Math 116 Midterm 1 Proof List

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1) Suppose $\|x\|$ denotes a real valued function of a vector x that satisfies all the requirements of a norm except perhaps the triangle inequality. Prove $\|x\|$ satisfies the triangle inequality iff $K = \{x \mid \|x\| \leq 1\}$ is convex.

Proof (=>) Let $\|x\|$ satisfy the triangle inequality. Let $x, y \in K$. Then consider $0 \leq \alpha \leq 1$.
$\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = |\alpha|\|x\| + (1-\alpha)\|y\| \leq \alpha + 1 - \alpha = 1$, so $\alpha x + (1-\alpha)y \in K$.

(<=) Let K be convex. Then consider $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ as vectors in K, and $\alpha = \frac{\|x\|}{\|x\|+\|y\|}$ s.t. $0 \leq \alpha \leq 1$. Then $\|\alpha x + (1-\alpha)y\| = \left\| \frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|} \right\| \leq 1$ since K is convex, so $\|x+y\| \leq \|x\| + \|y\|$.

2) Prove the interior of a convex set C is convex.

Proof Let $\mathring{C}$ denote the interior of convex set C. Let $x, y \in \mathring{C}$. We wish to prove, $\forall \alpha \in [0,1]$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(\alpha x + (1-\alpha)y) \subseteq C$. Because $x, y \in \mathring{C}$, $\exists\, \varepsilon_1, \varepsilon_2 > 0$ s.t. $B_{\varepsilon_1}(x), B_{\varepsilon_2}(y) \subseteq C$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Consider arbitrary w such that $\|w\| < \varepsilon$. Then $x+w, y+w \in C$, and $z = \alpha(x+w) + (1-\alpha)(y+w) = \alpha x + (1-\alpha)y + w \in C$, so $B_\varepsilon(\alpha x + (1-\alpha)y) \subseteq C \quad \forall \alpha \in [0,1]$.

3) Prove the closure of a convex set C is convex.

Proof Let $\bar{C}$ denote the closure of convex set C. Let $x, y \in \bar{C}$. Then, $\forall \varepsilon > 0$, $\exists\, x', y' \in C$ s.t. $x' \in B_\varepsilon(x)$ and $y' \in B_\varepsilon(y)$. Let $\alpha \in [0,1]$, and set $\varepsilon > 0$, and let x', y' be as mentioned. Then $\alpha x' + (1-\alpha)y' \in C$, and $\|\alpha x' + (1-\alpha)y' - (\alpha x + (1-\alpha)y)\| \leq \alpha\|x - x'\| + (1-\alpha)\|y - y'\| < \alpha\varepsilon + (1-\alpha)\varepsilon = \varepsilon$.
so $\alpha x' + (1-\alpha)y' \in B_\varepsilon(\alpha x + (1-\alpha)y)$, so $\alpha x + (1-\alpha)y \in \bar{C}$.

4) Prove if the sequence $\{x_n\}$ converges to x and to y, then $x = y$.

Proof Let $\varepsilon > 0$. Then
$$\begin{aligned} \|x - y\| &= \|x - x_n + x_n - y\| \\ &\leq \|x - x_n\| + \|x_n - y\| \\ &< \varepsilon/2 + \varepsilon/2 \qquad \text{for large enough } n \\ &< \varepsilon. \end{aligned}$$

Thus $\|x - y\| = 0 \Rightarrow x = y$.

5) Given that, for $a, b > 0$,

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

Prove if $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $x = \{\xi_1, \xi_2, \ldots\}$ and $y = \{\eta_1, \eta_2, \ldots\}$, $x \in \ell_p$, $y \in \ell_q$ then

$$\sum_{i=1}^{\infty} |\xi_i||\eta_i| \leq \|x\|_p \|y\|_q$$

Proof Let $a = \left(\frac{|\xi_i|}{\|x\|_p}\right)^p$, $b = \left(\frac{|\eta_i|}{\|y\|_q}\right)^q$, $\lambda = \frac{1}{p}$, $1-\lambda = \frac{1}{q}$. Applying the inequality,

$$\left[\left(\frac{|\xi_i|}{\|x\|_p}\right)^p\right]^{\frac{1}{p}} \left[\left(\frac{|\eta_i|}{\|y\|_q}\right)^q\right]^{\frac{1}{q}} \leq \frac{1}{p}\left(\frac{|\xi_i|}{\|x\|_p}\right)^p + \frac{1}{q}\left(\frac{|\eta_i|}{\|y\|_q}\right)^q$$

Summing over all i; we obtain

$$\frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^{\infty} |\xi_i||\eta_i| \leq \frac{1}{p} \underbrace{\sum_{i=1}^{\infty} \frac{|\xi_i|^p}{\|x\|_p^p}}_{1} + \frac{1}{q} \underbrace{\sum_{i=1}^{\infty} \frac{|\eta_i|^q}{\|y\|_q^q}}_{1}$$

which implies our result.

6) The Hölder inequality states that, for an n-dimensional vector space, if $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $x = \{\xi_1, \ldots, \xi_n\}$ and $y = \{\eta_1, \ldots, \eta_n\}$, $x \in \ell_p$, $y \in \ell_q$, then

$$\sum_{i=1}^{n} |\xi_i||\eta_i| \leq \|x\|_p \|y\|_q \qquad \text{where } \|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p}$$

Prove the Minkowski inequality using this, that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, for an n-dimensional vector space.

Proof

$$\sum_{i=1}^{n} |\xi_i + \eta_i|^p \leq \sum_{i=1}^{n} |\xi_i + \eta_i|^{p-1} |\xi_i| + \sum_{i=1}^{n} |\xi_i + \eta_i|^{p-1} |\eta_i|$$

$$\leq \left(\sum_{i=1}^{n} |\xi_i + \eta_i|^{(p-1)q}\right)^{1/q} \left(\left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |\eta_i|^p\right)^{1/p}\right)$$

since $(p-1)q = p$

$$\Rightarrow \left(\sum_{i=1}^{n} |\xi_i + \eta_i|^p\right)^{1-\frac{1}{q}} \leq \|x\|_p + \|y\|_q \quad \Rightarrow \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

7) Prove every convergent sequence is a Cauchy sequence and every Cauchy sequence is bounded.

Proof Let $\{x_n\}$ be a convergent sequence. Let $\varepsilon > 0$. Then $\exists N$ st $\forall n > N$, $\|x_n - x\| < \varepsilon/2$, where x is the limit of $\{x_n\}$. Thus, $\forall m > N$, $\|x_m - x_n\| = \|x_m - x + x - x_n\| \leq$ $\|x_m - x\| + \|x - x_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $\{x_n\}$ is Cauchy.

To prove boundedness, Fix $\varepsilon = 1$. Then $\exists N$ st. $\forall n, m > N$, $\|x_n - x_m\| < \varepsilon$

Thus, letting $m = N+1$, $\forall n > N$, $\|x_n\| = \|x_n - x_{N+1} + x_{N+1}\| \leq \|x_n - x_{N+1}\| + \|x_{N+1}\| < 1 + \|x_{N+1}\|$.

Thus, $\|x_k\| < \max(\|x_1\|, \|x_2\|, \ldots, \|x_N\|, \|x_{N+1}\|) + 1 \quad \forall k$.

8) Prove ℓ_p is a Banach space.

Proof The main problem is to show ℓ_p is complete. Let $\{x^{(n)}\}$ be a Cauchy sequence in ℓ_p, where $x^{(n)} = \{\xi_1^{(n)}, \xi_2^{(n)}, \ldots\}$. Let $\varepsilon > 0$. Then $\exists N$ st. $\forall n, m > N$,

$$\|x^{(n)} - x^{(m)}\|_p = \left(\sum_{i=1}^{\infty} |\xi_i^{(n)} - \xi_i^{(m)}|^p\right)^{1/p} < \varepsilon$$

$$\Rightarrow \quad |\xi_i^{(n)} - \xi_i^{(m)}| < \varepsilon$$ so for a fixed i, $\{\xi_i^{(n)}\}$ is Cauchy too.

Because $\xi_i^{(n)}$ are reals, we know $\xi_i^{(n)} \to \xi_i$. Let $x = \{\xi_1, \xi_2, \ldots\}$. I claim x is our desired limit for $\{x^{(n)}\}$. First, we show $x \in \ell_p$. Fix $k > 0$ (finite), and since $\{x^{(n)}\}$ is Cauchy, let M upper bound $\|x^{(n)}\|$.

$$\Rightarrow \sum_{i=1}^{k} |\xi_i^{(n)}|^p < M^p$$

$$\lim_{n\to\infty} \sum_{i=1}^{k} |\xi_i^{(n)}|^p = \sum_{i=1}^{k} \lim_{n\to\infty} |\xi_i^{(n)}|^p = \sum_{i=1}^{k} |\xi_i|^p \leq M^p$$

Now let $k \to \infty \Rightarrow \|x\|_p^p = \sum_{i=1}^{\infty} |\xi_i|^p \leq M^p$, so $x \in \ell_p$.

Now we show $x^{(n)} \to x$. Fix $k > 0$ (finite) again. Let $\varepsilon > 0$. $\exists N$ st $\forall m, n > N$,

$$\Rightarrow \left(\sum_{i=1}^{k} |\xi_i^{(n)} - \xi_i^{(m)}|^p\right)^{1/p} < \varepsilon/2$$

Letting $m \to \infty$, we have: $$\left(\sum_{i=1}^{k} |\xi_i^{(n)} - \xi_i|^p\right)^{1/p} \leq \varepsilon/2$$

Finally, letting $k \to \infty$: $$\left(\sum_{i=1}^{\infty} |\xi_i^{(n)} - \xi_i|^p\right)^{1/p} \leq \varepsilon/2 < \varepsilon \Rightarrow x^{(n)} \to x.$$

9) The Weierstrass polynomial theorem states for any $\varepsilon > 0$ and any continuous $x(t)$ on $[a,b]$, we can find polynomial $p(t)$ such that $\|x(t) - p(t)\| < \varepsilon$. Use this and the fact that the collection of all finite Cartesian products of a countable set is countable, prove $C[0,1]$ is separable.

Proof Let R be the set of all finite degree rational polynomials. This is countable because of the fact in the problem statement, and so it suffices to prove any $x(t) \in C[0,1]$ is sufficiently well approximated by some polynomial in R. Let $\varepsilon > 0$. By the Weierstrass polynomial theorem, $\exists$ $p(t)$ polynomial so $\|x(t) - p(t)\| < \varepsilon/2$. Now by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, if we let

$$p(t) = p_0 + p_1 t^2 + \dots + p_n t^n \qquad p_i \in \mathbb{R}$$

consider

$$r(t) = r_0 + r_1 t^2 + \dots + r_n t^n, \qquad r_i \in \mathbb{Q} \text{ s.t. } \|r_i - p_i\| < \frac{\varepsilon}{2(N+1)}$$

$$\Rightarrow \|p(t) - r(t)\| \le \|r_0 - p_0\| + \|r_1 - p_1\| t + \dots + \|r_n - p_n\| t^n$$

$$\le \|r_0 - p_0\| + \|r_1 - p_1\| + \dots + \|r_n - p_n\| \qquad \text{since } t \in [0,1]$$

$$< \frac{\varepsilon}{2(N+1)}(N+1) = \varepsilon/2$$

So overall,

$$\|x(t) - r(t)\| \le \|x(t) - p(t)\| + \|p(t) - r(t)\| < \varepsilon.$$

10) Suppose X is a Hilbert Space and M is a closed subspace of X. Let x be a vector in X and let m_0 be the closest vector to x in M. Prove m_0 exists and $x - m_0$ is orthogonal to M.

Proof Let $\delta = \inf_{m \in M} \|x - m\|$. Construct sequence $m_1, m_2, \dots \in M$ such that $\lim_{n \to \infty} \|m_n - x\| = \delta$.

I claim $\{m_i\}$ is Cauchy. Let $\varepsilon > 0$. Then by the parallelogram law,

$$\|m_i - m_j\|^2 + \|2(x - \tfrac{m_i + m_j}{2})\|^2 = 2\|x - m_i\|^2 + 2\|x - m_j\|^2$$

$\exists$ N s.t. $\forall$ $i,j > N$, $\|x - m_j\|^2 < \delta^2 + \varepsilon^2/4$ and $\|x - m_i\|^2 < \delta^2 + \varepsilon^2/4$. Note also that $\|x - \frac{m_i + m_j}{2}\|^2 \ge \delta^2$

$$\forall i,j > N \qquad \Rightarrow \|m_i - m_j\|^2 + 4\delta^2 < 2(\delta^2 + \varepsilon^2/4) + 2(\delta^2 + \varepsilon^2/4)$$

$$\Rightarrow \|m_i - m_j\|^2 < \varepsilon^2 \Rightarrow \|m_i - m_j\| < \varepsilon \quad \text{so } \{m_i\} \text{ is Cauchy.}$$

X is Hilbert and M is closed so m_i converges to m_0 in X, and $m_0 \in M$, which proves existence.

(11) Note that this proof also works if M is a closed convex subset of X, since $\frac{m_i + m_j}{2}$ is still in M.

Orthogonality (requires M to be closed subspace) Suppose for contradiction $x - m_0$ is not orthogonal to M, so $\exists$ unit vector m s.t. $(x - m_0 | m) = \delta > 0$. Let $m_1 = m_0 + \delta m$. Then $\|x - m_1\|^2 = \|x - m_0 + m_0 - m_1\|^2 = \|x - m_0\|^2 + \|m_0 - m_1\|^2 - 2(x - m_0 | m_1 - m_0)$ $= \|x - m_0\|^2 + \delta^2 - 2\delta^2 = \|x - m_0\|^2 - \delta^2$, contradicting minimality of $\|x - m_0\|^2$.

12) Let K be a closed convex subspace of Hilbert space H, and let x be a vector outside of K. Suppose $k_0 \in K$ minimizes $\|x-k\|$. Show. $(x-k_0 \mid k_1-k_0) \geq 0 \ \forall k_1 \in K$.

<u>Proof</u> Let $k_\alpha = \alpha k_0 + (1-\alpha)k_1 \in K$. Then we have that

$$
\begin{aligned}
\|x-k_\alpha\|^2 &= \|x-\alpha k_0-(1-\alpha)k_1\|^2 \\
&= \|\alpha(x-k_0)+(1-\alpha)(x-k_1)\|^2 \\
&= \|\alpha(x-k_0)\|^2+\|(1-\alpha)(x-k_1)\|^2+2(\alpha(x-k_0) \mid (1-\alpha)(x-k_1)) \\
&= \alpha^2\|x-k_0\|^2+(1-\alpha)^2\|x-k_1\|^2+2\alpha(1-\alpha)(x-k_0 \mid x-k_1)
\end{aligned}
$$

$$\frac{d}{d\alpha}\|x-k_\alpha\|^2 = 2\alpha\|x-k_0\|^2+2(\alpha-1)\|x-k_1\|^2+2(1-2\alpha)(x-k_0 \mid x-k_1)$$

Note that $\|x-k_\alpha\|^2$ is a convex function of α and has minimum at $\alpha=0$, so $\frac{d}{d\alpha}\|x-k_\alpha\|^2 \geq 0$ for all $\alpha \in [0,1]$. Thus, plugging in $\alpha=1$

$$\|x-k_0\|^2-(x-k_0 \mid x-k_1) \geq 0$$

$$\Rightarrow \quad (x-k_0 \mid (x-k_0-x+k_1)) \geq 0$$

$$\Rightarrow \quad (x-k_0 \mid k_1-k_0) \geq 0$$ as desired.