I}_i \mathbb{J}_i^A)\dot{\mathbf{q}} \\
T &= \frac{1}{2}\dot{\mathbf{q}}^T \mathbb{H} \dot{\mathbf{q}} \\
T &= \frac{1}{2}\begin{bmatrix} \dot{q}_1 & \ldots & \dot{q}_n \end{bmatrix} \begin{bmatrix} H_{11} & \ldots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \ldots & H_{nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \\
T &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}\dot{q}_i\dot{q}_j \\
\frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} H_{ij}\dot{q}_i\dot{q}_j \\
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) &= \sum_{j=1}^{n}(\frac{dH_{ij}}{dt}\dot{q}_i + H_{ij}\ddot{q}_i) \\
H_{ij} &= H_{ij}(q_1,\ldots,q_n) \\
\frac{dH_{ij}}{dt} &= \sum_{k=1}^{n}\frac{\partial H_{ij}}{\partial q_k}\frac{dq_k}{dt} = \sum_{k=1}^{n}\frac{\partial H_{ij}}{\partial q_k}\dot{q}_k \\
\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) &= \sum_{j=1}^{n}(\sum_{k=1}^{n}\frac{\partial H_{ij}}{\partial q_k}\dot{q}_k\dot{q}_i + H_{ij}\ddot{q}_i)
\end{aligned}
$$

Figure 1: A 3-Link Planar Robot Arm