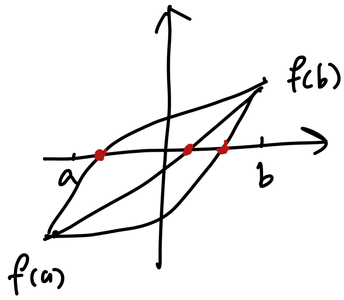
Midterm Review

1. Solution of Equations : solve for $f(x)=0$ for $x \in [a,b]$

① existence of such solution



- if $f(a)\cdot f(b) < 0$, then $\exists\ x \in [a,b]$ s.t. $f(x)=0$.
  (assume f is cts)

② Fixed point iteration.

Solving for $f(x)=0 \Leftrightarrow g(x)=x$, for suitable choice of g.

Given $x_0 \in [a,b]$, $x_{k+1} = g(x_k)$, $k=0,1,2,\cdots$

(2.1) Convergence of fpi

(strongest) If $\exists\ L<1$, s.t. $\forall\ x,y \in [a,b]$, $|g(x)-g(y)| \le L\cdot|x-y|$,

then it converges regardless of starting point.

If $g \in C^1$, then this is equivalent to $\sup\limits_{x\in[a,b]} |g'(x)| \le L$.

(weaker) Let x be the fixed point of g.

If $g \in C^1$ and $|g'(x)| < 1$, then it converges if we start sufficiently close to x.

Ex: Let $f(x) = xe^x - 1$. Show that $f(x)$ has unique root on $\mathbb{R}$.

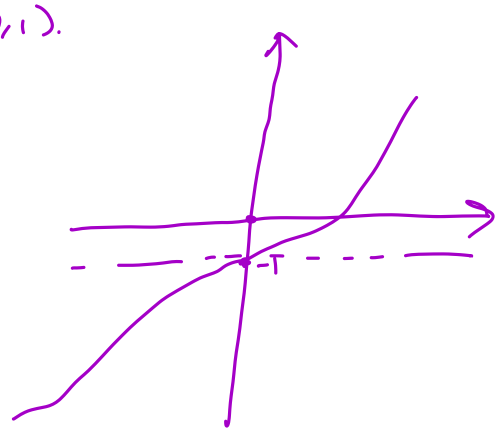
Sln: Existence of root: $f(0) = 0-1 = -1$

$f(1) = 1\cdot e^1 - 1 = e-1 > 0$.

Since $f(0)\cdot f(1) < 0$, $\exists$ a zero of f in $(0,1)$.

Uniqueness: For $x \le 0$, $xe^x \le 0$

$\Rightarrow f(x) = xe^x - 1 \le 0-1 = -1$



For $x>0$, $f'(x) = (xe^x-1)'$

$= e^x + xe^x = (x+1)e^x > e^x > 0$

$\Rightarrow f'(x) > 0 \Rightarrow f$ is always increasing on $(0,\infty)$.

$\Rightarrow f$ only crosses the x-axis once $\Rightarrow$ sln of $f(x)=0$ is unique.

$\varphi(x) = x - \lambda f'(x)$

ex: $x_{k+1} = \varphi(x_k) = x_k - \lambda f'(x_k)$, $\lambda > 0$, $f \in C^2$

$x_{k+1} = g(x_k)$

fixed pt.

$x = g(x)$

① What are fixed points of φ?

$x = \varphi(x)$

$x = x - \lambda f'(x) \Rightarrow f'(x) = 0$

② Assume $f''(x) > c > 0$ $\forall x$. What choice of λ will the fixed point iteration converges for any initial condition x_0?

Sln: We need $|\varphi'(x)| < 1$ $\forall x$

$$\varphi'(x) = 1 - \lambda f''(x) < 1 - \lambda \cdot c$$

$$\Rightarrow \text{need} \begin{cases} 1 - \lambda c < 1 & \Rightarrow \text{automatic} \quad (\lambda > 0,\ c > 0) \\ \underset{\sim\sim\sim}{1 - \lambda c > -1} & -\lambda c > -2 \Rightarrow \lambda < \frac{2}{c}. \end{cases}$$

④ Iterative method for root finding.

1° Newton's method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \cdots \qquad f'(x_k) \neq 0$$

If it converge, and $f'(x) \neq 0$, quadratic convergence speed.

- - - - - -, and $\underset{\sim\sim\sim}{f'(x)} = 0$, linear convergence.

x here satisfies $f(x) = 0$.

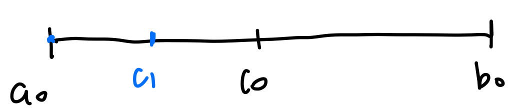
2° Secant Method.

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

3° Bisection method.

If $\exists$ a zero in some interval, then f evaluated at the interval endpoints have negative signs.

Idea: only focusing on intervals that have different signs at the endpoints. cut the interval half again.



2. Computer representation of number.

- fixed point: 0.000244 / 12345.789

- Floating point representation:

$$\begin{cases} 0.000244 = 2.44 \times 10^{-4} \\ 12345.789 = \underbrace{1.2345789}_{1 \le x < 10} \times 10^{4} \end{cases}$$

$$\underbrace{\pm}_{\text{sign}} \; \underbrace{m}_{\text{significand/mantissa.}} \times 2^{\overbrace{E}^{\text{exponent}}}, \quad 1 \le m < 2$$

- Machine precision: single: $2^{-23} \approx 10^{-7}$
  double: $2^{-52} \approx 10^{-16}$

3. Solutions of systems of linear equations.
   Solve the system $Ax = b$

① Existence of the solution:
   ↳ and unique if A is invertible.

② LU decomposition.

$$A = LU, \quad L = \begin{pmatrix} 1 & & & 0 \\ x & 1 & & \\ \vdots & & \ddots & \\ x & \cdots & x & 1 \end{pmatrix}, \quad U = \begin{pmatrix} x & x & \cdots & x \\ & \ddots & & \vdots \\ & & \ddots & \\ & & & x \end{pmatrix}$$

Given $A = LU$, $Ax = L\underbrace{Ux}_{y} = b$ $\begin{cases} Ly = b & \text{forward} \\ Ux = y. & \text{backward.} \end{cases}$

Cost of getting the LU: $\frac{2}{3}n^3 - \frac{1}{2}n^2 = O(n^3)$

--- solving forward/backward: $2n^2 + \cdots = O(n^2)$

Eg: Manually calculate LU.

$$A = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 4 & 0 \\ 0 & 4 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3/2 & 1 \end{pmatrix}$$

$$\frac{a_{21}}{a_{11}} = \frac{2}{4} = \frac{1}{2}$$

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix} \qquad \frac{a_{32}}{a_{22}} = \frac{4}{4} = 1$$

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix} \qquad \frac{a_{43}}{a_{33}} = \frac{3}{2}$$

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3/2 \end{pmatrix} = U$$

③ Pivoting: $PA = LU \iff A = P^T L U$ b/c $P^T = P^{-1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ swap the rows.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4. Norm and Conditioning.

Given $Ax=b$, how sensitive is x to changes in A / b?

① Conditioning: Condition numbers of matrix A.

↳ small changes in b could cause large changes in x.

$$A(x+\delta x) = b + \delta b.$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \le K(A) \cdot \frac{\|\delta b\|}{\|b\|}$$

↑ condition number of A.

$K(A) = \|A\| \cdot \|A^{-1}\|$, the choice of norm is 1-norm, 2-norm, ∞-norm.

② Is this a norm?

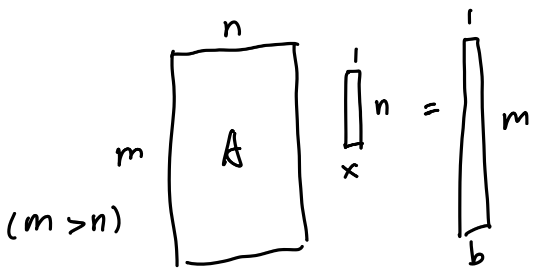
Is $|x_1|$ a norm? for $x \in \mathbb{R}^2$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

No. For norm, $\|x\| \ge 0$, and $\|x\| = 0$ iff $x = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

let $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $|x_1| = 0$, but $x \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

5 Least Squares

Solve the over-determined system $Ax=b$



① Least square formulation: $\min_x \|Ax-b\|_2$

$$\Leftrightarrow \min_x \|Ax-b\|_2^2$$

$$\Leftrightarrow \min_x \frac{1}{2}\|Ax-b\|_2^2$$

② Normal equations.

Sln of least square method solves the normal equation

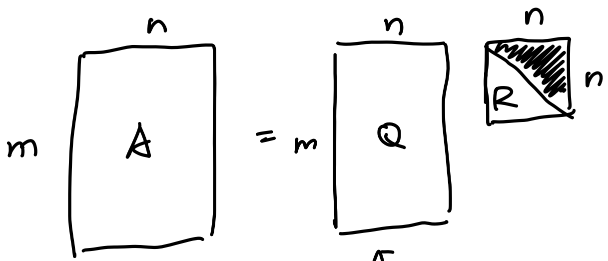
$$A^T A x = A^T b$$

→ if A is full rank, i.e, $\text{rank}(A) = n$, then $A^T A$ is invertible,

and $\exists$ unique sln x.

6. QR factorization

↖ done on rectangular / square matrix.



↖ Q has orthonormal columns.

If A is square, then Q is also square, Q is orthogonal,

$$Q^T Q = Q Q^T = \underline{I}$$

$$\Rightarrow Q^T = Q^{-1}$$

· obtaing QR: Gram Schmidt on columns of A.