Induction Proofs: Worked examples *(page 3 of 3)*

- **(*) For $n \geq 1$, $2 + 2^2 + 2^3 + 2^4 + ... + 2^n = 2^{n+1} - 2$**

  Let $n = 1$. Then:

  $$2 + 2^2 + 2^3 + 2^4 + ... + 2^n = 2^1 = 2$$

  ...and:

  $$2^{n+1} - 2 = 2^{1+1} - 2 = 2^2 - 2 = 4 - 2 = 2$$

  So (*) works for $n = 1$.

  Assume, for $n = k$, that (*) holds; that is, that

  $$2 + 2^2 + 2^3 + 2^4 + ... + 2^k = 2^{k+1} - 2$$

  Let $n = k + 1$.

  $$\begin{aligned} &2 + 2^2 + 2^3 + 2^4 + ... + 2^k + 2^{k+1} \\ &\quad = [2 + 2^2 + 2^3 + 2^4 + ... + 2^k] + 2^{k+1} \\ &\quad = [2^{k+1} - 2] + 2^{k+1} \\ &\quad = 2 \times 2^{k+1} - 2 \\ &\quad = 2^1 \times 2^{k+1} - 2 \\ &\quad = 2^{k+1+1} - 2 \\ &\quad = 2^{(k+1)+1} - 2 \end{aligned}$$

  Then **(*) works** for $n = k + 1$.

Note this common technique: In the "$n = k + 1$" step, it is usually a good first step to write out the whole formula in terms of $k + 1$, and then break off the "$n = k$" part, so you can replace it with whatever assumption you made about $n = k$ in the previous step. Then you manipulate and simplify, and try to rearrange things to get the right-hand side of the formula in terms of $k + 1$.

- **(*) For $n \geq 1$, $1 \times 2 + 2 \times 3 + 3 \times 4 + ... + (n)(n+1) = (n)(n+1)(n+2)/3$**

  Let $n = 1$.

  Then the left-hand side of (*) is $1 \times 2 = 2$
  and the right-hand side of (*) is $(1)(2)(3)/3 = 2$.

  So (*) holds for $n = 1$. Assume, for $n = k$, that (*) holds; that is, assume that

  $$1 \times 2 + 2 \times 3 + 3 \times 4 + ... + (k)(k+1) = (k)(k+1)(k+2)/3$$

  Let $n = k + 1$. The left-hand side of (*) then gives us:

  $$\begin{aligned} &1 \times 2 + 2 \times 3 + 3 \times 4 + ... + (k)(k+1) + (k+1)((k+1)+1) \\ &\quad = [1 \times 2 + 2 \times 3 + 3 \times 4 + ... + (k)(k+1)] + (k+1)((k+1)+1) \\ &\quad = [(k)(k+1)(k+2)/3] + (k+1)((k+1)+1) \\ &\quad = (k)(k+1)(k+2)/3 + (k+1)(k+2) \\ &\quad = (k)(k+1)(k+2)/3 + 3(k+1)(k+2)/3 \end{aligned}$$

$$= (k+3)(k+1)(k+2)/3$$
$$= (k+1)(k+2)(k+3)/3$$
$$= (k+1)((k+1)+1)((k+1)+2)/3$$

...which is the right-hand side of (*). Then **(*) works** for all $n \geq 1$.

- **(*) For $n \geq 5$, $4n < 2^n$.**

This one doesn't start at $n = 1$, and involves an inequality instead of an equation. (If you graph $4x$ and 2^x on the same axes, you'll see why we have to start at $n = 5$, instead of the customary $n = 1$.)

Let $n = 5$.

Then $4n = 4 \times 5 = 20$, and $2^n = 2^5 = 32$.

Since $20 < 32$, then (*) holds at $n = 5$.

Assume, for $n = k$, that (*) holds; that is, assume that $4k < 2^k$

Let $n = k + 1$.

The left-hand side of (*) gives us $4(k + 1) = 4k + 4$, and, by assumption,

$$[4k] + 4 < [2^k] + 4$$

Since $k \geq 5$, then $4 < 32 \leq 2^k$. Then we get

$$2^k + 4 < 2^k + 2^k = 2 \times 2^k = 2^1 \times 2^k = 2^{k+1}$$

Then $4(k+1) < 2^{k+1}$, and (*) holds for $n = k + 1$.

Then **(*) holds** for all $n \geq 5$.

- **(*) For all $n \geq 1$, $8^n - 3^n$ is divisible by 5.**

Let $n = 1$.

Then the expression $8^n - 3^n$ evaluates to $8^1 - 3^1 = 8 - 3 = 5$, which is clearly divisible by 5.

Assume, for $n = k$, that (*) holds; that is, that $8^k - 3^k$ is divisible by 5.

Let $n = k + 1$.

Then:

$$8^{k+1} - 3^{k+1} = 8^{k+1} - 3 \times 8^k + 3 \times 8^k - 3^{k+1}$$

$$= 8^k(8 - 3) + 3(8^k - 3^k) = 8^k(5) + 3(8^k - 3^k)$$

The first term in $8^k(5) + 3(8^k - 3^k)$ has 5 as a factor (explicitly), and the second term is divisible by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, $8^k(5) + 3(8^k - 3^k) = 8^{k+1} - 3^{k+1}$, must be divisible by 5.

Then **(*) holds** for $n = k + 1$, and thus for all $n \geq 1$.