Week 8: Double Integrals

Cavalieri's

Cavalier's Principle or The Slice Method (M&T p. 266)

Suppose that R is a region in $\mathbb{R}^3$ contained between $z = a$ and $z = b$.
Let $A(t) = \text{Area}(R \cap \{z = t\})$ be the area of the slice of R at height t.

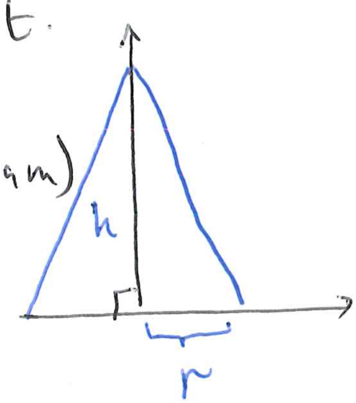
$$\text{Vol}(R) = \int_a^b A(t)dt = \text{“Volume is the integral of Area”}$$

Example: The Volume of a Cone

Find the volume of the right circular cone with base radius r and height h.

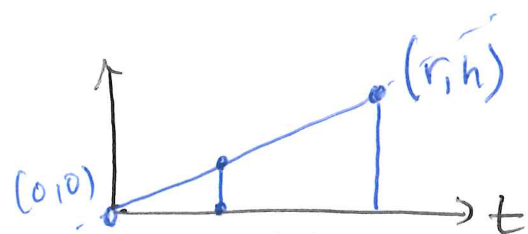
We need volume of slice at height t.

It is nicer to parametrize with t running horizontally. (2nd diagram)



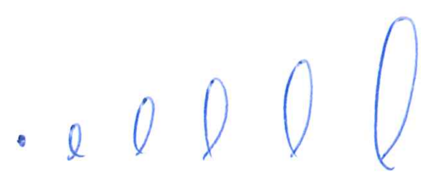
We get:

$$\text{Vol}(R) = \underbrace{\int_0^h \left[\left(\frac{r}{h}\right) t\right]^2 \pi}_{\text{Area of disk of radius } r(t).} dt$$



$r(t) = \frac{r}{h} t$

$$= \left[\left(\frac{r}{h}\right)^2 \pi \cdot \frac{1}{3} t^3\right]_{t=0}^{t=h}$$



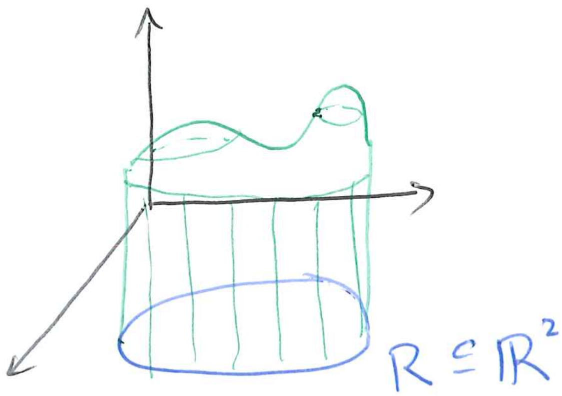
$$= \frac{1}{3}\left(\frac{r}{h}\right)^2 \cdot \pi \cdot h^3 = \frac{\pi r^2 h}{3}.$$

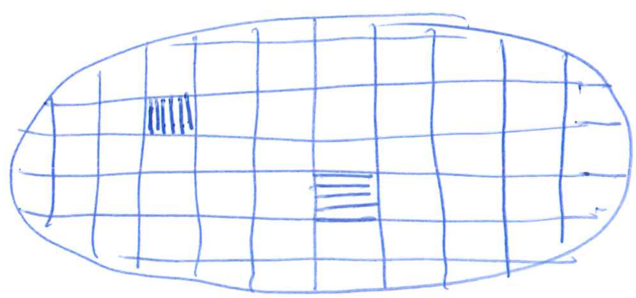
Double Integral (M&T p. 264)

Let $R \subset \mathbb{R}^2$ be a region and $f : R \to \mathbb{R}$ be a non-negative function: $f(x, y) \geq 0$. The volume above a region R and below f is given by the double integral.

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dxdy$$

The term $\boxed{dA = dxdy = dydx}$ is the area element of Cartesian coordinates.
We will say more about such terms in Weeks 11 & 12 when we introduce polar coordinates.





$dA = dxdy = dydx$

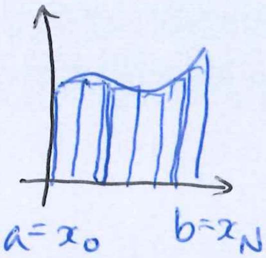
From the perspective of the region R, the order of integration dxdy or dydx does not matter.

Theorem: The Riemann Integral in One Dimension

The regular partition of $[a,b]$ in to N parts is the set of points $\{x_i\}_{i=0}^N$ where

$$a = x_0 < x_1 < \cdots < x_N = b$$

and $\Delta x_k = x_{k+1} - x_k = \dfrac{b-a}{N} = \dfrac{\text{“Length”}}{N}$.



A set of points $\{x_i^*\}_{i=0}^N$ where $x_i^* \in [x_i, x_{i+1}]$ is called a set of sample points.

Given a function $f : [a,b] \to \mathbb{R}$ we define its Riemann integral with sample points $\{x_i^*\}_{i=0}^N$ to be:

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*)\,\Delta x_i$$

It is an important theorem that for many functions this integral is independent of the choice of x_i^*.

Activity: Think-Pair-Share (3 minutes)

13:27

Picking the sample points x_i^* correctly is always subtle. Suppose that $\{x_i\}_{i=0}^N$ is the regular partition of $[a,b]$. Give a formula (in terms of a, b, i, N) for x_i^* when using the following sample points.

- Left endpoints
- Midpoints
- Right endpoints

Left: $x_i^* = a + i\Delta x_i = a + i\left(\frac{b-a}{N}\right)$

Middle: $x_i^* = \dfrac{(a + i\Delta x_i) + (a + (i+1)\Delta x_i))}{2}$

$= a + (i + \frac{1}{2})\Delta x_i = a + (i+\frac{1}{2})\left(\frac{b-a}{N}\right)$.

Right: $x_i^* = a + (i+1)\Delta x_i = a + (i+1)\left(\frac{b-a}{N}\right)$

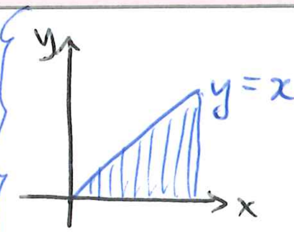
(!) For these last two x_N^* is not in the domain $[a,b]$.

Example: A Riemann Sum

Evaluate $\int_0^1 x\,dx$ using the Riemann sum definition of the definite integral with left-hand endpoints.

Consider the regular partition

$$x_i = \left(\frac{1-0}{N}\right)i = \frac{i}{N} \Rightarrow \Delta x_i = \frac{1}{N}$$



Area $= \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$.

We have left end points: $x_i^* = x_i$.

This gives:

$$\int_0^1 x\,dx = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*)\Delta x_i$$

$$= \lim_{N\to\infty} \sum_{i=0}^{N-1} x_i^* \Delta x_i \quad \# f(x) = x.$$

$$= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left(\frac{i}{N}\right)\left(\frac{1}{N}\right) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=0}^{N-1} i$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\left(0+1+2+\cdots+(N-1)\right).$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\left(\frac{(N-1)((N-1)+1)}{2}\right) \quad \# \text{ Summation formula}$$

$$= \lim_{N\to\infty} \frac{1}{N^2}\left(\frac{N^2+\cdots}{2}\right) = \frac{1}{2}.$$

Activity: Generalize (5 min) 13:45

The one-dimensional Riemann sum is defined on $[a, b]$. We want to define a two-dimensional Riemann sum on $[a, b] \times [c, d]$.

- What parts of the definition need to generalize?
- What are the two-dimensional analogues of those parts?

The Riemann Integral in Two Dimensions (M&T p. 272)

$$\iint_R f(x, y)\, dA = \lim_{N \to \infty} \sum_{i,j=0}^{N-1} f(\mathbf{c}_{ij}) \Delta x \Delta y$$

The Riemann Integral in Two Dimensions (M&T p. 272)

We say that f is integrable if the limit above exists and is independent of the choice of $\mathbf{c}_{ij}$.

- Regular partition.

$$a = x_0 < x_1 < \cdots < x_N = b$$
$$c = y_0 < y_1 < \cdots < y_N = d$$

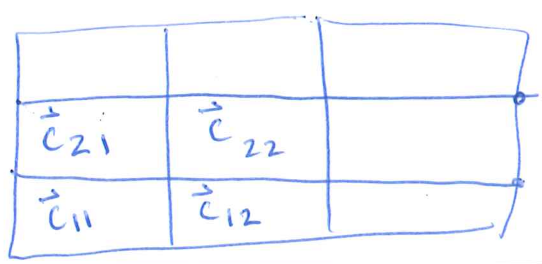
where $\Delta x_i = \frac{b-a}{N}$ ad $\Delta y_j = \frac{d-c}{N}$

We could make different amounts on x ad y-axes.

- Sample Points

$$\vec{c}_{ij} \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

The Picture



Example: A Riemann Double-Integral

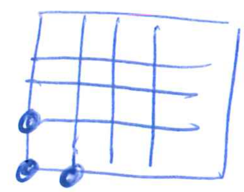
Integrate $\iint\limits_R 2x + 3y \, dxdy$ where $R = [0,1] \times [0,1]$ using a two-dimensional Riemann sum with lower-left-corner sample points.

We setup the Riemann sum.

$$\lim_{N\to\infty} \sum_{i,j=0}^{N-1} f(\vec{c}_{ij}) \Delta x_i \Delta y_j$$

$$\vec{c}_{ij} = (x_i, y_j) = \left(\frac{i}{N}, \frac{j}{N}\right)$$

$$= \lim_{N\to\infty} \sum_{i,j=0}^{N-1} f(x_i, y_j)\left(\frac{1-0}{N}\right)\left(\frac{1-0}{N}\right)$$



$$= \lim_{N\to\infty} \sum_{i,j=0}^{N-1} f\left(0+\frac{i}{N}, 0+\frac{j}{N}\right)\left(\frac{1}{N}\right)^2$$

$$= \lim_{N\to\infty} \sum_{i,j=0}^{N-1} \left[2\left(\frac{i}{N}\right) + 3\left(\frac{j}{N}\right)\right]\left(\frac{1}{N}\right)^2$$

$$= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left[\sum_{j=0}^{N-1} 2\left(\frac{i}{N}\right) + 3\left(\frac{j}{N}\right)\right]\left(\frac{1}{N}\right)^2$$

This is like $dxdy = dydx$

$$= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left[2\left(\frac{i}{N}\right)N + \sum_{j=0}^{N-1} 3\left(\frac{j}{N}\right)\right]\left(\frac{1}{N}\right)^2$$

Indep. of j.

$$= \lim_{N\to\infty} \sum_{i=0}^{N-1} \left[2(i) + 3 \cdot \frac{1}{N} \cdot \frac{(N-1)((N-1)+1)}{2}\right]\left(\frac{1}{N}\right)^2$$

See back of p. 97

$$= \lim_{N \to \infty} 3 \cdot \frac{1}{N} \cdot \frac{(N-1)((N-1)+1)}{2} \cdot N \cdot \left(\frac{1}{N}\right)^2$$

$$+ \sum_{i=0}^{N-1} 2(i) \left(\frac{1}{N}\right)^2$$

$$= \lim_{N \to \infty} \boxed{3 \cdots\cdots \left(\frac{1}{N}\right)^2}$$

$$+ 2 \frac{(N-1)((N-1)+1)}{2} \cdot \left(\frac{1}{N}\right)^2$$

$$= \lim_{N \to \infty} \frac{3}{2} \cdot \left(\frac{N^3 + \cdots}{N^3 + \cdots}\right) + \frac{2}{2}\left(\frac{N^2 + \cdots}{N^2}\right)$$

$$= \frac{3}{2} + \frac{2}{2} = \frac{5}{2}$$ ← The volume below

$z = 2x + 3y$

and above $[0,1] \times [0,1]$

is $V = \frac{5}{2}$.

Example: An Integral from Data

Suppose that you know the following data about a function.

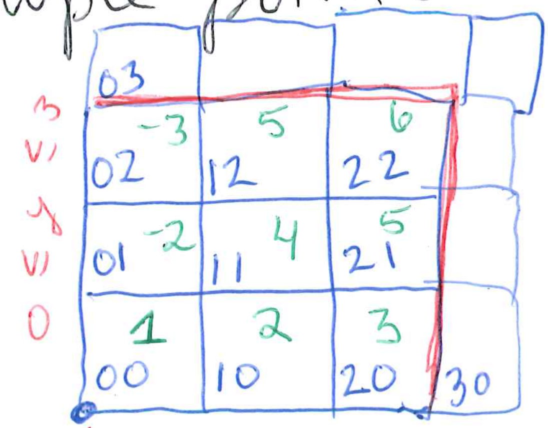
$f(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	1	2	3
$y=1$	−2	4	5
$y=2$	−3	5	6

The columns correspond to values of x and the rows correspond to values of y. The table shows the value of $f(x,y)$ at the point (x,y). For example, $f(0,2)=-3$ and $f(1,1)=4$. Use this data to estimate the following integral.

$$\int_0^3\int_0^3 f(x,y)\,dxdy$$

To estimate a Riemann sum from data:
Setup the Riemann sum such that the data points ARE the sample points.

$$\int_0^3\int_0^3 f(x,y)\,dx\,dy$$



$$\approx \sum_{i=0}^{2}\sum_{j=0}^{2} f(\vec{c}_{ij})\,\Delta x_i\,\Delta y_j$$

$0 \le x \le 3$

$\vec{c}_{ij}$ = lower-left corners

$$\approx \sum_{i,j=0}^{2} f(\vec{c}_{ij})$$

$\Delta x_i = \Delta y_j = 1.$

$$\approx (1+2+3)+(-2+4+5)+(-3+5+6) = 21.$$

Bounded Functions **(M&T p. 271)**

We say that a function $f(x, y)$ is <u>bounded</u> if there is a constant $M > 0$ such that $-M \leq f(x, y) \leq M$ for all points in the domain of f.

Example: Proving a Bound

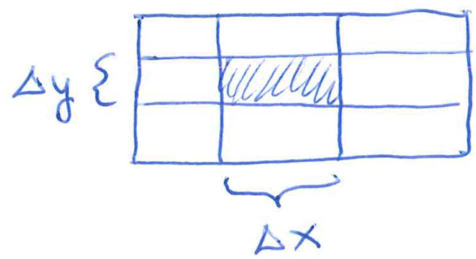
Suppose that R is a rectangle, $f(x, y)$ is integrable and bounded by $|f(x, y)| \leq M$ on R.

$$-M \text{ Area}(R) \leq \iint_R f(x, y) dA \leq M \text{ Area}(R)$$

Consider the Riemann sum with N^2 sub-rectangles

We have:

$$\sum_{i,j=0}^{N-1} f(\vec{c}_{ij}) \Delta x_i \Delta y_j$$



$$\leq \sum_{i,j=0}^{N-1} M \Delta x_i \Delta y_j = M \sum_{i,j=0}^{N-1} \Delta x_i \Delta y_j$$

This sum adds the area of All the sub-rectangles.

$$= M \text{ Area}(R)$$

The lower bound is similar. Taking the limit $N \to \infty$ gives:

$$-M \text{ Area}(R) \leq \iint_R f(x,y) dA \leq M \text{ Area}(R).$$

Properties of Integrals (M&T p. 275)

Suppose that f and g are integrable functions on R. (our setup has $f, g \geq 0$)

Linearity

$$\iint_R af(x,y) + bg(x,y)\ dA = a \iint_R f(x,y)\ dA + b \iint_R g(x,y)\ dA$$

Monotonicity If $f(x,y) \geq g(x,y)$ then:

$$\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$$

Additivity Suppose that R is the disjoint union of finitely many rectangles $R = \bigcup_{i=1}^n R_i$.

$$\iint_R f(x,y)\ dA = \sum_{i=1}^n \iint_{R_i} f(x,y)\ dA$$

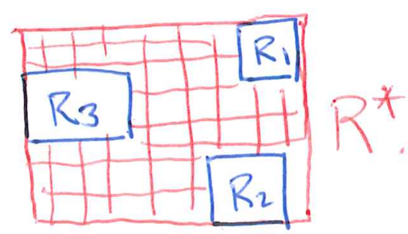
Activity: Think-Pair-Share (3 minutes)

Pick one of the properties above and convince your neighbour that it is true. 13:40

Let $R = \bigcup_{i=1}^n R_i$ be a disjoint union of rectangles.

We have that R is closed and bounded.

There is some R^* a rectangle which contains all R_i.



Consider the following

$$f^*(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in R \\ 0 & \text{if } \vec{x} \notin R \end{cases}$$

This is a bounded integrable on R^*.

See back of p.100.

We get:

$$\iint_{R^*} f^* \, dA = \lim_{N\to\infty} \sum_{i,j=0}^{N-1} f^*(\vec{c}_{ij}) \Delta x_i \Delta y_j$$

$$= \lim_{N\to\infty} \sum_{k=1}^{n} \sum_{\vec{c}_{ij}\in R_k} f^*(\vec{c}_{ij}) \Delta x_i \Delta y_j$$ # Split by rectangles

$$= \lim_{N\to\infty} \sum_{k=1}^{n} \sum_{\vec{c}_{ij}\in R_k} f(\vec{c}_{ij}) \Delta x_i \Delta y_j$$

$$= \sum_{k=1}^{n} \left(\lim_{N\to\infty} \sum_{\vec{c}_{ij}\in R_k} f(c_{ij}) \Delta x_i \Delta y_j \right)$$ # Finitely many rect.s.

$$= \sum_{k=1}^{n} \iint_{R_k} f \, dA$$ # Riemann sum on R_k

Iterated Integrals (M&T p. 267)

Suppose that $R = [a,b] \times [c,d]$ is a rectangle and f is integrable on R.
We apply Cavalier's principle, and get the following two results:
If we cut the volume perpendicular to the x-axis then

$$\iint_R f(x,y)\, dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

If we cut the volume perpendicular to the y-axis then

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

Example: An Integral by Slices

Calculate the following integral.

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\, dxdy$$

$$\int_0^{\frac{\pi}{2}} \left[-\cos(x+y)\right]_{x=0}^{x=\frac{\pi}{2}} dy$$

$$= \int_0^{\frac{\pi}{2}} \left[-\cos\left(\frac{\pi}{2}+y\right)+\cos(0+y)\right] dy$$

$$= \left[-\sin\left(\frac{\pi}{2}+y\right)+\sin(y)\right]_{y=0}^{y=\frac{\pi}{2}}$$

$$= \left(-\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)\right)-\left(-\sin\left(\frac{\pi}{2}\right)+\sin(0)\right)$$

$$= 2.$$